Solving a Lifecycle Model

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Econ 64200

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This homework is the first part of a multi-part homework. This homework starts us with a relatively simple deterministic lifecycle model and asks you to solve it in Matlab.

Deliverables

- You should have a word/LaTeXdocument that has three sections:
    1. Discusses the model and answers the questions I pose throughout.
    2. Contains the tables and figures you will produce.
    3. Contains a discussion of your programming choices if you had to make any.
- You should have a Matlab file or set of files (zipped) that contain **all** your programs and raw data. There should be a file called "Main.M" that produces everything I need in one click.

1 Model

Households have period utility over consumption c_t and labor L_t at time t: $u(c_t, L_t)$, which takes the form of balanced growth preferences with constant Frisch elasticity of labor supply:

$$u(c_t, L_t) = log(c_t) - \psi \frac{\epsilon}{1+\epsilon} L_t^{1+\epsilon}\epsilon$$

Letting s denote savings, w denote wages, r denote interest rates, households have period budget constraint:

$$c_t + s_t = w_t L_t + (1 + r_t)s_{t-1}$$

Households maximize the net present value of utility, discounted at a rate $0 < \beta < 1$. Assume that households live until period T.

Question 1a: Write out the household's net present value budget constraint.

Suggested Sol: Should look something like:

$$\sum_{t=1}^{T} \frac{c_t}{\prod_{\tau=1}^{t}(1+r_\tau)} - \sum_{t=1}^{T} \frac{w_t L_t}{\prod_{\tau=1}^{t}(1+r_\tau)}$$

Defining $r_1 = 0$. Could also play with indicies.

Question 1b: Write out the household's sequence problem.

Suggested Sol: Should look something like:

$$\max_{\{s_{t+1}, c_t, L_t\}_{t=0}^{\infty}} \sum_{t=1}^{T} \beta^t u(c_t, L_t)$$

subject to:

$$c_t + s_t = w_t L_t + (1+r_t) s_{t-1}$$

With s_0 given.

Question 1c: Write out the household's value function problem.

Suggested Sol: Should look something like:

$$V_t(s_t, w_t, r_t) = \max_{\{s_{t+1}, c_t, L_t\}} \{u(c_t, L_t) + \beta V_{t+1}(s_{t+1}, w_{t+1}, r_{t+1})\}$$

Note that because this is a lifecycle model, t is also a state variable, with subscript.

2 Basic Matlab

Now, let the following numerical assumptions hold:

Table 1: Calibration

Concept	Parameter	Value
Lifespan	T	45
Disutility of labor	ψ	10
Frisch elasticity of labor supply	ϵ	0.75
Discount factor	β	0.96
Wages: w_t, $t \neq 30$	$w_{t\neq30}$	1
Wages: w_30	$w_{t\neq30}$	1.05
Interest rates: r_t, $t \neq 20$	$r_{t\neq30}$	0.05
Interest rates r_20	$w_{t\neq20}$	1.1

Question 2: Write a Matlab program that does the following.

- Section 1a: Defines all exogenous parameters in Table 1, including making w_t and r_t 45-entry column vectors. (It is important to be consistent in terms of column vs. row).
- Section 1b: Defines a NPV discount rate for all 45 periods (e.g. contains $[1; 1/(1+r_1); 1/((1+r_1)*(1+r_2))]$and so on. You may find Matlab's cumulative product function "cumprod" useful.
- Section 2a: Define the utility function, which takes in c and L (as inputs) and spits out utility (given ψ and ϵ, which are set by Table 1. Write it so you can give the function a vector of c and L and it will spit out a vector of utilities.

$$ut(c, L) = log(c_t) - \psi \frac{\epsilon}{1+\epsilon} L_t^{1+\epsilon} \epsilon$$

- Section 2b: A function of the lifetime excess savings, which takes in a vector of c_t's and a vector of w_t's and L_t's, and spits out excess savings

$$bc\_lifetime(c, L) = \sum_{t=1}^{T} \frac{c_t}{\prod_{\tau=1}^{t}(1+r_\tau)} - \sum_{t=1}^{T} \frac{w_t L_t}{\prod_{\tau=1}^{t}(1+r_\tau)}$$

- Section 2c: A lifetime utility function which penalizes debt (a simple way of making constrained optimization unconstrained), e.g.:

$$ut\_lifetime(c, L) = sum(beta.^[0:T-1] ut(c, L)) - 10*(bc\_lifetime(c, L) > 0)*bc\_lifetime(c, L)$$

- Section 3: Create an initial guess, and pass the negative of the life to a *minimizer*, such as fminunc. You may have to think about how to pass the anonymous arguments.

- Section 4: Graph out c_t and L_t. Clearly label your graphs.
- Section 5: Answer the following questions:
  - What is the intertemporal elasticity of substitution suggested by consumption's response in period 20?
  - What is the Frisch elasticity of labor supply suggested by labor's response in period 30?
  - Why is consumption rising over time?
  - What, in the calibration and functional form choice drove the numerical answers to the previous three questions? What parameterization of β would make c_{t+1}/c_t flat?

Suggested Sol: See Hwk1Sol.m Below, I plot the results: Some interesting takeaways:

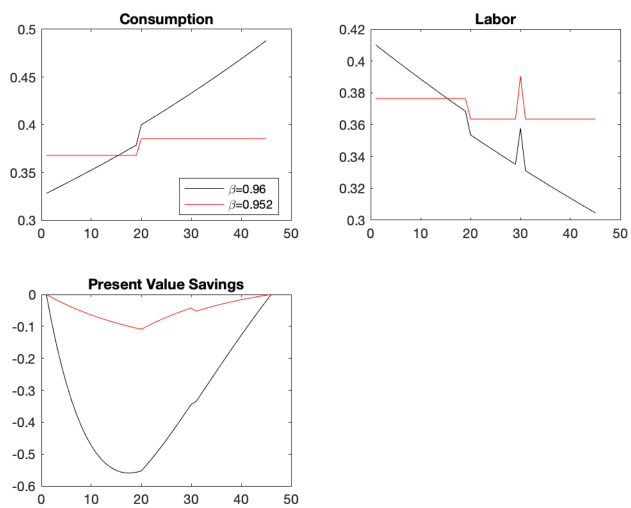

Figure 1: Consumption, labor, and savings for $\beta = 0.96$ and $\beta = 0.952$

- The intertemporal elasticity of substitution can be calculated by knowing one convenient definition:

$$\sigma = -\frac{d\log\left(\frac{c_{t+1}}{c_t}\right)}{dr}$$

  Put simply, the percentage change in consumption growth due to a percentage point change in interest rates. When there is no trend ($\beta = 0.952$) then we can just calculate the percent change: 0.385/0.368-1=0.046, divided by the interest rate change of 0.05 is 0.92, close to the one that log preferences would predict. If there is a trend, then we have to difference it out: (0.40/0.379)-(0.379/0.376)=0.047, or $\sigma = 0.94$.

- The Frisch elasticity of substitution is also quite close to the calibrated value. For no trend, wages went up by 10%, while labor went up by (0.358/0.335)=0.074, for a Frisch elasticity of 0.74, close to the 0.75 the preferences would predict (in a perfect experiment). When $\beta = 0.96$, we take out the labor trend (-0.6% decline), so that the 6.8% increase is really a 0.074% increase, giving again a Frisch elasticity of 0.75.

- Consumption goes up because the Euler equation is:

$$\frac{c_{t+1}}{c_t} = \beta(1 + r_{t+1})$$

If your indexing was different so that consumption in the first period was discounted by interest rates but utility wasn't multiplied by β, you might have had a different trend.

Table 2: Consumption and Labor

Consumption			Labor		
Time	$\beta = 0.96$	$\beta = 0.952$	Time	$\beta = 0.96$	$\beta = 0.952$
c_{18}	0.376	0.368	L_{28}	0.337	0.364
c_{19}	0.379	0.368	L_{29}	0.335	0.364
c_{20}	0.400	0.385	L_{30}	0.358	0.391

3 Research

You now have a very simple deterministic numerical model of people's labor over the lifecycle. However, the model is too simple to be particularly interesting. Change the model in an interesting way, and report your results. This is your opportunity to actually think, so please take time with this: how can you adjust the model, parameterization, etc. to make this model (a) more interesting and (b) more realistic?

Suggested Sol: Many possibilities! One is to include "learning by doing." I did so in a primitive way by making the change in wage equal to the cumulative hours spent working divided by 10, minus a quadratic trend denoting declining labor productivity at older ages. When I do so (with $\beta = 0.952$) I get flat consumption, a u-shaped labor supply (at first age-related depreciation is small, so learning by doing dominates, then it rises). I then get a u-shaped lifetime labor curve.

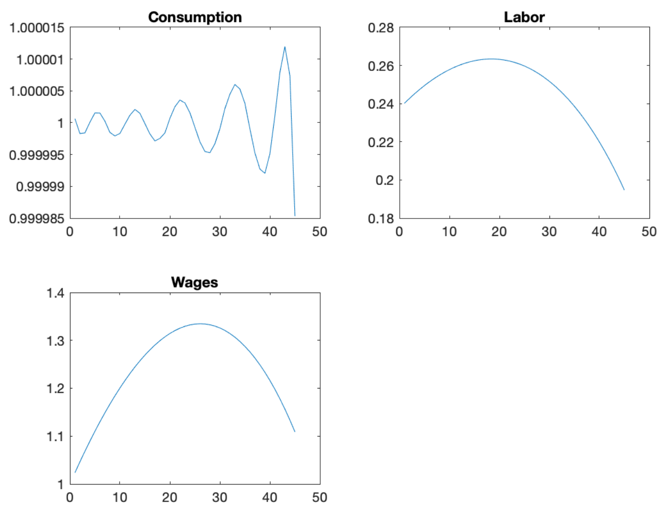

Figure 2: Consumption, labor, and wages in a learning-by-doing model. Note the y-axis on consumption (consumption is smoothed over the lifecycle).