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Group 5

Worksheet 3 - MATH223

The colours are helpful!

1) $\dfrac{y^2}{9} + \dfrac{z^2}{4} = 1 - x^2$

$z = 4\left(1 - x^2 - \dfrac{y^2}{9}\right)$

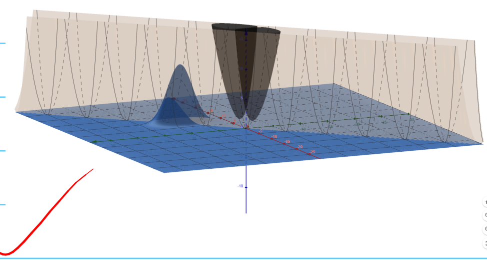
$z^2 = 4 - 4x^2 - \dfrac{4y^2}{9}$

$2z\dfrac{\partial z}{\partial x} = -8x \qquad \dfrac{\partial z}{\partial x} = \dfrac{-8x}{2z} \qquad \dfrac{\partial z}{\partial x} = \dfrac{-4x}{z}$

$2z\dfrac{\partial z}{\partial y} = -\dfrac{8y}{9} \qquad \dfrac{\partial z}{\partial y} = \dfrac{-8y}{18} \qquad \dfrac{\partial z}{\partial y} = \dfrac{-4y}{9}$

$f(x,y) = \dfrac{1}{x^2} + y^2 + (y-x)^2$ $\qquad v(t,s) = \langle 10\exp(-0.3(t-4)^2 - 0.1(s-8)^2), (t-s)^2 \rangle$

2) a) The first component of v corresponds to the x input of f, while the second component of v corresponds to y. x is represented by an equation in the form e^{-x^2}, where graphs the normal distribution, or a curve with a constant area. This means that regardless of how much sleep or stress the friend has, the relationship will still be proportional and continue to be the same. The scalars and constants in the exponent exist to position and scale the hump graphed. The y term is squared so that the y term is never negative. This makes sense in the context of the problem because stress levels cannot be negative. The graph is included here.



The blue graphs the x component of v, orange is the y, black is $f(x,y)$

b) $\nabla f = \langle f_x, f_y \rangle$, $f(x,y) = x^{-2} + y^2 + (y-x)^2$

$f_x = -2x^{-3} + 2(y-x)(-1)$

$\quad = -2x^{-3} - 2(y-x)$

$f_y = 2y + 2(y-x)$

$\nabla f(x,y) = \left\langle \dfrac{-2}{x^3} - 2(y-x),\ 2y + 2(y-x) \right\rangle$

c) $v(10,5) = \langle 10\exp(-0.3(10-9)^2 - 0.1(5-8)^2), (10-5)^2 \rangle$

$= \langle 10\exp(-0.3 - 0.1(9)), 25 \rangle = \langle 10\exp(-0.3 - 0.9), 25 \rangle$

$= \langle 10\exp(-1.2), 25 \rangle = \langle 3.01, 25 \rangle$

To the nearest integer: $\langle 3, 25 \rangle$ energy level $= 3$

stress level $= 25$

d) $\nabla(f \circ v) = \nabla f \cdot J_v$ $\qquad J_v = \left[\frac{\partial v}{\partial t} \;\middle|\; \frac{\partial v}{\partial s} \right]$

$\nabla f = \langle \frac{-2}{x^3} - 2(y-x),\ 2y + 2(y-x) \rangle$

$\nabla f(3, 25) = \langle -44.07, 94 \rangle$

$$J_v = \begin{bmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \end{bmatrix} \Rightarrow \begin{array}{l} \frac{\partial x}{\partial t} = (10\exp(-0.3(t-9)^2 - 0.1(s-8)^2))(-0.6(t-9)) \\ \frac{\partial y}{\partial t} = 2(t-s) \\ \frac{\partial x}{\partial s} = (10\exp(-0.3(t-9)^2 - 0.1(s-8)^2))(-0.2(s-8)) \\ \frac{\partial y}{\partial s} = 2(t-s)(-1) = -2(t-s) \end{array}$$

$$J_v(10,5) = \begin{bmatrix} \frac{\partial x}{\partial t}(10,5) & \frac{\partial x}{\partial s}(10,5) \\ \frac{\partial y}{\partial t}(10,5) & \frac{\partial y}{\partial s}(10,5) \end{bmatrix} \qquad \begin{array}{l} \frac{\partial x}{\partial t}(10,5) = -1.8 \\ \frac{\partial y}{\partial t}(10,5) = 10 \\ \frac{\partial x}{\partial s}(10,5) = 1.8 \\ \frac{\partial y}{\partial s}(10,5) = -10 \end{array}$$

$J_v(10,5) = \begin{bmatrix} -1.8 & 1.8 \\ 10 & -10 \end{bmatrix}$ and $\nabla f(v(10,5)) = \langle -44.07, 94 \rangle$

$$\nabla(f \circ v) = \nabla f(v(t,s)) \cdot J_v = \langle -44.07, 94 \rangle \cdot \begin{bmatrix} -1.8 & 1.8 \\ 10 & -10 \end{bmatrix}$$

$$= \langle 1019.326, -1019.326 \rangle$$

$$\nabla(f \circ v) = \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle = \langle 1019.326, -1019.326 \rangle$$

e) The friend is equally affected by the time of day and sleep. The two values in the gradient have equal absolute values, but the friend's procrastination is impacted differently at $t=10$ and $s=5$. They will procrastinate more with a change of time of day, and will procrastinate less with more hours of sleep. Their procrastination will be affected by an amount of equal magnitude, in opposite ways.

3) a) $f(x,y) = x^y = e^{y \ln x}$

$f_x = e^{y\ln x}\left(\frac{y}{x}\right) = x^y\left(\frac{y}{x}\right) = x^y y x^{-1} = yx^{y-1}$ ← power rule $\qquad \nabla f = \langle yx^{y-1}, x^y \ln x \rangle$

$f_y = e^{y \ln x}(\ln x) = x^y(\ln x)$ ← $\frac{d}{dx}(a^x)$ derivative

b) $f(x,y) \approx \nabla f(a,b) \cdot (\langle x,y \rangle - \langle a,b \rangle) + f(a,b)$, given $(a,b) = (1,3)$, approx. $\left(\frac{\pi}{3}\right)^e$

$f(1,3) = 1^3 = 1$, $\nabla f(1,3) = \langle 3(1)^2, 1 \ln 1 \rangle = \langle 3, 0 \rangle$

$f(\frac{\pi}{3}, e) = ?$ $\quad f(\frac{\pi}{3}, e) \approx \nabla f(1,3) \cdot (\langle \frac{\pi}{3}, e \rangle - \langle 1,3 \rangle) + f(1,3)$ $\qquad f(\frac{\pi}{3}, e) \approx \pi - 2$

$$= \langle 3, 0 \rangle \cdot \langle \tfrac{\pi}{3} - 1, e - 3 \rangle + 1$$

$$= (\pi - 3) + 0 + 1 = \pi - 2$$

c) via calculator: $f(\frac{\pi}{3}, e) = (\frac{\pi}{3})^e = 1.1336$

approximation of $f(\frac{\pi}{3}, e) = 1.14159$

The error (|difference|) is about 0.00804, very low!

4) $f(x,y) = x^2 + 2xy + y^2$

a) find $f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$

$f_x = 2x + 2y$ $\qquad f_{xx} = 2$

$f_y = 2x + 2y$ $\qquad f_{yy} = 2$

$f_{xy} = 2$ $\qquad f_{yx} = 2$

b) $z = 0, 1, 2, 3$, $A: (0.5, 0.5)$ $\qquad z = x^2 + 2xy + y^2$

$0 = x^2 + 2xy + y^2$

$$x = \frac{-2y \pm \sqrt{4y^2 - 4(1)(y^2)}}{2} = \frac{-2y \pm \sqrt{0}}{2}$$

$x = -y$, $y = -x$

$1 = x^2 + 2xy + y^2$

$$x = \frac{-2y \pm \sqrt{4y^2 - 4(1)(y^2 - 1)}}{2} = \frac{-2y \pm \sqrt{4}}{2}$$

$x = -y \pm 1$, $y = -x \pm 1$

$2 = x^2 + 2xy + y^2$

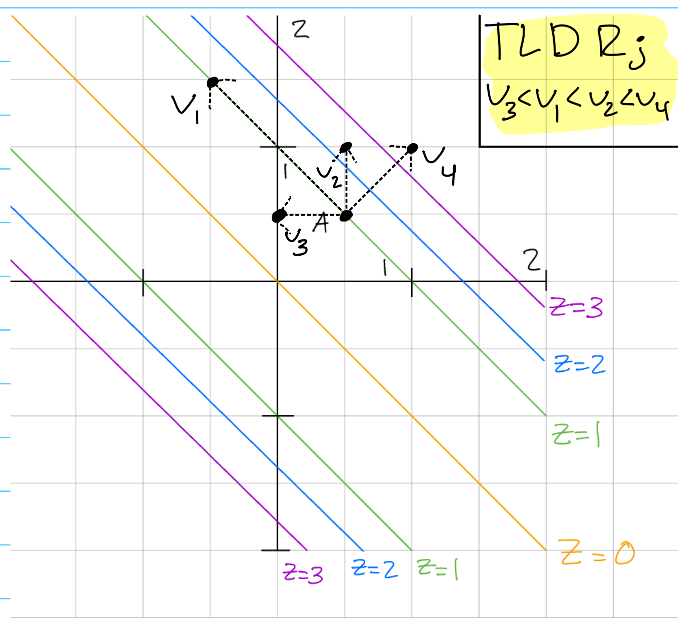
$$x = \frac{-2y \pm \sqrt{4y^2 - 4(1)(y^2 - 2)}}{2} = \frac{-2y \pm \sqrt{8}}{2}$$

$x = -y \pm \sqrt{2}$, $y = -x \pm \sqrt{2}$

$3 = x^2 + 2xy + y^2$

$$x = \frac{-2y \pm \sqrt{4y^2 - 4(1)(y^2 - 3)}}{2} = \frac{-2y \pm \sqrt{12}}{2}$$

$x = -y \pm \sqrt{3}$ $\quad y = -x \pm \sqrt{3}$



TLDR;
$v_3 < v_1 < v_2 < v_4$

c) $v_1 = \langle -2, 2 \rangle$, $v_2 = \langle 0, 1 \rangle$

$v_3 = \langle -1, 0 \rangle$, $v_4 = \langle 1, 1 \rangle$

i) As seen by the plot to the left, v_1 points in the direction of the contour line $z = 1$. z is constant along this direction, so this directional derivative should equal 0. v_2 and v_4 both point in directions where the z value is increasing, as seen where the z value goes from 1 to 2 for v_2 and 1 to 3+ for v_4. Due to the contour values increasing more at a more rapid pace for v_4, it is where the Directional derivative is most positive. v_3 is in the direction of $z = 0$, a lower contour line, so this is where the directional derivative is most negative.

ii) Let $\vec{u}_n$ be the unit vectors of Defined vectors, $\vec{v}_n$.

$\vec{u}_1 = \dfrac{\langle -2, 2\rangle}{\sqrt{4+4}} = \dfrac{\langle -2, 2\rangle}{\sqrt{8}} = \left\langle -\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right\rangle$

$\vec{u}_2 = \langle 0, 1\rangle$

$\vec{u}_3 = \langle -1, 0\rangle$

These are already unit vectors

$\vec{u}_4 = \dfrac{\langle 1, 1\rangle}{\sqrt{1+1}} = \dfrac{\langle 1, 1\rangle}{\sqrt{2}} = \left\langle \dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right\rangle$

$\nabla f = \langle 2x+2y, 2y+2x\rangle$ using part A

$\nabla f(A) = \nabla f(0.5, 0.5)$

$= \langle 2(0.5)+2(0.5), 2(0.5), 2(0.5)\rangle$

$\nabla f(0.5, 0.5) = \langle 2, 2\rangle$

$D_{\vec{u}_1} f = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle \cdot \langle 2, 2\rangle$

$= -\sqrt{2} + \sqrt{2} = 0$

$D_{\vec{u}_2} f = \langle 0, 1\rangle \cdot \langle 2, 2\rangle = 2$

$D_{\vec{u}_3} f = \langle -1, 0\rangle \cdot \langle 2, 2\rangle = -2$

$D_{\vec{u}_4} f = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle \cdot \langle 2, 2\rangle$

$= \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

$D_{\vec{u}_3} f < D_{\vec{u}_1} f < D_{\vec{u}_2} f < D_{\vec{u}_4} f$

Because unit vectors $\vec{u}_n$ correspond to those defined in 4c, my assumptions of the directional derivatives is true.

5) Challenge Problem

$f(x,y,z) = c$ for $c \in \mathbb{R}$, Prove $\dfrac{\partial z}{\partial x} = -\dfrac{f_x}{f_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{f_y}{f_z}$

a) Let $z = f(x,y)$ and $v(x,y) = (x, y, z(x,y))$

$$J_v = \left[\frac{\partial v}{\partial x} \;\middle|\; \frac{\partial v}{\partial y} \right] = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix}$$

b) $\nabla(f \circ v) = \nabla f \cdot J_v$

$\nabla f = (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$

$$\nabla(f \circ v) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix}$$

$$\nabla(f \circ v) = \left\langle f_x(x,y,z) + f_z(x,y,z)\frac{\partial z}{\partial x}, f_y(x,y,z) + f_z(x,y,z)\frac{\partial z}{\partial y}\right\rangle$$

c) It is known that $f(x,y,z) = c$, a constant, so $f \circ v$ is also constant as v controls the inputs to f. We can now set $\nabla(f \circ v) = 0$ and solve for $\left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle$.

$\langle 0, 0\rangle = \left\langle f_x + f_z \frac{\partial z}{\partial x}, f_y + f_z \frac{\partial z}{\partial y}\right\rangle$

Note: Partials from B have been condensed to save space

$0 = f_x + f_z \frac{\partial z}{\partial x}$ $\qquad 0 = f_y + f_z \frac{\partial z}{\partial y}$

$-f_x = f_z \frac{\partial z}{\partial x}$ $\qquad -f_y = f_z \frac{\partial z}{\partial y}$

$\dfrac{\partial z}{\partial x} = \dfrac{-f_x}{f_z}$ $\qquad \dfrac{\partial z}{\partial y} = \dfrac{-f_y}{f_z}$

Solving for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ proves the shortcut for implicit differentiation