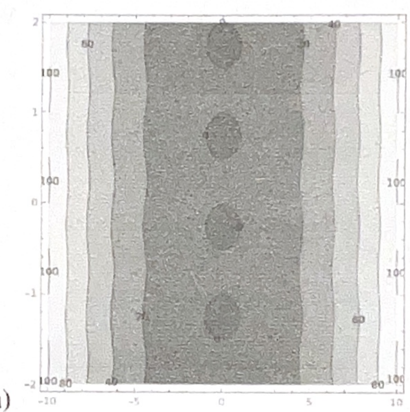
Trevor Swan
Group 4

MATH 223 — Worksheet 2

Due: 2023–09–22

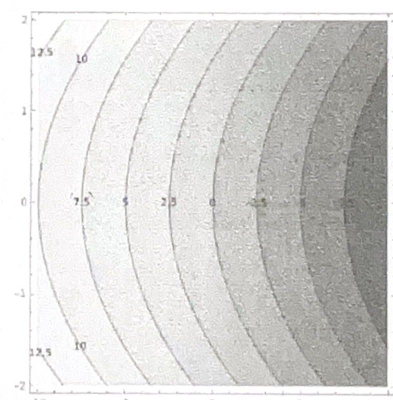
1. For each of the following multivariate functions, indicate the corresponding contour plot, and explain how you know.

- Lake Erie.

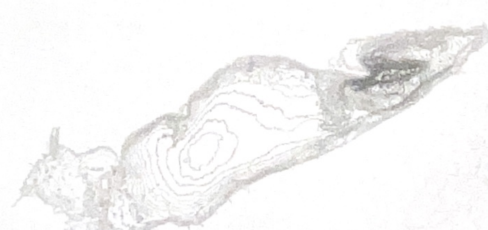


(a)

- $f_1(x, y) = y^2 - x$.



(b)



- $f_2(x, y) = x^2 + \sin(2\pi y)$

(c)

2. A lot of systems in physics are proportional to the inverse square of some property (e.g., gravity, Coulomb's Law).

   (a) Pick at least four values of z equally spaced. Carefully draw a contour map of $f(x, y) = \dfrac{1}{x^2 + y^2}$ with level curves corresponding to your chosen z values. Label the axes and the curves.

   (b) Based on the contour map, does the direction you approach the origin affect the limit of f? What does this tell you about the limit of f at the origin?

   (c) Pick at least four values of z equally spaced. Carefully draw a contour map of $g(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ with level curves corresponding to your chosen z values. Label the axes and the curves.

   (d) Based on the contour map, does the direction you approach the origin affect the limit of g? What does this tell you about the limit of g at the origin?

   (e) Convert f and g to polar coordinates. What does this tell you about the limits of f and g at the origin?

3. For each of the following functions, decide whether the function is continuous or not. Explain.

   (a) $f_1(x, y, z) = \dfrac{xy + z \cdot \sin(y)}{x^2 + 1}$.

   (b) $f_2(x, y, z) = \exp(x^2 - y^2)$.

   (c) $f_3(x, y, z) = \dfrac{x - 1}{xy - y + xz^2 - z^2}$.

Trevor Swan
Group 4

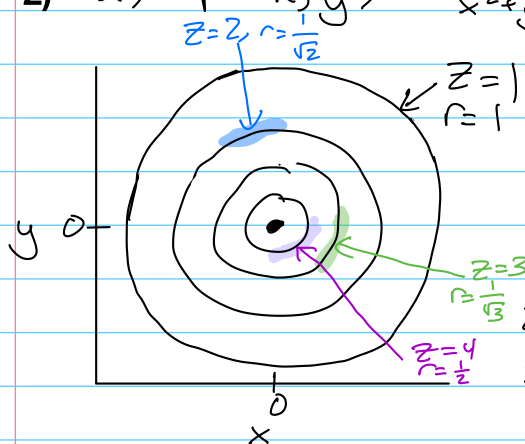
Worksheet 2 - MATH223

1) • Lake Erie must be "c" because it is abstract in nature and cannot be easily explained by a single equation

• $f_1(x,y) = y^2 - x$ corresponds to "b" due to the following: if x is constant, increasing y-values causes the function value, $f_1(x,y)$, to increase as well. Similarly, if y is held constant, increasing x-values causes $f_1(x,y)$ to decrease. Trying these scenarios out on the given graphs reveals this relationship to hold true on "b".

• $f_2(x,y) = \frac{1}{x^2+y^2}$ corresponds to "c" and this can be proved by the same reasoning mentioned above. A more robust, yet less fun method, is the process of elimination. A graph on mathematica also reveals this correspondence.

2) a) $f(x,y) = \frac{1}{x^2+y^2}$, chosen z values: $z = \{1, 2, 3, 4$



• Centered @ (0,0)

• Radius decreases as z increases

$z_0 = 1$, $r_0 = 1$

$z_1 = 2$, $r_1 = \frac{1}{\sqrt{2}}$

$z_2 = 3$, $r_2 = \frac{1}{\sqrt{3}}$

$z_3 = 4$, $r_3 = \frac{1}{2}$

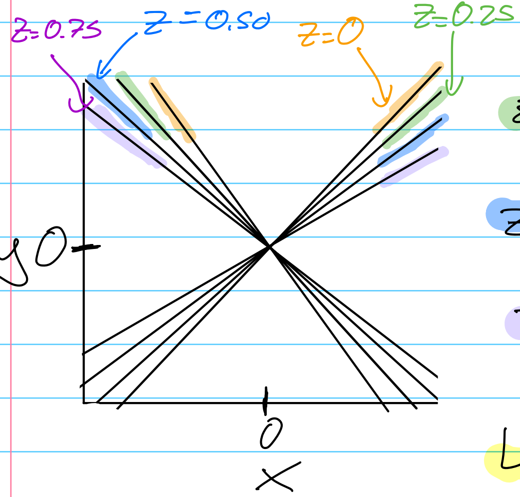
$$z = \frac{1}{x^2+y^2}$$

$$x^2+y^2 = \frac{1}{z}, \quad r = \frac{\sqrt{1}}{\sqrt{z}} = \frac{1}{\sqrt{z}}$$

b) No, because the contour plot consists of circles that have constant, decreasing radii. The direction/angle you approach the origin does not effect the limit of f. Because the radius can never be equal to 0, the limit of f approaches infinity (or does not exist).

c) $g(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$, chosen z values: $z = \{0, 0.25, 0.50, 0.75$

Centered @ (0,0)



$z = \frac{x^2 - y^2}{x^2 + y^2}$, $z = 0$, $0 = \frac{x^2-y^2}{x^2+y^2}$ $\quad x^2 - y^2 = 0$

$y^2 = x^2 \Rightarrow y = \pm x$

$z = .25$, $0.25(x^2+y^2) = x^2 - y^2$

$0.75x^2 = 1.25y^2$ $\quad y^2 = \frac{0.75}{1.25}x^2$

$y = \pm .775x$

$z = 0.50$, $0.50(x^2+y^2) = x^2 - y^2$ $\quad y^2 = \frac{0.50}{1.50}x^2$

$0.50x^2 = 1.50y^2$ $\quad y = \pm 0.577x$

$z = 0.75$, $.75(x^2+y^2) = x^2 - y^2$ $\quad y^2 = \frac{0.28}{1.75}x^2$

$0.25x^2 = 1.75y^2$ $\quad y = \pm 0.378x$

Lines w/ tips highlighted the same color have the same z

d) Yes, the direction you approach the origin does effect the limit of g. If you approach from the y direction, where x is held constant, the limit is -1. When y is held constant and the origin is approached from the x direction, the limit 1. Due to the graph having two different limit values at the origin, the limit does not exist.

e) $f(x,y) = \frac{1}{x^2+y^2} \Rightarrow F(r,\theta) = \frac{1}{r^2}$

$g(x,y) = \frac{x^2-y^2}{x^2+y^2} \Rightarrow G(r,\theta) = \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2}$

$G(r,\theta) = \cos^2\theta - \sin^2\theta$

F only has r in the polar equation, so this means limits are only determined by the radius. This supports the previous conclusion as when $r \to 0$, $f \to \infty$ (or DNE).

G is dependent on θ, so the limit of G at the origin is also dependent on θ. The limit changes based on the angle you approach the origin, so the limit does not exist.

3) Key Idea: Numerator and Denominator must be cts. and the denom must never equal 0 to be cts.

a) $f_1(x,y,z) = \dfrac{xy + z \cdot \sin y}{x^2+1}$

x, y, and z are all continuous input in the domain of f_1, so $xy + z\sin y$ must also be cts. because xy and $z\sin y$ are defined for all values of x, y, and z. x^2+1 is also a cts function because x itself is cts. Because x is squared and added to 1, the denom. can never equal zero. All criteria have been met $\therefore$ f_1 is cts.

b) $f_2(x,y,z) = \exp(x^2 - y^2)$

x and y are both continuous in the domain of f, so x^2 and y^2 are also cts for all x and all y. $x^2 - y^2$ is also cts. because x^2 and y^2 are cts. e to the power of any real number yields an output, so it is cts for all x and y. All criteria have been met $\therefore$ f_2 is cts.

c) $f_3(x,y,z) = \dfrac{x-1}{xy - y + xz^2 - z^2}$

Denom: $y(x-1) + z^2(x-1) \Rightarrow (y+z^2)(x-1)$

f_3 can be proven discontinuous assuming x, y, and z are cts variables, which is true. When $x=1$, f_3 is undefined as it will be equal to $0/0$. Another discontinuity can be found when $x=0$, $y=0$, and $z=0$, where the output will be $-1/0$. This yields yet another divide by zero error. Finally, the graph is discontinuous for all y when $y = z^2$. Although only one instance of discontinuity was needed, these instances prove f_3 is dis cts.