



Chapters 1 & 2 - Concepts

Key Terminology

Six Sigma: A data-driven management strategy

Quality: Fitness for use

Two aspects: a) Quality of design - high quality?

b) Quality of conformity - is the high quality consistent?

Characteristics or "dimensions of quality" that affect fitness for use as a product

(i) Performance performs required job (ii) Aesthetics product appearance

(iii) Reliability how often will it fail (iv) Features what the product does

(v) Durability lifetime → will it meet it (vi) Perceived Quality Reputation

(vii) Scrutability easy to repair (viii) Conformance to Standards

Producing a Service... list those dimensions of quality

(i) Responsiveness time to reply to requests

(ii) Professionalism knowledge, timeliness

(iii) Attentiveness caring, personalized attention

Hidden Factors: The part of the plant the customer doesn't see which affects w/bd quality

Quality Engineering: Consists of activities to ensure that ① that quality characteristics are at desired levels, and ② variability is at its minimum

→ **Quality Characteristics:** Important elements of a product/service to the customer

Variability: Describes or quantifies differences within a set of data or parts or c2qc

→ **Statistical Methods:** Analysis, interpretation, and presentation of numerical data

→ **Spec. Rating:** Desired measurements for quality characteristics This is your target value

Nonconformity: Failure to meet spec

Defect: Nonconforming that hurts usc

Variability

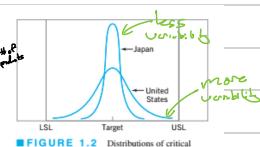
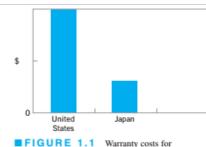
• Quality is inversely proportional to variability

◦ Variability will impact all quality dimensions

• Variability leads to greater cost & waste

example to right compares domestic vs. imported car quality

• Why variability leads to more cost for a company



Quality Costs

Prevention Costs	Internal Failure Costs
Quality planning and engineering	Scrap
New products review	Rework
Product/process design	Restest
Process control	Failure analysis
Burn-in 6-Sigma comes 20 or less	Downtime
Training & Process	Yield losses
Quality data acquisition and analysis	Downgrading (off-specing)
Appraisal Costs	External Failure Costs
Inspection and test of incoming material	Complaint adjustment
Product inspection and test	Returned product/material
Materials and services consumed	Warranty charges
Maintaining accuracy of test equipment	Liability costs
	Indirect costs

- Products found in plant outside of specifications result in higher costs and waste
- Prevention & Appraisal costs are cheaper costs and result in more consistent products and cheaper costs down the line

Internal Failure & External Failure costs >> Prevention & appraisal costs!

Example: Cost of Variance ← Find manufacturing cost per good product

Illustrative Example: We are making 100 mechanical components per day where 75% of the parts conform to specifications, and the other 25% are non-conforming. Of the 25% of nonconforming, 60% can be reworked and the other 40% are scraped. Each part costs \$20 to manufacture and an additional \$4 to be reworked.

$$\frac{\text{cost}}{\text{good}} = \frac{\text{cost to make} (\# \text{ made total}) + \text{cost to rework} (\# \text{ reworked})}{\# \text{ of goods sold}}$$

$$= \frac{\$20(100) + \$4(100)(0.25 \cdot 0.60)}{70 + 15} = \$22.69$$

Now, the Benefit of Improvement ← Find manufacturing cost per good product

Illustrative Example: We improve the previous process such that now 95% of the parts conform to specifications, and the other 5% are non-conforming. Similar to before, of the 5% of nonconforming, 60% can be reworked and the other 40% are scraped. Each part costs \$20 to manufacture and an additional \$4 to be reworked.

$$\frac{\text{cost}}{\text{good}} = \frac{\$20(100) + \$4(100)(0.05)(0.60)}{95 + 3}$$

$$= \$20.53$$

Quality Engineering

Process: A system of inputs and outputs

Example: Doing a process

Example: A hospital is trying to increase the quality of drug administration. To do this, it is considering providing patients with bar-coded wristbands to help guide workers. Your team is charged with studying the effects of bar-coding by carefully watching 250 episodes in which drugs are given to patients without bar coding and 250 episodes with bar coding. Every time a drug is administered you will check the amount, if any, of discrepancy between what was supposed to be given and what was given.

Control Charts: A process monitoring technique which tracks averages in a quality characteristic with time or sample number

- **Control:** Where the characteristic should be; find abnormal sources of variation
- **Control Limit:** Defined by statistics. Alert the user to unusual variability
↳ Different than spec limits

Designed experiments: Discovers very variable factors have a significant influence on quality characteristics in a process

Six Sigma

Broader Definition: Statistical based, data driven, management approach and continuous quality improvement methodology for eliminating defects in a product, process, or service

↳ Reduces variability to a level where defects are unlikely

Three Elements of Six Sigma

(i) **Quality Planning:** Listening to the voice of your customer

(ii) **Quality Assurance:** Establishing a system to prevent quality issues from arising

(iii) **Quality Control & Improvement:** A set of specific steps and tools to ensure products meet requirements and are continuously improved

Name Meaning: Refers to the quality level target such that sigma (σ , or variance σ^2) should be small enough so that 6σ is within the specification limits

Sigma Quality Level: The Probability that product/service is non-defective

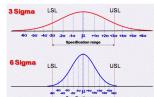
Why is Six Sigma Selected?

Suppose 100 parts: 3σ : $(.9973)^{100} = .7631$ is probability non-defective $\rightarrow 24\%$ defects
 6σ : $(.9999998)^{100} = .99999998$ non-defective $\rightarrow \sim .000002\%$ defects

Some companies use other multiples of σ , depending on what required by company/customers

Example of Six Sigma (will be on HW)

Illustrative Example: Let's say you are making a part that has a radius of 10 mm. Specification limits are between 9 (LSL) and 11 (USL) mm. Before you applied quality improvement, the standard deviation (sigma - a measure of variability) was 1 mm, and after you applied improvements, the standard deviation (sigma) decreased to 0.1. Is the new process 6 sigma quality?



$$\frac{|USL \text{ or } LSL - \text{target}|}{\sigma \text{ (standard deviation)}} = \frac{|10 \text{ mm} - 10 \text{ mm}|}{1 \text{ mm}} = 1 \quad \text{For old 1-sigma level}$$

$$= \frac{|10 \text{ mm} - 10 \text{ mm}|}{0.1 \text{ mm}} = 10 \quad \text{For new 10-sigma level}$$

Lean Manufacturing

Def: Another management philosophy aimed at eliminating waste (e.g., time and materials)

Process Cycle Efficiency (PCE): $(\text{Value added time}) / (\text{Process time})$

$\hookrightarrow \text{Value added time} = \text{time spent making the product more valuable}$

Process Cycle Time (PCT): $(\text{Number of items in progress}) / (\text{Completion rate})$ \hookleftarrow backlog issues as an example

Why is a slow process (low PCE or high PCT) an expensive process?

- Hidden Factory →

- (i) Customers don't like waiting
- (ii) More handling = more personnel
- (iii) More opportunity for damage or loss
- (iv) Inventory has to be higher
- (v) More documentation

DMACC Process

Def: A Project-based 5-step process and problem-solving procedure

$\hookrightarrow \text{Define} \rightarrow \text{Measure} \rightarrow \text{Analyze} \rightarrow \text{Improve} \rightarrow \text{Control}$

Toll gates: Where projects are reviewed to ensure on track

5 stages of DMACC (will be on HW too)

(i) Define Stage

- Main Obj: Identify a great opportunity and verify that it represents a legitimate value opportunity or breakthrough

◦ Breakthrough: Major improvement to a process or product

◦ Value Opportunity: The financial opportunity at stake

- Tools used

◦ SIPOC diagram: High level map of process

\rightarrow Suppliers: Those who provide information, material or other items that are used in the process

\rightarrow Input: The actual information or material provided

\rightarrow Process: The set of steps required to do the work

\rightarrow Output: The product delivered to the customer \rightarrow Other dept. in same company

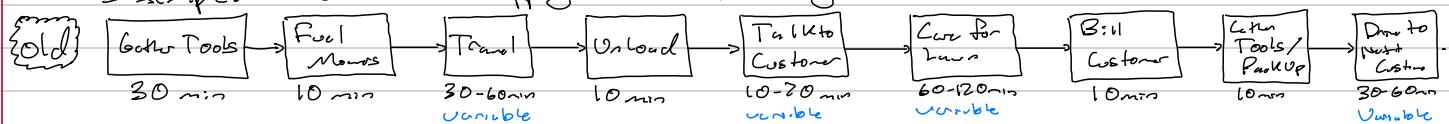
\rightarrow Customers: who buys the product (internal or external)

- Process Mapping: Involves creating flow diagrams for systems

◦ Value stream mapping: A variant of process mapping where engineers focus on steps which could be eliminated or simplified to reduce waste

- Value stream: The min amount of processing steps from raw material to customer

Example: Value Stream Mapping (Lawn Mowing)



Possible Improvements: Using customer's lawnmower, store SOP (home on truck), protocol, personnel/intense, eliminate/reduce customer interactions, optimize route (graph shortest distance)/logistics

- Pareto charting: Plots frequency (# of parts) or cost (time/money) against causes

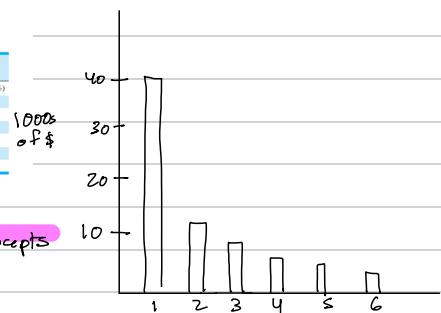
◦ Pareto Rule: 80% of the effects come from 20% of the causes

• Graph Pareto by Hand (Example)

Often in order from largest cost to smallest

Cost or % defects is on y-axis, category is on the x-axis

Monthly Quality-Costs Information for Assembly of Printed Circuit Boards		
Type of Defect	Percentage of Total Defects	Scrap and Rework Costs
1 Insufficient solder	42%	\$37,500.00 (52%)
2 Missing components	20	12,000.00
3 Defective components	15	8,000.00
4 Missing components	10	5,000.00
5 Cold solder joints	7	3,000.00
6 All other causes	5	4,600.00
Total:	100%	\$72,200.00



Can also do this with Lean concepts

Project Charter: Short document
with clear focus to product.

Example to right, but worse across companies

Business Case	Opportunity Statement																					
<ul style="list-style-type: none"> This project supports the business quality goals, namely (a) reduce customer resolution cycle time by x% and (b) improve customer satisfaction by y%. 	<ul style="list-style-type: none"> An opportunity exists to close the gap between our customer expectations and our actual performance by reducing the cycle time of the customer return process. 																					
Goal Statement	Project Scope																					
<ul style="list-style-type: none"> Reduce the overall response cycle time for returned product from our customers by x% year to year. 	<ul style="list-style-type: none"> Overall response cycle time is measured from the receipt of a product return to the time that either the customer has the product replaced or the customer is reimbursed. 																					
Project Plan	Team																					
<table border="1"> <tr> <th>Activity</th> <th>Start</th> <th>End</th> </tr> <tr> <td>Define</td> <td>6/04</td> <td>6/30</td> </tr> <tr> <td>Measure</td> <td>6/18</td> <td>7/06</td> </tr> <tr> <td>Analyze</td> <td>7/15</td> <td>8/09</td> </tr> <tr> <td>Improve</td> <td>8/15</td> <td>9/30</td> </tr> <tr> <td>Control</td> <td>9/15</td> <td>10/30</td> </tr> <tr> <td>Track Benefits</td> <td>11/01</td> <td></td> </tr> </table>	Activity	Start	End	Define	6/04	6/30	Measure	6/18	7/06	Analyze	7/15	8/09	Improve	8/15	9/30	Control	9/15	10/30	Track Benefits	11/01		<ul style="list-style-type: none"> Team Sponsor Team Leader Team Members
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(i:) Measure Stage

- Objectives:** Explain & understand the current state of the process
- Tools Used:** Quantify what the current process is performing at
- Measurement Capability:** Assess if the current measurement systems are adequate
- Measure Performance:** Check to understand variability
- Tools Used:** Histograms, bar charts, run charts, scatter diagrams

(ii:) Analyze Stage

- Objectives:** Use data to begin to determine cause and effect relationships and sources of variation

Common Causes of Variability: Internal Sources (noise)

Assignable Causes of Variability: External Sources

Tools Used

- Control Charts:** Used to separate the common and assignable causes of variation
- Hypothesis Testing & Confidence Intervals:** Used to determine if different conditions produce significant results
- Regression Analysis:** Allows models of the process to be built
- Failure Modes & Effects Analysis (FMEA):** A method for identifying and prioritizing sources of variability or defects in a process or product

↳ For each item, we determine:

(i) Likelihood something will go wrong | (not likely) - 10

(ii) Ability to detect the failure or defect | (likely) - 10

(iii) Severity | (not severe) - 10

Risk Priority #: (i) x (ii) x (iii)

Example

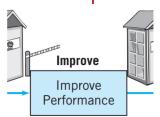
- Severity, 1 (not severe) - 10
- Probability of occurrence, 1 (not likely) - 10
- Probability of detection, 1 (likely) - 10

RPN = A X B X C



RPN = Risk Priority Number

FAILURE MODE & EFFECTS ANALYSIS (FMEA)					Date: 1/1/2018
Process Name: Left Front Seat Belt Install Process Number: SBT 445					Revision: 1.3
Failure Mode	A) Severity	B) Probability of Occurrence	C) Probability of Detection	Risk Preference Number (RPN)	
1) Select Wrong Color Seat Belt	5	4	3	60	Audited
2) Seat Belt Bolt Not Fully Tightened	9	2	8	144	
3) Trim Cover Clip Misaligned	2	3	4	24	



(i) Improve Stage

- Objectives
 - Generate and quantify potential solutions
 - Evaluate and select final solution
 - Verify and gain approval for final solution

• Objectives: Use creative thinking & info gathered previously to make specific suggestions that will lead to desired solution process. Confirmation is important

• Tools Used

- Mistake Proofing (example)
- Optimization
- Pilot Testing
- Conformance Experiments

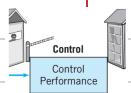
- Documentation
 - (i) Solution obtained
 - (ii) Alternatives Considered
 - (iii) Confirmation tests
 - (iv) Risk Analysis

(ii) Control Stage

• Objectives: Complete all remaining work and hand off to improved process so gains in progress are institutionalized

• Tools Used

- Control Charts
- Transition Plans



- Objectives
 - Develop ongoing process management plans
 - Mistake-proof process characteristics
 - Develop out-of-control action plans

Chapter 3 - Quantifying Stats in Quality Characteristics

Statistics

Sample Mean: Average Value

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{\sum_{i=1}^n x_i}{n}$$

where n is the number of data points in the sample
and $x_1, x_2, x_3, \dots, x_n$ are data points

For this data set: 12, 12, 15, 18, 21, 22, 22, 23, 25, 25, 26, 27, 28, 29, 30, 30, 31

$$\bar{x} = \frac{12+12+15+\dots+31}{17} = \frac{396}{17} = 23$$

Sample Variance:

{Example var: $\sum(x_i - \bar{x})^2$ }

$$s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}$$

Standard Deviation: $s = \sqrt{s^2}$

{Example var: $\sum(x_i - \bar{x})^2$ }

For this data set: 12, 12, 15, 18, 21, 22, 22, 23, 25, 25, 26, 27, 28, 29, 30, 30, 31

$$s^2 = \frac{(12-23)^2 + (12-23)^2 + \dots + (31-23)^2}{17-1} = 37$$

$$s = \sqrt{37} \approx 6.08$$

MINITAB: Stat > Basic Statistics > Display Descriptive Statistics > In Variables add data

Illustrative Example

Sample 1	Sample 2
$x_1 = 1$	$x_1 = 1$
$x_2 = 3$	$x_2 = 5$
$x_3 = 5$	$x_3 = 9$
$\bar{x} = 3$	$\bar{x} = 5$

Sample 3
$x_1 = 101$
$x_2 = 103$
$x_3 = 105$
$\bar{x} = 103$

Sample 1: $s = 2$

Sample 2: $s = 4$

Sample 3: $s = 2$

Tools to Describe Data Numerically and Visualize Variables

↳ Often easier to arrange sorted (lowest to highest) data

- **Percentiles:** Tells us how much of the data is below a certain point

12, 12, 15, 18, 21, 22, 22, 23, 25, 25, 26, 27, 28, 29, 30, 30, 31

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

↳ Values

↳ Rank (1-indexed)

- **Lower Quartile (Q_1 , or 25th perc.):** Value with rank = $(n+1)(\frac{1}{4})$

here: $\frac{17+1}{4} = \text{rank of } Q_1 \rightarrow \frac{18+1}{2} = 10.5$

- **Middle Quartile (Q_2 , or median, or 50th perc.):** Value with rank = $(n+1)(\frac{1}{2})$

here: $\frac{17+1}{2} = \text{rank of } Q_2 \rightarrow 10.5$

- **Upper Quartile (Q_3 , or 75th perc.):** Value with rank = $(n+1)(\frac{3}{4})$

here: $(17+1)(\frac{3}{4}) = \text{rank of } Q_3 \rightarrow 28.5$

- **Interquartile Range: $Q_3 - Q_1$:**

here: $Q_3 - Q_1 = 28.5 - 10.5 = 18$

- **Whisker Limits:** Max Limit = $Q_3 + 1.5(Q_3 - Q_1)$

Min Limit = $Q_1 - 1.5(Q_3 - Q_1)$

here: Max Limit = 43, Min Limit = 6

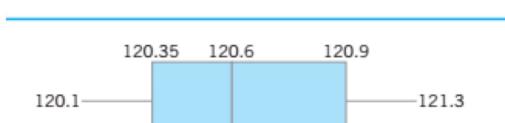
MINITAB: Stat > Basic Statistics > Display Descriptive Statistics > In Variables add data

- **Box Plot Features**

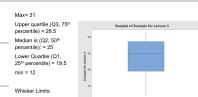
Displays 25th, 50th and 75th quartiles (ends and middle of the box)

- 25th and 75th are the ends of the box
- Median (50th percentile) is a single line through box
- The min and max of data (whiskers)
- Limits of whiskers (check) that the max and min do not go past (points that fall outside this can be outliers, represented as single data points and the new min or max is the next data point within the limit) – limit formula on next slide

TABLE 3.4 Hole Diameters (in mm) in Wing Leading Edge Ribs		
120.5	120.4	120.7
120.9	120.2	121.1
120.3	120.1	120.9
121.3	120.5	120.8



here:



Minitab: Graph > Boxplot > Simple > input data column into graph variables

Excel

- Stdev is for sample data (use most often)
- Stdevp is for entire population

• **Histograms:** Plots of frequencies of data occurring w.r.t. the data sorted into bins/intervals

↳ **Bins:** The range that data will be sorted into $\# \text{ bins} = \text{ceil}(\sqrt{n}) \leftarrow \text{round up}$ $\text{bin size} = \frac{\text{max-min}}{\# \text{ bins}}$

- i. Need to determine the bin range for continuous data
- ii. Best if bins are the same size
- iii. Rule of thumb: number of bins = square root of number of observations (round up if not a whole number)

Note: for categorical data or discrete data, each category is its own bin

For this data set: 12, 12, 15, 18, 21, 22, 23, 25, 25, 26, 27, 28, 29, 30, 30, 31

$$\# \text{ of bins} = \sqrt{17} = 4.1 \rightarrow 5$$

$$\text{bin size} = \frac{31-12}{5} = 3.8$$

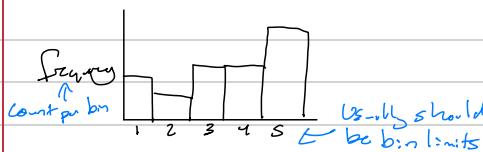
Bin 1: min + bin size $\rightarrow 12 + 3.8 \rightarrow [12, 15.8)$ Inclusive below, Exclusive above

Bin 2: bin 1 end + bin size $\rightarrow 15.8 + 3.8 \rightarrow [15.8, 19.6)$

Bin 3: $[19.6, 25.4)$

Bin 4: $[25.4, 29.2)$

Bin 5: $[29.2, 31]$



MINITAB: Graph > histogram > simple > add your column to graph variables
To change bins: double click on histogram bars (twice), select bins tab, change number of intervals to number of bins (Cut point is how we graphed in class)

↳ Final bin should be inclusive above

Significant Digits / Figures: Numbers of digits contributing to measurement resolution

Rules:

- (i) Non-zero digits always significant

(ii) Any zeros between two significant digits are significant (leading zeros are not)

(iii) Trailing zeros in the decimal portion only are significant (unless indicated by a bar)

Examples:

How many sig. figs in each number?	0.00500 = 3
0.03040 = 4	10 ² = 4

How would you write this # with the correct sig. figs?

0.899923 (report 2 sig. figs)
0.40
23000.0 (report 3 sig. figs)
23000

Will take away points for tons of sig figs!

Just pay attention to them!

For your sample means: keep as many significant figures as the number with the lowest significant digits in your data set (same for median and quartiles)

For your standard deviation: report to the last place that is significant, but no more

Your Average and SD should have some # of places!

E.g. Sample mean = 4.09 Sample Standard Deviation = 0.00923. What do I report? 4.09 ± 0.01

Probability Distributions Introductory

• **Sample:** Collection of measurements from a larger source or population

• **Population:** All of the elements from a group

• **Probability Distribution:** Mathematical Model that relates value of a variable with its probability of occurrence

• **Continuous Distribution:** The variable is expressed on a continuous scale

• **Discrete Distribution:** The variable can only take on certain values

• **Writing out Probability Statements**

◦ **Discrete:** The probability that x is equal to x_i ?

$$P\{x=x_i\} = p(x_i)$$

• **Calculating mean and variance**

$$\mu = \begin{cases} \int_{-\infty}^{\infty} xf(x) dx, x \text{ continuous} \\ \sum_{i=1}^{\infty} x_i p(x_i), x \text{ discrete} \end{cases}$$

◦ **Continuous:** The probability that x is between a and b ?

$$P\{a \leq x \leq b\} = \int_a^b f(x) dx$$

$$\sigma^2 = \begin{cases} \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx, x \text{ continuous} \\ \sum_{i=1}^{\infty} (x_i - \mu)^2 p(x_i), x \text{ discrete} \end{cases}$$

• **Median:** 50th percentile

• **Mean:** Central Tendency (Average)

• **Variance:** Spread of the distribution

M, σ^2 (population model parameters)
 \bar{x}, s^2 (sample data notation)

Ray attention to this notation

The Normal Distribution

- For a random normal variable x , the probability distribution of x is: $P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

↳ Short hand $N(\mu, \sigma^2)$

○ Continuous, symmetric, "bell-shaped"

•

$$\text{General Formula: } P\{\mu - 3\sigma \leq x \leq \mu + 3\sigma\} = \int_{\mu-3\sigma}^{\mu+3\sigma} f(x) dx$$



In a Probability Band:

$$P\{\mu - \sigma \leq x \leq \mu + \sigma\} = 0.68$$

Due to symmetry:

$$P\{x > \mu\} = P\{x < \mu\} = 0.5$$

Holds for any normally distributed variable 'x'

- Got to these properties: Set $\sigma=1$, $\mu=0$ so $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$\left\{ \begin{array}{l} a = \mu - 1\sigma = -1 \\ b = \mu + 1\sigma = 1 \end{array} \right.$$

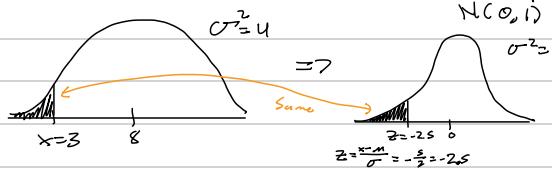
$$\int_{-1}^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \text{(using mathematica)} 0.68$$

• Standardization: $N(\mu, \sigma^2) \Rightarrow N(0, 1)$

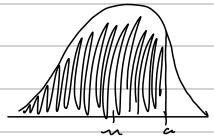
• Standard Normal Distribution: $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

• To standardize any random variable: $z = \frac{x-\mu}{\sigma}$ z is a standard normal variable

• Drawing out normal distributions: Suppose $N(8, 4)$

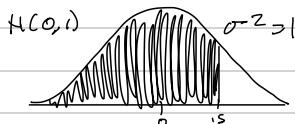


• Cumulative Normal Distribution: The probability that the variable x is less than or equal to some value a



Example: Assume Z is a standard normal random variable, and you want to find the probability that Z is less than or equal to 1.5

$$P\{Z \leq 1.5\} = 0.93$$

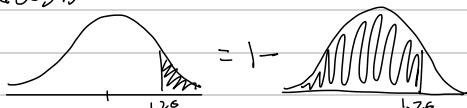


General Probability Demonstration

$$(i) P(Z > 1.26) = 1 - P(Z \leq 1.26)$$

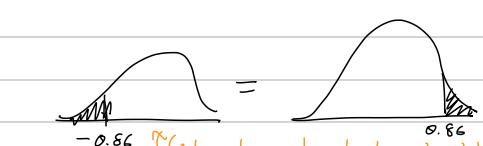
$$(ii) P(Z \leq -0.86) = P(Z > 0.86) = 1 - P(Z \leq 0.86)$$

$N(0, 1)$

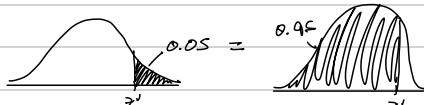


$$= 1 - 0.89616 = 0.1039$$

0.10



$$(iii) P(Z > Z') = 0.10$$



Chapter 4 - Hypothesis Testing

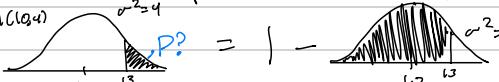
Example: Suppose the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 millamps and a variance of 4 millamps². What is the probability that the measurement will exceed 13 millamps?

1) Convert the word problem to mathematical expression and sketch the situation.

$$\mu = 10$$

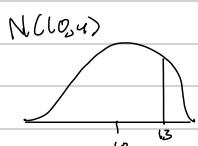
$$\sigma^2 = 4$$

$$P(x \geq 13)$$

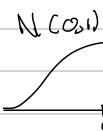


3) Convert sketch and formula to standard normal (if using tables in back of book) and a cumulative normal distribution (so it is compatible with the table in the book and Minitab)

Minitab: Calc > Probability dist. > Normal



=>



$$z = \frac{x - \mu}{\sigma} = \frac{13 - 10}{2} = 1.5$$

$$1 - P(z < 1.5) = 0.07$$

$$\rightarrow P(z \geq 1.5)$$

→ What about the integral?

$$P(z \geq 1.5) = \int_{1.5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \quad \text{or} \quad P(x \geq 13) = \int_{13}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Checking for normality

→ How do we know if the normal distribution is a reasonable model for our data?

- **Probability Plotting:** Graphical method for determining whether sample data conform to a hypothesized distribution based on subjective visual examination.

- **Constructing a normal prob. plot by hand:** using $z_i = \frac{x_i - \bar{x}}{s}$

How-to (Steps)

1) Ranking data and ordering it $(x_{(1)}, x_{(2)}, \dots, x_{(n)})$, i = the ranked value, n = total number of values

2) Calculating cumulative frequency $F_i = i/n$ for each data point

to tell us the % of data that lie below certain rank i : $F_i = \frac{i}{n}$

$3(z_i - \bar{z}) = \frac{F_i - 0.5}{s} = 0.25$

3) Find a z_i value for each rank using the cumulative frequency value is the probability

Find z_3 s.t. $P(z \leq z_3) = \frac{3-0.5}{n} = 0.25$. Tells us the value of z_3 if the data are normally distributed

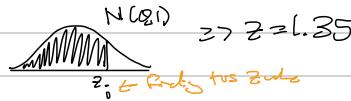
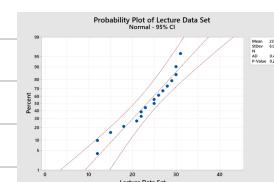
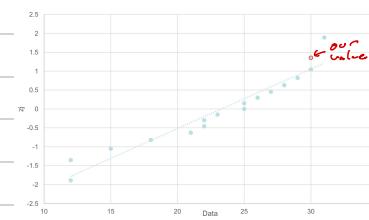
4) Plot the actual data (x -axis) vs. corresponding z (y -axis)

Normal Probability Plot in Minitab - Minitab: Graph > probability plot > graph variables, choose column with desired data

Note: each value is plotted versus the cumulative frequency with the y-axis is transformed so that the fitted distribution forms a straight line if normally distributed.

$$C. \text{ Freq } \left(\frac{i-0.5}{n} \right)$$

$$\text{For rank } 16: \frac{16-0.5}{17} = 0.91 \rightarrow z_{16}: P(z < z_{16}) = 0.91$$



$P(z > 2) \approx 0.02$

\rightarrow P-value < α means not normal

In literature,

$$\alpha = 0.05$$

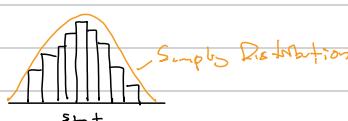
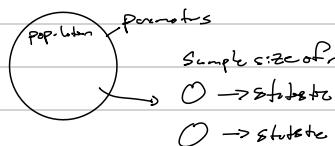
$$\text{or } \alpha = 0.01$$

Statistics B-School

- **Statistical Inference:** Drawing conclusions about a population based on sample from the population (of them selected randomly)

- **Statistics:** Any function of sample data that doesn't contain unknown parameters

- **Sampling Distribution:** The probability distribution of statistics



Suppose x_1, x_2, \dots, x_n are normally distributed random variable w/ given μ and σ^2

→ If x_1, x_2, \dots, x_n is a random sample of size n , the distribution of the sample mean is:

Normally distributed with mean of μ and covariance of $\frac{\sigma^2}{n}$

$$M_{\bar{x}} = \mu$$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

$$N(M, \frac{\sigma^2}{n}) \text{ shorthand}$$

- **Central Limit Theorem:** Implies that the sample mean will be approximately normally distributed for large sample sizes, regardless of the distribution you are sampling from.

→ **(Large) Sample size:** Some suggest $n > 20$ (non-normal), $n = 3-4$ (closer to normal)

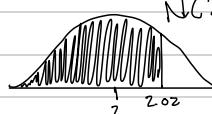
• Example

Electrodes are manufactured with a population mean thickness of 2.0 mm and a standard deviation of 0.048 mm. Find the probability that a random sample of $n = 16$ electrodes will have a sample mean diameter less than or equal to 2.02 mm? (keep 2 sig figs)

↳ Central Limit Theorem Applies

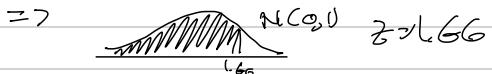
$$\bar{X} \Rightarrow N(\mu, \frac{\sigma^2}{n}) \quad n=16, \mu=2.0 \text{ mm}, \sigma=0.048 \text{ mm}, \sigma^2=0.0023$$

$$P(\bar{X} \leq 2.02)$$



$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

Now!
For samples



$$P(Z \leq 1.66) = 0.45$$

Hypothesis Testing

- **Parameters:** What probability distributions we described with

e.g. normal dist. is described with μ, σ

- **Statistical Hypothesis:** A statement about values of parameters of a probability distribution

- **Null Hypothesis:** States no statistical difference exists (H_0)

- **Alternative Hypothesis:** Contradicts to null hypothesis (H_1)

Example: Suppose we think the mean inside diameter of bearing is 1.500 in

$$H_0: \mu = 1.500 \quad H_1: \mu \neq 1.500 \quad \text{two-sided}$$

$$\left\{ \begin{array}{l} \mu < 1.500 \\ \mu > 1.500 \end{array} \right. \quad \text{one-sided}$$

- Two outcomes of a hypothesis test:

(i) Reject H_0 or (ii) Fail to reject H_0

- **Test Statistic:** Used to test a hypothesis and is computed from a random sample of the population

- **Sampling Distribution (Null Distribution):** The sampling distribution of the test statistic assuming that H_0 (the null hypothesis) is true

- **Critical Region:** Sets values of the test statistic that lead to the rejection of the null hypothesis

↳ Can either fail to reject or reject H_0 , critical region defines these cutoffs

- Two types of errors

- (i) **Type I errors:** If the null hypothesis is rejected - but it is true

↳ Defn: $\alpha = P\{\text{type I error}\} = P\{\text{reject } H_0 | H_0 \text{ is true}\}$ This is specified

- (ii) **Type II errors:** If the null hypothesis is not rejected - but it is false

↳ Defn: $\beta = P\{\text{type II error}\} = P\{\text{fail to reject } H_0 | H_0 \text{ is false}\}$ Hard to control

○ Type II errors are uncontrollable & unknown, so we try to minimize + avoid it

- **Conclusions:** Must be worded very clearly

- **Rejection of Null:** A strong conclusion because it controls

○ to be small

- **Failing to Reject:** Weak conclusion because it has to control β

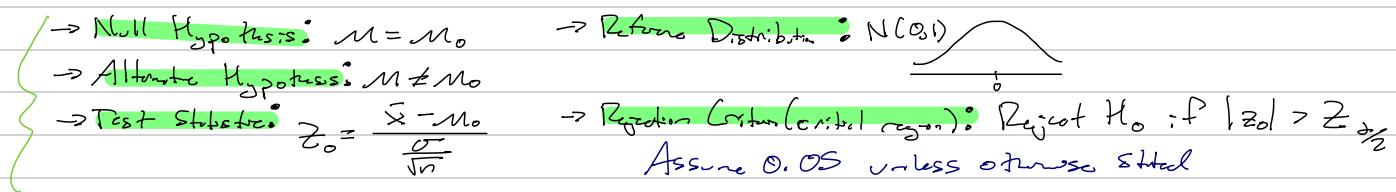
↳ to control β

The Claim is the	The Decision is	The Conclusion is
Null Hypothesis	Fail to Reject the Null Reject the Null	The evidence is <i>not sufficient</i> to reject the claim. The evidence is <i>sufficient</i> to reject the claim.
Alternative Hypothesis	Fail to Reject the Null Reject the Null	There is <i>insufficient</i> evidence to support the claim. There is <i>sufficient</i> evidence to support the claim.

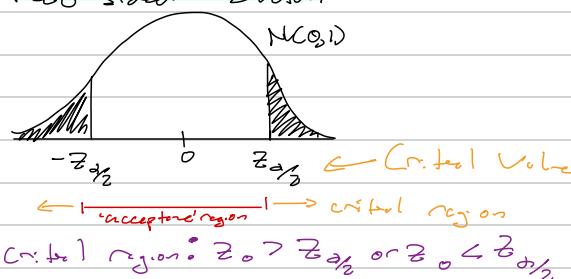
We never say accept the null hypothesis

One Sample Z-test & Inference on the Mean of a Population, Variance Known

Suppose x is a random variable with unknown mean μ and a known variance σ^2 and we wish to test the hypothesis that the population mean is equal to a standard value (H_0).



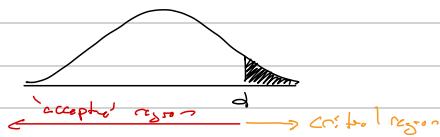
Two-sided Sketch



One-sided Test

$$H_0: \mu = \mu_0$$

$$H_1: \mu > \mu_0$$



General Procedure C for this class!

- Parameters of Interest: From the problem context, identify known/unknown parameters of interest, the test you're going to do and identify type of problem.
- Null Hypothesis: State the null hypothesis H_0 .
- Alternative Hypothesis: State the alternative hypothesis H_1 .
- Test Statistic: Define an appropriate test statistic & reference distribution.
- Reject H_0 : State rejection criterion for the null hypothesis. Should be drawn on plot w/ test statistic.
- Computations: Compute necessary sample quantities, substitute those into eqn for test statistic, and compute test value. Compare to rejection criteria.
- Draw Conclusion: Decide whether or not H_0 should be rejected and report that in the problem context. Phrase appropriately!

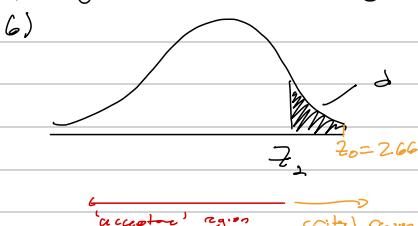
Example

Example: You are testing the response time of a distributed computer system. The system manager wants to know whether the mean response time to a specific command exceeds 75 ms. Previous experience has determined the standard deviation is 8 ms. Use $\alpha = 0.05$. A test is conducted executing the command 25 times, and the mean response time is 79.25 ms.

4) Test Statistic: $Z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{79.25 - 75}{8/\sqrt{25}} = 2.66$

Reference Dist: $N(0,1)$

5) Rejection criterion: $|Z_{\text{reject}}| > Z_{\alpha/2} = Z_{0.05}$



1) Parameters of Interest: H_0 : inferring mean of population, known variance mean response time

↳ Test: One-sample z-test

2) $H_0: \mu = 75 \text{ ms}$

3) $H_1: \mu > 75 \text{ ms}$ (one-sided)

7) Conclusion: $Z_0 > Z_{\alpha}$

Reject the null Hypothesis. Conclude that the mean response time exceeds 75 ms.

$P(Z < Z_{\alpha}) = 0.95$

$Z_{\alpha} = 1.65$ critical value

How do I do this in Minitab? Minitab>Stat>Basic Stats>OneSampleZ – click summarized data, but you could also highlight a column of data, options will allow us to set the alpha value and the alternative hypothesis

Actual Example/Application

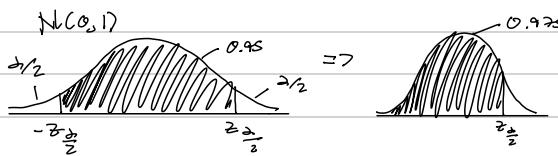
We pull a sample of 20 electrodes from an assembly line of a process with known variance in electrode thickness to be 0.64 mm². The average thickness of the sample is 7.9 mm. Construct a 95% confidence interval.

$$n = 20$$

$$\sigma^2 = 0.64 \text{ mm}^2 \rightarrow \sigma = 0.8$$

$$\bar{x} = 7.9 \text{ mm}, s = 0.08$$

$$N(0, 1)$$



$$L = 7.4 - 1.96 \frac{0.8}{\sqrt{20}} = 7.5$$

$$U = 7.9 + 1.96 \frac{0.8}{\sqrt{20}} = 8.3$$

Often our result would be said like this:

~~"We are 95% confident that the mean of the population is between 7.5 and 8.3."~~

~~Not this!~~

Perhaps a better description would be:

"In repeated sampling, this method will produce intervals that capture the true population mean about 95% of the time."

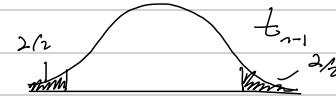
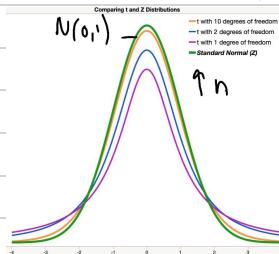
Use this front bearing exam!"

One Sample t-test

- Inference on mean distribution, known variance.
- Suppose x is a normal random variable with unknown mean μ and unknown variance σ^2 .
- Null Hypothesis: $H_0: \mu = \mu_0$
- Alternative Hypothesis: $H_1: \mu \neq \mu_0$
- Test Statistic: $T_0 = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}$
- Rejection Criterion: $|T_0| > t_{\alpha/2, n-1}$

Two-tailed degrees of freedom

Rejection Criterion: $|T_0| > t_{\alpha/2, n-1}$



For the t-distribution:

- Is a continuous probability distribution of the z-score when the estimated standard deviation is used in the denominator rather than the true standard deviation.
- It is a type of normal distribution, except it accounts for the fact that as sample size increases, variability decreases.
- The t-distribution, like the normal distribution, is bell-shaped and symmetric about zero, but it has heavier tails, which means it tends to produce values that fall farther from its mean.
- As the sample size increases, the t-distribution becomes more similar to a normal distribution. The shape depends on n !! And this is why you MUST write down n each time you draw out a t-distribution.

Summary

Testing Hypotheses on the Mean of a Normal Distribution, Variance Unknown

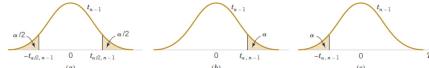
Null hypothesis: $H_0: \mu = \mu_0$

$$\text{Test statistic: } T_0 = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}$$

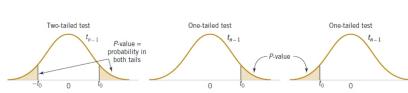
t dist with $n-1$ D.F

Alternative Hypotheses	P-Value	Rejection Criterion for Fixed-Level Tests
$H_1: \mu \neq \mu_0$	Probability above $ T_0 $ and probability below $- T_0 $	$T_0 > t_{\alpha/2, n-1}$ or $T_0 < -t_{\alpha/2, n-1}$
$H_1: \mu > \mu_0$	Probability above T_0	$T_0 > t_{\alpha, n-1}$
$H_1: \mu < \mu_0$	Probability below T_0	$T_0 < -t_{\alpha, n-1}$

Fixed Significance Test by



P-Value



t-test Assumes

- Data are discrete.
- Data are normally distributed.
- Data are random samples.

Example

Inference on the mean of a distribution, unknown variance

Rubber can be added to asphalt to reduce road noise. The viscosity (cP) of 15 specimens of asphalt is measured. To be suitable for use, the mean should be equal to 3200 cP. Test the hypothesis using alpha=0.05.

Using this data
 $n=15$

cP
3193
3124
3153
3145
3093
3466
3355
3079
3182
3227
3256
3332
3204
3282
3170

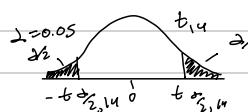
- Inference on mean, un known variance, one-sample
- One-sample t-test

$$1) H_0: \mu = 3200 \text{ cP} = \mu_0 \quad 3) H_1: \mu \neq 3200 \text{ cP}$$

$$4) T_0 = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}, \quad \bar{x} = 3193 + 3124 + \dots + 3170 = 3217$$

$$\text{Plug & Chug} = 0.64 \quad S = \sqrt{\frac{(3193 - 3217)^2 + \dots + (3170 - 3217)^2}{15}} = 106$$

$$5) \text{Reject if } |T_0| > t_{\alpha/2, 14} \quad 7) \text{Conclusion}$$



$$t_{\alpha/2, 14} = 2.145$$

(P+2) The critical value is not sufficient to reject the claim that the mean is equal to 3200 cP.
(Q+2) The test statistic does not fall in the rejection region, so we do not have enough evidence to conclude that the mean is not equal to 3200 cP.

CI for One-sample t-test

Suppose x is a normal random variable with unknown mean μ and unknown variance σ^2 . From a sample of n observations, the sample mean \bar{x} and sample variance s^2 are computed. Then:

$$\bar{x} - t_{\alpha/2, n-1} \frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{\alpha/2, n-1} \frac{s}{\sqrt{n}}$$

Is a 100(1-alpha)% two-sided CI

where $t_{\alpha/2, n-1}$ is the value of the t distribution with $n-1$ degrees of freedom such that $P(t_{n-1} \geq t_{\alpha/2, n-1}) = \alpha/2$

$$-\infty < \mu \leq \bar{x} + t_{\alpha, n-1} \frac{s}{\sqrt{n}}$$

Is a 100(1-alpha)% upper confidence bound

where $t_{\alpha, n-1}$ is the percentage point of the t distribution with $n-1$ degrees of freedom such that $P(t_{n-1} \geq t_{\alpha, n-1}) = \alpha$

$$\bar{x} - t_{\alpha, n-1} \frac{s}{\sqrt{n}} \leq \mu \leq 0$$

Is a 100(1-alpha)% lower confidence bound

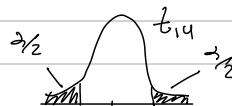
where $t_{\alpha, n-1}$ is the percentage point of the t distribution with $n-1$ degrees of freedom such that $P(t_{n-1} \geq t_{\alpha, n-1}) = \alpha$

→ For our previous example

$$\bar{x} = 3217$$

$$s = 106$$

$$n = 15$$



$$t_{0.025, 14} = 2.145$$

$$L = 3217 - 2.145 \frac{106}{\sqrt{15}} = 3158$$

$$U = 3217 + 2.145 \frac{106}{\sqrt{15}} = 3276$$

$$3158 \leq \mu \leq 3276$$

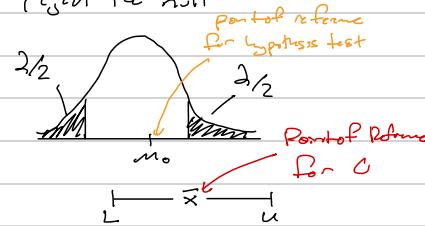
CI Relationship w/ Hypothesis Test

- Both do make inferences about a population parameter (but in different ways)
- A CI provides a range of values likely containing the true population parameter based on a set of data
- A hypothesis test determines whether a specific claim about the population parameter is supported by the data
- Difference in emphasis of the point of reference

Minitab: Stat>Basic Stat>1-sample t-test, samples in columns (drop down), double click data, check hypothesis test select type and in options select alpha value

Illustration

If the value in your H_0 is in your CI, you likely cannot reject the null



Inference or decisions on Normal Data (Chi-square test)

Suppose we wish to test the hypothesis that the variance of a normal population (σ^2) equals a specified value, say σ_0^2 , or that the standard deviation of the population (σ) is equal to σ_0 . Let x_1, x_2, \dots, x_n be a random sample of n observations from this population to test:

$$H_0: \sigma^2 = \sigma_0^2$$

$$H_a: \sigma^2 \neq \sigma_0^2$$

$$\text{Test Statistic: } \chi^2 = \frac{(n-1)s^2}{\sigma_0^2} \quad s = \text{sample variance with } n \text{ observations}$$

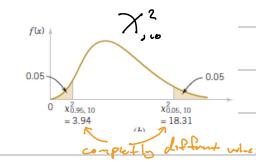
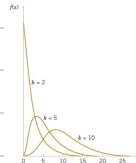
Reference Distribution: Chi-Squared Distribution

Useful for many controls
about the variability of your process

Which Hypothesis should I do?

Test	# of Populations	Variance Known?	Parameter of interest?	Reference Dist.
One-sample Z-test	1	Yes	Mean	Standard normal
Two-sample Z-test	2	Yes	Difference between means	Standard normal
One-sample t-test	1	No	Mean	t-distribution
Two-sample (pooled) t-test (equal variance)	2	No	Means of two populations	t-distribution
Two-sample t-test (unequal variance)	2	No	Means of two populations	t-distribution
Paired t-test	2 (simplifies to 1)	No	Difference between paired means	t-distribution
1 Variance test	1	No	Variance of one population	chi-squared distribution

If both mean and variance of populations are known, it's a probability problem

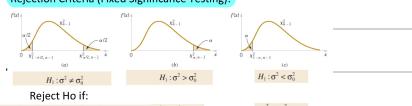


Chi-squared Distribution

- χ^2 has a chi-squared distribution if true null is true
- Very Features:
 - (i) Depends on $n-1$ degrees of freedom
 - (ii) Not symmetric
 - (iii) Not centered at 0
 - (iv) Total Area under curve is still 1!

Rejections

Rejection Criteria (Fixed Significance Testing):



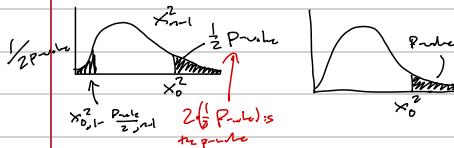
Summary

Null hypothesis: $H_0: \sigma^2 = \sigma_0^2$
Test statistic: $\chi^2_0 = \frac{(n-1)s^2}{\sigma_0^2}$

Alternative Hypothesis

Rejection Criteria

- | | |
|---------------------------------|--|
| $H_1: \sigma^2 \neq \sigma_0^2$ | $\chi^2_0 > \chi^2_{\alpha/2, n-1}$ or $\chi^2_0 < \chi^2_{1-\alpha/2, n-1}$ |
| $H_1: \sigma^2 > \sigma_0^2$ | $\chi^2_0 > \chi^2_{\alpha, n-1}$ |
| $H_1: \sigma^2 < \sigma_0^2$ | $\chi^2_0 < \chi^2_{1-\alpha, n-1}$ |



Ref. Dist: Chi-Square

Check Normality! It is sensitive to this

Example

Inference on the variance of a distribution

Rubber can be added to asphalt to reduce road noise. The viscosity (cP) of 15 specimens of asphalt is measured. s is calculated to be 106. Let's say we are interested in knowing if the variance less than 30,000.

Use $\alpha = 0.05$

1) Parameter of interest: $\sigma^2 \rightarrow$ 1-variance test

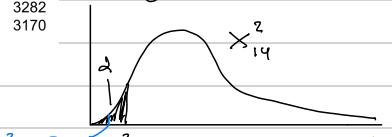
2) $H_0: \sigma^2 = 30,000 \text{ cP}^2$

3) $H_a: \sigma^2 < 30,000 \text{ cP}^2$

$$4) \chi^2_0 = \frac{(n-1)s^2}{\sigma^2} = \frac{(15-1)(106)^2}{30,000} = 5.2$$

Reference Distribution: χ^2_{14}

5) Rejection Criteria: Reject H_0 if $\chi^2_0 < \chi^2_{1-\alpha, n-1}$

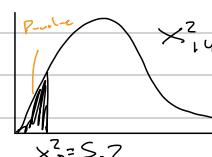


6) Computation: $5.2 < 6.57$

7) Conclusion: Reject H_0 . We have evidence to reject the claim that the variance is 30,000 cP². Having evidence to accept the hypothesis that variance is less than 30,000 cP². We accept H_1 . (or accept H_1 , but not accept H_0)

purple

Reject: If $\chi^2_0 < \chi^2_{1-\alpha, n-1}$



$$\chi^2_0 = 5.2$$

$\chi^2_{1-\alpha, n-1}$ levels below

$$0.490 \text{ and } 0.975$$

\rightarrow purple is below 1-tail

$$(0.01 < p < 0.025) \rightarrow \text{both} < 2 \text{ so reject } H_0!$$

Fixed-Significance Level

critical $\chi^2_{0.05, 14}$ 'accept' region

How do I do this in Minitab? Stat > basic stat > 1 variance

Confidence Intervals for Inference on the Variance (χ^2)

Suppose x is a normal random variable with unknown mean μ and unknown variance σ^2 . From a sample of n observations, the sample mean \bar{x} and sample variance s^2 are computed. Then:

(1 - α). Two-sided CI

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2, n-1}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha/2, n-1}}$$

Where $\chi^2_{\alpha/2, n-1}$ denotes the percentage point of the chi-square distribution such that $P(\chi^2_{n-1} \geq \chi^2_{\alpha/2, n-1}) = \alpha/2$ and $\chi^2_{1-\alpha/2, n-1}$ denotes the percentage point of the chi-square distribution such that $P(\chi^2_{n-1} \geq \chi^2_{1-\alpha/2, n-1}) = 1 - \alpha/2$

(100 - α). Upper bound

$$\sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha, n-1}}$$

Where $\chi^2_{1-\alpha, n-1}$ denotes the percentage point of the chi-square distribution such that $P(\chi^2_{n-1} \geq \chi^2_{1-\alpha, n-1}) = 1 - \alpha$

(100 - α). Lower bound

$$\frac{(n-1)s^2}{\chi^2_{\alpha, n-1}} \leq \sigma^2$$

Where $\chi^2_{\alpha, n-1}$ denotes the percentage point of the chi-square distribution such that $P(\chi^2_{n-1} \geq \chi^2_{\alpha, n-1}) = \alpha$

For our example (Upper bound)

$$\sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha, n-1}} \rightarrow \sigma^2 \leq \frac{(15-1)(106)^2}{\chi^2_{0.05, 14}}$$

$$\chi^2_{1-\alpha, n-1} = \chi^2_{0.05, 14} = 6.57$$

$$\Rightarrow \sigma^2 \leq \frac{(14)(106)^2}{6.57} = 234.42 \text{ cP}^2$$

6-sigma Example/Application

Inference on the variance of a distribution

Rubber can be added to asphalt to reduce road noise. The viscosity (cP) of 15 specimens of asphalt is measured. The asphalt specification limit is 1100 cP away from the mean in either direction – what is the sigma level of this process given that we know $\sigma = 173$ (from our hypothesis test) and we assume the mean of the population is on target?

$$\text{Sigma level} = \frac{\text{Spec limit} - \text{target}}{\sigma} = 1100$$

$$= 6.35 \rightarrow \text{at least } 6.35$$

Data

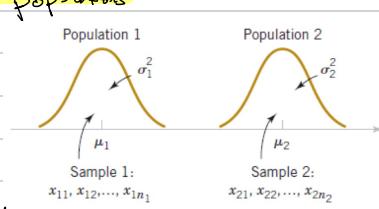
3193
3124
3153
3145
3093
3466
3355
3079
3182
3227
3256
3332
3204
3282
3170

1-variance Test Assumptions

- The data are continuous.
- The data are randomly sampled.
- The data are normally distributed.

Inference on means of two normally distributed populations

Let's say population 1 has a mean of μ_1 and a variance of σ_1^2 and population 2 has a mean of μ_2 and a variance of σ_2^2 and inferences are based on two random samples of size n_1 and n_2 respectively.



Want to know: is there a difference between two populations

Two-Sample Z-test

Statistical inference on the difference in means $\mu_1 - \mu_2$ of the populations below with known variances of σ_1^2 and σ_2^2

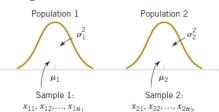
$$\text{Null: } H_0: \mu_1 - \mu_2 = \Delta_0$$

Refers Dist: $N(0,1)$

$$\text{Alt: } H_1: \mu_1 - \mu_2 \neq \Delta_0 \quad (\text{two-tail})$$

Rejection Criteria: Same as one-sample Z-test

$$\text{Test statistic: } Z_0 = \frac{\bar{x}_1 - \bar{x}_2 - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$



Summary

$$\text{Null hypothesis: } H_0: \mu_1 - \mu_2 = \Delta_0$$

$$\text{Ref: } N(0,1) \quad (10-2)$$

$$\text{Test statistic: } Z_0 = \frac{\bar{x}_1 - \bar{x}_2 - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Alternative Hypotheses	P-Value	Rejection Criterion for Fixed-Level Tests
$H_1: \mu_1 - \mu_2 \neq \Delta_0$	Probability above $ z_0 $ and probability below $- z_0 $, $P = 2[1 - \Phi(z_0)]$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1: \mu_1 - \mu_2 > \Delta_0$	Probability above z_0 , $P = 1 - \Phi(z_0)$	$z_0 > z_\alpha$
$H_1: \mu_1 - \mu_2 < \Delta_0$	Probability below z_0 , $P = \Phi(z_0)$	$z_0 < -z_\alpha$

Assumptions

- (1) $X_{11}, X_{12}, \dots, X_{1n_1}$ is a random sample from population 1.
- (2) $X_{21}, X_{22}, \dots, X_{2n_2}$ is a random sample from population 2.
- (3) The two populations represented by X_1 and X_2 are independent.
- (4) Both populations are normal.

- (5) The variances of the populations are known

Samples from two populations are independent
if the samples selected from one population have no relationship with the samples selected from the other population.

Example

Paint Drying Time A product developer is interested in reducing the drying time of a primer paint.

Two formulations of the paint are tested; formulation 1 is the standard chemistry, and formulation 2 has a new drying ingredient that should reduce the drying time. From experience, it is known that the standard deviation of drying time is 8 minutes, and this inherent variability should be unaffected by the addition of the new ingredient. Ten specimens are painted with formulation 1, and another 10 specimens are painted with formulation 2; the 20 specimens are painted in random order. The two sample average drying times are $\bar{x}_1 = 121$ minutes and $\bar{x}_2 = 112$ minutes, respectively. What conclusions can the product developer draw about the effectiveness of the new ingredient, using $\alpha = 0.05$?

$$H_0: \mu_1 - \mu_2 = 0 \quad (n_1 = n_2)$$

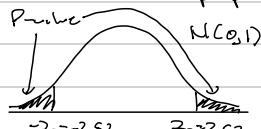
$$H_1: \mu_1 - \mu_2 \neq 0 \quad (n_1 \neq n_2)$$

$$Z_0 = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{121 - 112}{\sqrt{\frac{8^2}{10} + \frac{8^2}{12}}} = 2.52$$

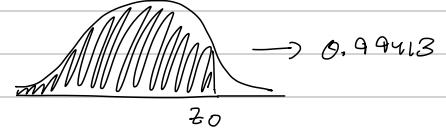
$$P\text{-value: Reject } H_0 \text{ if } p\text{-value} < \alpha = 0.05$$

$$P\text{-value} = 0.012 < \alpha = 0.05$$

Reject H_0 and accept H_1 , and conclude there is a difference btwn the drying time of the two



$$p\text{-value} = 2(1 - 0.9943) = 0.012$$



Confidence Intervals (do they cross zero?)

Confidence Intervals (do they cross zero!?)

100(1- α)% two-sided confidence interval on $\mu_1 - \mu_2$

$$\bar{x}_1 - \bar{x}_2 - z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

where $Z_{\alpha/2}$ is the value of the $N(0,1)$ such that $P(z \geq Z_{\alpha/2}) = \alpha/2$

$$\mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + z_\alpha \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$\bar{x}_1 - \bar{x}_2 - z_\alpha \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2$$

where Z_α is the value of the $N(0,1)$ such that $P(z \geq Z_\alpha) = \alpha$

Two-Sample t-test (variance unknown)

Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$ \leftarrow unknown var, but assumed true

Suppose we have two independent populations with unknown means μ_1 and μ_2 and equal variances $\sigma_1^2 = \sigma_2^2 = \sigma^2$ and we wish to know if the means are different:

H₀: $\mu_1 - \mu_2 = \Delta_0$ ($\mu_1 = \mu_2$)

H_a: $\mu_1 - \mu_2 \neq \Delta_0$ ($\mu_1 \neq \mu_2$)

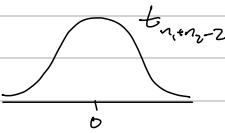
$$\text{Test Statistic: } T_0 = \frac{\bar{x}_1 - \bar{x}_2 - \Delta_0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Since $n_1 = n_2$

Pooling Sample Variance

$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1 + n_2 - 2}$$

Weighted average of two sample variances



Rejection Region: See below exercise

t-dist with $n_1 + n_2 - 2$ degrees of freedom

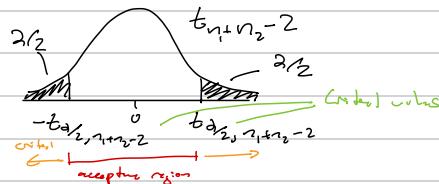
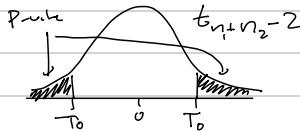
Exercise: Sketch out fixed rejection criteria & p-value for two-sided H_a

P-value:

Reject H₀: P p-value < α

Fixed:

Reject: P |T₀| > t_{α/2, n₁+n₂-2} or T₀ < -t_{α/2, n₁+n₂-2}



Example

Yield from a Catalyst Two catalysts are being analyzed to determine how they affect the mean yield of a chemical process. Specifically, catalyst 1 is currently used; but catalyst 2 is acceptable. Because catalyst 2 is cheaper, it should be adopted, if it does not change the process yield. A test is run in the pilot plant

Observation Number	Catalyst 1	Catalyst 2
1	91.50	89.19
2	94.18	90.95
3	92.18	90.46
4	95.39	93.21
5	91.79	97.19
6	89.07	97.04
7	94.72	91.07
8	89.21	92.75
	$\bar{x}_1 = 92.255$	$\bar{x}_2 = 92.733$
	$s_1 = 2.39$	$s_2 = 2.98$

1) Interest: \rightarrow difference between 2 means, specifically yield of catalyst 1 and catalyst 2, variance unknown \rightarrow two-sample t-test
 \hookrightarrow Assume equal variances \leftarrow rule of thumb later

2) H₀: $\mu_1 - \mu_2 = 0$

3) H_a: $\mu_1 - \mu_2 \neq 0$

$$4) T_0 = \frac{\bar{x}_1 - \bar{x}_2 - 0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{92.255 - 92.733}{2.7 \sqrt{\frac{1}{8} + \frac{1}{8}}} = -0.35$$

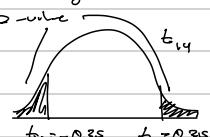
$$5) S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1 + n_2 - 2} = \frac{7 \cdot 2.39^2 + 7 \cdot 2.98^2}{8+8-2} = 2.3$$

$$\rightarrow S_p = \sqrt{S_p^2} = \sqrt{2.3} = 2.3$$

Checking in Minitab: stat -> basic stat -> 2 sample t -> input columns and other info (CLICK ASSUME EQUAL VARIANCE UNDER OPTIONS!)

6) Reject: P p-value < → 0.05

$$7) t_0 = -0.35$$



7) p-value > α

Fail to reject H₀, redundant new evidence to say the yield from catalyst 1 is different from catalyst 2.

0.5 < p-value < 0.8