



# Midterm Exam 1

---



# Chapters 1 & 2 - Concepts

## Key Terminology

**Six Sigma:** A data-driven management strategy

**Quality:** Fitness for use

**Two aspects:** a) Quality of design - high quality?

b) Quality of conformity - is the high quality consistent?

**Characteristics** or "dimensions of quality" that affect fitness for use as a product

(i) Performance performs required job (ii) Aesthetics product appearance

(iii) Reliability how often will it fail (iv) Features what the product does

(v) Durability lifetime → will it meet it (vi) Perceived Quality Reputation

(vii) Scrutability easy to repair (viii) Conformance to Standards

**Producing a Service...** list those dimensions of quality

(i) Responsiveness time to reply to requests

(ii) Professionalism knowledge, timeliness

(iii) Attentiveness caring, personalized attention

**Hidden Factors:** The part of the plant the customer doesn't see which affects w/bd quality

**Quality Engineering:** Consists of activities to ensure that ① that quality characteristics are at desired levels, and ② variability is at its minimum

→ **Quality Characteristics:** Important elements of a product/service to the customer

**Variability:** Describes or quantifies differences within a set of data or parts or c2qc

→ **Statistical Methods:** Analysis, interpretation, and presentation of numerical data

→ **Spec-Reading:** Desired measurements for quality characteristics This is your target value

**Nonconformity:** Failure to meet spec

**Defect:** Nonconforming that hurts usc

## Variability

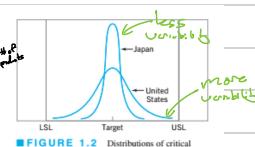
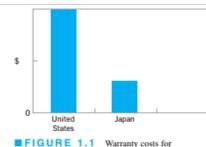
• Quality is inversely proportional to variability

◦ Variability will impact all quality dimensions

• Variability leads to greater cost & waste

example to right compares domestic vs. imported car quality

• Why variability leads to more cost for a company



## Quality Costs

Prevention Costs	Internal Failure Costs
Quality planning and engineering	Scrap
New products review	Rework
Product/process design	Restest
Process control	Failure analysis
Burn-in 6-Sigma comes 20 or less	Downtime
Training & Process	Yield losses
Quality data acquisition and analysis	Downgrading (off-specing)
Appraisal Costs	External Failure Costs
Inspection and test of incoming material	Complaint adjustment
Product inspection and test	Returned product/material
Materials and services consumed	Warranty charges
Maintaining accuracy of test equipment	Liability costs
	Indirect costs

- Products found in plant outside of specifications result in higher costs and waste
- Prevention & Appraisal costs are cheaper costs and result in more consistent products and cheaper costs down the line

**Internal Failure & External Failure costs >> Prevention & appraisal costs!**

**Example: Cost of Variance** ← Find manufacturing cost per good product

Illustrative Example: We are making 100 mechanical components per day where 75% of the parts conform to specifications, and the other 25% are non-conforming. Of the 25% of nonconforming, 60% can be reworked and the other 40% are scraped. Each part costs \$20 to manufacture and an additional \$4 to be reworked.

$$\frac{\text{cost}}{\text{good}} = \frac{\text{cost to make} (\# \text{made total}) + \text{cost to rework} (\# \text{reworked})}{\# \text{of goods sold}}$$

$$= \frac{\$20(100) + \$4(100)(0.25 \cdot 0.60)}{70 + 15} = \$22.69$$

**Now, the Benefit of Improvement** ← Find manufacturing cost per good product

Illustrative Example: We improve the previous process such that now 95% of the parts conform to specifications, and the other 5% are non-conforming. Similar to before, of the 5% of nonconforming, 60% can be reworked and the other 40% are scraped. Each part costs \$20 to manufacture and an additional \$4 to be reworked.

$$\frac{\text{cost}}{\text{good}} = \frac{\$20(100) + \$4(100)(0.05)(0.60)}{95 + 3}$$

$$= \$20.53$$

## Quality Engineering

**Process:** A system of inputs and outputs

### Example: Doing a process

**Example:** A hospital is trying to increase the quality of drug administration. To do this, it is considering providing patients with bar-coded wristbands to help guide workers. Your team is charged with studying the effects of bar-coding by carefully watching 250 episodes in which drugs are given to patients without bar coding and 250 episodes with bar coding. Every time a drug is administered you will check the amount, if any, of discrepancy between what was supposed to be given and what was given.

**Control Charts:** A process monitoring technique which tracks averages in a quality characteristic with time or sample number

- **Control:** Where the characteristic should be; find abnormal sources of variation
- **Control Limit:** Defined by statistics. Alert the user to unusual variability  
↳ Different than spec limits

**Designed experiments:** Discovers very variable factors have a significant influence on quality characteristics in a process

## Six Sigma

**Broader Definition:** Statistical based, data driven, management approach and continuous quality improvement methodology for eliminating defects in a product, process, or service

↳ Reduces variability to a level where defects are unlikely

### Three Elements of Six Sigma

- (i) **Quality Planning:** Listening to the voice of your customer
- (ii) **Quality Assurance:** Establishing a system to prevent quality issues from arising
- (iii) **Quality Control & Improvement:** A set of specific steps and tools to ensure products meet requirements and are continuously improved

**Name Meaning:** Refers to the quality level target such that sigma ( $\sigma$ , or variance  $\sigma^2$ ) should be small enough so that  $6\sigma$  is within the specification limits

**Sigma Quality Level:** The Probability that product/service is non-defective

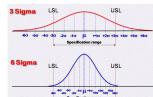
Why is Six Sigma Selected?

Suppose 100 parts:  $3\sigma$ :  $(.9973)^{100} = .7631$  is probability non-defective  $\rightarrow 24\%$  defects  
 $6\sigma$ :  $(.9999998)^{100} = .99999998$  non-defective  $\rightarrow \sim .000002\%$  defects

Some companies use other multiples of  $\sigma$ , depending on what required by company/customers

### Example of Six Sigma (will be on HW)

**Illustrative Example:** Let's say you are making a part that has a radius of 10 mm. Specification limits are between 9 (LSL) and 11 (USL) mm. Before you applied quality improvement, the standard deviation (sigma - a measure of variability) was 1 mm, and after you applied improvements, the standard deviation (sigma) decreased to 0.1. Is the new process 6 sigma quality?



$$\frac{|USL - LSL|}{\sigma \text{ (standard deviation)}} = \frac{|10mm - 10mm|}{1mm} = 1 \quad \text{For old 1\sigma level}$$

$$= \frac{|10mm - 10mm|}{0.1mm} = 10 \quad \text{For new 10\sigma level}$$

## Lean Manufacturing

**Def:** Another management philosophy aimed at eliminating waste (e.g., time and materials)

**Process Cycle Efficiency (PCE):**  $(\text{Value added time}) / (\text{Process time})$

$\hookrightarrow \text{Value added time} = \text{time spent making the product more valuable}$

(Number of units)

**Process Cycle Time (PCT):**  $(\text{Number of items in progress}) / (\text{Completion rate})$   $\hookleftarrow$  backlog issues as an example

Why is a slow process (low PCE or high PCT) an expensive process?

- Hidden Factory  $\rightarrow$

- (i) Customers don't like waiting
- (ii) More handling = more personnel
- (iii) More opportunity for damage or loss
- (iv) Inventory has to be higher
- (v) More documentation

## DMAIC Process

**Def:** A Project-based 5-step process and problem-solving procedure

$\hookrightarrow$  Define  $\rightarrow$  Measure  $\rightarrow$  Analyze  $\rightarrow$  Improve  $\rightarrow$  Control

**Toll gates:** Where projects are reviewed to ensure on track

**5 stages of DMAIC** (will be on HW too)

### (i) Define Stage

- Main Obj: Identify a great opportunity and verify that it represents a legitimate value opportunity or breakthrough

◦ Breakthrough: Major improvement to a process or product

◦ Value Opportunity: The financial opportunity at stake

- Tools used

◦ SIPOC diagram: High level map of process

$\rightarrow$  Suppliers: Those who provide information, material or other items that are used in the process

$\rightarrow$  Input: The actual information or material provided

$\rightarrow$  Process: The set of steps required to do the work

$\rightarrow$  Output: The product delivered to the customer  $\rightarrow$  Other dept. in same company

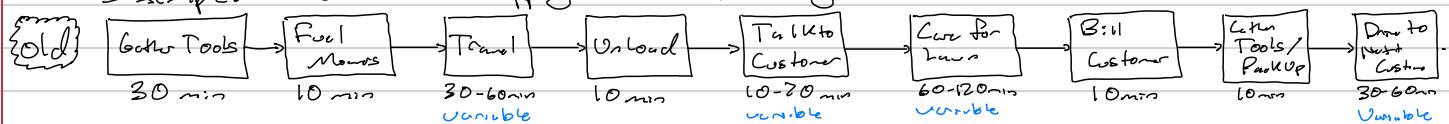
$\rightarrow$  Customers: who buys the product (internal or external)

- Process Mapping: Involves creating flow diagrams for systems

◦ Value stream mapping: A variant of process mapping where engineers focus on steps which could be eliminated or simplified to reduce waste

- Value stream: The min amount of processing steps from raw material to customer

**Example: Value Stream Mapping (Lawn Mowing)**



**Possible Improvements:** Using customer's lawnmower, store SOP (come on track), protocol, personnel/intens., eliminate/reduce customer interactions, optimize route (graph shortest distance)/logistics

- Pareto charting: Plots frequency (# of parts) or cost (time/money) against causes

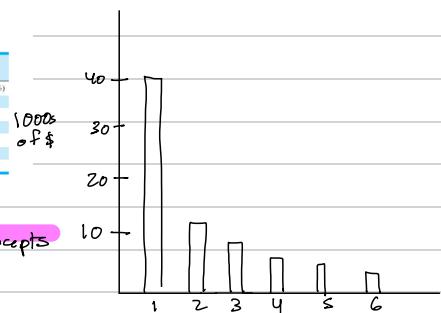
◦ Pareto Rule: 80% of the effects come from 20% of the causes

## • Graph Pareto by Hand (Example)

Often in order from largest cost to smallest

Cost or % defects is on y-axis, category is on the x-axis

Monthly Quality-Costs Information for Assembly of Printed Circuit Boards		
Type of Defect	Percentage of Total Defects	Scrap and Rework Costs
1 Insufficient solder	42%	\$37,500.00 (52%)
2 Missing components	20	12,000.00
3 Defective components	15	8,000.00
4 Missing components	10	5,000.00
5 Cold solder joints	7	3,000.00
6 All other causes	5	4,600.00
<b>Total:</b>	<b>100%</b>	<b>\$72,200.00</b>



Can also do this with Lean concepts

Project Charter: Short document  
with clear focus to product.

Example to right, but worse across companies

<b>Business Case</b>	<b>Opportunity Statement</b>																					
<ul style="list-style-type: none"> <li>This project supports the business quality goals, namely (a) reduce customer resolution cycle time by x% and (b) improve customer satisfaction by y%.</li> </ul>	<ul style="list-style-type: none"> <li>An opportunity exists to close the gap between our customer expectations and our actual performance by reducing the cycle time of the customer return process.</li> </ul>																					
<b>Goal Statement</b>	<b>Project Scope</b>																					
<ul style="list-style-type: none"> <li>Reduce the overall response cycle time for returned product from our customers by x% year to year.</li> </ul>	<ul style="list-style-type: none"> <li>Overall response cycle time is measured from the receipt of a product return to the time that either the customer has the product replaced or the customer is reimbursed.</li> </ul>																					
<b>Project Plan</b>	<b>Team</b>																					
<table border="1"> <tr> <th>Activity</th> <th>Start</th> <th>End</th> </tr> <tr> <td>Define</td> <td>6/04</td> <td>6/30</td> </tr> <tr> <td>Measure</td> <td>6/18</td> <td>7/06</td> </tr> <tr> <td>Analyze</td> <td>7/15</td> <td>8/09</td> </tr> <tr> <td>Improve</td> <td>8/15</td> <td>9/30</td> </tr> <tr> <td>Control</td> <td>9/15</td> <td>10/30</td> </tr> <tr> <td>Track Benefits</td> <td>11/01</td> <td></td> </tr> </table>	Activity	Start	End	Define	6/04	6/30	Measure	6/18	7/06	Analyze	7/15	8/09	Improve	8/15	9/30	Control	9/15	10/30	Track Benefits	11/01		<ul style="list-style-type: none"> <li>Team Sponsor</li> <li>Team Leader</li> <li>Team Members</li> </ul>
Activity	Start	End																				
Define	6/04	6/30																				
Measure	6/18	7/06																				
Analyze	7/15	8/09																				
Improve	8/15	9/30																				
Control	9/15	10/30																				
Track Benefits	11/01																					

## (i:) Measure Stage

- Objectives:** Explain & understand the current state of the process
- Tools Used:** Quantify what the current process is performing at
- Measurement Capability:** Assess if the current measurement systems are adequate
- Measure Performance:** Check to understand variability
- Tools Used:** Histograms, bar charts, run charts, scatter diagrams

## (ii:) Analyze Stage

- Objectives:** Use data to begin to determine cause and effect relationships and sources of variation

### Common Causes of Variability: Internal Sources (noise)

### Assignable Causes of Variability: External Sources

#### Tools Used

- Control Charts:** Used to separate the common and assignable causes of variation
- Hypothesis Testing & Confidence Intervals:** Used to determine if different conditions produce significant results
- Regression Analysis:** Allows models of the process to be built
- Failure Modes & Effects Analysis (FMEA):** A method for identifying and prioritizing sources of variability or defects in a process or product

↳ For each item, we determine:

(i) Likelihood something will go wrong | (not likely) - 10

(ii) Ability to detect the failure or defect | (likely) - 10

(iii) Severity | (not severe) - 10

Risk Priority #: (i) x (ii) x (iii)

## Example

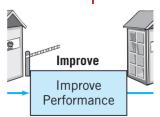
- Severity, 1 (not severe) - 10
- Probability of occurrence, 1 (not likely) - 10
- Probability of detection, 1 (likely) - 10

$$RPN = A \times B \times C$$



$RPN = RPK$  Priority Number

FAILURE MODE & EFFECTS ANALYSIS (FMEA)					Date: 1/1/2018
Process Name: Left Front Seat Belt Install Process Number: SBT 445					Revision: 1.3
Failure Mode	A) Severity	B) Probability of Occurrence	C) Probability of Detection	Risk Preference Number (RPN)	
1) Select Wrong Color Seat Belt	5	4	3	60	Audited
2) Seat Belt Bolt Not Fully Tightened	9	2	8	144	
3) Trim Cover Clip Misaligned	2	3	4	24	



### (i) Improve Stage

#### Objectives

- Generate and quantify potential solutions
- Evaluate and select final solution
- Verify and gain approval for final solution

• **Objective:** Use creative thinking & info gathered previously to make specific suggestions that will lead to desired solution process. Confirmation is important

#### Tools Used

- Mistake Proofing (example)
- Optimization
- Pilot Testing
- Conformance Experiments

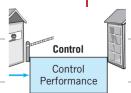
- Documentation
  - (i) Solution obtained
  - (ii) Alternatives Considered
  - (iii) Confirmation tests
  - (iv) Risk Analysis

### (ii) Control Stage

• **Objective:** Complete all remaining work and hand off to improved process so gains in progress are institutionalized

#### Tools Used

- Control Charts
- Transition Plans



#### Objectives

- Develop ongoing process management plans
- Mistake-proof process
- Monitor and control critical process characteristics
- Develop out-of-control action plans

# Chapter 3 - Quantifying Stats in Quality Characteristics

## Statistics

### Sample Mean: Average Value

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{\sum_{i=1}^n x_i}{n}$$

where  $n$  is the number of data points in the sample  
and  $x_1, x_2, x_3, \dots, x_n$  are data points

For this data set: 12, 12, 15, 18, 21, 22, 22, 23, 25, 25, 26, 27, 28, 29, 30, 30, 31

$$\bar{x} = \frac{12+12+15+\dots+31}{17} = \frac{396}{17} = 23$$

### Sample Variance:

{Example var:  $\sum(x_i - \bar{x})^2$ }

$$s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}$$

### Standard Deviation: $s = \sqrt{s^2}$

{Example var:  $\sum(x_i - \bar{x})^2$ }

For this data set: 12, 12, 15, 18, 21, 22, 22, 23, 25, 25, 26, 27, 28, 29, 30, 30, 31

$$s^2 = \frac{(12-23)^2 + (12-23)^2 + \dots + (31-23)^2}{17-1} = 37$$

$$s = \sqrt{37} \approx 6.08$$

MINITAB: Stat > Basic Statistics > Display Descriptive Statistics > In Variables add data

### Illustrative Example

Sample 1	Sample 2
$x_1 = 1$	$x_1 = 1$
$x_2 = 3$	$x_2 = 5$
$x_3 = 5$	$x_3 = 9$
$\bar{x} = 3$	$\bar{x} = 5$

Sample 3
$x_1 = 101$
$x_2 = 103$
$x_3 = 105$
$\bar{x} = 103$

Sample 1:  $s = 2$

Sample 2:  $s = 4$

Sample 3:  $s = 2$

## Tools to Describe Data Numerically and Visualize Variables

↳ Often easier to arrange sorted (lowest to highest) data

- **Percentiles:** Tells us how much of the data is below a certain point

12, 12, 15, 18, 21, 22, 22, 23, 25, 25, 26, 27, 28, 29, 30, 30, 31

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

↳ Values

↳ Rank (1-indexed)

- **Lower Quartile ( $Q_1$ , or 25<sup>th</sup> perc.):** Value with rank =  $(n+1)(\frac{1}{4})$

here:  $\frac{17+1}{4} = \text{rank of } Q_1 \rightarrow \frac{18+1}{2} = 10.5$

- **Middle Quartile ( $Q_2$ , or median, or 50<sup>th</sup> perc.):** Value with rank =  $(n+1)(\frac{1}{2})$

here:  $\frac{17+1}{2} = \text{rank of } Q_2 \rightarrow 10.5$

- **Upper Quartile ( $Q_3$ , or 75<sup>th</sup> perc.):** Value with rank =  $(n+1)(\frac{3}{4})$

here:  $(17+1)(\frac{3}{4}) = \text{rank of } Q_3 \rightarrow 28.5$

- **Interquartile Range:  $Q_3 - Q_1$ :**

here:  $Q_3 - Q_1 = 28.5 - 10.5 = 18$

- **Whisker Limits:** Max Limit =  $Q_3 + 1.5(Q_3 - Q_1)$

Min Limit =  $Q_1 - 1.5(Q_3 - Q_1)$

here: Max Limit = 43, Min Limit = 6

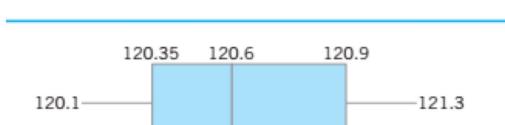
MINITAB: Stat > Basic Statistics > Display Descriptive Statistics > In Variables add data

- **Box Plot Features**

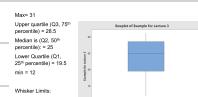
Displays 25<sup>th</sup>, 50<sup>th</sup> and 75<sup>th</sup> quartiles (ends and middle of the box)

- 25<sup>th</sup> and 75<sup>th</sup> are the ends of the box
- Median (50<sup>th</sup> percentile) is a single line through box
- The min and max of data (whiskers)
- Limits of whiskers (check) that the max and min do not go past (points that fall outside this can be outliers, represented as single data points and the new min or max is the next data point within the limit) – limit formula on next slide

TABLE 3.4 Hole Diameters (in mm) in Wing Leading Edge Ribs		
120.5	120.4	120.7
120.9	120.2	121.1
120.3	120.1	120.9
121.3	120.5	120.8



here:



Minitab: Graph > Boxplot > Simple > input data column into graph variables

## Excel

- Stdev is for sample data (use most often)
- Stdevp is for entire population

• **Histograms:** Plots of frequencies of data occurring w.r.t. the data sorted into bins/intervals

↳ **Bins:** The range that data will be sorted into  $\# \text{ bins} = \text{ceil}(\sqrt{n}) \leftarrow \text{round up}$   $\text{bin size} = \frac{\text{max-min}}{\# \text{ bins}}$

- i. Need to determine the bin range for continuous data
- ii. Best if bins are the same size
- iii. Rule of thumb: number of bins = square root of number of observations (round up if not a whole number)

For this data set: 12, 12, 15, 18, 21, 22, 23, 25, 25, 26, 27, 28, 29, 30, 30, 31

$$\# \text{ of bins} = \sqrt{17} = 4.1 \rightarrow 5$$

$$\text{bin size} = \frac{31-12}{5} = 3.8$$

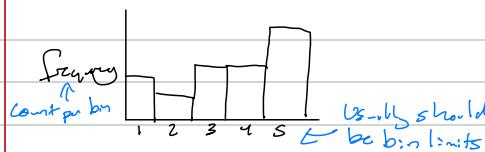
**Bin 1:** min + bin size  $\rightarrow 12 + 3.8 \rightarrow [12, 15.8)$  Inclusive below, Exclusive above

**Bin 2:** bin 1 end + bin size  $\rightarrow 15.8 + 3.8 \rightarrow [15.8, 19.6)$

**Bin 3:**  $[19.6, 25.4)$

**Bin 4:**  $[25.4, 29.2)$

**Bin 5:**  $[29.2, 31]$



Can be fraction w.r.t. count  
but data must not overlap

MINITAB: Graph > histogram > simple > add your column to graph variables  
To change bins: double click on histogram bars (twice), select bins tab, change number of intervals to number of bins (Cut point is how we graphed in class)

↳ Final bin should be inclusive above

**Significant Digits / Figures:** Numbers of digits contributing to measurement resolution

Rules:

- (i) Non-zero digits always significant

(ii) Any zeros between two significant digits are significant (leading zeros are not)

(iii) Trailing zeros in the decimal portion only are significant (unless indicated by a bar)

Examples:

How many sig. figs in each number?	0.00500 = 3
0.03040 = 4	10 <sup>2</sup> = 4

How would you write this # with the correct sig. figs?

0.899923 (report 2 sig. figs)
0.40
23000.0 (report 3 sig. figs)
23000

Will take away points for tons of sig. figs!

Just pay attention to them!

For your sample means: keep as many significant figures as the number with the lowest significant digits in your data set (same for median and quartiles)

For your standard deviation: report to the last place that is significant, but no more

Your Average and SD should have some # of places!

E.g. Sample mean = 4.09 Sample Standard Deviation = 0.00923. What do I report?  $4.09 \pm 0.01$

## Probability Distributions Introductory

• **Sample:** Collection of measurements from a larger source or population

• **Population:** All of the elements from a group

• **Probability Distribution:** Mathematical Model that relates value of a variable with its probability of occurrence

• **Continuous Distribution:** The variable is expressed on a continuous scale

• **Discrete Distribution:** The variable can only take on certain values

• **Writing out Probability Statements**

◦ **Discrete:** The probability that  $x$  is equal to  $x_i$ ?

$$P\{x=x_i\} = p(x_i)$$

• **Calculating mean and variance**

$$\mu = \begin{cases} \int_{-\infty}^{\infty} xf(x) dx, x \text{ continuous} \\ \sum_{i=1}^{\infty} x_i p(x_i), x \text{ discrete} \end{cases}$$

◦ **Continuous:** The probability that  $x$  is between  $a$  and  $b$ ?

$$P\{a \leq x \leq b\} = \int_a^b f(x) dx$$

$$\sigma^2 = \begin{cases} \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx, x \text{ continuous} \\ \sum_{i=1}^{\infty} (x_i - \mu)^2 p(x_i), x \text{ discrete} \end{cases}$$

• **Median:** 50<sup>th</sup> percentile

• **Mean:** Central Tendency (Average)

• **Variance:** Spread of the distribution

$M, \sigma^2$  (population model parameters)  
 $\bar{x}, s^2$  (sample data notation)

Ray attention to this notation

## The Normal Distribution

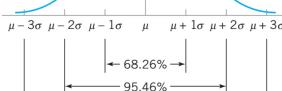
- For a random normal variable  $x$ , the probability distribution of  $x$  is:  $P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

↳ Short hand  $N(\mu, \sigma^2)$

○ Continuous, symmetric, "bell-shaped"

•

$$\text{General Formula: } P\{\mu - 3\sigma \leq x \leq \mu + 3\sigma\} = \int_{\mu-3\sigma}^{\mu+3\sigma} f(x) dx$$



In a Probability Band:

$$P\{\mu - \sigma \leq x \leq \mu + \sigma\} = 0.68$$

Due to symmetry:

$$P\{x > \mu\} = P\{x < \mu\} = 0.5$$

Holds for any normally distributed variable 'x'

- Got to these properties: Set  $\sigma=1$ ,  $\mu=0$  so  $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$\left\{ \begin{array}{l} a = \mu - 1\sigma = -1 \\ b = \mu + 1\sigma = 1 \end{array} \right.$$

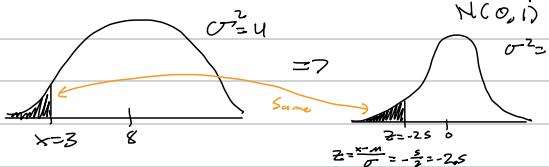
$$\int_{-1}^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \text{(using mathematica)} 0.68$$

• Standardization:  $N(\mu, \sigma^2) \Rightarrow N(0, 1)$

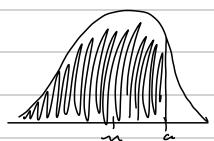
• Standard Normal Distribution:  $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

• To standardize any random variable:  $z = \frac{x-\mu}{\sigma}$   $z$  is a standard normal variable

• Drawing out normal distributions: Suppose  $N(8, 4)$

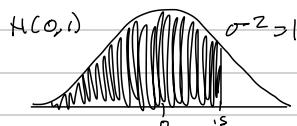


• Cumulative Normal Distribution: The probability that the variable  $x$  is less than or equal to some value  $a$



Example: Assume  $Z$  is a standard normal random variable, and you want to find the probability that  $Z$  is less than or equal to 1.5

$$P\{Z \leq 1.5\} = 0.93$$

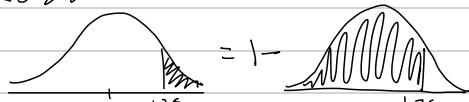


General Probability Demonstration

$$(i) P(Z > 1.26) = 1 - P(Z \leq 1.26)$$

$$(ii) P(Z \leq -0.86) = P(Z > 0.86) = 1 - P(Z \leq 0.86)$$

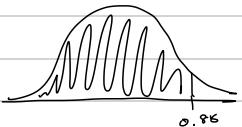
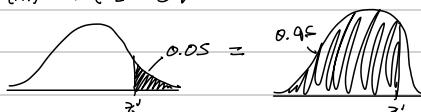
$N(0, 1)$



$$= 1 - 0.89616 = 0.1039$$

0.10

$$(iii) P(Z > Z') = 0.05$$



# Chapter 4 - Hypothesis Testing

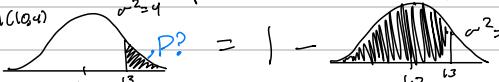
**Example:** Suppose the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 millamps and a variance of 4 millamps<sup>2</sup>. What is the probability that the measurement will exceed 13 millamps?

1) Convert the word problem to mathematical expression and sketch the situation.

$$\mu = 10$$

$$\sigma^2 = 4$$

$$P(x \geq 13)$$

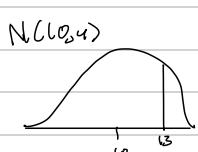


$$P? = 1 -$$

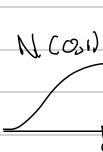
$$P(x \leq 13)$$

3) Convert sketch and formula to standard normal (if using tables in back of book) and a cumulative normal distribution (so it is compatible with the table in the book and Minitab)

Minitab: Calc > Probability dist. > Normal



=>



$$z = \frac{x - \mu}{\sigma} = \frac{13 - 10}{2} = 1.5$$

$$1 - P(z \leq 1.5) = 0.07$$

$$\rightarrow P(z \geq 1.5)$$

→ What about the integral?

$$P(z \geq 1.5) = \int_{1.5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \quad \text{or} \quad P(x \geq 13) = \int_{13}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

## Checking for normality

→ How do we know if the normal distribution is a reasonable model for our data?

- **Probability Plotting:** Graphical method for determining whether sample data conform to a hypothesized distribution based on subjective visual examination.

- **Constructing a normal prob. plot by hand:** using  $z_i = \frac{x_i - \bar{x}}{s}$

## How-to (Steps)

1) Ranking data and ordering it  $(x_1, x_2, \dots, x_n)$ ,  $i$  = the ranked value,  $n$  = total number of values

2) Calculating cumulative frequency  $F_i = i/n$  for each data point

to tell us the % of data that lie below certain ranked item

$z_i = \frac{x_i - \bar{x}}{s} = \frac{x_i - 25}{4} = 0.25$

3) Find a  $z_j$  value for each rank using the cumulative frequency value is the probability

Find  $z_j$  s.t.  $P(z \leq z_j) = \frac{i-0.5}{n}$ . Tells us the value of  $z_j$  if the data are normally distributed

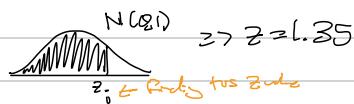
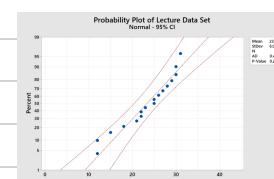
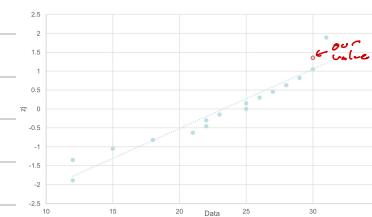
4) Plot the actual data ( $x$ -axis) vs. corresponding  $z$  ( $y$ -axis)

Normal Probability Plot in Minitab - Minitab: Graph > probability plot > graph variables, choose column with desired data

Note: each value is plotted versus the cumulative frequency with the y-axis is transformed so that the fitted distribution forms a straight line if normally distributed.

$$C. \text{ Freq } \left( \frac{i-0.5}{n} \right)$$

$$\text{For rank } 16: \frac{16-0.5}{17} = 0.91 \rightarrow z_j: P(z \leq z_j) = 0.91$$



$z_j \approx$  body to bone

$P(z \leq z_j) \approx 0.91$

In literature,

$$\alpha = 0.05$$

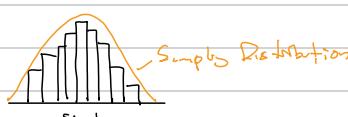
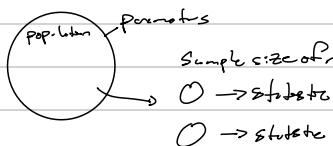
$$\text{or } \alpha = 0.01$$

## Statistics Basic

- **Statistical Inference:** Drawing conclusions about a population based on sample from the population (of them selected randomly)

- **Statistics:** Any function of sample data that doesn't contain unknown parameters

- **Sampling Distribution:** The probability distribution of statistics



Suppose  $x_1, x_2, \dots, x_n$  are normally distributed random variable w/ given  $\mu$  and  $\sigma^2$

→ If  $x_1, x_2, \dots, x_n$  is a random sample of size  $n$ , the distribution of the sample mean is:

Normally distributed with mean of  $\mu$  and covariance of  $\frac{\sigma^2}{n}$

$$M_{\bar{x}} = \mu$$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

$$N(M, \frac{\sigma^2}{n}) \text{ shorthand}$$

- **Central Limit Theorem:** Implies that the sample mean will be approximately normally distributed for large sample sizes, regardless of the distribution you are sampling from.

→ **(Large) Sample size:** Some suggest  $n > 20$  (non-normal),  $n = 3-4$  (closer to normal)

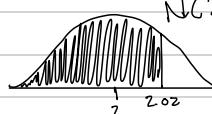
### • Example

Electrodes are manufactured with a population mean thickness of 2.0 mm and a standard deviation of 0.048 mm. Find the probability that a random sample of  $n = 16$  electrodes will have a sample mean diameter less than or equal to 2.02 mm? (keep 2 sig figs)

↳ Central Limit Theorem Applies

$$\bar{X} \Rightarrow N(\mu, \frac{\sigma^2}{n}) \quad n=16, \mu=2.0 \text{ mm}, \sigma=0.048 \text{ mm}, \sigma^2=0.0023$$

$$P(\bar{X} \leq 2.02)$$



$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

Now!  
For samples



$$P(Z \leq 1.66) = 0.45$$

### Hypothesis Testing

- **Parameters:** What probability distributions we described with

e.g. normal dist. is described with  $\mu, \sigma$

- **Statistical Hypothesis:** A statement about values of parameters of a probability distribution

- **Null Hypothesis:** States no statistical difference exists ( $H_0$ )

- **Alternative Hypothesis:** Contradicts to null hypothesis ( $H_1$ )

**Example:** Suppose we think the mean inside diameter of bearing is 1.500 in

$$H_0: \mu = 1.500 \quad H_1: \mu \neq 1.500 \quad \text{two-sided}$$

$$\left\{ \begin{array}{l} \mu < 1.500 \\ \mu > 1.500 \end{array} \right. \quad \text{one-sided}$$

- Two outcomes of a hypothesis test:

(i) Reject  $H_0$  or (ii) Fail to reject  $H_0$

- **Test Statistic:** Used to test a hypothesis and is computed from a random sample of the population

- **Sampling Distribution (Null Distribution):** The sampling distribution of the test statistic assuming that  $H_0$  (the null hypothesis) is true

- **Critical Region:** Sets values of the test statistic that lead to the rejection of the null hypothesis

↳ Can either fail to reject or reject  $H_0$ , critical region defines these cutoffs

- Two types of errors

- (i) **Type I errors:** If the null hypothesis is rejected - but it is true

↳ Defn:  $\alpha = P[\text{type I error}] = P[\text{reject } H_0 | H_0 \text{ is true}]$  This is specified

- (ii) **Type II errors:** If the null hypothesis is not rejected - but it is false

↳ Defn:  $\beta = P[\text{type II error}] = P[\text{fail to reject } H_0 | H_0 \text{ is false}]$  Hard to control

○ Type II errors are uncontrollable & unknown, so we try to minimize + avoid it

- **Conclusions:** Must be worded very clearly

- **Rejection of Null:** A strong conclusion because it's controlled

○ to be small

- **Failing to Reject:** Weak conclusion because it's hard to control  $\beta$

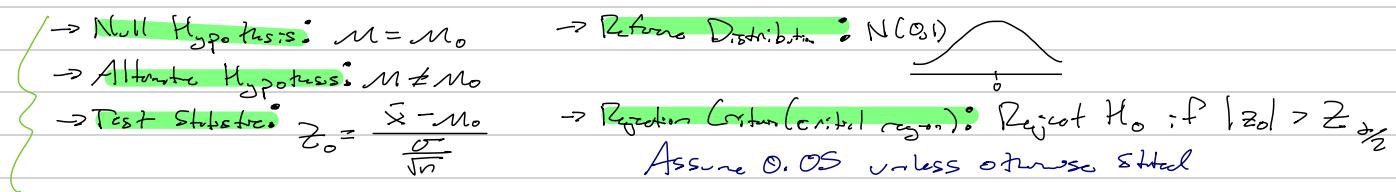
↳ to control  $\beta$

The Claim is the	The Decision is	The Conclusion is
Null Hypothesis	Fail to Reject the Null Reject the Null	The evidence is <i>not sufficient</i> to reject the claim. The evidence is <i>sufficient</i> to reject the claim.
Alternative Hypothesis	Fail to Reject the Null Reject the Null	There is <i>insufficient</i> evidence to support the claim. There is <i>sufficient</i> evidence to support the claim.

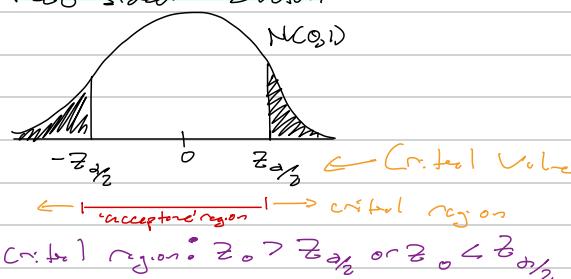
We never say accept the null hypothesis

## One Sample Z-test & Inference on the Mean of a Population, Variance Known

Suppose  $x$  is a random variable with unknown mean  $\mu$  and a known variance  $\sigma^2$  and we wish to test the hypothesis that the population mean is equal to a standard value ( $H_0$ ).



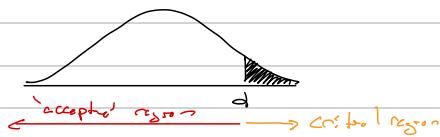
### Two-sided Sketch



### One-sided Test

$$H_0: \mu = \mu_0$$

$$H_1: \mu > \mu_0$$



### General Procedure C for this class!

- Parameters of Interest: From the problem context, identify known/unknown parameters of interest, the test you're going to do and identify type of problem.
- Null Hypothesis: State the null hypothesis  $H_0$ .
- Alternative Hypothesis: State the alternative hypothesis  $H_1$ .
- Test Statistic: Define an appropriate test statistic & reference distribution.
- Reject  $H_0$ : State rejection criterion for the null hypothesis. Should be drawn on plot w/ test statistic.
- Computations: Compute necessary sample quantities, substitute those into eqn for test statistic, and compute test value. Compare to rejection criteria.
- Draw Conclusion: Decide whether or not  $H_0$  should be rejected and report that in the problem context. Phrase appropriately!

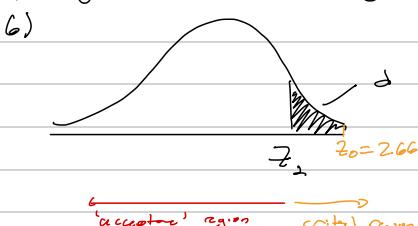
### Example

Example: You are testing the response time of a distributed computer system. The system manager wants to know whether the mean response time to a specific command exceeds 75 ms. Previous experience has determined the standard deviation is 8 ms. Use  $\alpha = 0.05$ . A test is conducted executing the command 25 times, and the mean response time is 79.25 ms.

4) Test Statistic:  $Z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{79.25 - 75}{8/\sqrt{25}} = 2.66$

Reference Dist:  $N(0,1)$

5) Rejection criterion:  $|Z_{\text{reject}}| > Z_{\alpha/2} = Z_{0.05}$



1) Parameters of Interest:  $\mu$ , inferring mean of population, known average mean response time

↳ Test: One-sample z-test

2)  $H_0: \mu = 75 \text{ ms}$

3)  $H_1: \mu > 75 \text{ ms}$  (one-sided)

7) Conclusion:  $Z_0 > Z_{\alpha}$

Reject the null Hypothesis. Conclude that the mean response time exceeds 75 ms.

How do I do this in Minitab? Minitab>Stat>Basic Stats>OneSampleZ – click summarized data, but you could also highlight a column of data, options will allow us to set the alpha value and the alternative hypothesis



## Actual Example/Application

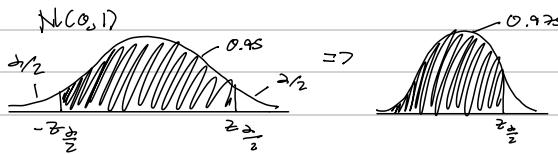
We pull a sample of 20 electrodes from an assembly line of a process with known variance in electrode thickness to be 0.64 mm<sup>2</sup>. The average thickness of the sample is 7.9 mm. Construct a 95% confidence interval.

$$n = 20$$

$$\sigma^2 = 0.64 \text{ mm}^2 \rightarrow \sigma = 0.8$$

$$\bar{x} = 7.9 \text{ mm}, s = 0.08$$

$$N(0, 1)$$



$$P\left\{\bar{x} - \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + \frac{\sigma}{\sqrt{n}}\right\} = 1 - \alpha$$

$$L = 7.9 - 1.96 \frac{0.8}{\sqrt{20}} = 7.5$$

$$U = 7.9 + 1.96 \frac{0.8}{\sqrt{20}} = 8.3$$

Often our result would be said like this:

~~"We are 95% confident that the mean of the population is between 7.5 and 8.3."~~

~~Not this!~~

Perhaps a better description would be:

*"In repeated sampling, this method will produce intervals that capture the true population mean about 95% of the time."*

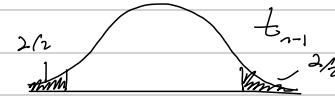
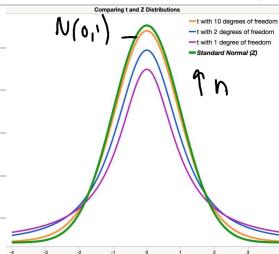
*Use this front bearing exam!"*

## One Sample t-test

- Inference on mean distribution, unknown variance.
- Suppose  $x$  is a normal random variable with unknown mean  $\mu$  and unknown variance  $\sigma^2$ .
- Null Hypothesis:  $H_0: \mu = \mu_0$
- Alternative Hypothesis:  $H_1: \mu \neq \mu_0$
- Test Statistic:  $T_0 = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}$
- Rejection Criterion:  $|T_0| > t_{\alpha/2, n-1}$

Two-tailed degrees of freedom

Rejection Criterion:  $|T_0| > t_{\alpha/2, n-1}$



## For the t-distribution:

- Is a continuous probability distribution of the z-score when the estimated standard deviation is used in the denominator rather than the true standard deviation.
- It is a type of normal distribution, except it accounts for the fact that as sample size increases, variability decreases.
- The t-distribution, like the normal distribution, is bell-shaped and symmetric about zero, but it has heavier tails, which means it tends to produce values that fall farther from its mean.
- As the sample size increases, the t-distribution becomes more similar to a normal distribution. The shape depends on  $n$ !! And this is why you MUST write down  $n$  each time you draw out a t-distribution.

## Summary

### Testing Hypotheses on the Mean of a Normal Distribution, Variance Unknown

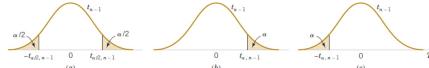
Null hypothesis:  $H_0: \mu = \mu_0$

$$\text{Test statistic: } T_0 = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}$$

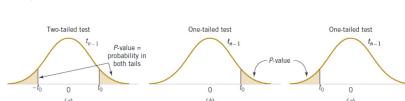
t dist with  $n-1$  D.F

Alternative Hypotheses	P-Value	Rejection Criterion for Fixed-Level Tests
$H_1: \mu \neq \mu_0$	Probability above $ T_0 $ and probability below $- T_0 $	$T_0 > t_{\alpha/2, n-1}$ or $T_0 < -t_{\alpha/2, n-1}$
$H_1: \mu > \mu_0$	Probability above $T_0$	$T_0 > t_{\alpha, n-1}$
$H_1: \mu < \mu_0$	Probability below $T_0$	$T_0 < -t_{\alpha, n-1}$

## Fixed Significance Test by



## P-Value



## t-test Assumptions

- Data are discrete.
- Data are roughly distributed.
- Data are random samples.

## Example

### Inference on the mean of a distribution, unknown variance

Rubber can be added to asphalt to reduce road noise. The viscosity (cP) of 15 specimens of asphalt is measured. To be suitable for use, the mean should be equal to 3200 cP. Test the hypothesis using alpha=0.05.

Use this data  
 $n=15$

3193  
3124  
3153  
3145  
3093  
3466  
3355  
3079  
3182  
3227  
3256  
3332  
3204  
3282  
3170

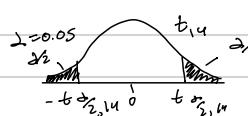
- Inference on mean, unknown variance, one-sample
- One-sample t-test

$$1) H_0: \mu = 3200 \text{ cP} = \mu_0 \quad 3) H_1: \mu \neq 3200 \text{ cP}$$

$$4) T_0 = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}, \quad \bar{x} = \frac{3193 + 3124 + \dots + 3170}{15} = 3217$$

$$\text{Plug & Chug} = 0.64 \quad S = \sqrt{\frac{(3193 - 3217)^2 + \dots + (3170 - 3217)^2}{15}} = 106$$

$$5) \text{Reject if } |T_0| > t_{\alpha/2, 14} \quad 7) \text{Conclusion}$$



$$t_{\alpha/2, 14} = 2.145$$

(P+2) The critical value is not sufficient to reject the claim that the mean viscosity is equal to 3200 cP.  
(Q+2) The null hypothesis is not supported because the mean is not equal to 3200 cP.

## CI for One-sample t-test

Suppose  $x$  is a normal random variable with unknown mean  $\mu$  and unknown variance  $\sigma^2$ . From a sample of  $n$  observations, the sample mean  $\bar{x}$  and sample variance  $s^2$  are computed. Then:

$$\bar{x} - t_{\alpha/2, n-1} \frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{\alpha/2, n-1} \frac{s}{\sqrt{n}}$$

Is a 100(1-alpha)% two-sided CI

where  $t_{\alpha/2, n-1}$  is the value of the t distribution with  $n-1$  degrees of freedom such that  $P(t_{n-1} \geq t_{\alpha/2, n-1}) = \alpha/2$

$$-\infty < \mu \leq \bar{x} + t_{\alpha, n-1} \frac{s}{\sqrt{n}}$$

Is a 100(1-alpha)% upper confidence bound

where  $t_{\alpha, n-1}$  is the percentage point of the t distribution with  $n-1$  degrees of freedom such that  $P(t_{n-1} \geq t_{\alpha, n-1}) = \alpha$

$$\bar{x} - t_{\alpha, n-1} \frac{s}{\sqrt{n}} \leq \mu \leq 0$$

Is a 100(1-alpha)% lower confidence bound

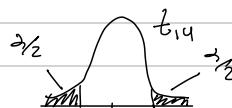
where  $t_{\alpha, n-1}$  is the percentage point of the t distribution with  $n-1$  degrees of freedom such that  $P(t_{n-1} \geq t_{\alpha, n-1}) = \alpha$

→ For our previous example

$$\bar{x} = 3217$$

$$s = 106$$

$$n = 15$$



$$t_{0.025, 14} = 2.145$$

$$L = 3217 - 2.145 \frac{106}{\sqrt{15}} = 3158$$

$$U = 3217 + 2.145 \frac{106}{\sqrt{15}} = 3276$$

$$3158 \leq \mu \leq 3276$$

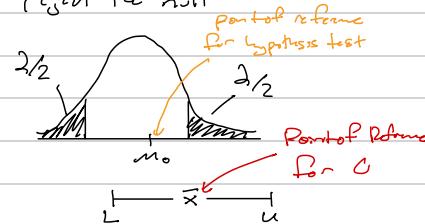
## CI Relationship w/ Hypothesis Test

- Both do make inferences about a population parameter (but in different ways)
- A CI provides a range of values likely containing the true population parameter based on a set of data
- A hypothesis test determines whether a specific claim about the population parameter is supported by the data
- Difference in emphasis of the point of reference

Minitab: Stat>Basic Stat>1-sample t-test, samples in columns (drop down), double click data, check hypothesis test select type and in options select alpha value

### Illustration

If the value in your  $H_0$  is in your CI, you likely cannot reject the null



## Inference or decisions on Normal Data (Chi-square test)

Suppose we wish to test the hypothesis that the variance of a normal population ( $\sigma^2$ ) equals a specified value, say  $\sigma_0^2$ , or that the standard deviation of the population ( $\sigma$ ) is equal to  $\sigma_0$ . Let  $x_1, x_2, \dots, x_n$  be a random sample of  $n$  observations from this population to test:

$$H_0: \sigma^2 = \sigma_0^2$$

$$H_a: \sigma^2 \neq \sigma_0^2$$

$$\text{Test Statistic: } \chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

*s = sample variance with n observations*

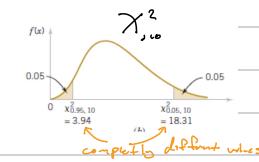
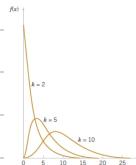
Reference Distribution: Chi-Squared Distribution

Useful for many controls  
about the variability of your process

## Which Hypothesis should I do?

Test	# of Populations	Variance Known?	Parameter of interest?	Reference Dist.
One-sample Z-test	1	Yes	Mean	Standard normal
Two-sample Z-test	2	Yes	Difference between means	Standard normal
One-sample t-test	1	No	Mean	t-distribution
Two-sample (pooled) t-test (equal variance)	2	No	Means of two populations	t-distribution
Two-sample t-test (unequal variance)	2	No	Means of two populations	t-distribution
Paired t-test	2 (simplifies to 1)	No	Difference between paired means	t-distribution
1 Variance test	1	No	Variance of one population	chi-squared distribution

If both mean and variance of populations are known, its a probability problem

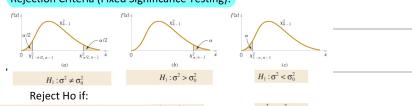


## Chi-squared Distribution

- $\chi^2$  has a chi-squared distribution if true null is true
- Very Features:
  - (i) Depends on  $n-1$  degrees of freedom
  - (ii) Not symmetric
  - (iii) Not centered at 0
  - (iv) Total Area under curve is still 1!

## Rejections

Rejection Criteria (Fixed Significance Testing):



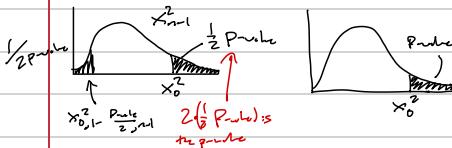
## Summary

Null hypothesis:  $H_0: \sigma^2 = \sigma_0^2$   
Test statistic:  $\chi^2_0 = \frac{(n-1)s^2}{\sigma_0^2}$

Alternative Hypothesis

Rejection Criteria

- |                                 |  |
|---------------------------------|--|
| $H_1: \sigma^2 \neq \sigma_0^2$ | $\chi^2_0 > \chi^2_{\alpha/2, n-1}$ or $\chi^2_0 < \chi^2_{1-\alpha/2, n-1}$ |
| $H_1: \sigma^2 > \sigma_0^2$    | $\chi^2_0 > \chi^2_{\alpha, n-1}$  |
| $H_1: \sigma^2 < \sigma_0^2$    | $\chi^2_0 < \chi^2_{1-\alpha, n-1}$  |



Check Homogeneity! It is sensitive to ties

## Example

### Inference on the variance of a distribution

Rubber can be added to asphalt to reduce road noise. The viscosity (cP) of 15 specimens of asphalt is measured.  $s$  is calculated to be 106. Let's say we are interested in knowing if the variance less than 30,000.

Use  $\alpha = 0.05$

1) Parameter of interest:  $\sigma^2 \rightarrow$  1-sided test

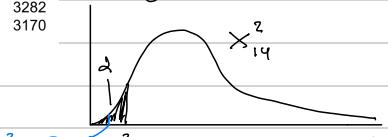
2)  $H_0: \sigma^2 = 30,000 \text{ cP}^2$

3)  $H_a: \sigma^2 < 30,000 \text{ cP}^2$

$$4) \chi^2_0 = \frac{(n-1)s^2}{\sigma^2} = \frac{(15-1)(106)^2}{30,000} = 5.2$$

Reference Distribution:  $\chi^2_{14}$

5) Rejection Criteria: Reject  $H_0$  if  $\chi^2_0 < \chi^2_{1-\alpha, n-1}$

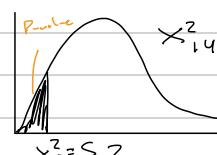


6) Computation:  $5.2 < 6.57$

7) Conclusion: Reject  $H_0$ . We have evidence to reject the claim that the variance is 30,000 cP<sup>2</sup>. Having evidence to accept the hypothesis that variance is less than 30,000 cP<sup>2</sup>. We accept  $H_1$ . (or accept  $H_1$ , but not accept  $H_0$ )

## purple

Reject  $H_0$ : purple < 2



$$\chi^2_0 = 5.2$$

$$\chi^2_{1-\alpha, 14} = 6.57$$

area to the left of 6.57 is 0.95

'accept' region

critical region

Fixed-Significance Level

$\chi^2_{1-\alpha, 14} = 6.57$

0.490 and 0.975

$\rightarrow$  purple is below 1-tuse

$(0.01 < p < 0.025) \rightarrow$  both < 2 so reject  $H_0$ !

How do I do this in Minitab? Stat > basic stat > 1 variance

## Confidence Intervals for Inference on the Variance ( $\chi^2$ )

Suppose  $x$  is a normal random variable with unknown mean  $\mu$  and unknown variance  $\sigma^2$ . From a sample of  $n$  observations, the sample mean  $\bar{x}$  and sample variance  $s^2$  are computed. Then:

### (1 - $\alpha$ ). Two-sided CI

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2, n-1}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha/2, n-1}}$$

Where  $\chi^2_{\alpha/2, n-1}$  denotes the percentage point of the chi-square distribution such that  $P(\chi^2_{n-1} \geq \chi^2_{\alpha/2, n-1}) = \alpha/2$  and  $\chi^2_{1-\alpha/2, n-1}$  denotes the percentage point of the chi-square distribution such that  $P(\chi^2_{n-1} \geq \chi^2_{1-\alpha/2, n-1}) = 1 - \alpha/2$

### (100 - $\alpha$ ). Upper bound

$$\sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha, n-1}}$$

Where  $\chi^2_{1-\alpha, n-1}$  denotes the percentage point of the chi-square distribution such that  $P(\chi^2_{n-1} \geq \chi^2_{1-\alpha, n-1}) = 1 - \alpha$

### (100 - $\alpha$ ). Lower bound

$$\frac{(n-1)s^2}{\chi^2_{\alpha, n-1}} \leq \sigma^2$$

Where  $\chi^2_{\alpha, n-1}$  denotes the percentage point of the chi-square distribution such that  $P(\chi^2_{n-1} \geq \chi^2_{\alpha, n-1}) = \alpha$

## For our example (Upper bound)

$$\sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha, n-1}} \rightarrow \sigma^2 \leq \frac{(15-1)(106)^2}{\chi^2_{0.95, 14}}$$

$$\chi^2_{1-\alpha, n-1} = \chi^2_{0.95, 14} = 6.57$$

$$\Rightarrow \sigma^2 \leq \frac{(14)(106)^2}{6.57} = 234.42 \text{ cP}^2$$

## 6-sigma Example/Application

### Inference on the variance of a distribution

Rubber can be added to asphalt to reduce road noise. The viscosity (cP) of 15 specimens of asphalt is measured. The asphalt specification limit is 1100 cP away from the mean in either direction – what is the sigma level of this process given that we know  $\sigma = 173$  (from our hypothesis test) and we assume the mean of the population is on target?

$$\text{Sigma level} = \frac{\text{Spec limit} - \text{target}}{\sigma} = \frac{1100}{173} = 6.35 \rightarrow \text{at least } 6.35$$

Data

3193

3124

3153

3145

3093

3466

3355

3079

3182

3227

3256

3332

3204

3282

3170

## 1-sigma Test Assumptions

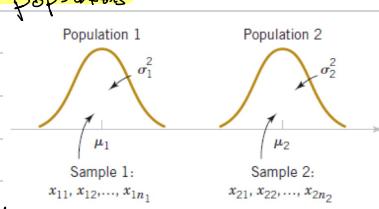
1. The data are continuous.

2. The data are randomly sampled.

3. The data are normally distributed.

## Inference on means of two normally distributed populations

Let's say population 1 has a mean of  $\mu_1$  and a variance of  $\sigma_1^2$  and population 2 has a mean of  $\mu_2$  and a variance of  $\sigma_2^2$  and inferences are based on two random samples of size  $n_1$  and  $n_2$  respectively.



Want to know: is there a difference between two populations

## Two-Sample Z-test

Statistical inference on the difference in means  $\mu_1 - \mu_2$  of the populations below with known variances of  $\sigma_1^2$  and  $\sigma_2^2$

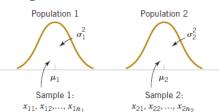
$$\text{Null: } H_0: \mu_1 - \mu_2 = \Delta_0$$

Refers Dist:  $N(0,1)$

$$\text{Alt: } H_1: \mu_1 - \mu_2 \neq \Delta_0 \quad (\text{two-tail})$$

Rejection Criteria: Same as one-sample Z-test

$$\text{Test statistic: } Z_0 = \frac{\bar{x}_1 - \bar{x}_2 - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$



## Summary

$$\text{Null hypothesis: } H_0: \mu_1 - \mu_2 = \Delta_0$$

$$\text{Ref: } N(0,1)$$

$$\text{Test statistic: } Z_0 = \frac{\bar{x}_1 - \bar{x}_2 - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$(10-2)$$

Alternative Hypotheses	P-Value	Rejection Criterion for Fixed-Level Tests
$H_1: \mu_1 - \mu_2 \neq \Delta_0$	Probability above $ z_0 $ and probability below $- z_0 $ , $P = 2[1 - \Phi( z_0 )]$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1: \mu_1 - \mu_2 > \Delta_0$	Probability above $z_0$ , $P = 1 - \Phi(z_0)$	$z_0 > z_\alpha$
$H_1: \mu_1 - \mu_2 < \Delta_0$	Probability below $z_0$ , $P = \Phi(z_0)$	$z_0 < -z_\alpha$

## Assumptions

- (1)  $X_{11}, X_{12}, \dots, X_{1n_1}$  is a random sample from population 1.
- (2)  $X_{21}, X_{22}, \dots, X_{2n_2}$  is a random sample from population 2.
- (3) The two populations represented by  $X_1$  and  $X_2$  are independent.
- (4) Both populations are normal.

- (5) The variances of the populations are known

Samples from two populations are independent  
if the samples selected from one population have no relationship with the samples selected from the other population.

## Example

Paint Drying Time A product developer is interested in reducing the drying time of a primer paint.

Two formulations of the paint are tested; formulation 1 is the standard chemistry, and formulation 2 has a new drying ingredient that should reduce the drying time. From experience, it is known that the standard deviation of drying time is 8 minutes, and this inherent variability should be unaffected by the addition of the new ingredient. Ten specimens are painted with formulation 1, and another 10 specimens are painted with formulation 2; the 20 specimens are painted in random order. The two sample average drying times are  $\bar{x}_1 = 121$  minutes and  $\bar{x}_2 = 112$  minutes, respectively. What conclusions can the product developer draw about the effectiveness of the new ingredient, using  $\alpha = 0.05$ ?

$$H_0: \mu_1 - \mu_2 = 0 \quad (n_1 = n_2)$$

$$H_1: \mu_1 - \mu_2 \neq 0 \quad (n_1 \neq n_2)$$

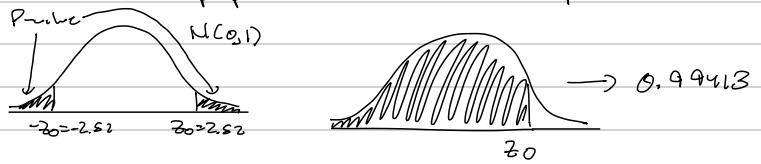
$$Z_0 = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{\frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}}} = \frac{121 - 112}{\sqrt{\frac{8^2}{10} + \frac{8^2}{12}}} = 2.52$$

$$P\text{-value: Reject } H_0 \text{ if } p\text{-value} < \alpha = 0.05$$

$$P\text{-value} = 0.012 < \alpha = 0.05$$

Reject  $H_0$  and accept  $H_1$ , and conclude there is a difference btwn the drying time of the two

1) Parameter of interest is the difference in the mean drying time ( $\mu_1 - \mu_2 = 0$ ), and we know the standard deviation is 8 min for both populations. → Two-Sample Z-test



$$p\text{-value} = 2(1 - 0.9943) = 0.012$$

## Confidence Intervals (do they cross zero?)

Confidence Intervals (do they cross zero!?)

100(1- $\alpha$ )% two-sided confidence interval on  $\mu_1 - \mu_2$

$$\bar{x}_1 - \bar{x}_2 - Z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + Z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

where  $Z_{\alpha/2}$  is the value of the  $N(0,1)$  such that  $P(z \geq Z_{\alpha/2}) = \alpha/2$

$$\mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + Z_\alpha \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$\bar{x}_1 - \bar{x}_2 - Z_\alpha \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2$$

where  $Z_\alpha$  is the value of the  $N(0,1)$  such that  $P(z \geq Z_\alpha) = \alpha$

## Two-Sample t-test (variance unknown)

Case 1:  $\sigma_1^2 = \sigma_2^2 = \sigma^2$   $\leftarrow$  unknown var, but assumed true

Suppose we have two independent populations with unknown means  $\mu_1$  and  $\mu_2$  and equal variances  $\sigma_1^2 = \sigma_2^2 = \sigma^2$  and we wish to know if the means are different:

H<sub>0</sub>:  $\mu_1 - \mu_2 = \Delta_0$  ( $\mu_1 = \mu_2$ )

H<sub>a</sub>:  $\mu_1 - \mu_2 \neq \Delta_0$  ( $\mu_1 \neq \mu_2$ )

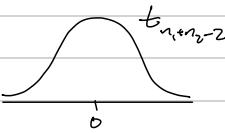
$$\text{Test Statistic: } T_0 = \frac{\bar{x}_1 - \bar{x}_2 - \Delta_0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Since  $n_1 = n_2$

Pooling Sample Variance

$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}$$

Weighted average of two sample variances



Rejection Region: See below exercise

t-dist with  $n_1+n_2-2$  degrees of freedom

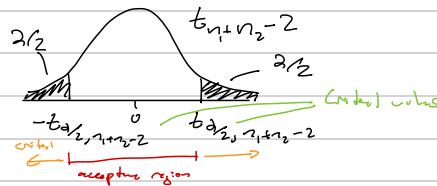
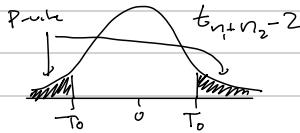
Exercise: Sketch out fixed rejection criteria & p-value for two-sided H<sub>a</sub>

P-value:

Reject H<sub>0</sub>: P p-value < α

Fixed:

Reject: P |T<sub>0</sub>| > t<sub>α/2, n\_1+n\_2-2</sub> or T<sub>0</sub> < -t<sub>α/2, n\_1+n\_2-2</sub>



### Example

**Yield from a Catalyst** Two catalysts are being analyzed to determine how they affect the mean yield of a chemical process. Specifically, catalyst 1 is currently used; but catalyst 2 is acceptable. Because catalyst 2 is cheaper, it should be adopted, if it does not change the process yield. A test is run in the pilot plant

Observation Number	Catalyst 1	Catalyst 2
1	91.50	89.19
2	94.18	90.95
3	92.18	90.46
4	95.39	93.21
5	91.79	97.19
6	89.07	97.04
7	94.72	91.07
8	89.21	92.75
	$\bar{x}_1 = 92.255$	$\bar{x}_2 = 92.733$
	$s_1 = 2.39$	$s_2 = 2.98$

1) Interest:  $\rightarrow$  difference between 2 means, specifically yield of catalyst 1 and catalyst 2, variance unknown  $\rightarrow$  two-sample t-test  
 $\hookrightarrow$  Assume equal variances  $\leftarrow$  rule of thumb later

2) H<sub>0</sub>:  $\mu_1 - \mu_2 = 0$

3) H<sub>a</sub>:  $\mu_1 - \mu_2 \neq 0$

$$4) T_0 = \frac{\bar{x}_1 - \bar{x}_2 - 0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{92.255 - 92.733}{2.7 \sqrt{\frac{1}{8} + \frac{1}{8}}} = -0.35$$

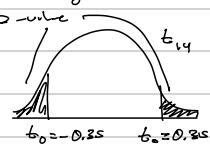
$$5) S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2} = \frac{7 \cdot 2.39^2 + 7 \cdot 2.98^2}{8+8-2} = 2.3$$

$$\rightarrow S_p = \sqrt{S_p^2} = \sqrt{2.3} = 2.3$$

Checking in Minitab: stat -> basic stat -> 2 sample t -> input columns and other info (CLICK ASSUME EQUAL VARIANCE UNDER OPTIONS!)

6) Reject: P p-value < → 0.05

$$7) t_0 = -0.35$$



$$P\text{-value} > \alpha$$

Fail to reject H<sub>0</sub>, redundant new evidence to say the yield from catalyst 1 is different from catalyst 2.

0.5 < p-value < 0.8

## Two Sample t-test (variances unknown & unequal)

Case 1:  $\sigma_1^2 \neq \sigma_2^2$

Suppose we have two independent populations with unknown means  $\mu_1$  and  $\mu_2$  and unequal variances  $\sigma_1^2 \neq \sigma_2^2$  and we wish to test if the means are different:

Example

Calculation:  $s_1 = 7.63$

$s_2 = 15.3$

$n_1 = n_2 = 10$

$$H_0: H_a \text{ are same as } 2 \text{ sample t-test w/ equal variance}$$

$$T_0^* = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

Reference Df for t-dist with  $\checkmark$   
degrees of freedom  
when  $v = \left( \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right)^2$   
If  $v$  is not an integer, round down

$$\frac{(s_1^2/n_1)^2 + (s_2^2/n_2)^2}{n_1 - 1} \quad \frac{(s_1^2/n_1)^2 + (s_2^2/n_2)^2}{n_2 - 1}$$

Summary: two-sample t-test (unequal variance)

Null hypothesis:  $H_0: \mu_1 - \mu_2 = \Delta_0$

Ref.: t-dist with

Test statistic:  $T_0^* = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$

v dof (10-14)

Alternative Hypotheses

$H_1: \mu_1 - \mu_2 \neq \Delta_0$

P-Value

Probability above  $|t_0|$  and  
probability below  $-|t_0|$

Rejection Criterion for  
Fixed-Level Tests

$t_0 > t_{\alpha/2, v}$  OR  $\checkmark$

$t_0 < -t_{\alpha/2, v}$   $\checkmark$

$H_1: \mu_1 - \mu_2 > \Delta_0$

Probability above  $t_0$

$t_0 > t_{\alpha, v}$   $\checkmark$

$H_1: \mu_1 - \mu_2 < \Delta_0$

Probability below  $t_0$

$t_0 < -t_{\alpha, v}$   $\checkmark$

Note: replace  $n_1+n_2$  with  $v$  in rejection criteria!

Checking in Minitab: stat -> basic stat ->

2 sample t -> input columns and in  
options deselect assume equal variance

$$v = \frac{\left( \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right)^2}{\frac{s_1^2}{n_1} / n_1 + \frac{s_2^2}{n_2} / n_2}$$

If  $v$  is not an integer, round down to the nearest integer.

## Assumptions

- (1)  $X_{11}, X_{12}, \dots, X_{1n_1}$  is a random sample from population 1.
  - (2)  $X_{21}, X_{22}, \dots, X_{2n_2}$  is a random sample from population 2.
  - (3) The two populations represented by  $X_1$  and  $X_2$  are independent.
  - (4) Both populations are normal.
- (5) The variances of the populations are unequal

If you're in doubt, assume unequal

## How do I know which t-test to use?

- 1) Are measurements being made on the exact same object? (Paired)
- 2) If the ratio of the larger variance to the smaller variance is less than 4 then we can assume the variances are approximately equal (use pooled t-test)  
OR you could run a two-variance test (will learn about in the future!)

- Remember variance is standard deviation squared!!
- This rule of thumb can vary widely

↳ Use this for exam!

## Confidence Intervals

### Equal Variance Two-sided

$$\bar{X}_1 - \bar{X}_2 - t_{\alpha/2, n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{X}_1 - \bar{X}_2 + t_{\alpha/2, n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

### Unequal Variance Two-sided

$$\bar{X}_1 - \bar{X}_2 - t_{\alpha/2, v} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{X}_1 - \bar{X}_2 + t_{\alpha/2, v} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

## Paired t-test

A special case of the two-sample t-tests when observations on the two populations are observed in pairs where each pair of observations is taken under homogenous conditions

To compare differences, we define  $D_j$ :

$$D_j = X_{1j} - X_{2j}, \quad j = 1, 2, \dots, n. \quad n = \text{# of observations}$$

where  $D_j$ 's are assumed to be normally distributed with mean  $\mu_D$  and variance  $\sigma_D^2$

$n$  is the number of paired observations ( $j$  is the paired observation number)

1 and 2 represents the paired data we are comparing

## Confidence Interval

$$H_0: \mu_D = \Delta_0$$

$$\bar{d} - t_{\alpha/2, n-1} s_D / \sqrt{n} \leq \mu_D \leq \bar{d} + t_{\alpha/2, n-1} s_D / \sqrt{n}$$

$$H_a: \mu_D \neq \Delta_0$$

$$\text{Test Statistic: } T_0 = \frac{\bar{D} - \Delta_0}{S_D / \sqrt{n}}$$

where  $\bar{D} = \text{sample average of } n \text{ differences}$

$S_D = \text{sample standard deviation of } n \text{ differences}$

Reference Dist: t-distr with  $n-1$  degrees of freedom

Rejection Criteria: Same as one sample t-test

## Summary

Null hypothesis:  $H_0: \mu_D = \Delta_0$

Ref: t-dist w/  $n-1$  dof

$$\text{Test statistic: } T_0 = \frac{\bar{D} - \Delta_0}{S_D / \sqrt{n}} \quad (10-24)$$

### Alternative Hypotheses

### P-Value

### Rejection Criterion for Fixed-Level Tests

$H_1: \mu_D \neq \Delta_0$	Probability above $ t_0 $ and probability below $- t_0 $	$t_0 > t_{\alpha/2, n-1}$ or $t_0 < -t_{\alpha/2, n-1}$
$H_1: \mu_D > \Delta_0$	Probability above $t_0$	$t_0 > t_{\alpha, n-1}$
$H_1: \mu_D < \Delta_0$	Probability below $t_0$	$t_0 < -t_{\alpha, n-1}$

### Assumptions:

- (1) Differences are continuous
- (2) Differences are normally distributed
- (3) Differences are obtained from a random sample
- (4) Differences are obtained from homogenous pairs

Minitab: stat -> basic stat-> paired t

or input differences data into one sample t-test

## Example

TABLE 4.3 Paired Tensile Strength Data for Example 4.11			
Specimen	Machine 1	Machine 2	Difference
1	74	78	-4
2	76	79	-3
3	74	75	-1
4	69	66	3
5	58	63	-5
6	71	70	1
7	66	66	0
8	65	67	-2

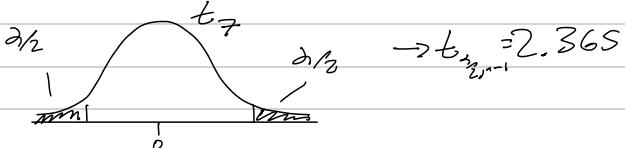
Mean difference: -1.38

Standard deviation of differences: 2.67

Alpha = 0.05

Do these machines yield the same tensile strength values on the same sample?

- (i) In testing the difference in mean tensile strength measured by two devices, same sample  $\rightarrow$  paired diff  $\Rightarrow$  paired t-test
- (ii)  $H_0: \mu_D = 0$       (iii)  $H_D \neq 0$
- (iv)  $t_0 = \frac{\bar{D}}{S_D / \sqrt{n}} = \frac{-1.38}{2.67 / \sqrt{8}} = -1.46$
- (v) Reject  $H_0: t_0 > t_{\alpha/2, n-1}$  or  $t_0 < -t_{\alpha/2, n-1}$



Conclusion: F: I to reject null ( $H_0$ ), we do not have sufficient evidence to reject the claim that the machines measure differently

End of Exam 1  
Motivation

# Exam 1 Summary

Which Hypotheses do I use?

Test	# of Populations	Variance Known?	Parameter of interest?	Reference Dist.
One-sample Z-test	1	Yes	Mean	Standard normal
Two-sample Z-test	2	Yes	Difference between means	Standard normal
One-sample t-test	1	No	Mean	t-distribution
Two-sample (pooled) t-test (equal variance)	2	No	Means of two populations	t-distribution
Two-sample t-test (unequal variance)	2	No	Means of two populations	t-distribution
Paired t-test	2 (simplifies to 1)	No	Difference between paired means	t-distribution
1 Variance test	1	No	Variance of one population	chi-squared distribution

How to Represent your data

- 1) Data are often represented as the mean  $\pm$  the standard deviation and you should tell people how many repeats are in your sample ( $n$ ) (write this in your caption as "Data are represented as the mean  $\pm$  the standard deviation with  $n=X$ ".)
- 2) Statistical significance (when the null is rejected) is represented by a symbol (usually  $*$ ), accompanied by the alpha level, and statistical test with appropriate detail. E.g.,  $^{**}$  represents p-value  $< 0.05$  compared to  $X$ , using a X-sided, X test"
- 1) Describe the test! E.g. two-sided or one sided? One sample or two sample? T=t-test or z=z-test? If t-test, is it assuming equal variance, if it is a z-test what is the variance?

Test 1

- Open book open note and class materials – you can have a calculator, but no access to internet (PdL notes are ok!)
- BRING YOUR STATISTICAL TABLES!!! (They will not be provided for the exam, but you will need them)
- You will be asked to sketch out the distributions and label things
- How can I do well?
  - Make sure you know how to do the homework – review the correct answers if something went wrong
  - Do the practice exam (to be posted later)
- What is going to be on the exam?
  - Anything in lectures 1-10 and HW1-5 is fair game, and the types of problems you see in the practice exam
  - Note: the test can't cover every single type of problem and you won't see the exact same problems in the practice exam on Test 1!

You can check hypothesis tests with your calculator just know what you're doing, and show all your work!

Topics

(Not necessarily comprehensive but good things to study)

- vocabulary and concepts regarding statistical process control, six sigma, lean manufacturing, and the DMAIC process
- tools to represent data, and check assumptions (e.g., normality)
- probability problems, properties of probability distributions
- correctly performing a hypothesis testing procedure, picking the correct test and alternative hypothesis (see list of tests)
- how to correctly phrase conclusions in a hypothesis test
- how to correctly phrase what the p-value is
- how to correctly phrase what confidence intervals are representing, and calculate them
- how to sketch hypothesis tests on a reference (sampling) distribution
- concepts surrounding what type I and type II error are, and why they allow us to make conclusions in a hypothesis test
- concepts around the central limit theorem, and sampling distributions vs. population distributions

Present conclusions

## ECHE 313 Practice Exam 1

Solutions on canvas

Write your name:

Write the following statement, and sign your name acknowledging your agreement:

"I neither received nor gave external assistance on this exam"

*Don't pay attention to this question*

The exam layout is such that all problems add up to 100 points. This point total is meant to guide you in time allocation for this test, will be adjusted such that this exam is 25% of your total class grade. You may use your class materials, class notes, homework, book, and Minitab or Excel to aid in the completion of the exam. The exam is designed to be completed in 1hr and 15 min, but you will have between 10 AM 2/27 to 10 AM 2/28 to complete the exam. All of the necessary tables are provided in the back of your book. Please show your work and all steps necessary to arrive at your answer. Note that it is expected you do this entire test by hand, using a calculator and the tables found in the back of your book. So while you are allowed to use Minitab if you wish to check your answers, it should not replace the work you have done by hand using the tables for your answers. Where applicable, it is best to write out the general formulas you want to use first, then write in your substituted values so it is clear. In addition, for this exam, we can use the rule of thumb that if the sample variance ratio (larger to smaller) is lower than 4, you can assume equal variance for a t-test (if needed). Also, when reporting numbers, keep the same number of decimals as the data you are working with unless there are specific directions on how many sig figs to report.

Good luck!

**Premise: You are a process engineer working for a lean six-sigma pharmaceutical company that makes advanced protein-based biologics. Unless otherwise stated, your company uses  $\alpha=0.01$ .**

- 1) (12 points) You are finishing up a project where you focused on increasing process cycle efficiency and are ready to hand over the improved process.
  - a. (2 points) What step of the DMAIC process are you in?

Control Stage

- b. (2 points) Name at least one tool that is used in this stage.

Transition Plan Control Chart

- c. (4 points) Explain what process cycle efficiency is and how it contributes to lean manufacturing.

$$PCB = \frac{\text{Value added time}}{\text{Process Time}}$$

See Key

- d. (4 points) Explain why a slow process is an expensive process with at least 3 specific reasons.

See Key

- 2) (39 points) Your next project is to determine if the bioreactor time can be reduced by getting a newer bioreactor system. You are able to demo a new unit and track the reactor time on 5 randomly selected batches in both the old and new bioreactors. You obtain the following data:

The following table shows the time it took in minutes for the batch to reach the designated amount of bacteria:

$$\bar{x}_1 = 262, s_1 = \sqrt{132}$$

$$\bar{x}_2 = 149, s_2 = \sqrt{94}$$

Old Bioreactor Time (min)	New Bioreactor Time (min)
245	165
270	141
262	151
275	142
260	147

- a. (3 points) List the name of the hypothesis test you would use

Two sample t-test (pooled) assuming equal variance  
 Use rule of thumb

- b. (3 points) Name the parameter of interest in the context of the problem statement

Mean Bioreactor time of the old and new (say time difference)

- c. (2 points) Write out the correct null hypothesis for this test

$$H_0: \mu_1 - \mu_2 = 0$$

- d. (2 points) Write out the correct alternative hypothesis for this test

$H_a: \mu_1 - \mu_2 \neq 0$  two-sided Can be interpreted as one-sided

- e. (7 points) Write the formula of the test statistic and calculate (report 3 sig figs)

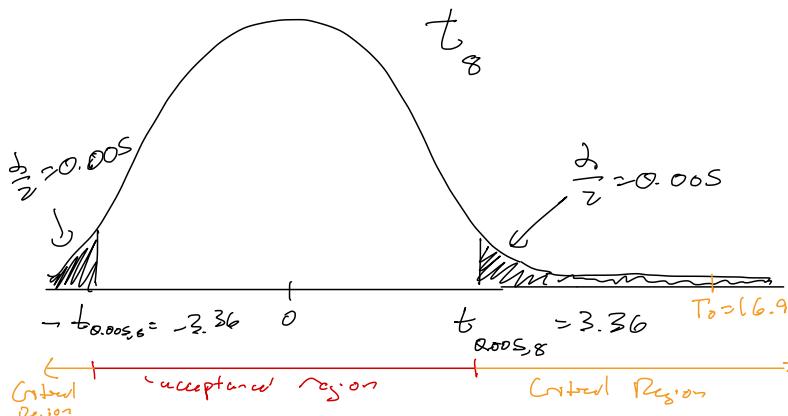
$$T_0 = \frac{\bar{x}_1 - \bar{x}_2 - 0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{262 - 149}{10.62 \sqrt{\frac{1}{5} + \frac{1}{5}}} = 16.9$$

$\uparrow$   
 $S_p = 10.62$

- f. (4 points) Write out the correct rejection criteria for fixed significance level testing and the critical value (3 sig figs)

$$\text{Reject } H_0 : \begin{cases} t_0 > t_{\alpha/2, n_1 + n_2 - 2} \\ t_0 < t_{\alpha/2, n_1 + n_2 - 2} \end{cases} \quad t_{0.005, 8} = \pm 3.36$$

- g. (9 points) Sketch out your reference distribution making sure to fully define the distribution you are sketching on with the correct label, drawing it with the correct shape and where zero is. In addition, label 1) acceptance region, 2) the rejection region (also known as the critical region), 3)  $\alpha$ , 4) the critical value and 5) the test statistic.



- h. (3 points) State your conclusion and remember to word your conclusion appropriately in the context of the problem statement.

$$T_0 < t_{0.005, 8}$$

$H_0$  Reject  $H_0$ , Accept  $H_A$ , Context  $\rightarrow$  Morrows news lower!

See key

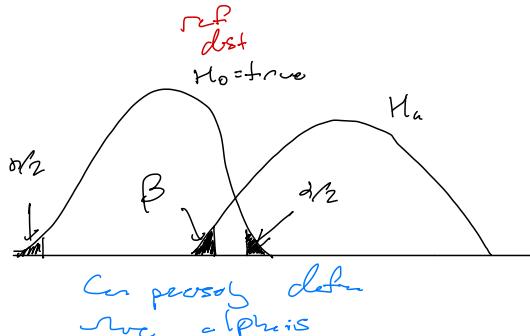
- i. (3 points) Explain what plot you would construct to check if the new bioreactor data are normally distributed (what feature(s) of the plot would you check?)

Normal probability plot, see F:ts linear (looks distinct non-linear shape)

- j. (3 points) Define in words what ( $\alpha$ ) is (what does it represent, what is its definition?) and explain why it helps us make strong conclusions.

$\alpha$  is Type I error =

See Key



- 3) (32 points) To determine if the new bioreactor is suitable for the process, the team also needs to know if the new bio reactor produces a yield that is greater than 810 mg/L of product. Twelve random samples are taken from batches in the piolet run:

Yield (mg/L)
807
825
816
805
815
820
822
817
823
817
810
812

$n \approx 12$

Sample Ave. 815.75

Sample Standard Deviation: 6.31

- a. (2 points) List the *name* of the hypothesis test you would use

One sample t-test

- b. (3 points) Name the parameter of interest in the context of the problem statement

Mean process yield

- c. (2 points) Write out the correct null hypothesis for this test

$H_0: \mu = 810 \text{ mg/L}$

- d. (2 points) Write out the correct alternative hypothesis for this test

$$H_a: \mu > \mu_0 = 810 \text{ mg/L}$$

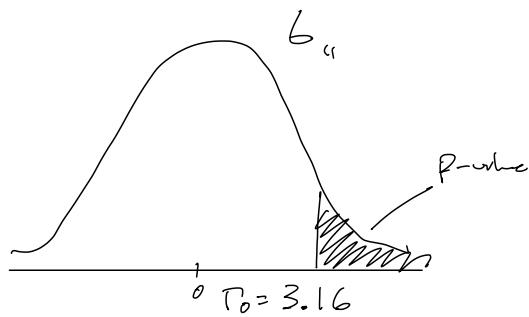
- e. (3 points) Write the formula of the test statistic and calculate (report 3 sig figs)

$$T_0 = \frac{\bar{x} - \mu_0}{S/\sqrt{n}} = \frac{815.78 - 810}{\frac{6.31}{\sqrt{12}}} = 3.16$$

- f. (2 points) Write out the correct rejection criteria using the p-value method

Reject if  $p\text{-value} < \alpha = 0.01$

- g. (6 points) Sketch out your reference distribution making sure to fully define the distribution you are sketching on with the correct label, drawing it with the correct shape and where zero is. In addition, label 1) the test statistic and 2) the p-value.



- h. (2 points) Find the p-value using the tables in the back of your book

- i. (2 points) Explain what the p-value is in words (i.e. what is the definition of a p-value?)

See  
Very

- j. (3 points) State your conclusion and remember to word your conclusion appropriately in the context of the problem statement.

*Reject*

- k. (2 points) You calculate your 99% confidence interval and show it to your boss. Explain what the 99% confidence interval really represents generally, in words (Hint: do not just say it means you are 99% confident that the mean lies in the CI range). You do not need to actually calculate the CI to answer this question correctly.

- l. (3 points) If the manufacturer told you the standard deviation on the yield coming from these new units is known to be 4.00 mg/L, and you wanted to know if the yield was *different* than 810 mg/L, name the hypothesis test you would conduct, and if your alternative hypothesis would be 1 or 2 sided.

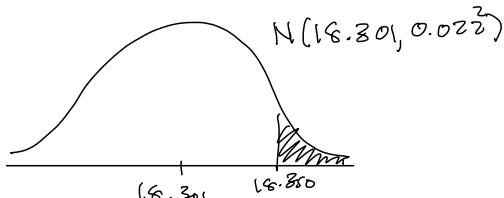
*One-sample z-test, and z-sid test due to df P-value*

- 4) (17 points) Your boss was so impressed with the projects you have been doing that she asked you to help the manufacturing floor with some questions regarding their packaging unit, which boxes up the biologics with instructional materials. Weight measurements are taken on each and every product automatically, thus it is known that the population mean  $\mu$  is 18.301 g and the population standard deviation is  $\sigma$  is 0.022 g and is normally distributed. New weight restrictions are being instated and the manufacturing floor wants to know the probability that a product will have a weight above a new limit of 18.350 g.

*Probability  
problem!  
Not hypothesis  
Test!*

- a. (11 points) Write out the probability equation you wish to solve, including the integral (but you do not need to solve the integral just set it up). Also include a probability distribution sketch with the 1) probability you wish to find shaded in 2) the correct shape, 3) correct label of the distribution, 4) where 18.350g is and 5) where zero is.

$$P(x > 18.350) = \int_{18.301}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma}\right)^2} dx$$



- b. (6 points) Solve the probability problem using the *tables in the back of the book*. Just like in class, sketches can help solve these problems, especially when using the tables. Report an appropriate number of sig figs based on the numbers given in the problem statement.

$$Z = \frac{16.380 - 16.38}{\pm 0.022} = 223$$

$$\rightarrow P-value \approx 0.013$$

## Midterm Exam 2

---



# Chapter 4 Continued - Test 2 Material

Inference on Two populations' variances

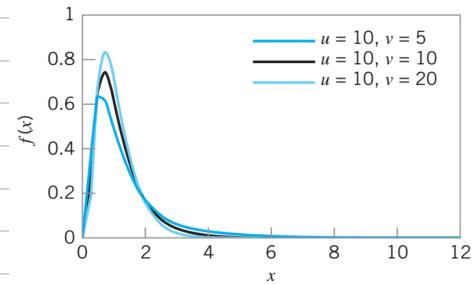
→ Two independent, normally distributed populations

$$\text{Null: } H_0: \sigma_1^2 = \sigma_2^2$$

$$\text{Alt: } H_a: \sigma_1^2 \neq \sigma_2^2 \text{ tested}$$

$$\text{Test Stat: } F_0 = \frac{S_1^2}{S_2^2} \leftarrow \text{DoF}_1 = n_1 - 1$$

Sensitive to normality – should check!



Reference distribution: F-distribution

(i) The curve is not symmetric (skewed right)

(ii) DF quantiles for each set of DoFs

(iii) F statistic  $\geq 0$

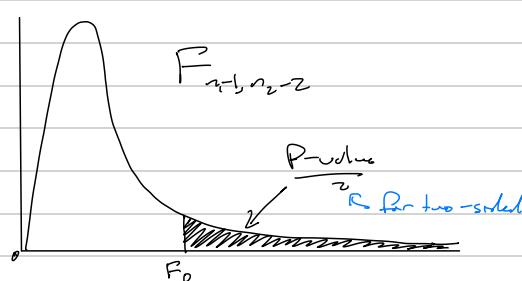
(iv) As DoF increase curve gets more round

$$\text{Note: } F_{0.99, v_1, v_2} = 1 / F_{0.01, v_2, v_1}$$

Rejection Criterion: (two-sided)

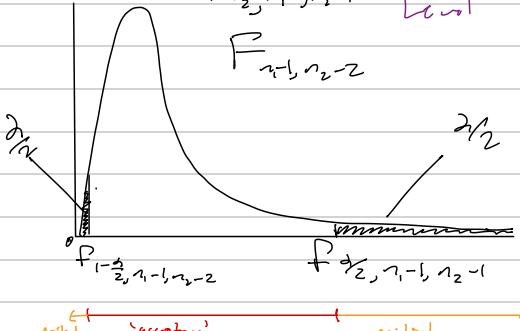
$F_{\alpha/2}$

Practical  $< 2$



$$F_0 \geq F_{\alpha/2, n_1-1, n_2-1}$$

$$\text{or } F_0 \leq F_{1-\alpha/2, n_1-1, n_2-1}$$



Fixed  
Sig. Lvl

## Summary

Null hypothesis:  $H_0: \sigma_1^2 = \sigma_2^2$

$$\text{Test statistic: } F_0 = \frac{S_1^2}{S_2^2}$$

(10-31)

### Alternative Hypotheses

$$H_1: \sigma_1^2 \neq \sigma_2^2$$

### Rejection Criterion

$$f_0 > f_{\alpha/2, n_1-1, n_2-1} \text{ or } f_0 < f_{1-\alpha/2, n_1-1, n_2-1}$$

$$H_1: \sigma_1^2 > \sigma_2^2$$

$$f_0 > f_{\alpha, n_1-1, n_2-1}$$

$$H_1: \sigma_1^2 < \sigma_2^2$$

$$f_0 < f_{1-\alpha, n_1-1, n_2-1}$$

## Assumptions

Grab from posted notes

## Example

**Example 10-13 Semiconductor Etch Variability** Oxide layers on semiconductor wafers are etched in a mixture of gases to achieve the proper thickness. The variability in the thickness of these oxide layers is a critical characteristic of the wafer, and low variability is desirable for subsequent processing steps. Two different mixtures of gases are being studied to determine whether one is superior in reducing the variability of the oxide thickness. Sixteen wafers are etched in each gas. The sample standard deviations of oxide thickness are  $s_1 = 1.96$  angstroms and  $s_2 = 2.13$  angstroms, respectively. Is there any evidence to indicate that either gas is preferable? Use a fixed-level test with  $\alpha = 0.05$ .

Minitab: Stat-> Basic Stat -> Two-sample Variance

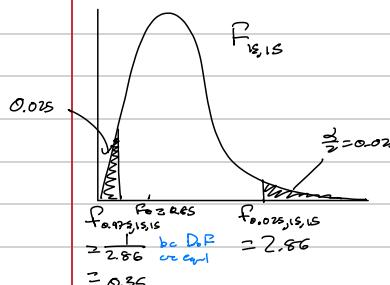
1) Parameters of interest: Variances oxide thickness for two mixtures of gases, variance variances

∴ Two-variance test

$$2) H_0: \sigma_1^2 = \sigma_2^2 \quad 4) F_0 = \frac{S_1^2}{S_2^2} = \frac{1.96^2}{2.13^2} = 0.85$$

$$3) H_a: \sigma_1^2 \neq \sigma_2^2$$

$$5) \text{ Reject if: } F_0 > F_{0.025, 15, 15} \text{ or } F_0 < F_{0.975, 15, 15}$$



To reject  $H_0$ , we don't have enough evidence to reject the claim that the variances of the oxide layer produced by the different wafers are different.

$$= 0.85$$

## Single Factor Analysis of Variance

• Analysis of Variance: Used for comparing means when there are more than two levels in a single factor

→ Factor = Entity being tested

→ Level = Settings within a factor

• Replicates: # of repeats taken at each level

• Randomization: Randomly order data such that nuisance variables are eliminated

Factor levels	Tensile Strength of Paper (psi)						Totals	Averages
	Observations							
Hardwood Concentration (%)	1	2	3	4	5	6		
5	7	8	15	11	9	10	60	10.00
10	12	17	13	18	19	15	94	15.67
15	14	18	19	17	16	18	102	17.00
20	19	25	22	23	18	20	127	21.17
							383	15.96

# of R...

Factors = 1

Levels = 4

Replicates ≥ 6

## Defining and Development of ANOVA

Typical Data for a Single-Factor Experiment

Treatment	Observations						Totals	Averages
1	$y_{11}$	$y_{12}$	...	$y_{1n}$	$\bar{y}_{1\cdot}$			
2	$y_{21}$	$y_{22}$	...	$y_{2n}$	$\bar{y}_{2\cdot}$			
...	...	...	...	...	...	...		
...	...	...	...	...	...	...		
$a$	$y_{a1}$	$y_{a2}$	...	$y_{an}$	$\bar{y}_{a\cdot}$			
					$\bar{y}_{\cdot\cdot}$			

$a = \# \text{ of treatments}$  ( $\sum_{i=1}^a$  total) ( $\bar{y}_{\cdot\cdot}$  grand mean)

$y_{ij} = j^{\text{th}}$  observation of  $i^{\text{th}}$  treatment

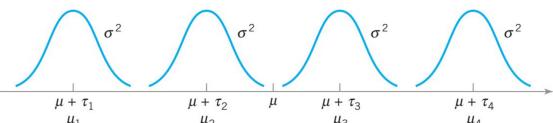
$n = \# \text{ of observations at each level}$

### Linear Statistical Model

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij} \quad \begin{cases} i = 1, 2, \dots, a \\ j = 1, 2, \dots, n \end{cases}$$

where  $\tau_i = i^{\text{th}}$  treatment effect

$\varepsilon_{ij} = \text{random error}$



## Hypothesis Testing for Equality of Treatment Effects

Null:  $H_0: \tau_1 = \tau_2 = \tau_3 = \dots = \tau_a = 0$

Alt:  $H_a: \tau_i \neq 0$  for at least one  $i$

Test Statistic:  $F_o = \frac{SS_{\text{treatments}}/a}{SS_E/(n-a)}$  where  $a = \# \text{ of treatments}$

$(n-a) = \# \text{ of replicates per each treatment level}$

Refined Distribution: F-distribution with  $(a-1)$  [top] and  $a(n-1)$  [bottom] DoF

Rejection Criteria:  $H_0: F_o > F_{\alpha/2, a-1, a(n-1)}$  This is always one-sided

## Calculations for ANOVA

The Analysis of Variance for a Single-Factor Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	$F_o$
Treatments	$SS_{\text{Treatments}}$	$a - 1$	$MS_{\text{Treatments}}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Error	$SS_E$	$a(n - 1)$	$MS_E$	
Total	$SS_T$	$an - 1$		

$$SS_T = SS_{\text{Treatments}} + SS_E$$

$$F_o = \frac{MS_{\text{Treatments}}}{MS_E} = \frac{SS_{\text{Treatments}}/a}{SS_E/(n-a)}$$

$$SS_T = \sum_{i=1}^a \sum_{j=1}^n (y_{ij} - \bar{y}_{\cdot\cdot})^2 = \text{total sum of squares}$$

$n \times a$  squared terms

$$SS_{\text{Treatments}} = n \sum_{i=1}^a (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 = \text{treatment sum of squares}$$

$a : \# \text{ of squared terms}$

$$\bar{y}_i = \frac{1}{n} \sum_{j=1}^n y_{ij} \quad \bar{y}_{i\cdot} = y_i/n \quad i = 1, 2, \dots, a$$

$$\bar{y}_{\cdot\cdot} = \frac{1}{n} \sum_{i=1}^a \sum_{j=1}^n y_{ij} \quad \bar{y}_{\cdot\cdot} = y_{\cdot\cdot}/N$$

## Deriving & Development of ANOVA

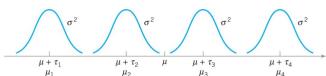
a = number of treatments

$y_{ij}$  = the j<sup>th</sup> observation of the i<sup>th</sup> treatment

n = number of observations for each level

Linear statistical model:  $y_{ij} = \mu + \tau_i + \epsilon_{ij}$   $\begin{cases} i=1,2,\dots,a \\ j=1,2,\dots,n \end{cases}$

Where:  $y_{ij}$  = observation  
 $\mu$  = overall mean (common to all treatments)  
 $\tau_i$  = the i<sup>th</sup> treatment effect  
 $\epsilon_{ij}$  = random error



## Hypotheses testing the equality of treatment effects

Null and Alternative Hypothesis:

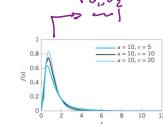
$$H_0: \tau_1 = \tau_2 = \dots = \tau_a = 0 \quad H_1: \tau_i \neq 0 \text{ for at least one } i$$

Test Statistic: (reference distribution is the F-distribution)

$$F_0 = \frac{SS_{\text{Treatments}} / (a-1)}{SS_E / [a(n-1)]} = \frac{MS_{\text{Treatments}}}{MS_E}$$

Rejection Criteria:

$$F_0 > f_{(a-1, n-1)}$$



## Calculations for ANOVA (same n)

The Analysis of Variance for a Single-Factor Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F <sub>0</sub>
Treatments	SS <sub>Treatments</sub>	a - 1	MS <sub>Treatments</sub>	$\frac{MS_{\text{Treatments}}}{MS_E}$
Error	SS <sub>E</sub>	a(n - 1)	MS <sub>E</sub>	
Total	SS <sub>T</sub>	an - 1		

$$SS_T = SS_{\text{Treatments}} + SS_E$$

$$SS_T = \sum_{i=1}^a \sum_{j=1}^n (y_{ij} - \bar{y}_{..})^2 \text{ = total sum of squares}$$

$$SS_{\text{Treatments}} = n \sum_{i=1}^a (\bar{y}_i - \bar{y}_{..})^2 \text{ = treatment sum of squares}$$

$$\bar{y}_i = \sum_{j=1}^n y_{ij} \quad \bar{y}_{..} = \bar{y}_i / n \quad i = 1, 2, \dots, a$$

$$\bar{y}_{..} = \sum_{i=1}^a \sum_{j=1}^n y_{ij} / n \quad \text{Total number of observations}$$

(green)

i = treatment

j = observation

bars = a var sc

dots = total

~ = replicates

a = # of treatments

$y_{ij}$  = j<sup>th</sup> observation of i<sup>th</sup> treatment

$\bar{y}_i$  = grand mean of all observations

$\bar{y}_{..}$  = total of all observations

$\bar{y}_i$  = average of observations @ treatment i

## What does the Calculation look like?

Tensile Strength of Paper (psi)

Factor	Observations						$\bar{y}_{..}$	$\bar{y}_i$	$SS_T = SS_{\text{Treatments}} + SS_E$
	1	2	3	4	5	6			
1	5	7	8	15	11	9	10	60	10.00
2	10	12	17	13	18	19	15	94	15.67
3	15	14	18	19	17	16	18	102	17.00
4	20	19	25	22	23	18	20	127	21.17
							383	15.96	

all data points

$$SS_T = SS_{\text{Treatments}} + SS_E$$

$$SS_E = \sum_{i=1}^a \sum_{j=1}^n (y_{ij} - \bar{y}_{..})^2 = (7-15.96)^2 + (8-15.96)^2 + \dots + (20-15.96)^2 = 513$$

$$SS_{\text{Treatments}} = n \sum_{i=1}^a (\bar{y}_i - \bar{y}_{..})^2 = 6[(10-15.96)^2 + (15.67-15.96)^2 + (17-15.96)^2 + (21.17-15.96)^2] = 382.8$$

$$\Rightarrow SS_B = 513 - 382.8 = 130.2$$

Hypothesis Test? Does hardwood concentration impact the tensile strength of paper?  $\alpha = 0.01$

1) Parameter of interest: Multiple pop. means of tensile strength  $\rightarrow$  ANOVA!

2)  $H_0: \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$

3)  $\exists i \in \{1, 2, 3, 4\}$  s.t.  $\tau_i \neq 0$

$$4) F_0 = \frac{\frac{SS_{\text{Treatments}}}{(a-1)}}{\frac{SS_E}{(a-1)}} = \frac{\frac{130.2}{3}}{\frac{382.8}{20}} = 19.6$$

5) Reject: if  $F_0 > F_{(a-1, a(n-1))}$

$$F_{(a-1, a(n-1))} \approx 4.94 \text{ from charts}$$



6) Conclusions: Reject  $H_0$  as  $F_0 > F_{(a-1, a(n-1))}$

At least one treatment has an impact on the tensile strength of paper. More generally,

Hardwood Concentration impacts paper tensile strength!

Minitab: Stat  $\rightarrow$  ANOVA  $\rightarrow$  one-way

What do you do when the null is rejected?

Multiple Comparison Methods - (post-hoc tests) is performed to find which means are different after ANOVA has been performed and  $H_0$  is rejected.

### Fisher's Least Significant Difference (LSD) Test:

$$\text{Declare Significantly Different if: } |\bar{y}_{ij} - \bar{y}_{jk}| > LSD$$

$$\rightarrow LSD = t_{\alpha/2, n(n-1)} \sqrt{\frac{2MSE}{n}}$$

### Example

Hardwood Concentration (%)	Observations						Totals	Averages
	1	2	3	4	5	6		
5	7	8	15	11	9	10	60	10.00
10	12	17	13	18	19	15	94	15.67
15	14	18	19	17	16	18	102	17.00
20	19	25	22	23	18	20	127	21.17
							383	15.96

Minitab: Stat -> ANOVA -> one-way (click the comparisons button - error rate)

$$S_{us.10} : |10 - 15.67| = 5.67^*$$

$$S_{us.15} : |10 - 17| = 7^*$$

$$S_{us.20} : |10 - 21.17| = 11.17^*$$

$$10 \text{ vs } 15 : |10 - 17| = 7$$

$$10 \text{ vs. } 20 : |10 - 21.17| = 11.17$$

$$15 \text{ vs. } 20 : |15 - 21.17| = 6.17$$

$$LSD = t_{0.005, 20} \sqrt{\frac{2MSE}{n}}$$

$$= 2.845 \sqrt{2 \left( \frac{6.5}{6} \right)}$$

$$= 4.19$$

20 : C <sup>10 vs 20 is equal</sup>  
 15 : C <sup>10 vs 15 is equal</sup>  
 10 : B <sup>10 vs 20 is equal</sup>  
 S : A <sup>Different</sup>

### Assumptions

- (i) Normal
- (ii) No hidden variables

### Summary

Declare significantly different if:  $|\bar{y}_{ij} - \bar{y}_{jk}| > LSD$

where LSD, the least significant difference, is

$$LSD = t_{\alpha/2, n(n-1)} \sqrt{\frac{2MSE}{n}}$$

Better notes on LSD test:

- Do this ONLY after ANOVA null hypothesis is rejected
- Find differences between all of the treatments
- Compare differences to the LSD to see if they are significantly different

Mini tab might produce an output like this:  
 If a pair has the same letter, they are not significantly different

Factor	N	Mean	Grouping
20%	6	21.17	A
15%	6	17.00	A B
10%	6	15.67	B
5%	6	10.00	C

## Fisher's LSD Test

Declare significantly different if:  $|\bar{y}_i - \bar{y}_j| > LSD$   
where LSD, the least significant difference,

$$LSD = t_{\alpha/2,a(n-1)} \sqrt{\frac{2MS_E}{n}}$$

Better notes on LSD test:

- Do this after ANOVA null hypothesis is rejected
- Find differences between all of the treatments
- Compare differences to the LSD to see if they are significantly different

Minitab might produce an output like this:

If a pair has the same letter, they are not significantly different

Sometimes funny overlap occurs – for example, 15% is not different than 20% or 10% but, 10% and 20% are different

Factor	N	Mean	Grouping
20%	6	21.17 A	
15%	6	17.000 A B	
10%	6	15.67 B	
5%	6	10.00 C	

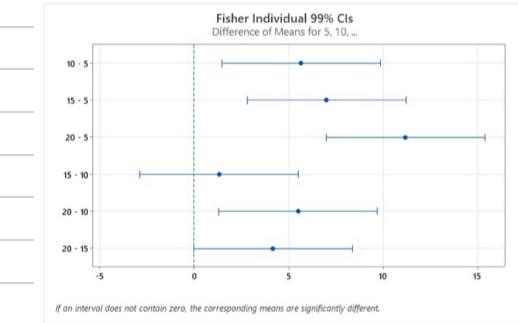
## Confidence Intervals

A  $100(1 - \alpha)\%$  confidence interval on the mean of the  $i$ th treatment  $\mu_i$  is

$$\bar{y}_{i \cdot} - t_{\alpha/2,a(n-1)} \sqrt{\frac{MS_E}{n}} \leq \mu_i \leq \bar{y}_{i \cdot} + t_{\alpha/2,a(n-1)} \sqrt{\frac{MS_E}{n}}$$

A  $100(1 - \alpha)$  percent confidence interval on the difference in two treatment means  $\mu_i - \mu_j$  is

$$\bar{y}_{i \cdot} - \bar{y}_{j \cdot} - t_{\alpha/2,a(n-1)} \sqrt{\frac{2MS_E}{n}} \leq \mu_i - \mu_j \leq \bar{y}_{i \cdot} - \bar{y}_{j \cdot} + t_{\alpha/2,a(n-1)} \sqrt{\frac{2MS_E}{n}}$$



## ANOVA Assumptions

- Errors (and observations) are normally distributed
- Each treatment has the same variance (equal variance assumption)
- Errors and observations are independently distributed

## Residual

The difference between an observation ( $y_{ij}$ ) and its estimated (or fitted) value from the statistical model being studied

## Normal Probability Plot of Residuals

(i) Rank data

(ii) Calc cumulative Frequency

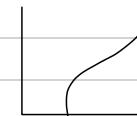
(iii) Find  $Z_{ij}$ 's

(iv) Plot residuals vs.  $Z_{ij}$ 's

Looking for straight line

## Minitab

Can ask Minitab to plot a normal probability plot after pasting in residuals into one column, or use ANOVA function, and select graphs, then select – normal probability plot of residuals



## Checking equal Variance at each Factor Level

→ Minitab does not automatically generate

Look for: even residuals, if not...

Rule of thumb we used for t-tests applies, or, you could check using your 2 variance test!

Can generate in Excel or Minitab: Residuals in one column, factor level in another, using scatter plot

## Checking Equal Variance as a function of predicted value

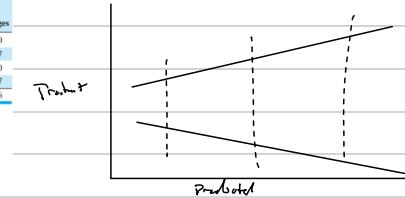
### Residuals plotted as a function of predicted value

Minitab: use ANOVA function, and select graphs, then select - residuals vs. fits

Can also generate in Excel or Minitab: Residuals in one column, predicted value in the other, using scatter plot

What if there is a pattern?

- Data transformations
- Residual Analysis



## Checking Independence

No formal plot... but:

- 1) Addressed via experimental design (randomize!)
- 2) Can plot residuals versus any suspected or nuisance variables
- 1) Example: plot residuals versus run order (run order, temp, pH...)

When is my data not independent?

- (1) repeated measurements are taken on the same subject → paired
- (2) residuals or observations are correlated with a nuisance variable

## Simple Linear Regression

- Use simple linear regression for building an empirical model
- Understand how the method of least squares is used to estimate the parameters in a linear regression model
- Analyze residuals to determine whether the regression model is an adequate fit to the data or whether assumptions are violated
- Test statistical hypotheses and construct confidence intervals on regression model parameters
- Apply the correlation model
- Use simple transformations to achieve a linear regression model

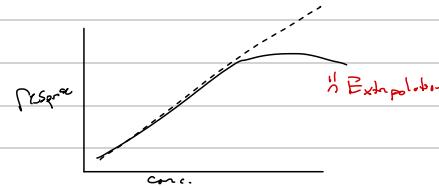
## Regression Modeling

- **Regression Modeling:** A mathematical model fit to a set of data relating a response variable ( $y$ ) to an independent variable ( $x_1, x_2, \dots, x_n$ )
- **Empirical Models:** Models where the structure is determined by the observed relationship among experimental data, but not necessarily mechanistically relevant
- **Simple Linear Regression Model:** There is only one indep. or predictor variable ( $x$ ) and the relationship with response ( $y$ ) is assumed to be linear

$$y = \beta_0 + \beta_1 x + \epsilon$$

$y$  = expected response  
 $\beta_0$  = intercept  
 $\beta_1$  = slope

$x$  = predictor variable  
 $\epsilon$  = random error term

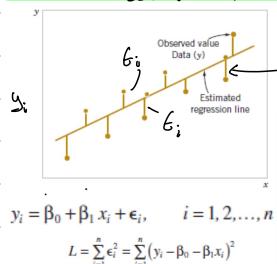


- **Pitfalls:** Empirical Models

→ Correlation does not equal causation!!!

→ Valid in the range of original data (be careful with extrapolation)

- How to estimate the coefficients in simple linear regression



**Method of Least Squares**  
Minimize the sum of squares of vertical deviations

The least squares estimates of the intercept and slope in the simple linear regression model are

$$\text{Avg of all response values} \rightarrow \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \quad (11-7)$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (y_i - \bar{y})(\sum_{i=1}^n x_i)}{\sum_{i=1}^n x_i^2} S_{xy} \quad (11-8)$$

where  $\bar{y} = (1/n) \sum_{i=1}^n y_i$  and  $\bar{x} = (1/n) \sum_{i=1}^n x_i$ .

Fitted or Estimated Regression Line

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

Predicted Value

$$y_i = \hat{\beta}_0 + \hat{\beta}_1 x + \epsilon_i$$

Each pair satisfies this

Example: Fit a simple linear regression line to these data.

TABLE • 11-1 Oxygen and Hydrocarbon Levels

Observation Number	Hydrocarbon Level $x(\%)$	Purity $y(\%)$
1	0.99	90.01
2	1.02	89.05
3	1.15	91.43
4	1.29	93.74
5	1.46	96.73
6	1.36	94.45
7	0.87	87.59
8	1.23	91.77
9	1.55	99.42
10	1.40	93.65
11	1.19	93.54
12	1.15	92.52
13	0.98	90.56
14	1.01	89.54
15	1.11	89.85
16	1.20	90.39
17	1.26	93.25
18	1.32	93.41
19	1.43	94.98
20	0.95	87.33

$$n=20$$

$$\sum x_i = 23.42 \quad \sum y_i = 1843.21$$

$$\sum x_i^2 = 24.28 \quad \sum y_i^2 = 170044$$

The least squares estimates of the intercept and slope in the simple linear regression model are

$$\hat{y}_0 = \bar{y} - \hat{B}_1 \bar{x} \quad (11-7)$$

$$\hat{B}_1 = \frac{\sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad (11-8)$$

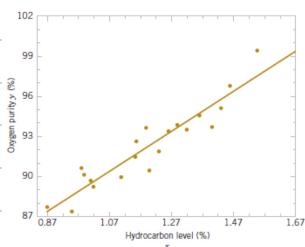
where  $\bar{y} = (1/n) \sum_{i=1}^n y_i$  and  $\bar{x} = (1/n) \sum_{i=1}^n x_i$ .

$$S_{xx} = 24.28 - \frac{(23.42)^2}{20} = 0.68$$

$$S_{xy} = 2214 - \frac{23.42(1843.21)}{20} = 10.18$$

$$\tilde{B}_1 = \frac{10.18}{0.68} = 14.95$$

$$\tilde{B}_0 = \bar{y} - \tilde{B}_1 \bar{x} = 92.1605 - (14.95)(1.196) \\ = 74.28$$



$$\hat{y} = 74.28 + 14.95x$$

Minitab: Stat > Regression -> Regression -> Fit Regression Model

Minitab: Stat > Regression -> Fitted Line -> choose x, y and linear

## ANOVA for Linear Regression

- Null & Alternative Hypotheses:

$$H_0: \beta_1 = 0$$

$$H_1: \beta_1 \neq 0$$

- Test Statistic

$$F_0 = \frac{SS_R / 1}{SS_E / n-2} = \frac{MS_R}{MS_E}$$

Signal  
noise

- Rejection Distribution:  $F$ -dist

↳ Want to use:  $F_{1,\nu_{df}}$

- Rejection Criteria:

$$\text{Reject } F_0 \quad F_0 > F_{1,\nu_{df}}$$

or  $p\text{-value} < \alpha$

## Calculating the Error

$$\text{Result: } S_{ST} = S_{S_R} + S_{S_E}$$

$$\bullet \text{Residual: } e_i = y_i - \hat{y}_i$$

Actual  $\Rightarrow$  predicted

$$\bullet \text{Error Sum of Squares: } S_{S_E} = \sum_{i=1}^n e_i^2 = \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

$$\bullet \text{Mean Squared Error: } S_{S_E} / n-2$$

$$\bullet \text{Sum of Squares Total: } S_{S_T} = \sum_{i=1}^n (y_i - \bar{y})^2$$

$$\bullet \text{Regression Sum of Squares: } S_{S_R} = \sum_{i=1}^n (\hat{y}_i - \bar{y})^2$$

Minitab: Stat>Regression>Fit Regression Model (options sets the alpha)

## Example

TABLE 11-1 Oxygen and Hydrocarbon Levels		
Observation Number	Hydrocarbon Level $x(\%)$	Purity $y(\%)$
1	1.00	90.56
2	1.02	89.05
3	1.15	91.43
4	1.29	93.74
5	1.46	96.78
6	1.36	94.45
7	0.87	87.59
8	1.23	91.77
9	1.55	99.42
10	1.40	93.65
11	1.19	93.54
12	1.15	92.52
13	0.98	90.56
14	1.01	93.42
15	1.11	89.35
16	1.20	90.39
17	1.26	93.25
18	1.32	93.41
19	1.43	94.98
20	0.95	87.33

1) Parameter of Interest: Slope of Linear Model; one regression  $\rightarrow$  simple linear regression

2)  $H_0: \beta_1 = 0$

$H_a: \beta_1 \neq 0$

3)  $F_0 = SSe_{\text{R}} / 1$

$SSe_{\text{E}} / n-2$

4) Reject if  $F_0 > F_{0.05, 18}$

$\bar{y} = 92.16$

$\hat{y} = \beta_0 + \beta_1 x_i \Rightarrow \hat{y} = 74.28 + 14.947x$

$n=1 \Rightarrow \hat{y} = 74.28 + 14.947(0.49) = 89.08$

: all data points

$n=20 \Rightarrow \hat{y} = 74.28 + 14.947(0.95) = 88.48$

$SS_{\text{E}} = \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 = (89.08 - 92.16)^2 + \dots + (88.48 - 92.16)^2 = 152.17$

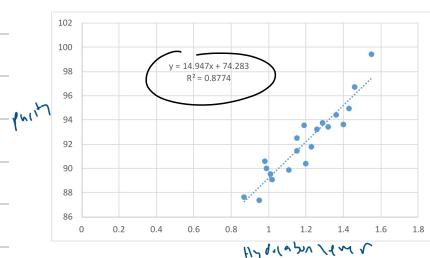
$SS_{\text{R}} = \sum_{i=1}^n (y_i - \bar{y})^2 = (90.01 - 92.16)^2 + \dots + (82.33 - 92.16)^2 = 173.38$

$SS_{\text{B}} = SS_{\text{R}} - SS_{\text{E}} = 21.2$

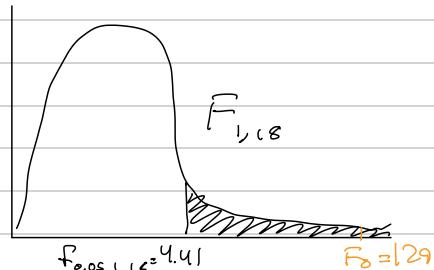
$\Rightarrow F_0 = \frac{SS_{\text{B}} / 1}{SS_{\text{E}} / (n-2)} = 129 \quad \left. \begin{array}{l} 129 > 4.41 \rightarrow \text{Reject } H_0 \\ \text{There is a significant linear relationship between hydrocarbon level and purity} \end{array} \right\}$

$F_{0.05, 18} = 4.41 \text{ from Table}$

$R^2 = \frac{SS_{\text{B}}}{SS_{\text{R}}} = 1 - \frac{SS_{\text{E}}}{SS_{\text{R}}} = \frac{152}{173.38} = 0.88$



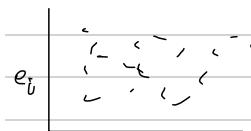
$R^2$  represents how much error is accounted for by the model



## Checking Assumptions

- Errors (and observations) are normally distributed
- Errors (and observations) are independently distributed
- Variance is constant

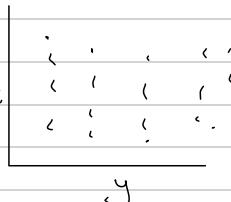
$e_i = y_i - \hat{y}_i \quad i = 1, 2, \dots, n$   
residuals vs. estimated values



Linear?  
Normality  
Plot residuals  
on Normal Probability Plot

Pathogenic?  
Independent Variance  
Plot residuals vs.  
Random order or  
measured variables

Pathogenic or Spurious  
Constant Variance  
① Plot residuals  
vs. predicted (constant)  
② Plot vs. independent  
variable (X)





## Multiple Linear Regression

**Multiple Regression Model:** A linear regression that contains more than one regressor ( $X$ )

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k + \epsilon$$

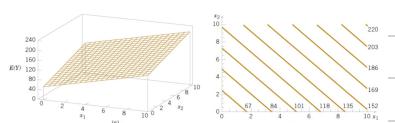


FIGURE 12.1 (a) The regression plane for the model  $E(Y) = 50 + 10x_1 + 7x_2$ . (b) The contour plot.

## Non-linear Surfaces

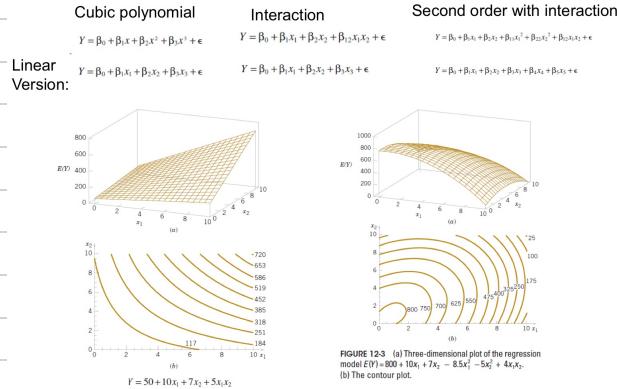


FIGURE 12.3 (a) Three-dimensional plot of the regression model  $E(Y) = 800 + 10x_1 + 7x_2 - 8.5x_1^2 + 4x_1x_2$ ; (b) The contour plot.

## Estimated Coefficients and Variance

**Coefficients:** Found by method of least squares

$\hookrightarrow$  Residuals:  $e_i = y_i - \hat{y}_i$  square of these minimized

Since concepts simple  
just a lot more terms

## Variance from different sources

• Variance from Random Error:  $SS_E$  from calculation of residuals

$$SS_T = SS_R + SS_E \quad \hookrightarrow SS_E = \sum e_i^2$$

• Variance from Regression:  $SS_R$  = variance from regression

• Variance from individual regression terms:  $SS_p(\beta_k) =$  variance coming from specific regression term

• Partial F-test: Components

## Hypothesis Testing

### Recall

$n = \# \text{ of data points}$

$k = \# \text{ of regression terms}$

$n-k-1 = \# \text{ of degrees of freedom}$

### Test for significance of regression:

$$H_0: \beta_1 = \beta_2 = \dots = \beta_k = 0$$

$$H_1: \beta_i \neq 0 \text{ for at least one } i$$

Test Statistic:

$$F_0 = \frac{SS_R/k}{SS_E/(n-k-1)}$$

### Partial F-test:

$$H_0: \beta_i = 0$$

$$H_1: \beta_i \neq 0$$

$$\text{Test Statistic: } F_0 = \frac{\frac{SS_{R2}(\beta_i)}{1}}{\frac{SS_E}{n-k-1}}$$

Reference Distribution:

$$F_{\text{dist}}$$

$$F_{k, n-k-1}$$

Rejection criteria:  $F_{\text{dist}} > F_0$

Reject  $H_0$  if  $F_0 > F_{\text{dist}}$

$$F_0 > F_{2, k, n-k-1}$$

or  $p\text{-value} < \alpha$

Reference Distribution:

$$F_{\text{dist}}$$

$$F_{1, n-k-1}$$

Rejection criteria:

$$F_0 > F_{2, 1, n-k-1}$$

### Nicely Typed Notes

#### Test for significance of regression:

$$H_0: \beta_1 = \beta_2 = \dots = \beta_k = 0$$

$$H_1: \beta_j \neq 0 \text{ for at least one } j$$

$$F_0 = \text{SSR}/k / \text{SSE}/(n-k-1)$$

Reference Distribution:  
F-distribution: F  $k, n-k-1$

Rejection criteria:  
Reject  $H_0$  if  $F_0$  exceeds F alpha,  $k, n-k-1$   
Or if p-value is  $< \alpha$

#### Partial F-test:

$$H_0: \beta_i = 0$$

$$H_1: \beta_i \neq 0$$

Reference Distribution:  
F-distribution: F 1,  $n-k-1$

$$F_0 = \text{SSR}(\beta_i)/1 / \text{SSE}/(n-k-1)$$

Rejection criteria:  
Reject  $H_0$  if  $F_0$  exceeds F alpha, 1,  $n-k-1$   
Or if p-value is  $< \alpha$

## Adjusted $R^2$

$$SS_T = SS_R + SS_E \Rightarrow R^2 = \frac{SS_R}{SS_T} \text{ Good for simple linear regression}$$

Use Adj.  $R^2$  for multiple!  $R^2 = \frac{SS_R}{k} / \frac{SS_E}{n-1}$

Avoids overfitting

Number of DoF avoids invalid  $R^2$  arbitrarily!

## Example

Find a statistically satisfactory (alpha = 0.05) model for:

Brake Horsepower (y) rpm	Road Octane Number	Compression
225	2000	90
212	1800	94
229	2400	88
222	1900	91
219	1800	96
278	2500	96
246	3000	94
237	3200	90
233	2800	88
224	3400	86
223	1800	90
230	2500	89
		104

## Method

Plot: Graph > Scatter plot > plot response against all X's

Fit Model: Stat > Regression > Fit Regression Model (add Response, and Predictors by double click) (Determining model by button "Model" to add or subtract terms)

Contour Plots: Stat - regression > regression > contour or surface plot

1) Plot the data

2) Fit a multiple linear regression model. Check the ANOVA results, R2adj, residuals, normality

3) Based on results, consider adding or removing regressors

4) If model is changed – repeat 2-3

5) Generate final model and contour plots

## Final Result

### Regression Equation

$$\text{Brake Horsepower} = -266.0 + 0.01071 \text{ rpm} + 3.135 \text{ Road Octane Number}$$

*Does low  $\rightarrow$  Sgn. B + U*

Goal: simplest model which significantly describes the data

# Control Charts

- Standard Process Control: A collection of problem-solving tools useful in achieving process stability and reducing variability

- The Magnificent Seven

1. Histogram or stem-and-leaf plot
2. Check sheet
3. Pareto chart
4. Cause-and-effect diagram
5. Defect concentration diagram
6. Scatter diagram
7. Control chart

## Showchart's Control Charts & Types of Variability

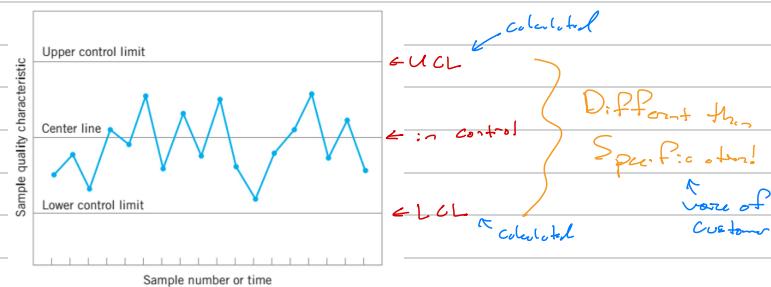
- Chance causes of variation: Inherent variability in a process or background noise - a process operating with only chance causes is in statistical control
- Assignable Causes of Variation: Sources of variability that are not due to chance - a process operating outside of assignable ranges is an out-of-control process
  - Sources: Improperly adjusted/controlled machines
  - Operator errors
  - Defective raw material \*

→ Control Charts: Online process monitoring technique

## Control Chart Uses

- Monitoring: Using the control chart to determine if the process is in control, and quickly detect any assignable causes
- Estimating: Using the control chart to estimate process parameters (mean or variance)
- Improving: Using the control chart to improve process parameters

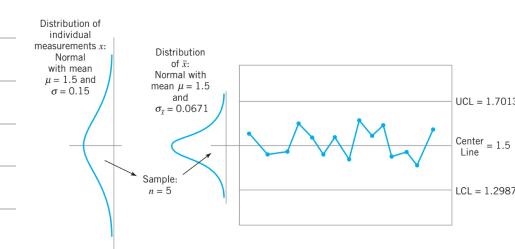
## Features of a Control Chart



## Relationship to Hypothesis Testing

Null Hypothesis:  $M = M_0$  in statistical control

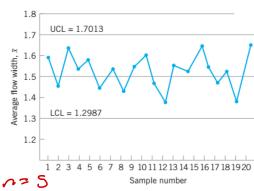
Alternative:  $M \neq M_0$



Type I error: Rejecting  $H_0$  when  $H_0$  is true; concluding the process is out of control when it is not

Type II error: Not rejecting  $H_0$  when it is false; assuming a process is in control when it really is out of control

## Example



Control chart: uses Sample average to monitor a process mean

$M = 1.5$   
 $\sigma = 0.15$   
 $100(1-\alpha)\%$  of Sample means should fall between control limits

Three Sigma Control limits: Define  $Z_{\alpha/2} = 3 \Rightarrow \alpha = 0.002 \Rightarrow$

$$\text{Control limits: } \bar{x} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

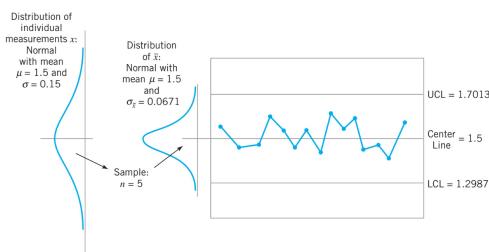
(Known values)

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{0.15}{\sqrt{5}} = 0.0671$$

$$UCL = \bar{x} + 3(0.0671) = 1.7013$$

$$LCL = \bar{x} - 3(0.0671) = 1.2987$$

## General Model



$w$  = quality characteristic of interest

$M_w$  = mean of  $w$

$\sigma_w$  = std dev of  $w$

$L$  = distance from centerline expressed in std dev

$$UCL = M_w + L \sigma_w$$

$$Center = M_w$$

$$LCL = M_w - L \sigma_w$$

## Design of a Control Chart

### Solving Control Limits:

- Set alpha (Europe)

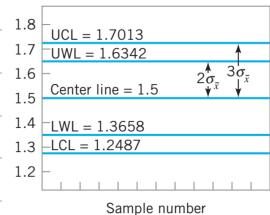
- Set Sigma Level ( $U_3$ )

$U_3$  is pretty standard compromise  
balancing type I and II error

### Two Limit Control Charts

- Action limits - outer limits that trigger a search for assignable cause

- Warning limits - inner limits which raise alarms



Average Run Length (ARL) - the average # of points that must be plotted before a point will indicate an out-of-control condition

$$ARL = \frac{1}{p} ; p = \alpha, \text{ probability the point exceeds control limit}$$

Average Run Signal (ARS) - the average amount of time between out-of-control signals

$$ARS = ARL \times (\text{time between samples})$$

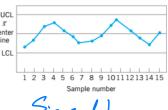
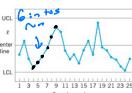
Analysis of Patterns A process can be out of control even if all points are within limits!

Run: A series of observations of the same type

Run Length: How many points are in the series

Pattern: Non-random behavior

Zone Rules: Criteria to help identify patterns



## Dotsy Patterns and Sensitizing Rules

Standard Action Signal:	1. One or more points outside of the control limits 2. Two of three consecutive points outside the two-sigma warning limits but still inside the control limits 3. Four or five consecutive points beyond the one-sigma limits 4. A run of eight consecutive points on one side of the center line	Western Electric Rules
Sensitizing Rules:	Out of control if yes	
If yes, investigate more, get more data	9. An unusual or nonrandom pattern in the data 10. One or more points near a warning or control limit	
Raises alarm		

## Sensitizing Rules: Criteria to Increase Sensitivity



⇒ Out of Control example.

X-Chart: Average of samples

R-Chart: the difference between mean values of a sample

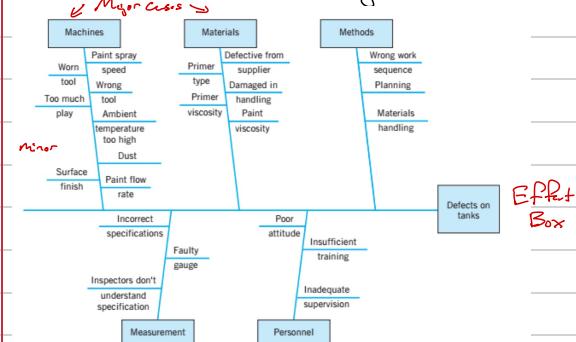
## Out of Control Action Plans

OCAP: A flowchart or text-based description of the sequence of actions that will take place following an activity event

Checkpoints: Potential assignable causes

Terminos: Action taken to Resolve

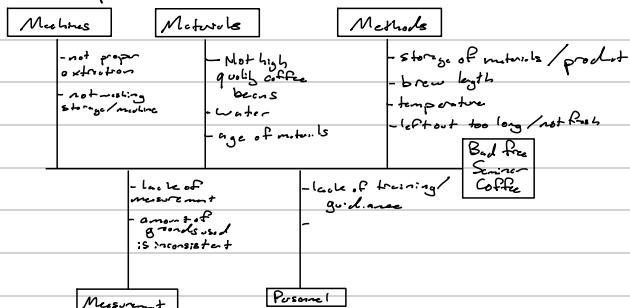
## Cause and Effect Diagram



## How to construct a C&E diagram

1. Define the problem or effect to be analyzed.
2. Form the team to perform the analysis. Often the team will uncover potential causes through brainstorming.
3. Draw the effect box and the center line.
4. Specify the major potential cause categories and join them as boxes connected to the center line.
5. Identify the possible causes and classify them into the categories in step 4. Create new categories, if necessary.
6. Rank order the causes to identify those that seem most likely to impact the problem.
7. Take corrective action.

## Example: Free Seminar Coffee





# Test 2 Review

## Topics

### General:

- how to correctly phrase conclusions in a hypothesis test
- how to correctly phrase what the p-value is
- how to correctly phrase what confidence intervals are representing
- how to sketch hypothesis tests on a reference (sampling) distribution

### Inferring the variance of two populations:

- how to conduct the hypothesis test
- how to draw conclusions

### ANOVA:

- how to conduct the test (by hand and interpret Minitab) and draw conclusions
- what situations ANOVA is used in (when to pick it as the statistical test)
- what, generally, the test statistic represents
- what the reference distribution is as related to the test statistic and when the test statistic follows the reference distribution
- know how to check assumptions and what assumptions are made
- how and when to perform a posthoc test
- how and when to transform data

### Linear Regression:

- know how to perform simple linear regression by hand (how to perform the hypothesis test – won't ask the regression terms by hand)
- how to set up equations and hypothesis for multiple and simple linear regression
- know how to interpret results from Minitab output on simple and multiple linear regression
- explain what  $R^2$  is (+adjusted), what it can and cannot be used for, and how to calculate by hand for simple and multiple linear regression
- know how to check assumptions and what assumptions are made
- how and when to transform data
- describe what the confidence intervals on the response and coefficients are

### Control Charts:

- Be able to explain what impact wide or narrow control limits have on type I and type II error, and be able to define type I and type II errors for control charts
- Be able to explain in words and sketches why it is important to use control charts for both process mean and variability
- Calculate control limits and warning limits for  $\bar{x}$  and R charts
- Be able to identify and out of control process
- ARL calculation and concept