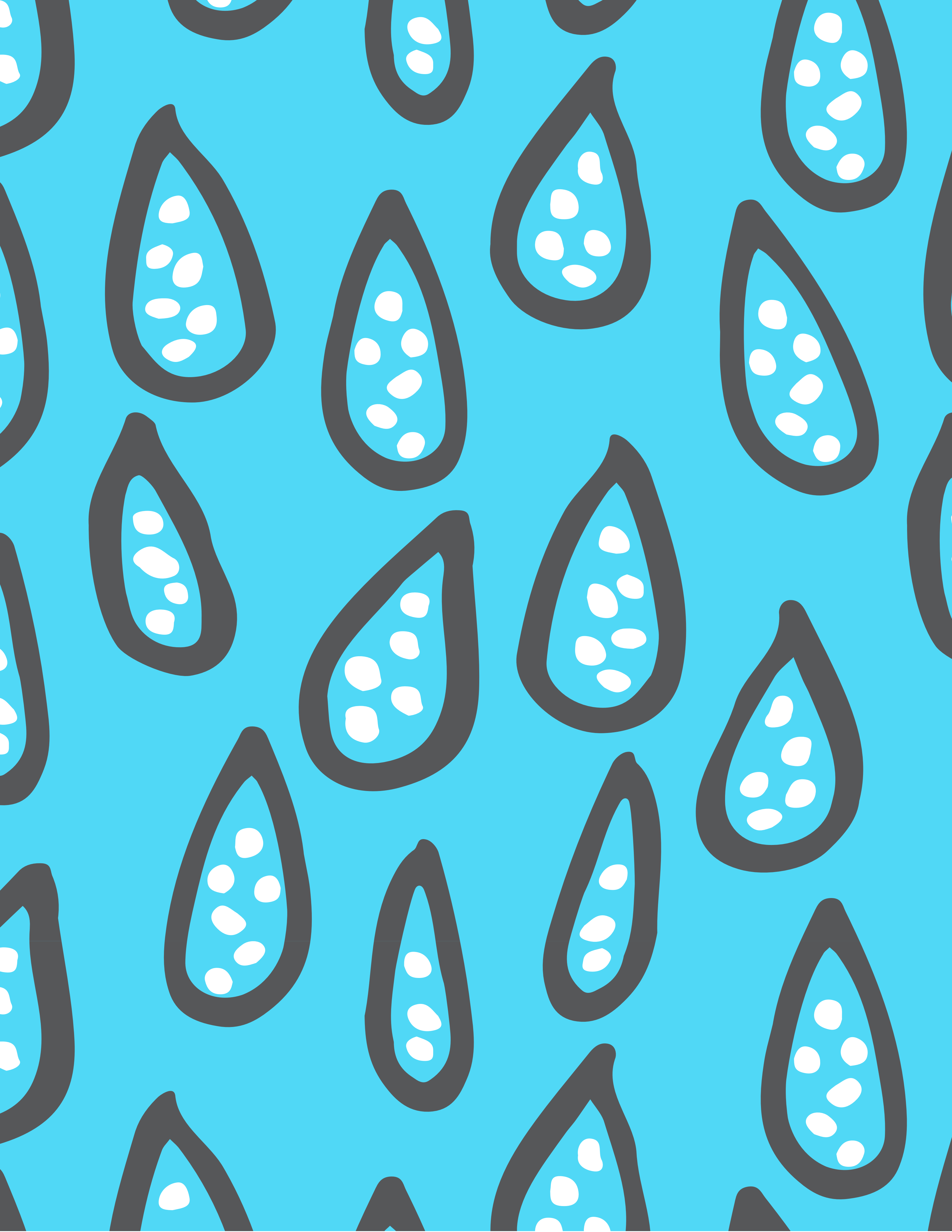


Midterm Exam 1

Chapters 1 & 2 - Concepts

Key Terminology

Six Sigma: A data-driven management strategy

Quality: Fitness for use

Two aspects:
a) Quality of design - high quality?
b) Quality of conformity - is the high quality consistent?

- "Characteristics" or "dimensions of quality" that affect fitness for use as a product
  - (i) Performance — performs required job
  - (ii) Reliability — how often will it fail
  - (iii) Durability — lifetime → will it meet it
  - (iv) Serviceability — easy to repair
  - (v) Aesthetics — product's appearance
  - (vi) Features — what the product does
  - (vii) Perceived Quality — Reputation
  - (viii) Conformance to Standards
- Producing a Service... list those dimensions of quality
  - (i) Responsiveness — time to reply to requests
  - (ii) Professionalism — Knowledge, timeliness
  - (iii) Attentiveness — caring, personalized attention

Hidden Factory: The part of the plant the customer doesn't see which deals w/ bad quality

Quality Engineering: Consists of activities to ensure that ① that quality characteristics are at desired levels, and ② variability is at its minimum

will call c2qc → Sometimes called 'critical to quality characteristics' → **Quality Characteristics:** Important elements of a product/service to the customer

Variability: Describes or quantifies differences within a set of data or parts or c2qc

Statistical Methods: Analyses, interpretation, and presentation of numerical data

Often an upper and lower limit → **Specifications:** Desired measurements for quality characteristics. This is your target value

Nonconformity: Failure to meet spec

Defect: Nonconformity that limits use

Variability

- Quality is inversely proportional to variability
  - Variability will impact all quality dimensions
- Variability leads to greater cost & waste

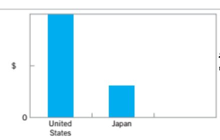


FIGURE 1.1 Warranty costs for transmissions.

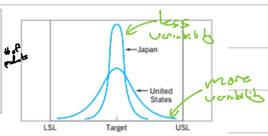


FIGURE 1.2 Distributions of critical dimensions for transmissions.

example to right examines domestic vs. imported car quality

- Why variability leads to more cost for a company

Quality Costs

Prevention Costs	Internal Failure Costs
Quality planning and engineering	Scrap
New products review	Rework
Product/process design	Retest
Process control	Failure analysis
Burn-in ← JR's company 20 yr test	Downtime
Training ← Personnel	Yield losses
Quality data acquisition and analysis	Downgrading (off-specing)
Appraisal Costs	**External Failure Costs**
Inspection and test of incoming material	Complaint adjustment
Product inspection and test	Returned product/material
Materials and services consumed	Warranty charges
Maintaining accuracy of test equipment	Liability costs
	Indirect costs

- Products found in plant outside of specifications results in higher costs and waste
- Prevention & Appraisal costs are cheaper costs and result in more consistent products and cheaper costs down the line

Internal Failure & External Failure costs >> Prevention & appraisal costs!

Example: Cost of Variance ← Find manufacturing cost per good product

Illustrative Example: We are making 100 mechanical components per day where 75% of the parts conform to specifications, and the other 25% are non-conforming. Of the 25% of nonconforming, 60% can be reworked and the other 40% are scraped. Each part costs \$20 to manufacture and an additional \$4 to be reworked.

$$\frac{\text{cost}}{\text{good}} = \frac{\text{cost to make (\# made total)} + \text{cost to rework (\# reworked)}}{\text{\# of goods sold}}$$

$$= \frac{\$70(100) + \$4(100)(0.25 \cdot 0.60)}{70 + 15} = \boxed{\$22.89}$$

Now, the Benefit of Improvement ← Find manufacturing cost per good product

Illustrative Example: We improve the previous process such that now 95% of the parts conform to specifications, and the other **5% are non-conforming**. Similar to before, of the 5% of nonconforming, 60% can be reworked and the other 40% are scraped. Each part costs \$20 to manufacture and an additional \$4 to be reworked.

$$\frac{\text{cost}}{\text{good}} = \frac{\$70(100) + \$4(100)(0.05)(0.60)}{95 + 3}$$

$$= \boxed{\$20.53}$$

Quality Engineering

Process: A system of inputs and outputs

Example, Defining a process

Example: A hospital is trying to increase the quality of drug administration. To do this, it is considering providing patients with bar-coded wristbands to help guide workers. Your team is charged with studying the effects of bar-coding by carefully watching 250 episodes in which drugs are given to patients without bar coding and 250 episodes with bar coding. Every time a drug is administered you will check the amount, if any, of discrepancy between what was supposed to be given and what was given.

- Controllable input variable: barcoding
- Key output variable: right amount of drug

Control Charts: A process monitoring technique which tracks averages in a quality characteristic with time or sample number

- Centerline: where the characteristic should be if no abnormal sources of variation
- Control Limit: Defined by statistics. Alert the user to unusual variability
  - ↳ Different than spec limits

Designed experiments: discovers key variable that have a significant influence on quality characteristics in a process

Six Sigma

Broader Definition: Statistical based, data driven management approach and continuous quality improvement methodology for eliminating defects in a product, process, or service

↳ Reduces variability to a level where defects are unlikely

Three Elements of Six Sigma

(i) Quality Planning: Listening to the voice of your customer

(ii) Quality Assurance: Establishing a system to prevent quality issues from arising

(iii) Quality Control & Improvement: A set of specific steps and tools to ensure products meet requirements and are continuously improved

Name Meaning: Refers to the quality level target such that sigma (σ, or variance σ^2) should be small enough so that $6\cdot\sigma$ is within the specification limits

Sigma Quality Level: The Probability that product/service is non-defective

Why is Six Sigma Selected?

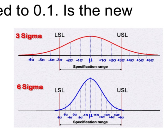
Suppose 100 parts: 3σ : $(.9973)^{100} = .7631$ is probability non-defective $\rightarrow \sim 24\%$ defective

6σ : $(.9999998)^{100} = .99999998$ non-defective $\rightarrow \sim .000002\%$ defective

Some companies use other multiples of σ, depending on what required by company/customers

Example of Six Sigma (will be on HW)

Illustrative Example: Let's say you are making a part that has a radius of 10 mm. Specification limits are between 9 (LSL) and 11 (USL) mm. Before you applied quality improvement, the standard deviation (sigma – a measure of variability) was 1 mm, and after you applied improvements, the standard deviation (sigma) decreased to 0.1. Is the new process 6 sigma quality?



$$\frac{|USL \text{ or } LSL - \text{target}|}{\sigma(\text{standard deviation})} = \frac{|9\text{ mm} - 10\text{ mm}|}{1\text{ mm}} = 1 \quad \text{for old} \quad 1\sigma \text{ level!!}$$

$$= \frac{|9\text{ mm} - 10\text{ mm}|}{0.1\text{ mm}} = 10 \quad \text{for new} \quad 10\sigma \text{ level!!}$$

Lean Manufacturing

Def: Another management philosophy aimed at eliminating waste (e.g., time and materials)

Process Cycle Efficiency (PCE): (Value added time) / (Process time)

↳ **Value added time** = time spent making the product more valuable

Process Cycle Time (PCT): (Number of items in progress) / (completion rate) (number / unit time) ← backlog issues as an example

Why is a slow process (low PCE or high PCT) an expensive process?

- Hidden Factory →
  - (i) Customers don't like waiting
  - (ii) More handling = more personel
  - (iii) More opportunity for damage or loss
  - (iv) Inventory has to be higher
  - (v) More documentation

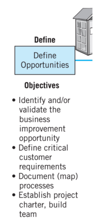
DMAIC Process

Def: A Project-based 5-step process and problem-solving procedure

↳ Define → Measure → Analyze → Improve → Control

Tollgates: where projects are reviewed to ensure on track

5 stages of DMAIC (will be on HW too)



(i) **Define Stage**

- **Main Obj:** Identify a project opportunity and verify that it represents a legitimate value opportunity or breakthrough
  - **Breakthrough:** Major improvement to a process or product
  - **Value Opportunity:** The financial opportunity at stake
- **Tools used**
  - **SIPOC diagram:** High level map of process
    - → Suppliers: Those who provide information, material or other items that are worked on in the process
    - → Input: The actual information or material provided
    - → Process: The set of steps required to do the work
    - → Output: The product delivered to the customer → Other depts in same company
    - → Customers: who buys the product (internal or external)
- **Process Mapping:** Involves creating flow diagrams for systems
- lean concept → **Value stream mapping:** A variant of process mapping where engineers focus on steps which could be eliminated or simplified to reduce waste
- **Value stream:** The min amount of processing steps from raw material to customer

Example: Value Stream Mapping (Lawn Mowing)

{Old}: Gather Tools (30 min) → Fuel Mowers (10 min) → Travel (30-60 min, variable) → Unload (10 min) → Talk to Customer (10-20 min, variable) → Care for Lawn (60-120 min, variable) → Bill Customer (10 min) → Load Tools / Pack Up (10 min) → Drive to next Customer (30-60 min, Variable) …

Possible Improvements: Using customer's Lawnmower, storage SOP (leave on truck), prefuel, personal/interns, eliminate/reduce customer interactions, optimize route (graph shortest distance) / logistics

- **Pareto charting:** Plots frequency (# of parts) or cost (time/money) against causes
  - **Pareto Rule:** 80% of the effects come from 20% of the causes

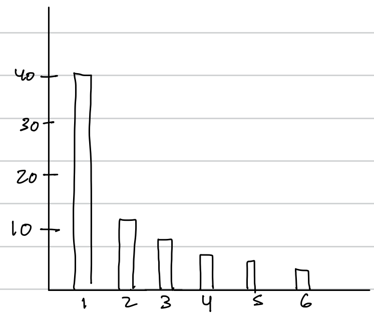
• Graph Pareto by Hand (Example)

Often in order from largest cost to smallest

Cost or % defects is on y-axis, category is on the x-axis

Monthly Quality-Costs Information for Assembly of Printed Circuit Boards

	Type of Defect	Percentage of Total Defects	Scrap and Rework Costs
1	Insufficient solder	42%	$37,500.00 (52%)
2	Misaligned components	21	12,000.00
3	Defective components	15	8,000.00
4	Missing components	10	5,100.00
5	Cold solder joints	7	5,000.00
6	All other causes	5	4,600.00
	Totals	100%	$72,200.00



Can also do this with Lean concepts

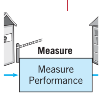
Using Minitab

1) Input data into Minitab by Type of Defect, and Cost (or other)
2) Go to Stat -> Quality Tools -> Pareto Chart
3) Specify which column the defects or attribute data are in (these are the labels with names)
4) Specify what column the "frequencies" are in (this can be costs, or number of defects – the numerical data)
5) Click ok and check that the chart makes sense

Project Charter: Short document which describes the product.

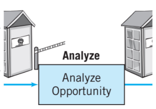
Example to right, but varies accross companies

Business Case • This project supports the business quality goals, namely (a) reduce customer resolution cycle time by x% and (b) improve customer satisfaction by y%.	Opportunity Statement • An opportunity exists to close the gap between our customer expectations and our actual performance by reducing the cycle time of the customer return process.
Goal Statement • Reduce the overall response cycle time for returned product from our customers by x% year to year.	Project Scope • Overall response cycle time is measured from the receipt of a product return to the time that either the customer has the product replaced or the customer is reimbursed.
Project Plan • Activity — Start — End Define — 6/04 — 6/30 Measure — 6/18 — 7/30 Analyze — 7/15 — 8/30 Improve — 8/15 — 9/30 Control — 9/15 — 10/30 Track Benefits — 11/01	Team • Team Sponsor • Team Leader • Team Members



Objectives
- Determine what to measure
- Manage measurement data collection
- Develop and validate measurement systems
- Determine Sigma performance level

(ii) Measure Stage
- Objective: Evaluate & understand the current state of the process
- Baseline Performance: Quantify what the current process is performing at
- Measurement Capability: Quantify if the current measurement systems are adequate
- Tools Used: Charts to understand variability
  Histograms, boxplots, run charts, scatter diagrams



Objectives
- Analyze data to understand reasons for variation and identify potential root causes
- Determine process capability, throughput, cycle time
- Formulate, investigate, and verify root cause hypotheses

(iii) Analyze Stage
- Objective: Use data to begin to determine cause and effect relationships and sources of variability
- Common Causes of Variability: Internal Sources (noise)
- Assignable Causes of variability: External Sources
- Tools Used
  - Control Charts: Used to separate the common and assignable causes of variation
  - Hypothesis Testing & Confidence Intervals: Used to determine if different conditions produce significant results
  - Regression Analysis: Allows models of the process to be built
  - Failure Modes & Effects Analysis (FMEA): A method for identifying and prioritizing sources of variability, errors or defects in a process or product
    ↳ For each item, we determine:
      (i) Likelihood something will go wrong | (not likely) - 10
      (ii) Ability to detect the failure or defect | (likely) - 10
      (iii) Severity | (not severe) - 10

Risk Priority #: (i) × (ii) × (iii)

Example

1) Severity, 1 (not severe) – 10
2) Probability of occurrence, 1 (not likely)-10
3) Probability of detection, 1 (likely) -10

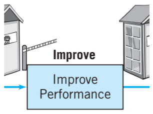
RPN = A X B X C

RPN = Risk Priority Number

FAILURE MODE & EFFECTS ANALYSIS (FMEA) Date: 1/1/2018 Revision: 1.3

Process Name: Left Front Seat Belt Install Process Number: SBT 445

Failure Mode	A) Severity Rate 1-10 10=Most Severe	B) Probability of Occurence Rate 1-10 10=Highest Probability	C) Probability of Detection Rate 1-10 10=Lowest Probability	Risk Preference Number (RPN) AxBxC
1) Select Wrong Color Seat Belt	5	4	3	60
2) Seat Belt Bolt Not Fully Tightened	9	2	8	144
3) Trim Cover Clip Misaligned	2	3	4	24

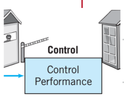


Objectives

- Generate and quantify potential solutions
- Evaluate and select final solution
- Verify and gain approval for final solution

(iv) Improve Stage

- Objective: Use creative thinking & info gathered previously to make specific suggestions that will have the desired effect on process. Confirmation is important
- Tools Used
  - Mistake Proofing (example)
  - Optimization
  - Pilot Testing
  - Confirmation Experiments
  - Documentation
    - (i) Solution obtained
    - (ii) Alternatives Considered
    - (iii) Confirmation tests
    - (iv) Risk Analysis



Objectives

- Develop ongoing process management plans
- Mistake-proof process
- Monitor and control critical process characteristics
- Develop out-of-control action plans

(v) Control Stage

- Objective: Complete all remaining work and hand off to improved process so gains in progress are institutionalized
- Tools Used
  - Control Charts
  - Transition Plan

Chapter 3 - Quantifying Stats in Quality Characteristics

Statistics

Sample Mean: Average Value

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

where n is the number of data points in the sample and $x_1, x_2, x_3, \ldots x_n$ are data points

For this data set: 12, 12, 15, 18, 21, 22, 22, 23, 25, 25, 26, 27, 28, 29, 30, 30, 31

$$\bar{x} = \frac{12+12+15+\ldots+31}{17} = \frac{396}{17} = 23$$

Sample Variance: {(Sample units)^2}

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}$$

Standard Deviation: $s = \sqrt{s^2}$ {Sample units}

For this data set: 12, 12, 15, 18, 21, 22, 22, 23, 25, 25, 26, 27, 28, 29, 30, 30, 31

$$s^2 = \frac{(12-23)^2 + (12-23)^2 + \ldots + (31-23)^2}{17-1} = 37 \qquad s = \sqrt{37} = 6.08$$

MINITAB: Stat > Basic Statistics > Display Descriptive Statistics > In Variables add data

Illustrative Example

Sample 1	Sample 2
$x_1 = 1$	$x_1 = 1$
$x_2 = 3$	$x_2 = 5$
$x_3 = 5$	$x_3 = 9$
$\bar{x} = 3$	$\bar{x} = 5$

Sample 3
$x_1 = 101$
$x_2 = 103$
$x_3 = 105$
$\bar{x} = 103$

Sample 1: $s = 2$
Sample 2: $s = 4$
Sample 3: $s = 2$

Tools to Describe Data Numerically and Visualize Variability

↳ Often easier to arranged sorted (lowest → highest) data

- **Percentiles:** Tell us how much of the data is below a certain point

12	12	15	18	21	22	22	23	25	25	26	27	28	29	30	30	31	← Values
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	← Rank (1-indexed)

- **Lower Quartile** (Q_1 or 25^{th} perc.): Value with rank $= (n+1)\left(\frac{1}{4}\right)$
  here: $\frac{17+1}{4}$ = rank of 4.5 → $\frac{18+21}{2} = \boxed{19.5}$
- **Middle Quartile** (Q_2, or median, or 50^{th} perc.): Value with rank $= (n+1)\left(\frac{1}{2}\right)$
  here: $\frac{17+1}{2}$ = rank of 9 → $\boxed{25}$
- **Upper Quartile** (Q_3, or 75^{th} perc.). Value with rank $= (n+1)\left(\frac{3}{4}\right)$
  here: $(17+1)\left(\frac{3}{4}\right)$ = rank of 13.5 → $\boxed{28.5}$
- **Interquartile Range:** $Q_3 - Q_1$
  here: $Q_3 - Q_1 = 28.5 - 19.5 = 9$
- **Whisker Limits:** Max Limit $= Q_3 + 1.5(Q_3 - Q_1)$
  Min Limit $= Q_1 - 1.5(Q_3 - Q_1)$
  here: Max Limit = 43, Min Limit = 6

MINITAB: Stat > Basic Statistics > Display Descriptive Statistics > In Variables add data

- **Box Plot Features**

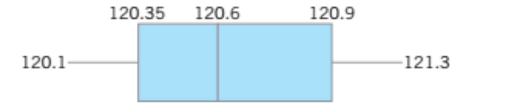
Displays 25^{th}, 50^{th} and 75^{th} quartiles (ends and middle of the box)

- 25^{th} and 75^{th} are the ends of the box
- Median (50^{th} percentile) is a single line through box
- The min and max of data (whiskers)
- Limits of whiskers (check) that the max and min do not go past

(points that fall outside this can be outliers, represented as single data points and the new min or max is the next data point within the limit) – limit formula on next slide

TABLE 3.4
Hole Diameters (in mm) in Wing Leading Edge Ribs

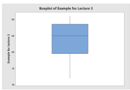
120.5	120.4	120.7
120.9	120.2	121.1
120.3	120.1	120.9
121.3	120.5	120.8



here:

Max= 31
Upper quartile (Q3, 75th percentile) = 28.5
Median is (Q2, 50th percentile): = 25
Lower Quartile (Q1, 25th percentile) = 19.5
min = 12

Whisker Limits:



Minitab: Graph > Boxplot > Simple > input data column into graph variables

Excel
- Stdev.s is for sample data (use most of time)
- Stdev.p is for entire population

- Histograms: Plots of frequencies of data occurring vs. the data sorted into bins/intervals
  - ↳ Bins: The range that data will be sorted into   # bins = ceil($\sqrt{n}$) ← round up   bin size = $\frac{\max - \min}{\# \text{ of bins}}$

i. Need to determine the bin range for continuous data
ii. Best if bins are the same size
iii. Rule of thumb: number of bins = square root of number of observations (round up if not a whole number)

Note: for categorical data or discrete data, each value or category is its own bin

For this data set: 12, 12, 15, 18, 21, 22, 22, 23, 25, 25, 26, 27, 28, 29, 30, 30, 31

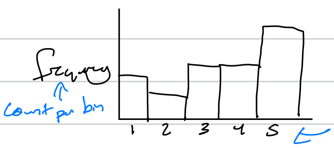
# of bins = $\sqrt{17} = 4.1 \rightarrow 5$

bin size = $\frac{31-12}{5} = 3.8$

Bin 1: min + bin size → 12+3.8 → [12, 15.8)   Inclusive below, Exclusive above

Can be flexible w/ rounding but data must not overlap

Bin 2: bin 1 end + bin size → 15.8+3.8 → [15.8, 19.6)



Usually should be bin limits

Bin 3: [19.6, 23.4)
Bin 4: [23.4, 27.2)
Bin 5: [27.2, 31]
↳ Final bin should be inclusive above

MINITAB: Graph > histogram > simple > add your column to graph variables
To change bins: double click on histogram bars (twice), select bins tab, change number of intervals to number of bins (Cut point is how we graphed in class)

Significant Digits / Figures: Number of digits contributing to measurement resolution
Rules: (i) Non-zero digits always significant
(ii) Any zeros between two significant digits are significant (leading zeros are not)
(iii) Trailing zeros in the decimal portion only are significant (unless indicated by a bar)

Examples:

How many sig. figs in each number?
0.00500 = 3
0.03040 = 4
102$\bar{0}$0 = 4

How would you write this # with the correct sig. figs?
0.899923 (report 2 sig. figs) 0.90
23000.0 (report 3 sig. figs) 23$\bar{0}$00

Will take away points for tons of sig figs!   Just pay attention to them!

For your sample means: keep as many significant figures as the number with the lowest significant digits in your data set (same for median and quartiles)

For your standard deviation: report to the last *place* that is significant, but no more

Your Average and SD should have same # of places!

E.g. Sample mean = 4.09 Sample Standard Deviation = 0.00923. What do I report? 4.09 ± 0.01

Probability Distributions Introduction
- Sample: Collection of measurements from a larger source or population
- Population: All of the elements from a group
- Probability Distribution: Mathematical Model that relates value of a variable with its probability of occurence
- Continuous Distribution: The variable is expressed on a continuous scale
- Discrete Distribution: The variable can only take on certain values
- Writing out Probability Statements
  - Discrete: The probability that x is equal to x_i ?

$$P\{x = x_i\} = p(x_i)$$

  - Continuous: The probability that x is between a and b?

$$P\{a \le x \le b\} = \int_a^b f(x)\,dx$$

- Calculating mean and variance

$$\mu = \begin{cases} \int_{-\infty}^{\infty} x f(x)\, dx, x \text{ continuous} \\ \sum_{i=1}^{\infty} x_i p(x_i), x \text{ discrete} \end{cases}$$

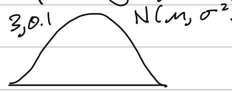
$$\sigma^2 = \begin{cases} \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx, x \text{ continuous} \\ \sum_{i=1}^{\infty} (x_i - \mu)^2 p(x_i), x \text{ discrete} \end{cases}$$

- Median: 50th percentile
- Mean: Central Tendency (average)
- Variance: Spread of the distribution

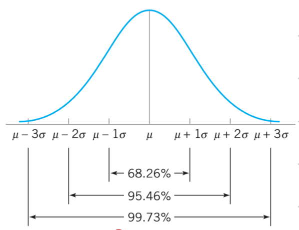
μ, σ^2 (Population model params)
$\bar{x}, s^2$ (Sample data notation)
Pay attention to this notation

The Normal Distribution

- For a random, normal variable x, the probability distribution of x is: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$
  - ↳ Short hand $N(\mu, \sigma^2)$



  - o Continuous, symmetric, "bell-shaped"



Holds for any normally distributed variable, 'x'

General Formula: $P\{a \le x \le b\} = \int_a^b f(x)\,dx$

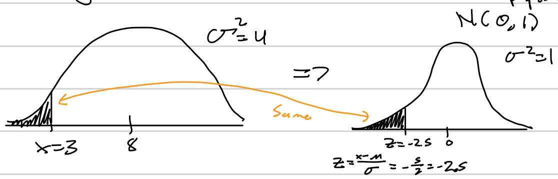
In a Probability Function: $P\{(\mu - \sigma) \le x \le (\mu + \sigma)\} = 0.68$

Due to symmetry: $P\{x > \mu\} = P\{x < \mu\} = 0.5$

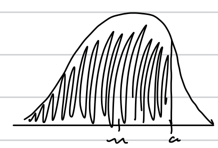
  - o Getting these percentages: Set $\sigma = 1$, $\mu = 0$ so $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $\begin{cases} a = \mu - 1\sigma = -1 \\ b = \mu + 1\sigma = 1 \end{cases}$

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \text{(using mathematica)} \; 0.68$$

- Standardization: $N(\mu, \sigma^2) \Rightarrow N(0, 1)$
- Standard Normal Distribution: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
- To standardize any random variable x: $z = \frac{x - \mu}{\sigma}$ z is a standard normal variable
- Drawing out normal distributions: Suppose $N(8, 4)$



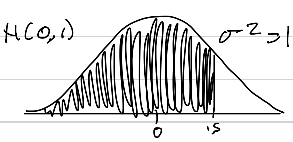
- Cummulative Normal Distribution: The probability that the variable x is less than or equal to some value a



↳ Standardization: $P\{x \le a\} = P\{z \le \frac{a-\mu}{\sigma}\}$ ← Go to Appendix II

Example: Assume Z is a standard normal random variable, and you want to find the probability that Z is less than or equal to 1.5

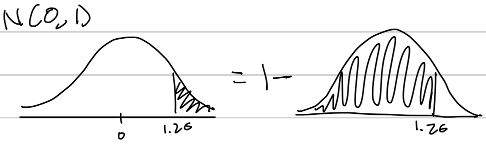
$P\{z \le 1.5\} = 0.93$



General Probability Demonstration

(i) $P(z > 1.26) = 1 - P(z < 1.26)$

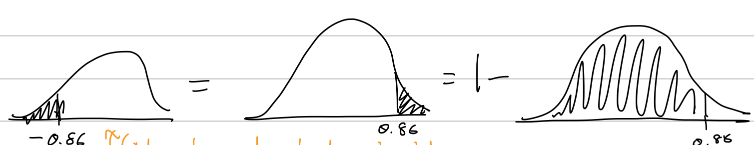
N(0,1)



$= 1 - 0.89616 = 0.1039$

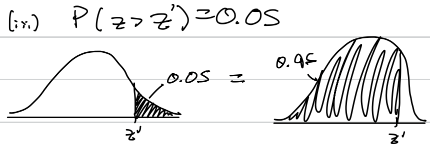
$\boxed{0.10}$

(ii) $P(z < -0.86) = P(z > 0.86) = 1 - P(z < 0.86)$



↑(Negatives not in back of book)

(iii) $P(z > z') = 0.05$



Chapter 4 - Hypothesis Testing

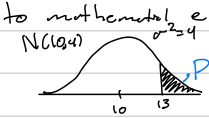
Example: Suppose the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 milliamps and a variance of 4 milliams^2. What is the probability that the measurement will exceed 13 milliamps?

1) Convert the word problem to mathematical expression and sketch the situation.

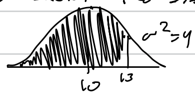
$\mu = 10$

$\sigma^2 = 4$

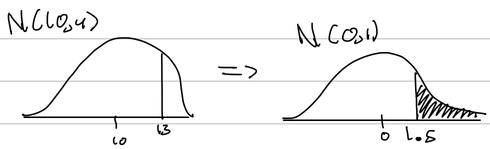
$P(x \geq 13)$



$= 1 -$



3) Convert sketch and formula to standard normal (if using tables in back of book) and a cumulative normal distribution (so it is compatible with the table in the book and Minitab)
Minitab: Calc > Probability dist. > Normal



$\rightarrow P(z \geq 1.5)$

$z = \frac{x-\mu}{\sigma} = \frac{13-10}{2} = 1.5$

$1 - P(z < 1.5) = \boxed{0.07}$

→ what about the integral?

$$P(z \geq 1.5) = \int_{1.5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \quad \text{or} \quad P(x \geq 13) = \int_{13}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Checking for normality

→ How do we know if the normal distribution is a reasonable model for our data?

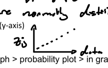
- **Probability Plotting:** Graphical method for determining whether sample data conform to a hypothesized distribution based on subjective visual examination.
- **Constructing a normal prob. plot by hand:** using

12, 12, 15, 18, 21, 22, 22, 23, 25, 25, 26, 27, 28, 29, 30, 30, 31 ← Data
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 ← Rank (j)

How-to (Steps)

1) Ranking data and ordering it ($x_1, x_2 \ldots x_j \ldots x_n$), j= the ranked value, n = total number of values
2) Calculating cumulative frequencies (j-0.5)/n for each data point
   ↳ tells us the % of data that lie below a certain ranked value
   $3\,(n=10) = \frac{3-.5}{10} = 0.25$

3) Find a zj value for each rank using the cumulative frequency value is the probability
Find z_j s.t. $P(z < z_j) = \frac{j-0.5}{n}$. Tells us the value of z_j if the data are normally distributed
4) Plot the actual data (x-axis) vs. corresponding zj (y-axis)

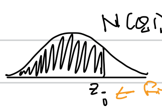


Normal Probability Plots in Minitab – Minitab: Graph > probability plot > in graph variables, choose column with desired data

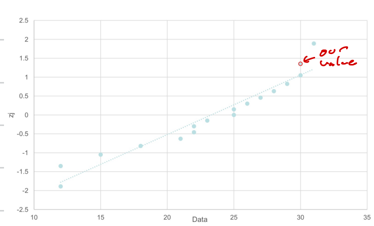
Note: each value is plotted versus the cumulative frequency with the y-axis is transformed so that the fitted distribution forms a straight line if normally distributed.

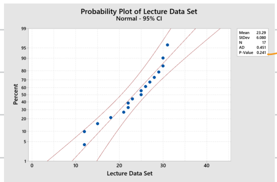
C. Freq $\left(\frac{j-0.5}{n}\right)$

For rank 16: $\frac{16-0.5}{17} = 0.91 \rightarrow z_j$: $P(z < z_j) = 0.91$



$\Rightarrow z = 1.35$





P-value < α means not normal

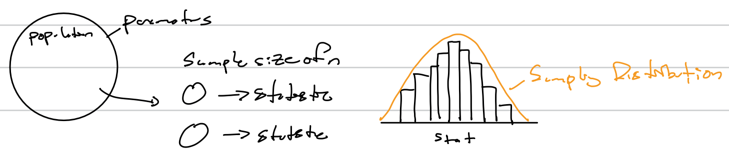
In literature,

α = 0.05

or α = 0.01

Statistics Basically

- **Statistical Inference:** Drawing conclusions about a population based on a sample from that population (often them selected randomly)
- **Statistics:** Any function of sample data that doesn't contain unknown parameters
- **Sampling Distributions:** The probability distribution of a statistic



Suppose x is a normally distributed random variable w/ given μ and σ^2

→ If $x_1, x_2, \ldots x_n$ is a random sample of size n, the distribution of the sample mean is:
Normally distributed with a mean of μ and a variance of $\frac{\sigma^2}{n}$

$\mu_{\bar{x}} = \mu$

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$

$N\left(\mu, \frac{\sigma^2}{n}\right)$ shorthand

- **Central Limit Theorem:** Implies that the sample mean will be approximately normally distributed for large sample sizes, regardless of the distribution you are sampling from.
  - → 'Large' Sample size: some suggest $n > 20$ (non-normal), $n = 3-4$ (close to normal)

- **Example**

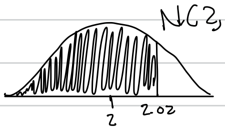
Electrodes are manufactured with a population mean thickness of 2.0 mm and a standard deviation of 0.048 mm. Find the probability that a random sample of n = 16 electrodes will have a sample mean diameter less than or equal to 2.02 mm? (keep 2 sig figs)

↳ Central Limit Theorem Applies

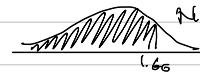
$\bar{x} \Rightarrow N(\mu, \frac{\sigma^2}{n})$ $\quad n = 16, \mu = 2.0\,m, \sigma = 0.048\,mm, \sigma^2 = 0.0023$

$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$ New! For samples

$P(\bar{x} \leq 2.02)$ $\quad N(2, \frac{0.0023}{16} = 0.00014)$



$\Rightarrow$



$N(0,1)$ $\quad z = 1.66$

$P(z \leq 1.66) = 0.95$

Hypothesis Testing

- **Parameters:** what probability distributions are described with
  - e.g. normal dist. is described with μ, σ
- **Statistical Hypothesis:** A statement about values of parameters of a probability distribution
- **Null Hypothesis:** States no statistical difference exists (H_0)
- **Alternative Hypothesis:** Contradicts the null hypothesis (H_1)

Example: Suppose we think the mean inside diameter of bearing is 1.500 in

$H_0: \mu = 1.500$ in $\qquad H_1: \mu \neq 1.500$ two sided

$\begin{cases} \mu < 1.500 \\ \mu > 1.500 \end{cases}$ one-sided

- **Two outcomes of a hypothesis test:**
  - (i) Reject H_0 or (ii) Fail to reject H_0
- **Test Statistic:** Used to test a hypothesis and is computed from a random sample of the population
- **Reference Distribution (Null distribution):** The sampling distribution of the test statistic assuming that H_0 (the null hypothesis) is true
- **Critical Region:** Sets values of the test statistic that lead to the rejection of the null hypothesis
  - ↳ Can either fail to reject or reject H_0, critical region defines these cutoffs
- **Two types of errors**
  - (i) **Type I errors:** If the null hypothesis is rejected when it is true
    - ↳ Defined: $\alpha = P\{\text{type I error}\} = P\{\text{reject } H_0 \mid H_0 \text{ is true}\}$ This is specified
  - (ii) **Type II errors:** If the null hypothesis is not rejected when it is false
    - ↳ Defined: $\beta = P\{\text{type II errors}\} = P\{\text{fail to reject } H_0 \mid H_0 \text{ is false}\}$ Hard to control
  - ○ Type II errors are uncontrolled & unknown, so we try to minimize it / avoid it
- **Conclusions:** Must be worded very clearly
  - ○ **Rejection of Null:** A strong conclusion because we can control α to be small
  - ○ **Failing to Reject:** Weak conclusion bc it is hard to control β

The Claim is the	The Decision is	The Conclusion is
Null Hypothesis	Fail to Reject the Null	The evidence is *not sufficient* to *reject* the claim.
	Reject the Null	The evidence is *sufficient* to *reject* the claim.
Alternative Hypothesis	Fail to Reject the Null	There is *insufficient* evidence to *support* the claim.
	Reject the Null	There is *sufficient* evidence to *support* the claim.

We never say accept the null hypothesis

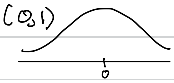
One sample Z-test ← Inference on the Mean of a Population, Variance Known

Suppose x is a random variable with unknown mean μ and a known variance σ^2 and we wish to test the hypothesis that the population mean is equal to a standard value (μ_o).

These are defined

- → Null Hypothesis: $\mu = \mu_0$
- → Alternate Hypothesis: $\mu \neq \mu_0$
- → Test Statistic: $Z_0 = \dfrac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$

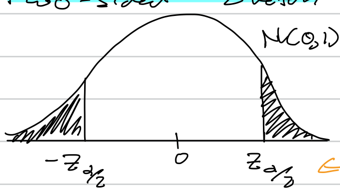
→ Reference Distribution: N(0,1)



→ Rejection Criterion (critical region): Reject H_0 if $|Z_0| > Z_{\alpha/2}$

Assume 0.05 unless otherwise stated

∘ Two-sided Sketch



← Critical Value

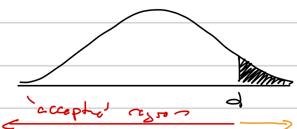
← 'acceptance' region → critical region

Critical region: $Z_0 > Z_{\alpha/2}$ or $Z_0 < Z_{\alpha/2}$

∘ One-sided Test

$H_0: \mu = \mu_0$

$H_1: \mu > \mu_0$



← 'acceptance' region → critical region

General Procedure (for this class!)

(i) Parameter of Interest: From the problem context, identify known/unknown parameter of interest, the test you're going to do and identify type of problem.

(ii) Null Hypothesis: State the null hypothesis H_0

(iii) Alternate Hypothesis: State the alternative hypothesis H_a

(iv) Test Statistic: Determine an appropriate test statistic & reference distributions

(v) Reject H_0 if: State rejection criterion for the null hypothesis. Should be drawn on plot w/ test statistic

(vi) Computations: Compute remaining sample quantities, substitute these into the equation for the test statistic, and compute that value. Compare to rejection criteria

(vii) Draw Conclusions: Decide whether or not H_0 should be rejected and report that in the problem context. Phrase appropriately!

Example

Example: You are testing the response time of a distributed computer system. The system manager wants to know whether the mean response time to a specific command exceeds 75 ms. Previous experience has determined the standard deviation is 8 ms. Use α = 0.05. A test is conducted executing the command 25 times, and the mean response time is 79.25 ms.

1) Parameter of Interest: μ, inferring mean of population, known variance, mean response time

 ↳ Test: One-sample Z-test

2) $H_0: \mu = 75$ ms

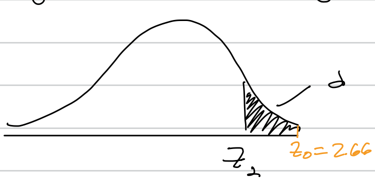
3) $H_1: \mu > 75$ ms (one-sided)

4) Test Statistic: $Z_0 = \dfrac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \dfrac{79.25 - 75}{\frac{8}{\sqrt{25}}} = 2.66$

Reference Dist: N(0,1)

5) Rejection criteria: Reject if $Z_0 > Z_\alpha = Z_{0.05}$

6)



← 'acceptance' region → critical region

$P(Z < Z_\alpha) = 0.95$

$Z_\alpha = 1.65$ critical value

7) Conclusion: $Z_0 > Z_\alpha$

Reject the null Hypothesis. Conclude that the mean response time exceeds 75 ms.

How do I do this in Minitab? Minitab> Stat>Basic Stats>OneSampleZ – click summarized data, but you could highlight a column of data, options will allow us to set the alpha value and the alternative hypothesis

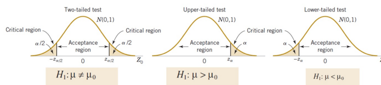
Summary of Z-tests

Testing Hypotheses on the Mean, Variance Known (Z-Tests)

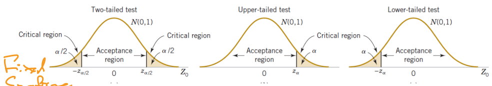
Null hypothesis: $H_0: \mu = \mu_0$

Test statistic: $Z_0 = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$

Alternative Hypotheses	*P*-Value	Rejection Criterion for Fixed-Level Tests
$H_1: \mu \neq \mu_0$	Probability above $\lvert z_0 \rvert$ and probability below $-\lvert z_0 \rvert$, $P = 2\left[1 - \Phi(\lvert z_0 \rvert)\right]$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1: \mu > \mu_0$	Probability above z_0, $P = 1 - \Phi(z_0)$	$z_0 > z_\alpha$
$H_1: \mu < \mu_0$	Probability below z_0, $P = \Phi(z_0)$	$z_0 < -z_\alpha$

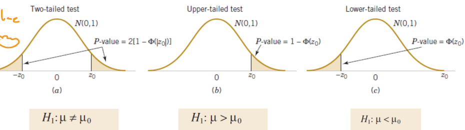


Fixed Significance Level testing



Critical Values Determined by H1 and α

P-value testing



$\Phi(z_0) = P(z < z_0)$

P-values determined by H1 and Test Statistic

Hypothesis Testing Insights

- Making statements about population probability distribution parameters (e.g. mean or variance)
- Testing the statements using a test statistic – calculated from a randomly selected sample
- Determining criteria for rejection of a null hypothesis - defined by Type I error level and H_1
- Remembering when making conclusions that the test is designed such that rejection of null is strong, not rejecting null – not so strong
- General: so many different hypothesis tests!!

P-values

P-value: The probability of obtaining a sample outcome at least as extreme as the one obtained given the value stated if the null hypothesis is true

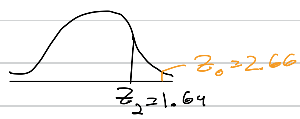
=> Reporting the compatibility of the data with the null hypothesis

Rejection Criterion: Reject H_0 if the p-value $< \alpha$

Back to our example, but now with p-values!

The response time of a distributed computer system is an CTQ. The system manager wants to know whether the mean response time to a specific command exceeds 75 ms. Previous experience has determined the standard deviation is 8.0 ms. A test is conducted executing the command 25 times, and the mean response time is 79.25 ms. Compute the P-value appropriate for this situation.

Rejection Criterion: Reject H_0 if p-value $< \alpha$



$$\text{P-value} = 1 - \Phi(z_0)$$
$$= 1 - P\{z < z_0\}$$

Z-test Assumes

- Cts data
- Normally dist. Data
- Random Samples
- Population Std dev is known

Confidence Intervals

The interval between two statistics that includes the true value of that parameter with some probability

→ $100(1-\alpha)\%$ confidence interval

$\mu: P(L \le \mu \le U) = 1 - \alpha$ where the statistics $\begin{cases} L = \text{Lower confidence limit} \\ U = \text{Upper confidence limit} \end{cases}$

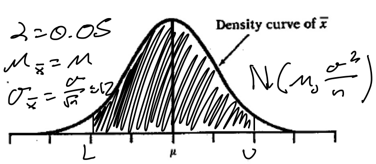
Example

$\mu = 90, \sigma = 36, n = 9$

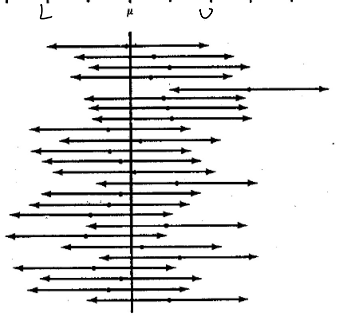
$\alpha = 0.05$

$\mu_{\bar{x}} = \mu$

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = 12$



$N(\mu, \frac{\sigma^2}{n})$



Find L and U s.t.

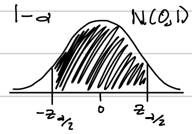
$P(L \le 90 \le \mu) = 0.95$

$\rightarrow L = 66.5, U = 113.5$

Assuming L, U are equidistant from the sample mean

$\rightarrow L, U$ are 23.5 units from μ

Generally



$$P(L \le \mu \le U) = 1 - \alpha$$

$$P\left(-z_{\frac{\alpha}{2}} \le \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \le z_{\frac{\alpha}{2}}\right)$$

$$\rightarrow P\left(\bar{X} - z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

L ↑ U ↑

CI includes infinity

One sided: $\mu \le \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}$ or $\bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}} \le \mu$

Recall

- $z_{\frac{\alpha}{2}}$ is value of the N(0,1) s.t. $P(z \ge z_{\frac{\alpha}{2}}) = \frac{\alpha}{2}$
- z_α is value of the N(0,1) s.t. $P(z \ge z_\alpha) = \alpha$

Minitab: Stat > Basic Stats > One sample Z, click summarized data and no hypothesis test, click options to select the % of the CI and if it is two sided or one sided

Actual Example/Application

We pull a sample of 20 electrodes from an assembly line of a process with known variance in electrode thickness to be 0.64 mm². The average thickness of the sample is 7.9 mm. Construct a 95% confidence interval.

$$P\left\{\bar{x} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right\} = 1-\alpha$$

$n = 20$

$\sigma^2 = 0.64\ mm^2 \rightarrow \sigma = 0.8$

$\bar{x} = 7.9\ mm,\ \alpha = 0.05$

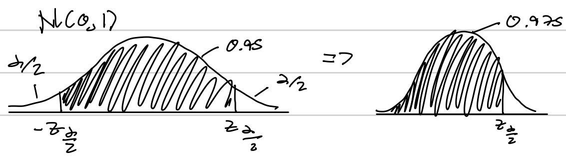
$L = 7.9 - 1.96\frac{0.8}{\sqrt{20}} = 7.5$

$U = 7.9 + 1.96\frac{0.8}{\sqrt{20}} = 8.3$

~~Often our result would be said like this:~~

~~"We are 95% confident that the mean of the population is between 7.5 and 8.3~~

Not this!



$z_{\frac{\alpha}{2}} = 1.96$ from tables

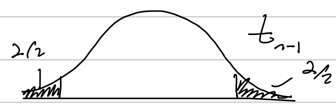
Perhaps a better description would be:

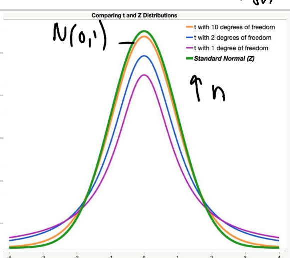
"In repeated sampling, this method will produce intervals that capture the true population mean about 95% of the time."

Use this! (might be on an exam!!)

One sample t-test

- Inference on the mean distribution, unknown variance.
- Suppose x is a normal random variable with unknown mean μ and unknown variance σ^2
- Null Hypothesis: $\mu = \mu_0$
- Alternative Hypothesis: $\mu \neq \mu_0$
- Test Statistic: $T_0 = \dfrac{\bar{x} - \mu_0}{S/\sqrt{n}}$
- Reference distribution
  - ↳ t-distribution w/ n-1 degrees of freedom
- Two-sided: Rejection Criterion: $|t_0| > t_{\alpha/2, n-1}$





For the t-distribution:

- Is a a continuous probability distribution of the z-score when the estimated standard deviation is used in the denominator rather than the true standard deviation.
- It is a type of normal distribution, except it accounts for the fact that as sample size increases, variability decreases.
- The t-distribution, like the normal distribution, is bell-shaped and symmetric about zero, but it has heavier tails, which means it tends to produce values that fall farther from its mean.
- As the sample size increases, the *t*-distribution becomes more similar to a normal distribution. *The shape depends on n!!* And this is why you MUST write down n each time you draw out a t-distribution.

Summary

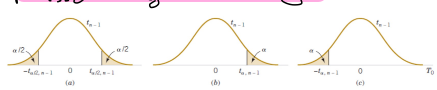
Testing Hypotheses on the Mean of a Normal Distribution, Variance Unknown

Null hypothesis: $H_0: \mu = \mu_0$

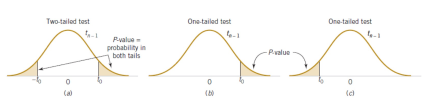
Test statistic: $T_0 = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}$ — t dist with n-1 DOF

Alternative Hypotheses	P-Value	Rejection Criterion for Fixed-Level Tests
$H_1: \mu \neq \mu_0$	Probability above $\lvert t_0 \rvert$ and probability below $-\lvert t_0 \rvert$	$t_0 > t_{\alpha/2, n-1}$ or $t_0 < -t_{\alpha/2, n-1}$
$H_1: \mu > \mu_0$	Probability above t_0	$t_0 > t_{\alpha, n-1}$
$H_1: \mu < \mu_0$	Probability below t_0	$t_0 < -t_{\alpha, n-1}$

Fixed Sig Value Testing



P-Value



t-test Assumptions
- Data are cts.
- Data are normally distributed
- Data are random samples

Example

Inference on the mean of a distribution, unknown variance

Rubber can be added to asphalt to reduce road noise. The viscosity (cP) of 15 specimens of asphalt is measured. To be suitable for use, the mean should be equal to 3200 cP. Test the hypothesis using alpha=0.05.

cP
3193
3124
3153
3145
3093
3466
3355
3079
3182
3227
3256
3332
3204
3282
3170

Use these data

$n = 15$

1) Inference on mean, unknown variance, one-sample
   ↳ One-sample t-test

2) $H_0: \mu = 3200\ cP = \mu_0$ 3) $H_1: \mu \neq 3200\ cP$

4) $T_0 = \dfrac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$, $\bar{x} = \dfrac{3193 + 3124 + \ldots + 3170}{15} = 3217$

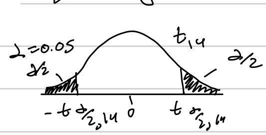
Plug & Chug $= 0.64$ $\quad s = \sqrt{\dfrac{(3193-3217)^2 + \ldots + (3170-3217)^2}{n-1}} = 106$

5) Reject if $\begin{cases} t_0 > t_{0.025, 14} \\ t_0 < -t_{0.025, 14} \end{cases}$

7) Conclusion

(Opt 1) The evidence is not sufficient to reject the claim that the mean is equal to 3200 cP

(Opt 2) There is insufficient evidence to support the mean not being equal to 3200 cP.



$t_{\alpha/2} = 2.145$

CI for One-Sample t-test → For our previous example

Suppose x is a normal random variable with unknown mean μ and unknown variance σ^2. From a sample of n observations, the sample mean $\bar{x}$ and sample variance s^2 are computed. Then:

$$\bar{x} - t_{\alpha/2,\, n-1}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + t_{\alpha/2,\, n-1}\frac{s}{\sqrt{n}}$$ **Is a 100 (1-alpha)% two-sided CI**

where $t_{\alpha/2,\, n-1}$ is the value of the t distribution with n-1 degrees of freedom such that $P(t_{n-1} \ge t_{\alpha/2,\, n-1}) = \alpha/2$

$$-\infty \le \mu \le \bar{x} + t_{\alpha,n-1}\frac{s}{\sqrt{n}}$$ **Is a 100(1-alpha)% upper confidence bound**

where $t_{\alpha,\, n-1}$ is the percentage point of the t distribution with n-1 degrees of freedom such that $P(t_{n-1} \ge t_{\alpha,\, n-1}) = \alpha$

$$\bar{x} - t_{\alpha,n-1}\frac{s}{\sqrt{n}} \le \mu \le \infty$$ **Is a 100(1-alpha)% lower confidence bound**

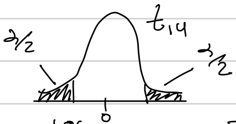
where $t_{\alpha,\, n-1}$ is the percentage point of the t distribution with n-1 degrees of freedom such that $P(t_{n-1} \ge t_{\alpha,\, n-1}) = \alpha$

$\bar{x} = 3217$

$s = 106$

$n = 15$

$t_{0.025,14} = 2.145$



$$\left.\begin{array}{l} L = 3217 - 2.145\,\dfrac{106}{\sqrt{15}} = 3158 \\ U = 3217 + 2.145\,\dfrac{106}{\sqrt{15}} = 3276 \end{array}\right\} 3158 \le \mu \le 3276$$

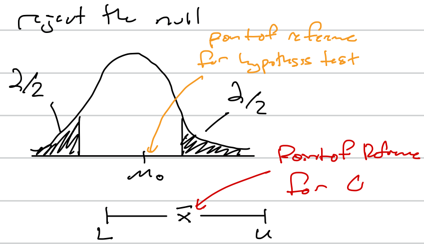
Minitab: Stat>Basic Stat>1-sample t-test, samples in columns (drop down), double click data, check hypothesis test select type and in options select alpha value

CI Relationship w/ Hypothesis Testing

- Both do make interfaces about a population parameter (but in different ways)
- A CI provides a range of values likely containing the true population parameter based on a set of data
- A hypothesis test determines whether a specific claim about the population parameter is supported by the data
- Difference in emphasis of the point of reference

Illustration

If the value in your Ho is in your CI, you likely cannot reject the null



Inference on the variance of Normal Dist. (1-variance test)

Suppose we wish to test the hypothesis that the variance of a normal population (σ^2) equals a specified value, say σ_0^2, or that the standard deviation of the population (σ) is equal to σ_0. Let $x_1, x_2, \ldots x_n$ be a random sample of n observations from this population to test:

Useful for making conclusions about the variability of your process

$H_0: \sigma^2 = \sigma_0^2$

$H_a: \sigma^2 \neq \sigma_0^2$

Test Statistic: $\chi_0^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$ s = sample variance with n observations

Reference Distribution: Chi-Squared Distribution

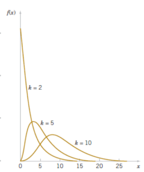
Which Hypothesis should I do?

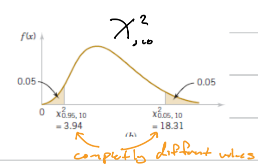
Test	# of Populations	Variance Known?	Parameter of interest?	Reference Dist.
One-sample Z-test	1	Yes	Mean	Standard normal
Two-sample Z-test	2	Yes	Difference between means	Standard normal
One-sample t-test	1	No	Mean	t-distribution
Two-sample (pooled) t-test (equal variance)	2	No	Means of two populations	t-distribution
Two-sample t-test (unequal variance)	2	No	Means of two populations	t-distribution
Paired t-test	2 (simplifies to 1)	No	Difference between paired means	t-distribution
1 Variance test	1	No	Variance of one population	chi-squared distribution

If both mean and variance of the population are known, its a probability problem

Chi-squared Distribution

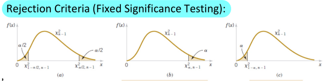
- χ_0^2 has a chi-squared distribution if the null is true
- Key features:
  - (i) Depends on n-1 degrees of freedom
  - (ii) Not symmetric
  - (iii) Not centered @ 0
  - (iv) Total Area under curve is still 1!





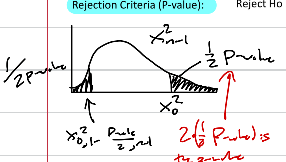
Rejections

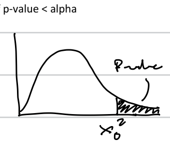
Rejection Criteria (Fixed Significance Testing):



$H_1: \sigma^2 \neq \sigma_0^2$	$H_1: \sigma^2 > \sigma_0^2$	$H_1: \sigma^2 < \sigma_0^2$
Reject Ho if:		
$\chi_0^2 > \chi^2_{\alpha/2,n-1}$ or $\chi_0^2 < \chi^2_{1-\alpha/2,n-1}$	$\chi_0^2 > \chi^2_{\alpha,n-1}$	$\chi_0^2 < \chi^2_{1-\alpha,n-1}$

Rejection Criteria (P-value): Reject Ho if p-value < alpha





Summary

Null hypothesis: $H_0: \sigma^2 = \sigma_0^2$

Test statistic: $\chi_0^2 = \dfrac{(n-1)S^2}{\sigma_0^2}$

Ref. Dist: Chi-square

Alternative Hypothesis	Rejection Criteria
$H_1: \sigma^2 \neq \sigma_0^2$	$\chi_0^2 > \chi^2_{\alpha/2,n-1}$ or $\chi_0^2 < \chi^2_{1-\alpha/2,n-1}$
$H_1: \sigma^2 > \sigma_0^2$	$\chi_0^2 > \chi^2_{\alpha,n-1}$
$H_1: \sigma^2 < \sigma_0^2$	$\chi_0^2 < \chi^2_{1-\alpha,n-1}$

Check Normality! Its Sensitive to this

Example

Inference on the variance of a distribution

Rubber can be added to asphalt to reduce road noise. The viscosity (cP) of 15 specimens of asphalt is measured. s is calculated to be 106. Let's say we are interested in knowing if the variance less than 30,000.

Use $\alpha = 0.05$

Data
3193
3124
3153
3145
3093
3466
3355
3079
3182
3227
3256
3332
3204
3282
3170

1) Parameter of interest: σ^2 → 1-variance test

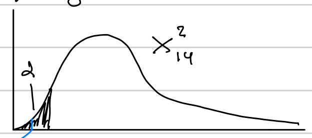
2) $H_0: \sigma_0^2 = 30{,}000\ cP^2$

3) $H_a: \sigma_0^2 < 30{,}000\ cP^2$

4) $\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(15-1)(106)^2}{30{,}000} = 5.2$

Reference Distribution: χ^2_{14}

5) Rejection Criteria: Reject if $\chi_0^2 < \chi^2_{1-\alpha, n-1}$



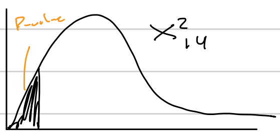
6) Computations: $5.2 < 6.57$

7) Conclusion: Reject H_0. We have evidence to reject the claim that the variance is 30,000 cP^2, having evidence that the viscosity test variance is less than 30,000 cP^2.

We accept H_1. ← Can accept H_1, but not accept H_0

p-value

Reject if: p-value < α



5.2 lands btwn 0.990 and 0.975

→ p-value is btwn 1-tail

(0.01 < p < 0.025) → both < α so reject H_0!

Fixed-Significance Level

How do I do this in Minitab? Stat > basic stat > 1 variance

Confidence Intervals for Inference on the variance (χ^2)

Suppose x is a normal random variable with unknown mean μ and unknown variance σ². From a sample of n observations, the sample mean $\bar{x}$ and sample variance s² are computed. Then:

$(100-\alpha)\%$ two-sided CI

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}$$

Where $X^2_{\alpha/2,\, n-1}$ denotes the percentage point of the chi-square distribution such that $P(X^2_{n-1} \ge X^2_{\alpha/2,\, n-1}) = \alpha/2$ and $X^2_{1-\alpha/2,\, n-1}$ denotes the percentage point of the chi-square distribution such that $P(X^2_{n-1} \ge X^2_{1-\alpha/2,\, n-1}) = 1-\alpha/2$

$(100-\alpha)\%$ upper bound

$$\sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha,n-1}}$$

Where $X^2_{1-\alpha,\, n-1}$ denotes the percentage point of the chi-square distribution such that $P(X^2_{n-1} \ge X^2_{1-\alpha,\, n-1}) = 1-\alpha$

$(100-\alpha)\%$ lower bound

$$\frac{(n-1)s^2}{\chi^2_{\alpha,n-1}} \le \sigma^2$$

Where $X^2_{\alpha,\, n-1}$ denotes the percentage point of the chi-square distribution such that $P(X^2_{n-1} \ge X^2_{\alpha,\, n-1}) = \alpha$

For our example (upper bound)

$$\sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha,n-1}} \rightarrow \sigma^2 \le \frac{(15-1)(106)^2}{\chi^2_{1-\alpha,n-1}}$$

$$\chi^2_{1-\alpha,n-1} = \chi^2_{0.95,14} = 6.57$$

$$\Rightarrow \sigma^2 \le \frac{(14)(106)^2}{6.57} = 2394.42\ cP^2$$

6-sigma Example/Application

Inference on the variance of a distribution

Rubber can be added to asphalt to reduce road noise. The viscosity (cP) of 15 specimens of asphalt is measured. The asphalt specification limit is 1100 cP away from the mean in either direction – what is the sigma level of this process given that we know $\sigma \le$ 173 (from our hypothesis test) and we assume the mean of the population is on target?

Data
3193
3124
3153
3145
3093
3466
3355
3079
3182
3227
3256
3332
3204
3282
3170

$$\text{Sigma level} = \frac{|\text{spec limit} - \text{target}|}{\sigma} = \frac{1100}{173}$$

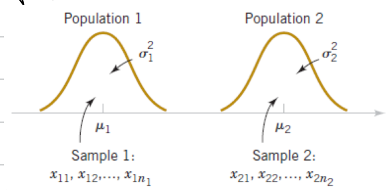
$= 6.35$ → at least 6.35

1-variance Test Assumptions

1. The data are continuous.
2. The data are randomly sampled.
3. The data are normally distributed.

Inference on means of two normally distributed populations

Let's say population 1 has a mean of μ_1 and a variance of σ_1^2 and population 2 has a mean of μ_2 and a variance of σ_2^2 and inferences are based on two random samples of size n_1 and n_2 respectively.



Want to know: is there a difference between their parameters

Two-Sample Z-test

Statistical inference on the difference in means $\mu_1 - \mu_2$ of the populations below with known variances of σ_1^2 and σ_2^2

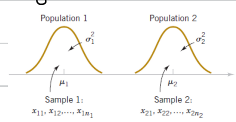
Null: $H_0: \mu_1 - \mu_2 = \cancel{\Delta_0}\ 0$

Alt: $H_1: \mu_1 - \mu_2 \neq 0$ (two-sided)

Test statistic: $Z_0 = \dfrac{\bar{x}_1 - \bar{x}_2 - \cancel{\Delta_0}\ 0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$

Reference Dist: $N(0,1)$

Rejection Criteria: Same as one-sample z-test



Summary

Null hypothesis: $H_0: \mu_1 - \mu_2 = \Delta_0$

Test statistic: $$Z_0 = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad (10\text{-}2)$$

Ref: $N(0,1)$

Alternative Hypotheses	P-Value	Rejection Criterion for Fixed-Level Tests
$H_1: \mu_1 - \mu_2 \neq \Delta_0$	Probability above $\lvert z_0 \rvert$ and probability below $-\lvert z_0 \rvert$, $P = 2[1 - \Phi(\lvert z_0 \rvert)]$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1: \mu_1 - \mu_2 > \Delta_0$	Probability above z_0, $P = 1 - \Phi(z_0)$	$z_0 > z_\alpha$
$H_1: \mu_1 - \mu_2 < \Delta_0$	Probability below z_0, $P = \Phi(z_0)$	$z_0 < -z_\alpha$

Assumptions

(1) $X_{11}, X_{12}, \ldots, X_{1n_1}$ is a random sample from population 1.
(2) $X_{21}, X_{22}, \ldots, X_{2n_2}$ is a random sample from population 2.
(3) The two populations represented by X_1 and X_2 are independent.
(4) Both populations are normal.

(5) The variances of the populations are known

→ Samples from two populations are independent if the samples selected from one population have no relationship with the samples selected from the other population.

Example

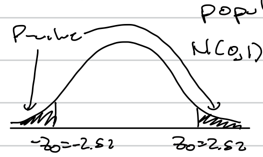
Paint Drying Time A product developer is interested in reducing the drying time of a primer paint. Two formulations of the paint are tested; formulation 1 is the standard chemistry, and formulation 2 has a new drying ingredient that should reduce the drying time. From experience, it is known that the standard deviation of drying time is 8 minutes, and this inherent variability should be unaffected by the addition of the new ingredient. Ten specimens are painted with formulation 1, and another 10 specimens are painted with formulation 2; the 20 specimens are painted in random order. The two sample average drying times are $\bar{x}_1 = 121$ minutes and $\bar{x}_2 = 112$ minutes, respectively. What conclusions can the product developer draw about the effectiveness of the new ingredient, using $\alpha = 0.05$?

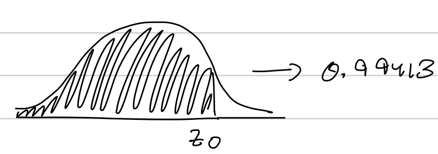
1) Parameter of interest is the difference in the mean drying time ($\mu_1 - \mu_2 = 0$), and we know the standard deviation is 8 min for both populations. → two-sample Z-test

2) $H_0: \mu_1 - \mu_2 = 0$ $(\mu_1 = \mu_2)$

3) $H_a: \mu_1 - \mu_2 \neq 0$ $(\mu_1 \neq \mu_2)$

4) $Z_0 = \dfrac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \dfrac{121 - 112}{\sqrt{\frac{8^2}{10} + \frac{8^2}{12}}} = 2.52$





5) P-value: Reject H_0 if p-value < $\alpha = 0.05$

p-value = $2(1 - 0.99413) = 0.012$

6) P-value = $0.012 < \alpha = 0.05$

7) Reject H_0 and accept H_1 and conclude there is a difference btwn the drying time of the two

Confidence Intervals (do they cross zero?)

Confidence Intervals (do they cross zero!?)

100 (1-α)% two-sided confidence interval on $\mu_1 - \mu_2$

$$\bar{x}_1 - \bar{x}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

where $Z_{\alpha/2}$ is the value of the N(0,1) such that $P(z \ge Z_{\alpha/2}) = \alpha/2$

$$\mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + z_\alpha\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

100 (1-α)% one-sided upper confidence bound on $\mu_1 - \mu_2$ (interval includes - infinity)

$$\bar{x}_1 - \bar{x}_2 - z_\alpha\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2$$

100 (1-α)% one-sided lower confidence bound on $\mu_1 - \mu_2$ (interval includes + infinity)

where Z_α is the value of the N(0,1) such that $P(z \ge Z_\alpha) = \alpha$

Two-Sample t-test (variances unknown)

Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$ ← unknown values, but assumed true

Suppose we have two independent populations with unknown means μ_1 and μ_2 and equal variances $\sigma_1^2 = \sigma_2^2 = \sigma^2$ and we wish to know if the means are different:

Null: $H_0: \mu_1 - \mu_2 = \Delta_0$ $(\mu_1 = \mu_2)$

Alt: $H_a: \mu_1 - \mu_2 \neq \Delta_0$ $(\mu_1 \neq \mu_2)$

Test Statistic: $T_0 = \dfrac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$

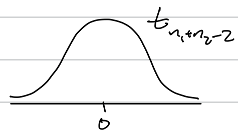
Pooling Sample Variance — Square root of this

$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}$$

Weighted average of two sample variances

Reference Distribution:

t-dist with n_1+n_2-2 degrees of freedom

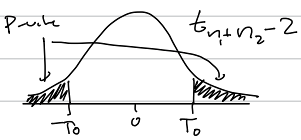


Rejection Criteria: See below exercise

Exercise: Sketch out Fixed rejection criteria & p-value for two-sided H_1

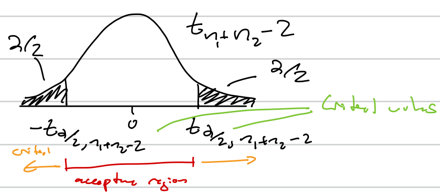
P-value:

Reject H_0 if p-value $< \alpha$



Fixed:

Reject if $T_0 > t_{\alpha/2, n_1+n_2-2}$ or $T_0 < t_{\alpha/2, n_1+n_2-2}$



Example

Yield from a Catalyst Two catalysts are being analyzed to determine how they affect the mean yield of a chemical process. Specifically, catalyst 1 is currently used; but catalyst 2 is acceptable. Because catalyst 2 is cheaper, it should be adopted, if it does not change the process yield. A test is run in the pilot plant

Observation Number	Catalyst 1	Catalyst 2
1	91.50	89.19
2	94.18	90.95
3	92.18	90.46
4	95.39	93.21
5	91.79	97.19
6	89.07	97.04
7	94.72	91.07
8	89.21	92.75
	$\bar{x}_1 = 92.255$	$\bar{x}_2 = 92.733$
	$s_1 = 2.39$	$s_2 = 2.98$

Checking in Minitab: stat -> basic stat -> 2 sample t -> input columns and other info (CLICK ASSUME EQUAL VARIANCE UNDER OPTIONS!)

1) Interested in difference between 2 means, specifically yield of catalyst 1 and catalyst 2, variance unknown → two-sample t-test
   ↳ Assume equal variance ← rule of thumb later

2) $H_0: \mu_1 - \mu_2 = 0$

3) $H_a: \mu_1 - \mu_2 \neq 0$

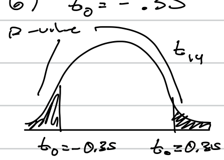
4) $t_0 = \dfrac{\bar{x}_1 - \bar{x}_2 - 0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \dfrac{92.255 - 92.733}{2.7\sqrt{\frac{1}{8} + \frac{1}{8}}} = -0.35$

$S_p^2 = \dfrac{(n_1-1)s_1^2 - (n_2-1)s_2^2}{n_1+n_2-2} = \dfrac{7 \cdot 2.39^2 + 7 \cdot 2.98^2}{8+8-2} = 7.3$

$\to S_p = \sqrt{S_p^2} = \sqrt{7.3} = 2.7$

5) Reject if p-value $< \alpha = 0.05$

6) $t_0 = -.35$



7) p-value $> \alpha$

Fail to reject H_0, we don't have enough evidence to say the yield from catalyst 1 is different from catalyst 2.

$0.5 <$ p-value $< .8$

Two sample t-test (variances unknown & unequal)

Case 1: $\sigma_1^2 \neq \sigma_2^2$

Suppose we have two independent populations with unknown means μ_1 and μ_2 and unequal variances $\sigma_1^2 \neq \sigma_2^2$ and we wish to test if the means are different:

H_0, H_a are same as 2 sample t-test w/ equal variance

$$T_0^* = \frac{\bar{X}_1 - \bar{X}_2 - \cancel{\Delta_0}^{\,0}}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

Reference Dist: t-dist with v^* degrees of freedom

where
$$v = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{\left(S_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(S_2^2/n_2\right)^2}{n_2 - 1}}$$

If v is not an integer, round down

Example v calculation: $s_1 = 7.63$, $S_2 = 15.3$, $n_1 = n_2 = 10$ } $v = 13.2 \to 13$

Summary: **two-sample t-test (unequal variance)**

Null hypothesis: $H_0: \mu_1 - \mu_2 = \Delta_0$.

Test statistic:
$$T_0^* = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \qquad (10\text{-}14)$$

Ref. : t-dist with v dof

Alternative Hypotheses	P-Value	Rejection Criterion for Fixed-Level Tests
$H_1: \mu_1 - \mu_2 \neq \Delta_0$	Probability above $\|t_0\|$ and probability below $-\|t_0\|$	$t_0 > t_{\alpha/2,\cancel{n_1+n_2-2}}$ or $t_0 < -t_{\alpha/2,\cancel{n_1+n_2-2}}$
$H_1: \mu_1 - \mu_2 > \Delta_0$	Probability above t_0	$t_0 > t_{\alpha,\cancel{n_1+n_2-2}}$
$H_1: \mu_1 - \mu_2 < \Delta_0$	Probability below t_0	$t_0 < -t_{\alpha,\cancel{n_1+n_2-2}}$

Note: replace n1+n2 with v in rejection criteria!

$$v = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(s_2^2/n_2\right)^2}{n_2 - 1}}$$

If v is not an integer, round down to the nearest integer.

Checking in Minitab: stat -> basic stat -> 2 sample t -> input columns and in options deselect assume equal variance

Assumptions

(1) $X_{11}, X_{12}, \ldots, X_{1n_1}$ is a random sample from population 1.
(2) $X_{21}, X_{22}, \ldots, X_{2n_2}$ is a random sample from population 2.
(3) The two populations represented by X_1 and X_2 are independent.
(4) Both populations are normal.

(5) The variances of the populations are unequal

If you're in doubt, assume unequal

How do I know which t-test to use?

1) Are measurements being made on the exact same object? (Paired)
2) **If the ratio of the larger variance to the smaller variance is less than 4** then we can assume the variances are approximately equal (use pooled t-test) OR you could run a two-variance test (will learn about in the future!)

- Remember variance is standard deviation squared!!
- This rule of thumb can vary widely

↳ Use this for exam 1

Confidence Intervals

Equal Variance Two sided

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha/2, n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \le \mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + t_{\alpha/2, n_1+n_2-2}\, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Unequal Variance Two-sided

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha/2, v} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + t_{\alpha/2, v} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Paired t-test

A special case of the two-sample t-tests when observations of on the two populations are observed in pairs where each pair of observations is taken under homogenous conditions

To compare differences, we define Dj:

$D_j = X_{1j} - X_{2j},\ j = 1, 2, \ldots, n.$ n = # of observations

where Dj's are assumed to be normally distributed with mean u_D and variance σ^2_D

n is the number of paired observations (j is the paired observation number)

1 and 2 represents the paired data we are comparing

$H_0: \mu_D = 0 = \cancel{\Delta_0}$

$H_a: \mu_D \neq \cancel{\Delta_0} = 0$

Test Statistic: $T_0 = \dfrac{\bar{D} - \Delta_0}{\frac{S_D}{\sqrt{n}}}$

where $\bar{D}$ = Sample average of n differences

S_D = Sample standard deviation of n differences

Reference Dist: t-dist with $n-1$ degrees of freedom

Rejection Criterion: Same as one sample t-test

Confidence Interval

$$\bar{d} - t_{\alpha/2,n-1}\, s_D/\sqrt{n} \le \mu_D \le \bar{d} + t_{\alpha/2,n-1}\, s_D/\sqrt{n}$$

Summary

Null hypothesis: $H_0: \mu_D = \Delta_0$ Ref: t-dist w/ $n-1$ dof

Test statistic: $T_0 = \dfrac{\bar{D} - \Delta_0}{S_D/\sqrt{n}}$ (10-24)

Alternative Hypotheses	P-Value	Rejection Criterion for Fixed-Level Tests
$H_1: \mu_D \neq \Delta_0$	Probability above $\lvert t_0\rvert$ and probability below $-\lvert t_0\rvert$	$t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$
$H_1: \mu_D > \Delta_0$	Probability above t_0	$t_0 > t_{\alpha,n-1}$
$H_1: \mu_D < \Delta_0$	Probability below t_0	$t_0 < -t_{\alpha,n-1}$

Assumptions:

(1) Differences are continuous

(2) Differences are normally distributed

(3) Differences are obtained from a random sample

(4) Differences are obtained from homogonous pairs

Minitab: stat -> basic stat-> paired t

or input differences data into one-sample-t-test

Example

■ TABLE 4.3 ($n=8$)

Paired Tensile Strength Data for Example 4.11

Specimen	Machine 1	Machine 2	Difference
1	74	78	−4
2	76	79	−3
3	74	75	−1
4	69	66	3
5	58	63	−5
6	71	70	1
7	66	66	0
8	65	67	−2

Get $\bar{D}$ and S_D from these differences

Mean difference: -1.38 $\bar{D}$

Standard deviation of differences: 2.67 S_D

Alpha = 0.05

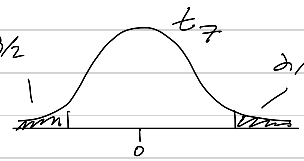
Do these machines yield the same tensile strength values on the same sample?

(i) Interested in the difference in mean tensile strength measured by two devices, same samp → paired data ⇒ paired t-test

(ii) $H_0: \mu_D = 0$ (iii) $\mu_D \neq 0$

(iv) $t_0 = \dfrac{\bar{d}}{\frac{S_d}{\sqrt{n}}} = \dfrac{-1.38}{\frac{2.67}{\sqrt{8}}} = -1.46$

(v) Reject if $t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$



$\rightarrow t_{\alpha/2,n-1} = 2.365$

Conclusion: Fail to reject null (H_0), we do not have sufficient evidence to reject the claim that the machines measure differently

End of Exam 1 Material

Exam 1 Summary

Which Hypothesis do I use?

Test	# of Populations	Variance Known?	Parameter of interest?	Reference Dist.
One-sample Z-test	1	Yes	Mean	Standard normal
Two-sample Z-test	2	Yes	Difference between means	Standard normal
One-sample t-test	1	No	Mean	t-distribution
Two-sample (pooled) t-test (equal variance)	2	No	Means of two populations	t-distribution
Two-sample t-test (unequal variance)	2	No	Means of two populations	t-distribution
Paired t-test	2 (simplifies to 1)	No	Difference between paired means	t-distribution
1 Variance test	1	No	Variance of one population	chi-squared distribution

How to Represent your data

1) Data are often represented as the mean ± the standard deviation and you should tell people how many repeats are in your sample (n) (write this in your caption as "Data are represented as the mean ± the standard deviation with n=X".)
2) Statistical significance (when the null is rejected) is represented by a symbol (usually *), accompanied by the alpha level, and statistical test with appropriate detail. E.g., "* represents p-value < 0.05 compared to X, using a X-sided, X test"
    1) Describe the test! E.g. two-sided or one sided? One sample or two sample? T=test or z=test? If t-test, is it assuming equal variance, if it is a z-test what is the variance?

Test 1

- Open book open note and class materials – you can have a calculator, but no access to internet iPad notes are ok
- BRING YOUR STATISTICAL TABLES!!! (They will not be provided for the exam, but you will need them)
- You will be asked to sketch out the distributions and label things
- How can I do well?
    - Make sure you know how to do the homework – review the correct answers if something went wrong
    - Do the practice exam (to be posted later)
- What is going to be on the exam?
    - Anything in lectures 1-10 and HW1-5 is fair game, and the types of problems you see in the practice exam
    - Note: the test can't cover every single type of problem and you won't see the exact same problems in the practice exam on Test 1!

You can check hypothesis tests with your calculator, just know what you're doing, and show all your work!

Topics

(Not necessarily comprehensive but good things to study)

- vocabulary and concepts regarding statistical process control, six sigma, lean manufacturing, and the DMAIC process
- tools to represent data, and check assumptions (e.g., normality)
- probability problems, properties of probability distributions
- correctly performing a hypothesis testing procedure, picking the correct test and alternative hypothesis (see list of tests)
- how to correctly phrase conclusions in a hypothesis test
- how to correctly phrase what the p-value is
- how to correctly phrase what confidence intervals are representing, and calculate them
- how to sketch hypothesis tests on a reference (sampling) distribution
- concepts surrounding what type I and type II error are, and why they allow us to make conclusions in a hypothesis test
- concepts around the central limit theorem, and sampling distributions vs. population distributions

Presenting Conclusions

ECHE 313 Practice Exam 1 Solutions on Canvas

Write your name:

Write the following statement, and sign your name acknowledging your agreement:

don't pay attention to fuss, this year was take-home

"“I neither received nor gave external assistance on this exam "

The exam layout is such that all problems add up to 100 points. This point total is meant to guide you in time allocation for this test, will be adjusted such that this exam is 25% of your total class grade. You may use your class materials, class notes, homework, book, and Minitab or Excel to aid in the completion of the exam. The exam is designed to be completed in 1hr and 15 min, but you will have between 10 AM 2/27 to 10 AM 2/28 to complete the exam. All of the necessary tables are provided in the back of your book. Please show your work and all steps necessary to arrive at your answer. Note that it is expected you do this entire test by hand, using a calculator and the tables found in the back of your book. So while you are allowed to use Minitab if you wish to check your answers, it should not replace the work you have done by hand using the tables for your answers. Where applicable, it is best to write out the general formals you want to use first, then write in your subbed in values so it is clear. In addition, for this exam, we can use the rule of thumb that if the sample variance ratio (larger to smaller) is lower than 4, you can assume equal variance for a t-test (if needed). Also, when reporting numbers, keep the same number of decimals as the data you are working with unless there are specific directions on how many sig figs to report.

Good luck!

Premise: You are a process engineer working for a lean six-sigma pharmaceutical company that makes advanced protein-based biologics. Unless otherwise stated, your company uses α=0.01.

1) (12 points) You are finishing up a project where you focused on increasing process cycle efficiency and are ready to hand over the improved process.
   a. (2 points) What step of the DMAIC process are you in?

      Control Stage

   b. (2 points) Name at least one tool that is used in this stage.

      Transition Plan, Control Chart

   c. (4 points) Explain what process cycle efficiency is and how it contributes to lean manufacturing.

$$PCE = \frac{\text{Value added time}}{\text{Process Time}}$$

See Key

d. (4 points) Explain why a slow process is an expensive process with at least 3 specific reasons.

See key

2) (39 points) Your next project is to determine if the bioreactor time can be reduced by getting a newer bioreactor system. You are able to demo a new unit and track the reactor time on 5 randomly selected batches in both the old and new bioreactors. You obtain the following data:

The following table shows the time it took in minutes for the batch to reach the designated a amount of bacteria:

$\bar{x}_1 = 262$, $s_1 = \sqrt{132}$

$\bar{x}_2 = 149$, $s_2 = \sqrt{94}$

Old Bioreactor Time (min)	New Bioreactor Time (min)
245	165
270	141
262	151
275	142
260	147

a. (3 points) List the *name* of the hypothesis test you would use

Two sample t-test (pooled) assuming equal variance (Use rule of thumb)

b. (3 points) Name the parameter of interest in the context of the problem statement

Mean Bioreactor time of the old and new (or you can say their difference)

c. (2 points) Write out the correct null hypothesis for this test

$H_0: \mu_1 - \mu_2 = 0$

d. (2 points) Write out the correct alternative hypothesis for this test

$H_a: \mu_1 - \mu_2 \neq 0$ two-sided Can be interpreted as one-sided

e. (7 points) Write the formula of the test statistic and calculate (report 3 sig figs)

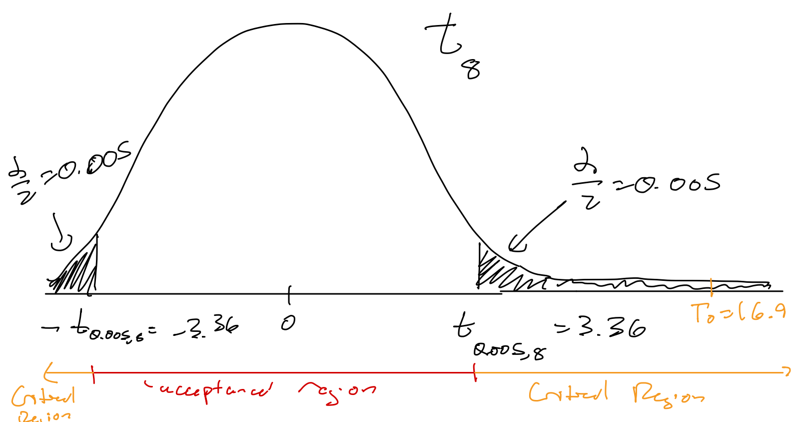
$$T_0 = \frac{\bar{x}_1 - \bar{x}_2 - 0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{262 - 149}{10.62\sqrt{\frac{1}{5} + \frac{1}{5}}} = 16.9$$

$S_p = 10.62$

f. (4 points) Write out the correct rejection criteria for fixed significance level testing and the critical value (3 sig figs)

Reject if: $\begin{cases} t_0 > t_{\alpha/2, n_1+n_2-2} \\ t_0 < t_{\alpha/2, n_1+n_2-2} \end{cases}$ $t_{0.005,8} = \pm 3.36$

g. (9 points) Sketch out your reference distribution making sure to <u>fully define the distribution you are sketching on with the correct label</u>, drawing it with the <u>correct shape</u> and <u>where zero is</u>. In addition, label 1) acceptance region, 2) the rejection region (also known as the critical region), 3) α, 4) the critical value and 5) the test statistic.



h. (3 points) State your conclusion and remember to word your conclusion appropriately in the context of the problem statement.

$T_0 < t_{0.005,8}$

Reject Ho, Accept Ha, Context → Morcom's mean is lower!

See key

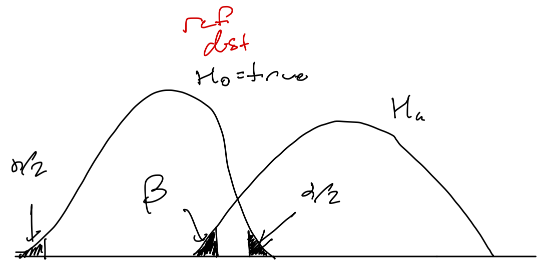
i. (3 points) Explain what plot you would construct to check if the new bioreactor data are normally distributed (what feature(s) of the plot would you check?)

Normal probability plot, see if its linear (looks distinct non-linear shape)

j. (3 points) Define in words what (α) is (what does it represent, what is it's definition?) and explain why it helps us make strong conclusions.

α is Type I error =

See Key



Can precisely define
have alpha is

3) (32 points) To determine if the new bioreactor is suitable for the process, the team also needs to know if the new bio reactor produces a yield that is greater than 810 mg/L of product. Twelve random samples are taken from batches in the piolet run:

Yield (mg/L)
807
825
816
805
815
820
822
817
823
817
810
812

$n=12$

Sample Ave. 815.75

Sample Standard Deviation: 6.31

a. (2 points) List the *name* of the hypothesis test you would use

One sample t-test

b. (3 points) Name the parameter of interest in the context of the problem statement

Mean process yield

c. (2 points) Write out the correct null hypothesis for this test

$H_0: \mu = 810$ mg/L

d. (2 points) Write out the correct alternative hypothesis for this test

$H_a: \mu > \mu_0 = 810$ mg/L

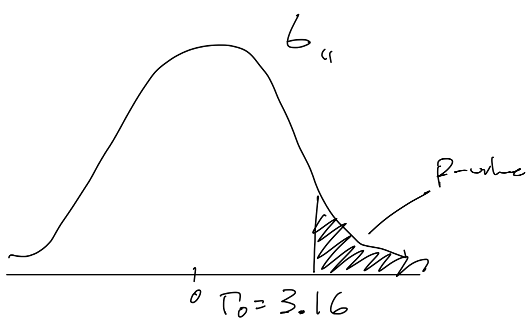
e. (3 points) Write the formula of the test statistic and calculate (report 3 sig figs)

$$T_0 = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = \frac{815.78 - 810}{\frac{6.31}{\sqrt{12}}} = 3.16$$

f. (2 points) Write out the correct rejection criteria using the p-value method

Reject if p-value < $\alpha = 0.01$

g. (6 points) Sketch out your reference distribution making sure to fully define the distribution you are sketching on with the correct label, drawing it with the correct shape and where zero is. In addition, label 1) the test statistic and 2) the p-value.



See Key

h. (2 points) Find the p-value using the tables in the back of your book

i. (2 points) Explain what the p-value is in words (i.e. what is the definition of a p-value?)

j. (3 points) State your conclusion and remember to word your conclusion appropriately in the context of the problem statement.

Reject

k. (2 points) You calculate your 99% confidence interval and show it to your boss. Explain what the 99% confidence interval really represents generally, in words (Hint: do not just say it means you are 99% confident that the mean lies in the CI range). You do not need to actually calculate the CI to answer this question correctly.

l. (3 points) If the manufacturer told you the standard deviation on the yield coming from these new units is known to be 4.00 mg/L, and you wanted to know if the yield was *different* than 810 mg/L, name the hypothesis test you would conduct, and if your alternative hypothesis would be 1 or 2 sided.

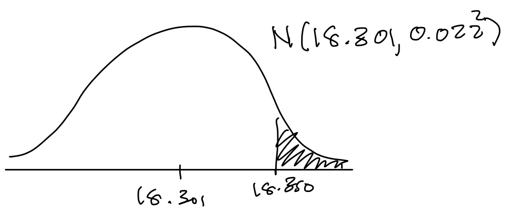
One-sample z-test, and 2-sided due to different

4) (17 points) Your boss was so impressed with the projects you have been doing that she asked you to help the manufacturing floor with some questions regarding their packaging unit, which boxes up the biologics with instructional materials. Weight measurements are taken on each and every product automatically, thus it is known that the population mean μ is 18.301 g and the population standard deviation is σ is 0.022 g and is normally distributed. New weight restrictions are being instated and the manufacturing floor wants to know the probability that a product will have a weight above a new limit of 18.350 g.

Probability problem! Not hypothesis Test!

a. (11 points) A write out the probability equation you wish to solve, including the integral (but you do not need to solve the integral just set it up). Also include a probability distribution sketch with the 1) probability you wish to find shaded in 2) the correct shape, 3) correct label of the distribution, 4) where 18.350g is and 5) where zero is.

$$P(x > 18.350\,g) = \int_{18.350}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$



b. (6 points) Solve the probability problem using the *tables in the back of the book*. Just like in class, sketches can help solve these problems, especially when using the tables. Report and appropriate number of sig figs based on the numbers given in the problem statement.

$$Z = \frac{16.350 - 16.30}{0.022} = 2.23$$

$\rightarrow$ P-value ≈ 0.013

Midterm Exam 2

Chapter 4 Continued - Test 2 Material

Inference on Two populations' variances

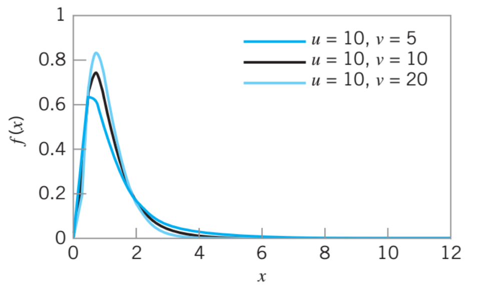
Sensitive to normality – should check!

→ Two independent, normally distributed populations

Null: $H_0: \sigma_1^2 = \sigma_2^2$

Alt: $H_a: \sigma_1^2 \neq \sigma_2^2$ two sided

Test Stat: $F_0 = \dfrac{S_1^2}{S_2^2}$ ← DoF$_1 = n_1 - 1$ ← DoF$_2 = n_2 - 1$



Reference distribution: F - distribution

(i) The curve is not symmetric (skewed right)

(ii) Different curve for each set of DoFs

(iii) F statistic ≥ 0

(iv) As DoF increase curve gets more normal

Defined by two DoFs

Note: $F_{0.99, \nu_1, \nu_2} = 1 / F_{0.01, \nu_2, \nu_1}$

Rejection Criteria: (two-sided)

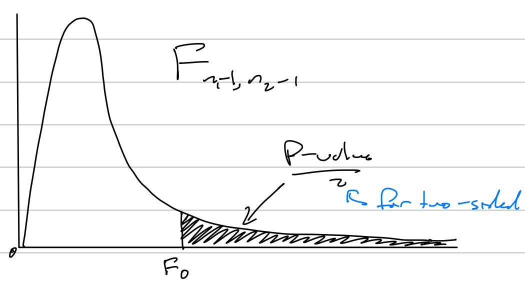
P-value

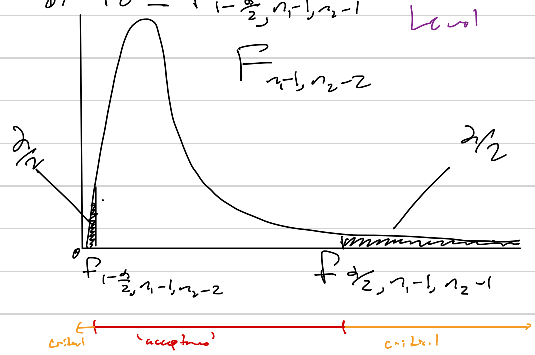
P-value < α

$f_0 \geq f_{\alpha/2, n_1-1, n_2-1}$

or $f_0 \leq f_{1-\frac{\alpha}{2}, n_1-1, n_2-1}$

Fixed Signif. Level





Summary

Null hypothesis: $H_0: \sigma_1^2 = \sigma_2^2$

Test statistic: $F_0 = \dfrac{S_1^2}{S_2^2}$ (10-31)

Alternative Hypotheses	Rejection Criterion
$H_1: \sigma_1^2 \neq \sigma_2^2$	$f_0 > f_{\alpha/2, n_1-1, n_2-1}$ or $f_0 < f_{1-\alpha/2, n_1-1, n_2-1}$
$H_1: \sigma_1^2 > \sigma_2^2$	$f_0 > f_{\alpha, n_1-1, n_2-1}$
$H_1: \sigma_1^2 < \sigma_2^2$	$f_0 < f_{1-\alpha, n_1-1, n_2-1}$

Assumptions

- The data are normally distributed
- Samples are independent

Sensitive to normality!

Example

Example 10-13 **Semiconductor Etch Variability** Oxide layers on semiconductor wafers are etched in a mixture of gases to achieve the proper thickness. The variability in the thickness of these oxide layers is a critical characteristic of the wafer, and low variability is desirable for subsequent processing steps. Two different mixtures of gases are being studied to determine whether one is superior in reducing the variability of the oxide thickness. Sixteen wafers are etched in each gas. The sample standard deviations of oxide thickness are $s_1 = 1.96$ angstroms and $s_2 = 2.13$ angstroms, respectively. Is there any evidence to indicate that either gas is preferable? Use a fixed-level test with $\alpha = 0.05$.

Minitab: Stat-> Basic Stat -> Two-sample Variance

1) Parameter of interest: Variances oxide thickness for two mixtures of gases, unknown variances

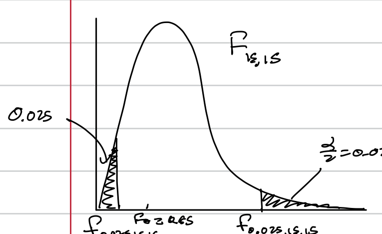
∴ Two-variance test

2) $H_0: \sigma_1^2 = \sigma_2^2$

3) $H_a: \sigma_1^2 \neq \sigma_2^2$

4) $F_0 = \dfrac{s_1^2}{s_2^2} = \dfrac{1.96^2}{2.13^2} = 0.85$

5) Reject if: $F_0 > f_{0.025, 15, 15}$ or $f_0 < f_{0.975, 15, 15}$



$f_{0.975,15,15} = \dfrac{1}{2.86}$ bc DoF are equal $= 0.35$

$f_{0.025,15,15} = 2.86$

Fail to reject H_0, we don't have enough evidence to reject the claim the variances of the oxide layer produced by the different mixtures are different.

Single Factor Analysis of Variance

- Analysis of Variance: Used for comparing means when there are more than two levels in a single factor
  - → Factor: Entity being tested
  - → Level: Settings within a factor
- Replicates: # of repeats taken at each level
- Randomization: Randomly gather data such that nuisance variables are eliminated

Factor → Levels, Repeats

Tensile Strength of Paper (psi)

Hardwood Concentration (%)	Observations 1	2	3	4	5	6	Totals	Averages
5	7	8	15	11	9	10	60	10.00
10	12	17	13	18	19	15	94	15.67
15	14	18	19	17	16	18	102	17.00
20	19	25	22	23	18	20	127	21.17
							383	15.96

\# of...
Factors = 1
Levels = 4
Replicates = 6

Defining and Development of ANOVA

Typical Data for a Single-Factor Experiment

Treatment	Observations				Totals	Averages
1	y_{11}	y_{12}	...	y_{1n}	$y_{1\cdot}$	$\bar{y}_{1\cdot}$
2	y_{21}	y_{22}	...	y_{2n}	$y_{2\cdot}$	$\bar{y}_{2\cdot}$
.	.	.	...	.	.	.
.	.	.	...	.	.	.
.	.	.	...	.	.	.
a	y_{a1}	y_{a2}	...	y_{an}	$y_{a\cdot}$	$\bar{y}_{a\cdot}$
					$y_{\cdot\cdot}$	$\bar{y}_{\cdot\cdot}$

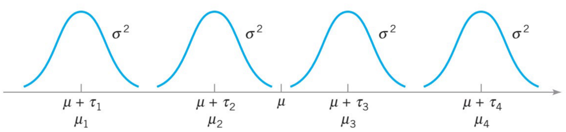
$y_{\cdot\cdot}$ (grand total) $\bar{y}_{\cdot\cdot}$ (grand mean)

a = # of treatments
y_{ij} = j^{th} observation of i^{th} treatment
n = # of observations at each level

Linear Statistical Model

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, n \end{cases}$$

where $\begin{cases} \tau_i = i^{th} \text{ treatment effect} \\ \varepsilon_{ij} = \text{random error} \end{cases}$



Hypothesis Testing the Equality of Treatment Effects

Null: $H_0: \tau_1 = \tau_2 = \tau_3 = \ldots = \tau_a = 0$

Alt: $H_a: \tau_i \neq 0$ for at least one i

Test Statistic: $F_0 = \dfrac{SS_{treatments}/a-1}{SS_E/a(n-1)}$ where $\begin{cases} a = \text{\# of treatments} \\ n = \text{\# of replicates @ each treatment level} \end{cases}$

Reference Distribution: F-dist w/ $(a-1)$ [top] and $a(n-1)$ [bottom] DoF

Rejection Criteria: H_0 if $F_0 > F_{\alpha, a-1, a(n-1)}$ this is always one-sided

Calculations for ANOVA

The Analysis of Variance for a Single-Factor Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}}$	$a-1$	$MS_{\text{Treatments}}$	$\dfrac{MS_{\text{Treatments}}}{MS_E}$
Error	SS_E	$a(n-1)$	MS_E	
Total	SS_T	$an-1$		

$$F_0 = \frac{MS_{trt}}{MS_E} = \frac{SS_{trt}/DoF}{SS_E/DoF}$$

$$SS_T = SS_{\text{Treatments}} + SS_E$$

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{\cdot\cdot}\right)^2 = \text{total sum of squares}$$ $n \times a$ squared terms

$$SS_{\text{Treatments}} = n\sum_{i=1}^{a}\left(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}\right)^2 = \text{treatment sum of squares}$$ a: # of squared terms

$$y_{i\cdot} = \sum_{j=1}^{n} y_{ij} \qquad \bar{y}_{i\cdot} = y_{i\cdot}/n \qquad i = 1, 2, \ldots, a$$

$$y_{\cdot\cdot} = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij} \qquad \bar{y}_{\cdot\cdot} = y_{\cdot\cdot}/N$$

Defining & Development of ANOVA

a=number of treatments

y_{ij} = the j 'th observation of the i 'th treatment

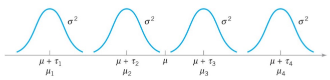
n = number of observations for each level

Typical Data for a Single-Factor Experiment

Treatment	Observations				Totals	Averages
1	y_{11}	y_{12}	...	y_{1n}	$y_{1\cdot}$	$\bar{y}_{1\cdot}$
2	y_{21}	y_{22}	...	y_{2n}	$y_{2\cdot}$	$\bar{y}_{2\cdot}$
.	.	.	...	.	.	.
.	.	.	...	.	.	.
.	.	.	...	.	.	.
a	y_{a1}	y_{a2}	...	y_{an}	$y_{a\cdot}$	$\bar{y}_{a\cdot}$
					$y_{\cdot\cdot}$	$\bar{y}_{\cdot\cdot}$

Linear statistical model: $y_{ij} = \mu + \tau_i + \varepsilon_{ij} \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,n \end{cases}$

Where: y_{ij} = observation
μ = overall mean (common to all treatments)
τ_i = the i 'th treatment effect
ε_{ij} = random error



Hypothesis testing the equality of treatment effects

Null and Alternative Hypothesis:

$H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0 \qquad H_1: \tau_i \neq 0$ for at least one i

Test Statistic: (reference distribution is the F-distribution)

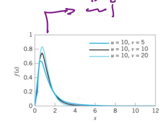
$$F_0 = \frac{SS_{\text{Treatments}} / \overset{\text{DOF}}{(a-1)}}{SS_E / \underbrace{[a(n-1)]}_{\text{DOF}}} = \frac{MS_{\text{Treatments}}}{MS_E}$$

signal / noise

Rejection Criteria:

$f_0 > f_{\alpha, a-1, a(n-1)}$

F_{v_1, v_2}



Calculations for ANOVA (same n)

The Analysis of Variance for a Single-Factor Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}}$	$a-1$	$MS_{\text{Treatments}}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Error	SS_E	$a(n-1)$	MS_E	
Total	SS_T	$an-1$		

$SS_T = SS_{\text{Treatments}} + SS_E$

$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{\cdot\cdot})^2$ = total sum of squares

$SS_{\text{Treatments}} = n\sum_{i=1}^{a}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2$ = treatment sum of squares

$y_{i\cdot} = \sum_{j=1}^{n} y_{ij} \qquad \bar{y}_{i\cdot} = y_{i\cdot}/n \qquad i = 1,2,\ldots,a$

$y_{\cdot\cdot} = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij} \qquad \bar{y}_{\cdot\cdot} = y_{\cdot\cdot}/N \rightarrow$ Total number of observations

General

i = treatment

j = observation

bar = average

dots = total

n = replicates

a = # of treatments

y_{ij} = j^{th} observation of i^{th} treatment

$\bar{y}$ = grand mean of all observations

$\dot{y}$ = total of all observations

$\bar{y}_i$ = average of observations @ treatment i

What does the Calculation look like

Tensile Strength of Paper (psi)

Factor Hardwood Concentration (%)	Observations 1	2	3	4	5	6 (n=6)	Totals $y_{i\cdot}$	Averages $\bar{y}_{i\cdot}$
1: 5	7	8	15 (y_{13})	11	9	10	60	10.00
2: 10	12	17	13	18	19	15	94	15.67
3: 15	14	18	19	17	16	18	102	17.00
4: 20	19	25	22	23	18	20	127	21.17
							383 ($y_{\cdot\cdot}$)	15.96 ($\bar{y}_{\cdot\cdot}$)

Treatment Level

$a=4$

$SS_T = SS_{\text{Treatments}} + SS_E$

$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{\cdot\cdot})^2$ = total sum of squares

$SS_{\text{Treatments}} = n\sum_{i=1}^{a}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2$ = treatment sum of squares

$$SS_T = SS_{\text{Treatments}} + \overset{\text{error}}{SS_E}$$

all data points

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n}(y_{ij} - \bar{y}_{\cdot\cdot})^2 = (7-15.96)^2 + (8-15.96)^2 + \ldots + (20-15.96)^2 = 513$$

$$SS_{\text{Treatments}} = n\sum_{i=1}^{a}(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 = 6\left[(10-15.96)^2 + (15.67-15.96)^2 + (17-15.96)^2 + (21.17-15.96)^2\right] = 382.8$$

$$\Rightarrow SS_E = 518 - 382.8 = 130.2$$

Hypothesis Test

Does hardwood concentration impact the mean tensile strength of paper? $\alpha = 0.01$

1) Parameter of Interest: Multiple pop. means of tensile strength → ANOVA!

2) $H_0: \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$

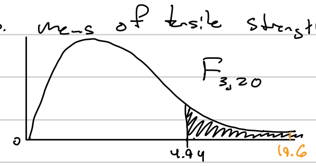
3) $\exists\, i \in [1,4]$ s.t. $\tau_i \neq 0$

4) $F_0 = \dfrac{\frac{SS_{\text{Treatments}}}{(a-1)}}{\frac{SS_E}{a(n-1)}} = \dfrac{\frac{512.96}{3}}{\frac{130.2}{20}} = 19.6$

5) Reject if $F_0 > F_{\alpha, a-1, a(n-1)}$

$F_{0.01, 3, 20} = 4.94$ from charts

6)



Minitab: Stat -> ANOVA -> one-way

7) Conclusions: Reject H_0 as $F_0 > F_{0.01, 3, 20}$

At least one treatment has an impact on the tensile strength of paper. More generally, Hardwood Concentration impacts paper tensile strength!

What do you do when the null is rejected

Multiple Comparison Methods – (post-hoc tests) is performed to find which means are different after ANOVA has been performed and H_0 is rejected

Fischer's Least Significant Difference (LSD) Test:

Declare Significantly Different if: $|\bar{y}_{i\cdot} - \bar{y}_{j\cdot}| > LSD$

$\rightarrow LSD = t_{\alpha/2, a(n-1)} \sqrt{\frac{2MS_E}{n}}$

Example

Tensile Strength of Paper (psi)

Hardwood Concentration (%)	Observations 1	2	3	4	5	6	Totals	Averages
5	7	8	15	11	9	10	60	10.00
10	12	17	13	18	19	15	94	15.67
15	14	18	19	17	16	18	102	17.00
20	19	25	22	23	18	20	127	21.17
							383	15.96

Minitab: Stat -> ANOVA -> one-way (click the comparisons button – error rate)

5 vs. 10 : $|10 - 15.67| = 5.67^*$

5 vs. 15 : $|10 - 17| = 7^*$

5 vs. 20 : $= 11.17^*$

10 vs 15 : $= 1.33$

10 vs. 20 : $= 5.5^*$

15 vs. 20 : $= 4.17$

$LSD = t_{0.005, 20} \sqrt{\frac{2MS_E}{n}}$

$= 2.845 \sqrt{2\left(\frac{6.5}{6}\right)}$

$= 4.19$

20 : C

15 : C B

10 : B

5 : A

10 vs 20 is ok

10 vs 15 is ok

Different than others

Assumptions

(i) Normal

(ii) No hidden variables

Summary

Declare significantly different if: $|\bar{y}_{i\cdot} - \bar{y}_{j\cdot}| > \text{LSD}$

where LSD, the least significant difference, is

$$\text{LSD} = t_{\alpha/2, a(n-1)} \sqrt{\frac{2MS_E}{n}}$$

Better notes on LSD test:

- Do this ONLY after ANOVA null hypothesis is rejected
- Find differences between all of the treatments
- Compare differences to the LSD to see if they are significantly different

Mini tab might produce an output like this:
If a pair has the same letter, they are not significantly different

Factor	N	Mean	Grouping		
20%	6	21.17	A		
15%	6	17.000	A	B	
10%	6	15.67		B	
5%	6	10.00			C

Fischer's LSD Test

Declare significantly different if: $|\bar{y}_{i.} - \bar{y}_{j.}| > \text{LSD}$

where LSD, the least significant difference, is

$$\text{LSD} = t_{\alpha/2,a(n-1)}\sqrt{\frac{2MS_E}{n}}$$

Better notes on LSD test:

- Do this after ANOVA null hypothesis is rejected
- Find differences between all of the treatments
- Compare differences to the LSD to see if they are significantly different

Mini tab might produce an output like this:
If a pair has the same letter, they are not significantly different
Sometimes funny overlap occurs – for example,15% is not different than 20% or 10% but, 10% and 20% are different

Factor	N	Mean	Grouping		
20%	6	21.17	A		
15%	6	17.000	A	B	
10%	6	15.67		B	
5%	6	10.00			C

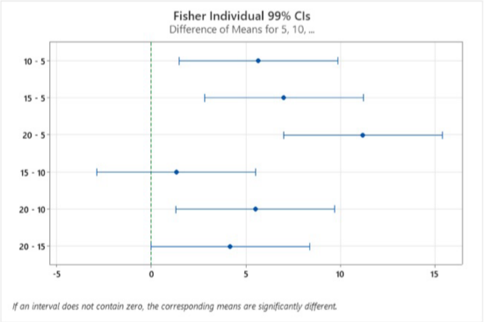
Confidence Intervals

A 100(1 – α)% confidence interval on the mean of the ith treatment μ_i is

$$\bar{y}_{i.} - t_{\alpha/2,a(n-1)}\sqrt{\frac{MS_E}{n}} \le \mu_i \le \bar{y}_{i.} + t_{\alpha/2,a(n-1)}\sqrt{\frac{MS_E}{n}}$$

A 100(1 – α) percent confidence interval on the difference in two treatment means $\mu_i - \mu_j$ is

$$\bar{y}_{i.} - \bar{y}_{j.} - t_{\alpha/2,a(n-1)}\sqrt{\frac{2MS_E}{n}} \le \mu_i - \mu_j \le \bar{y}_{i.} - \bar{y}_{j.} + t_{\alpha/2,a(n-1)}\sqrt{\frac{2MS_E}{n}}$$



ANOVA Assumptions

(i) Errors (and observations) are normally distributed
(ii) Each treatment has the same variance (equal variance assumption)
(iii) Errors and observations are independently distributed

$y_{ij} = \mu_i + \epsilon_{ij}$
obs. estimated

$E_{ij} = y_{ij} - \bar{y}_{i.}$
residual obs

Tensile Strength of Paper (psi)

Hardwood Concentration (%)	Observations 1	2	3	4	5	6	Totals	Averages
5	7	8	15	11	9	10	60	10.00
10	12	17	13	18	19	15	94	15.67
15	14	18	19	17	16	18	102	17.00
20	19	25	22	23	18	20	127	21.17
							383	15.96

Hardwood Concentration (%)	Residuals					
5	–3.00	–2.00	5.00	1.00	–1.00	0.00
10	–3.67	1.33	–2.67	2.33	3.33	–0.67
15	–3.00	1.00	2.00	0.00	–1.00	1.00
20	–2.17	3.83	0.83	1.83	–3.17	–1.17

Residual

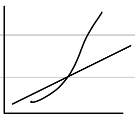
The difference between an observation (y_{ij}) and its estimated (or fitted) value from the statistical model being studied

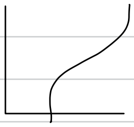
Normal Probability Plot of Residuals

(i) Rank data
(ii) Calc commulative frequency
(iii) Find z_j's
(iv) Plot residuals vs. z_j's
Looking for straight line

Minitab

Can ask Minitab to plot a normal probability plot after pasting in residuals into one column, **or** use ANOVA function, and select graphs, then select – normal probability plot of residuals





Checking equal Variance at each Factor Level

→ Minitab does not automatically generate

Look for: even residuals, if not...
Rule of thumb we used for t-tests applies, or, you could check using your 2 variance test!

Can generate in Excel or Minitab: Residuals in one column, factor level in another, using scatter plot

Checking Equal Variance as a function of predicted value

Residuals plotted as a function of predicted value

Minitab: use ANOVA function, and select graphs, then select – residuals vs. fits

Can also generate in Excel or Minitab: Residuals in one column, predicted value in the other, using scatter plot

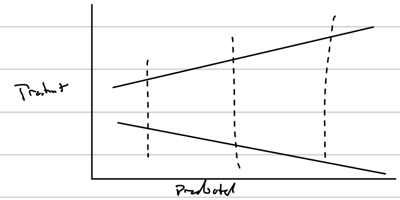
Hardwood Concentration (%)	Residuals					
5	–3.00	–2.00	5.00	1.00	–1.00	0.00
10	–3.67	1.33	–2.67	2.33	3.33	–0.67
15	–3.00	1.00	2.00	0.00	–1.00	1.00
20	–2.17	3.83	0.83	1.83	–3.17	–1.17

Tensile Strength of Paper (psi)

Hardwood Concentration (%)	Observations 1	2	3	4	5	6	Totals	Averages
5	7	8	15	11	9	10	60	10.00
10	12	17	13	18	19	15	94	15.67
15	14	18	19	17	16	18	102	17.00
20	19	25	22	23	18	20	127	21.17
							383	15.96

What if there is a pattern?

- Data transformation
- Ratio Analysis



Checking Independence

No formal plot… but:

1) Addressed via experimental design (randomize!)
2) Can plot residuals versus any suspected or nuisance variables
   1) Example: plot residuals versus run order (or other) (run order, temp, pH..)

When is my data not independent?

(1) repeated measurements are taken on the same subject – paired
(2) residuals or observations are correlated with a nuisance variable

Simple Linear Regression

- Use simple linear regression for building an empirical model
- Understand how the method of least squares is used to estimate the parameters in a linear regression model
- Analyze residuals to determine whether the regression model is an adequate fit to the data or whether assumptions are violated
- Test statistical hypotheses and construct confidence intervals on regression model parameters
- Apply the correlation model
- Use simple transformations to achieve a linear regression model

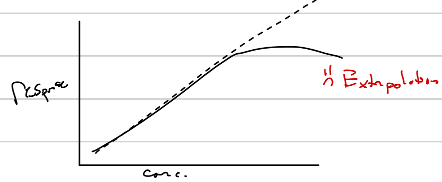
Regression Modeling

- **Regression Modeling:** A mathematical model fit to a set of data relating a response variable (y) to an independent variable ($x_1, x_2, \ldots, x_n$)
- **Empirical Models:** Models where the structure is determined by the observed relationship among experimental data, but not necessarily mechanistically relevant
- **Simple Linear Regression Model:** There is only one indep. or predictor variable (x) and the relationship with response (y) is assumed to be linear

$$y = \beta_0 + \beta_1 x + \epsilon$$

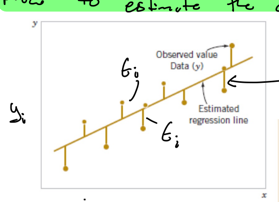
y = expected response
β_0 = intercept
β_1 = slope
$\}$ regression coef.

x = predictor variable
ϵ = random error term



- **Pitfalls:** Empirical Models
  → Correlation does not equal causation!!!
  → Valid in the range of original data (be careful with extrapolation)

- How to estimate the coefficients in simple linear regression



$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \qquad i = 1, 2, \ldots, n$

$L = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$

Method of Least squares

Minimize the sum of squares of vertical deviation

The **least squares estimates** of the intercept and slope in the simple linear regression model are

Avg of all response values → $\bar{y}$; Avg of all x's → $\bar{x}$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \qquad (11\text{-}7)$$

$$\hat{\beta}_1 = \frac{\overbrace{\sum_{i=1}^{n} y_i x_i - \frac{\left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} x_i\right)}{n}}^{S_{xy}}}{\underbrace{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}_{S_{xx}}} \qquad (11\text{-}8)$$

where $\bar{y} = (1/n)\sum_{i=1}^{n} y_i$ and $\bar{x} = (1/n)\sum_{i=1}^{n} x_i$.

Fitted or Estimated Regression Line

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

$\hat{y}$: Predicted Value

$$y_i = \hat{\beta}_0 + \hat{\beta}_1 x + \epsilon_i$$

y_i: Each data; ϵ_i: residual

← Each Pair Satisfies this

Example: Fit a simple linear regression line to these data

TABLE • 11-1 Oxygen and Hydrocarbon Levels

Observation Number	Hydrocarbon Level $x(\%)$	Purity $y(\%)$
1	0.99	90.01
2	1.02	89.05
3	1.15	91.43
4	1.29	93.74
5	1.46	96.73
6	1.36	94.45
7	0.87	87.59
8	1.23	91.77
9	1.55	99.42
10	1.40	93.65
11	1.19	93.54
12	1.15	92.52
13	0.98	90.56
14	1.01	89.54
15	1.11	89.85
16	1.20	90.39
17	1.26	93.25
18	1.32	93.41
19	1.43	94.98
20	0.95	87.33

$n = 20$

$\sum x_i = 23.92 \qquad \sum y_i = 1843.21 \qquad \sum x_i y_i = 2214$

$\sum x_i^2 = 29.28 \qquad \sum y_i^2 = 170044$

The **least squares estimates** of the intercept and slope in the simple linear regression model are

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \qquad (11\text{-}7)$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} y_i x_i - \frac{\left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} x_i\right)}{n}}{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}} \quad \begin{matrix} S_{xy} \\ S_{xx} \end{matrix} \qquad (11\text{-}8)$$

where $\bar{y} = (1/n)\sum_{i=1}^{n} y_i$ and $\bar{x} = (1/n)\sum_{i=1}^{n} x_i$.

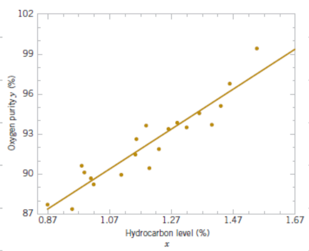
$S_{xx} = 29.28 - \frac{(23.92)^2}{20} = 0.68$

$S_{xy} = 2214 - \frac{23.92(1843)}{20} = 10.18$

$\tilde{\beta}_1 = \frac{10.18}{0.68} = 14.95$

$\tilde{\beta}_0 = \bar{y} - \tilde{\beta}_1 \bar{x} = 92.1605 - (14.95)(1.196)$

$= 74.28$



$\boxed{\hat{y} = 74.28 + 14.95x}$

Minitab: Stat > Regression -> Regression -> Fit Regression Model
Minitab: Stat > Regression -> Fitted Line -> choose x, y and linear

ANOVA for Linear Regression

- Null & Alternative Hypotheses:
  $H_0 : \beta_1 = 0$
  $H_1 : \beta_1 \neq 0$
- Test Statistic

$$F_0 = \frac{SS_R / 1}{SS_E / n-2} = \frac{MS_R}{MS_E} \qquad \frac{\text{signal}}{\text{noise}}$$

- Reference Distribution: F-dist
  ↳ Written as: $F_{1,n-2}$
- Rejection Criteria:
  Reject if $F_0 > F_{\alpha,1,n-2}$
  or p-value $< \alpha$

Calculating the Error

Recall: $SS_T = SS_R + SS_E$

- Residual: $e_i = y_i - \hat{y}_i$
  residual, response, predicted
- Error Sum of Squares: $SS_E = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$
- Mean Squared Error: $SS_E / n-2$
- Sum of Squares Total: $SS_T = \sum_{i=1}^{n} (y_i - \bar{y})^2$
- Regression Sum of Squares: $SS_R = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2$

Minitab: Stat>Regression>Fit Regression Model (options sets the alpha)

Example

TABLE • 11-1 Oxygen and Hydrocarbon Levels

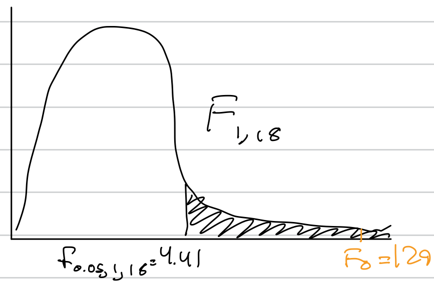
Observation Number	Hydrocarbon Level $x(\%)$	Purity $y(\%)$
1	0.99	90.01
2	1.02	89.05
3	1.15	91.43
4	1.29	93.74
5	1.46	96.73
6	1.36	94.45
7	0.87	87.59
8	1.23	91.77
9	1.55	99.42
10	1.40	93.65
11	1.19	93.54
12	1.15	92.52
13	0.98	90.56
14	1.01	89.54
15	1.11	89.85
16	1.20	90.39
17	1.26	93.25
18	1.32	93.41
19	1.43	94.98
20 n	0.95	87.33

1) Parameter of Interest: Slope of Linear Model; one regressor → simple linear regression

2) $H_0: \beta_1 = 0$ $H_a: \beta_1 \neq 0$

4) $F_0 = \dfrac{SS_R/1}{SS_E/n-2}$

5) Reject if $F_0 > f_{0.05,1,18}$



$\bar{y} = 92.161$

$\hat{y} = \hat{\beta}_0 + \beta_1 x_i \Rightarrow \hat{y} = 74.28 + 14.947x$

$n = 1 \Rightarrow \hat{y} = 74.28 + 14.947(0.99) = 89.08$

⋮ all data points

$n = 20 \Rightarrow \hat{y} = 74.28 + 14.947(0.95) = 88.48$

$SS_R = \sum_{i=1}^{n} (\hat{y} - \bar{y})^2 = (89.08 - 92.16)^2 + \overset{\text{all points}}{\cdots} + (88.48 - 92.16)^2 = 152.17$

$SS_T = \sum_{i=1}^{n} (y_i - \bar{y})^2 = (90.01 - 92.16)^2 + \overset{\text{all points}}{\cdots} + (87.33 - 92.16)^2 = 173.38$

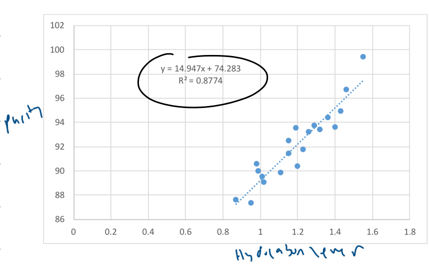
$SS_E = SS_T - SS_R = 21.2$

$\Rightarrow F_0 = \dfrac{152.17/1}{21.2\ /(20-2)} = 129$

$f_{0.05,1,18} = 4.41$ from tables

$129 > 4.41 \rightarrow$ Reject H_0
There is a significant linear relationship between hydrocarbon level and purity

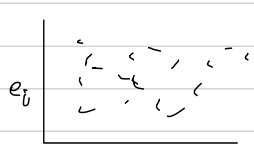
$R^2 = \dfrac{SS_R}{SS_T} = 1 - \dfrac{SS_E}{SS_T} = \dfrac{152}{173} = 0.88$



R^2 represents how much error is accounted for by the model

Checking Assumptions

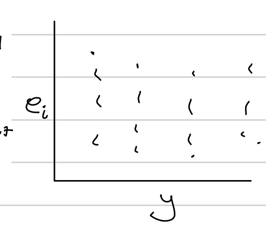
1) Errors (and observations) are normally distributed
2) Errors (and observations) are independently distributed
3) Variance is constant



$e_i = y_i - \hat{y}_i$ $i = 1, 2 \ldots n$

residual, data pt response, estimated values

Linear? Normality	Patterns? Independent Variance	Patterns or Spread Constant Variance
Plot residuals on Normal Probability Plot	Plot residuals vs. Run order or nuisance variables	① Plot residuals vs. predicted (common) ② Plot vs. independent variable (x)



Minitab: Regression -> Regression -> Fit Regression Model (select graphs button)
Choose “Four in One” … need to plot plot residuals vs. X separately (residuals vs. X)
In Minitab, Storage Button can give you fits and residuals

Confidence Intervals around Coefficients

Under the assumption that the observations are normally and independently distributed, a $100(1-\alpha)\%$ **confidence interval on the slope** β_1 in simple linear regression is

$$\hat{\beta}_1 - t_{\alpha/2,n-2}\sqrt{\frac{\hat{\sigma}^2}{S_{xx}}} \le \beta_1 \le \hat{\beta}_1 + t_{\alpha/2,n-2}\sqrt{\frac{\hat{\sigma}^2}{S_{xx}}} \quad (11\text{-}29)$$

Similarly, a $100(1-\alpha)\%$ **confidence interval on the intercept** β_0 is

$$\hat{\beta}_0 - t_{\alpha/2,n-2}\sqrt{\hat{\sigma}^2\left[\frac{1}{n}+\frac{\bar{x}^2}{S_{xx}}\right]} \le \beta_0 \le \hat{\beta}_0 + t_{\alpha/2,n-2}\sqrt{\hat{\sigma}^2\left[\frac{1}{n}+\frac{\bar{x}^2}{S_{xx}}\right]} \quad (11\text{-}30)$$

How to write and interpret:

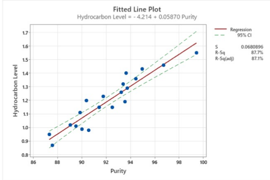
In repeated sampling the intervals calculated capture the true slope or intercept $(1-\alpha)\%$ of the time

Notes on CIs around Response

- CIs (confidence intervals) around a response: give a range of values associated with the true response for a given x
- You can construct confidence limits of your regression line by repeating the CIs for different values of x

How to Write?

CIs: In repeated experiments, there 95% chance that the CI on the responses calculated contain the true responses



How to generate?

To get the bounds on a graph: Minitab: Stat > Regression -> Fitted Line -> choose x, y and linear (Options tab: display CIs)

You can also use the "predict" function to have Minitab give you the CI's for any given x

Minitab: Regression -> Regression -> Fit Regression Model (select Results button)
Need to choose "expanded tables" from dropdown menu

Confidence Intervals around the Response

A $100(1-\alpha)\%$ **confidence interval on the mean response** at the value of $x = x_0$, say $\mu_{Y|x_0}$, is given by

$$\hat{\mu}_{Y|x_0} - t\alpha/2, n-2\sqrt{\hat{\sigma}^2\left[\frac{1}{n}+\frac{(x_0-\bar{x})^2}{S_{xx}}\right]}$$

$$\le \mu_{Y|x_0} \le \hat{\mu}_{Y|x_0} + t_{\alpha/2,n-2}\sqrt{\hat{\sigma}^2\left[\frac{1}{n}+\frac{(x_0-\bar{x})^2}{S_{xx}}\right]} \quad (11\text{-}31)$$

where $\hat{\mu}_{Y|x_0} = \hat{\beta}_0 + \hat{\beta}_1 x_0$ is computed from the fitted regression model.

(Nicely worded) ANOVA for Linear Regression

$H_0: \beta_1 = \beta_2 = \cdots = \beta_k = 0$
$H_1: \beta_j \neq 0 \quad$ for at least one j

Test Statistic:

$$F_0 = \frac{SS_R/1}{SS_E/(n-2)} = \frac{MS_R}{MS_E}$$

Reference Distribution:

F-distribution
Write as: $F_{1,n-2}$

Rejection Criteria: reject H_0 if $f_0 > f_{\alpha,1,n-2}$

Regression on Transformed Variables

Intrinsically Linear Models: Models that can be transformed into linear equations

Transformation is useful when: Relationship isn't linear and/or when residuals show a pattern

(i) All statistical Analyses should be done and interpreted on transformed variables

(ii) Back calculations can be done to report values

Examples

$Y = \beta_0 e^{\beta_1 x}\epsilon \;\rightarrow\; \ln Y = \ln \beta_0 + \beta_1 x + \ln \epsilon$

$Y = \beta_0 + \beta_1\left(\frac{1}{x}\right) + \epsilon \;\rightarrow\; Y = \beta_0 + \beta_1 z + \epsilon$

Example

Find a statistically satisfactory (alpha = 0.05) model for:

Observation Number, i	Wind Velocity (mph), xi	DC Output, yi	Observation Number, i	Wind Velocity (mph), xi	DC Output, yi
1	5.00	1.582	14	5.80	1.737
2	6.00	1.822	15	7.40	2.088
3	3.40	1.057	16	3.60	1.137
4	2.70	0.500	17	7.85	2.179
5	10.00	2.236	18	8.80	2.112
6	9.70	2.386	19	7.00	1.800
7	9.55	2.294	20	5.45	1.501
8	3.05	0.558	21	9.10	2.303
9	8.15	2.166	22	10.20	2.310
10	6.20	1.866	23	4.10	1.194
11	2.90	0.653	24	3.95	1.144
12	6.35	1.930	25	2.45	0.123
13	4.60	1.562			

Final Model: $y = \beta_0 + \beta_1\left(\frac{1}{x}\right) + \epsilon$

Minitab

Transform: Calc > Calculator (input your function)

For Residuals: Run regression using "Fit Regression Model" – click storage button and select residuals

Scatterplot: Graph>Scatterplot> select x and y data

Steps

1) Plot data out
2) Fit data with a model we think could work (try linear)
3) Analyze the residuals (ANOVA, R2 – plot residuals, normality)
4) Transform the data if there is a problem and regress data again
5) Check the results again (ANOVA, R2, residuals, normality)
6) Determine if model is better
7) Write out final model if: ANOVA is significant, and assumptions are met

We will rely on Minitab a lot more!

1) Scatter Plot doesn't look linear

2-3) Tried Linear → Residuals vs Fit results in clear pattern
↳ Equal variance assumption violated, try a different model

4) Try reciprocal of data

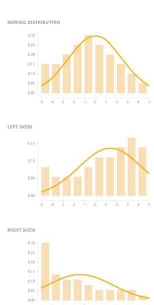
5-6) Results look good now! If they didn't, just try a different model

7) y = DC output, x = wind velocity } $y = 2.974 - 6.935\left(\frac{1}{x}\right)$ Always specify variables

Conclusions: The transformed model for DC output and wind velocity is significant.

Common Transformations

Method	Math Operation	Good for:	Bad for:
Log	ln(x) $\log_{10}(x)$	Right skewed data $\log_{10}(x)$ is especially good at handling higher order powers of 10 (e.g. 1000, 100000)	Zero values Negative values
Square root	$\sqrt{x}$	Right skewed data	Negative values
Square	x^2	Left skewed data	Negative values
Cube root	$x^{1/3}$	Right skewed data Negative values	Not as effective at normalizing as log transform
Reciprocal	1/x	Making small values bigger and big values smaller	Zero values Negative values



Multiple Linear Regression

Multiple Regression Models: A linear regression that contains more than one regressor (k)

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \epsilon$$

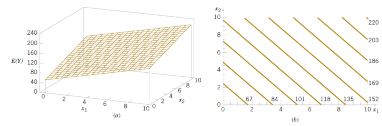
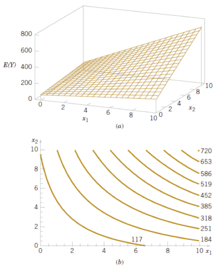


FIGURE 12-1 (a) The regression plane for the model $E(Y) = 50 + 10x_1 + 7x_2$. (b) The contour plot.

Non Linear Surfaces

	Cubic polynomial	Interaction	Second order with interaction
	$Y = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \epsilon$	$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \epsilon$	$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{12} x_1 x_2 + \epsilon$
Linear Version:	$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \epsilon$	$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \epsilon$	$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \epsilon$



$Y = 50 + 10x_1 + 7x_2 + 5x_1x_2$

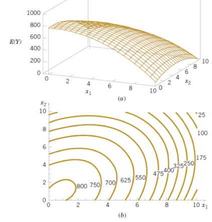


FIGURE 12-3 (a) Three-dimensional plot of the regression model $E(Y) = 800 + 10x_1 + 7x_2 - 8.5x_1^2 - 5x_2^2 + 4x_1x_2$. (b) The contour plot.

Estimated Coefficients and Variance

Same concepts as simple just a lot more terms

Coefficients: Found by method of least squares

↳ Residuals: $e_i = y_i - \hat{y}_i$ — Square of these minimized

Variances from different sources

- **Variance from Random Error:** SS_E from calculation of residuals

$$SS_T = SS_R + SS_E \qquad SS_E = \sum e_i^2$$

Goto Causey Notes

- **Variation from Regression:** SS_R = variance from regression
- **Variance from individual regression terms:** $SS_R(\beta_k)$ = variance coming from specific regressed term
- **Partial F-test:** Compares

Hypothesis Testing

Recall

n = # of data points

k = # of regressor terms

Test for significance of regression:

Ho: $\beta_1 = \beta_2 = \ldots = \beta_k = 0$

H1: $\beta_i \neq 0$ for at least one i

Reference Distribution:

F-dist

$F_{k, n-k-1}$

Test Statistic:

$$F_0 = \frac{SS_R / k}{SS_E / (n-k-1)}$$

Rejection criteria: (Fixed)

Reject if:

$F_0 > F_{\alpha, k, n-k-1}$

or p-value < α

Partial F-test:

Ho: $\beta_i = 0$

H1: $\beta_i \neq 0$

Reference Distribution:

F-dist

$F_{1, n-k-1}$

Test Statistic:

$$F_0 = \frac{SS_R(\beta_j)/1}{SS_E / n-k-1}$$

Rejection criteria:

$F_0 > F_{\alpha, 1, n-k-1}$

Nicole's Typed Notes

Test for significance of regression:

$H_0: \beta_1 = \beta_2 = \ldots = \beta_k = 0$

$H_1: \beta_j \neq 0$ for at least one j

Reference Distribution:
F-distribution: F k, n-k-1

F0 = SSR/k / SSE/(n-k-1)

Rejection criteria:
Reject Ho if Fo exceeds F alpha, k, n-k-1
Or if p-value is <alpha

Partial F-test:

Ho: Bj = 0
H1: Bj DNE 0

Reference Distribution:
F-distribution: F 1, n-k-1

F0 = SSR (Bj)/1 / SSE/(n-k-1)

Rejection criteria:
Reject Ho if Fo exceeds F alpha, 1, n-k-1
Or if p-value is <alpha

Adjusted R^2

$$SS_T = SS_R + SS_E \Rightarrow R^2 = \frac{SS_R}{SS_T}$$

Good for simple linear regression

Use Adj. R^2 for multiple!

$$R^2 = \frac{SS_E / k}{SS_T / n-1}$$

Avoids overfitting

Normalizing by DoF avoids increasing R^2 arbitrarily!

Example

Find a statistically satisfactory (alpha = 0.05) model for:

Brake Horsepower (y)	rpm	Road Octane Number	Compression
225	2000	90	100
212	1800	94	95
229	2400	88	110
222	1900	91	96
219	1600	86	100
278	2500	96	110
246	3000	94	98
237	3200	90	100
233	2800	88	105
224	3400	86	97
223	1800	90	100
230	2500	89	104

1) Plot the data
2) Fit a multiple linear regression model, Check the ANOVA results, R2adj, residuals, normality
3) Based on results, consider adding or removing regressors
4) If model is changed – repeat 2-3
5) Generate final model and contour plots

Goal: simplest model which significantly describes the data

Minitab

Plot: Graph > Scatter plot > plot response against all X's

Fit Model: Stat > Regression > Fit Regression Model (add Response, and Predictors by double click) (Determine model by button 'Model" to add or subtract terms)

Contour Plots: Stat- regression > regression > contour or surface plot

Final Result

Regression Equation

Brake Horsepower = -266.0 + 0.01071 rpm + 3.135 Road Octane Number + 1.867 Compression

p-values low → significant :)

Control Charts

- Statistical Process Control: A collection of problem-solving tools useful in achieving process stability and reducing variability

The Magnificent Seven

1. Histogram or stem-and-leaf plot
2. Check sheet
3. Pareto chart
4. Cause-and-effect diagram
5. Defect concentration diagram
6. Scatter diagram
7. Control chart

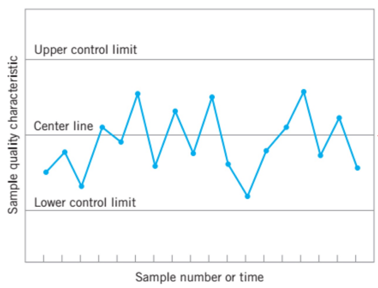
Shewhart's Control Charts & Theory of Variability

- Chance causes of variation: Inherent variability in a process or background noise – a process operating with only chance causes is in statistical control
- Assignable Causes of Variation: Sources of variability that are not due to chance – a process operating outside of assignable causes is an out-of-control process
  - → Sources: Improperly adjusted/controlled machines
    Operator errors
    Defective raw material
  - → Control Chart: Online process monitoring technique

Control Chart Uses

- Monitoring: Using the control chart to determine if the process is in control, and quickly detect any assignable causes
- Estimating: Using the control chart to estimate process parameters (mean or variance)
- Improving: Using the control chart to improve process parameters

Features of a Control Chart



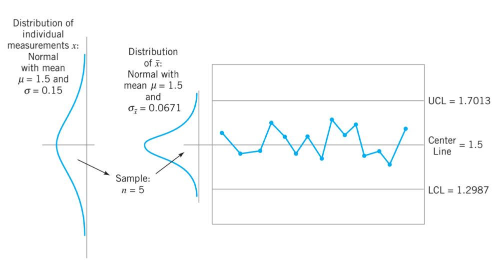
calculated
← UCL
← in control
← LCL
calculated

Different than Specification!
voice of Customer

Relationship to Hypothesis Testing

Null Hypothesis: $\mu = \mu_0$ in statistical control

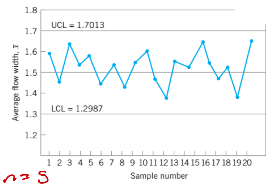
Alternative: $\mu \neq \mu_0$



Type I error: Rejecting H_0 when H_0 is true; concluding the process is out of control when it is not

Type II error: Not rejecting null H_0 when it is false; Assuming a process is in control when it really is out of control

Example



$\mu_0 = 1.5$
$\sigma = 0.15$
$n = 5$

$\bar{x}$ Control chart: uses sample average to monitor a process mean

$100(1-\alpha)\%$ of sample means should fall between control limits

Three Sigma Control Limits: Define $Z_{\alpha/2} = 3 \Rightarrow \alpha = 0.0027$

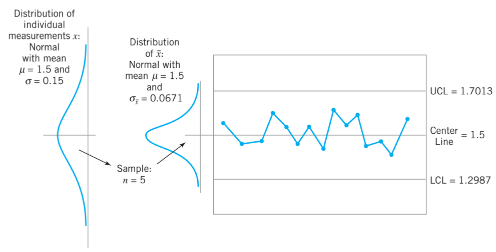
Control Limits (Known variance): LCL, UCL $\bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{0.15}{\sqrt{5}} = 0.0671$$

$$UCL = 1.5 + 3(.0671) = 1.7013$$

$$LCL = 1.5 - 3(.0671) = 1.2987$$

General Model

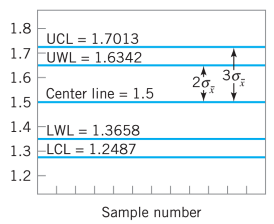


w = quality characteristic of interest
μ_w = mean of w
σ_w = std dev of w
L = distance from centerline expressed in std dev
$UCL = \mu_w + L\sigma_w$
Center $= \mu_w$
$LCL = \mu_w - L\sigma_w$

Design of a Control Chart

- Selecting Control Limits:
  - Set alpha (Europe)
  - Set Sigma level (US)

3 is pretty standard compromise balancing type I and II error



- Two Limit Control Charts
  - Action limits - outer limits that trigger search for assignable cause
  - Warning limits - inner limits - which raise alarms

Average Run Length (ARL) - the average # of points that must be plotted before a point will indicate an out of control condition

$ARL = \frac{1}{p}$; $p = \alpha$ probability the point exceeds control limit

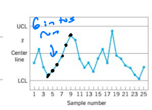
Average Run Signal (ARS) - the average amount of time between out of control signals

$ARS = ARL \times$ (time between samples)

Analysis of Patterns

A process can be out of control even if all points are within limits!

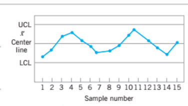
Run: A series of observations of the same type



Run length: How many points are in the series

Pattern: Non-random behavior

Zone Rules: Criteria to help identify patterns/runs



Sinusoidal

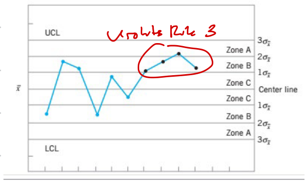
Detecting Patterns and Sensitizing Rules

Standard Action Signal:	1. One or more points outside of the control limits	Western Electric Rules
	2. Two of three consecutive points outside the two-sigma warning limits but still inside the control limits	
	3. Four of five consecutive points beyond the one-sigma limits	
	4. A run of eight consecutive points on one side of the center line	
Sensitizing Rules:	5. Six points in a row steadily increasing or decreasing	
	6. Fifteen points in a row in zone C (both above and below the center line)	
	7. Fourteen points in a row alternating up and down	
	8. Eight points in a row on both sides of the center line with none in zone C	
	9. An unusual or nonrandom pattern in the data	
	10. One or more points near a warning or control limit	

Out of control if yes

If yes, investigate more, get more data
Raises alarm

Sensitizing Rules: Criteria to Increase Sensitivity



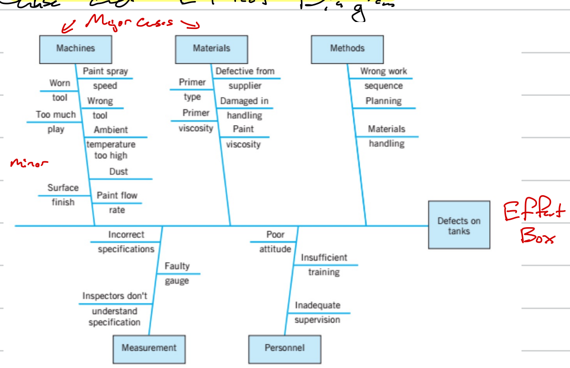
=> Out of Control action plan

$\bar{x}$ chart: Average of samples
R-Chart: the difference between min & max reading of a sample

Out of Control Action Plans

OCAP: A Flow chart or text-based description of the
abbrev. Sequence of actions that take place following an
activity event
Checkpoints: Potential assignable causes
Terminators: Action taken to Resolve

Cause and Effect Diagram

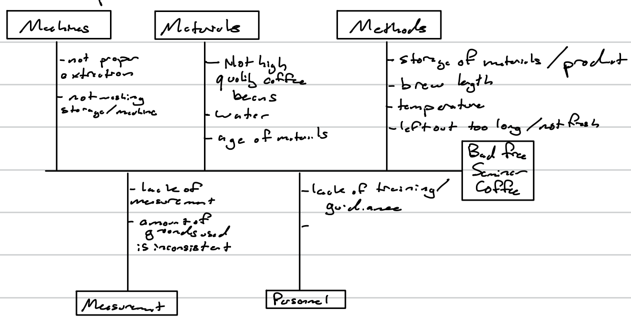


How to construct a C&E diagram

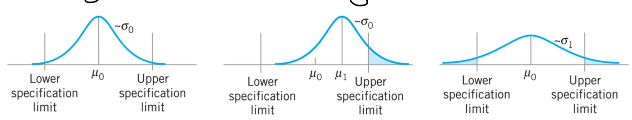
How to Construct a Cause-and-Effect Diagram

1. Define the problem or effect to be analyzed.
2. Form the team to perform the analysis. Often the team will uncover potential causes through brainstorming.
3. Draw the effect box and the center line.
4. Specify the major potential cause categories and join them as boxes connected to the center line.
5. Identify the possible causes and classify them into the categories in step 4. Create new categories, if necessary.
6. Rank order the causes to identify those that seem most likely to impact the problem.
7. Take corrective action.

Example: Free Seminar Coffee



Controlling Mean & Variability



In control — Mean shifts — σ shifts

Why use both $\bar{x}$ and R charts

Mean Shifts: $\bar{x}$ shows change, R chart won't

Variance Shifts: R chart shows change, $\bar{x}$ won't

Known Population and Variance

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$$\mu + Z_{\alpha/2}\sigma_{\bar{x}} = \mu + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \text{and} \quad \mu - Z_{\alpha/2}\sigma_{\bar{x}} = \mu - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

LSL — USL

Parameters Unknown

Estimated from "in control" samples of at least 20-25 (m) each with at least 4-5 replicates (n)

where m is the total # of samples of size n

$$\mu = \bar{\bar{x}} = \frac{\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_m}{m}$$

$$\bar{R} = \frac{R_1 + R_2 + \cdots R_m}{m} \rightarrow R_m = x_{max} - x_{min}$$

for any sample

Designed to give limits 3 standard deviations away from mean:

Control Limits for the $\bar{x}$ Chart

$$UCL = \bar{\bar{x}} + A_2\bar{R}$$
$$\text{Center line} = \bar{\bar{x}} \qquad (6.4)$$
$$LCL = \bar{\bar{x}} - A_2\bar{R}$$

The constant A_2 is tabulated for various sample sizes in Appendix Table VI.

Control Limits for the R Chart

$$UCL = D_4\bar{R}$$
$$\text{Center line} = \bar{R} \qquad (6.5)$$
$$LCL = D_3\bar{R}$$

The constants D_3 and D_4 are tabulated for various values of n in Appendix Table VI.

Example

EXAMPLE 6.1 $\bar{x}$ **and R Charts for a Manufacturing Process**

A hard-bake process (see Section 5.3.1) is used in conjunction with photolithography in semiconductor manufacturing. We wish to establish statistical control of the flow width of the resist in this process using $\bar{x}$ and R charts. Twenty-five samples, each of size five wafers, have been taken when we think the process is in control. The interval of time between samples or subgroups is one hour. The flow width measurement data (in microns) from these samples are shown in Table 6.1.

TABLE 6.1
Flow Width Measurements (microns) for the Hard-Bake Process

Sample Number	Wafers 1	2	3	4	5	$\bar{x}_i$	R_i
1	1.3235	1.4128	1.6744	1.4573	1.6914	1.5119	0.3679
2	1.4314	1.3592	1.6075	1.4666	1.6109	1.4951	0.2517
3	1.4284	1.4871	1.4932	1.4324	1.5674	1.4817	0.1390
4	1.5028	1.6352	1.3841	1.2831	1.5507	1.4712	0.3521
5	1.5604	1.2735	1.5265	1.4363	1.6441	1.4882	0.3706
6	1.5955	1.5451	1.3574	1.3281	1.4198	1.4492	0.2674
7	1.6274	1.5064	1.8366	1.4177	1.5144	1.5805	0.4189
8	1.4190	1.4303	1.6637	1.6067	1.5519	1.5343	0.2447
9	1.3884	1.7277	1.5355	1.5176	1.3688	1.5076	0.3589
10	1.4039	1.6697	1.5089	1.4627	1.5220	1.5134	0.2658
11	1.4158	1.7667	1.4278	1.5928	1.4181	1.5242	0.3509
12	1.5821	1.3355	1.5777	1.3908	1.7559	1.5284	0.4204
13	1.2856	1.4106	1.4447	1.6398	1.1928	1.3947	0.4470
14	1.4951	1.4036	1.5893	1.6458	1.4969	1.5261	0.2422
15	1.3589	1.2863	1.5996	1.2497	1.5471	1.4083	0.3499
16	1.5747	1.5301	1.5171	1.1839	1.8662	1.5344	0.6823
17	1.3680	1.7269	1.3957	1.5014	1.4449	1.4874	0.3589
18	1.4163	1.3864	1.3057	1.6210	1.5573	1.4573	0.3153
19	1.5796	1.4185	1.6541	1.5116	1.7247	1.5777	0.3062
20	1.7106	1.4412	1.2361	1.3820	1.7601	1.5060	0.5240
21	1.4371	1.5051	1.3485	1.5670	1.4880	1.4691	0.2185
22	1.4738	1.5936	1.6583	1.4973	1.4720	1.5390	0.1863
23	1.5917	1.4333	1.5551	1.5295	1.6866	1.5592	0.2533
24	1.6399	1.5243	1.5705	1.5563	1.5530	1.5688	0.1156
25	1.5797	1.3663	1.6240	1.3732	1.6887	1.5264	0.3224
						$\Sigma\bar{x}_i = 37.6400$	$\Sigma R_i = 8.1302$
						$\bar{\bar{x}} = 1.5056$	$\bar{R} = 0.32521$

R-chart

$$\bar{R} = \frac{0.3679 + .2517 + \dots + R_{25}}{25} = 0.32521$$

$$LCL = \bar{R}D_3 = 0$$

$$UCL = \bar{R}D_4 = .32521(2.114) = 0.68749$$

$\bar{x}$ Chart

$$\bar{\bar{x}} = \frac{1.5119 + \dots R_{25}}{25} = 1.5056$$

$$UCL = \bar{\bar{x}} + A_2\bar{R} = 1.5056 + .577(.3252) = 1.69325$$

$$LCL = \bar{\bar{x}} - A_2\bar{R} = 1.5056 - .577(.3252) = 1.31795$$

Minitab

Stat -> Control Charts -> Variables Charts for Subgroups -> Select Xbar or R -> dropdown to observations from subgroup are in one row of columns -> select all columns with data (n=number of selections)

Estimates from Control Charts

Estimating process standard deviation:

$$\hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{.32821}{2.326} = 0.1348$$

Estimate of process mean

$$\bar{\bar{\mu}} = \mu$$

Specs are not the same as control limits

Note: Can chart versus specifications, but be careful

Test 2 Review

Topics

General:

- how to correctly phrase conclusions in a hypothesis test
- how to correctly phrase what the p-value is
- how to correctly phrase what confidence intervals are representing
- how to sketch hypothesis tests on a reference (sampling) distribution

Inferring the variance of two populations:

- how to conduct the hypothesis test
- how to draw conclusions

ANOVA:

- how to conduct the test (by hand and interpret Minitab) and draw conclusions
- what situations ANOVA is used in (when to pick it as the statistical test)
- what, generally, the test statistic represents
- what the reference distribution is as related to the test statistic and when the test statistic follows the reference distribution
- know how to check assumptions and what assumptions are made
- how and when to perform a posthoc test
- how and when to transform data

Linear Regression:

- know how to perform simple linear regression by hand (how to perform the hypothesis test – won't ask the regression terms by hand)
- how to set up equations and hypothesis for multiple and simple linear regression
- know how to interpret results from Minitab output on simple and multiple linear regression
- explain what R^2 is (+adjusted), what it can and cannot be used for, and how to calculate by hand for simple and multiple linear regression
- know how to check assumptions and what assumptions are made
- how and when to transform data
- describe what the confidence intervals on the response and coefficients are

Control Charts:

- Be able to explain what impact wide or narrow control limits have on type I and type II error, and be able to define type I and type II errors for control charts
- Be able to explain in words and sketches why it is important to use control charts for both process mean and variability
- Calculate control limits and warning limits for xbar and R charts
- Be able to identify and out of control process
- ARL calculation and concept

ECHE 313 Exam 2

Write your name: ______________________

Write the following statement, and sign your name acknowledging your agreement:

“"I neither received nor gave external assistance on this exam "

____

The exam layout is such that all problems add up to 100 points. This point total is meant to guide you in time allocation for this test, will be adjusted such that this exam is 25% of your total class grade. You may use your class materials, class notes, homework, book, and Minitab to aid in the completion of the exam, but you are <u>not allowed to use any outside resources</u>. The exam is designed to be completed in 1 hr and 15 min, but you will have between 10 AM 4/9 to 10 AM 4/10 to complete the exam. <u>Note: Once you start the exam, you have 4 hours to complete the test</u>. All of the necessary tables are provided in the back of your book. Please show your work and all steps necessary to arrive at your answer. It is expected you do this entire test by hand, using a calculator and the tables found in the back of your book. So while you are allowed to use Minitab if you wish to check your answers, it should not replace the work you have done by hand using the tables for your answers. Where applicable, it is best to write out the general formals you want to use first, then write in your subbed in values so it is clear. Also, when reporting numbers, keep the same number of decimals as the data you are working with unless there are specific directions on how many sig figs to report. Good luck!

Premise: You work as a process engineer for a company that makes eclipse glasses. What a great time to be in this business! These glasses work by filtering light, reducing light exposure to the eye. <u>Unless otherwise stated, your company uses α=0.05</u>.

Problem 1 (67 points total) Your team wants to change the concentration of materials that are used to block light in your product to save money. Material B is particularly expensive and is typically set at 5.00 wt% in the material. You want to know if the concentration of material B has an impact on the % transmittance of light, so you vary material B and test the amount of light that is transmitted through the material, and get the following data:

	n=3 % Transmittance			$\bar{y}_{i\cdot}$	
	Replicate				
Material B (wt%)	1	2	3	Ave.	Standard Deviation
(typical) 5.00	0.0023	0.0023	0.0024	0.0023	0.000058
4.00	0.0023	0.0025	0.0028	0.0025	0.000252
3.00	0.0027	0.0024	0.0021	0.0024	0.000300
2.00	0.0032	0.0038	0.0035	0.0035	0.000300
1.00	0.0041	0.0039	0.0045	0.0042	0.000306

a=5

$\bar{y}_{..} = \frac{0.0023 + 0.0025 + 0.0024 + 0.0035 + 0.0045}{5} = 0.00298$

a. (1 points) List the *name* of the hypothesis test you would use

ANOVA

b. (2 points) Name the parameter of interest in the context of the problem statement

Multiple means, specifically % transmittance at different levels of material B.

c. (2 points) Write out the correct null hypothesis for this test

$H_0: \tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau_5 = 0$

d. (2 points) Write out the correct alternative hypothesis for this test

$H_a: \tau_i \neq 0$ for some $i \in (1, 5)$

e. (6 points) Write the formula of the test statistic, and other formulas needed to fully calculate the test statistic from the data above, and calculate its value given that SS_E is 0.00000068

$$F_0 = \frac{SS_{Treatments}/(a-1)}{SS_E\,(a(n-1))} \qquad \begin{cases} n=5 \\ a=3 \end{cases}$$

$$SS_{treatments} = n \sum (\bar{y}_{i.} - \bar{y}_{..})^2$$

$$= 3\left[(0.0023-0.00298)^2 + (0.0025-0.00298)^2 + (0.0024-0.00298)^2 + (0.0035-0.00298)^2 + (0.0042-0.00298)^2\right]$$

$$= 8.364 \times 10^{-6}$$

If you use excel/exact values, you get to 0.0000079

$$F_0 = \frac{(8.364 \times 10^{-6})/(5-1)}{(0.00000068)/(5(3-1))} = 30.75$$

$F_0 = 30.75$

$F_0 = 29$ if using exact values

f. (4 points) Describe what the test statistic represents (specifically the numerator and denominator of the test statistic as they related to sources of variation) and what it means if the test statistic is very large versus very small

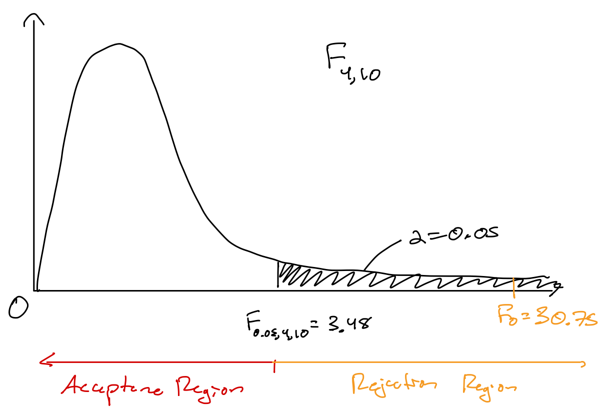
F_0 is effectively $\frac{\text{signal}}{\text{noise}}$. The numerator describes variance from treatments and the denominator describes variance from random error. If F_0 is very large, variance is coming from treatments (but hypothesis test determine significance). If this is small, variance is comparable to random error. DoF has an impact on the value.

g. (4 points) Define your general rejection criteria using fixed significance level testing, sub in values and write out the critical value

Reject if: $F_0 > F_{\alpha, a-1, a(n-1)}$

$\Rightarrow F_0 > F_{0.05, 4, 10} = 3.48$

h. (10 points) Sketch out the reference distribution for this problem and label: the type of reference distribution used and any defining characteristics (shape, DOF if applicable, and where zero is), the critical value(s), the test statistic, alpha, the acceptance region, and the rejection region.



i. (3 points) Perform any remaining calculations and state your conclusions. Remember to state them in the context of the problem statement, and phrase them appropriately

$F_0 = 30.75 > F_{crit} = 3.48 \rightarrow$ Reject H_0.

Reject the null and accept the alternative. We have significant evidence to indicate that the wt% of material B impacts % transmittance.

j. (7 points) Describe when it is appropriate to do post-hoc testing, and assuming it is appropriate in this case, conduct a Fisher's post hoc test and identify which groups are significantly different than the normal process setting (Material B at 5%).

A post-hoc test is only appropriate once the ANOVA H_0 is rejected!

Declare significantly different if: $|\bar{y}_{i.} - \bar{y}_{j.}| > LSD$

5 vs. 4: $|0.0023 - 0.0028| = 0.0002$

5 vs. 3: $|0.0023 - 0.0024| = 0.0001$

5 vs. 2: $|0.0023 - 0.0038| = 0.0012$

5 vs. 1: $|0.0023 - 0.0042| = 0.0019$

$$LSD = t_{\alpha/2,\, a(n-1)} \sqrt{\frac{2MSE}{n}}$$

$$= t_{0.025,10} \sqrt{\frac{2\left(\frac{0.00000068}{10}\right)}{3}}$$

$$= 2.228\,(2.128 \times 10^{-4})$$

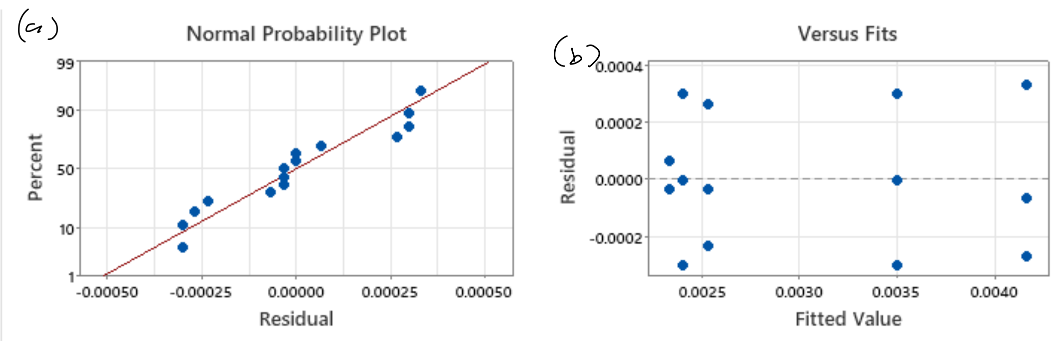
$$= 4.74 \times 10^{-4}$$

$$= 0.000474$$

1 and 2 wt% are significantly different than 5 wt% (norm)

k. (4 points) If these are the residual plots:

(a) (b)



1. State what assumptions are being checked for each plot

(a) Errors and observations are normally distributed

(b) Constant variance

2. Describe what you should do if the assumptions are violated

Transform the data and perform ANOVA again.

I. You are concerned that the variance of the data for 5 wt% treatment is different than the other treatments.

1. (1 points) What is the name of the hypothesis test I should perform if I want to know if the variance from 5% treatment and 4% treatment are different?

   2-variance test!

2. (2 points) Write out the correct null and alternative hypotheses for this test

   $H_o: \sigma_1^2 = \sigma_2^2$ where 1 = 5wt%, and 2 = 4wt%

   $H_a: \sigma_1^2 \neq \sigma_2^2$

3. (5 points) Calculate the test statistic (report 3 significant figures)

Write out the general formula:

$$F_0 = \frac{s_1^2}{s_2^2}$$

Sub in values and calculate:

$$F_0 = \frac{0.000058^2}{0.000252^2} = .052973$$

4. (8 points) 1. Write out the rejection critical using p-value method and 2. determine the p-value from your test statistic (report 2 sig figs in your p-value). 3. Show the p-value graphically. In your sketch, you should have the reference distribution labeled with any defining characteristics (shape, zero, and degrees of freedom if applicable), the test statistic, and the p-value).

   (i) Reject if p-value < α = 0.05

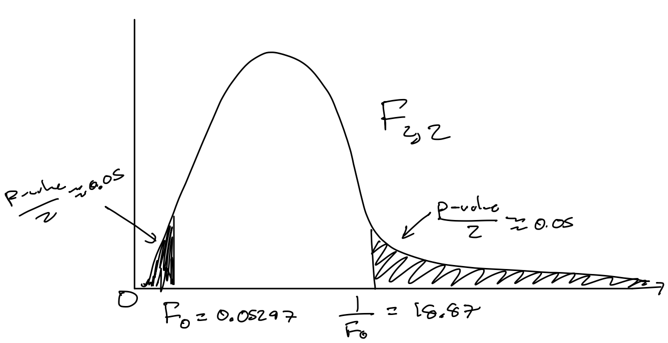
   (ii) $F_{2,2}$ (F_{n_1-1, n_2-1})

   $19.00 = F_{0.05,2,2}$

   $\therefore$ p-value ≈ 2(0.05) = 0.10

   → p-value ≈ 0.10

(iii)



5. (3 points) State your conclusions. Remember to state them in the context of the problem statement, and phrase them appropriately. Also use your conclusion to answer: is there cause for concern?

P-value = 0.1 > 0.05 => Fail to reject Ho. We do not have enough evidence to say that the variance at 5wt% is different than 4wt%. No cause for concern as this implies constant variance is still valid.

m. (3 points) Provide and answer to your boss who was wondering: "Why is the rejection criteria for ANOVA one sided, whereas sometimes an F-test can be one- or two-sided?"

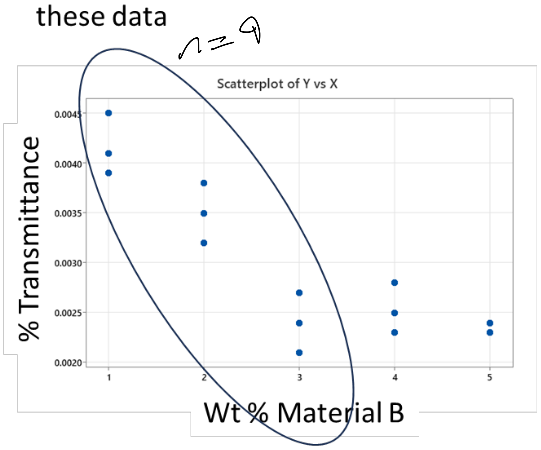
ANOVA is looking to see if a "signal" is larger than "noise". We therefore only care if the variance of the "signal" is larger than random error, thus it is one sided. If using an F-test we are determining if two variances are different, thus we need a two-sided test.

Problem 2. (22 points total) You suspect that the Material B wt% (from 1 to 3 wt% - predictor variable) is linearly correlated to % Transmittance (response variable). You want to perform a hypothesis test to see if the linear correlation is significant.

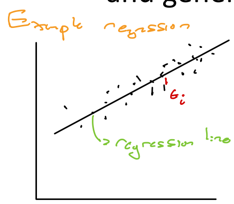
Performing linear regression on these data

k = # of regression terms

n = # of data points



a. (3 points) You perform the liner regression and your estimated coefficient for the slope is -0.00088 and the intercept is 0.0051. What is the name of the method that is used to estimate these coefficients, and describe generally how that method works. (Hint: I am <u>NOT</u> looking for how to calculate S_{xx} or S_{xy} here). You can use a diagram and general formulas for your answer if needed.



The method of <u>least squares</u> is used to estimate the coefficients in linear regression. In this method, the vertical deviations from data and the regression line (ϵ_i's) are minimized. These are the residuals, and the model aims to minimize $\sum_{i=1}^{n} \epsilon_i^2$.

b. This is your Minitab output from the simple linear regression:

Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value
Regression		0.00000468167		51.16	0.000
Error		0.00000064056			
Total					

(1) (2 points) State the degrees of freedom for the $SS_{Regression}$ and the SS_{Error}

$SS_{Regression}$: $k = 1$

SS_{Error} : $n - k - 1 = n - 2 = 7$

(2) (2 points) Write out the formula used and calculate the SS_{Total} using information given in the table above

$$SS_{Total} = SS_R + SS_E$$
$$= 0.00000468167 + 0.00000064056$$
$$= 0.0000053$$

(3) (2 points) Write out the formula for and calculate R^2

$$\frac{SS_R}{SS_T} = \frac{0.00000468167}{0.0000053} = 0.88$$

(4) (3 points) State your conclusions. Remember to state them in the context of the problem statement, and phrase them appropriately. Assume that no issues were noted with the residual plots.

p-value = 0 < α = 0.05 → Reject H_0 and accept H_a. We have significant evidence to reject the claim that there is no linear relationship between material B from 1-3% and % transmittance. The wt% of material B from 1-3% is linearly correlated with % transmittance.

c. (10 points) You think there might be another variable that could be useful in your model (wt% of another material, Material A). You add in this variable and run multiple linear regression to get the following output:

Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value
Regression	2	0.000005	0.000002	26.21	0.001
Wt% B	1	0.000002	0.000002	19.92	0.004
Wt% A	1	0.000000	0.000000	1.03	0.349
Error	6	0.000001	0.000000		
Total	8	0.000005			

Model Summary

S	R-sq	R-sq(adj)	R-sq(pred)
0.0003018	89.73%	86.30%	76.89%

Coefficients

Term	Coef	SE Coef	T-Value	P-Value	VIF
Constant	0.005844	0.000760	7.69	0.000	
X	-0.001100	0.000246	-4.46	0.004	4.00
X2	-0.000289	0.000285	-1.02	0.349	4.00

(1) (3 points) The R^2 increased after adding the new variable. Should we keep the variable in the model? Why or why not?

No. R^2 is <u>not</u> a valid tool for determining the significance of a variable in regression. In addition, the partial F-test shows p-value > 0.05 = α for wt% A, so there isn't enough evidence to say it has a significant relationship with % transmittance.

(2) (3 points) R^2 adjusted before adding Material A as a predictor was 86%. Explain why R^2 adjusted did not go up when adding the new variable. Use the formula in your answer.

$$R^2_{adj} = \frac{SS_R/k}{SS_T/n-1}$$

R^2_{adj} includes degrees of freedom which prevents k^2 from increasing simply by adding more regression terms.

(3) (4 points) State your conclusions appropriately in the context of the problem statement and write out the final model

Fail to reject Ho. we don't have enough evidence to say wt% A has a significant relationship with % transmittance. The final model is: $y = 0.0051 - 0.00088x$

y = % transmittance

x = wt% material B

Problem 3. (11 points total) Your company wants to establish and Xbar control chart on the process for component C wt%. There are 22 preliminary samples, each of size 8, on the internal diameter of the seal. The summary data (in mm) are as follows:

Sample #	Range (R)	Average (Xbar)
1	5	15
2	9	11
3	8	11
4	4	13
5	9	13
6	5	10
7	10	14
8	5	14
9	9	12
10	8	13
11	8	15
12	8	11
13	8	15
14	8	12
15	5	13
16	8	14
17	7	14
18	4	11
19	9	12
20	8	15
21	8	14
22	5	11
Sum	158	283

$\bar{R} = \frac{158}{22} = 7.18$ $\bar{\bar{x}} = \frac{283}{22} = 13.0$

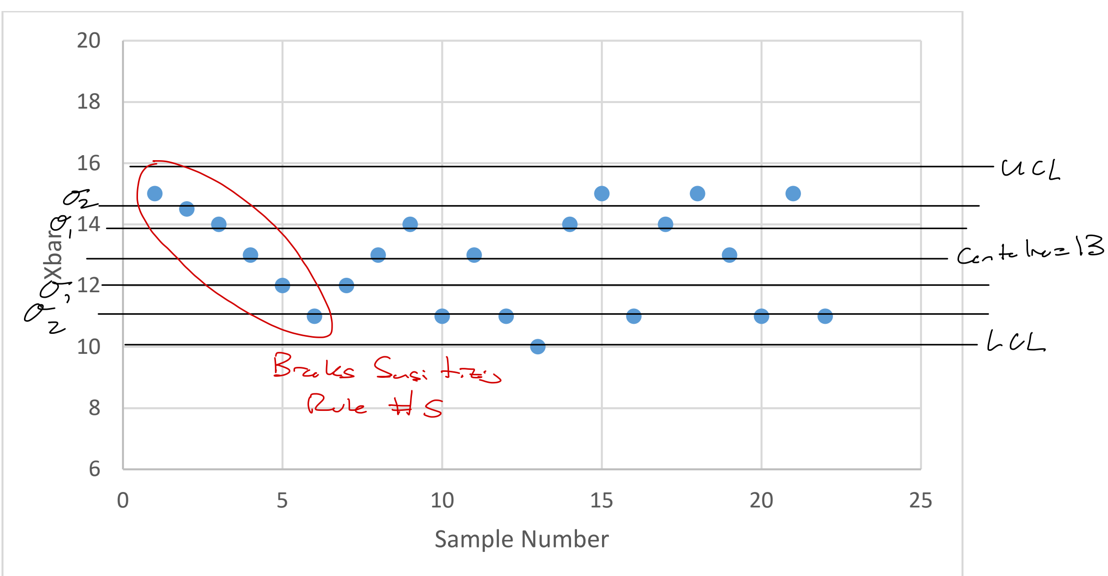
a. (8 Points) Write out the general formulas for and calculate the centerline as well as the three sigma control limits that should be used on the Xbar and R control chart

X_{bar}
$$\begin{cases} UCL = \bar{\bar{x}} + A_2\bar{R} = 13.0 + 0.373(7.18) = 15.7 \sim 16 \\ Center = \bar{\bar{x}} = 13.0 \sim 13 \\ LCL = \bar{\bar{x}} - A_2\bar{R} = 13.0 - 0.373(7.18) = 10.3 \sim 10 \end{cases}$$

R
$$\begin{cases} UCL = D_4\bar{R} = 1.864(7.18) = 13.4 \sim 13 \\ Center = \bar{R} = 7.18 \\ LCL = D_3\bar{R} = 0.136(7.18) = 0.98 \sim 1 \end{cases}$$

b. (3 points) In the next month, you take an additional 20 samples in a similar manner as before. Sketch in your sigma control limits (UCL and LCL) and centerline from part A, as well as the 1 sigma and 2 sigma warning limits on this graph of your data. After analyzing this plot, what is your recommendation?

In control, does not break rules 1-4

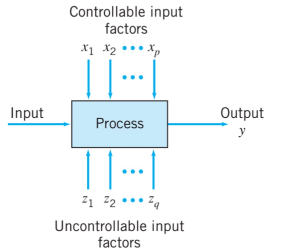


Process should be investigated as Sensitizing #5 is broken

Final Exam

Design of Experiments

- Improve Stage!
- Improvements are Project Driven
- Also useful for six sigma (product design)



Designed Experiment: A test or series of tests in which purposeful changes are made to input variables of a process so we may observe and identify changes in output

→ Objectives can be: (1) Determine which variables are most influential on y
(2) Determine where to set influential x's so y is near requirement/optimal
(3) Determine where to set x's so variability in y is small
(4) Determine where to set x's so impact of z's are minimal

→ Uses: (1) Identify variables to investigate in out of control process
(2) Determine optimum settings (starting or existing)
(3) Identify optimum spec limits

Guidelines for Designing Experiments

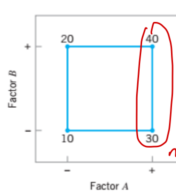
Pre-experimental planning [
1. Recognition of and statement of the problem
2. Choice of factors and levels] often done simultaneously, or in reverse order
3. Selection of the response variable]

4. Choice of experimental design
5. Performing the experiment
6. Data analysis
7. Conclusions and recommendations
8. Follow up runs confirming testing
9. Iteration

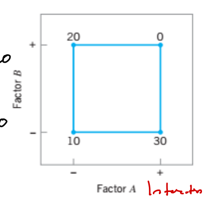
Factors: Controllable inputs to be varied
Levels: Specific levels @ which runs are made
Response Variable: Should provide useful info about process (help achieve goal)

Factorial Experiments

Factorial Experiment: All possible combinations of the levels of the factors investigated
Main Effect: The change in response produced by a change in the level of a primary factor
Interaction: Occurs when the difference in response between the levels of one factor is not the same at all levels of the others



Low B:
$A = 30 - 10 = 20$
High B:
$A = 40 - 20 = 20$



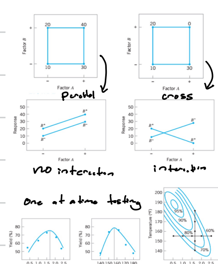
Low B:
$A = 30 - 10 = 20$
High B
$A = 0 - 20 = -20$

Run	A	B
1	+	+
2	+	−
3	−	+
4	−	−

Main Effect A:
$$A = \bar{y}_{A^+} - \bar{y}_{A^-} = \frac{30+40}{2} - \frac{20+10}{2} = 15$$

Importance of Interactions

Interactions can mask effects:
$$A = \frac{30-0}{2} - \frac{10+20}{2} = 0$$



Problems with one at a time testing:
- inefficient
- no assured you will get optimum results

2-Factor Experiments (ANOVA)

y = response

total # of observations = abn

Data for a Two-Factor Factorial Design

		Factor B			
		1	2	...	b
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$	...	$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$	...	$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮	⋮	⋮	⋮	⋮
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$	...	$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

Factors: A & B (2-factor)
Levels of Factor A = a
Levels of Factor B = b
Replicated n times

Effect of interaction between A and B

normal dist. random error

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

Overall mean effect

Effect of i^{th} level of Factor A

Effect of j^{th} level of factor B

Hypothesis Testing

Null & Alternative Hypothesis test for A, B, and AB:

$H_0: \tau_i = 0$ for all i $H_a: \tau_i \neq 0$ for any i

$H_0: \beta_j = 0$ for all j $H_a: \beta_j \neq 0$ for any j

$H_0: (\tau\beta)_{ij} = 0$ for all (ij) $H_a: (\tau\beta)_{ij} \neq 0$ for any ij

Total observations of ith level of factor A

$$y_{i..} = \sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{i..} = \frac{y_{i..}}{bn} \qquad i = 1, 2, \ldots, a \text{ — row average (A)}$$

Total observations of jth level of Factor B

$$y_{.j.} = \sum_{i=1}^{a}\sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{.j.} = \frac{y_{.j.}}{an} \qquad j = 1, 2, \ldots, b \text{ — column average (B)}$$

Total observations in the (ij)th cell

$$y_{ij.} = \sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{ij.} = \frac{y_{ij.}}{n} \qquad i = 1, 2, \ldots, a;\ j = 1, 2, \ldots, b \text{ — Cell average}$$

Grand total

$$y_{...} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{...} = \frac{y_{...}}{abn} \text{ — Grand Mean}$$

Test Statistic: Reference distribution is F-dist.

The ANOVA Table for a Two-Factor Factorial, Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A	SS_A	$a-1$	$MS_A = \frac{SS_A}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B	SS_B	$b-1$	$MS_B = \frac{SS_B}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n-1)$	$MS_E = \frac{SS_E}{ab(n-1)}$	
Total	SS_T	$abn-1$		

Rejection Criteria:

Reject if: $f_0 > f_\alpha$

Primer Type	Application Method: Dipping		Application Method: Spraying		$y_{i..}$
1	4.0, 4.5, 4.3	12.8	5.4, 4.9, 5.6	15.9	28.7
2	5.6, 4.9, 5.4	15.9	5.8, 6.1, 6.3	18.2	34.1
3	3.8, 3.7, 4.0	11.5	5.5, 5.0, 5.0	15.5	27.0
$y_{.j.}$	40.2		49.6		$89.8 = y_{...}$

$$\begin{cases} SS_T = SS_A + SS_B + SS_{AB} + SS_E \\ SS_T = \sum_i^a \sum_j^b \sum_k^c (y_{ijk} - \bar{y}_{...})^2 \\ SS_A = bn \sum_i^a (\bar{y}_{i..} - \bar{y}_{...})^2 \end{cases}$$

$$\begin{cases} SS_B = an \sum^b (\bar{y}_{.j.} - \bar{y}_{...})^2 \\ SS_{AB} = n \sum^a \sum^b (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2 \\ SS_E = \sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{ij.})^2 \end{cases}$$

Review – 2 Factor Experiments

Data for a Two-Factor Factorial Design

		Factor B: 1	2	…	b
	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$	…	$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
Factor A	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$	…	$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮	⋮	⋮	⋮	⋮
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$	…	$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

Factors : A and B

Levels of factor A = a

Levels of factor B = b

Replicated n times

Y = response, $y_{a,b,n}$

Number of observations = abn

Null and Alternative Hypothesis Tests for A, B and AB:

H_0: $T_i = 0$ (for all i), H_1: $T_i \neq 0$ (for some i)

H_0: $B_j = 0$ (for all j), , $H_1 : B_j \neq 0$ (for some j)

H_0: (TB)ij (for all ij), = 0, H_1: $(TB)_{ij} \neq 0$ (for some ij)

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

Test Statistic: (reference distribution is the F-distribution)

TABLE 13.3
The ANOVA Table for a Two-Factor Factorial, Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A	SS_A	$a-1$	$MS_A = \frac{SS_A}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B	SS_B	$b-1$	$MS_B = \frac{SS_B}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n-1)$	$MS_E = \frac{SS_E}{ab(n-1)}$	
Total	SS_T	$abn-1$		

Rejection Criteria:

Reject if $f_0 > f_\alpha$, DOF numerator, DOF denominator

$$y_{i..} = \sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk} \quad \bar{y}_{i..} = \frac{y_{i..}}{bn} \quad i = 1, 2, \ldots, a$$

$$y_{.j.} = \sum_{i=1}^{a}\sum_{k=1}^{n} y_{ijk} \quad \bar{y}_{.j.} = \frac{y_{.j.}}{an} \quad j = 1, 2, \ldots, b$$

$$y_{ij.} = \sum_{k=1}^{n} y_{ijk} \quad \bar{y}_{ij.} = \frac{y_{ij.}}{n} \quad i = 1, 2, \ldots, a;\ j = 1, 2, \ldots, b$$

$$y_{...} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk} \quad \bar{y}_{...} = \frac{y_{...}}{abn}$$

Data for a Two-Factor Factorial Design

		Factor B: 1	2	…	b
	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$	…	$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
Factor A	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$	…	$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮	⋮	⋮	⋮	⋮
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$	…	$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E$$

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{...})^2 = bn\sum_{i=1}^{a}(\bar{y}_{i..} - \bar{y}_{...})^2 + an\sum_{j=1}^{b}(\bar{y}_{.j.} - \bar{y}_{...})^2 + n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{ij.})^2$$

Example

Aircraft primer paints are applied to aluminum surfaces by two methods—dipping and spraying. The purpose of the primer is to improve paint adhesion; some parts can be primed using either application method. A team using the DMAIC approach has identified three different primers that can be used with both application methods. Three specimens were painted with each primer using each application method, a finish paint was applied, and the adhesion force was measured. The 18 runs from this experiment were run in random order. The resulting data are shown in Table 13.1. The circled numbers in the cells are the cell totals. The objective of the experiment was to determine which combination of primer paint and application method produced the highest adhesion force. It would be desirable if at least one of the primers produced high adhesion force regardless of application method, as this would add some flexibility to the manufacturing process. Alpha = 0.05

B

A: Primer Type	B: Application Method: Dipping		Application Method: Spraying		$y_{i..}$	$\bar{y}_{i..}$
1	4.0, 4.5, 4.3	12.8	5.4, 4.9, 5.6	15.9	28.7	4.7833
2	5.6, 4.9, 5.4	15.9	5.8, 6.1, 6.3	18.2	34.1	5.6833
3	3.8, 3.7, 4.0	11.5	5.5, 5.0, 5.0	15.5	27.0	4.5
$y_{.j.}$	40.2		49.6		$89.8 = y_{...}$	
$\bar{y}_{.j.}$	4.4667		5.511			

1) Parameter of Interest: Main effects and interactions of method of application and primer type

2) $H_0: \tau_1 = \tau_2 = \tau_3 = 0$

$\beta_1 = \beta_2 = 0$

$(\tau\beta)_{11} = (\tau\beta)_{12} = (\tau\beta)_{21} = (\tau\beta)_{22} = (\tau\beta)_{31} = (\tau\beta)_{32} = 0$

3) $H_a: \tau_i \neq 0$ for any i $(i = 1-3)$

$\beta_j \neq 0$ for any j $(j = 1-2)$

$(\tau\beta)_{ij} \neq 0$ for any ij $(i = 1-3, j = 1-2)$

4) Test Stats – See Summary

5) Rejection Criteria: Reject if: $F_0 > F_{\alpha, \text{DoF num}, \text{DoF den}}$

6) Calculations

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E$$

$$\bar{y}_{...} = \frac{89.8}{(3)(2)(3)} = 4.9889$$

$$SS_T = \sum_i^a \sum_j^b \sum_k^n (y_{ijk} - \bar{y}_{...})^2 = (4.0 - 4.9889)^2 + (4.5 - 4.9889)^2 + \underbrace{\ldots}_{\text{all points}} + (5.0 - 4.9889)^2 = 10.718$$

$$SS_A = bn \sum_i^a (\bar{y}_{i..} - \bar{y}_{...})^2 = 2(3)\left[(4.7823 - 4.9889)^2 + (5.6837 - 4.9889)^2 + (4.5 - 4.9889)^2\right] = 4.5881$$

$$SS_B = an \sum_j^b (\bar{y}_{.j.} - \bar{y}_{...})^2 = 3(3)\left[(4.4667 - 4.9889)^2 + (5.511 - 4.9889)^2\right] = 4.90889$$

$$SS_{AB} = n \sum_i^a \sum_j^b (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2$$

$$= 3\left[(4.266 - 4.7837 - 4.667 + 4.9889)^2 + (\text{all cells})\right] = 0.2411$$

$$SS_E = \sum_i^a \sum_j^b \sum_k^n (y_{ijk} - \bar{y}_{ij.})^2 = (4.0 - 4.266)^2 + (4.5 - 4.266)^2 + (\text{all data points}) = 0.98667$$

	Calcs		F_0	F_{crit}	
A	$F_0 = \dfrac{SS_A/a-1}{SS_E/ab(n-1)}$	$= \dfrac{45811/3-1}{.98667/3(2)(3-1)}$	$=$ 27.858	$F_{0.05,2,12} = 3.89$	✓ yes
B	$F_0 = \dfrac{SS_B/b-1}{SS_E/ab(n-1)}$	$= \dfrac{4.90829/2-1}{.98667/12}$	$=$ 59.7027	$F_{0.05,1,12} = 4.75$	✓ yes
AB	$F_0 = \dfrac{SS_{AB}/(a-1)(b-1)}{SS_E/ab(n-1)}$	$= \dfrac{0.2411/2(1)}{0.98667/12}$	$=$ 1.4667	$F_{0.05,2,12} = 3.89$	✗ no

Primer Type	Application Method: Dipping		Spraying		$y_{i..}$
1	4.0, 4.5, 4.3	12.8	5.4, 4.9, 5.6	15.9	28.7
2	5.6, 4.9, 5.4	15.9	5.8, 6.1, 6.3	18.2	34.1
3	3.8, 3.7, 4.0	11.5	5.5, 5.0, 5.0	15.5	27.0
$y_{.j.}$	40.2		49.6		89.8 = y

TABLE 13.3
The ANOVA Table for a Two-Factor Factorial, Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A	SS_A	$a-1$	$MS_A = \frac{SS_A}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B	SS_B	$b-1$	$MS_B = \frac{SS_B}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n-1)$	$MS_E = \frac{SS_E}{ab(n-1)}$	
Total	SS_T	$abn-1$		

7) Conclusions (assuming assumptions are valid)
We conclude that primer type & application method have a significant impact on adhesion force.
There isn't enough evidence to say there is an interaction between application method and primer type.
8) Plots – Main effects & Interaction
9) Residual Analysis (assumptions)
10) Model generation

Plots & Minitab and Excel

Interaction Plots: will plot average response at each combination of factor settings vs. factor setting
Main Effects Plots: will plot average response at each

Excel

1) Set up your data table
2) Calculate cell averages, row and column averages and grand averages
3) Calculate sum of squares terms (use the space!)
   a. SST and SSE – a term for each data point
   b. SSA and SSB – a term for each row or column average
   c. SSAB – a term for each averaged cell
4) Calculate Fo for each term (A, B, AB) and compare with critical values
5) Plot interaction and main effects plots
   a. Interaction plots – cell averages at each level
   b. Main effects plots – column or row averages at each level

Minitab: See Document on Canvas

Checking Assumptions

TABLE 13.6
Residuals for the Aircraft Primer Paint Experiment

Primer Type	Application Method: Dipping	Spraying
1	−0.26, 0.23, 0.03	0.10, −0.40, 0.30
2	0.30, −0.40, 0.10	−0.26, 0.03, 0.23
3	−0.03, −0.13, 0.16	0.34, −0.17, −0.17

$\varepsilon_{ijk} = y_{ijk} - \hat{y}_{ijk}$ ← predicted
residual
$= y_{ijk} - \bar{y}_{ij\cdot}$ ← cell avg

Assumptions and Plots:
1) The errors are normally distributed (plot residuals on normal probability plot)
2) The errors are independently distributed (plot residuals versus run order)
3) Each factor level has the same variance (residuals versus factor levels, both A and B levels in this case, and predicted values)

Checking Assumptions – Cont

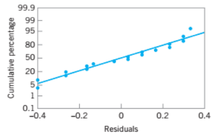


FIGURE 13.13 Normal probability plot of the residuals from Example 13.5.

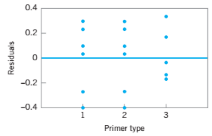


FIGURE 13.14 Plot of residuals versus primer type.

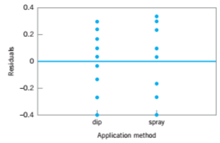


FIGURE 13.15 Plot of residuals versus application method.

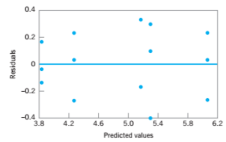


FIGURE 13.16 Plot of residuals versus predicted values.

Model Generation

1) Conduct the DOE analysis outlined for the full model (ANOVA, residuals and plots)
2) Based on results, consider removing terms from the model that are above the alpha level in the ANOVA table
3) Repeat 2-3 till all of the factors are significant
4) Generate final model and contour plots (if applicable)

Process is called backward elimination (automatically done via):

Stat-> DOE-> Factorial > Analyze Factorial Design > Select "stepwise" button > Select "backward elimination"

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \quad \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,b \\ k = 1,2,\ldots,n \end{cases}$$

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \quad \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,b \\ k = 1,2,\ldots,n \end{cases}$$

Y = grand mean + (grand mean – average at t1) t1 +
(grand mean – average at t2) t2 +
(grand mean – average at t3) t3 +
(grand mean – average at B1) B1 +
(grand mean – average at B2) B2 +

(Note: t1, t2, B1.. And so on… these are 0 or 1 depending on the conditions)

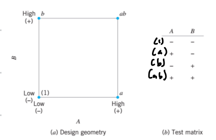
If interactions were present, it would be each combination of factor levels, with coefficients given by: (grand mean + cell average – row average A – column average B)

The 2^k Experiment

2^k Factorial Design - A designed experiment with k factors at 2 levels
- Forms the basis for many other DoEs
- Simplified Analyses
- # of runs = $n2^k$ — n = replicates, k = # of factors

ex: If an experiment has 4 factors, replicated twice, how many runs are there

# of runs = $2(2^4) = 2 \cdot 16 = 32$



- Each corner is a run
- Lower case letter = high setting
- no letter = low setting
- (1) = all low settings

Let (1), a, b, and ab = totals of all n observations taken at the design points

→ Main Effect of A:

$$A = \bar{y}_{A+} - \bar{y}_{A-}$$

$$= \frac{a+ab}{2n} - \frac{b+(1)}{2n} = \frac{1}{2n}\big(\underbrace{a+ab-b-(1)}_{\text{contrast}}\big)$$

→ Interaction AB

$$AB = \frac{1}{2n}\left(ab+(1)-a-b\right)$$

→ Main Effect of B:

$$B = \bar{y}_{B+} - \bar{y}_{B-}$$

$$= \frac{b+ab}{2n} - \frac{a+(1)}{2n} = \frac{1}{2n}\left(b+ab-a-(1)\right)$$

Contrasts in 2^k Factorial Design

TABLE 13.7
Signs for Effects in the 2^2 Design

Run		Factorial Effect I	A	B	AB
1	(1)	+	−	−	+
2	a	+	+	−	−
3	b	+	−	+	−
4	ab	+	+	+	+

Contrast: The total of all n observations at each combination of factor levels, multiplied by the corresponding sign of that factor or interaction

$$\text{Effect} = \frac{\text{Contrast}}{n2^{k-1}}$$

$$SS = \frac{(\text{Contrast})^2}{n2^k}$$

ex: Contrast A = $a + ab - b - (1)$
Contrast AB = $ab + (1) - a - b$ ↑ totals

Sum of squares for this example:

Error DoF: (Total DoF) - (all Main effects) / (interactions DoF)

$$SS_A = \frac{[a+ab-b-(1)]^2}{4n}$$
$$SS_B = \frac{[b+ab-a-(1)]^2}{4n} \quad (13.15)$$
$$SS_{AB} = \frac{[ab+(1)-a-b]^2}{4n}$$

2^k experiment: total DoF = $n2^k - 1$

DoF: A = 1
B = 1
AB = 1
... (each main effect/interaction)

$$SS_T = \sum_{\text{all points}} \left(\text{each data point} - \bar{y}_{...}\right)^2$$ ($\bar{y}_{...}$: grand mean)

SS_E = can be same as before if you have replicated

Example

A router is used to cut registration notches in printed circuit boards. The average notch dimension is satisfactory, and the process is in statistical control (see the $\bar{x}$ and R control charts in Figure 13.18), but there is too much variability in the process. This excess variability leads to problems in board assembly. The components are inserted into the board using automatic equipment, and the variability in notch dimension causes improper board registration. As a result, the auto-insertion equipment does not work properly. How would you improve this process?

Since the process is in statistical control, the quality improvement team assigned to this project decided to use a designed experiment to study the process. The team considered two factors: bit size (A) and speed (B). Two levels were chosen for each factor (bit size A at $\frac{1}{16}''$ and $\frac{1}{8}''$ and speed B at 40 rpm and 80 rpm), and a 2^2 design was set up. Since variation in notch dimension was difficult to measure directly, the team decided to measure it indirectly. Sixteen test boards were instrumented with accelerometers that allowed vibration on the (X, Y, Z) coordinate axes to be measured. The resultant vector of these three components was used as the response variable. Since vibration at the surface of the board when it is cut is directly related to variability in notch dimension, reducing vibration levels will also reduce the variability in notch dimension.

Four boards were tested at each of the four runs in the experiment, and the resulting data are shown in Table 13.8.

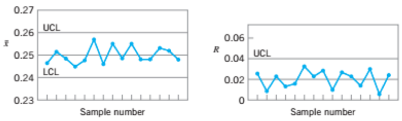


TABLE 13.8
Data from the Router Experiment

AB	Run		Factors A	Factors B	Vibration				Total
+	1	(1)	−	−	18.2	18.9	12.9	14.4	64.4
−	2	a	+	−	27.2	24.0	22.4	22.5	96.1
−	3	b	−	+	15.9	14.5	15.1	14.2	59.7
+	4	ab	+	+	41.0	43.9	36.3	39.9	161.1

1) Parameters of Interest: Main Effects and interactions of bit size (A) and speed on variation (B)

2) H_0: $A=0$, $B=0$, $AB=0$

3) H_1: $A\neq0$, $B\neq0$, $AB\neq0$

$n=4$
$k=2$
DoF: $A=B=AB=1$
Total $= n2^k - 1 = 4\times4-1 = 15$
Error $= 12$

Effects:

$$A = \frac{1}{2n}\left(a+ab-b-(1)\right) = \frac{1}{2(4)}\left(96.1+161.1-59.7-64.4\right)$$

$$= \frac{133.1}{8} = 16.64$$ — contrast

$$B = \frac{1}{2n}\left(b+ab-a-(1)\right) = \frac{1}{8}\left(59.7+161.1-96.1-64.4\right)$$

$$= \frac{60.3}{8} = 7.54$$

$$AB = \frac{1}{2n}\left(ab+(1)-a-b\right) = \frac{1}{8}\left(161.1+64.4-96.1-59.7\right)$$

$$= \frac{69.7}{8} = 8.71$$

$$SS_A = \frac{133.1^2}{4(4)} = 1107 \qquad SS_{AB} = \frac{69.7^2}{4(4)} = 304$$

$$SS_B = \frac{60.3^2}{4(4)} = 227$$

$$SS_E = SS_T - SS_A - SS_B - SS_{AB}$$
$$= 1710 - 1107 - 227 - 304$$
$$= 71.7$$

Reject if: $F_0 > F_{\alpha,\ \text{DoF Top},\ \text{DoF Bottom}}$

Top: $\dfrac{SS_x}{DoF_x}$ Bottom: $\dfrac{SS_E}{DoF_E}$

TABLE 13.9
Analysis of Variance for the Router Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-value
Bit size (A)	1,107.226	1	1,107.226	185.25	1.17×10^{-8}
Speed (B)	227.256	1	227.256	38.03	4.82×10^{-5}
AB	303.631	1	303.631	50.80	1.20×10^{-5}
Error	71.723	12	5.977		
Total	1,709.836	15			

Conclusions

Both bit size and speed have a significant impact on vibration, and there is a significant interaction between bitsize and speed.

Model & Residuals

TABLE 13.8
Data from the Router Experiment

Run		Factors A	Factors B	Vibration				Total
1	(1)	−	−	18.2	18.9	12.9	14.4	64.4
2	a	+	−	27.2	24.0	22.4	22.5	96.1
3	b	−	+	15.9	14.5	15.1	14.2	59.7
4	ab	+	+	41.0	43.9	36.3	39.9	161.1

$$y = \beta_0 + \beta_1x_1 + \beta_2x_2 + \dots \beta_{12}x_1x_2 + \varepsilon$$

y = response variable
x's = Coded variables $(-1, 1)$
= interaction terms are products
β terms = coefficients for model $\rightarrow \dfrac{\text{Effect}}{2}$
β_0 = grand average

For Our above example:

$$\hat{y} = \underbrace{23.83}_{\beta_0\text{ grand mean}} + \underbrace{\left(\frac{16.64}{2}\right)}_{\beta_1\text{ main effect } A}x_1 + \underbrace{\left(\frac{7.54}{2}\right)}_{\beta_2\text{ main effect } B}x_2 + \underbrace{\left(\frac{8.71}{2}\right)}_{\text{effect of interaction } AB}x_1x_2$$

$\varepsilon_{ij} = y_{ij} - \bar{y}_{ij}$

residual = data point − predicted value

ex: (small bit, low speed) → $(-1)(-1)$

Estimate: $\hat{y} = 23.83 + \frac{(16.64)}{2}(-1) + \frac{7.54}{2}(-1) + \left(\frac{8.71}{2}\right)(-1)(-1) = 16.64$

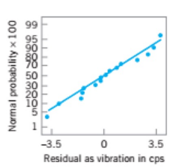


FIGURE 13.19 Normal probability plot, Example 13.6.

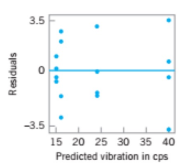


FIGURE 13.20 Plot of residuals versus $\hat{y}$, Example 13.6.

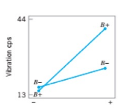


FIGURE 13.21 AB interaction plot.

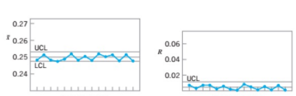
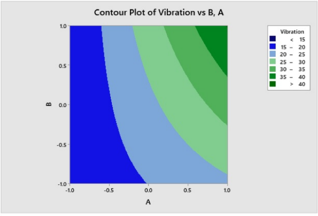
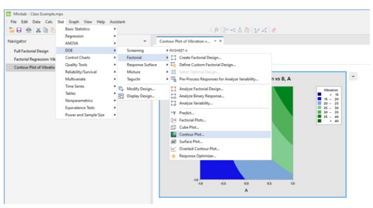


FIGURE 13.22 $\bar{x}$ and R charts for the router process after the experiment.

Concern: if speed (B) is too slow will slow down production – this tells us that we can have high B if we have low A



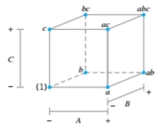


Minitab: The directions for this are similar to the directions before, **except you should select a 2-level factorial** in the menu. You can also generate a contour plot with the model to help you determine the optimum settings.

Summary

- 2^k is a special DOE that makes it easy to calculate main effects, interactions and develop a model and number of runs
- Same step by step process should be used as in Lecture 20 (including plotting residuals, main effects and interactions)
- Differences are noted in DOF vs. the general 2 factor factorial from last lecture

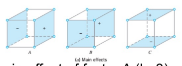
Example for 2^3 $= 8$



	A	B	C
(1)	−	−	−
a	+	−	−
b	−	+	−
ab	+	+	−
c	−	−	+
ac	+	−	+
bc	−	+	+
abc	+	+	+

(a) Design geometry (b) Test matrix

FIGURE 13.23 The 2^3 factorial design.



(a) Main effects

For main effect of factor A (k=3)

$$A = \bar{y}_{A^+} - \bar{y}_{A^-} = \frac{1}{4n}[a + ab + ac + abc - b - c - bc - (1)] \quad (13.16)$$

Generally (for all 2^k):

Analysis Procedure for Factorial Designs

1. Estimate the factor effects + SS
2. Form preliminary model
3. Test for significance of factor effects
4. Analyze residuals
5. Refine model, if necessary
6. Interpret results

Number of combinations = 2^k

$$\text{Effect} = \frac{\text{Contrast}}{n2^{k-1}} \qquad SS = \frac{(\text{Contrast})^2}{n2^k}$$

Regression Coefficients = Effect/2

DOF: 1 for each term (A, B, C, AB… ABC)
Total: $n2^k - 1$

SSterm/1 / SSe/DOF error = Fo

Model:

$$y = \mu + A + B + C + AB + AC + BC + ABC + \varepsilon$$

Example for 2^3

EXAMPLE 13.7 A 2^3 Factorial Design

An experiment was performed to investigate the surface finish of a metal part. The experiment is a 2^3 factorial design in the factors feed rate (A), depth of cut (B), and tool angle (C), with $n = 2$ replicates. Table 13.12 presents the observed surface-finish data for this experiment, and the design is shown graphically in Figure 13.25. Analyze and interpret the data from this experiment.

$n=2$ $k=3$

TABLE 13.12 Surface-Finish Data for Example 13.7

Run		Design Factors A	B	C	Surface Finish	Totals
1	(1)	−1	−1	−1	9, 7	16
2	a	1	−1	−1	10, 12	22
3	b	−1	1	−1	9, 11	20
4	ab	1	1	−1	12, 15	27
5	c	−1	−1	1	11, 10	21
6	ac	1	−1	1	10, 13	23
7	bc	−1	1	1	10, 8	18
8	abc	1	1	1	16, 14	30

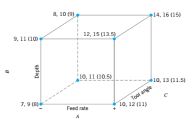


FIGURE 13.25 2^3 design for the surface-finish experiment in Example 13.7 (the numbers in parentheses are the average responses at each design point).

$B = 1.625$ $\quad SS_B = 10.5625$
$C = 0.875$ $\quad SS_C = 3.0625$
$AB = 1.375$ $\quad SS_{AB} = 7.5625$
$AC = 0.125$ $\quad SS_{AC} = 0.0625$
$BC = -0.625$ $\quad SS_{BC} = 1.5625$
$ABC = 1.125$ $\quad SS_{ABC} = 5.0625$

TABLE 13.13 Analysis of Variance for the Surface-Finish Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-value
A	45.5625	1	45.5625	18.69	2.54×10^{-3}
B	10.5625	1	10.5625	4.33	0.07
C	3.0625	1	3.0625	1.26	0.29
AB	7.5625	1	7.5625	3.10	0.12
AC	0.0625	1	0.0625	0.03	0.88
BC	1.5625	1	1.5625	0.64	0.45
ABC	5.0625	1	5.0625	2.08	0.19
Error	19.5000	8	2.4375		
Total	92.9375	15			

Reject if P-value $< \alpha = 0.15$

$$A = \frac{1}{4n}\left[a + ab + ac + abc - (1) - b - c - bc\right]$$
$$= \frac{1}{4(2)}\left[22 + 27 + 23 + 30 - 20 - 21 - 18 - 16\right]$$
$$= \frac{1}{8}(27) = 3.375$$

contrast

$$SS_A = \frac{\text{Contrast}^2}{n2^k} = \frac{27^2}{2(8)} = 45.56$$

Repeat for all others (B, AB, C, ...)

If we pick an alpha level of 0.15, what model would you use to represent this process? (Given the grand average is 11.06). Also, what surface finish would you predict at the lowest settings?

$$SS_T = \sum(\text{each point} - \text{grand mean})^2$$

$$SS_E = SS_T - (\text{all SS terms})$$

A, B, AB are significant, should redo with only these terms

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \varepsilon$$

grand mean A B AB

Coded variables

$$\Rightarrow \hat{y} = 11.065 + 1.6875 x_1 + 0.805 x_2 + 0.6875 x_1 x_2$$

final model of figs

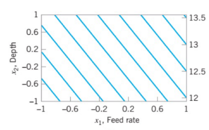


FIGURE 13.28 (a) Response surface for the model $\hat{y} = 11.0625 + 1.6875x_1 + 0.8125x_2$. (b) The contour plot.

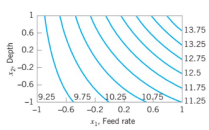


FIGURE 13.29 (a) Response surface for the model $\hat{y} = 11.0625 + 1.6875x_1 + 0.8125x_2 + 0.6875x_1x_2$. (b) The contour plot.

Minitab: The directions for this are similar to the directions before, **except you should select a 2-level factorial** in the menu. You can also generate a contour plot with the model to help you determine the optimum settings (see handout)

For residuals and predicted values go to analyze DOE, storage, and select residuals and fits by checking their boxes Also see Canvas document

New model: $SS_E = SS_{pure\,error} + SS_A$

Check on canvas to see if missing

Single Replicate Example

Sparsity of effects principle: The system is usually dominated by the main effects and low order interactions (two-way) $K \geq 4$ or 5, run only one replicate

EXAMPLE 13.8 Characterizing a Plasma Etching Process

An article in *Solid State Technology* ("Orthogonal Design for Process Optimization and Its Application in Plasma Etching," May 1987, pp. 127–132) describes the application of factorial designs in developing a nitride etch process on a single-wafer plasma etcher. The process uses C_2F_6 as the reactant gas. It is possible to vary the gas flow, the power applied to the cathode, the pressure in the reactor chamber, and the spacing between the anode and the cathode (gap). Several response variables would usually be of interest in this process, but in this example we will concentrate on etch rate for silicon nitride. Perform an appropriate experiment to characterize the performance of this etching process with respect to the four process variables.

Design Factor Level	Gap A (cm)	Pressure B (m Torr)	C_2F_6 Flow C (SCCM)	Power D (W)
Low (−)	0.80	450	125	275
High (+)	1.20	550	200	325

The 2^4 Design for the Plasma Etch Experiment

Run	A (Gap)	B (Pressure)	C (C_2F_6 flow)	D (Power)	Etch Rate (Å/min)
1	−1	−1	−1	−1	550
2	1	−1	−1	−1	669
3	−1	1	−1	−1	604
4	1	1	−1	−1	650
5	−1	−1	1	−1	633
6	1	−1	1	−1	642
7	−1	1	1	−1	601
8	1	1	1	−1	635
9	−1	−1	−1	1	1,037
10	1	−1	−1	1	749
11	−1	1	−1	1	1,052
12	1	1	−1	1	868
13	−1	−1	1	1	1,075
14	1	−1	1	1	860
15	−1	1	1	1	1,063
16	1	1	1	1	729

Contrast Constants for the 2^4 Design

Run		A	B	AB	C	AC	BC	ABC	D	AD	BD	ABD	CD	ACD	BCD	ABCD
1	(1)	−	−	+	−	+	+	−	−	+	+	−	+	−	−	+
2	a	+	−	−	−	−	+	+	−	−	+	+	+	+	−	−
3	b	−	+	−	−	+	−	+	−	+	−	+	+	−	+	−
4	ab	+	+	+	−	−	−	−	−	−	−	−	+	+	+	+
5	c	−	−	+	+	−	−	+	−	+	+	−	−	+	+	−
6	ac	+	−	−	+	+	−	−	−	−	+	+	−	−	+	+
7	bc	−	+	−	+	−	+	−	−	+	−	+	−	+	−	+
8	abc	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−
9	d	−	−	+	−	+	+	−	+	−	−	+	−	+	+	−
10	ad	+	−	−	−	−	+	+	+	+	−	−	−	−	+	+
11	bd	−	+	−	−	+	−	+	+	−	+	−	−	+	−	+
12	abd	+	+	+	−	−	−	−	+	+	+	+	−	−	−	−
13	cd	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+
14	acd	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
15	bcd	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
16	abcd	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

$$SS = \frac{(\text{Contrast})^2}{n2^k} \qquad (13.24)$$

SSABC = -125^2/16 = 976.5

SSABD = 33^2/16 = 68

SSACD = 45^2/16 = 126

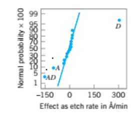
SSBCD = -203^2/16 = 2575

SSABCD = -321^2/16 = 6440

■ TABLE 13.17
Analysis of Variance for the Plasma Etch Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A	41,310.563	1	41,310.563	20.28
B	10.563	1	10.563	< 1
C	217.563	1	217.563	< 1
D	374,850.063	1	374,850.063	183.99
AB	248.063	1	248.063	< 1
AC	2,475.063	1	2,475.063	1.21
AD	94,402.563	1	99,402.563	46.34
BC	7,700.063	1	7,700.063	3.78
BD	1.563	1	1.563	< 1
CD	18.063	1	18.063	< 1
Error	10,186.815	5	2,037.363	
Total	531,420.938	15		

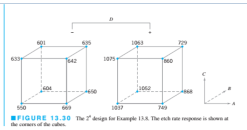
$A = -101.625$ $AD = -153.625$
$B = -1.625$ $BD = -0.625$
$AB = -7.875$ $ABD = 4.125$
$C = 7.375$ $CD = -2.125$
$AC = -24.875$ $ACD = 5.625$
$BC = -43.875$ $BCD = -25.375$
$ABC = -15.625$ $ABCD = -40.125$
$D = 306.125$



■ FIGURE 13.31 Normal probability plot of effects, Example 13.8.

The 2^4 Design for the Plasma Etch Experiment

Run	A (Gap)	B (Pressure)	C (C_2F_6 flow)	D (Power)	Etch Rate (Å/min)
1	−1	−1	−1	−1	550
2	1	−1	−1	−1	669
3	−1	1	−1	−1	604
4	1	1	−1	−1	650
5	−1	−1	1	−1	633
6	1	−1	1	−1	642
7	−1	1	1	−1	601
8	1	1	1	−1	635
9	−1	−1	−1	1	1,037
10	1	−1	−1	1	749
11	−1	1	−1	1	1,052
12	1	1	−1	1	868
13	−1	−1	1	1	1,075
14	1	−1	1	1	860
15	−1	1	1	1	1,063
16	1	1	1	1	729



■ FIGURE 13.30 The 2^4 design for Example 13.8. The etch rate response is shown at the corners of the cubes.

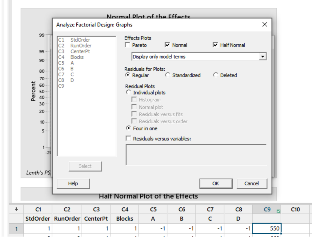
$$\text{Effect} = \frac{\text{Contrast}}{n2^{k-1}} \qquad (13.23)$$

Contrast Constants for the 2^4 Design

Run		A	B	AB	C	AC	BC	ABC	D	AD	BD	ABD	CD	ACD	BCD	ABCD
1	(1)	−	−	+	−	+	+	−	−	+	+	−	+	−	−	+
2	a	+	−	−	−	−	+	+	−	−	+	+	+	+	−	−
3	b	−	+	−	−	+	−	+	−	+	−	+	+	−	+	−
4	ab	+	+	+	−	−	−	−	−	−	−	−	+	+	+	+
5	c	−	−	+	+	−	−	+	−	+	+	−	−	+	+	−
6	ac	+	−	−	+	+	−	−	−	−	+	+	−	−	+	+
7	bc	−	+	−	+	−	+	−	−	+	−	+	−	+	−	+
8	abc	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−
9	d	−	−	+	−	+	+	−	+	−	−	+	−	+	+	−
10	ad	+	−	−	−	−	+	+	+	+	−	−	−	−	+	+
11	bd	−	+	−	−	+	−	+	+	−	+	−	−	+	−	+
12	abd	+	+	+	−	−	−	−	+	+	+	+	−	−	−	−
13	cd	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+
14	acd	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
15	bcd	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
16	abcd	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

$A = -101.625$ $AD = -153.625$
$B = -1.625$ $BD = -0.625$
$AB = -7.875$ $ABD = 4.125$
$C = 7.375$ $CD = -2.125$
$AC = -24.875$ $ACD = 5.625$
$BC = -43.875$ $BCD = -25.375$
$ABC = -15.625$ $ABCD = -40.125$ $= 10187$
$D = 306.125$

$SS_E = SS_{ABC} + SS_{ABD} + SS_{ACD} + SS_{BCD} + SS_{ABCD} = SS_{\text{Lack of fit}}$

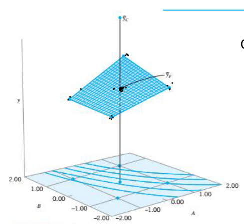


You can ask Minitab to create a normal plot of the effects in the Graphs menu (Stat->DOE->Factorial->Analyze Factorial) – or calculate effects by hand and input into Minitab and ask it to do a normal probability plot!

Addition of Centerpoints

quadratics!!

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \varepsilon$$



■ FIGURE 13.34 A 2^2 design with center points.

Centerpoints:

Points at $x_i = 0$ for all factors ($K = 1, 2, 3, \ldots$) with n_c replicates

- Tests for curvature
- Independent estimate of error

■ TABLE 13.18
The 2^4 Design for the Plasma Etch Experiment

Run	A (Gap)	B (Pressure)	C (C_2F_6 flow)	D (Power)	Etch Rate (Å/min)
1	−1	−1	−1	−1	550
2	1	−1	−1	−1	669
3	−1	1	−1	−1	604
4	1	1	−1	−1	650
5	−1	−1	1	−1	633
6	1	−1	1	−1	642
7	−1	1	1	−1	601
8	1	1	1	−1	635
9	−1	−1	−1	1	1,037
10	1	−1	−1	1	749
11	−1	1	−1	1	1,052
12	1	1	−1	1	868
13	−1	−1	1	1	1,075
14	1	−1	1	1	860
15	−1	1	1	1	1,063
16	1	1	1	1	729
17	0	0	0	0	706
18	0	0	0	0	764
19	0	0	0	0	780
20	0	0	0	0	761

$n_c = 4$

$$SS_{\text{pure quadratic}} = \frac{n_F n_C (\bar{y}_F - \bar{y}_C)^2}{n_F + n_C}$$

$\bar{y}_F$ = average of runs at factorial points

$\bar{y}_C$ = average of runs at centerpoints

- If $\bar{y}_F - \bar{y}_C$ is small, lie on or near the plane passing through factorial points (no curvature)
- If $\bar{y}_F - \bar{y}_C$ is large, curvature more likely

For our example ($n_c = 4$)

Pure Error: Sample variance from centerpoints

$= \text{Sum}\{(\text{centerpoint} - \text{average of centerpoints})^2\}$

DoF $= n_c - 1$ ← over each centerpoints

$SS_{\text{pure error}} = (706 - 752.28)^2 + (764 - 752.28)^2 + (780 - 752.25)^2 + (761 - 752.25)^2$

$= 3122.7$

$\bar{y}_F = 776.062$ $\bar{y}_C = 752.75$

$$SS_{\text{pure quadratic}} = \frac{n_F n_C (\bar{y}_F - \bar{y}_C)^2}{n_{F+C}}$$

$$= \frac{16(4)(776.06 - 752.25)^2}{16 + 4}$$

$= 1739.1$

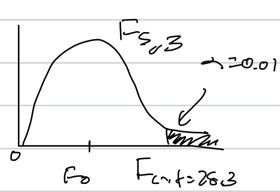
DoF $= n_c$

$SS_{\text{Lack of fit}} = 10187$ higher order terms

are higher terms significant?

$$F_0 = \frac{SS_{\text{Lack of fit}} / \text{DoF}}{SS_{\text{pure error}} / \text{DoF}} = \frac{10187/5}{3122.7/3} = 1.96$$

$F_{crit} = F_{0.01,5,3} = 28.23$



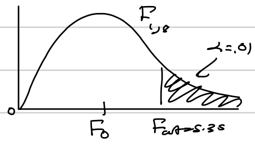
Conclusions: We don't have enough evidence to say the higher order terms is significant, so we can add them to the estimate of error.

Example Contd

$SS_{Residual} = SS_{pure\ error} + SS_{lack\ of\ fit}$
$= 10187 + 3122$

$F_{crit} = F_{0.01,1,8} = 5.35$

$$F_0 = \frac{SS_{curvature}/DoF}{SS_{Residual}/DoF} = \frac{12391}{\frac{10187+3122}{5+3}} = 1.05$$



Fail to reject Ho → $H_0: \beta_{11} \dots = 0$

quadratic terms

We do not have enough evidence to say curvature is significant.

What Next?

If Curvature is Significant:

- linear assumption invalid for 2^k
- Add more designed experiment points (Ch. 14)

Central Composite Design

Review:

Pure error = estimate from center points replicates

Lack-of-fit error = SS of all of the lower order terms you assume to be insignificant

Residual error = Error used to test curvature is combination of lack-of-fit and and pure error
Pure error + lack of fit
(Residual Error just called error in Minitab – is your SSE term)

Benefits of Centerpoints:

Gives you an estimate of Pure error!

1) You can test for curvature
2) You can test sparcity of effects principle
3) You don't need many centerpoints

Fractional Factorial Concepts

Plus and Minus Signs for the 2^3 Factorial Design

	Factorial Effect							
Run	***I***	***A***	***B***	***C***	***AB***	***AC***	***BC***	***ABC***
a	+	+	–	–	–	–	+	+
b	+	–	+	–	–	+	–	+
c	+	–	–	+	+	–	–	+
abc	+	+	+	+	+	+	+	+
ab	+	+	+	–	+	–	–	–
ac	+	+	–	+	–	+	–	–
bc	+	–	+	+	–	–	+	–
(1)	+	–	–	–	+	+	+	–

- used when we have many factors
- Screening experiments
- Aliasing (confounding): When the estimate of one effect includes the influence of one or more other effects
- If you drop a factor in a half factorial → design turns into a full