Topics and Sample Questions for EECS 302 Exam 1:

1. **Topics:** Sets, Cartesian Products, Relations and Functions

   Sample Problem: Given A = {1, 2, 4, 7} and B = {2, 4, 5}, check ALL boxes that are applicable in describing the following sets and briefly justify your answers

The set in each row represents…..	Relation	Function	Neither	Justification
V: {(1,2),(4,2},(7,5),(4,2)}	✓			$V \subset A \times B$ (relation) but $2 \in A$ is not mapped to B (not function)
W: {(1,2),(4,2),(2,5), (7,5)}	✓	✓		$W \subset A \times B$ (relation) and each member of A has exactly one map to B (function)
X: {(1,2),(4,2),(2,3), (7,5)}			✓	$X \not\subset A \times B$ (not a relation hence not a function)
Y: {(1,2),(4,2),(2,4), (7,5), (4,4)}	✓			$Y \subset A \times B$ (relation) by $4 \in A$ has two mapping in B (not a function)
Z: {(1,2),(4,2),(2,4), (7,5), (4,2)}	✓	✓		$Z \subset A \times B$ (relation) and each member of A has exactly one map to B (a function)

2. **Topics:** Statements (or propositions) and logical operators (~, ∧, ∨, →, ↔); Using logical operators to create negation of a statement, compound statements, and conditional statements; Truth values, truth tables and logical equivalences; Informal language and formal Language

   Sample Problems:

   2.1 Use logical identities (logical equivalence laws such as Associative laws, Commutative laws, Distributive laws, De Morgan's laws, Double negative laws, Idempotent laws, Absorption laws, and logical identities involving → and ↔) to show logical equivalences of the following conditional statements: Let *p, q,* and *r* be proposition variables. Then

$$p \vee {\sim}q \rightarrow r \vee q \equiv ({\sim}p \wedge q) \vee (r \vee q) \equiv \text{negation of } {\sim}({\sim}p \wedge q) \wedge ({\sim}r \wedge {\sim}q)$$

$p \vee {\sim}q \rightarrow r \vee q \equiv {\sim}(p \vee {\sim}q) \vee (r \vee q)$ Identity involving "→"

$\equiv ({\sim}p \wedge q) \vee (r \vee q)$ De Morgan's and Double negative laws

${\sim}({\sim}({\sim}p \wedge q) \wedge ({\sim}r \wedge {\sim}q)) \equiv {\sim}((p \vee {\sim}q) \wedge ({\sim}r \wedge {\sim}q)$ De Morgan's and Double negative laws

(negation of ${\sim}({\sim}p \wedge q) \wedge ({\sim}r \wedge {\sim}q)$) $= ({\sim}(p \vee {\sim}q)) \vee {\sim}({\sim}r \wedge {\sim}q)$ De Morgan's law

$\equiv ({\sim}p \wedge q) \vee (r \vee q)$ De Morgan's and Double negative laws

QED

   2.2 Use a truth table to reconfirm the logical equivalence of the first two statements above

p	*q*	*r*	~q	p∨~q	r∨q	p∨~q→r∨q	~p	~p∧q	(~p∧q)∨(r∨q)
T	T	T	F	T	T	T	F	F	T
T	T	F	F	T	T	T	F	F	T
T	F	T	T	T	T	T	F	F	T
T	F	F	T	T	F	F	F	F	F
F	T	T	F	F	T	T	T	T	T
F	T	F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	F	T
F	F	F	T	T	F	F	T	F	F

Same

$\therefore\ p \vee {\sim}q \rightarrow r \vee q \equiv ({\sim}p \wedge q) \vee (r \vee q)$.

3. **Topics:** Arguments: Proof of validity using (1) Truth Tables or (2) Rules of Inference/Valid Argument Forms (namely Modus Ponens, Modus Tollens,, Generalization, Specialization, Elimination, Transitivity, Proof by Division into Cases, and Contradiction Rule)

Sample problems:
Given the following argument in informal language (English)

> Sam is a CS major or an economics major.
> If Sam is a CS major, Sam is required to take EECS 302
> ∴ Sam is an economics major or Sam is not required to take EECS 302

3.1 Write the above argument using formal language (using appropriate logical operators).

Let $p \triangleq$ "Sam is a CS major"; $q \triangleq$ "Sam is an Econ major"; $r =$ "Sam takes EECS 302".

∴ Argument:
$$p \vee q$$
$$p \to r$$
$$\therefore q \vee \sim r$$

3.2 Use appropriate rules of inference to deduce whether the argument is valid. (If you think it is invalid you can provide a case to show invalidity).

The argument is invalid since the case of $(p, \sim q, r)$ makes both premises true but the conclusion $q \vee \sim r$ is false.
Although finding a case to show invalidity of the argument suffices, we will show how to use the rules of inference to do this

We will consider the case $\sim q$
We have $p \vee q$
$\sim q$
$\therefore p$ (by Elimination)

$p \to r$
p
$\therefore r$ (Modus Ponens)

and $q \vee \sim r$
$\sim q$ $\therefore \sim r$ (Elimination)

$\therefore$ $p \vee q$
$p \to r$
$q \vee \sim r$
$\sim q$
$\therefore r \wedge \sim r = C$

This shows the invalidity of the argument

3.3 Use a truth table to confirm the conclusion found in 3.2.

p	q	r	$p \vee q$	$p \to r$	$\sim r$	$q \vee \sim r$
T	T	T	T	T	F	T
T	T	F	T	F	T	
T	F	T	T	T	F	F ←
T	F	F	T	F	T	
F	T	T	T	T	F	T
F	T	F	T	T	T	T
F	F	T	F	T	F	
F	F	F	F	T	T	

This critical line leads to false conclusion
Hence the argument is invalid.

4. **Topics:** Quantified statements (including quantified conditional statements and multiply-quantified statements) and negation of quantified statements

Sample problem:

Consider the following (multiply quantified) statement S:

S: $\forall e \in Z^+, \exists m \in N$ such that $\forall n \in N$, if $n \geq m$, then $\forall x \in R$, $|f_n(x) - f(x)| < e$.

4.1. What is the negation of the statement S?

$\sim S = \exists e \in Z^+, \forall m \in N, \exists n \in N$ such that $n \geq m$ and $(\exists x \in R$ such that $|f_n(x) - f(x)| \geq e)$

4.2 What is the contrapositive form of the statement S? (Hint: Just concentrate on the if-then part)

Contrapositive form of S

$= \forall e \in Z^+, \exists m \in N, \forall n \in N$ such that if $\exists x \in R$ such that $|f_n(x) - f(x)| \geq e$, then $n < m$

5 **Topics:** Relationships among: i) logical conditioning quantifiers $\rightarrow$ and $\leftrightarrow$; (ii) conditional informal language operators ("if", "only if" and "if and only if"); (iii) implicit quantifiers $\Rightarrow$ and $\Leftrightarrow$; and (iv) necessary conditions, sufficient conditions and necessary and sufficient conditions

Sample problems:

Let $f(x)$ be a twice continuously differentiable real-valued function of a real variable x (i.e. the domain of f is R and the range (co-domain) of f is also R). The following results are well known:

a) A necessary condition for a real x^* to be a minimizer of f is that $f'(x^*) = 0$ and $f''(x^*) \geq 0$
b) A sufficient condition for a real x^* to be a strict local minimizer of f is that $f'(x^*) = 0$ and $f''(x^*) > 0$
c) If f is a convex function, then a necessary and sufficient condition for a real x^* to be a minimizer of f is that $f'(x^*) = 0$

Write each of the above statements in a formal form using first ($\rightarrow$ or $\leftrightarrow$) then ($\Rightarrow$ or $\Leftrightarrow$):

Answer:

a) $\forall x \in R$, x is a minimizer of $f(x) \rightarrow f'(x) = 0$ and $f''(x) \geq 0$
or
A real x is a minimizer of $f(x) \Rightarrow f'(x) = 0$ and $f''(x) \geq 0$

Now write similar formal statements for b) and c).

b) $\forall x \in R$, $f'(x) = 0$ and $f''(x) > 0 \longrightarrow x$ is a strict local minimizer of f.
or For a real x such that $f'(x) = 0$ and $f''(x) > 0 \Rightarrow x$ is a strict local minimizer of f

c) $\forall x \in R$, x is a minimizer of a convex function $f \longleftrightarrow f'(x) = 0$
or A real x is a minimizer of a convex function $f \Longleftrightarrow f'(x) = 0$

It is often easier to prove the above results by a contrapositive proof.

What is the contrapositive form of the necessary condition in part a) above?

Contrapositive form of the necessary condition in part (a) is
such that either $f'(x) \neq 0$ or $f''(x) < 0$, x is not a minimizer of $f(x)$

5. **Topics:** Direct proof, counterexample, proof by contradiction, contrapositive proof

Sample problems:

5.1 Prove or disprove the following statements/claims formally and briefly justify each step in the body of your proof. Also clearly specify which method of proof you are using.

a) Claim: There exists an integer $m \geq 4$ such that $2m^2 - 5m + 2$ is prime

We will disprove the claim by showing a counterexample i.e will show
$\forall$ integer $m \geq 4$, $2m^2 - 5m + 2$ is not a prime.
Since $2m^2 - 5m + 2 = (2m-1)(m-2)$ and both $(2m-1)$ and $(m-2)$ are integers greater than 1 for all $m \geq 4$.
$\therefore$ $2m^2 - 5m + 2$ is a composite. QED

b) Claim: The difference between any odd integer and an even integer is odd

We will use a direct proof.
Let n be an odd number and m be an even number.
$\therefore n = 2k_1 + 1$ and $m = 2k_2$ for some integers k_1 and k_2
$\therefore n - m = 2k_1 + 1 - 2k_2 = 2\underbrace{(k_1 - k_2)}_{\text{integer } r \text{ (axiom of algebra)}} + 1 = 2r + 1$
$\therefore n - m$ is odd by definition. QED

c) Claim: If a sum of two real numbers is less than 70, then at least one of the numbers is less than 35. We will prove by a contrapositive proof

Suppose x and y are real numbers greater than 35 (or equal to) i.e $x \geq 35$ and $y \geq 35$.
$\therefore x + y \geq 35 + 35 = 70$
$\therefore$ The supposition is wrong
$\therefore$ At least one of x and y must be less than 35. QED

d) Claim: If x and y are rational numbers with $y \neq 0$, and if z is an irrational number, then $x + yz$ is irrational We will prove this by contradiction.

Let x and y be rational numbers with $y \neq 0$
$\therefore x = \frac{m}{n}$ and $y = \frac{k}{l}$ for some integers m, n, k, l with $n \neq 0, k \neq 0$ & $l \neq 0$
Assume that $x + yz$ is rational (where z is irrational)
$\therefore x + yz = \frac{r}{s}$ for some integers r and s with $s \neq 0$
$\therefore \frac{m}{n} + \frac{k}{l} z = \frac{r}{s}$ $\therefore z = \left(\frac{r}{s} - \frac{m}{n}\right)\frac{l}{k} = \frac{\overbrace{l(rk - sm)}^{\text{integer } M}}{\underbrace{snk}_{\text{integer } N \neq 0}} = \frac{M}{N}$
$\therefore z$ is rational
A contradiction since z is irrational $\therefore x + yz$ is irrational QED

5.2 Provide a brief justification for each claim made in the following body of proof

Theorem: There are infinitely many primes.

Proof: Reason

Suppose there are finitely many prime with p being the largest prime. Contrapositive argument

Let $M = p! + 1$. Clearly every prime divides $p!$. Since every prime $\le p$ and is a factor of $p!$

But none divides M (Due to a theorem which says if $p|a$, $p \nmid a+1$.

However, since M is an integer greater than 1, M is divisible by a prime Due to a theorem which says every integer >1 is divisible by a prime

Let q be a prime that divides M. So $q > p$. Because all prime $\le p$ does not divide M.

So the supposition that p is the greatest prime is false. p cannot be the largest prime

Thus, there must be infinitely many primes. Hence the theorem is proved Contrapositive argument

6. **Topics:** Key results and concepts in Number Theory: the quotient-remainder theorem, div and mod, floor and ceiling, the uniqueness factorization theorem

Sample Problems:

6.1 For an odd integer n, show that $\left\lfloor \frac{n^2}{4} \right\rfloor = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)$

Let n be an odd integer

$\therefore n = 2k+1$ for some integer k

$\therefore \frac{n^2}{4} = \frac{4k^2+4k+1}{4} = k^2+k+\frac{1}{4} \Rightarrow \left\lfloor \frac{n^2}{4} \right\rfloor = k^2+k$. Since $k^2+k \le k^2+k < k^2+k+1$

But $k^2+k = k(k+1) = \frac{2k}{2} \cdot \frac{2k+2}{2} = \frac{2k+1-1}{2} \cdot \frac{2k+1+1}{2} = \frac{n-1}{2} \cdot \frac{n+1}{2}$

$\therefore \left\lfloor \frac{n^2}{4} \right\rfloor = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)$.

6.2 Show that if n mod 15 = 3, then $10n$ mod 15 = 0.

Since $n \bmod 15 = 3$, $n = 15k+3$ for some integer k (By QR Theorem)

$\therefore 10n = 150k + 30 = (10k+2)15 + 0$

$\therefore 10n \bmod 15 = 0$

6.3 If today is Tuesday, use the quotient-remainder theorem to find out the day of the week 127 days from now.

Let Tuesday = 0, Wednesday = 1, . . . , Monday = 6

Let d be the day of the week 127 days from now (which is Tuesday)

$\therefore d = 127 \bmod 7 = 1$

Thus 127 days from now will be Wednesday.

6.4 Find the smallest value of k such that $2^5.5^1.7^7.11^3.k$ is a perfect cube.

Let $k = 2^1.5^2.7^2.11^0 = 2450$

$\therefore 2^5.5^1.7^7.11^3.k = 2^6.5^3.7^9.11^3 = (2^2.5.7^3.11)^3 = (75460)^3$

Note any other number k that will make the given number a perfect cube will have to be larger than 2450 (by the unique factorization theorem)

$\therefore$ 2450 is the smallest required number.