Solution

Topics and Sample Questions for EECS 302/MATH 304 Exam 2:
Sections of the text covered in Exam 2 are 4.8, 5.1. 5.2, 5.3, 5.4, 5.6, 6.1, 6.2, 8.1, 8.2, 8.3

1. **Topics:** Division Algorithm, Euclidean Algorithm GCD and LCM

Sample Problem:
Compute the greatest common divisor (GCD) of 264 and 126 using: a) the Prime Factorization Theorem and b) The Euclidean Algorithm.
Then compute the least common multiple (LCM) of 264 and 126

GCD: ~~By Prime Factorization~~ : $\cancel{\left.\begin{array}{l} 264 = 2.2.2.3.11 = 2^3.3^1.7^0.11^1 \\ 126 = 2.3.3.7 = 2^1.3^2.7^1.11^0 \end{array}\right\} \Rightarrow GCD(264,126) = 2^1.3^1.7^0.11^0 = 6}$

✓ By Euclid Algorithm : $264 = 126*2+12$
$126 = 12*10+6$
$12 = 6*2+0$
$\therefore GCD(264,126) = 6$

LCM: $\cancel{LCM(a,b) = \frac{a*b}{GCD(a,b)} \quad \therefore LCM(264,126) = \frac{264*126}{6}} = 5544$

2. **Topics:** Sequences and Ordinary, Mathematical Induction

Sample Problems:

2.1 Given the sequence $S = \{2, 6, 12, 20, 30, 42, \ldots\}$, write the sequence S in form $\{a_n\}_{n\ge1}$, i.e. determine a formula for a_n

$2 = 1*2;\ 6 = 2*3;\ 12 = 3*4;\ 20 = 4*5;\ 30 = 5\times6;\ 42 = 6*7, \ldots \Rightarrow a_n = n(n+1)$

2.2 Given the sequence $S = \{a_n\}_{n\ge1}$ defined recursively as $a_1 = 1$, $a_2 = 1$ & $a_n = a_{n-1} + a_{n-2}$ for all $n \ge 3$
Write S in the expanded list form.

$a_1 = 1;\ a_2 = 1;\ a_3 = a_2 + a_1 = 1+1 = 2;\ a_4 = a_3 + a_2 = 2+1 = 3;\ a_5 = a_4 + a_3 = 3+2 = 5, \ldots$

$\therefore S = \{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$

2.3 Prove by mathematical induction that $\sum_{i=1}^{n} i(i+1) = n(n+1)(n+2)/3$ for all $n \ge 1$

Let P(n) be the statement "$\sum_{i=1}^{n} i(i+1) = n(n+1)(n+2)/3$"

Claim: P(n) is true for all $n \ge 1$

Proof (By Mathematical Induction)

Basic Step: P(1) is true since $\sum_{i=1}^{1} i(i+1) = 1(1+2) = 2$ and $\frac{1(1+1)(1+2)}{3} = 2$

Inductive Step: Assume P(k) is true for $k \ge 1$ (Induction Hypothesis)
That is, for some $k \ge 1$, $\sum_{i=1}^{k} i(i+1) = \frac{k(k+1)(k+2)}{3}$ (Definition of P(k))

Now consider P(k+1):
$$\sum_{i=1}^{k+1} i(i+1) = \sum_{i=1}^{k} i(i+1) + (k+1)(k+2) \quad \text{(Recursive form of "Sum")}$$
$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \text{(Since P(k) is true)}$$
$$= \frac{(k+1)(k+2)}{3}(k+3) = \frac{(k+1)(k+2)(k+3)}{3} \quad \text{(Algebraic manipulations)}$$

$\therefore$ P(k+1) is true

$\therefore$ P(n) is true for all $n \ge 1$ (by Math Induction Principle.)

QED.

3. **Topics:** Strong Mathematical Induction and the Well Ordering Principle for Integers

Sample problems:

3.1 Use strong mathematical induction to prove the following statement:
Given $a_1 = 1, a_2 = 3$, and $a_k = a_{k-2} + 2a_{k-1}$ for all $k \geq 3$, then for all integers $n \geq 1$, a_n is odd.

Let $P(n)$ be the statement " $a_1=1, a_2=3, a_k=a_{k-2}+2a_{k-1}$ for $k\geq 3$, then a_n is odd"

Claim: $P(n)$ is true for all integers $n \geq 1$

Proof (By strong mathematical induction with $a=1$ and $b=2$)

Basic Step: $P(1)$ is true (since $a_1 = 1$

$P(2)$ is true (since $a_2 = 3$, hence a_2 is odd)

Inductive Step: Assume, for some $k \geq 3$, $P(i)$ is true for all $i=1,2,\ldots,k$. (Induction Hypothesis)

That is a_i is odd for all $i=1,2,\ldots,k$. (Definition of $P(i)$)

Now consider a_{k+1} (or $P(k+1)$)

$a_{k+1} = a_{k-1} + 2a_k$ (Recursive definition of a_{k+1})

$= (2r_1+1) + 2(2r_2+1)$ for some integers r_1 & r_2 (Since a_{k-1} and a_k are odd according to the Induction Hypothesis

$= 2(r_1+r_2+1)+1$ (algebraic manipulations)

$= 2 \times \text{integer} + 1$ (Since sum of integers is integer)

$\therefore a_{k+1}$ is odd (Definition of "odd")

$\therefore P(k+1)$ is true (Definition of $P(k+1)$)

$\therefore P(n)$ is true for all $n \geq 1$ QED (By strong math induction principle)

3.2 Use the well-ordering principle ~~and~~ combined with a contradiction proof to prove the statement in 2.4, namely $\sum_{i=1}^{n} i(i+1) = n(n+1)(n+2)/3$ for all $n \geq 1$

Let $P(n)$ be the statement " $\sum_{i=1}^{n} i(i+1) = n(n+1)(n+2)/3$ "

Claim: $P(n)$ is true for all $n \geq 1$

Proof (By Well-Ordering Principle for Integers and contradiction)

Suppose there are some $n \geq 1$ such that $P(n)$ is not true (Starting statement of a proof by contradiction)

$\therefore S = \{n \in \mathbb{Z} \mid n \geq 1 \text{ and } P(n) \text{ is false}\}$ is not empty (Following the supposition above)

$\therefore S$ has a smallest member, say k (By the well-ordering principle for integers)

$\therefore k \geq 1$ and $P(k)$ is false (Since $k \in S$)

Now $P(1)$ is true and $1 \notin S$ (Since $\sum_{i=1}^{1} i(i+1) = 1$ and $\frac{1(1+1)(1+2)}{3} = 1 \Rightarrow P(1)$ is true

$\therefore k > 1$ and $k-1 \geq 1$ (Since $k \geq 1$ and $k \neq 1$ because $1 \notin S$ but $k \in S$

But $k-1 \notin S$ (Since $k-1 < k$ and k is the least element of S)

$\therefore P(k-1)$ is true i.e. $\sum_{i=1}^{k-1} i(i+1) = \frac{(k-1)k(k+1)}{3}$ (Since $k-1 \geq 1$ & $(k-1) \notin S$)

Now consider $\sum_{i=1}^{k} i(i+1) = \sum_{i=1}^{k-1} i(i+1) + k(k+1)$ (Recursive form of "sum")

$= \frac{(k-1)k(k+1)}{3} + k(k+1)$ (Since $P(k-1)$ is true)

$= \frac{k(k+1)}{3}(k-1+3) = \frac{k(k+1)(k+2)}{3}$ (Algebraic manipulations)

$\therefore P(k)$ is true, which is a contradiction ($P(k)$ is false and $P(k)$ is true is a contradiction)

$\therefore$ Supposition that $S \neq \emptyset$ is false (Contradictory argument)

$\therefore S = \emptyset$ and $P(n)$ is true for all $n \geq 1$ QED (S being empty implies there are no $n \geq 1$ such that $P(n)$ is false)

4 **Topics:** Set Theory, Equality of Sets, Operations on Sets, Empty set, Properties of Sets

Sample problems:

Let A, B, and C be any sets of elements of the universal set U

4.1. Show that $(A-B)-C = A\cap(B\cup C)^c$

First we prove that $(A-B)-C = A\cap(B\cup C)^c$

Let $x \in (A-B)-C$ (Starting statement in a proof of inclusion of sets)

$\therefore x \in (A\cap B^c)\cap C^c$ (Definition of "set difference", $A-B = A\cap B^c$)

$\therefore x \in A\cap(B^c\cap C^c)$ (Associative law)

$\therefore x \in A\cap(B\cup C)^c$ (De Morgan's law)

$\therefore \underline{(A-B)-C \subseteq A\cap(B\cup C)^c}$ (Definition of a subset)

Next we prove that $A\cap(B\cup C)^c \subseteq (A-B)-C$

Let $x \in A\cap(B\cup C)^c$ (Starting statement in a proof of inclusion of sets)

$\therefore x \in A\cap(B^c\cap C^c)$ (De Morgan's law)

$\therefore x \in (A\cap B^c)\cap C^c$ (Associative law)

$\therefore x \in (A-B)-C$ (Definition of "set difference", $A-B = A\cap B^c$)

$\therefore \underline{A\cap(B\cup C)^c \subseteq (A-B)-C}$ (Definition of a subset

$\therefore (A-B)-C = A\cap(B\cup C)^c$ as desired to be proved QED

4.2 Use appropriate "Properties of Sets" to show that $\left(A\cap(B\cup C)^c\right)\cap(B\cup A)\cap C = \phi$

$A\cap(B\cup C)^c = A\cap B^c\cap C^c$ (De Morgan's law)

$\therefore LHS = (A\cap B^c\cap C^c)\cap(B\cup A)\cap C$ (Substitution)

$= \left((A\cap B^c\cap C^c\cap B)\cup(A\cap B^c\cap C^c\cap A)\right)\cap C$ (Distributive law)

$= \left(\phi\cup(A\cap B^c\cap C^c)\right)\cap C$ (a combination of Associative law and $A\cap\phi = \phi$ and $A\cap(A\cap B) = A\cap B$)

$= A\cap B^c\cap C^c\cap C$ (Since $\phi\cup A = A$ Identity law)

$= \phi$ (Since $C^c\cap C = \phi$)

5 **Topics:** Relations on Sets, Directed Graphs, Inverse Relations, Reflexivity, Symmetry, Transitivity, Transitive Closure, Equivalence Relations, Equivalence classes

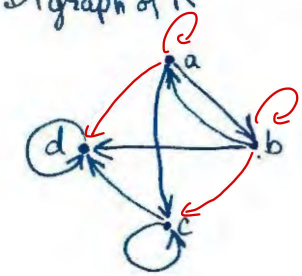
Sample problems:

5.1 Let A = {a, b, c, d} and let R be a relation on A defined as

R = {(a, b), (a, c), (b, a), (b, d), (c, c), (c, d), (d, d)}

Draw a digraph (directed graph) representing R, and determine whether R is reflexive, symmetric, and/or transitive. If not transitive, find its transitive closure. Also find inverse relation R^{-1}.

Digraph of R



Clearly R is not reflexive since no loop at a and b

R is not symmetric since return (c,a), (d,b) and (d,c)

R is not transitive

The transitive closure of R is

$$R^t = \{R,\ (a,d),\ (b,c),\ (a,a),\ (b,b)\}$$

$$R^{-1} = \{(b,a),\ (c,a),\ (a,b),\ (d,b),\ (c,c),\ (d,c),\ (d,d)\}$$

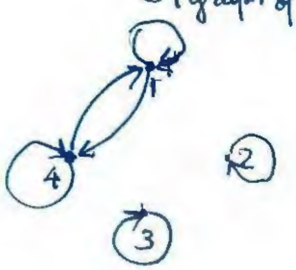
5.2 Let A = {1, 2, 3, 4} and let R be a relation on A defined as For all $x, y \in A$, $xRy \Leftrightarrow 3 \mid (x-y)$

List all elements of R and R^{-1}.

Draw a digraph representing R, and determine whether R is an equivalence relation on A. If so find all equivalence classes induced by R

$\therefore R = \{(1,1), (1,4), (2,2), (3,3), (4,1), (4,4)\}$

Digraph of R



Clearly R is reflexive, symmetric and transitive

$\therefore$ R is an equivalence relation on A

The equivalence classes induced by R are

$[1] = \{1, 4\}$

$[2] = \{2\}$

$[3] = \{3\}$

And $R^{-1} = \{(1,1), (4,1), (2,2), (3,3), (1,4), (4,4)\}$

$= R$.