Numerical Methods-Lecture III: Bellman Equations: Theory

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Preliminaries

- I assume you have seen Bellman equations before
- Seeing Bellman equations before is unnecessary
- I will introduce as if you have forgotten everything
- I leave the technical requirements to chapters 4 and 9 of Stokey & Lucas with Prescott, (RMED).
- Most of the problems we write down will be well-conditioned
- This class : DP math :: rocket engineering : physics

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- With that shallow warning, we'll be learning Monkey Bellmans

Sequence Problem

- We can formulate a number of very interesting problems as:

$$\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

$$\text{s.t. } x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2, \ldots$$

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- Too abstract for an engineer. What is this saying? Translate it.

Sequence Problem: Pieces

- $F(\cdot, \cdot)$ is something we're trying to maximize
- x_{t+1} is the thing we can control
- F is influenced by:
  - Something we control right now: x_{t+1}
  - Something that we don't control right now: x_t
  - But yesterday, we controlled it (except in first period)
- Our ability to control x_{t+1} is governed by x_t (via Γ)

Generality? Example 1: NCG

We get utility from consumption. Consumption is constrained by capital.

$$\max \text{ NPV of } \; U(c_t) \quad s.t. \quad Ak_t^\alpha = c_t + i_t, \qquad k_{t+1} = (1-\delta)k_t + i_t$$

- $U(c_t)$ is something we're trying to maximize
- k_{t+1} is the thing we can control
- F is influenced by:
  - Something we control right now: k_{t+1} (via LOM, c_t)
  - Something that we don't control right now: k_t
- Our ability to control k_{t+1} is governed by k_t, which gives our budget constraint

Generality? Example 2: McCall's Model-I

- Search for job
- Each period, one offer w from bounded wage distribution with CDF $F(w)$
- Can reject, get c
- Can accept, get w forever
- Utility is equal to whatever income you get (c or w).

Generality? Example 2: McCall's Model

$$\text{max NPV of } \; U(c_t) \quad s.t. \quad c_t = \begin{cases} w & \text{if accept w} \\ c & \text{if not accepted} \end{cases}$$

$$\begin{aligned} w_{t+1} = w & \quad \text{if accept w} \\ w_{t+1} \sim F(w) & \quad \text{if not accepted} \end{aligned}$$

- $U(c_t)$ is something we're trying to maximize
- We control if w or c
- F is influenced by:
  - Our choice of c vs. w if not yet chosen (memoryless)
  - Our past choice of w if accepted (no choice)
- Next w is influenced by past w (sometimes)

Generality? Example 3: Firm Investment

Firm wants to maximize profit given A and K, with profits:

$$a_i \in \{0, 1, ..., 100\}$$

$$Pr(A_{t+1} = a_i | A_t) \;\; \textit{given}$$

$$\pi_t = \begin{cases} -(A_t - K_t)^2 & \text{if no change} \\ -(A_t - K_t^*)^2 - C & \text{if change} \end{cases}$$

$$\begin{cases} K_{t+1} = K_t & \text{if no change} \\ K_{t+1} = K_t^* & \text{if change} \end{cases}$$

Where if change, K_t^* is chosen from the range of a_i.

The Bellman Equation - I

- Starting with a sequence problem:

$$\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

s.t. $x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2, ...$

$$x_0 \in X \text{ is given.}$$

- We can define an equation:

$$V(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\}$$

- Call x the state, y the control, V the value function

The Bellman Equation - II

$$V(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\}$$

- What are doing?
  - Want to maximize current and future F's through choice of all future x's
  - Simplify the problem: today, and the problem you wake up with tomorrow
  - Tomorrow, you maximize today, and the next day's problem
  - Define everything recursively: the NPV of happiness if you wake up with x and maximize is utility today plus the NPV of happiness when you wake up with whatever you have left tomorrow
- Call x our state, the thing we wake up with
- Call y our control, the thing we control
- Call Γ our law of motion

The Bellman Equation - III

$$V(x_t) = \max_{\{y_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} F(x_s, y_s)$$

Break the first part out:

$$V(x_t) = \max_{\{y_s\}_{s=t}^{\infty}} \left[\beta^{t-t} F(x_t, y_t) + \sum_{s=t+1}^{\infty} \beta^{s-t} F(x_s, y_s) \right]$$

Break up the optimization and take out a β:

$$V(x_t) = \max_{y_t} \left[F(x_t, y_t) + \beta \max_{\{y_s\}_{s=t+1}^{\infty}} \sum_{s=t+1}^{\infty} \beta^{s-t-1} F(x_s, y_s) \right]$$

The Bellman Equation - VI

$$V(x_t) = \max_{y_t} \left[F(x_t, y_t) + \beta \max_{\{y_s\}_{s=t+1}^{\infty}} \sum_{s=t+1}^{\infty} \beta^{s-t-1} F(x_s, y_s) \right]$$

Notice that if we replaced t with $t+1$ in the definition, we would have:

$$V(x_{t+1}) = \max_{\{y_s\}_{s=t+1}^{\infty}} \sum_{s=t+1}^{\infty} \beta^{s-t-1} F(x_s, y_s)$$

Plugging this in:

$$V(x_t) = \max_{y_t} [F(x_t, y_t) + \beta V(x_{t+1})]$$

Given an x_t, just need a y_t, and whatever the value of all the future decisions we make will be.

Summing Up

- Bellmans are a flexible way of describing a large number of dynamic problems
- We can write down problems easily
- Now we want to numerically solve them
- Next up: value function iteration (extremely easy and intuitive ways to solve)