Numerical Methods-Lecture IX: Reinforcement Learning

(See Powell Chapter 10)

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Motivation

- Previously, this lecture was devoted to quadrature, and specifically how to integrate sampling from Chebyshev nodes
- While valuable, it may be going out of style relative to modern methods
- Instead, we'll talk about Reinforcement Learning
- Could have had a whole course, but I want to give you the motivation and a crash course

Neural Networks

- Neural networks, for our purposes, are just very very flexible nonlinear functional forms
- Have y^{data} and want to fit $f(x|\theta)$ to it
- One way of producing a flexible function with easy derivatives (for fitting) is to stack logit functions
- Idea: $X = [x_1, x_2, ...]$ gets fed into multiple logits $f_1(X)$, $f_2(X)$, $f_3(X)$, each with its own logit weight on X: β_1^f, β_2^f, β_3^f.
- The outputs of these three $(+)$ logits then (possibly) get fed into another set of logits, $g_1(f_1, f_2, f_3)$, $g_2(f_1, f_2, f_3)$, each with their own set of weights β_1^g...
- Eventually these are summed up or scaled to a single (or multiple!) output, such as y^{fit}
- For most uses, NN just find θ like we would in any fitting problem, but use gradient descent (chain rule!) rather than Newton's method (too many cross-derivatives + many local minima)

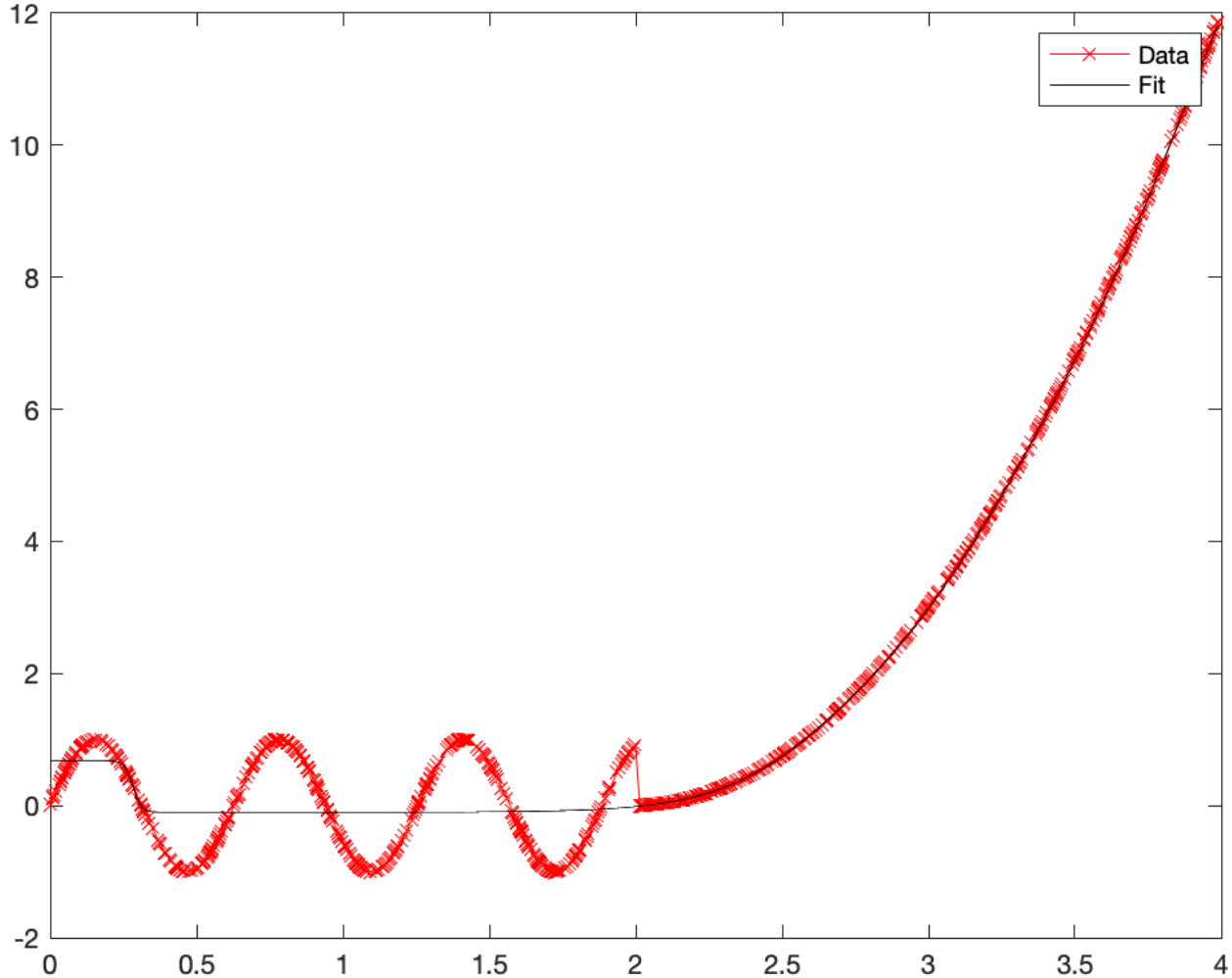
Neural Networks

- Concrete example: want to approximate $f(x)$
- Suggestion: one set of five logits, whose sum is then scaled:

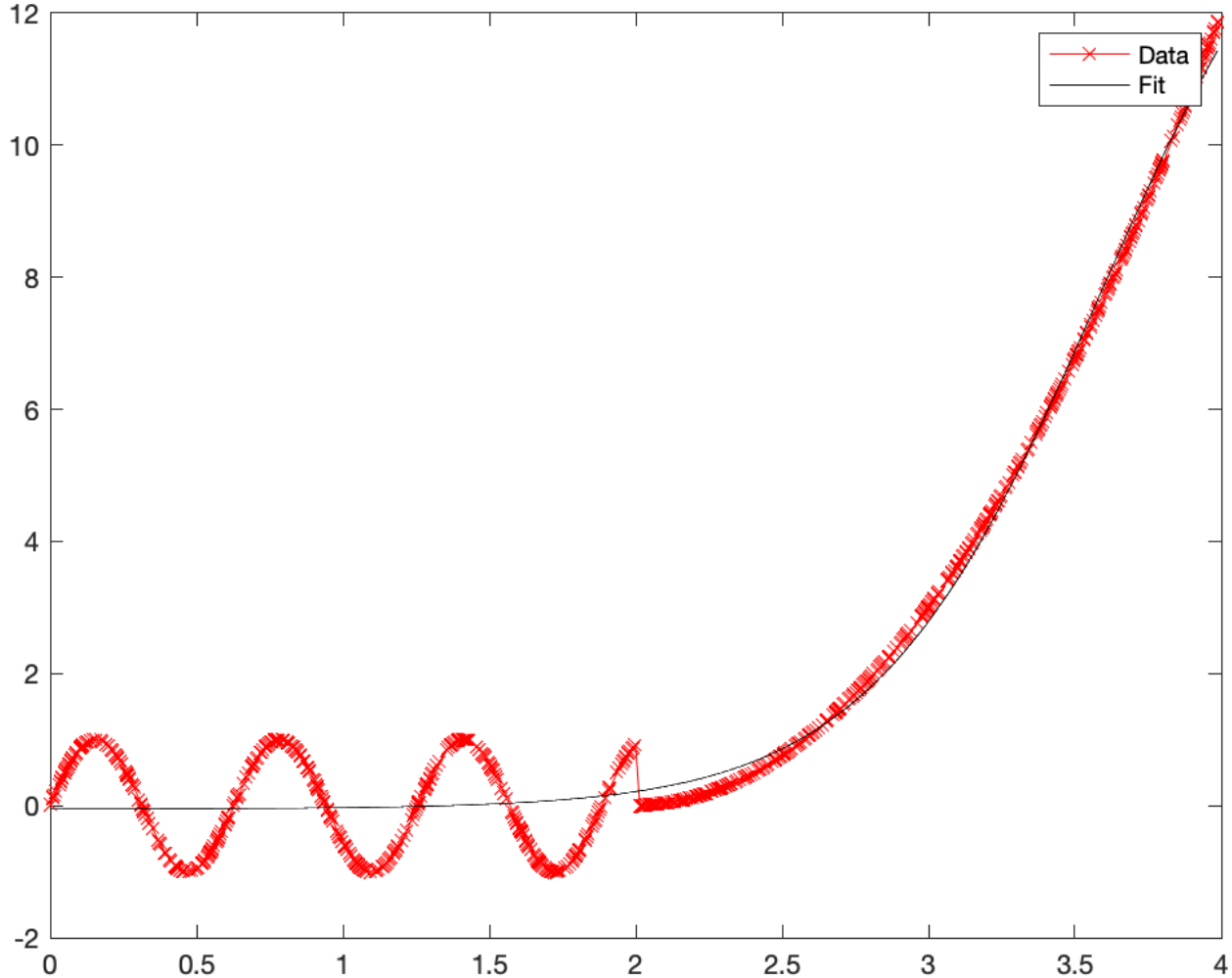
$$f(x) \approx \alpha_{L2} + \left(\sum_{i=1}^{5} \beta_{L2,i} \frac{1}{1 + e^{-(\alpha_{1,i} + \beta_{L1,i} x)}} \right)$$

- A little hard to read!
- See Simple_5_Main.m, and then Simple_N_Main.m for N logits

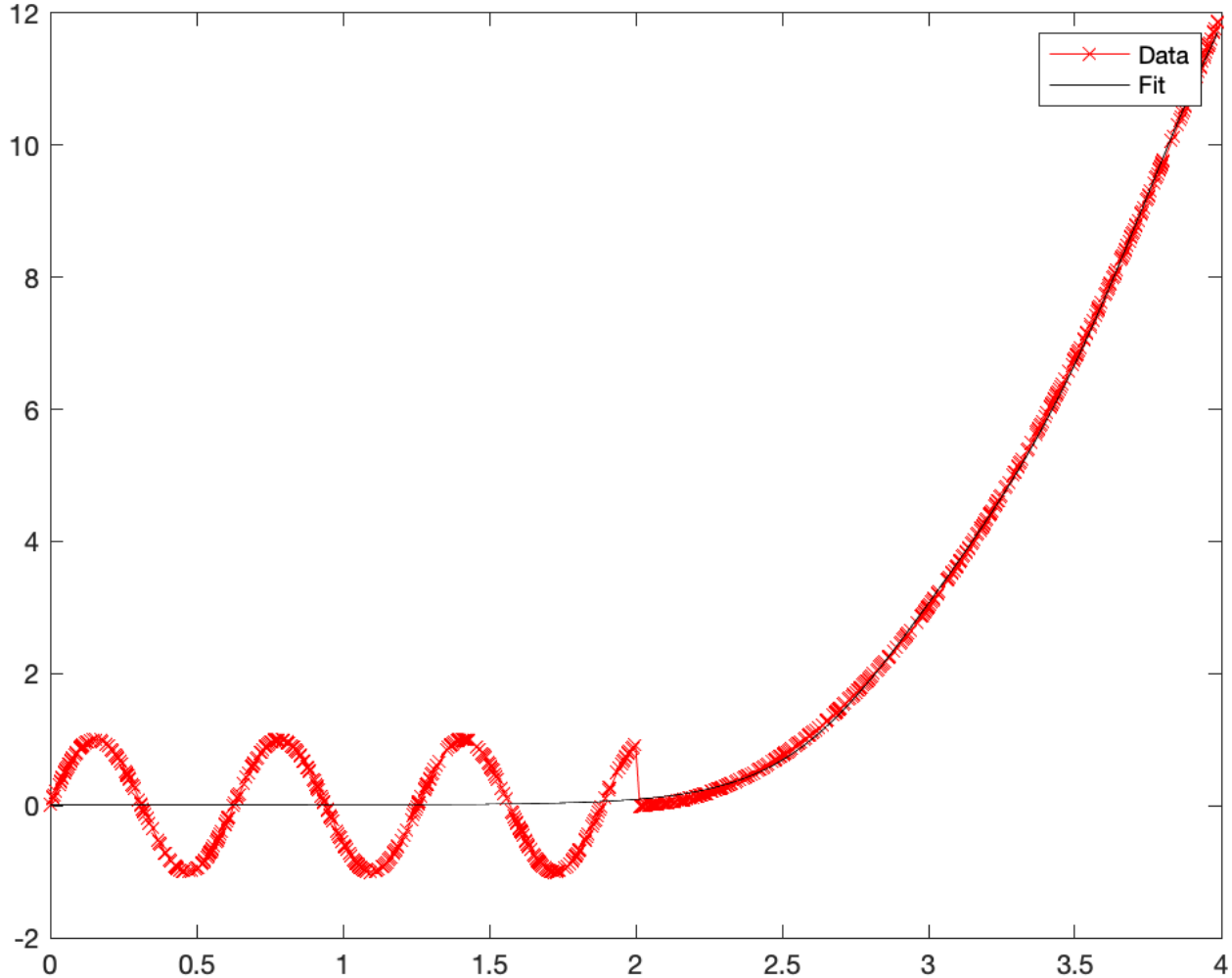
Neural Networks: 5 Logit+Linear Approx



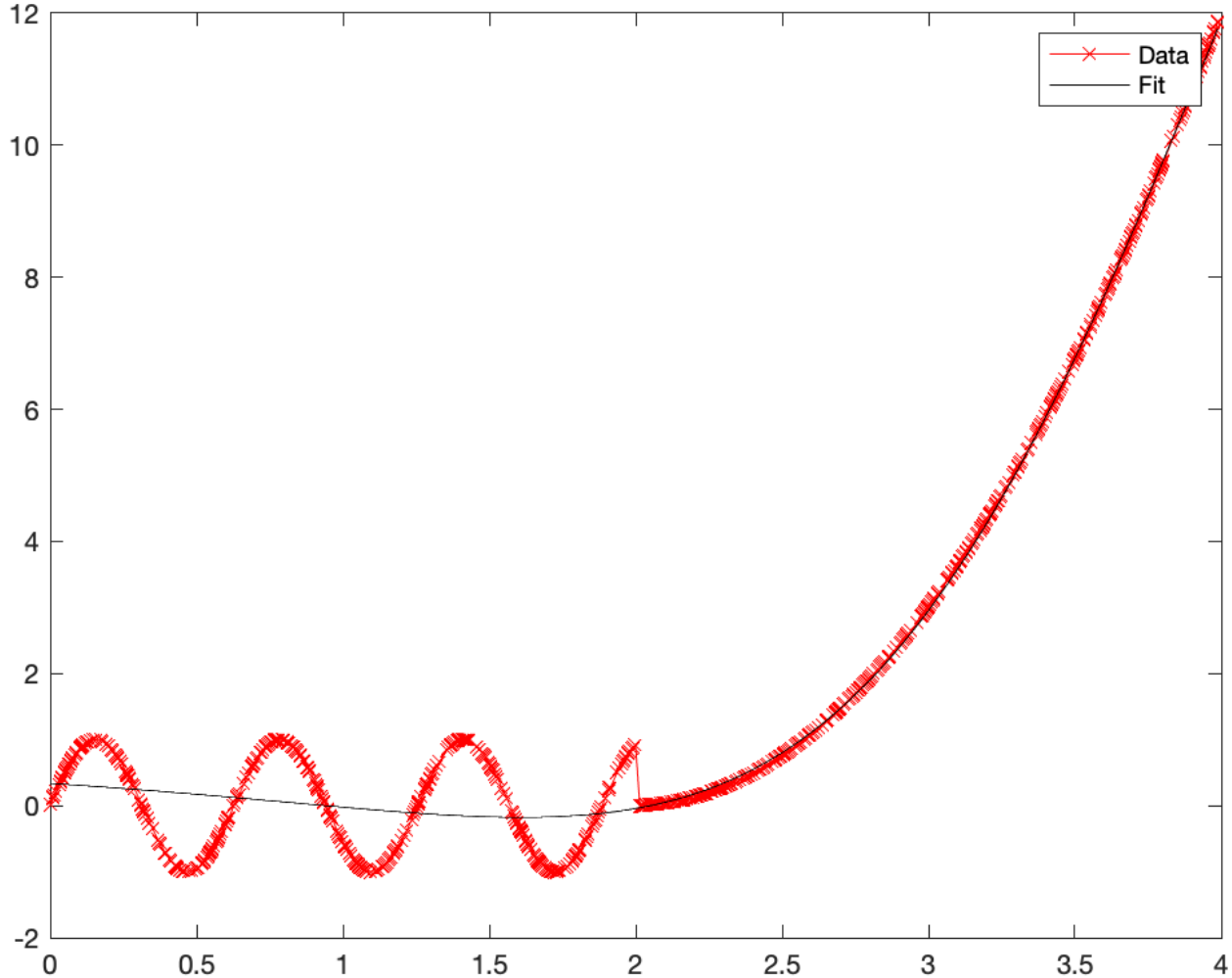
Neural Networks: 1 Logit+Linear Approx



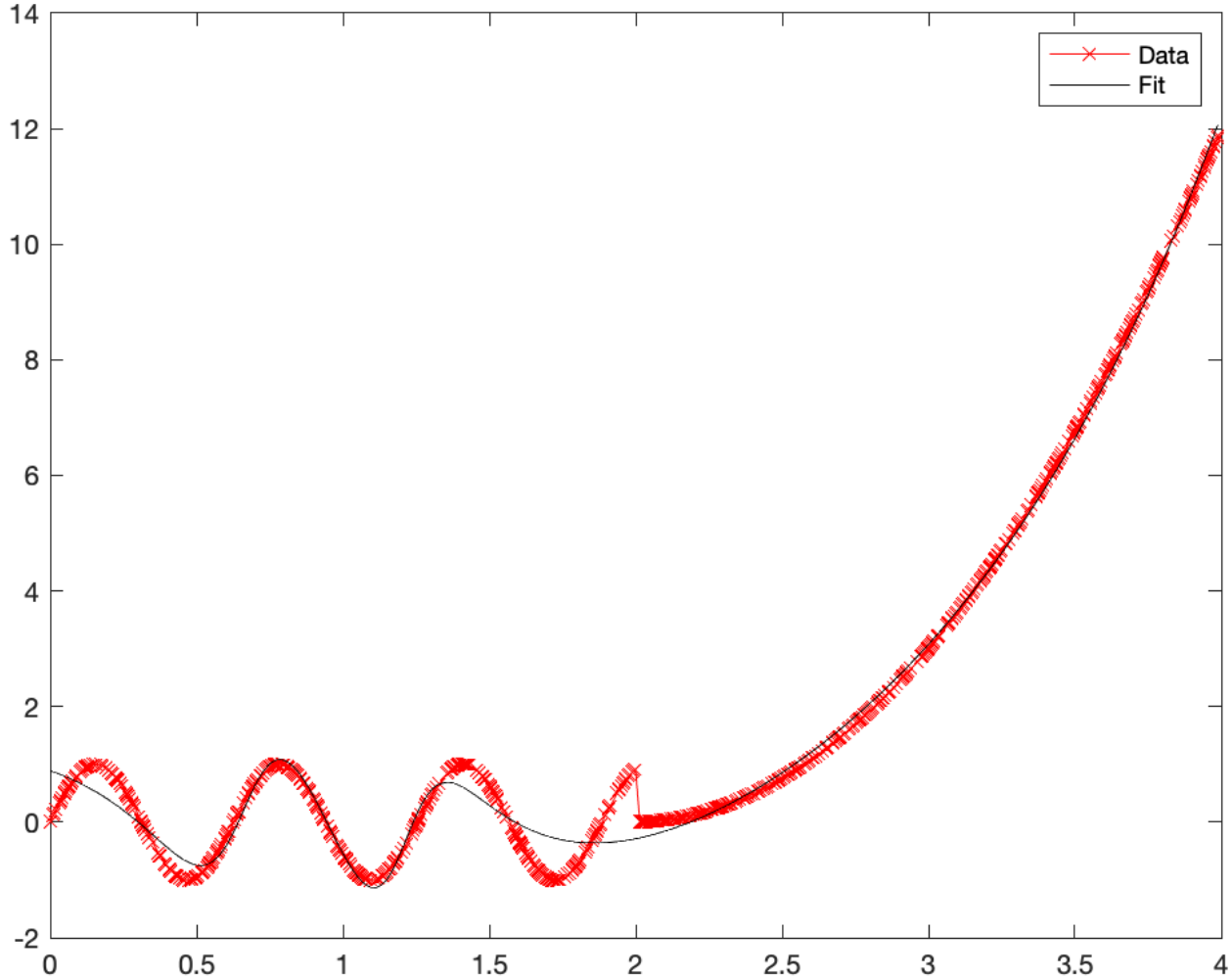
Neural Networks: 2 Logit+Linear Approx



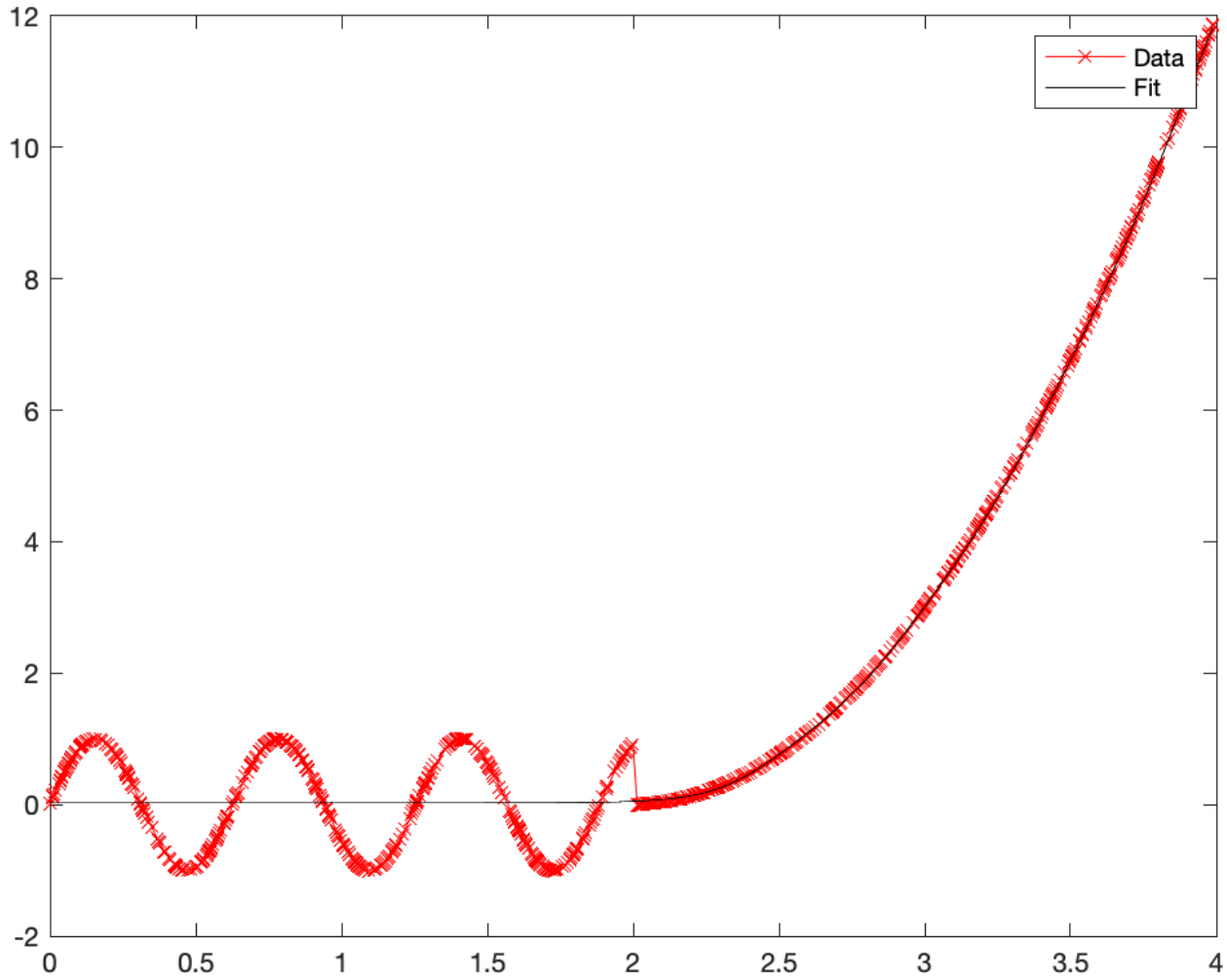
Neural Networks: 3 Logit+Linear Approx



Neural Networks: 4 Logit+Linear Approx

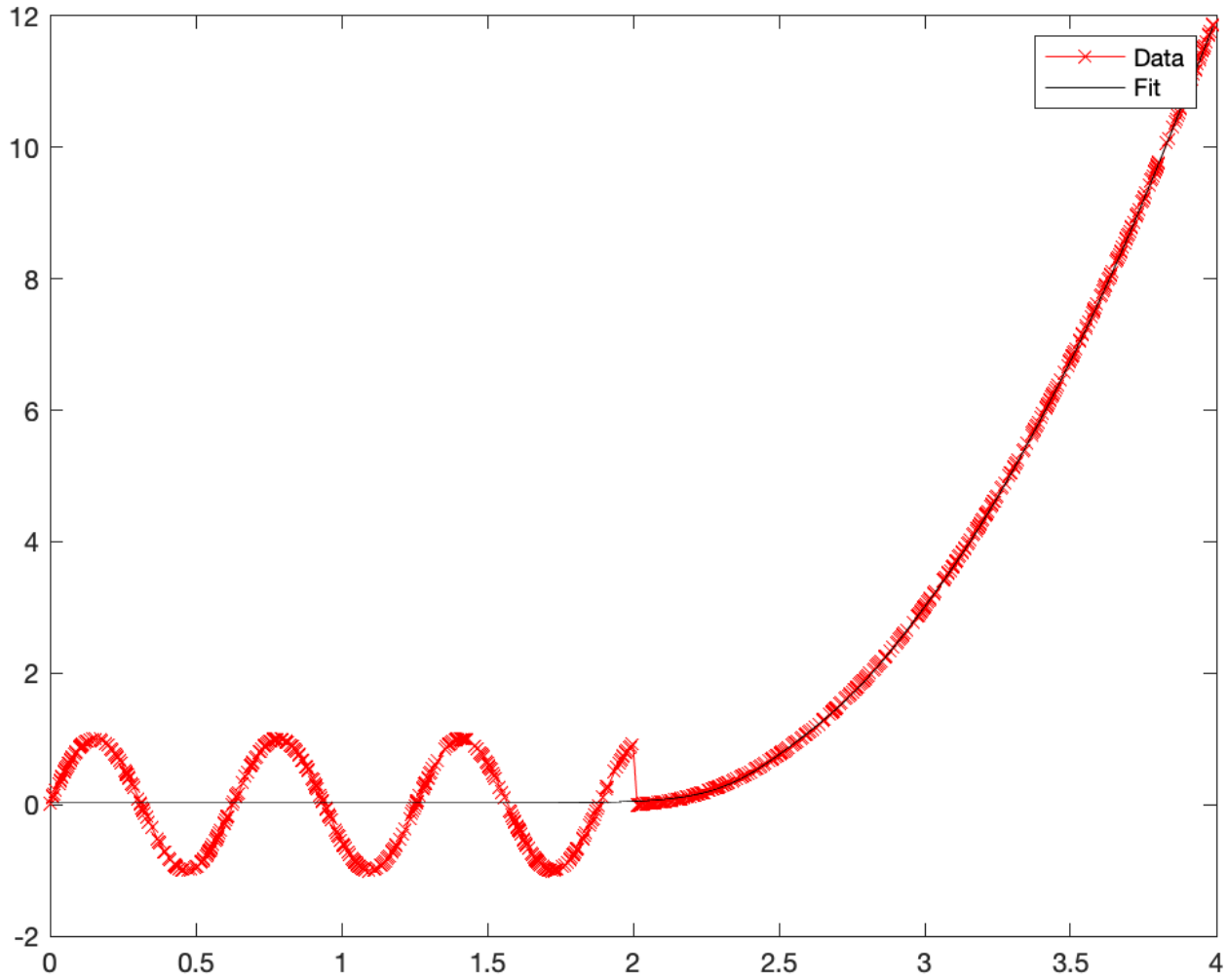


Neural Networks: 6 Logit+Linear Approx



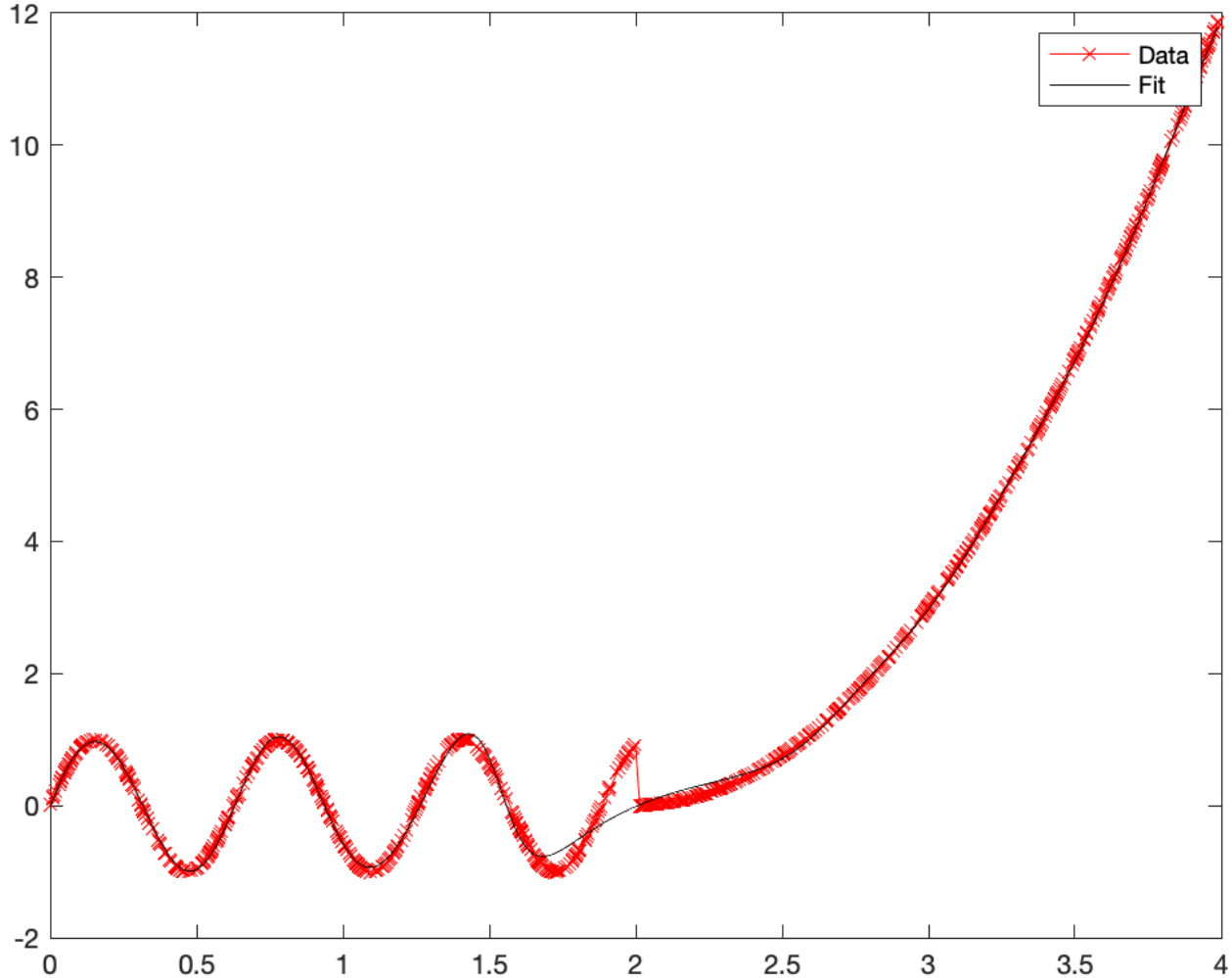
Got stuck at local minima! (Bad!)

Neural Networks: 7 Logit+Linear Approx

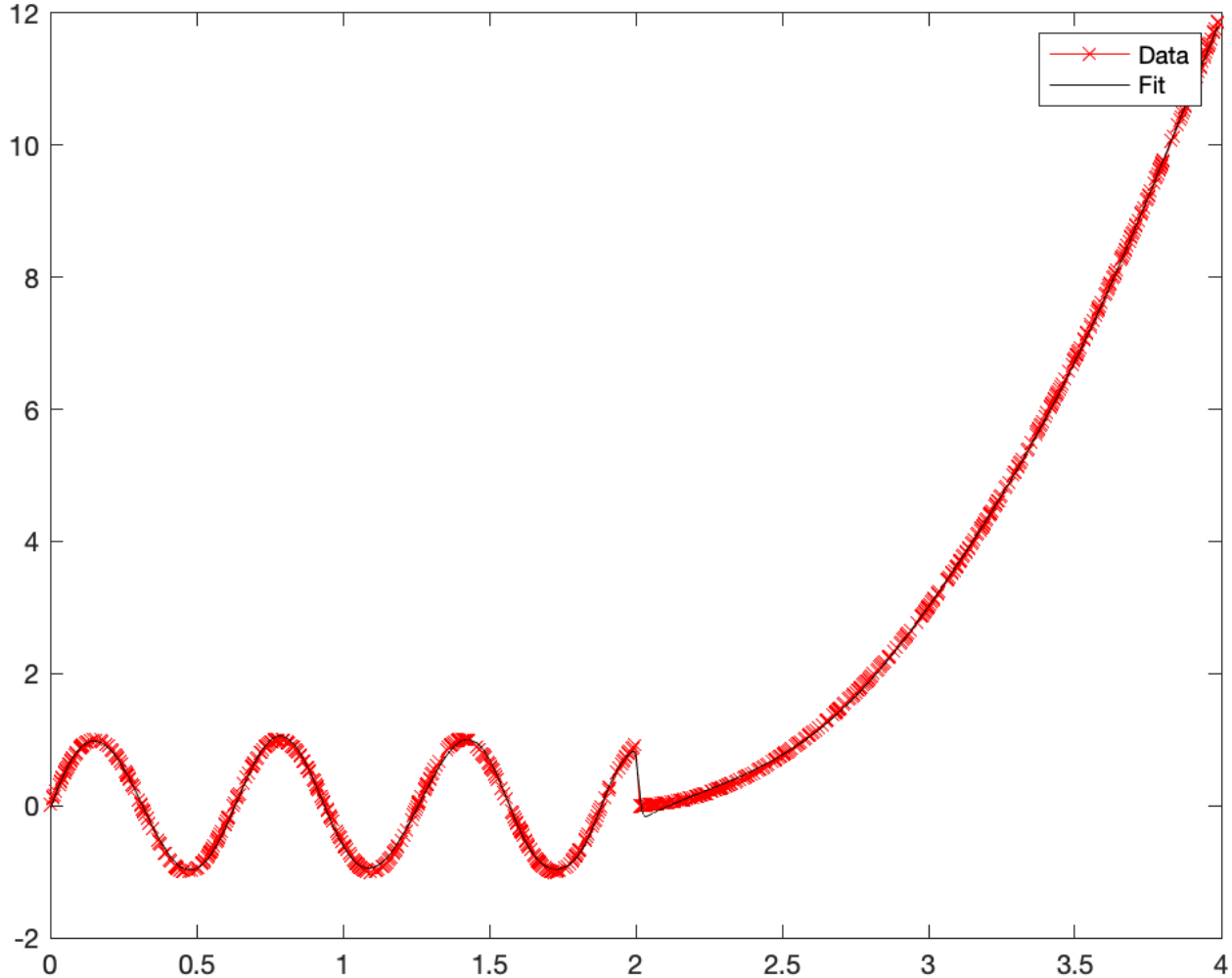


Got stuck at local minima! (Bad!)

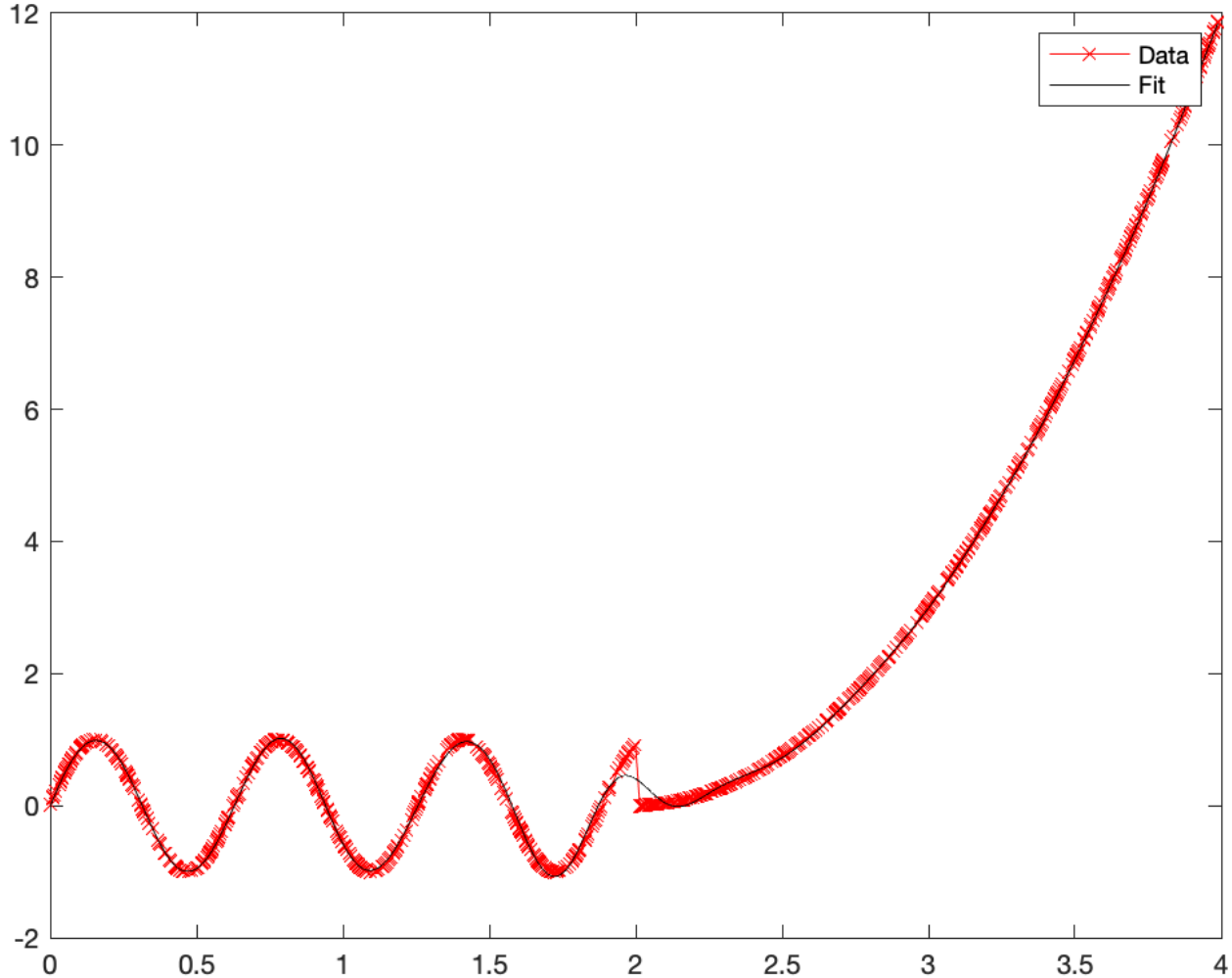
Neural Networks: 8 Logit+Linear Approx



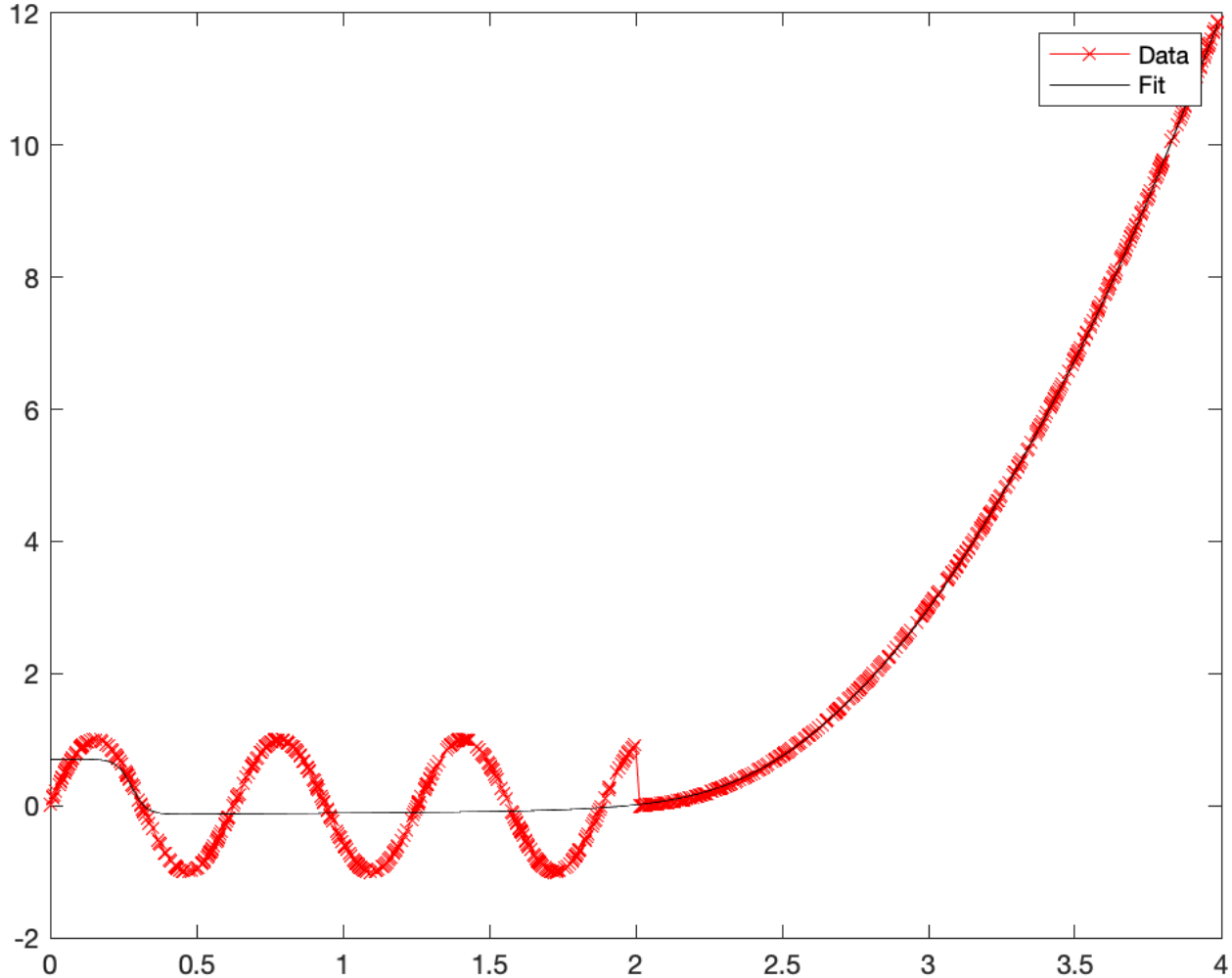
Neural Networks: 9 Logit+Linear Approx



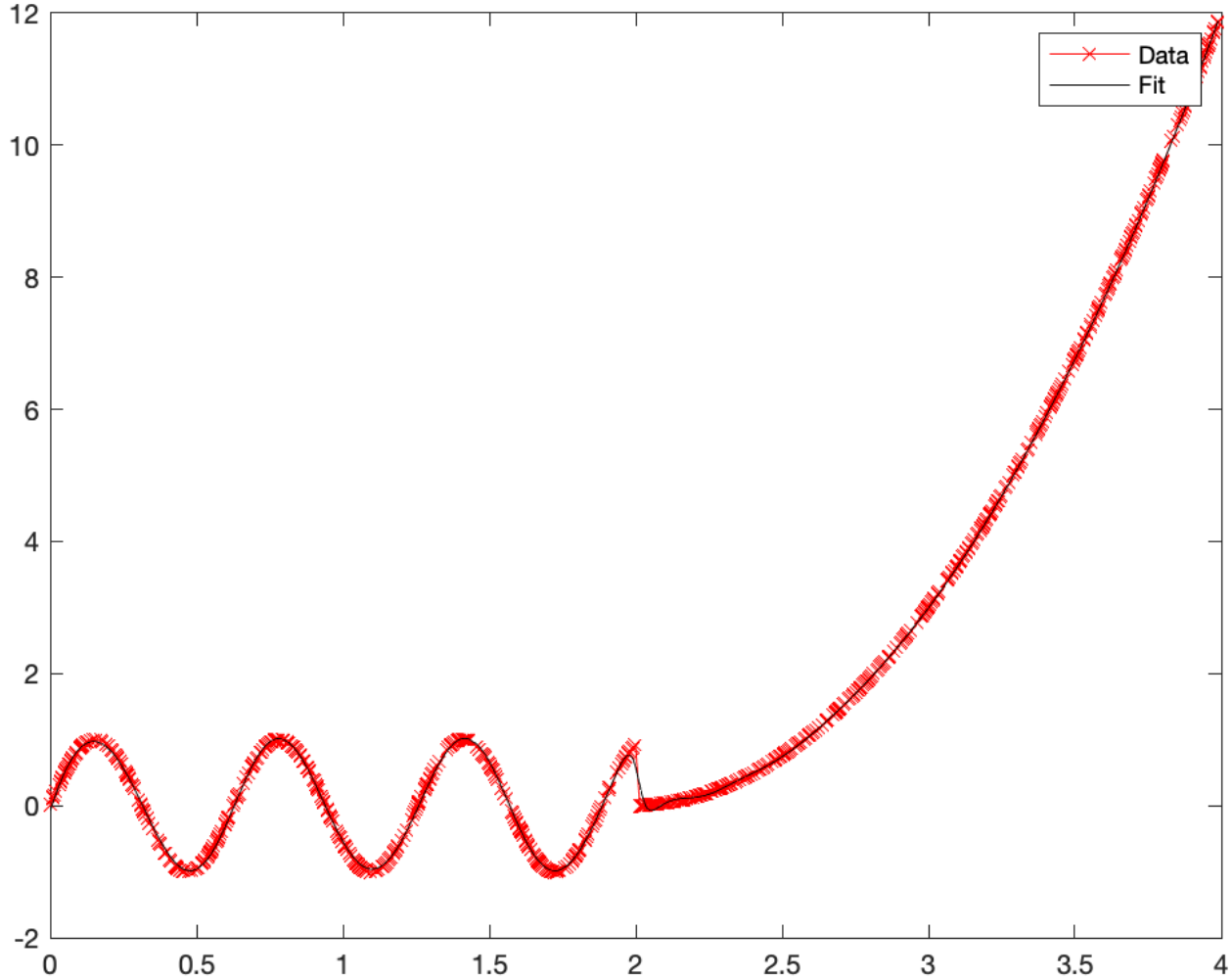
Neural Networks: 10 Logit+Linear Approx



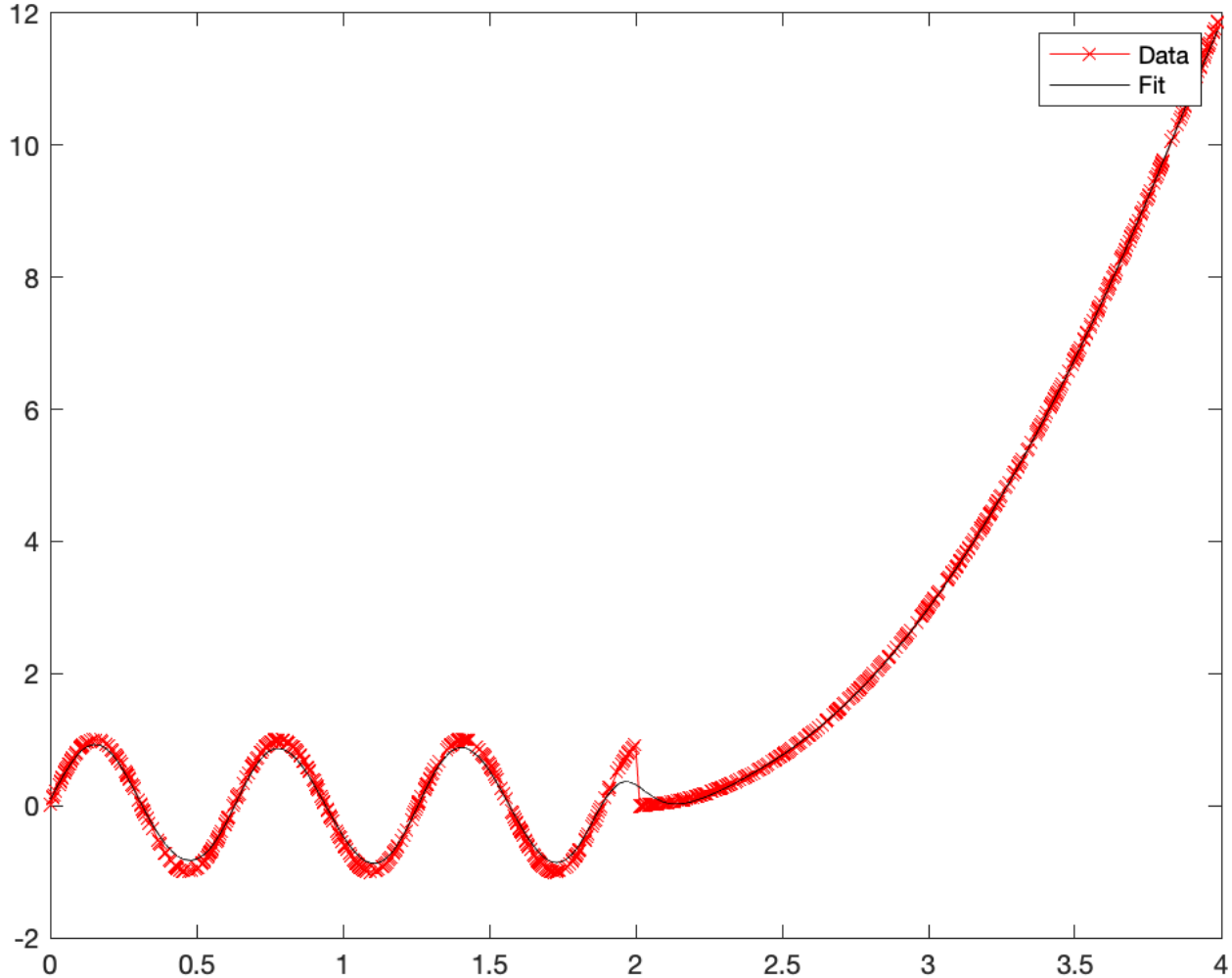
Neural Networks: 11 Logit+Linear Approx



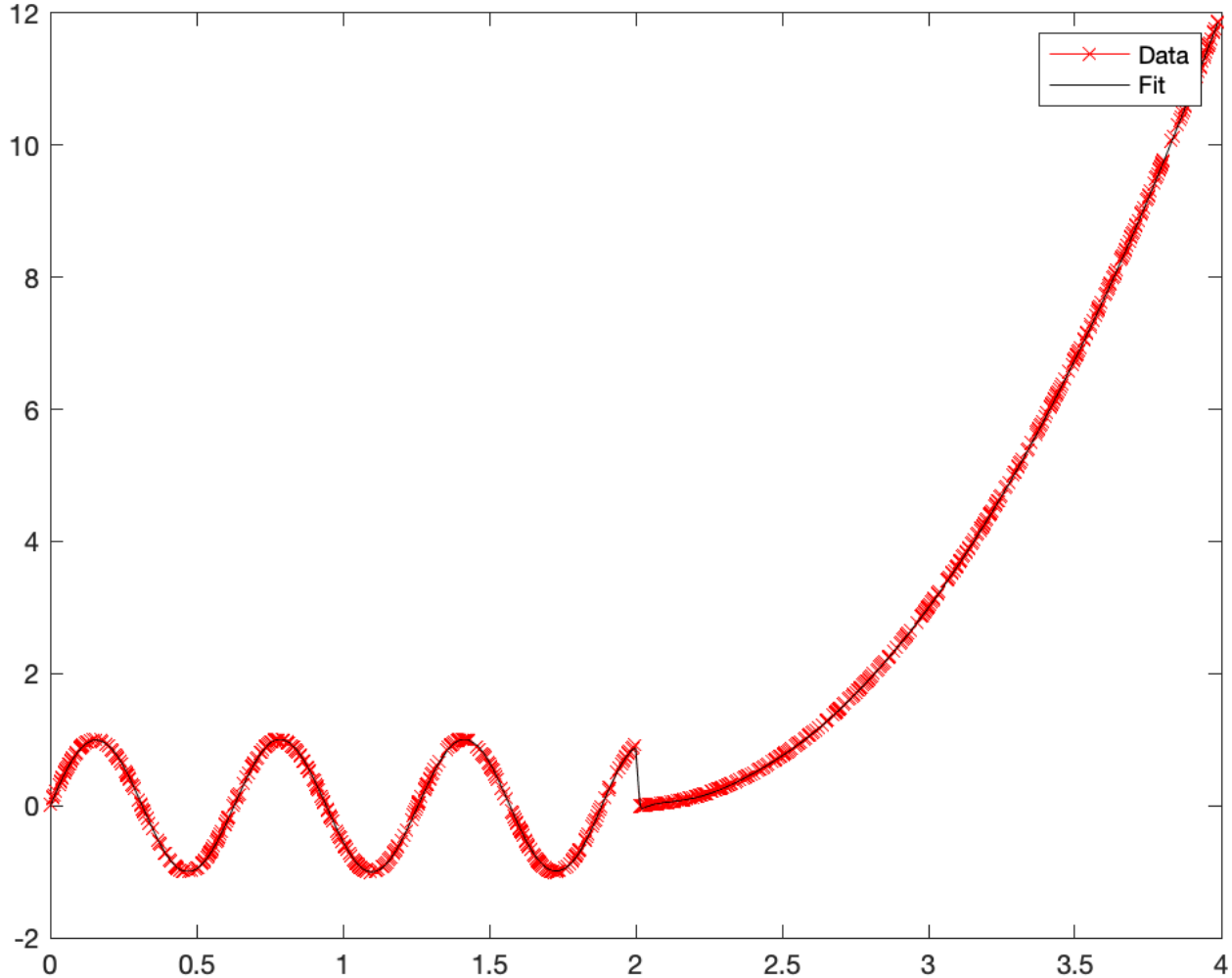
Neural Networks: 12 Logit+Linear Approx



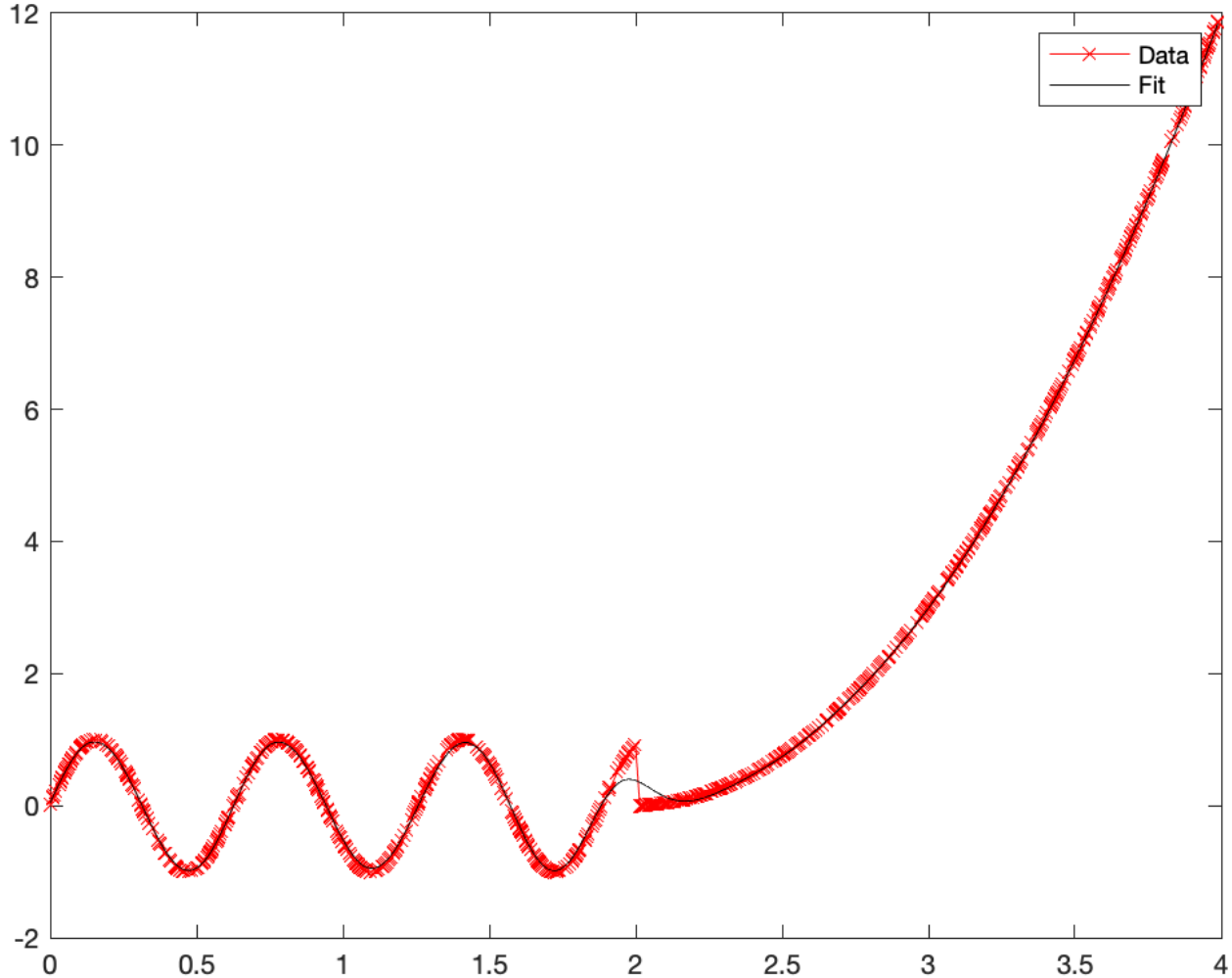
Neural Networks: 13 Logit+Linear Approx



Neural Networks: 14 Logit+Linear Approx



Neural Networks: 15 Logit+Linear Approx



Deep Learning

- I think of Neural Networks as really flexible sets of if statements combined with linear functions.

- Logits act as "if's"

- But could have a doubly-stacked if statement:
  - First layer:

$$f_{a,1}(x_1, x_2) = 1 \iff x_1 > 3$$

$$f_{a,2}(x_1, x_2) = 1 \iff x_2 < 1$$

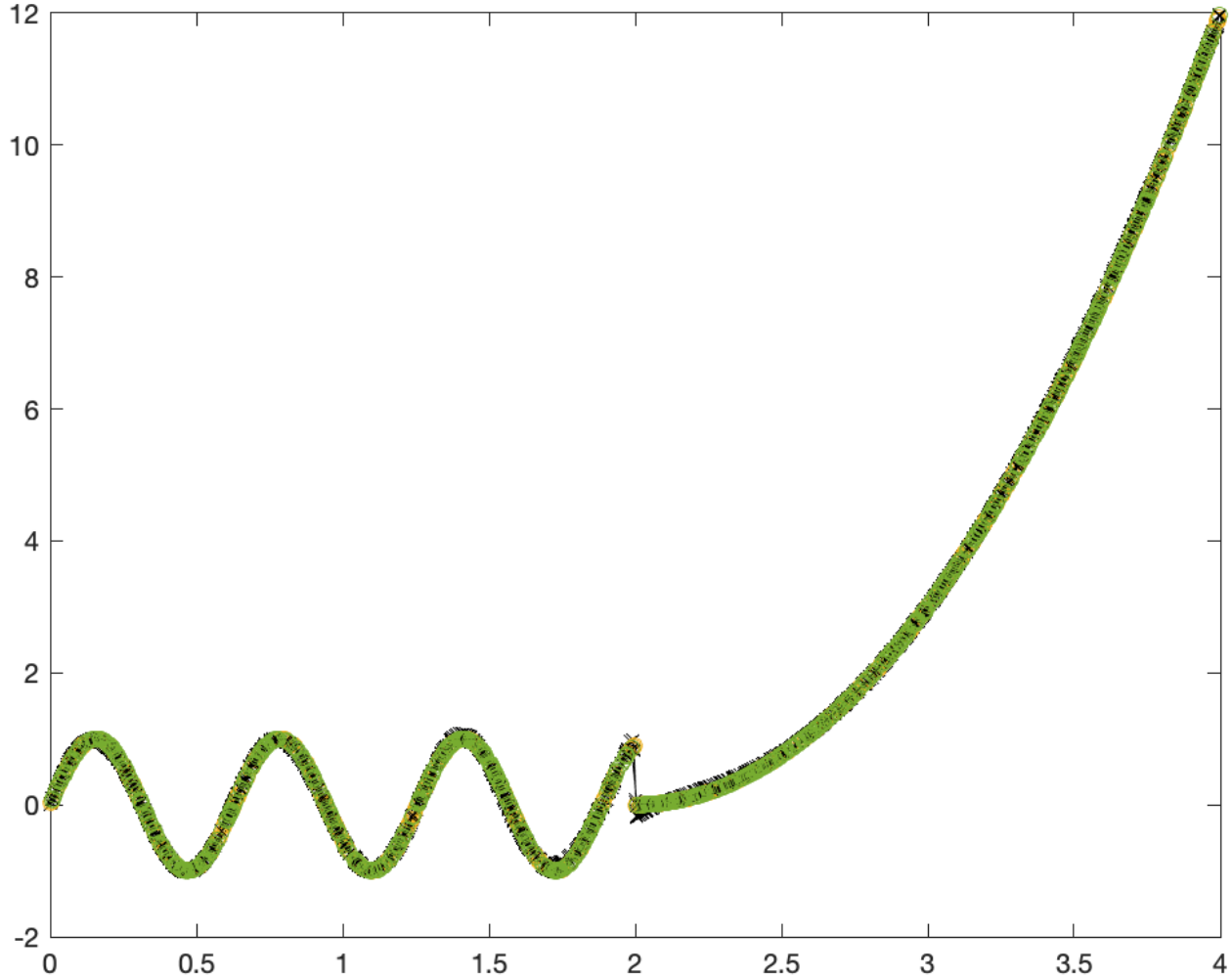
  - Second layer:

$$f_b(f_{a,1}, f_{a,2}) = 1 \iff f_{a,1} == 1 \ \& \ f_{a,2} == 1$$

- Idea: multiple layers give ability to encode complex "if" statements "easily"

- Let's see my stacked version! Deep_N_Main.m

Neural Networks: 5x5 Logit+Linear Approx



Neural Networks

- Can generalize this, add more layers of logits or other functions
- Because minimized with gradient descent, we want things that we can use chain rule on
- Because multiple minima, want things that are bounded
- Some common ones:
  - "Rectifier" $f(x) = max(0, x)$ (helps for piecewise)
  - Hyperbolic tangent $tanh(x)$ (similar to logit/"sigmoid", but -1 to 1)
  - Scaling layer: $\alpha + \beta X$
  - Softmax/"categorical distribution": vector output, takes vec x and spits out $\frac{exp(x_i)}{\sum_j exp(x_j)}$
  - Less relevant in most econ: convolution, pool "nearby" observations
  - LSTM: let value today affect fits for tomorrow (so $f(X_2)$ (X matrix at $t = 2$) is a function of X_2 and X_1), all the way back
  - Dropout layer: randomly drop inputs (train drunk!)s

Matlab: build in shallow NN

- Matlab has a lot of built-in neural network stuff
- Unfortunately(?) this is an evolving field, and so commands are changing, becoming obsolete, etc.
- Can be frustrating, but equivalent to Theano dying, Google transitioning from TensorFlow to JAX(?)
- But can be as simple as:

```
net = fitnet(netsize);
net = train(net,Xdata,Ydata,'UseParallel','yes');
```

- Just like we write before, a bunch of logits that are netsize, give it X and Y data, minimizes.
- Great for predicting, obviously **not solution for causality** (sorry micro ppl)

Example to fit: $x.^2 + y.^2 - 2sin(x. * y)$

- See Fitnet Example/Main.m for details!

The Three Curses of Dimensionality

- The three curses of dimensionality

  1. As n_{states} proliferates, # problems to solve explodes for VFI
  2. As $n_{actions}$ proliferates, # choices to compare explodes
  3. As $n_{outcomes}$ proliferates (particularly stochastic), space to integrate over explodes

$$V(x) = \max_{y \in \Gamma(X)} \left\{ F(x, y) + \beta E(V(x'(y))) \right\}$$

- This is a problem if you're trying to model lifecycle behavior...age, permanent wage, transitory wage, marital status, age of kids, occupation, health, etc.

- How can we solve?

One way to approximately solve the problem

- There are many flavors of solution, but we'll focus on the "Actor-Critic" method
- Have two functions, parameterized by θ and ϕ:
  - Actor function $\overline{\pi}(y|x;\theta)$ takes in state and spits out an action (possibly probabilistically)
  - Critic function $\overline{V}(x|\phi)$ takes in state and spits out value (traditional value function)
- We can represent Actor & Critic as flexible neural networks parameterized by θ and ϕ but how get values to fit?
- Good actor function embeds both reward and stochastic future (actions and integral!)
- *Given* θ and ϕ, can simulate an agent, get data to fit
- Need to find θ and ϕ

Actor-Critic Algorithm (Discussion)

- Start with a guess for θ and ϕ, and an initial state value
- Simulate the system many times (random draws & laws of motion for stochastic problems)
- Now we have a bunch of paths for a given θ and ϕ
- For every step t, compute the return G_t, sum of reward and discounted future reward, calculated with $\overline{V(x_{t+1})}$
- Calculate the "advantage function" $D_t = G_t - V(S_t|\phi)$, value of action vs value of what we think is best action embedded in V
- Calculate gradients for actor and critic networks:

$$d\theta = \sum_{i=1}^{N} \nabla)_\theta \log(\pi(A|x_t;\theta) D_t$$

$$d\phi = \sum_{t=1}^{N} \nabla_\phi (G_t - V(x_t;\phi))^2$$

- Idea: ∇_θ pushes us in direction of better choices/happiness, ∇_ϕ pushes us in direction of lower error in value function
- Update the parameters of the functions:

$$\theta = \theta + \alpha d\theta$$

$$\phi = \phi + \beta d\phi$$

- So we update the actor π using critic V, and update V using simulation

Explaining to Mum-I

- Start out with some surface that represents best action given state, and some surface that represents the value of being in that state
- Simulate
- Change value surface by comparing data with what you actually got (using initial surface for future, so change is slow), trying to match surface
- Change action surface by trying to increase received value vs value at best guess (small perturbations toward better actions)
- Repeat in tiny steps

Idea

- We try to toddle slowly to both how to evaluate our situation (V) and what to do (π)
- We learn about value function by exploring the space
- We learn about maximization (actor) by exploiting V and our actual actions
- We learn about expectation by simulation–enough simulations and we will explore the relevant space
- Additional cool (but dangerous) aspect: we only explore relevant functions of state space
- But how choose V and π?

Choosing V and π

- V and π could be any functional form (e.g. linear $V = \alpha + \beta \sum_{j=1}^{N} \phi_j x_j$)
- But we want an extremely functional flexible form
- Neural networks are (typically) just simple stacked functions interacted with one another–think very flexilble functions, with many parameters
- Advantage of NN-style flexibility is we can spend degrees of freedom in complicated areas and not in simple areas (as in sparse interpolation)
- I'm giving this short shrift (sorry), but we'll put together a neural network for V and π in Matlab

Problem to Solve

- We'll solve a simple finite-horizon NCG-style problem:

$$V(A, K, t) = \max_{K'} \left\{ \log(AK^{\alpha} - K') + \beta E(V(A', K', t+1)) \right\}$$

$$A' = (1 - \rho) + \rho A + \epsilon, \quad \epsilon \sim \mathcal{N}(0, 0.05)$$

- How will we set this up?

- Define a set of functions that:

  - A function that sets up actor and critic neural networks, defines observations, and trains (Main.m)

  - Initialize an agent (including random draws) (myResetFunction.m)

  - Step an agent forward in time, simulate draws (myStepFunction.m)

Reset Function

```
function [InitialObservation, LoggedSignal] =
myResetFunction()
 % Initial values
 S.K = 250+(rand(1,1)-0.5)*200;
 S.A = 1+rand(1,1)*0.05;
 S.step = 1
 % Return initial environment state variables as
logged signals.
 LoggedSignal.State = [S.K;S.A;S.step];
 LoggedSignal.C=NaN;
 InitialObservation = LoggedSignal.State;
```

Step Function

- See myStepFunction.m
- This is the simulator, it takes in signals and an action and simulates the environment, returns the observations, rewards, and whether or not it is done
- Note: logged signals are known to Matlab, but not agent. Observations are known to agent. In this simulation, they are the same thing. (But could have had hidden state)

Main.m

- See main.M
- Idea: set up a flexible function for the actor and critic, and then send to trainer

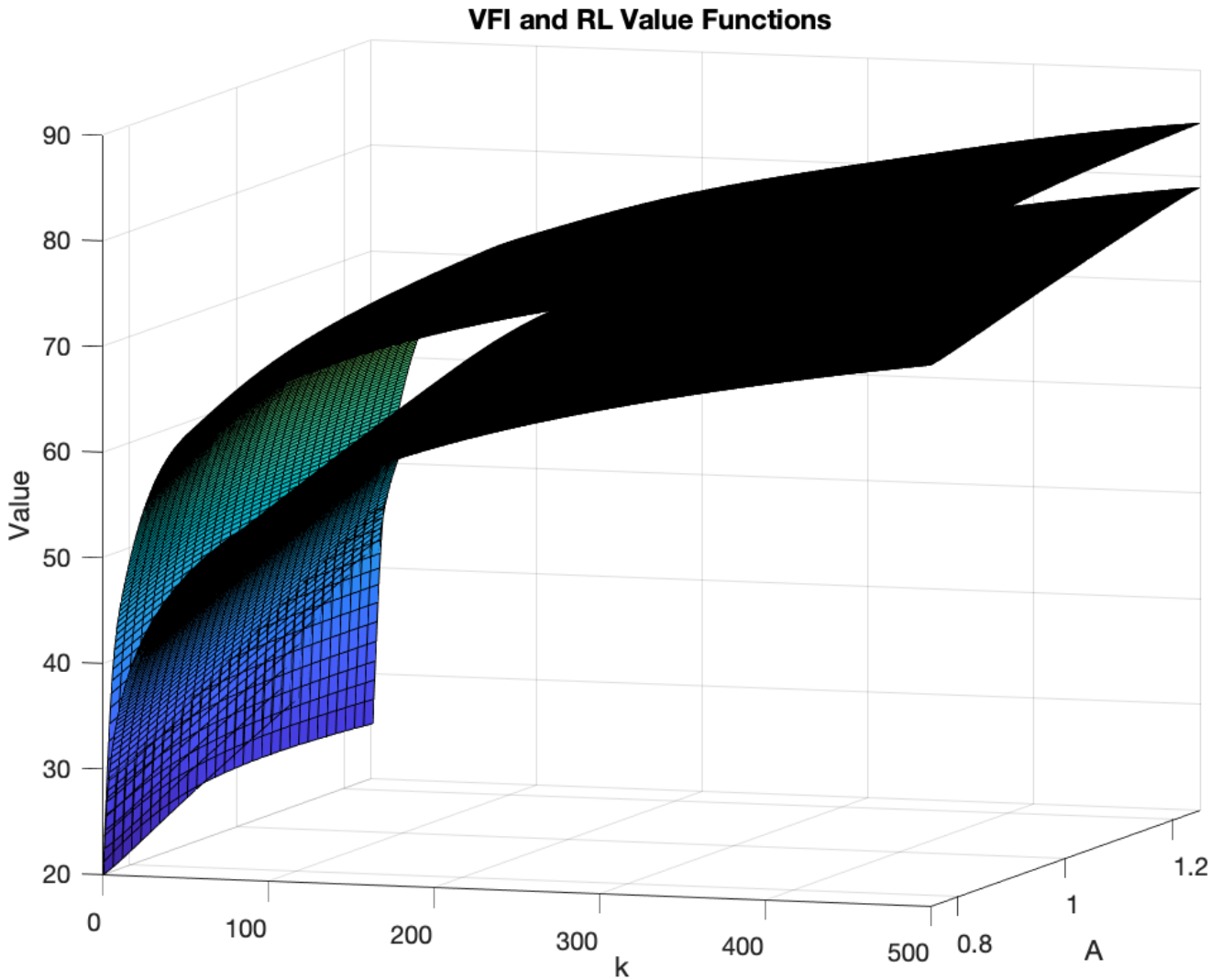
Comparing to Traditional VFI

- There are many options to maximize, can take more time, etc. but let's get a ballpark idea of how well this can do
- Hard to see, so let's look at GraphDiff.m

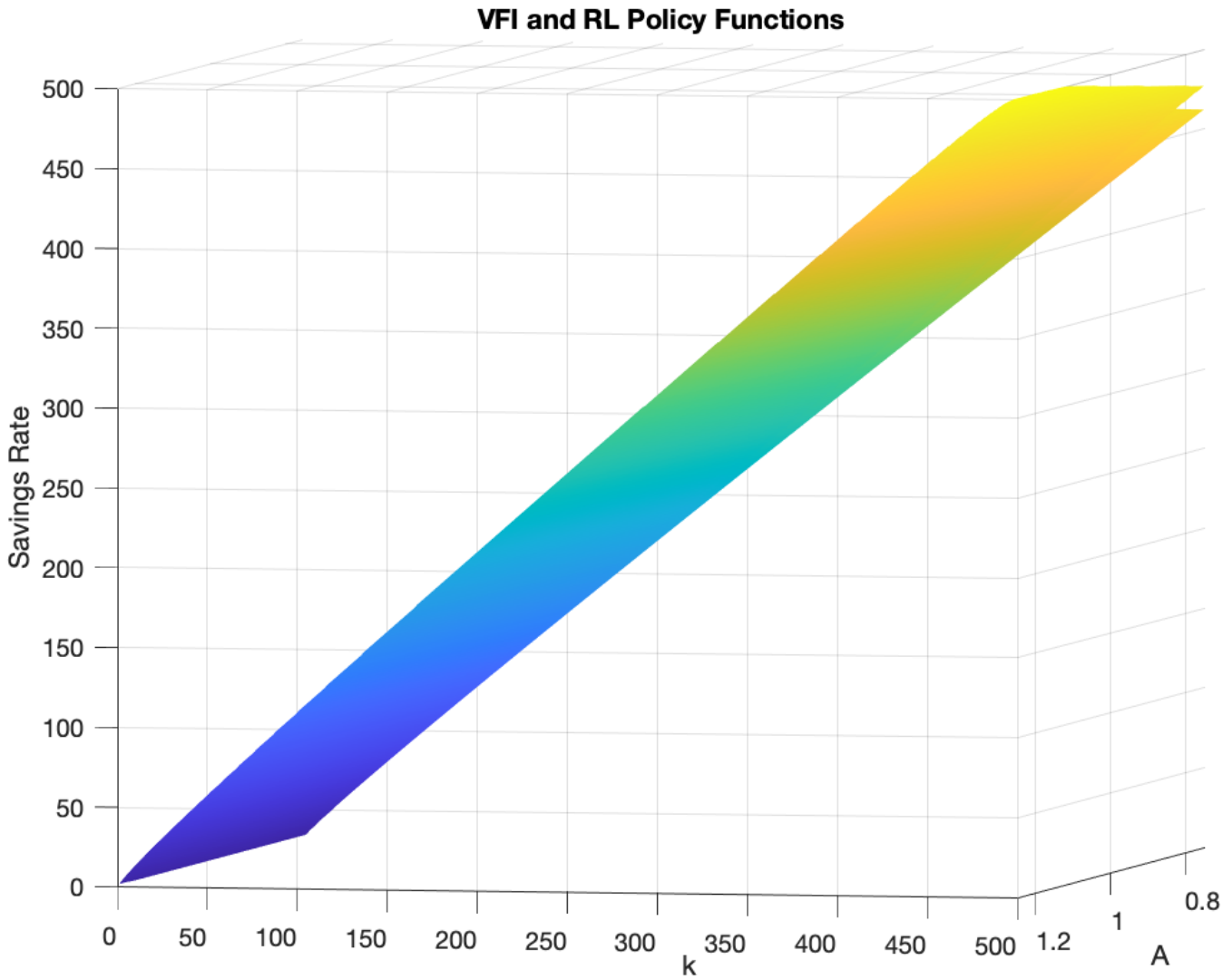
Value Functions are Similar

- There are many options to maximize, can take more time, etc. but let's get a ballpark idea of how well this can do after five hours
- Note: won't be perfect! Feel free to run longer and/or with more parameters
- Hard to see, so let's look at GraphDiff.m

Value Functions

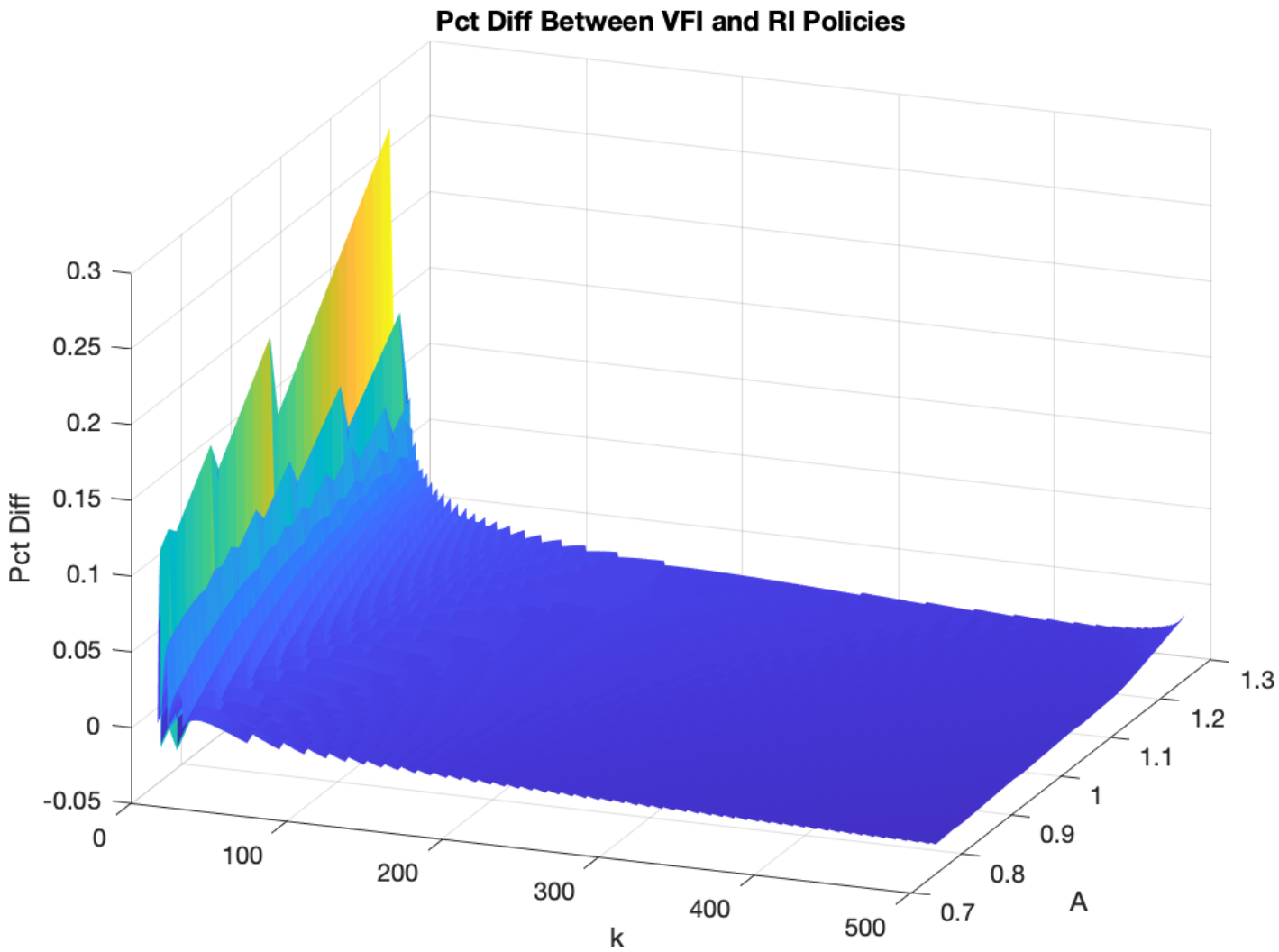


Policy Functions



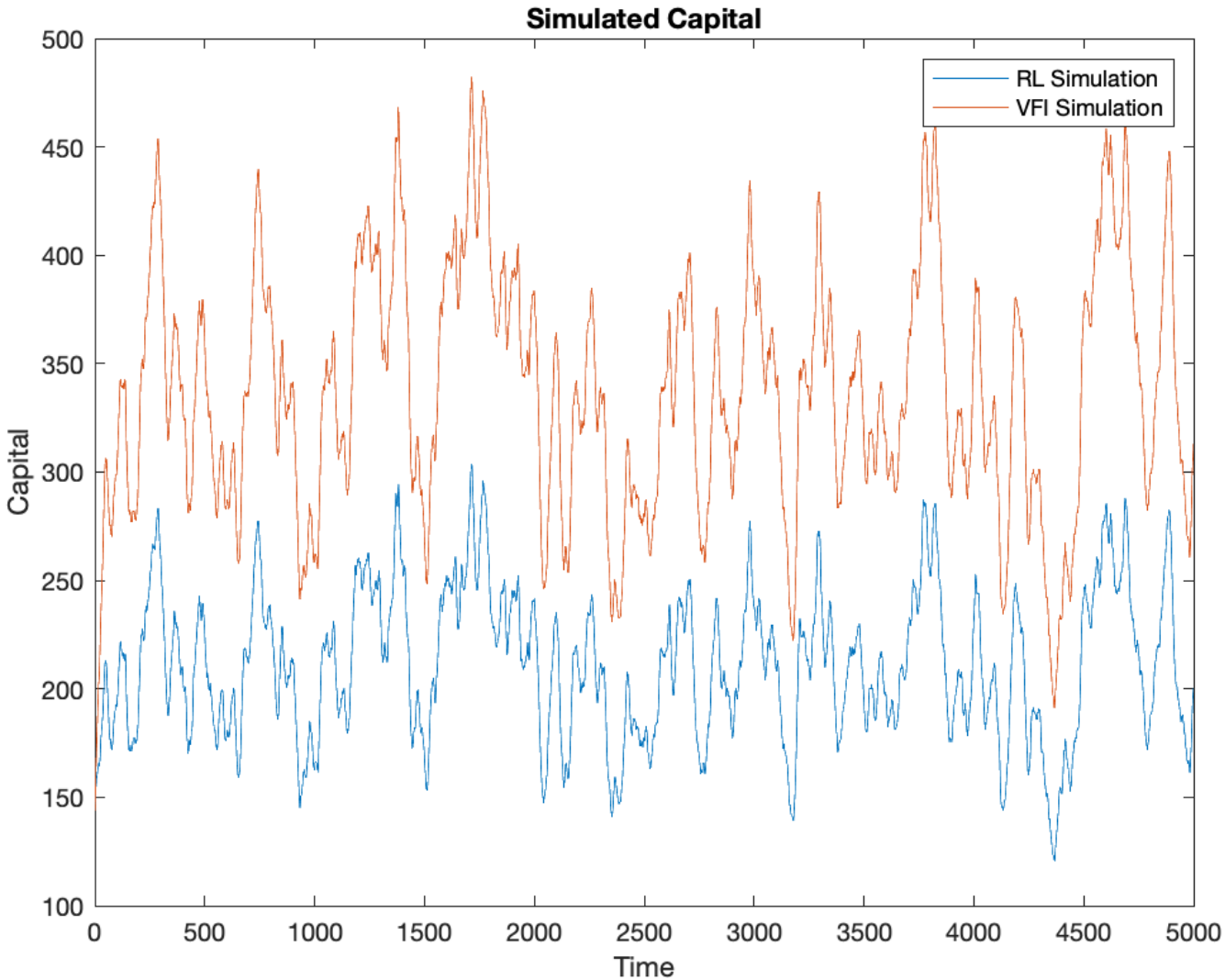
Hard to see, but right on top of one another

Policy Functions Difference



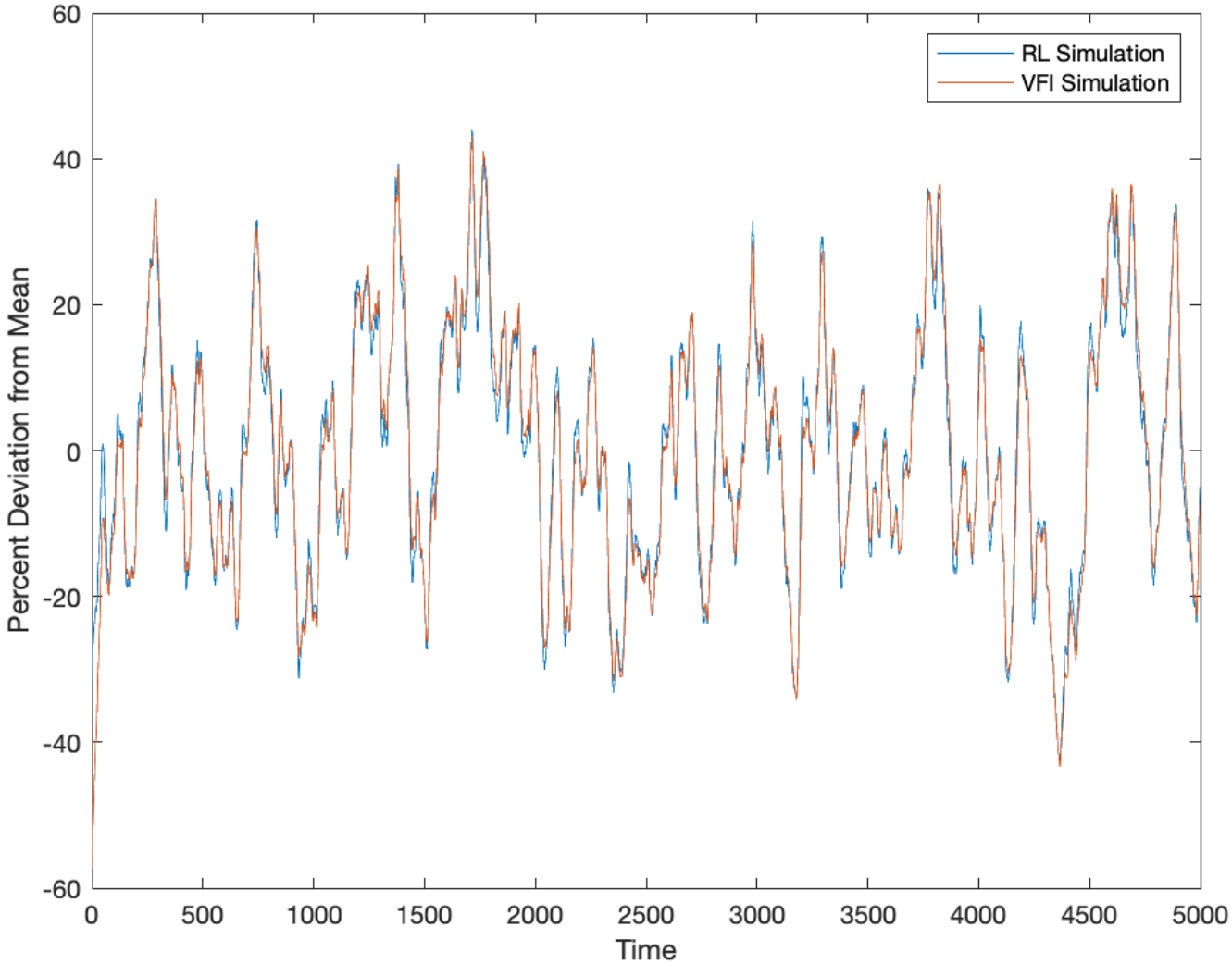
Mostly very small differences...except at k near zero (why?)

Simulation



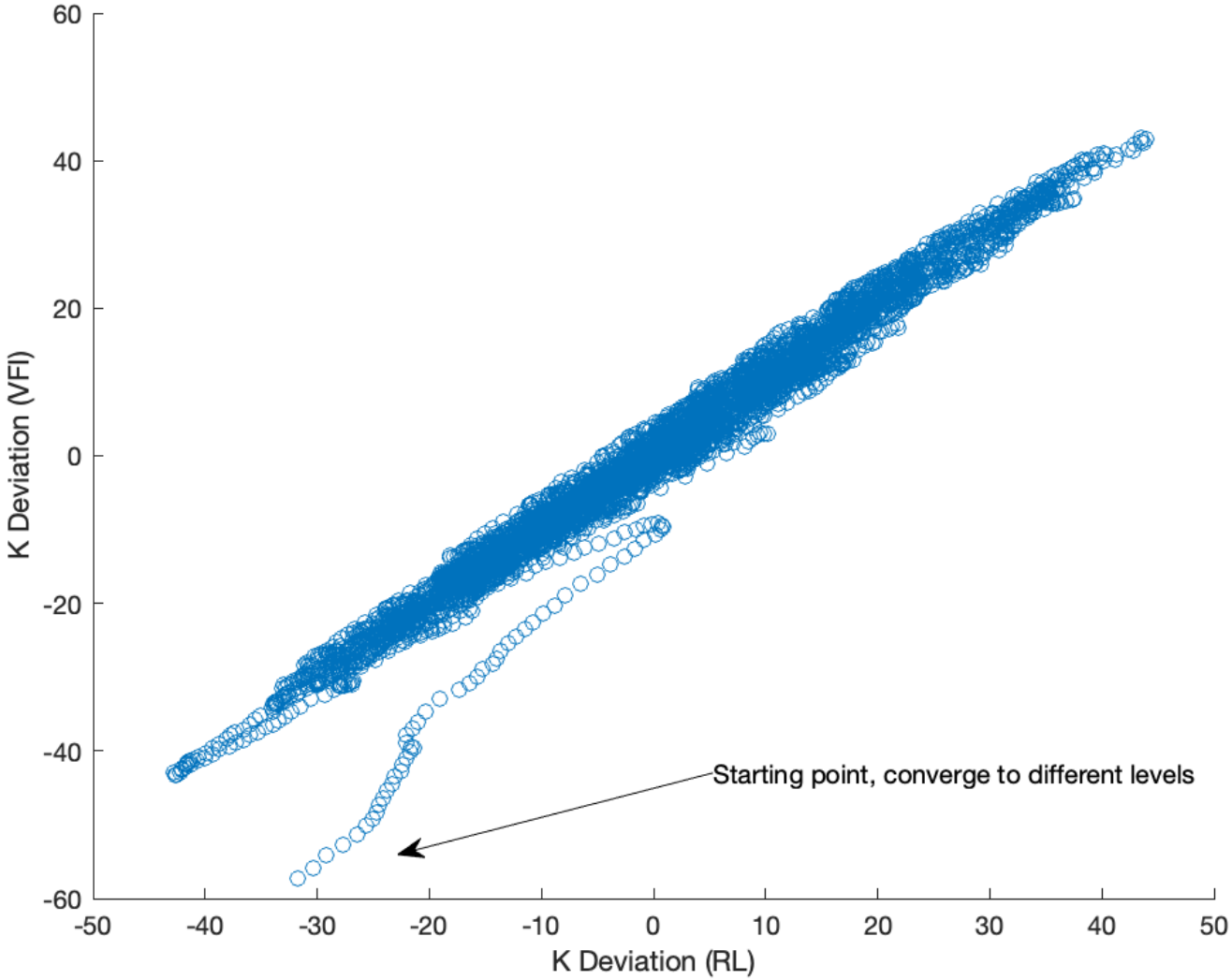
We get towards the right answer in percent deviations (let's check!)

Simulation



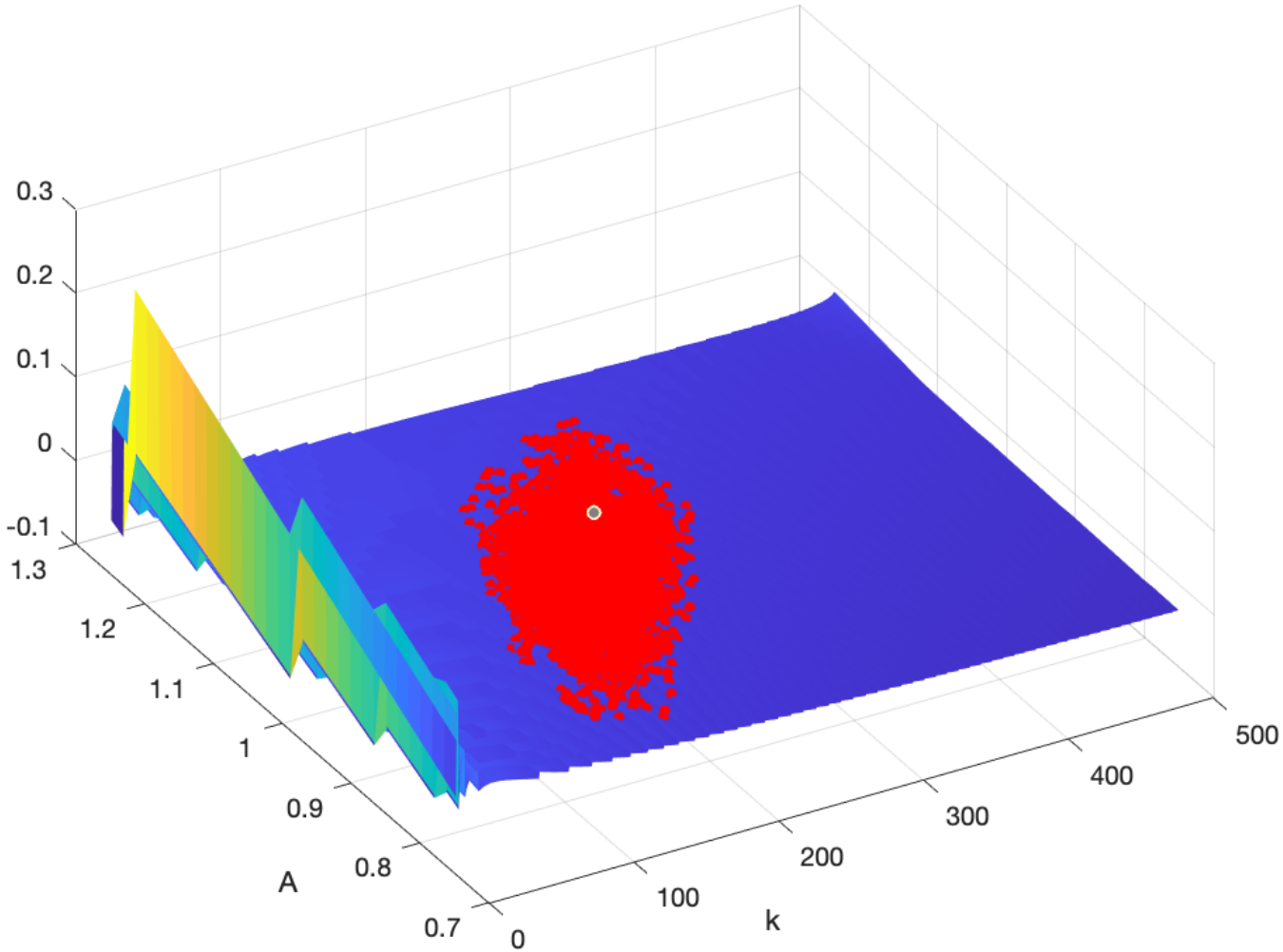
Mostly spot-on in differences

Simulation



Not terrible, but could use more running & debugging to get level

We mostly sample points in the red area, so most accurate there



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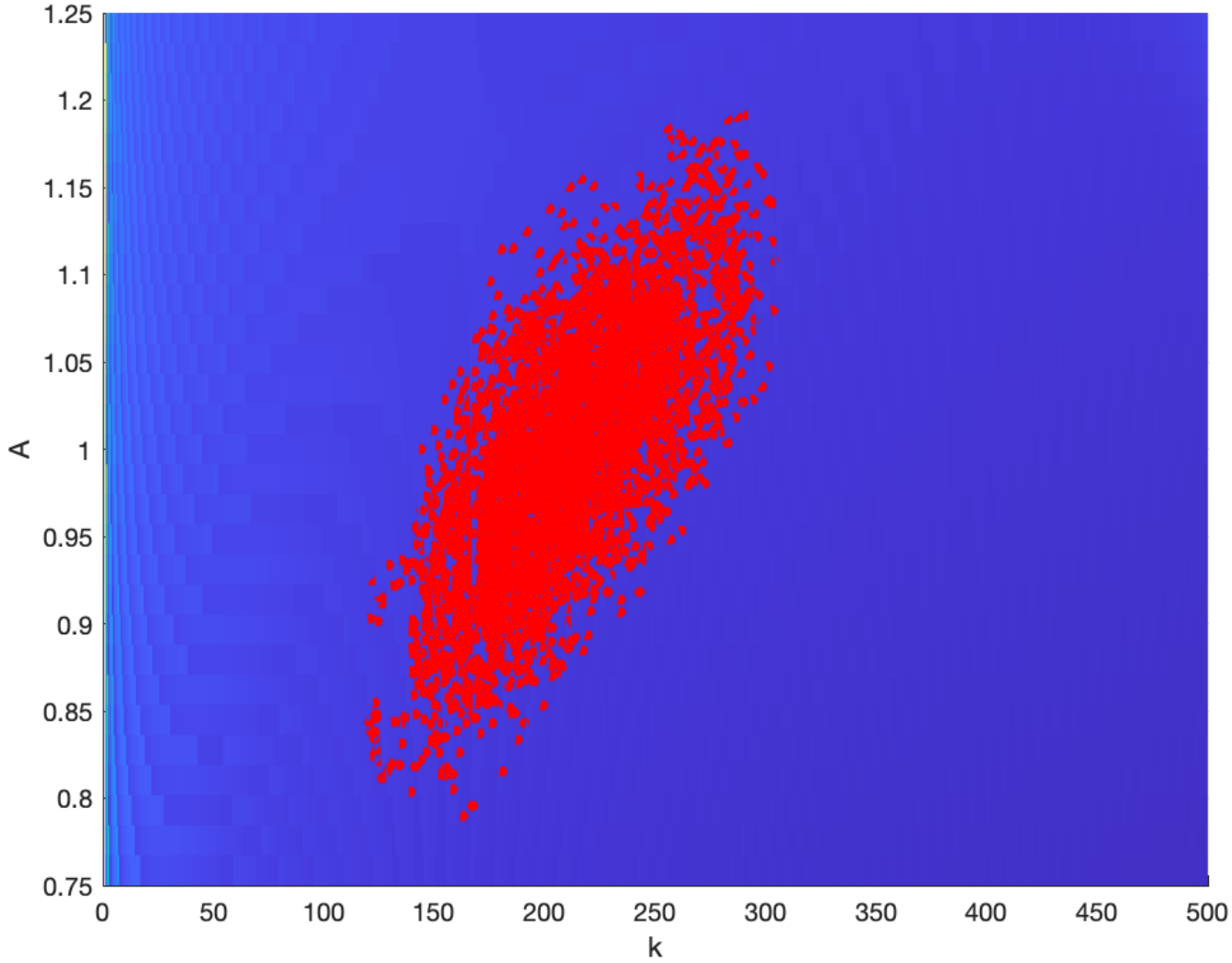
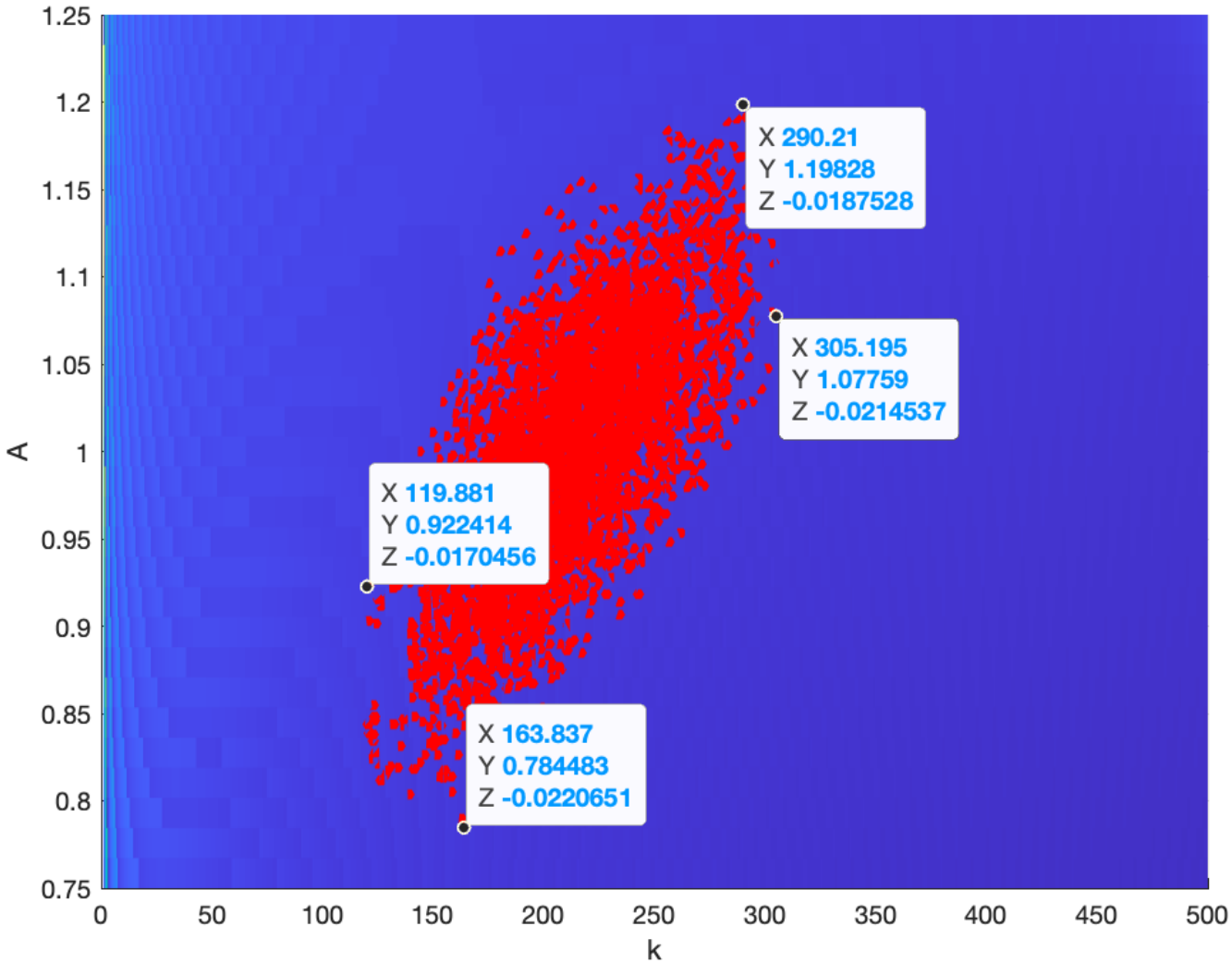


Some useful bounds



Summing Up

- Reinforcement learning incredibly useful
- Nothing spectacularly clever, just a combination of:
  - Flexible functional forms (so many state variables can be accomodated efficiently)
  - *Forward*-looking (average over many simulations is estimation)
  - Simple maximization (use gradient and flexible function, so don't have to solve perfectly and solutions potentially informative across state space)
- Could have gone much farther. Could have had parameters α, ρ, etc. be draws too (solve not only over k and A but over parameterization, so can solve for heterogeneous agents or estimate parameters easily! (VFI would require solving for each parameter set).
- In practical terms, you just have a setup function, and then a function that steps through time (simulates) and pass it off to solver

Last warning

- There's a lifetime of details in terms of efficiency, problem setup, solvers, etc., and the devil is in the details
  - Discrete, continuous
  - Shallow RL, deep RL (how setup?)
  - On/off policy
  - Delayed learning

- We went through one algorithm (actor-critic) and one example (continuous state & action space).

- There are a cornucopia of flavors & algorithms–I haven't yet found one that isn't intuitive & obvious in retrospect (once you grok the core idea behind approximate dynamic programming idea)

- However, economist notation and ML notation diverge a bit, so there's an investment in learning

Other ML Topics

- We talked about reinforcement learning: have a problem, throw it at computer over and over, have computer solve it
- There are other things you might care about: highly nonlinear functions more generally (Kriging/gaussian process regression "fitrgp")
- Classification learning (such as support vector machines "fitcsvm")
- Regression learner
- We won't talk about these, instead we'll start applying!