Numerical Methods-Lecture VIII: Interpolation

(See Judd Chapter 6)

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Warning

- Over last few years I've noticed students don't use interpolation I teach here
- Easier to use built-in Matlab
- I'll run you through it because it is good to know the vulnerabilities and how we can avoid them
- But I won't dwell on equations

Motivation

- Most solutions are functions
- Many functions are (potentially) high-dimensional
- Want a way to simplify
- A cloud of points and connecting the dots is one way
- How should we connect the dots (and choose where they are?)

The most basic

- The basics:
    - Evenly-spaced grid, Connect the dots
    - Linearly or quadratically summarize the line

Polynomial Basis

- We can summarize functions with different bases
- The most common is the polynomial basis:

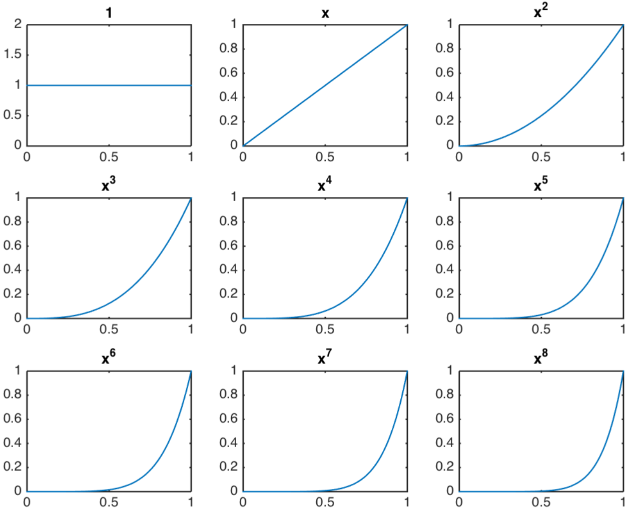
$$f(x) = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{bmatrix}$$

- This is really just the taylor polynomial:

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$$

- But the polynomials are really colinear
- This is pretty inefficient coding

Monomial Basis



Example: "Regression"

- Have a cloud of n points (data is Y), location in grid summarized by X
- Regress cloud on polynomial basis of dimension i, $i < n$:

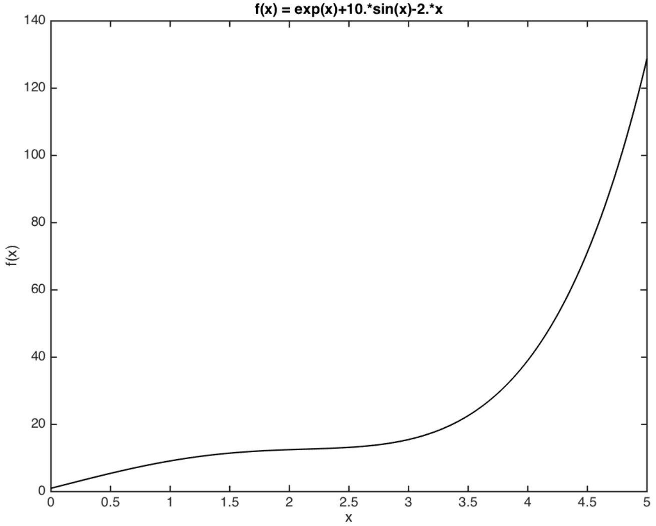
$$\hat{\beta} = (X' * X)^{-1} X' * Y$$

- Then your interpolation is always just:

$$\hat{Y} = \hat{\beta} X$$

- This is okay for a lot of problems

Example: "Regression"



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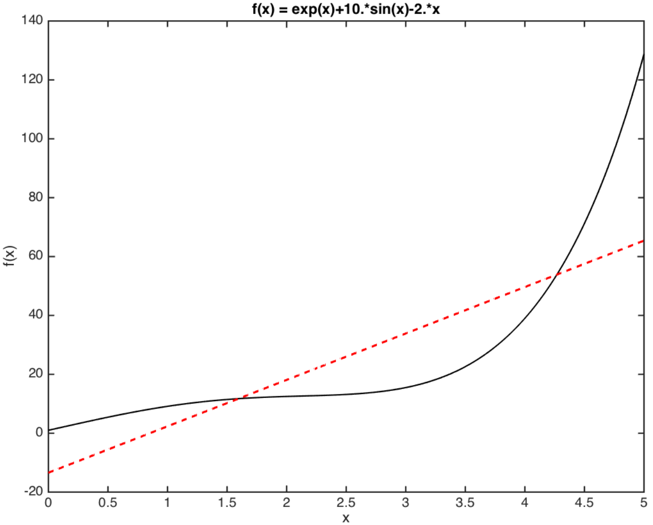

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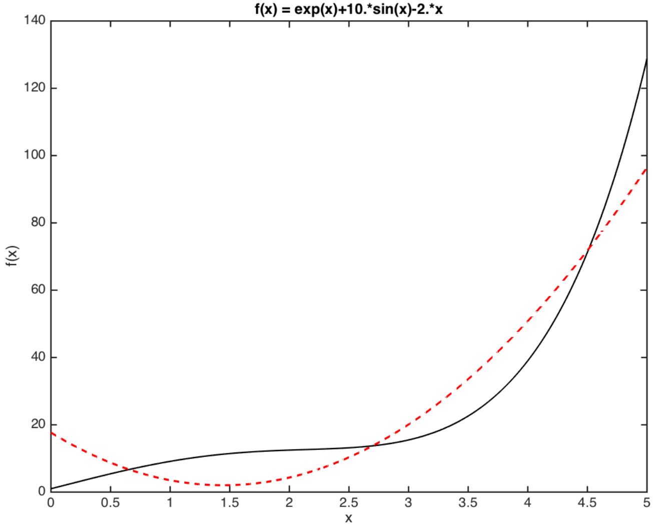

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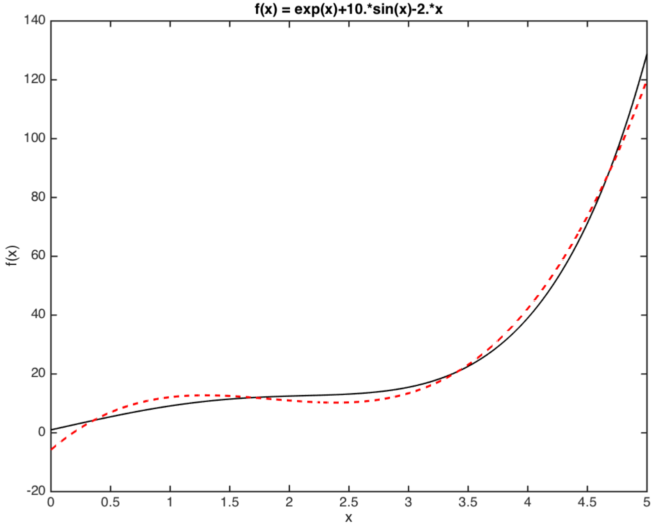

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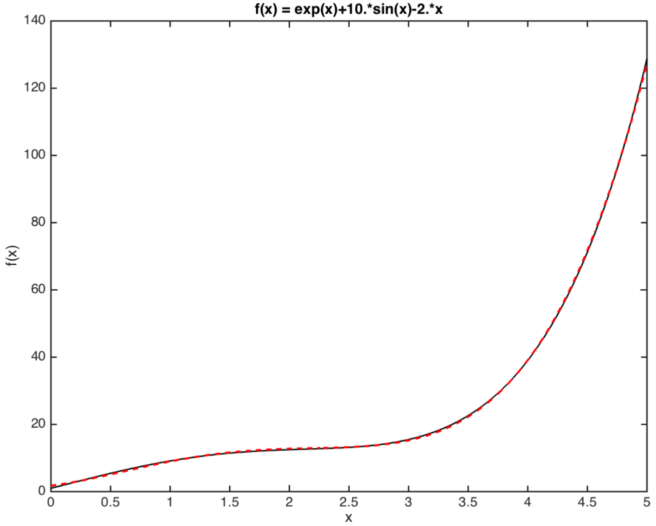

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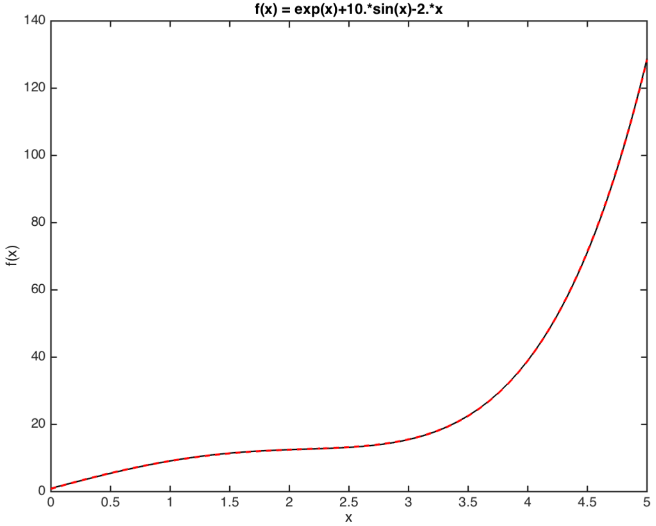
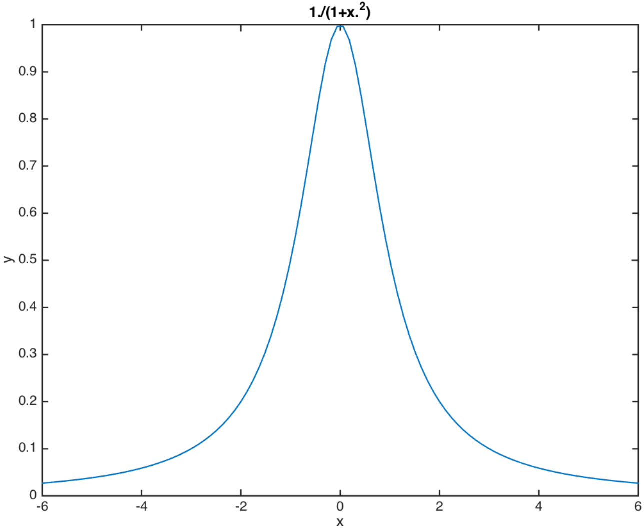
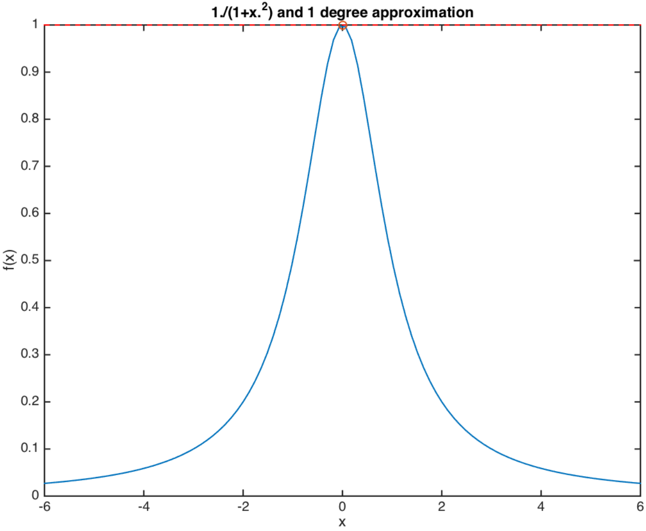


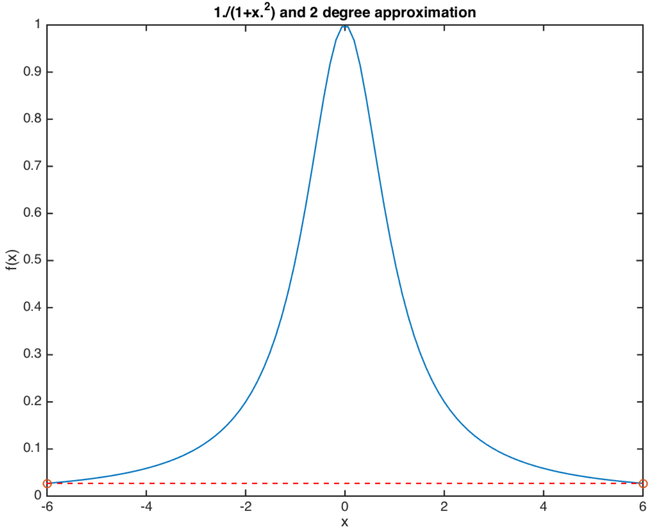
Runge's Phenomenon: $f(x) = \frac{1}{1+x.^2}$



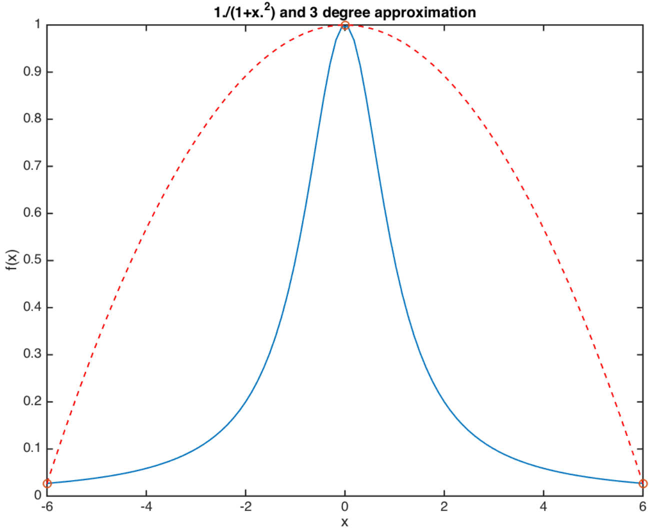
APPROX: $f(x) = \frac{1}{1+x.^2}$: 1-DEG. APPROX



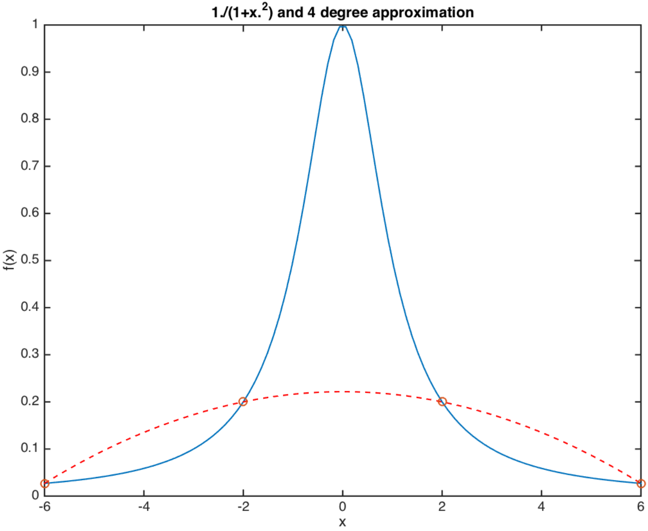
Approx: $f(x) = \frac{1}{1+x.^2}$: 2-deg. approx



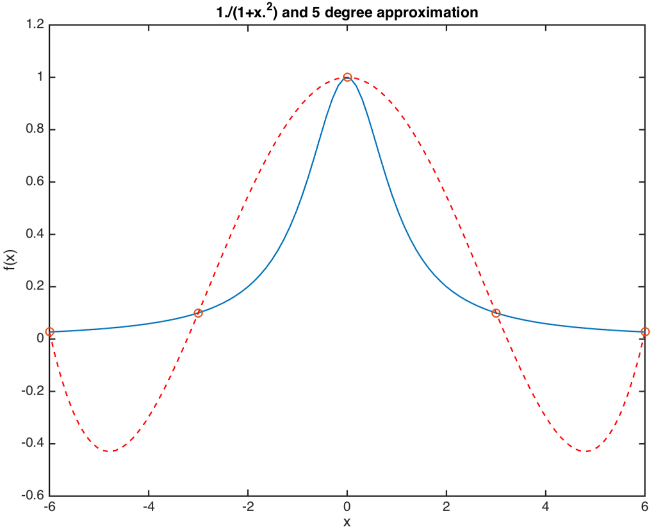
Approx: $f(x) = \frac{1}{1+x.^2}$: 3-deg. approx



APPROX: $f(x) = \frac{1}{1+x.^2}$: 4-DEG. APPROX



Approx: $f(x) = \frac{1}{1+x.^2}$: 5-deg. approx



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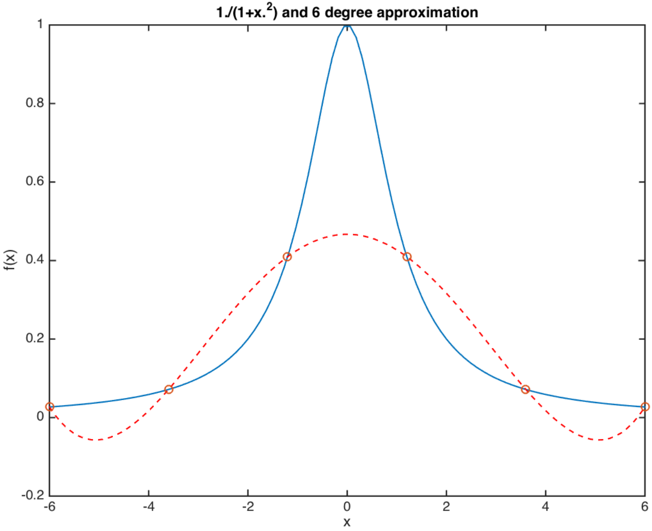
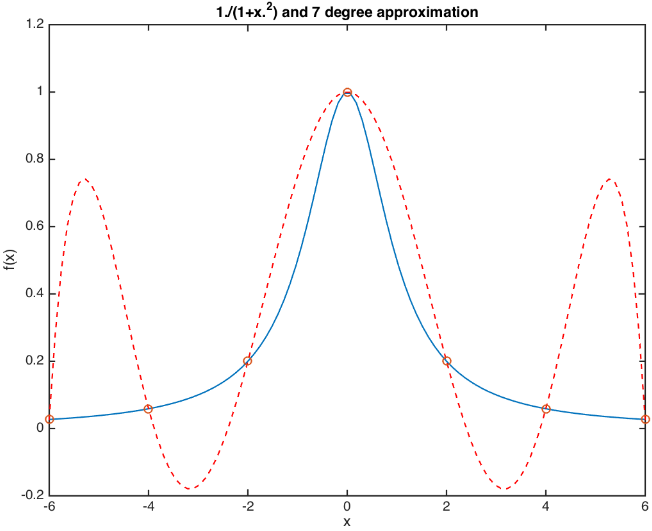


Approx: $f(x) = \frac{1}{1+x.^2}$: 7-deg. approx



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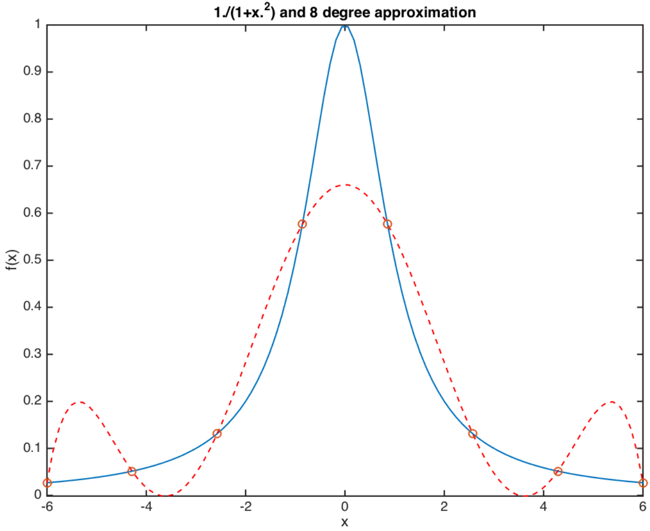

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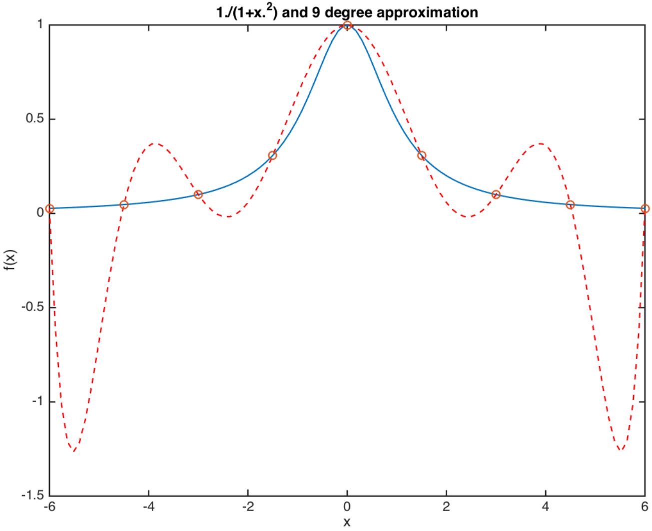

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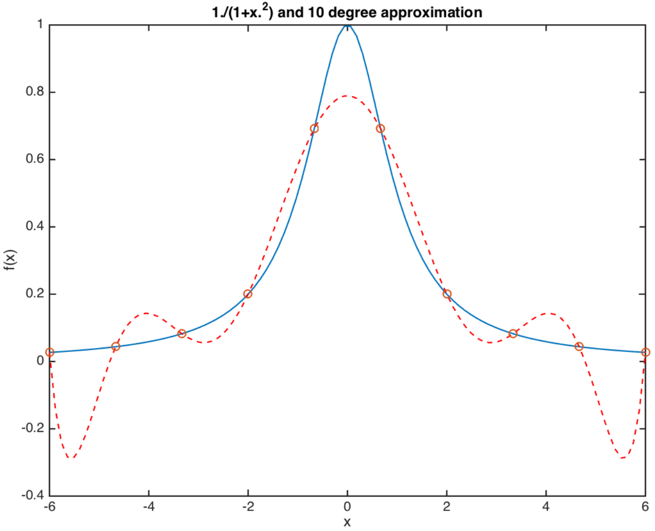

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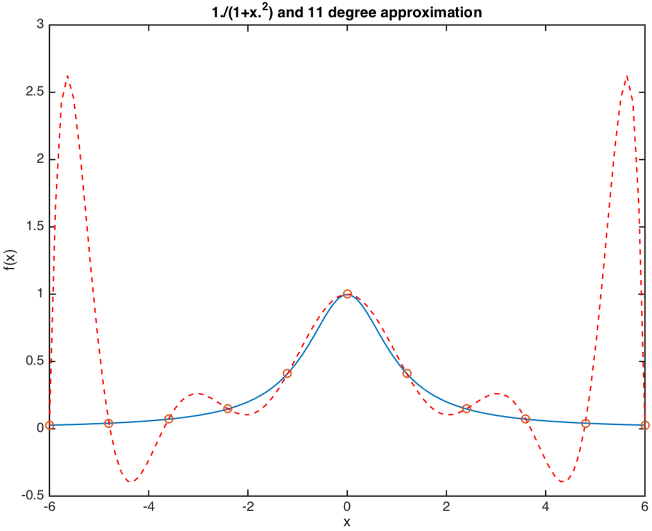
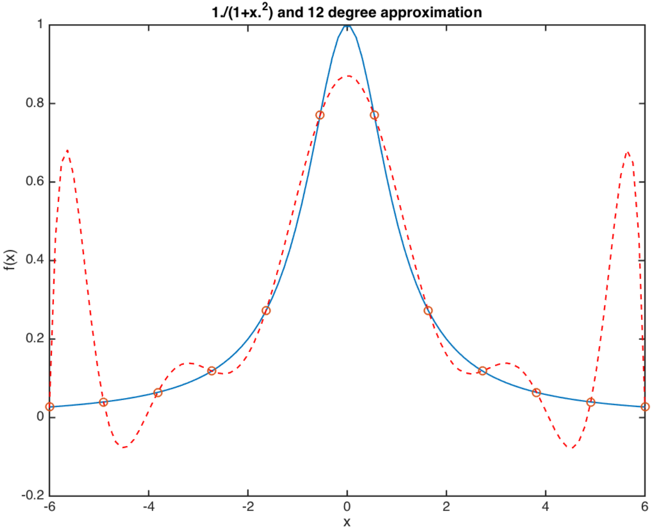


Approx: $f(x) = \frac{1}{1+x.^2}$: 12-deg. approx



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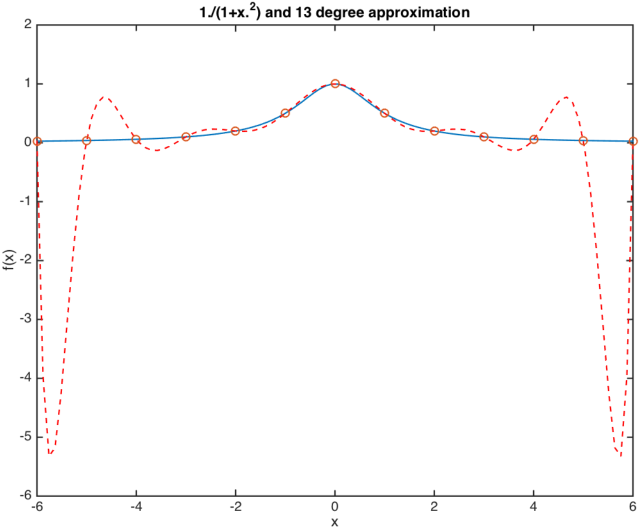
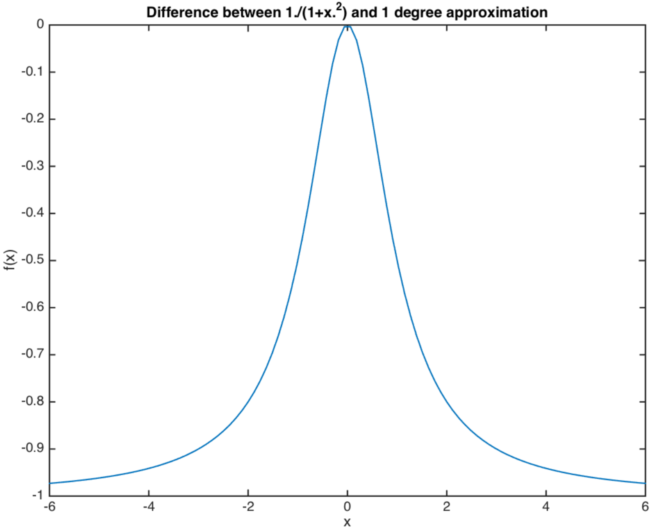


Errors: $f(x) = \frac{1}{1+x.^2}$: 1-deg. approx



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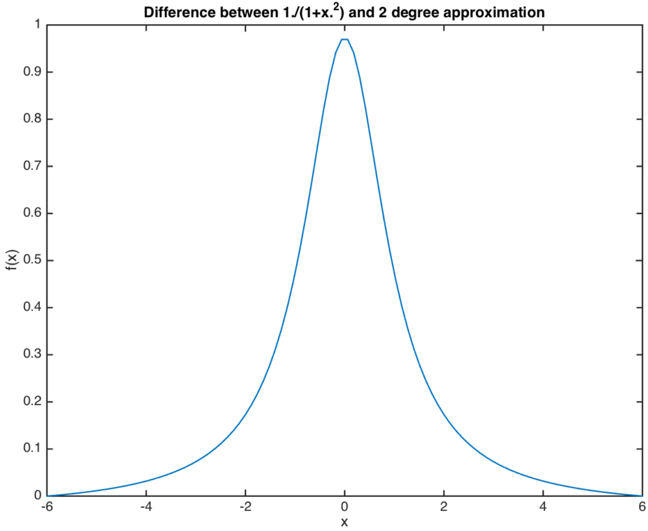

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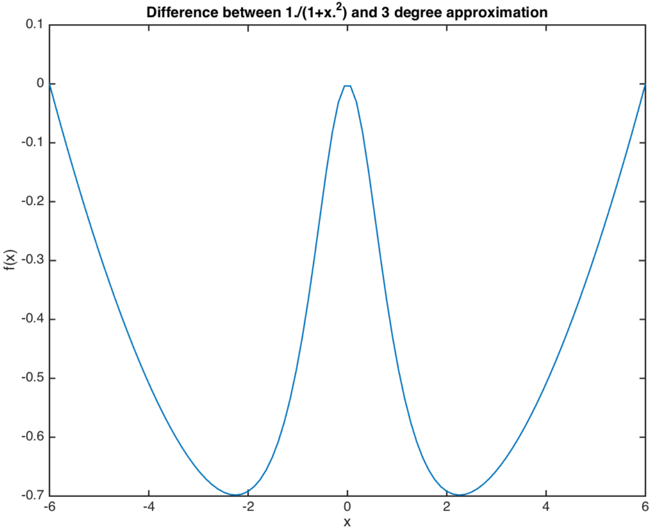

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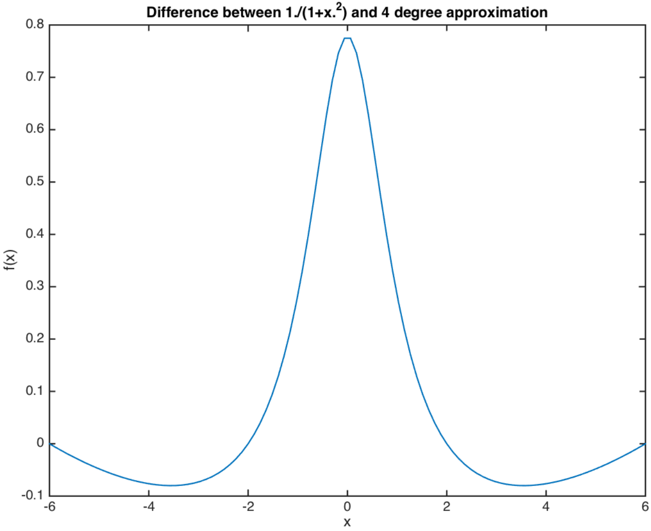

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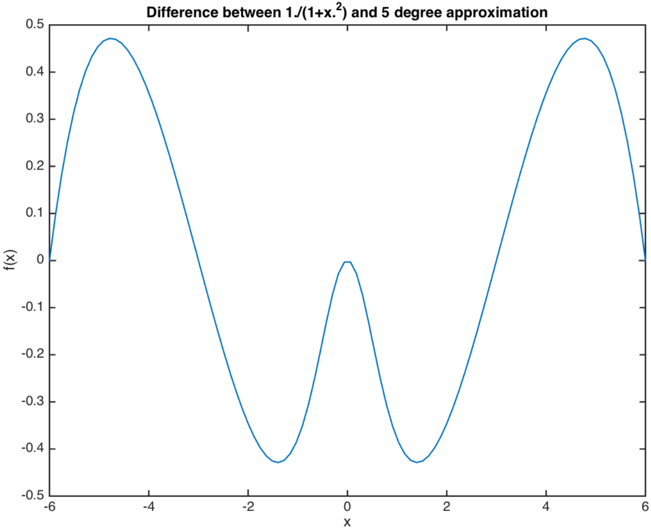

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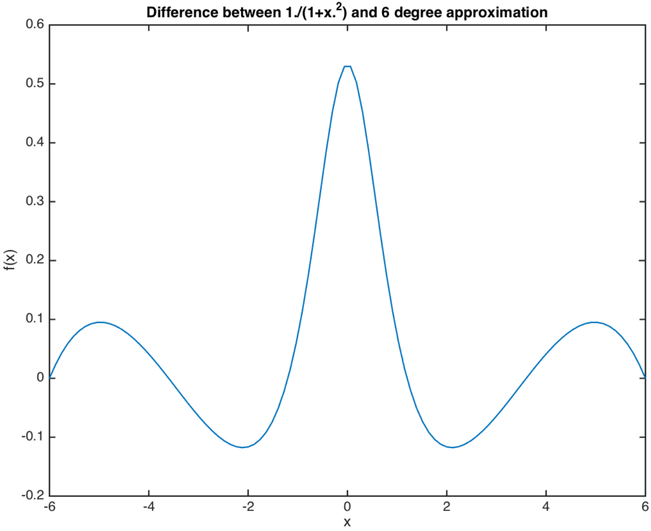

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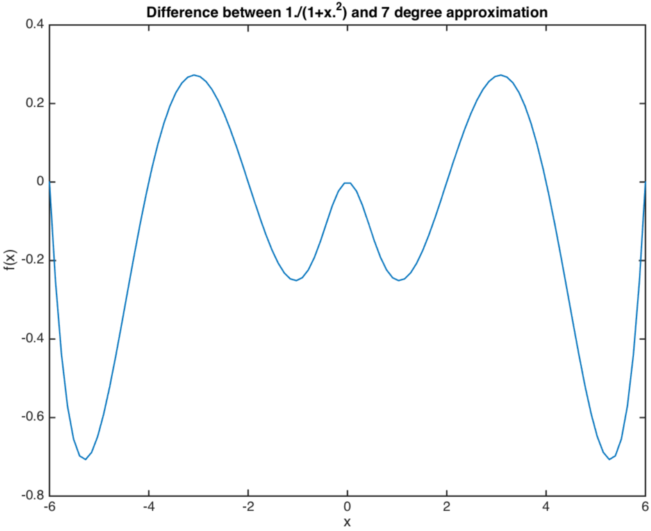

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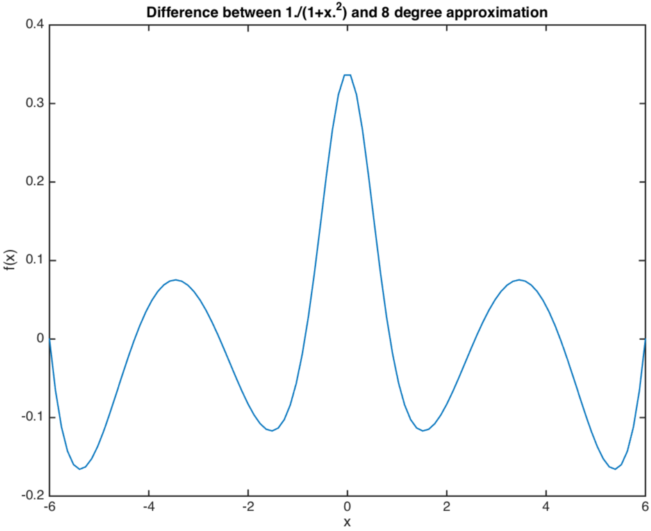

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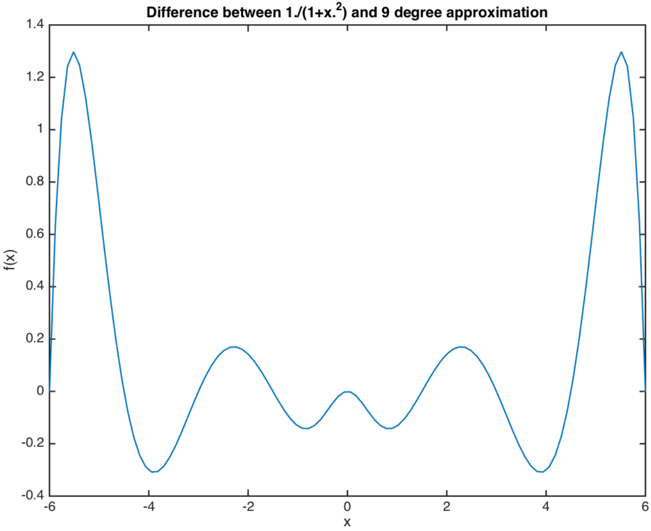

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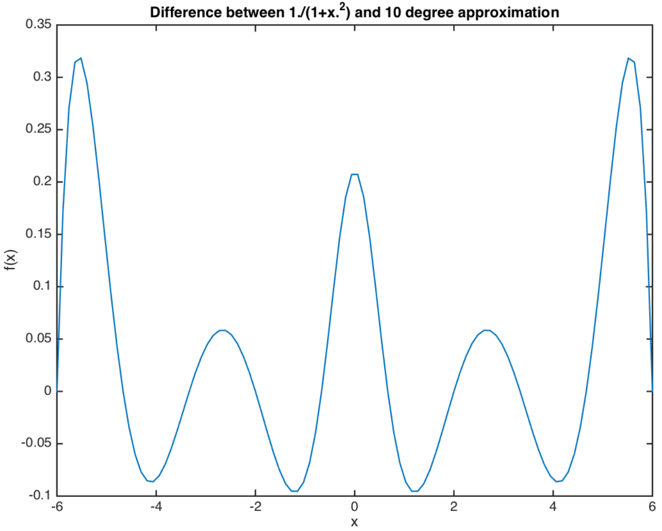

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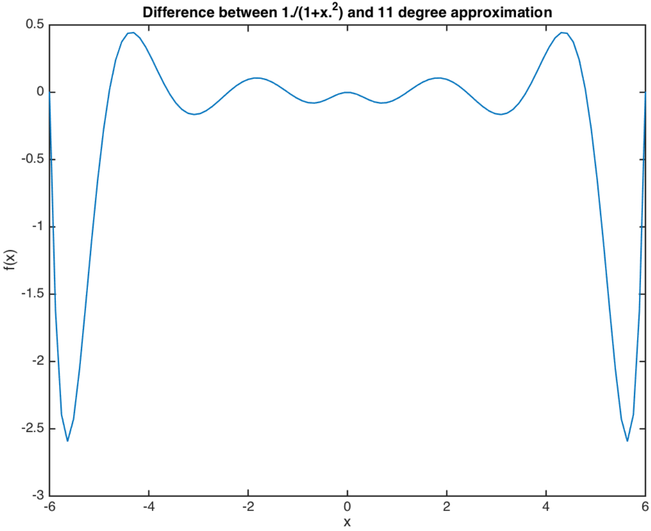

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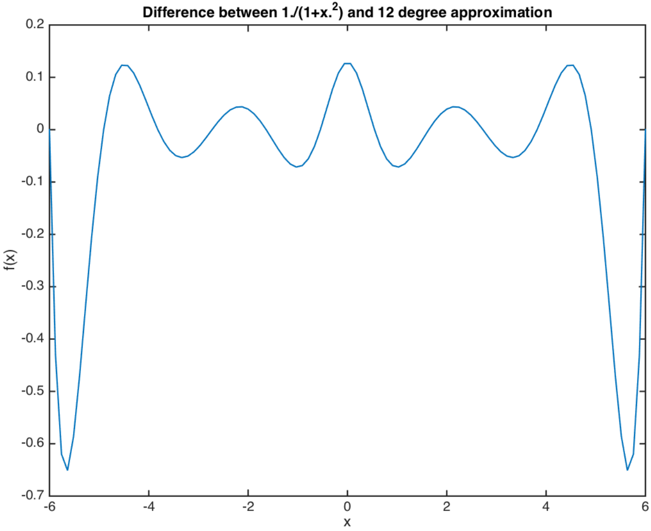

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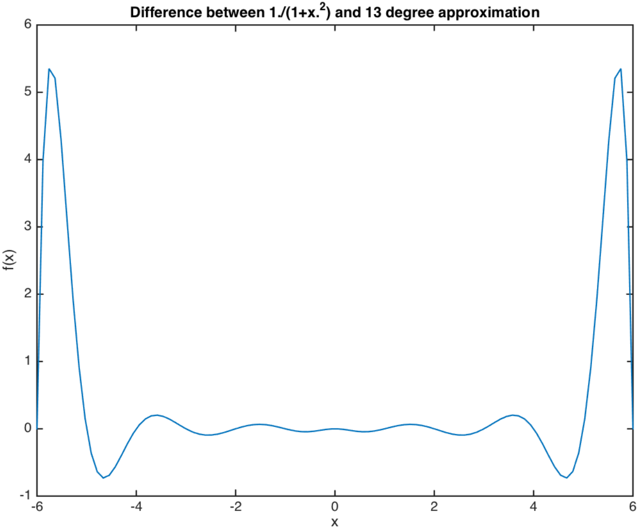

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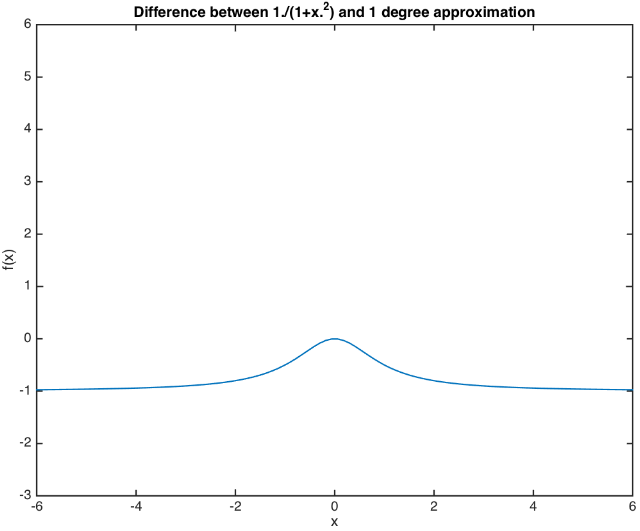

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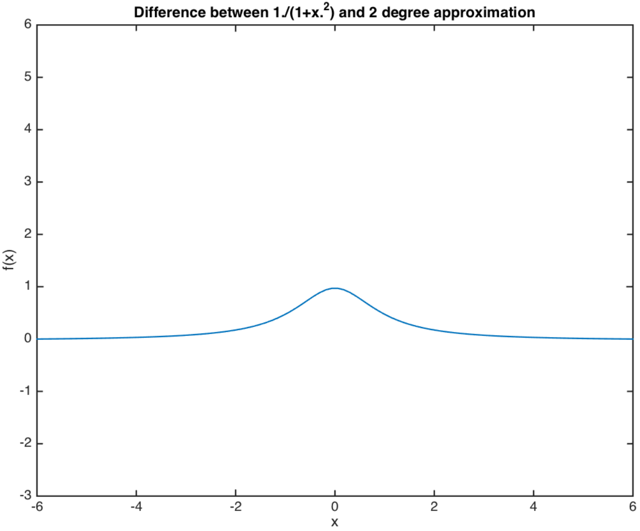

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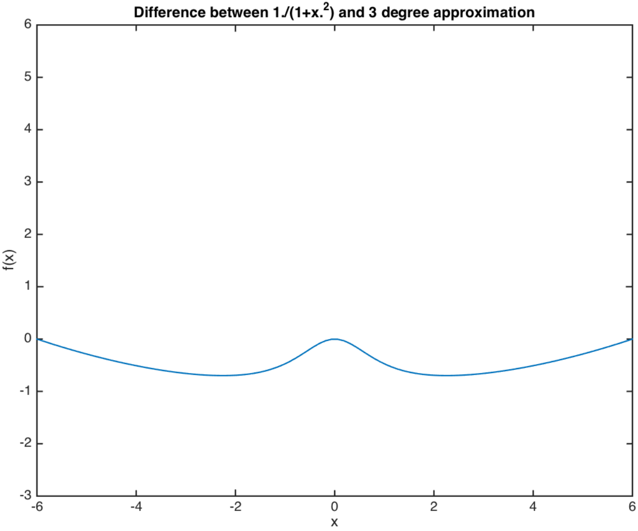

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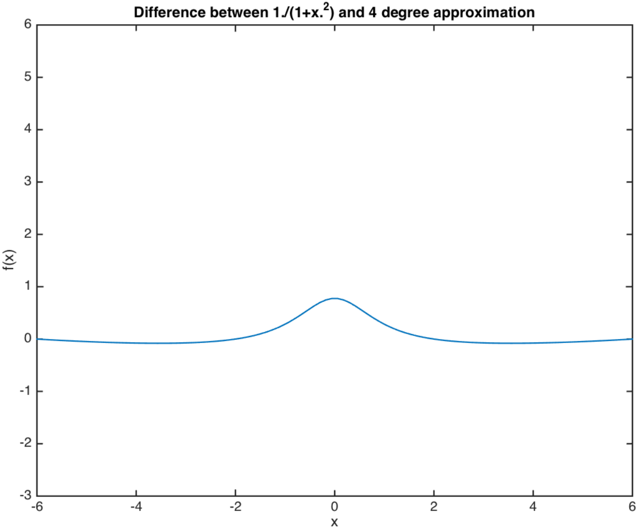

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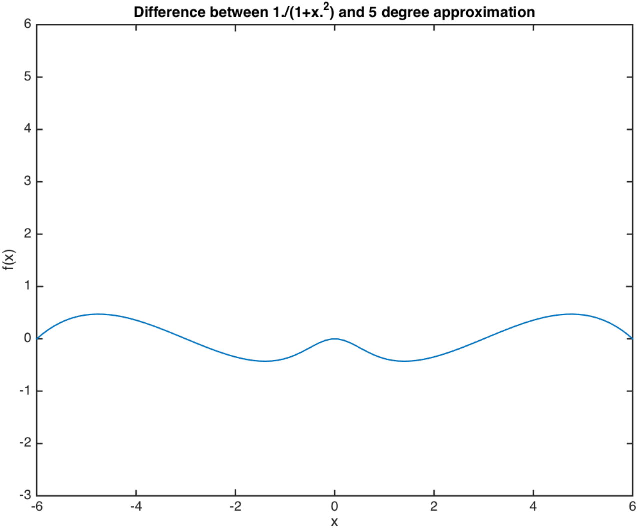

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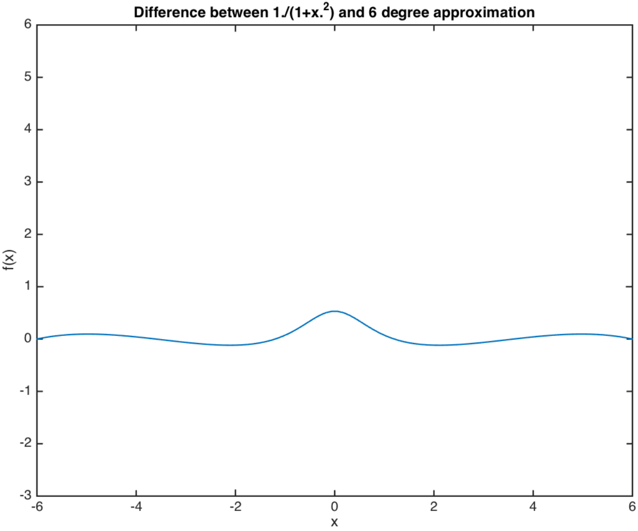

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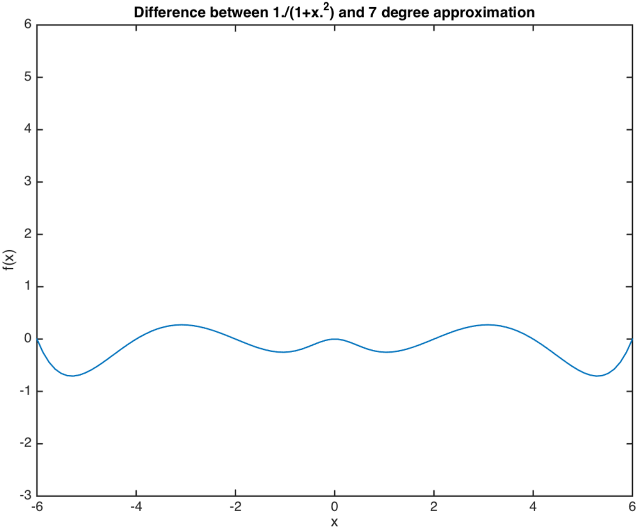

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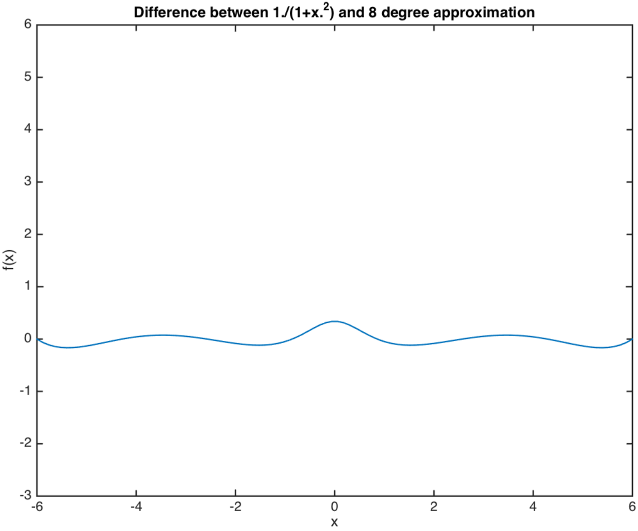

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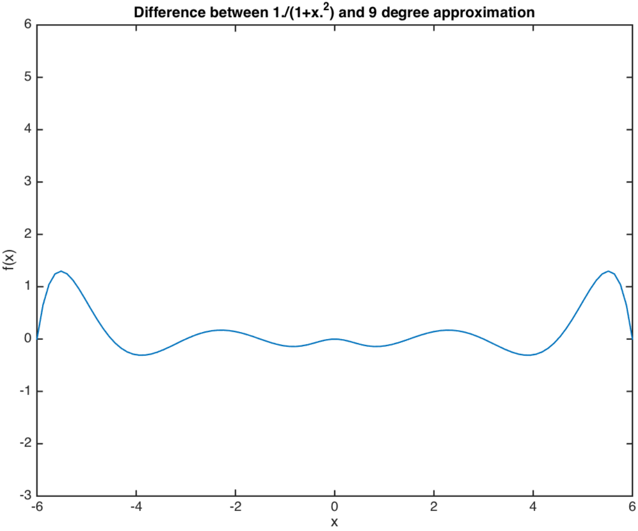

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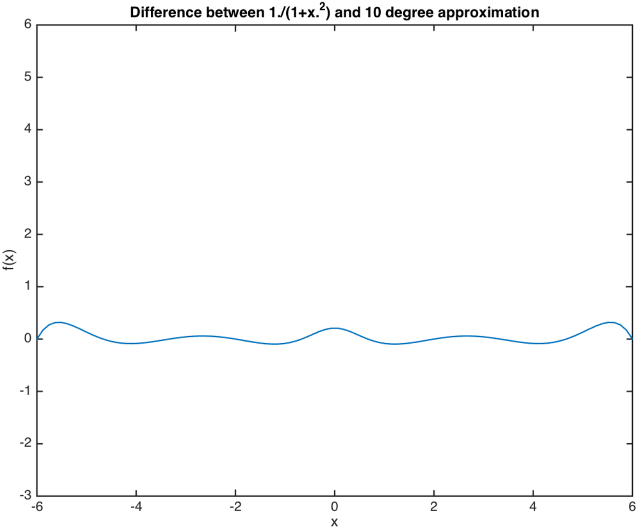

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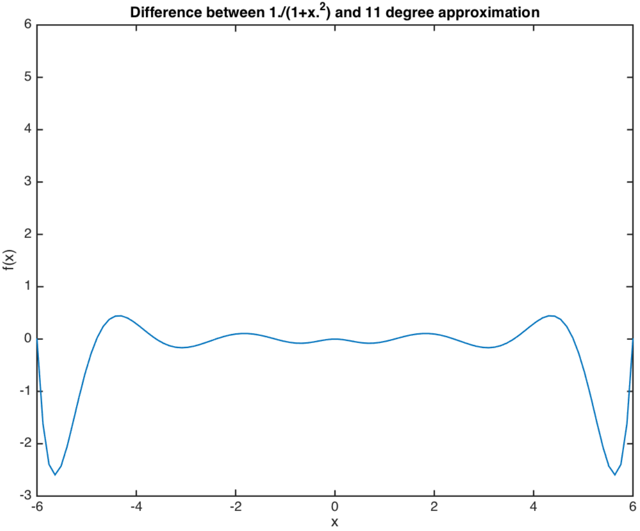

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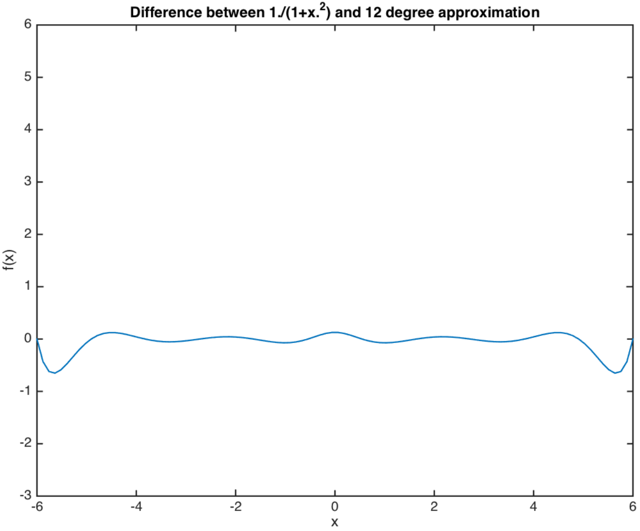

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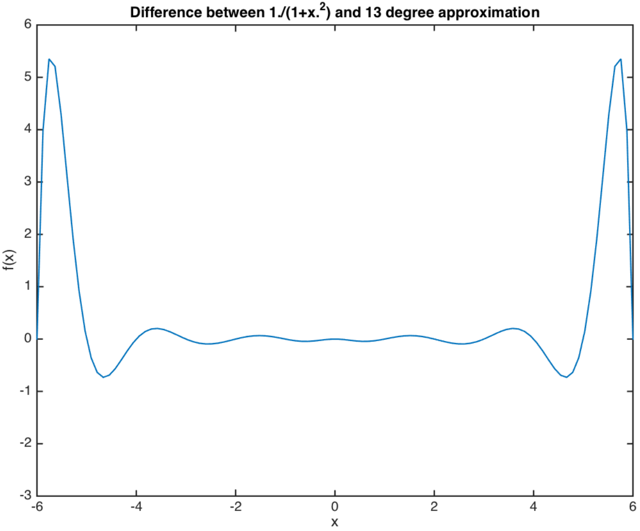


Chebychev Polynomials

- Less violent and colinear polynomials are desirable
- We also want something that avoids Runge's phenomenon if possible
- We also want something that's pretty "easy" to integrate
- It just so happens Chebychev polynomials are all of these things
- Defined recursively:

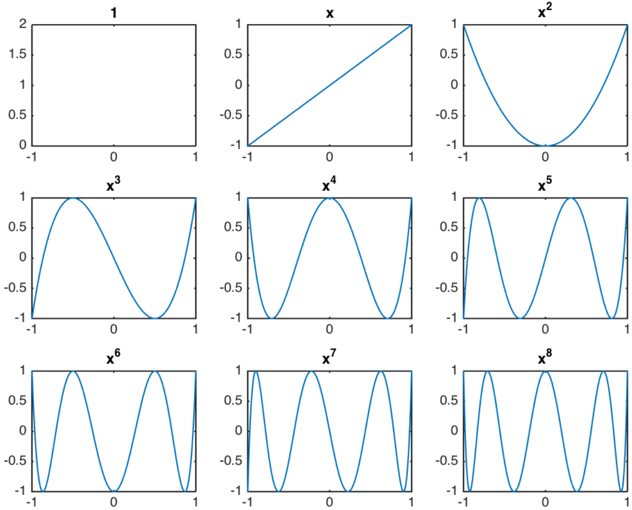
$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

- Defined over [-1,1]! Needs transformation.

Chebychev Basis



Properties

- Also can show that:

$$T_n(x) = cosh(n \cdot arccosh(x))$$

- And that the roots x_j^* T_n are:

$$x_j = cos\left(\frac{2j-1}{2n}\pi\right) \quad j = 1, ..., n$$

- It just so happens that sampling at Chebychev nodes help minimize the maximum error and make sure it's monotonically decreasing as n increases

Interpolation: 1-D

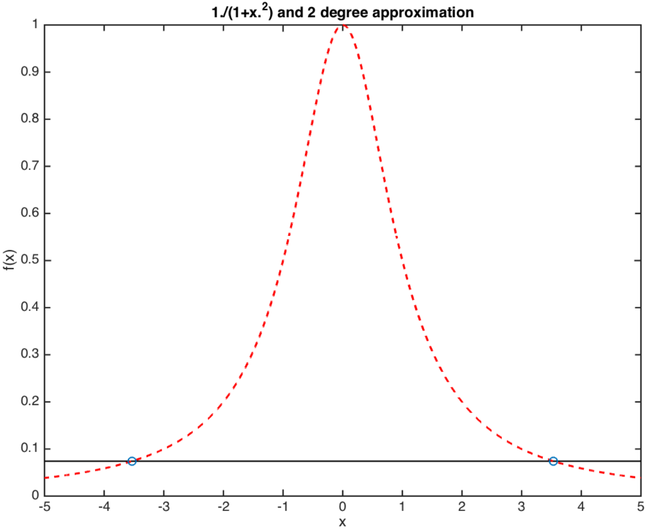
- You have some function $f(x)$ that you want to interpolate over $[a, b]$.
- Define a bridge between your range and the interpolating polynomial's range:
- Define: $z = 2(x-a)/(b-a) - 1$, or $x = a + \frac{(b-a)(z+1)}{2}$
- Then
  - $x = a \Rightarrow z = -1$
  - $x = b \Rightarrow z = 1$
  - $x = (a+b)/2 \Rightarrow z = 0.$
- Our goal will be, for interpolating polynomial $g(z)$ over $[-1, 1]$ and $f(x)$ over $[a, b]$, that:

$$g(2(x-a)/(b-a) - 1) = f(x) \quad \forall x$$

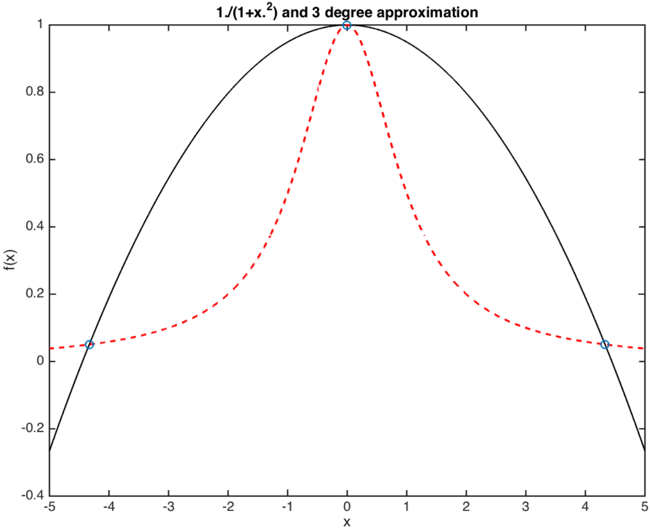
Interpolation: 1-d

1. Pick fcn. $f(x)$, degree n, bounds a, b for interpolation.
2. Find Chebychev nodes for $f(x)$ in "z" space.
$$z_j = -cos\left(\frac{2j-1j}{2n}\pi\right) \quad j = 1, ..., n$$
3. Convert nodes into "x" space:
$$x_j = a + \frac{(b-a)(z_j+1)}{2}$$
4. Calculate $y_j = f(x_j)$ at each x_j. This is the value of $g(z_j)$ at each of the corresponding nodes.
5. Given y_j, calculate the Chebychev coefficients c_j as:
$$c_j = \frac{2}{n+1}\sum_{k=0}^{n} y_j T_j(z_j)$$
6. Your approximation is given by:
$$\hat{f}(x) = \sum_j c_j T_j(x) \approx f(x)$$

FIT: $f(x) = \frac{1}{1+x.^2}$: 2-DEG. APPROX



Fit: $f(x) = \frac{1}{1+x.^2}$: 3-deg. approx



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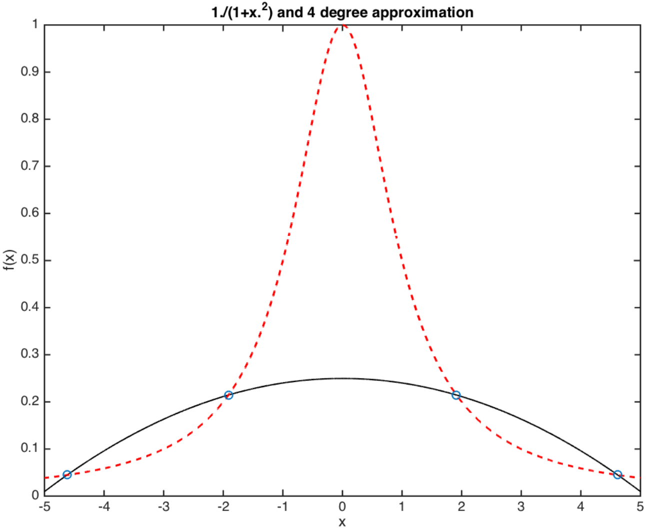

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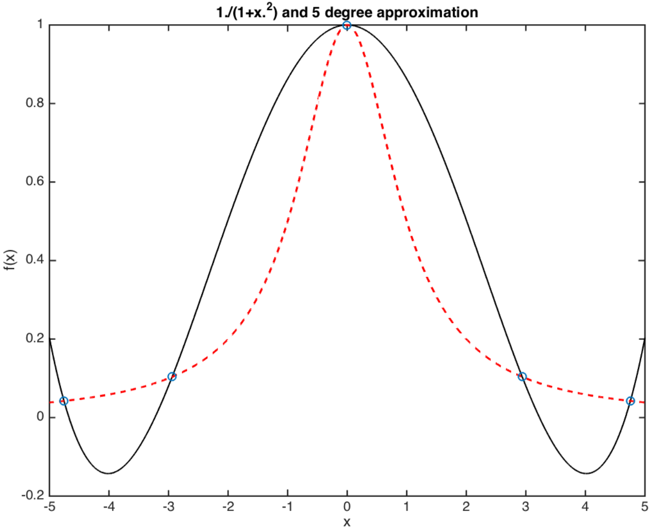

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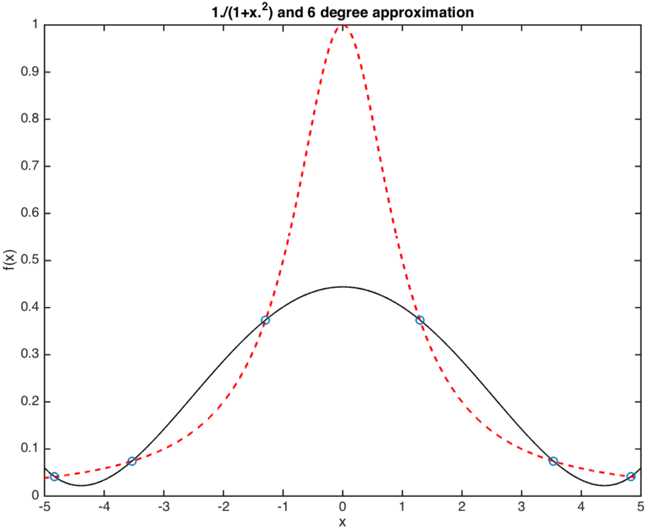
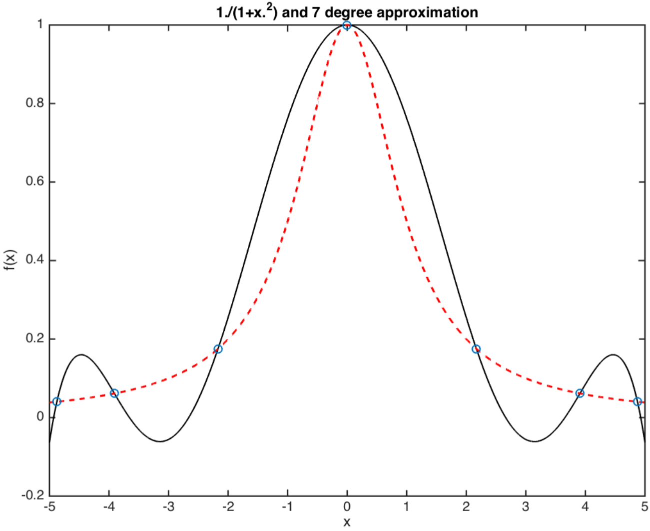
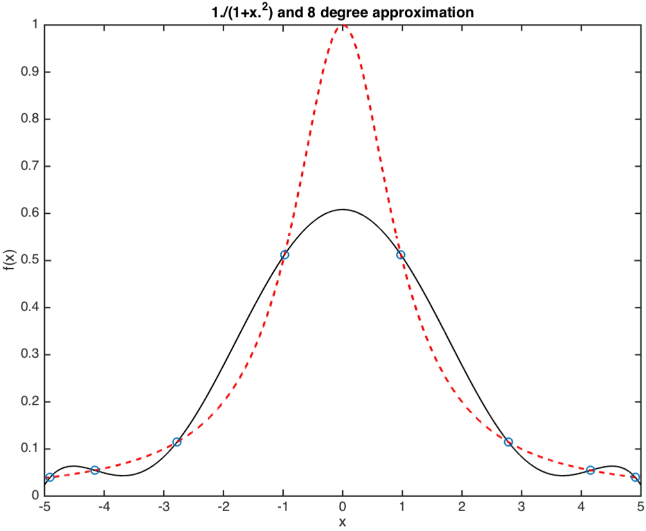


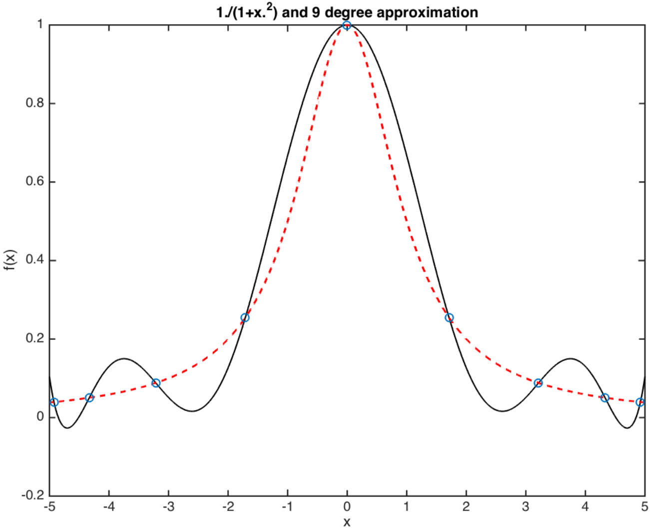
FIT: $f(x) = \frac{1}{1+x.^2}$: 7-DEG. APPROX



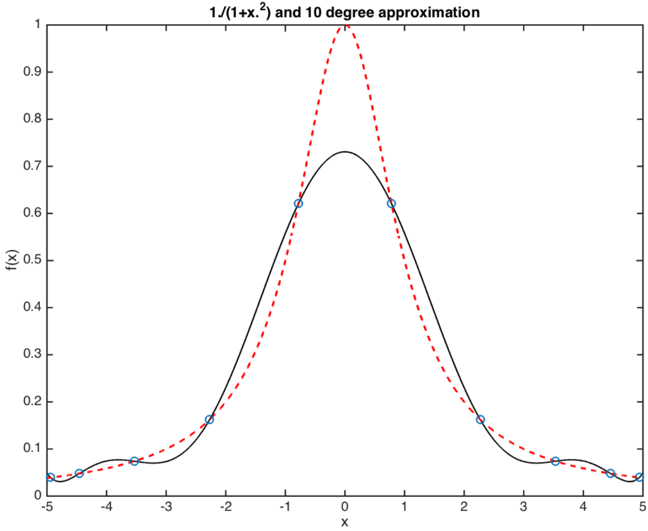
FIT: $f(x) = \frac{1}{1+x.^2}$: 8-DEG. APPROX



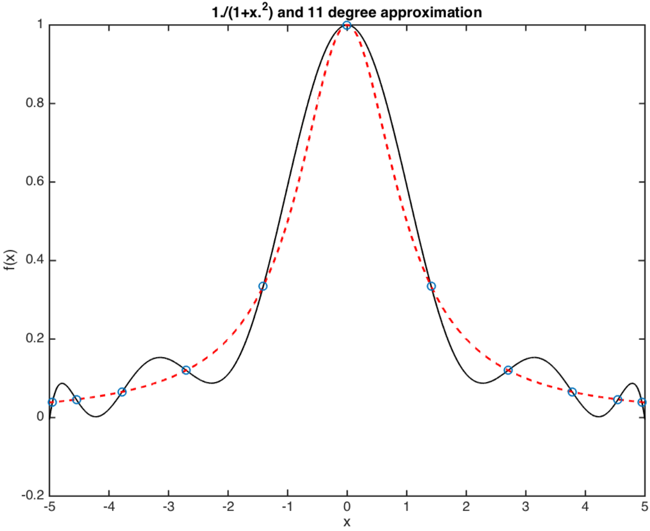
FIT: $f(x) = \frac{1}{1+x.^2}$: 9-DEG. APPROX



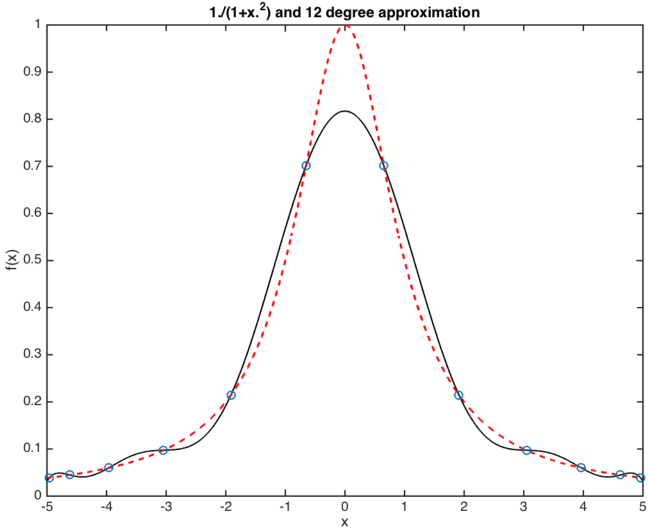
FIT: $f(x) = \frac{1}{1+x.^2}$: 10-DEG. APPROX



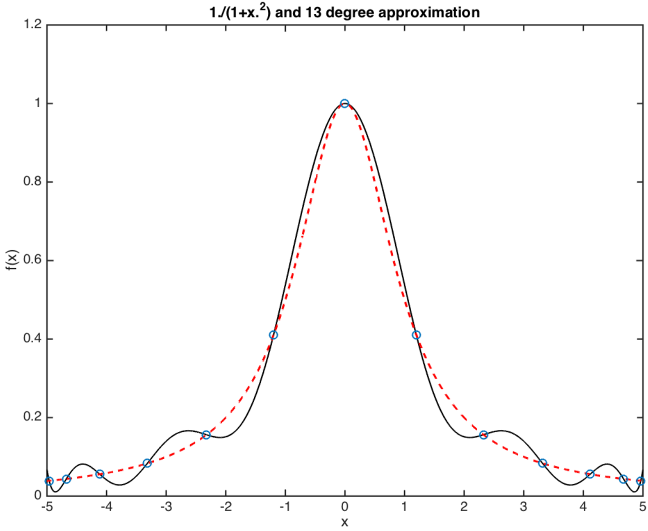
FIT: $f(x) = \frac{1}{1+x.^2}$: 11-DEG. APPROX



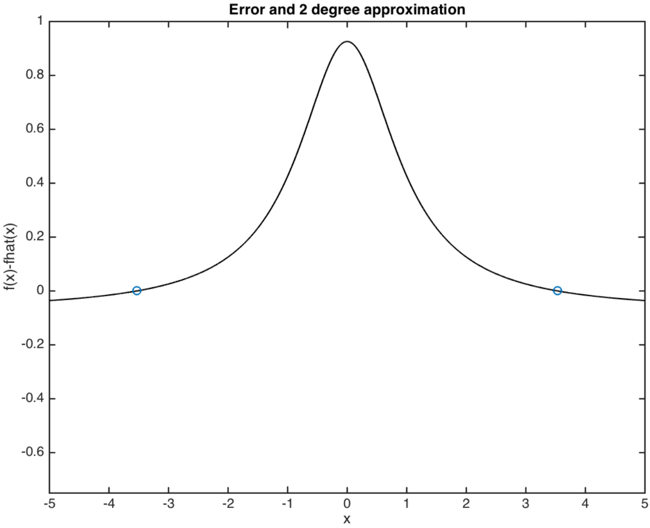
FIT: $f(x) = \frac{1}{1+x.^2}$: 12-DEG. APPROX



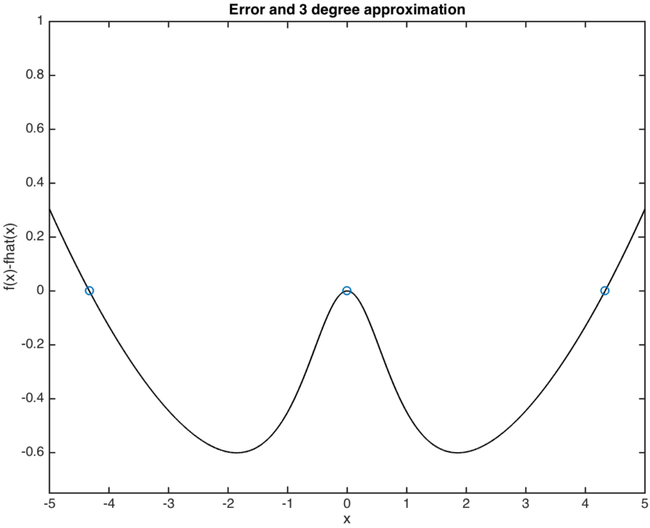
FIT: $f(x) = \frac{1}{1+x.^2}$: 13-DEG. APPROX



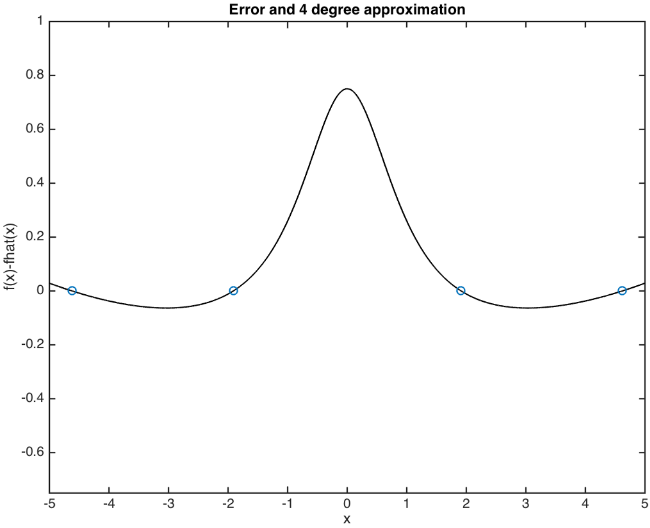
ERRORS: $f(x) = \frac{1}{1+x.^2}$: 2-DEG. APPROX



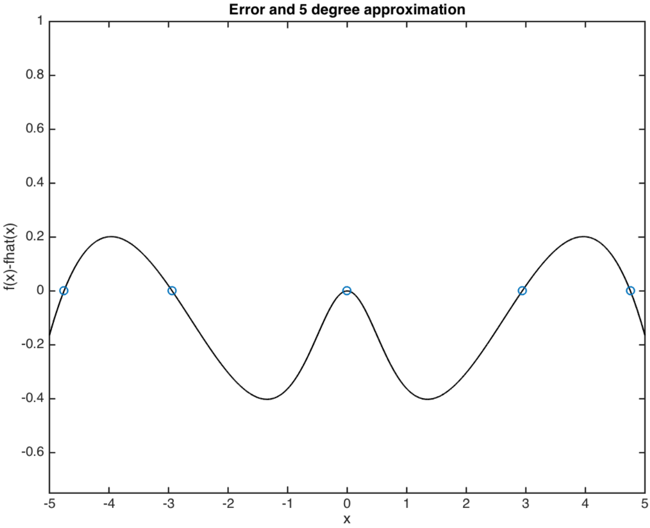
Errors: $f(x) = \frac{1}{1+x.^2}$: 3-deg. approx



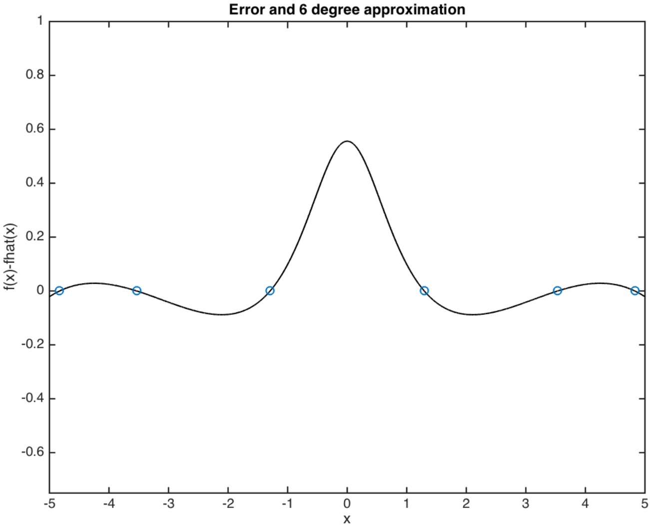
Errors: $f(x) = \frac{1}{1+x.^2}$: 4-deg. approx



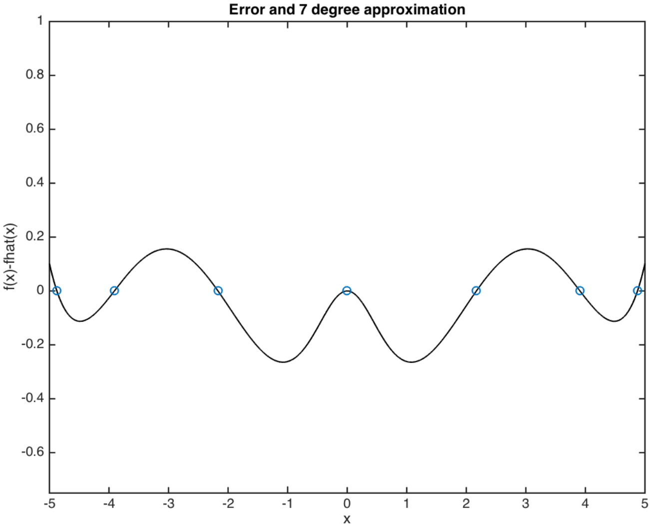
ERRORS: $f(x) = \frac{1}{1+x.^2}$: 5-DEG. APPROX



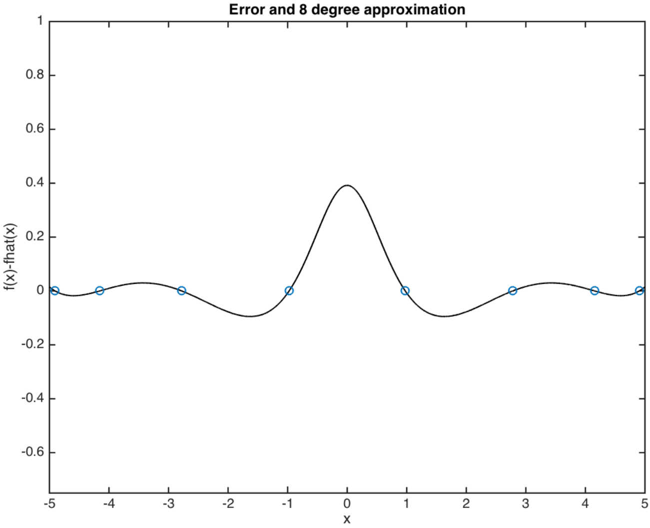
ERRORS: $f(x) = \frac{1}{1+x.^2}$: 6-DEG. APPROX



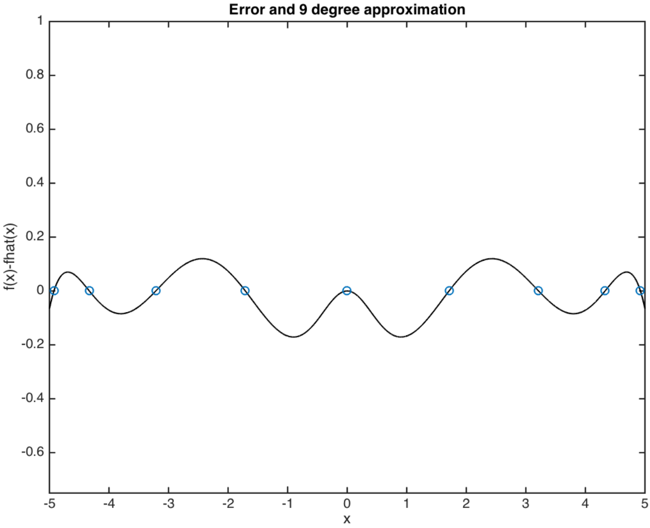
Errors: $f(x) = \frac{1}{1+x.^2}$: 7-deg. approx



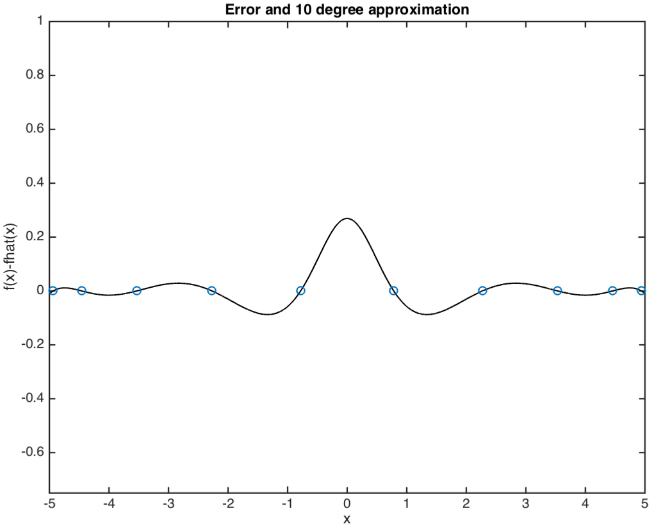
ERRORS: $f(x) = \frac{1}{1+x.^2}$: 8-DEG. APPROX



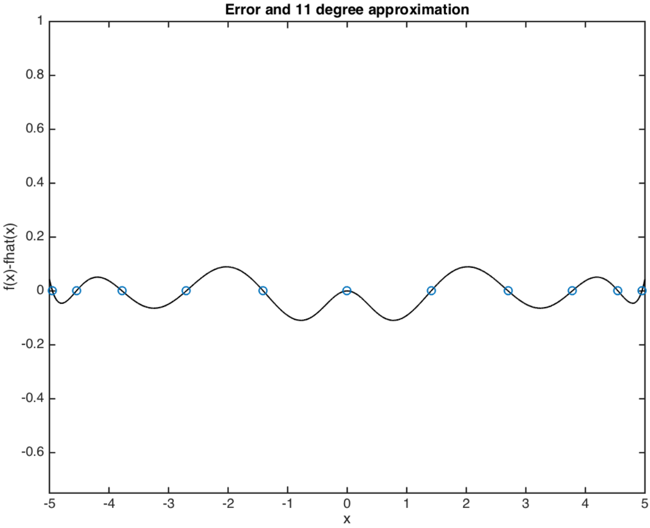
Errors: $f(x) = \frac{1}{1+x.^2}$: 9-deg. approx



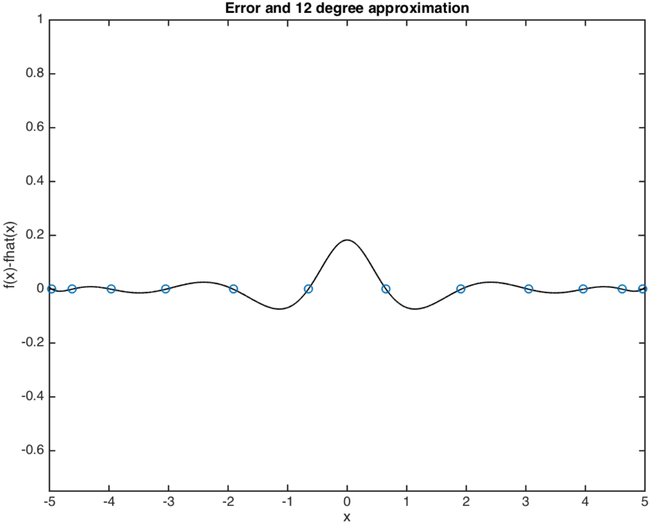
Errors: $f(x) = \frac{1}{1+x.^2}$: 10-deg. approx



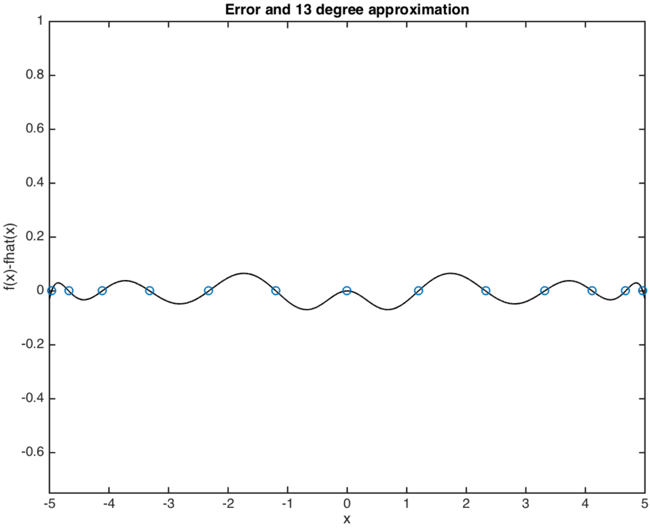
Errors: $f(x) = \frac{1}{1+x.^2}$: 11-deg. approx



Errors: $f(x) = \frac{1}{1+x.^2}$: 12-deg. approx



Errors: $f(x) = \frac{1}{1+x.^2}$: 13-deg. approx



Interpolation: 2-d

1. Pick fcn $f(x,y)$ deg. n, bounds a,b and c,d for interpolation.
2. Find Chebychev nodes for $f(x,y)$ in "$z^{[1]}$" and "$z^{[2]}$" space.

$$z_j^{[1]} = cos\left(\frac{2j-1j}{2n}\pi\right), \qquad z_j^{[2]} = cos\left(\frac{2j-1j}{2n}\pi\right), \quad j = 1, ..., n$$

3. Convert nodes into "x" space:

$$x_j = a + \frac{(b-a)(z_j^{[1]}+1)}{2}, \quad y_j = c + \frac{(d-c)(z_j^{[2]}+1)}{2}$$

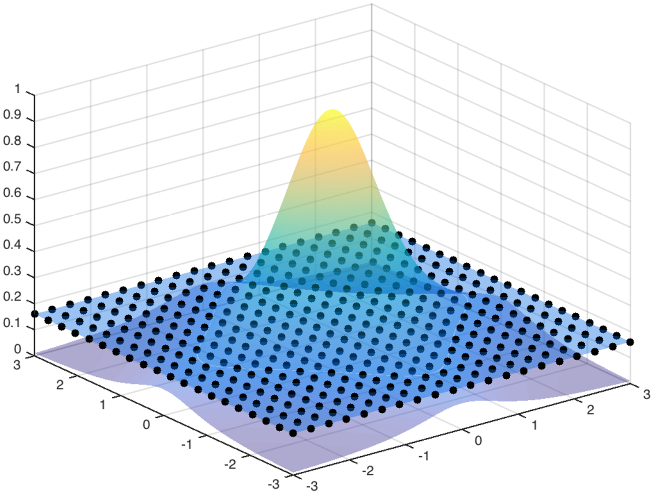
4. Calculate $\xi_{i,j} = f(x_i, y_j)$ for all points.
5. Given y_j, calculate the Chebychev coefficients c_j as:

$$c_{i,j} = \frac{\sum_{k=1}^{n} \sum_{l=1}^{n} \xi_{i,j} T_i(z_k^{[1]}) T_j(z_l^{[2]})}{\sum_{k=1}^{n} T_i(z_k^{[1]}) T_j(z_l^{[2]})}$$

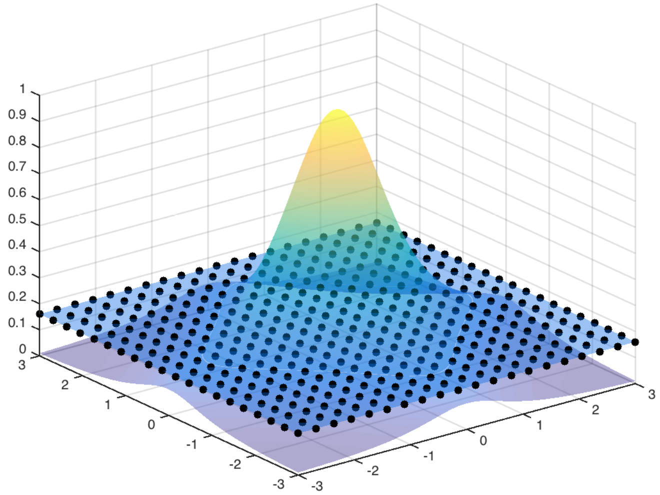
6. Approximation:

$$\hat{f}(x,y) \approx \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} T_i\left(2\frac{x-a}{b-a} - 1\right) T_j\left(2\frac{y-c}{d-b} - 1\right)$$

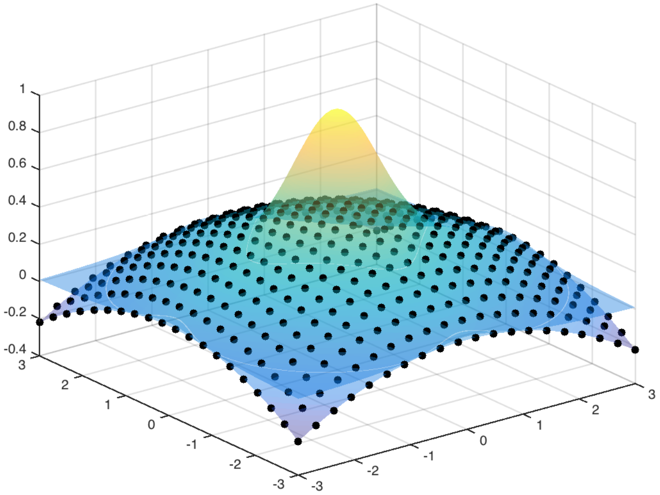
MONOM FIT: $f(x) = \frac{1}{1+x.^2}$: 1-DEG. APPROX



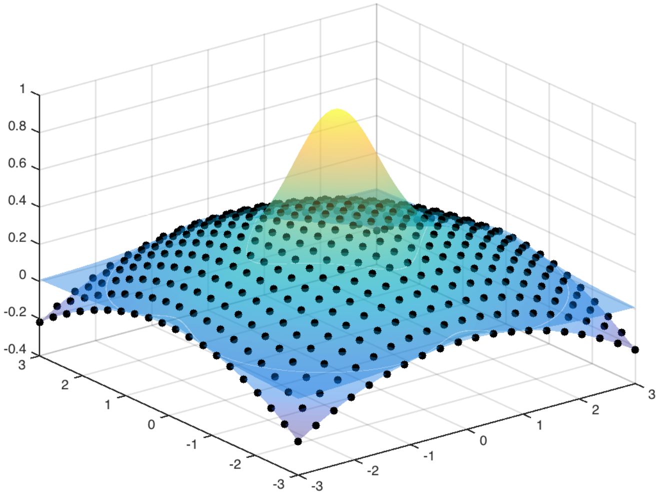
Monom Fit: $f(x) = \frac{1}{1+x.^2}$: 2-deg. approx



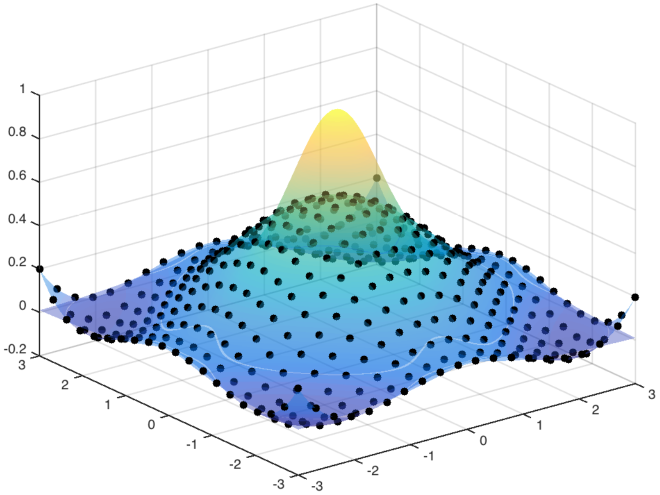
Monom Fit: $f(x) = \frac{1}{1+x.^2}$: 3-deg. approx



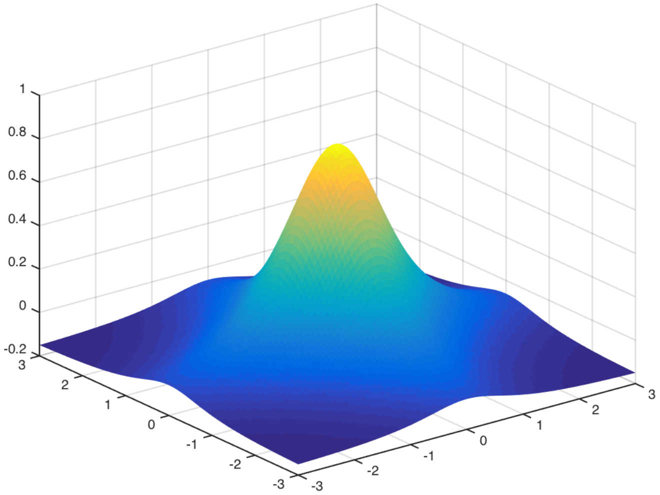
Monom Fit: $f(x) = \frac{1}{1+x.^2}$: 4-deg. approx



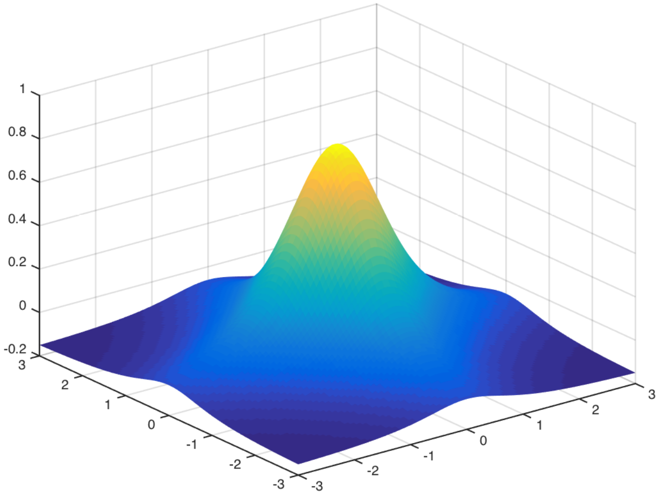
Monom Fit: $f(x) = \frac{1}{1+x.^2}$: 5-deg. approx



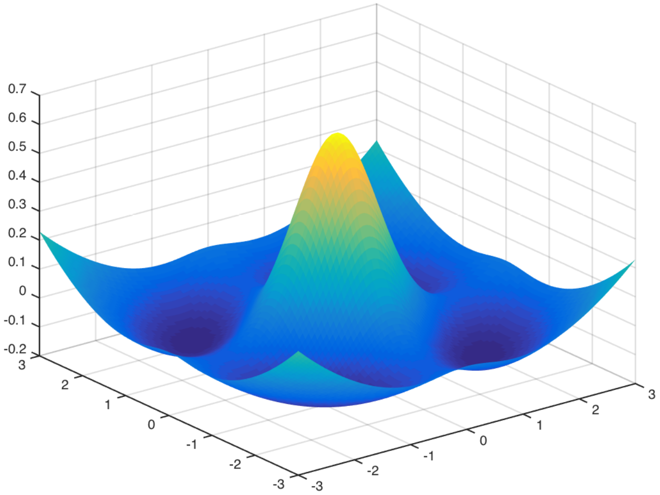
Monom Errors: $f(x) = \frac{1}{1+x.^2}$: 1-deg. approx



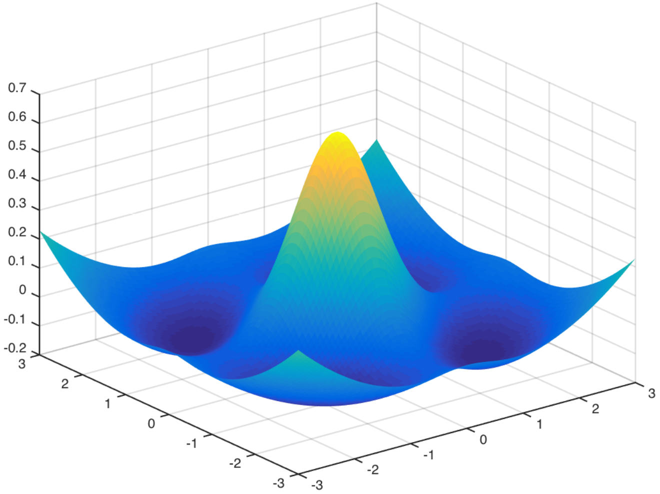
Monom Errors: $f(x) = \frac{1}{1+x.^2}$: 2-deg. approx



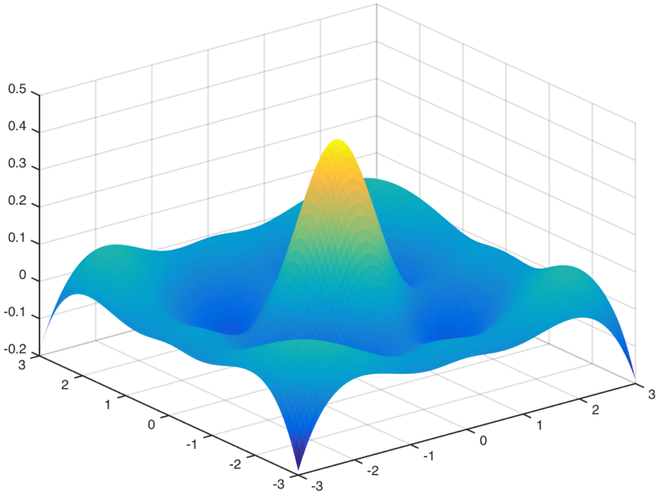
MONOM ERRORS: $f(x) = \frac{1}{1+x.^2}$: 3-DEG. APPROX



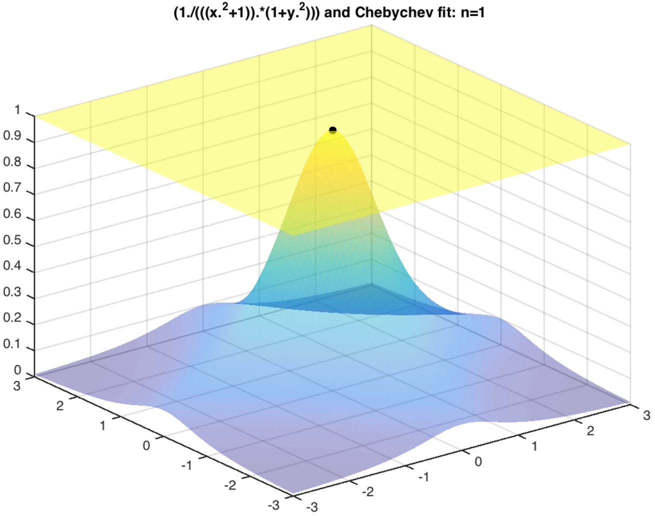
Monom Errors: $f(x) = \frac{1}{1+x.^2}$: 4-deg. approx



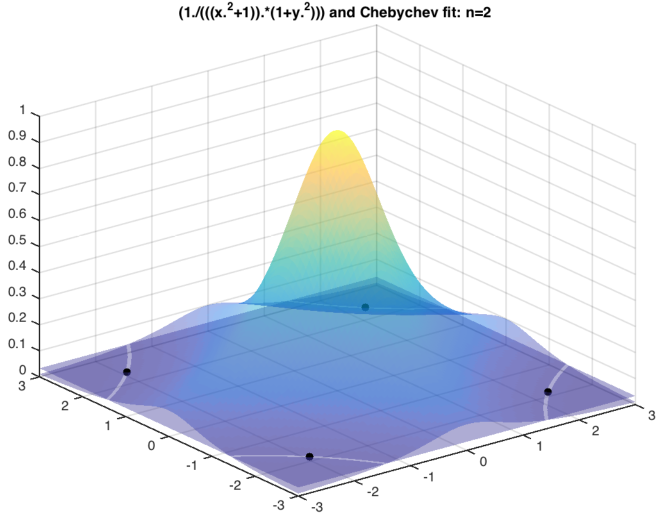
Monom Errors: $f(x) = \frac{1}{1+x.^2}$: 5-deg. approx



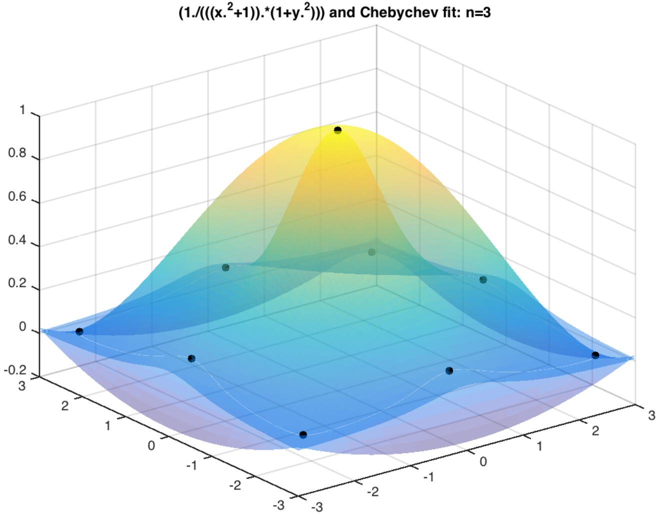
Cheby Fit: $f(x) = \frac{1}{1+x.^2}$: 1-deg. approx



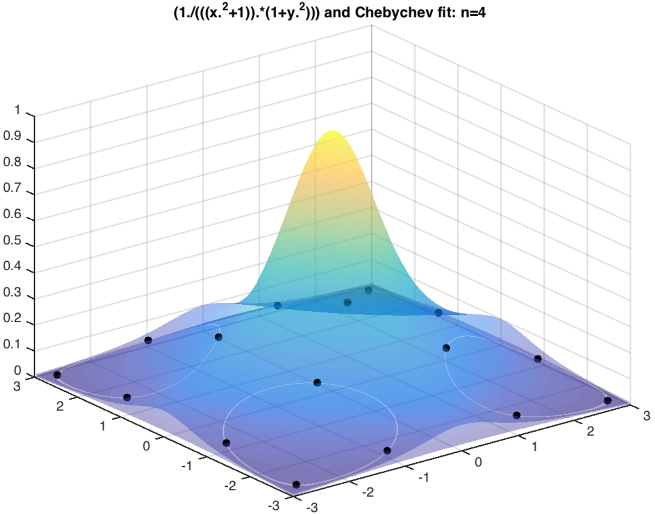
Cheby Fit: $f(x) = \frac{1}{1+x.^2}$: 2-deg. approx



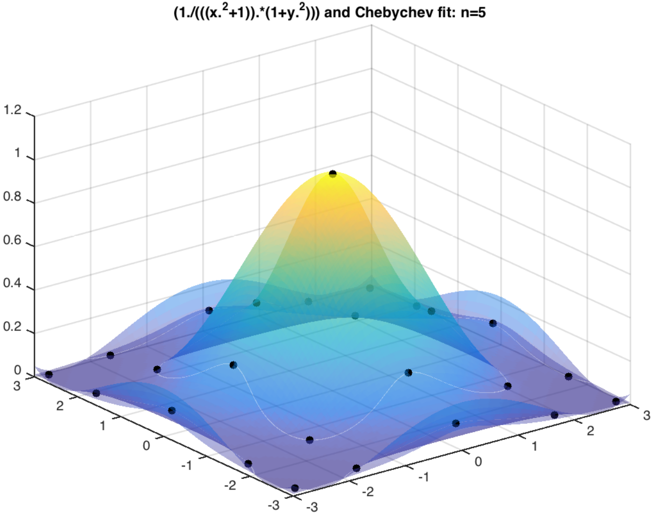
Cheby Fit: $f(x) = \frac{1}{1+x.^2}$: 3-deg. approx



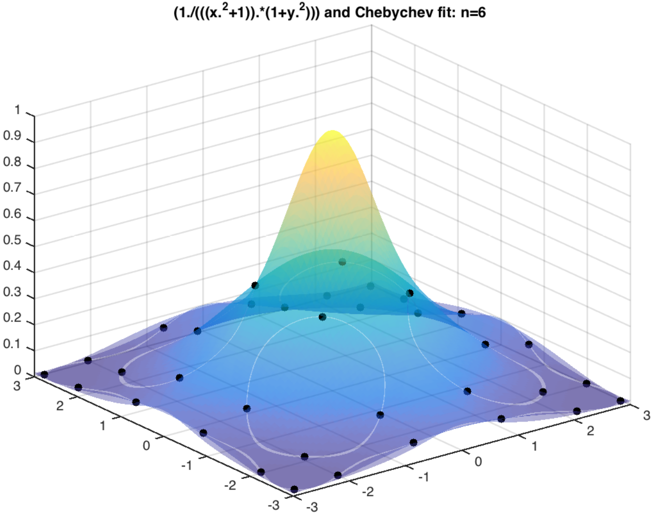
CHEBY FIT: $f(x) = \frac{1}{1+x.^2}$: 4-DEG. APPROX



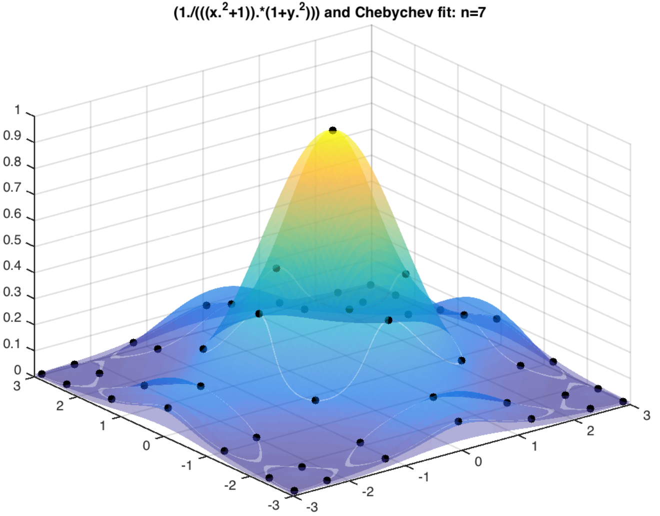
CHEBY FIT: $f(x) = \frac{1}{1+x.^2}$: 5-DEG. APPROX



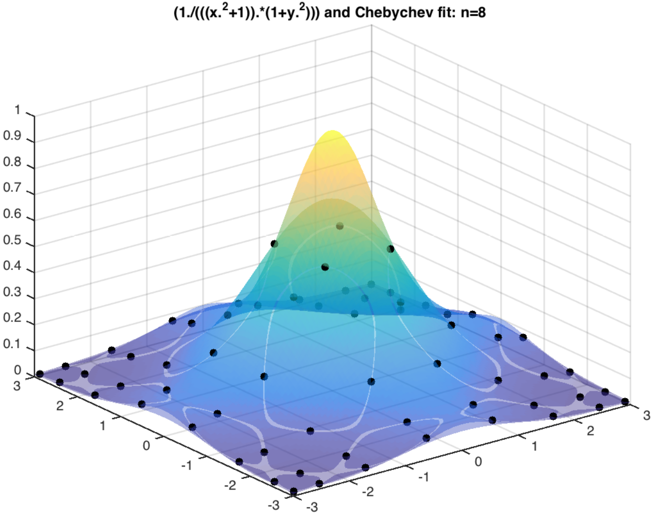
CHEBY FIT: $f(x) = \frac{1}{1+x.^2}$: 6-DEG. APPROX



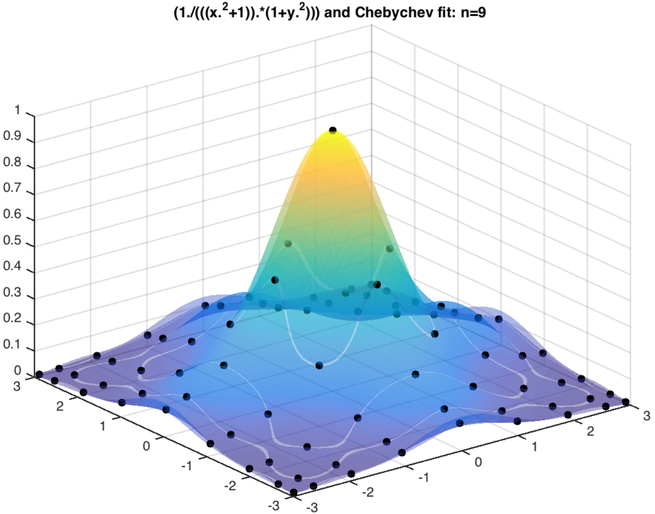
CHEBY FIT: $f(x) = \frac{1}{1+x.^2}$: 7-DEG. APPROX



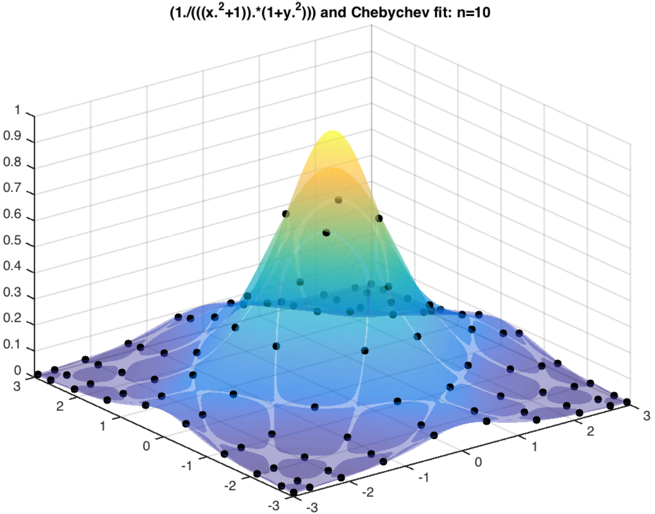
CHEBY FIT: $f(x) = \frac{1}{1+x.^2}$: 8-DEG. APPROX



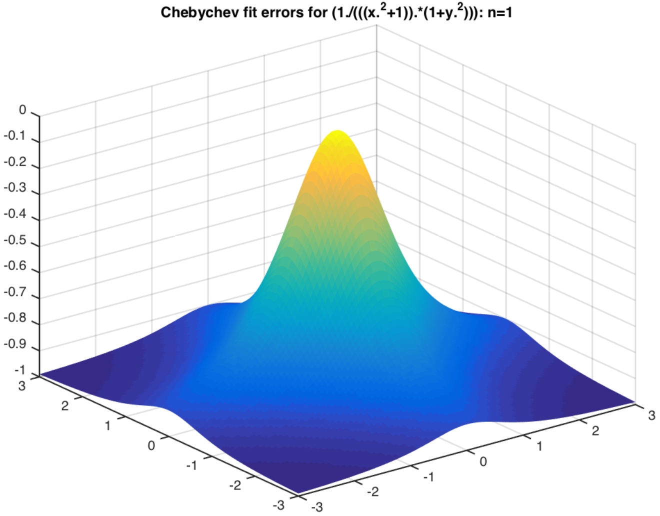
Cheby Fit: $f(x) = \frac{1}{1+x.^2}$: 9-deg. approx



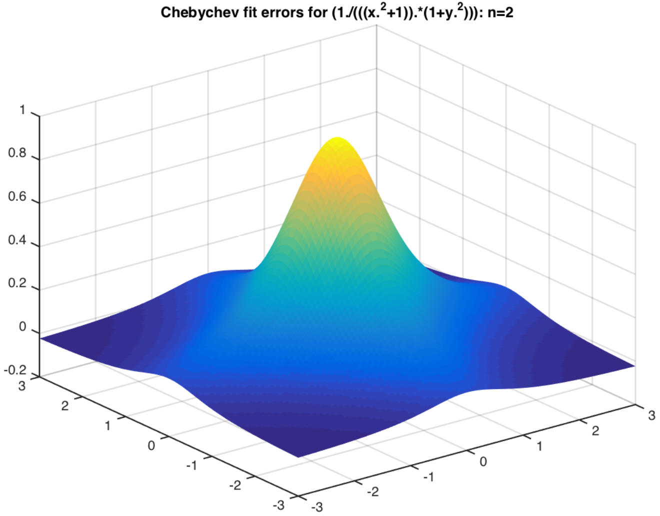
Cheby Fit: $f(x) = \frac{1}{1+x.^2}$: 10-deg. approx



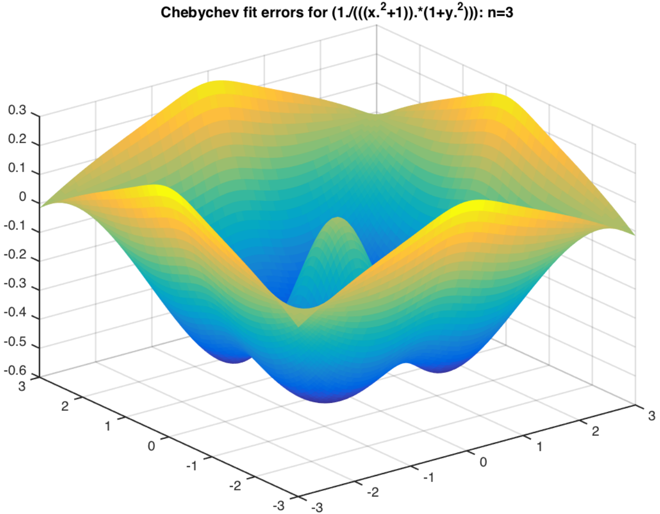
Cheby Errors: $f(x) = \frac{1}{1+x.^2}$: 1-deg. approx



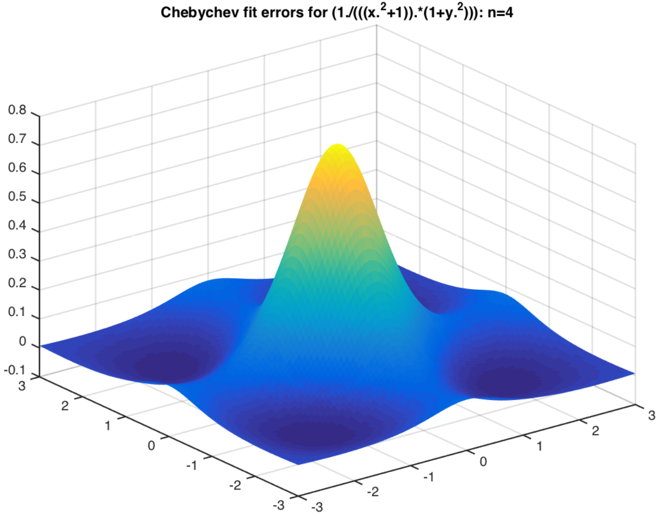
CHEBY ERRORS: $f(x) = \frac{1}{1+x.^2}$: 2-DEG. APPROX



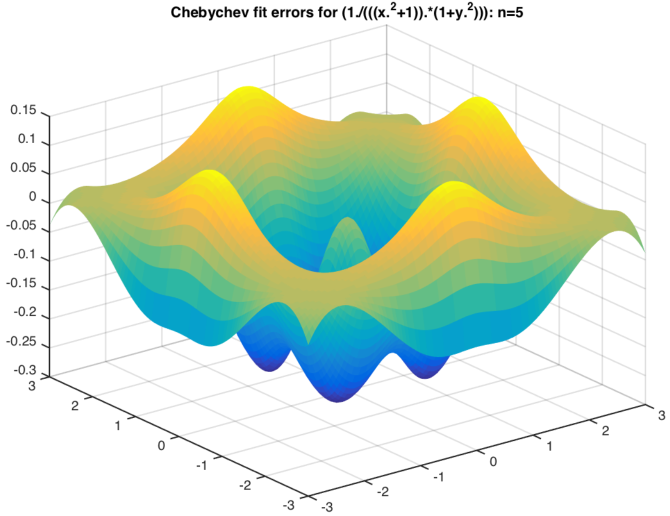
Cheby Errors: $f(x) = \frac{1}{1+x.^2}$: 3-deg. approx



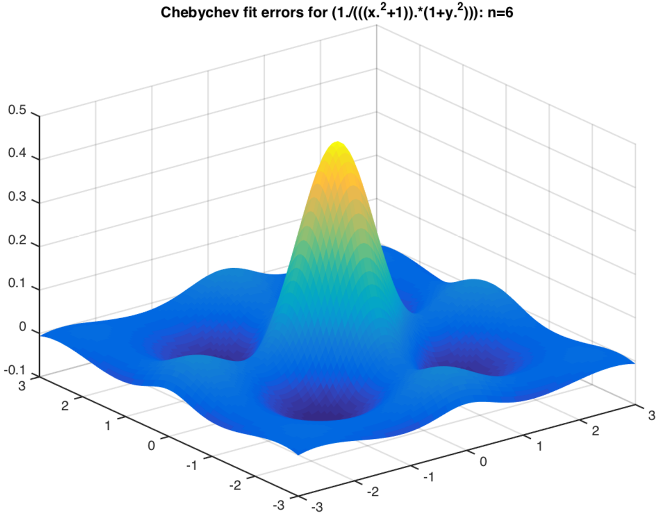
Cheby Errors: $f(x) = \frac{1}{1+x.^2}$: 4-deg. approx



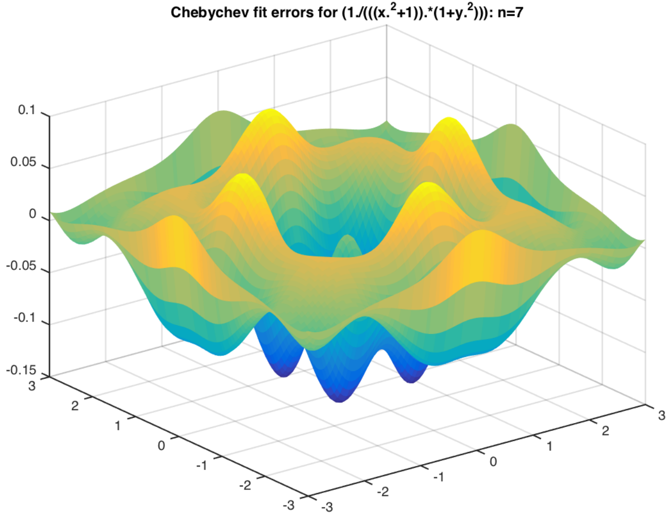
CHEBY ERRORS: $f(x) = \frac{1}{1+x.^2}$: 5-DEG. APPROX



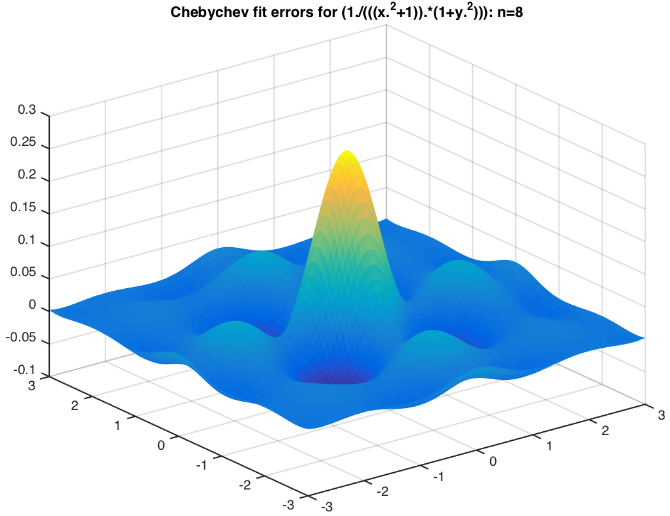
Cheby Errors: $f(x) = \frac{1}{1+x.^2}$: 6-deg. approx



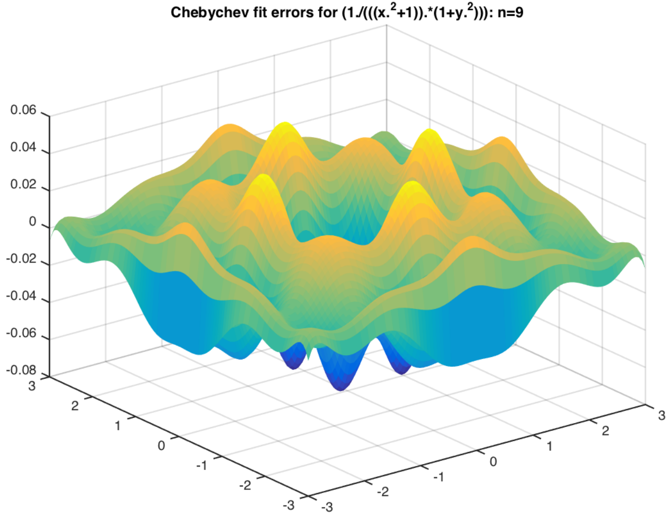
Cheby Errors: $f(x) = \frac{1}{1+x.^2}$: 7-deg. approx



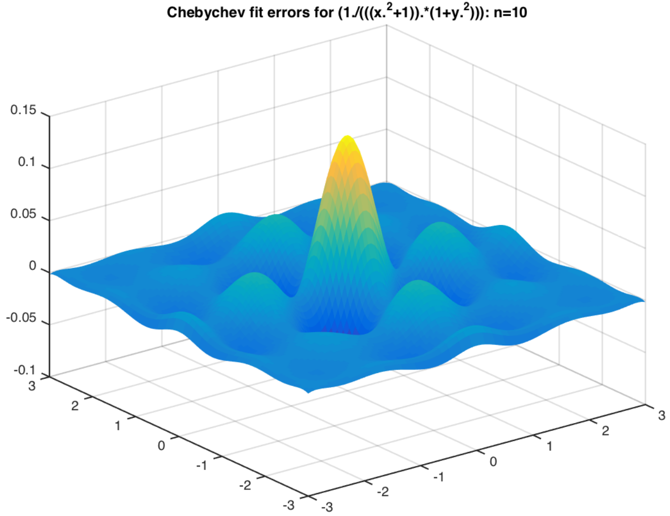
Cheby Errors: $f(x) = \frac{1}{1+x.^2}$: 8-deg. approx



CHEBY ERRORS: $f(x) = \frac{1}{1+x.^2}$: 9-DEG. APPROX



CHEBY ERRORS: $f(x) = \frac{1}{1+x.^2}$: 10-DEG. APPROX



Taste of some technical points

- If we have some function $f(x)$ and some class of approximating functions, Ω
- We want to find the function $\omega(x) \in \Omega$ such that:

$$\omega = \arg\min_{\omega \in \Omega} ||f - g||^2$$

- Interpolating at Chebychev nodes allows us to obtain a lower bound for the error:

$$|f(x) - \hat{f}_{n-1}(x)| \leq \frac{1}{2^{n-1} n!} \max_{\xi \in [-1,1]} |f^{(n)}(\xi)|$$

- For more see Judd Chapter 6: many ways to approximate functions.

Practical Note-I

- As mentioned, I've found that this discussion of interpolation is not frequently used by students in research
- Easier to use canned commands
- In matlab, interpolation is relatively easy
- Give it the basis for the function and the function, create an interpolating (and potentially extrapolating function)

Practical Note-II

- General Matlab interpolation methods
  - Linear - connect the dots with a line or plane
  - Nearest, next, previous - rarely used
  - Spline: Fit points with local polynomials of (typically) degree three
  - PChip: Spline but doesn't overshoot (good for non-smooth functions)
  - Makima: spline with orthogonal basis vectors

Practical Note-III

- How does it work?

- Let's say I want to define a value function in three dimensions:

```
wvec = linspace(0,10,5)
mvec = [0,1]
cvec = [0,1,2,3,4]
[mgrid,wgrid,cgrid] = ndgrid(wvec,mvec,cvec)
V=rand(size(wgrid))
Vfxn = griddedInterpolant(wgrid,mgrid,cgrid,V)
Vfxn(2,1,2)
```

- Could also use:
  `scatteredInterpolant(xvec1,xvec2,xvec3,yvec)`
  for scattered/non-gridded data.

Practical Note-IV

- Making grids in Matlab
- Two commands: ndgrid (interpolation) and meshgrid (ease of graphing)
- $[a, b] = meshgrid([1:2], [3:4])$ gives:

$$a = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$$

- $[a, b] = ndgrid([1:2], [3:4])$ gives:

$$a = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}$$

- Matlab expects things to be in ndgrid format for interpolation (while surf uses meshgrid)
- Can always transpose, but if things aren't working, check if this is a problem. Useful to have different $\#$'s of indicies so interp crashes if things are in wrong order