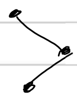
Graph Theory

$G = (V, E)$

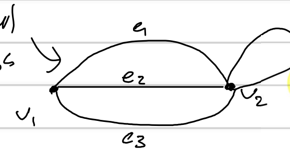
V is set of vertices / Nodes

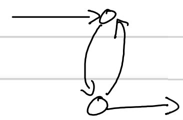
E is set of edges

Def: A graph is a simple graph if there is no loop or parallel edges



Parallel Edges





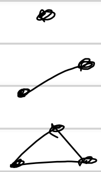
A simple graph is called a complete graph if every pair of vertices are connected by an edge

$G = (V, E)$

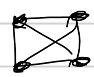
$|V| = n$

K_n

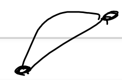
K_1 K_2 K_3



K_4



Parallel Edges



2 different edges between two nodes

How many edges for K_n?

$|E| = \binom{n}{2}$

Recall: $C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$

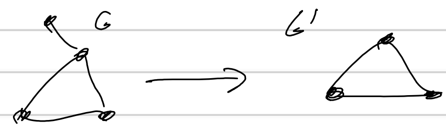
Def: Suppose $G = (V, E)$

$G' = (V', E')$ is a subgraph of G if $V' \subseteq V$, $E' \subseteq E$

Vertices subset of those in G, same with E

also a clique

Def: A complete subgraph is a clique



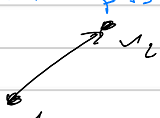
Many possible cliques here

Largest clique is K_3 for this G

Def: Degree of a vertex v

$deg(v)$ = number of edges that v is the ending vertex

A loop is counted twice

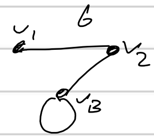


wrong
$deg(v_1) = 1$
$deg(v_2) = 0$

don't really care about di-graph degree in CSDS 302

Def: Degree of Graph G

$deg(G) = \sum_{i=1}^{n} deg(v_i)$



$deg(v_1) = 1$
$deg(v_2) = 2$
$deg(v_3) = 3$ (loop)

$\therefore deg(G) = 6$

Handshake Theorem

$deg(G) = 2 \times |E|$

degree of a graph is equal to 2 times the number of edges

Individual $deg(V)$ can be odd, but $deg(G)$ must be even

Holds true for non-digraphs

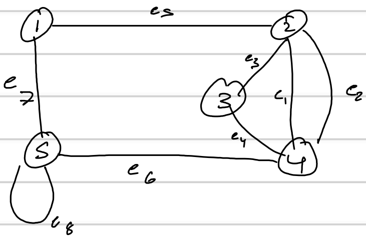
Does there exist a graph where $deg(G) = 5$?

$deg(G) \neq 5$, since deg must be even!

$2 \times |E|$ is even always since $|E| \in \mathbb{Z}^+$

{paths} ⊆ {trails} ⊆ {walks}

Def: A walk from v to w is a finite sequence of adjacent vertices



①⟶④ ?

① $\xrightarrow{e_5}$ ② $\xrightarrow{e_1}$ ④

How many possible walks from ① → ④

∞ many walks as you can go ①→④→①...

Def: A trail from v to w is a walk with all edges being distinct — Special walk

Def: A path from v to w is a trail with all vertices being distinct — Super special walk

①⟶⑤ ?

① $\xrightarrow{e_5}$ ② $\xrightarrow{e_1}$ ④ $\xrightarrow{e_2}$ ② $\xrightarrow{e_3}$ ③ $\xrightarrow{e_4}$ ④ $\xrightarrow{e_6}$ ⑤

This is a walk and a trail
all distinct edges

Not a path as ② and ④ being used twice

Def: A circuit is a trail from one vertex to itself, A specific trail!

A trail where v = w

Def: A simple circuit or a cycle is a circuit with all vertices being distinct

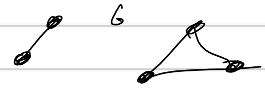
Like a path but violates first entry!

Def: Vertex v and w are connected if there is a walk from v to w

Def: G is a connected graph if $\forall v, w \in V$, v and w are connected

↳ All pairs of vertices are connected

G

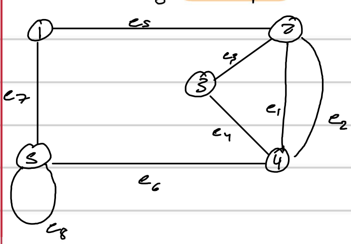


G is not connected, but has 2 connected components!

Euler Circuit

A Euler Circuit for graph G is a circuit that contains every vertex at least once and every edge exactly once.

Continue using Example



① $\xrightarrow{e_5}$ ② $\xrightarrow{e_3}$ ③ $\xrightarrow{e_4}$ ④

$\xrightarrow{e_1}$ ② $\xrightarrow{e_2}$ ④ $\xrightarrow{e_6}$ ⑤

$\xrightarrow{e_8}$ ⑤ $\xrightarrow{e_7}$ ①

This is generally very difficult

Not every Graph G has an Euler Circuit, like if e_3 didn't exist in the Example to the left

Theorem

Given a graph G, G has an Euler Circuit if $deg(V_i)$ = even and connected

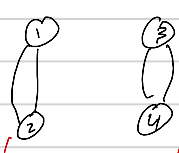
↑ Degree of each vertex is even

where $V_i \in V$ and $\Sigma V_i = V$

Examples of No Euler Circuit Graphs

1 Edge
3 Edges ✗
5 Edges

Exists for $deg(V_i)$ = even



Graph

Does not have Euler Circuit since it's not connected

Exists iff G is connected and

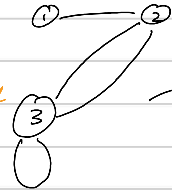
Divide into cases to prove things :)

Def: Let G be a graph with n vertices. Then adjacency matrix A of A is an $n \times n$ matrix where a_{ij} = number of edges between vertex i and vertex j

a_{ij} where i is row, j is column

Allows computers and algorithms to produce and manipulate graphs/pictures

G Undirected



$$\longrightarrow A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$ Symmetric

a_{11}: Edges from $v_1 \to v_1$

a_{12}: Edges from $v_1 \to v_2$

etc...

a_{ij} for Edges from $v_i \to v_j$

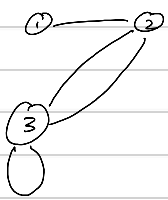
G Directed

$$\longrightarrow A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$ not symmetric

Def: A K-walk from v to w is a walk with length K. A walk with K # of edges.

Theorem 10.2.2 Let A be an adjacency matrix of G, then the number of K-walks from V_i to V_j is $a_{ij}^{(k)}$. i-j element of A^k

Example: 2 walks from v_1 to v_3



$$\longrightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

2-walk: $K=2$

$$\therefore A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 2 \\ 2 & 2 & 5 \end{bmatrix} \longrightarrow a_{13}^{(2)} = 2$$

Matrix Multiplication Review

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$