19 RBC Model with Growth

19.1 General Setup (aggregates)

So far we've only been dealing with an RBC model with stationary productivity and population. But obviously this isn't realistic, and if we want to put data to the model, we have to take this into account. From our growth analysis, we've seen that TFP seems to be the main driver of growth in per capital production.

Let's define *aggregate* (not per-capita) production, consumption, and investment in capitals. We have the budget constraint:

$$Y_t = C_t + I_t \tag{11}$$

And we have the Cobb-Douglas production:

$$Y_t = A_t A_0 \gamma_A^{(1-\alpha)t} K_t^{\alpha} H_t^{1-\alpha} \tag{12}$$

Where $A_t \gamma_A^{(1-\alpha)t}$ is TFP. A_t will be representing the percent *deviation* in trend, while $\gamma_A^{(1-\alpha)t}$ will represent the trend.

The *aggregate* capital stock depreciates geometrically:

$$K_{t+1} = (1-\delta)K_t + I_t \tag{13}$$

The household has dynastic preferences over consumption and hours of its members. Letting c_t and h_t be the *per capita* consumption and per capita fraction of time worked, e.g. $c_t = \frac{C_t}{N_t}$. The representative agent's utility is:

$$\sum_{t=0}^{\infty} \beta^t N_t \left[\left(\frac{C_t}{N_t}^{\lambda} v \left(\frac{H_t}{N_t} \right)^{1-\lambda} \right)^{\rho} - 1 \right] / \rho$$

You can see that we're weighting all members equally, but are discounting future utility over time.

In the long run, we claim that population grows at a constant rate:

$$N_{t+1} = \gamma_N N_t \quad \text{equivalently,} \quad N_t = N_0 \gamma^t$$

TFP deviation has the law of motion:

$$\log(A_{t+1}) = \rho \log(A_t) + \epsilon_t \qquad \epsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma_A^2) \tag{14}$$

This is our model. We have a utility function to be maximized, a budget constraint, a production function, and laws of motion for population, TFP deviation, and capital. All these things are exploding at constant rates in the long run, and it's hard to deal with exploding processes. We want to rewrite this system as deviation from trend. One way we'll detrend from population growth is to look at things in per capita units. The way we'll detrend from TFP growth is to divide by the growth rate of TFP. Table 19.1 displays what variables will be detrended:

Variable	Aggregate	Detrending	Detrended
Consumption	C_t	$\frac{C_t}{N_t \gamma_A^t}$	c_t
Labor	H_t	$\frac{H_t}{N_t}$	h_t
Production	Y_t	$\frac{Y_t}{N_t \gamma_A^t}$	y_t
Investment	I_t	$\frac{I_t}{N_t \gamma_A^t}$	i_t

19.2 Detrending

We aren't going to change our model. What we wrote down is going to be what we solve. What we're going to do is only transform our system. A solution to what we write down after detrending will represent a solution to what we had before. So, we'll take equations 11, 12, 13, and 14 and transform them.

Starting with the budget constraint:

$$\frac{Y_t}{N_t\gamma^t} = \frac{C_t}{N_t\gamma_A^t} + \frac{I_t}{N_t\gamma_A^t}$$

Gives us the transformed budget constraint:

$$y_t = c_t + i_t \tag{15}$$

And production function, dividing by $N_t\gamma_A^t$ and cleverly arranging different parts of it:

$$\frac{Y_t}{N_t\gamma_A^t} = A_0 A_t \frac{\gamma_A^{(1-\alpha)t}}{\gamma_A^{(1-\alpha)t}} \left(\frac{K_t}{N_t\gamma_A^t}\right)^\alpha \left(\frac{H_t}{N_t}\right)^{1-\alpha}$$

Gives us the transformed production function:

$$y_t = A_0 A_t k_t^\alpha h_t^{1-\alpha} \tag{16}$$

The aggregate capital stock law of motion becomes:

$$\frac{K_{t+1}}{N_{t+1}\gamma_A^{t+1}} = (1-\delta)\frac{K_t}{N_{t+1}\gamma_A^{t+1}} + \frac{I_t}{N_{t+1}\gamma_A^{t+1}}$$

Plugging in $N_{t+1} = \gamma_N N_t$ for one side and breaking up γ_A^{t+1}:

$$k_{t+1} = (1-\delta)\frac{K_t}{\gamma_N\gamma_A\gamma_A^t N_t} + \frac{I_t}{\gamma_N\gamma_A\gamma_A^t N_t}$$

Becomes:

$$k_{t+1} = (1-\delta)\frac{1}{\gamma_N\gamma_A}k_t + \frac{1}{\gamma_N\gamma_A}i_t$$

So the law of motion for capital becomes:

$$\gamma_N\gamma_A k_{t+1} = (1-\delta)k_t + i_t \tag{17}$$

Preferences:

$$\sum_{t=0}^{\infty}\beta^t N_t \left[\left(\frac{C_t}{N_t}^\lambda v\left(\frac{H_t}{N_t}\right)^{1-\lambda}\right)^\rho - 1\right]/\rho$$

Multiplying and dividing C_t by γ_A^t, and also using our constant population growth:

$$\sum_{t=0}^{\infty}\beta^t N_0\gamma_N^t\gamma_A^{\lambda\rho t}\left[\left(\frac{C_t}{N_t\gamma_A^t}^\lambda v\left(\frac{H_t}{N_t}\right)^{1-\lambda}\right)^\rho - \frac{1}{\gamma_A^{\lambda\rho t}}\right]/\rho$$

Simplifying:

$$\sum_{t=0}^{\infty}\beta^t N_0\gamma_N^t\left[\left(c_t^\lambda v(h_t)^{1-\lambda}\right)^\rho - \frac{1}{\gamma_A^{\lambda\rho t}}\right]/\rho$$

Taking the limit[17] as $\rho \to 0$ and taking $v(h_t) = 1 - h_t$, and dropping an innocuous constant term, we have utility proportional to:

$$\sum_{t=0}^{\infty}(\gamma_N\beta)^t\left[\lambda\log(c_t) + (1-\lambda)\log(1-h_t)\right] \tag{18}$$

[17] As an aside, note that if I let:

$$f(\rho) = \left[\left(A^\lambda B^{1-\lambda}\right)^\rho - 1/\gamma_A^{\lambda\rho t}\right] \quad \text{and} \quad g(\rho) = \rho$$

19.3 What is detrending doing?

We have four new equations that look like our old equations.

$$y_t = c_t + i_t$$

$$y_t = A_t k_t^{\alpha} h_t^{1-\alpha}$$

$$\gamma_A \gamma_N k_{t+1} = (1-\delta)k_t + i_t$$

$$\sum_{t=0}^{\infty} (\gamma_N \beta)^t \left[\lambda \log(c_t) + (1-\lambda)\log(1-h_t)\right]$$

What's going on? The first two equations look a lot like our old ones. The first changes the original equation "all production is either consumed or invested" to "all per-capita, detrended production is either consumed or invested." Makes sense. The second took production, which related labor capital and production, now expresses detrended production per capita in terms of detrended capital per capita and hours per capita. Where before it would be growing, holding K, H, and N constant, not it stays stationary. The third equation was previously talking about the capital stock. Now it's talking about the detrended per capita capital stock. The two numbers greater than one are saying that, to keep up with population growth or TFP growth, we need to be investing at a rate greater than just depreciation. Previously, if $i_t = \delta k_t$, then we were in a steady state. Now, investment must be $i_t = \gamma_N \gamma_A - 1 + \delta$ to keep per capita capital up with trend. Finally, our preferences are the same but for one small shift: before, we discounted the future. Now, we also recognize that there are more people tomorrow. While you have diminishing returns, utility is additive across each of your four children. More people tomorrow makes you act less impatient.

What was the point of all this? Our economy actually displays growth, and people will be maximizing within that framework. But the models we've dealt with so far all have a steady state. We've replaced the concept of a steady state with a balanced growth path. Where before all our variables would converge to the same value, now they all converge to their old path. Our model didn't actually change very much: our capital equation can be reinterpreted as investment giving us a little less, and capital depreciating a little faster, and our utility function discounts things at a greater rate, but we're going to be solving the same basic model. This is great news, and it's a defense to writing down models without growth: most of the time, a model with growth can be rewritten as a model without growth if you calibrate properly. Figure 35 displays the idea behind what we are doing: we can analyze the left hand, growing side, or the right hand, not growing side: they're equivalent. We've taken a growing process and transformed it to be stationary: a powerful and common trick in time series analysis.

19.4 Calibrating the system

Plugging in the first three equations:

$$c_t = (1 - \delta + A_t k_t^{\alpha-1} h_t^{1-\alpha})k_t - \gamma_A \gamma_N k_{t+1}$$

We can therefore write the Bellman:

$$V(k_t) = \max_{k_{t+1}, h_t} \left\{\lambda \log((1-\delta + A_t k_t^{\alpha-1} h_t^{1-\alpha})k_t - \gamma_A\gamma_N k_{t+1}) + (1-\lambda)\log(1-h_t) + \gamma_N \beta V(k_{t+1})\right\}$$

Then

$$\lim_{\rho\to 0} f(\rho) = 1 - 1 = 0 \qquad \text{and} \qquad \lim_{\rho \to 0} g(\rho) = 0$$

Taking derivatives of both:

$$f'(\rho) = t\gamma_A^{-t\rho}\log(\gamma_A) + (A^{\lambda}B^{1-\lambda})^{\rho}\log(A^{\lambda}B^{1-\lambda} \qquad g'(\rho) = 1$$

Then, applying L'Hôpital's rule:

$$\lim_{\rho\to 0}\frac{f(\rho)}{g(\rho)} = \lim_{\rho\to 0}\frac{f'(\rho)}{g'(\rho)} = \log(\gamma_A) + \log(A^{\lambda}B^{1-\lambda} \propto \lambda\log(A) + (1-\lambda)\log(B)$$

Taking the first order condition with respect to h_t:

$$\lambda \frac{(1-\alpha)A_t k_t^{\alpha} h_t^{-\alpha}}{c_t} = \frac{1-\lambda}{1-h_t}$$

$$\lambda \frac{w_t}{c_t} = \frac{1-\lambda}{1-h_t}$$

$$\lambda = \frac{c}{c+w(1-h)}$$

Taking the first order condition with respect to k_{t+1}:

$$\lambda \frac{\gamma_A \gamma_N}{c_t} = \gamma_N \beta V_{k_{t+1}}(k_{t+1})$$

Taking the Envelope condition with respect to k_t:

$$V_{k_t} = \lambda \frac{(1-\delta+\alpha A_t k_t^{\alpha-1} h_t^{1-\alpha})}{c_t}$$

Advancing the Envelope condition one period forward:

$$V_{k_{t+1}} = \lambda \frac{(1-\delta+\alpha A_{t+1} k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha})}{c_{t+1}}$$

Plugging this into the FOC:

$$\beta = \gamma_A \frac{1}{1-\delta+\alpha A_{t+1} k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha}} \frac{c_{t+1}}{c_t}$$

$$c_t = A_t k_t^{\alpha} h_t^{1-\alpha} - \gamma_A \gamma_N k_{t+1} - (1-\delta) k_t$$

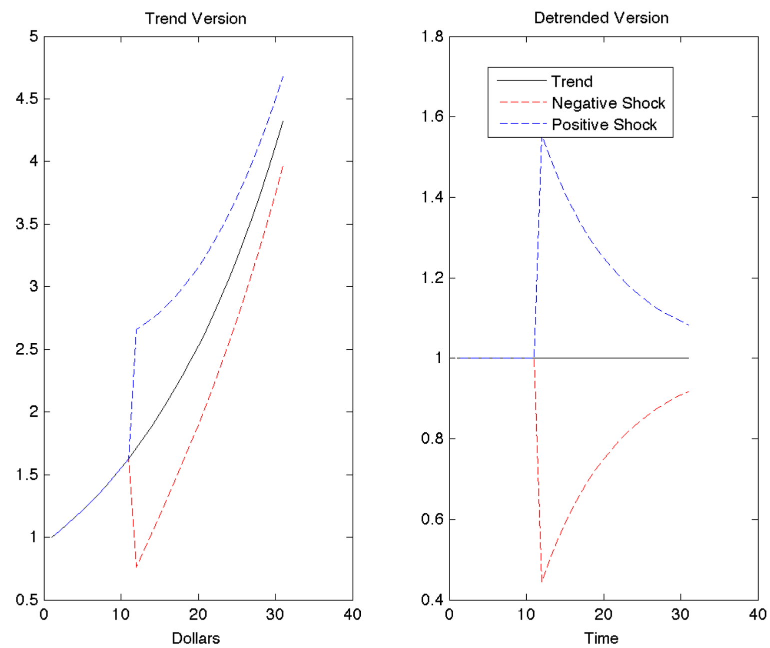

Figure 35: This figure displays a trend growth variable that suffered shocks and its detrended version. On the left hand side is some variable, such as capital or production, growing at its trend growth rate, along with possible shocks that decay over time. On the right hand side are those paths detrended.

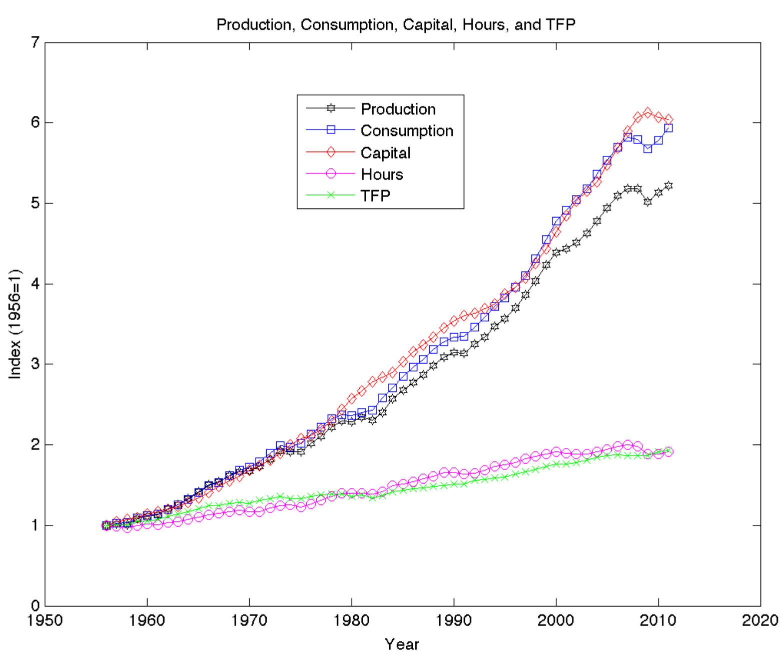

Figure 36: This figure depicts U.S. data on production, consumption, capital, hours, and TFP from 1955 to 2011.

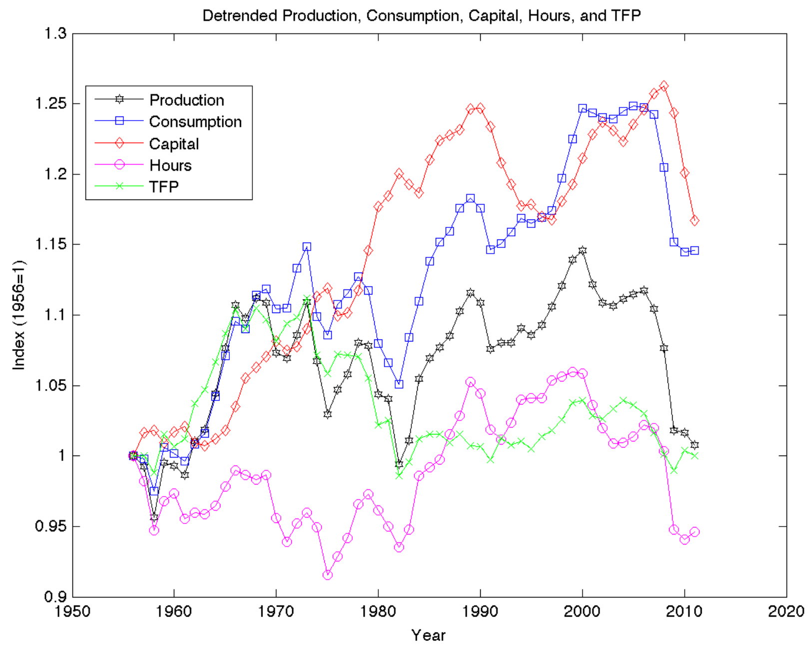

Figure 37: This figure detrends each variable according to theory. If nothing changed, if we were permanently on the same balanced growth path, then each of these would be a straight line out from 1. As we can see, we deviate a little from theory: the deviations present in this graph distinguish growth theory from business cycle theory. The dips and peaks here are the important, but they should be compared to the magnitudes in Figure 36.

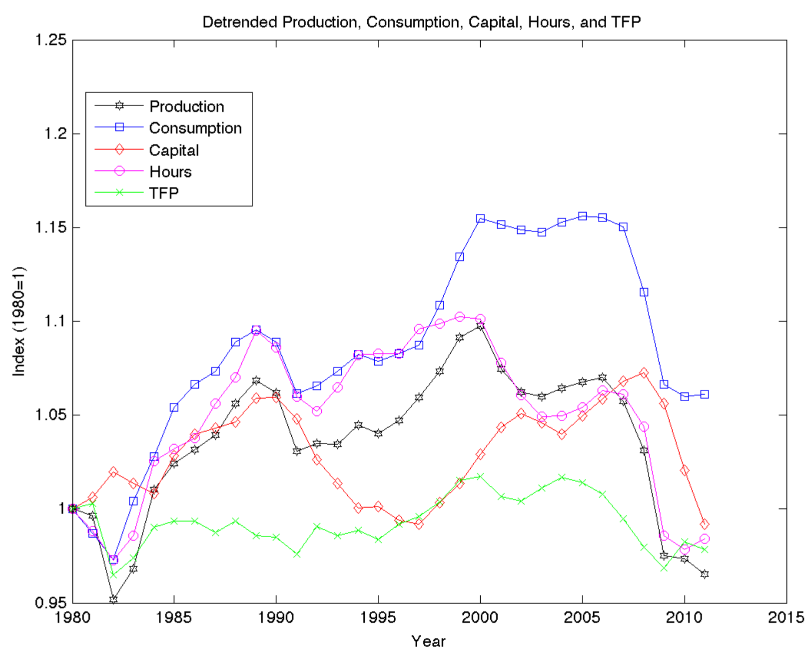

Figure 38: This figure displays the same data as figure 37, but from 1980.