# CSDS302 HW6 - Trevor Swan (*tcs94*)

# 5.2

# 3

For each positive integer $n$, let $P(n)$ be the formula

$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

## a

Write $P(1)$. Is $P(1)$ true?

**Answer**

The expression given can be written as the following;

$$1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

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- Therefore $P(1) = \sum_{i=1}^{1} i^2 = 1 \leftarrow \text{LHS}$
- The expression given evaluated for $P(1) = \frac{1(1+1)(21+1)}{6} = \frac{6}{6} = 1 \leftarrow \text{RHS}$

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The two results, or sides of the equation, are equal so $\boxed{P(1) \text{ is true}}$.

## b

Write $P(k)$

**Answer**

Generating this expression is as simple as substituting $k$ for $n$ in the given formula.

$$\boxed{1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}}$$

## c

Write $P(k+1)$

**Answer**

Generating this expression involves substituting $k+1$ in the formula given for every instance of the variable $n$.

$$\begin{aligned} 1^2 + 2^2 + \cdots + (k+1)^2 &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\ &= \boxed{\frac{(k+1)(k+2)(2k+3)}{6}} \end{aligned}$$

## d

In a proof by mathematical induction that the formula holds for every integer $n \geq 1$, what must be shown in the inductive step?

**Answer**

The inductive step must show that if $P(k)$ is true, then $P(k+1)$ is also true.

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This would involve assuming $1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$ is true and then using that statement to prove that \frac{(k+1)(k+2)(2k+3)}{6} is also true. The second statement is $P(k+1)$.

## 7

Prove the statement using mathematical induction. Do not derive them from theorem 5.2.1 or theorem 5.2.2.

For every integer $n \geq 1$
$1+6+11+16+...+(5n-4) = \frac{n(5n-3)}{2}$

Answer

**Step 1:** $a = 1 \to P(1) = 1 = \frac{1(5(1)-3)}{2}$

This expression can be simplified to $1 = 1$ which is true ✓

### Step 2: Assume $P(k)$ is true

$$\begin{aligned}
\text{Assume } 1+6+11+16+...+(5k-4) &= \frac{k(5k-3)}{2} \text{ is true} && \checkmark \\
\text{WTS: } 1+6+11+16+...+(5(k+1)-4) &= \frac{(k+1)(5(k+1)-3)}{2} && ? \\
1+6+11+16+...+(5k-4)+(5k+1) &= \frac{(k+1)(5k+2)}{2} \\
\text{Substitute Assumed: } \frac{k(5k-3)}{2}+(5k+1) &= \frac{(k+1)(5k+2)}{2} \\
\text{Rewrite: } \frac{k(5k-3)}{2}+\frac{2(5k+1)}{2} &= \frac{(k+1)(5k+2)}{2} \\
\text{Expand Terms: } \frac{5k^2-3k}{2}+\frac{10k+2}{2} &= \frac{(k+1)(5k+2)}{2} \\
\text{Simplify: } \frac{5k^2+7k+2}{2} &= \frac{(k+1)(5k+2)}{2} \\
\frac{(k+1)(5k+2)}{2} &= \frac{(k+1)(5k+2)}{2} && \checkmark
\end{aligned}$$

---

Because the two sides of the proof are equal, we can say that for every integer $n \geq 1$, $1+6+11+16+...+(5n-4) = \frac{n(5n-3)}{2}$ is true by mathematical induction.
$\square$

# 12

Prove the statement using mathematical induction.

$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{n(n+1)} = \frac{n}{n+1}$ for every integer $n \geq 1$

Answer

## Step 1: $a = 1 \rightarrow P(1) = \frac{1}{1 \cdot 2} = \frac{1}{1+1}$

This expression can be simplified to $\frac{1}{2} = \frac{1}{2}$ which is true ✓

## Step 2: Assume $P(k)$ is true

This assumption means that $\frac{k}{k+1}$ is true.

$$\text{RHS: } \frac{1}{n(n+1)} = \frac{n}{n+1} \bigg|_{(k+1)} = \frac{k+1}{(k+1)+1} = \frac{k+1}{k+2}$$

$$\text{WTS: } \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{(k+1)((k+1)+1)} \equiv \frac{k+1}{k+2}$$

---

Below I will refer to $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{(k+1)((k+1)+1)}$ as 'LHS'

$$\begin{aligned}
\text{LHS } &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{(k+1)(k+2)} \\
&= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\
\text{Use Assumption: } &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
\text{Rewrite: } &= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
\text{Simplify: } &= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

---

The expression derived above is equivalent to the substitution of $k+1$ in the statement that is assumed to be true. Becuase these statements are equal, we can say that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for every integer $n \geq 1$ is true through mathematical induction.

$\square$

# 29

Use the formula for the sum of the first $n$ integers and/or the formula for the sum of a geometric sequence to evaluate the sum or to write it in closed form.

$1 - 2 + 2^2 - 2^3 + \ldots + (-1)^n 2^n$ where $n$ is any positive integer

**Answer**

To start we can rewrite the expression in terms of powers of negative two.

$$1 - 2 + 2^2 - 2^3 + \ldots + (-1)^n 2^n \equiv (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + \cdots + (-2)^2$$

Therefore we can derive the following conclusion:

$$\begin{aligned}
1 - 2 + 2^2 - 2^3 + \ldots + (-1)^n 2^n &= (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + \cdots + (-2)^2 \\
&= \sum_{i=0}^{n} (-2)^n \\
\textit{Theorem 5.2.3:} \sum_{i=0}^{n} r^i &= \frac{r^{n+1} - 1}{r - 1} \\
\therefore \quad &= \frac{(-2)^{n+1} - 1}{(-2) - 1} \\
&= \frac{(-1)^{n+1} 2^{n+1} - 1}{-3} && \leftarrow \text{Exponent Rules} \\
&= \frac{1 - (-1)^{n+1} 2^{n+1}}{3} && \leftarrow \text{Simplify Negatives}
\end{aligned}$$

$\therefore$ The sum of the given formula is $\boxed{\dfrac{1 - (-1)^{n+1} 2^{n+1}}{3}}$

# 5.3

# 4

For each positive integer $n$, let $P(n)$ be the sentence that describes the following divisibility property:

$5^n - 1$ is divisible by 4.

## a

Write $P(0)$. Is $P(0)$ true?

**Answer**

$$P(0) = 5^0 - 1 = 1 - 1 = 0$$

0 is divisible by any integer, and the conclusion can be written as: $4|0$. Therefore the given statement evaluated at 0, $P(0)$ is true.

## b

Write $P(k)$

**Answer**

This statement can be derived by substituting $k$ in for $n$

$$P(k) = 5^k - 1 \text{ is divisble by } 4$$

## c

Write $P(k+1)$

**Answer**

This statement can be derived by substituting $(k+1)$ in for $n$

$$P(k) = 5^{k+1} - 1 \text{ is divisble by } 4$$

## d

In a proof by mathematical induction that this divisibility property holds for every integer $n \geq 0$, what must be shown in the inductive step?

**Answer**

The inductive step must show that $P(k+1)$ is true based on the assumption that $P(k)$ is true.

# 7

For each positive integer n, let P(n) be the sentence

In any round-robin tournament involving $n$ teams, the teams can be labeled $T_1, T_2, T_3, \ldots, T_n$, so that $T_i$ beats $T_{i+1}$ for every $i = 1, 2, \ldots, n$.

## a

Write $P(2)$. Is $P(2)$ true?

**Answer**

In any round-robin tournament involving 2 teams, the teams can be labeled $T_1, T_2$, so that $T_2$ beats $T_3$.

## b

Write $P(k)$.

**Answer**

In any round-robin tournament involving $k$ teams, the teams can be labeled $T_1, T_2, T_3, \ldots, T_k$, so that $T_k$ beats $T_{k+1}$ for every $i = 1, 2, \ldots, k$.

## c

Write $P(k+1)$

**Answer**

In any round-robin tournament involving $k+1$ teams, the teams can be labeled $T_1, T_2, T_3, \ldots, T_{k+1}$, so that $T_{k+1}$ beats $T_{k+2}$ for $i = 1, 2, \ldots, k+1$.

## d

In a proof by mathematical induction that $P(n)$ is true for each integer $n \geq 2$, what must be shown in the inductive step?

**Answer**

To prove this using mathematical induction, we would have to show that in any round-robin tournament involving $k+1$ teams, the teams can be labeled $T_1, T_2, T_3, \ldots, T_{k+1}$, so that $T_{k+1}$ beats $T_{k+2}$ for $i = 1, 2, \ldots, k+1$. This would have to be proved for all integers $k$ such that $k \geq 2$, in any round-robin tournament involving $k$ teams, the teams can be labeled $T_1, T_2, T_3, \ldots, T_k$, so that $T_k$ beats $T_{k+1}$ for every $i = 1, 2, \ldots, k$.

# 15

Prove the statement by mathematical induction

$n(n^2+5)$ is divisible by 6, for each integer $n \geq 0$

**Answer**

### Step 1: $a = 0 \rightarrow 6|0(0^2+5) = 6|0 \quad \checkmark$

0 is divisible by any number, so $6|0$ is true.

### Step 2: Assume $P(k)$ is true.

This means $P(k) = 6 \mid k(k^2+5)$ is true.
By the definition of divisibility, this means there exists an integer $m$ such that $k(k^2+5) = 6m$. We will assume the previous expression is true.

---

$$\text{WTS: } (k+1)((k+1)^2+5) = 6m \text{ for some integer m}$$

$$
\begin{aligned}
(k+1)((k+1)^2+5) =& (k+1)(k^2+2k+1+5) & \\
=& (k+1)(k^2+2k+6) & \text{Combine Terms} \\
=& k^3+2k^2+6k+k^2+2k+6 & \text{Distribute} \\
=& k^3+3k^2+8k+6 & \text{Combine Terms} \\
=& k^3+5k+3k^2+3k+6 & \text{Break up Terms} \\
=& k(k^2+5)+3k^2+3k+6 & \text{Split first terms} \\
=& 6m+3k^2+3k+6 & \text{Substitute Given} \\
=& 6m+3k(k+1)+6 & \text{Break up Terms} \\
\star_1 =& 6m+3(2l)+6 & \text{For some integer } l \\
=& 6m+6l+6 & \text{Multiply} \\
=& 6(m+l+6) & \text{Factor out 6}
\end{aligned}
$$

$\star_1$ : $k$ and $k+1$are consecutive so one must be even with a product $2l$

$6(m+l+1)$ is divisibe by 6 because $m+l+1$ is an integer so $6|6(m+l+1) \rightarrow 6(m+l+1)=6q$ where $q=m+l+1$, an integer. This is done through the definition of divisibility, proving that $P(k+1)$ is divisible by 6.

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Therefore, through mathematical induction, we can say that $n(n^2+5)$ is divisible by 6 for each integer $n \geq 0$.

□

# 5.4

# 10

The introductory example solved with ordinary mathematical induction in Section 5.3 can also be solved using strong mathematical induction. Let $P(n)$ be "any n¢ can be obtained using a combination of 3¢ and 5¢ coins." Use strong mathematical induction to prove that $P(n)$ is true for every integer $n \geq 8$.

Answer

### Step 1: Basis Step

Show $P(14), P(15),$ and , $P(16)$ are true

$$
\begin{array}{lll}
P(14): 14 = 3a + 5b & \text{True when } a = 3, b = 1 & \checkmark \\
P(15): 15 = 3a + 5b & \text{True when } a = 0, b = 5 & \checkmark \\
P(16): 16 = 3a + 5b & \text{Truw when } a = 2, b = 2 & \checkmark
\end{array}
$$

### Step 2: Inductive Step

*Show that for every integer $k \geq 16$, if $P(i)$ is true for each integer from 16 through k, then $P(k+1)$ is also true.*

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**Inductive Hypothesis:** Any $i$¢ can be obtained using a combination of 3¢ and 5¢ coins for all integers $i \geq 16$.
$P(k+1)$**:** We must show that any $(k+1)$¢ can be obtained using a combination of 3¢ and 5¢ coins.

---

Since $k \geq 16$, $k-2 \geq 14$ and $P(14)$ is true from earlier. This means $P(k-2)$ is true from the basis step. This means there are two positive integers $a$ and $b$ where $5a + 3b = k - 2$ is true.
We want to prove $k+1$ is true, so we will add 3 to each side of the equation.

$$
\begin{aligned}
5a + 3b &= k - 2 && \text{True} \\
5a + 3b + 3 &= k - 2 + 3 && \text{Add 3 to each side} \\
5a + 3(b+1) &= k + 1 && \text{Combine terms} \\
5a + 3c &= k + 1 && \text{For some integer c}
\end{aligned}
$$

---

Becuase $b$ is an integer, $b+1$ is also an integer. This means that $(k+1)$¢ can be written as a sum of 5¢ and 3¢ coins. By strong mathematical induction, we can thus say that "Any $n$¢ can be obtained using a combination of 3¢ and 5¢ coins."

# 13

Use strong mathematical induction to prove the existence part of the unique factorization of integers theorem (Theorem 4.4.5). In other words, prove that every integer greater than 1 is either a prime number or a product of prime numbers.

Answer

## Step 1: Basis Step

Show that $P(2)$ is true:
2 is either prime or a product of prime numbers.

$$P(2) : n = 2;\ 2 \text{ is prime so the given sentence is true.}$$

## Step 2: Inductive Step

*Show that for every integer $k \geq 2$, if $P(i)$ is true for each integer from 2 through k, then $P(k+1)$ is also true.*

---

**Inductive Hypothesis:** Any integer $i$ is either prime or a product of prime numbers if $2 \leq i < k$.
$P(k+1)$: We must show that any integer $k+1$ is either prime or a product of prime numbers.

---

### Case 1: $k+1$ is prime

$k+1$ is prime in this case so the given property holds true.

### Case 2: $k+1$ is not prime

$k+1$ is not prime, so it is composite. Composite numbers can be writen as $k+1 = a \cdot b$, for $1 < a, b < k+1$ by definition.
$a$ or $b \neq 1$ because $k+1$ is not a prime number in this case.
We will assume the inductive hypothesis is true, so $a$ and $b$ can be written as a product of prime numbers as follows:

$$\begin{aligned} a =& (p_1 \cdot p_2 \cdot p_3 \ldots p_i) \\ b =& (q_1 \cdot q_2 \cdot q_3 \ldots q_j) \end{aligned}$$

For some integers $i, j$ and primes $p_i, q_j$.

---

Therefore $k+1 = a \cdot b = (p_1 \cdot p_2 \cdot p_3 \ldots p_i) \cdot (q_1 \cdot q_2 \cdot q_3 \ldots q_j)$ is also a product of prime numbers. By strong mathematical induction, we can say that every integer greater than 1 is either a prime number or a product of prime numbers.

# 16

Use strong mathematical induction to prove that for every integer $n \geq 2$, if $n$ is even, then any sum of $n$ odd integers is even, and if $n$ is odd, then any sum of $n$ odd integers is odd.

**Answer**

## Step 1: Basis Step

Show that the statement is true for $P(2),$ and $P(3)$.

$$\begin{aligned}
&\text{For integers } q, r, s \text{ and the definition of even and odd:} \\
P(2) : (2q+1) + (2r+1) =& 2q + 2r + 2 = 2(q+r+1) \qquad && \text{Even} \\
P(3) : (2q+1) + (2r+1) + (2s+1) =& 2q + 2r + 2s + 2 + 1 \\
=& 2(q+r+s+1) + 1 && \text{Odd}
\end{aligned}$$

These two cases show that the statement holds true here.

## Step 2: Inductive Step

*Show that for every integer $k \geq 2$, if $P(i)$ is true for each integer from 2 through k, then $P(k+1)$ is also true.*

---

**Inductive Hypothesis:** For every integer $i \geq 2$,if $i$ is even, then any sum of $i$ odd integers is even, and if $i$ is odd, then any sum of $i$ odd integers is odd.
$P(k+1)$: We must show that the statement is true for some integer $k+1$

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### Case 1: $k+1$ is even

We can set up a summation to represent the sum of $k+1$ odd integers.
Becasue $k+1$ is even, we can define an integer $n$ where $2n = k+1$ by the definition of an even number.

$$\begin{aligned}\sum_{i=1}^{2n}(2i+1) =&(2\cdot 1+1)+(2\cdot 2+1)+\cdots+(2(2n)+1)\\ =&2(1+2+3+\cdots+2n)+2n\\ =&2(1+2+3+\cdots+i)+2n \quad \text{Let } i=2n\end{aligned}$$

Because $i$ is even, we can use the inductive hypothesis to say that the first $i$ odd integers is sum is even.

$$2(1+2+3+\cdots+i)+i=2a+i$$

This is true for some integer a. The sum of even integers is even, so we can say that the sum of an even number of odd integers is even.

### Case 2: $k+1$ is odd

We can set up a summation to represent the sum of $k+1$ odd integers. Because $k+1$ is odd, we can define an integer $m$ where $2m+1=k+1$ by the definition of an odd number.

$$\begin{aligned}\sum_{i=1}^{2m+1}(2i+1) =&(2\cdot 1+1)+(2\cdot 2+1)+\cdots+(2(2m+1)+1)\\ =&2(1+2+3+\cdots+(2m+1))+(2m+1)\\ =&2(1+2+3+\cdots+i)+(2m+1) \quad \text{Let } i=2m+1\end{aligned}$$

Because $i$ is odd, we can use the inductive hypothesis to say that the first $i$ odd integers sum is odd.

$$2(1+2+3+\dots)+2m+1=2b+i$$

This is true for some integer b. The sum of an even integer and an odd integer is always odd, so we can say that the sum of $i$ odd integers is odd.

---

The two cases presented lead to a true conclusion with respect to the assumption and $k+1$. This allows us to conclude that for every integer $n \geq 2$, if $n$ is even, then any sum of $n$ odd integers is even, and if $n$ is odd, then any sum of $n$ odd integers is odd.

# 17

Compute $4^1, 4^2, 4^3, 4^4, 4^5, 4^6, 4^7, 4^8$. Make a conjecture about the units digit of $4^n$ where $n$ is a positive integer. Use strong mathematical induction to prove your conjecture.

Answer

## Computation

$$\begin{aligned}4^1 &= 4\\ 4^2 &= 16\\ 4^3 &= 64\\ 4^4 &= 256\\ 4^5 &= 1024\\ 4^6 &= 4096\\ 4^7 &= 16384\\ 4^8 &= 65536\end{aligned}$$

## Conjecture

The units digit of $4^n$ for some positive integer $n$ is 4 when $n$ is odd and 6 when $n$ is even.

## Proof

Let $P(n)$ be the statement, if $n$ is odd, then $4^n$ has a units digit of 4, and the units digit is 6 is $n$ is even.

### Step 1: Basis Step

Using the computation above, we can see that $P(1) : 4^1 = 4$, when $n = 1$, an odd number. Therefore $P(1)$ is true. We can also see that $P(2) : 4^2 = 16$, when $n = 2$, an even number. Therefore $P(2)$ is also true.

### Step 2: Inductive Step

*Show that for every integer $k \geq 1$, if $P(i)$ is true for each integer from 1 through k, then $P(k+1)$ is also true.*

---

**Inductive Hypothesis:** For any integer $k \geq 2$, each integer $i$ such that $0 < i \leq k$, the units digit of $4^i$ is 4 if $i$ is odd and 6 if $i$ is even.
$P(k+1)$: We must show that $4^{k+1}$ has a units digit of 4 when $k+1$ is odd and 6 when $k+1$ is even.

---

**Case 1: $k+1$ is odd**

Let $k$ be some even integer so that $k+1$ is odd.

$$\begin{aligned} 4^{k+1} =& 4^k \cdot 4 && \text{Exponent Properties} \\ \star_1 =& (10q+6) \cdot 4 && \text{Inductive Hypothesis} \\ =& 40q + 24 && \text{Distribute} \\ =& 40q + 20 + 4 && \text{Split Terms} \\ =& 10(4q+2) + 4 && \text{Distributive} \\ =& 10r + 4 && \text{For some integer } r \end{aligned}$$

$\star_1$: Let q be some non-negative integer.

Because the expression can be written as $10r+4$, then the units digit must be 4 for some odd integer $k+1$.

**Case 2: $k+1$ is even**

Let $k$ be some odd integer so that $k+1$ is even.

$$\begin{aligned} 4^{k+1} =& 4^k \cdot 4 && \text{Exponent Rules} \\ \star_2 =& (10q+4) \cdot 4 && \text{Inductive Hypothesis} \\ =& 40l + 16 && \text{Distribute} \\ =& 40l + 10 + 6 && \text{Split Terms} \\ =& 10(4l+1) + 6 && \text{Distributive} \\ =& 10r + 6 && \text{For some integer } r \end{aligned}$$

$\star_2$: Let l be some non-negative integer.

Because the expression can be written as $10r+6$, then the units must be 6 for some even integer $k+1$.

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Because the two possible cases lead to correct conclusions with respect to the assumption, we can say that the units digit of $4^n$ for some positive integer $n$ is 4 when $n$ is odd and 6 when $n$ is even.

# 5.6

## 4

Find the first four terms of the sequence

$d_k = k(d_{k-1})^2$, for every integer $k \geq 1$
$d_0 = 3$

Answer

$$\begin{aligned} k = 1 : d_1 &= 1(d_0)^2, d_0 = 3 \to d_1 = 9 && (2) \\ k = 2 : d_2 &= 2(d_1)^2, d_1 = 9 \to d_2 = 162 && (3) \\ k = 3 : d_3 &= 3(d_2)^2, d_2 = 162 \to d_3 = 78732 && (4) \end{aligned}$$

$d_0 = 3$ is given, so the first 4 terms of the recursively defined sequence are:

$$\boxed{3, 9, 162, 78732}$$

# 12

Let $s_0, s_1, s_2, \ldots$ be defined by the formula $s_n = \frac{(-1)^n}{n!}$ for every integer $n \geq 0$. Show that this sequence satisfies the following recurrence relation for every integer $k \geq 1$:

$$s_k = \frac{-s_{k-1}}{k}$$

Answer

Let $k$ be an integer with $k \geq 1$. Then $k - 1 \geq 0$ is also true. Because $s_n = \frac{(-1)^n}{n!}$ is true for all integers $n \geq 0$, we can say that $s_n = \frac{(-1)^n}{n!}$ is true for both $n = k$ and $n = k - 1$. This observation yields the following expressions for $k \geq 1$:

$$s_k = \frac{(-1)^k}{k!} \quad \text{and} \quad s_{k-1} = \frac{(-1)^{k-1}}{(k-1)!}$$

**I will prove the recurrence relation with the brute force method.**

$$
\begin{aligned}
s_k =& \frac{(-1)^k}{k!} && \\
=& \frac{(-1)^k}{k(k-1)!} && \text{Factorial Rules} \\
=& \frac{(-1)^1(-1)^{k-1}}{k(k-1)!} && \text{Exponent Rules} \\
=& \frac{-(-1)^{k-1}}{k(k-1)!} && \text{Simplify} \\
=& \frac{-(-1)^{k-1}}{(k-1)!} \cdot \frac{1}{k} && \text{Factor out} \\
=& -s_{k-1} \cdot \frac{1}{k} && \text{Substitute } s_{k-1} \\
s_k =& \frac{-s_{k-1}}{k} && \text{Combine Terms}
\end{aligned}
$$

The conclusion derived above is equivalent to that proposed in the question, so the recurrence relation has been proved to be true.

# 38

*Compound Interest*: Suppose a certain amount of money is deposited in an account paying 3% annual interest compounded monthly. For each positive integer $n$, let $S_n =$ the amount on deposit at the end of the $n$th month, and let $S_0$ be the initial amount deposited.

## a

Find a recurrence relation for $S_0, S_1, S_2, \ldots$, assuming no additional deposits or withdrawals during the year. Justify your answer.

**Answer**

The interest is compounded monthly, so the monthly interest rate is:

$$\frac{3\%}{12} = 0.25\% = 0.0025$$

This means the amount deposited at the first month is $S_1 = S_0 + 0.0025 \cdot S_0$, where $S_0$ is the previous month.
This relationship is true for all months, so we can define the recurrence relation:

$$\boxed{S_k = S_{k-1} + 0.0025S_{k-1} = 1.0025S_{k-1}}$$

## b

If $S_0 = \$10{,}000$, find the amount of money on deposit at the end of one year.

**Answer**

The end of the year is the $12^{\text{th}}$month, so we can write the relation above as:

$$\begin{aligned} S_{12} =& 1.0025S_{11} \\ =& (1.0025)^{12} \cdot S_0 \\ =& (1.0025)^{12} \cdot 10000 \\ =& \boxed{\$10,304.16} \end{aligned}$$

## c

Find the APY for the account.

**Answer**

The Annual percentage yield (APY) is the percentage increase in the value of the account over a one-year period.

$$\frac{10,304.16 - 10,000}{10,000} = 0.030416 = \boxed{3.0416\%}$$

## Thank you to Trevor Nichols for copying textbook questions into a markdown format