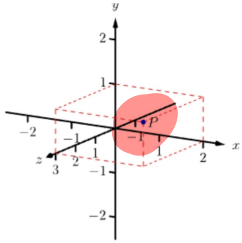
Chapter 11

Chapter 11.1 Notes

Fingers point towards positive direction

Right hand Rule - when the right hand index finger is extended in the direction of the $+x$-axis the perpendicularly bent middle finger represents the $+y$-axis, and the thumb in the direction of the $+z$-axis.

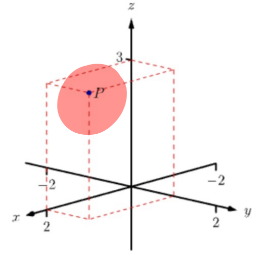
Point P is in the same spot!



Due to the difficulty of reading 3D planes on 2D displays, this convention is used to depict where a Point is.

8 quadrants in space... P is in first quad

This text displays 3D spaces with x, y being the "ground of a room" with z representing the height.



Distance in space - Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

The line segment that connects P and Q in space is $\overline{PQ}$, and the length is referred to as $\|\overline{PQ}\|$ (magnitude)

Circle - the set of all points in the plane equidistant from a given point

Sphere - the set of all points in space that are equidistant from a given point.
↓ Only difference

Point $C = (a, b, c)$, the center of a sphere with radius "r". If a point $P = (x, y, z)$ lies on the sphere, then P is r units from C

$$\|\overrightarrow{PC}\| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r$$

Standard equation of a sphere in space -

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Planes in space -

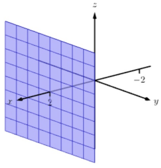
The coordinate axes define three planes...

xy-plane: Set of all points where $z = 0$.

xz-plane: Set of all points where $y = 0$.

yz-plane: Set of all points where $x = 0$.

Highlighted planes correspond with equations that characterize them.



The equation $x=2$ describes all points in space where the x-value is 2. This is a plane parallel to the yz coordinate plane.

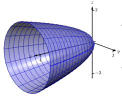
Cylinder- Let C be a curve in a plane and let L be a line not parallel to C. A cylinder is the set of all lines parallel to L that pass through C. The curve C is the **directrix** of the cylinder, and the lines are the rulings.

Directrix- A fixed line used in describing a curve or surface

For our purposes, curve "C" lies in a plane parallel to a coordinate plane, while lines "L" are perpendicular to these planes, forming right cylinders.

Consider the surface formed by revolving $y = \sqrt{x}$ about the x-axis...

- Cross-sections of this surface parallel to the yz plane are circles
- Each circle has equation in the form $y^2 + z^2 = r^2$ for some radius "r"
- The radius is a function of x, $r(x) = \sqrt{x}$
- The equation for the surface on the right is $y^2 + z^2 = (\sqrt{x})^2$



Key ideas

The equation of the surface formed by revolving...

- $y = r(x)$ or $z = r(x)$ about the x-axis is $y^2 + z^2 = r(x)^2$
- $x = r(y)$ or $z = r(y)$ about the y-axis is $x^2 + z^2 = r(y)^2$
- $x = r(z)$ or $y = r(z)$ about the z-axis is $x^2 + y^2 = r(z)^2$

Let $z = f(x)$, $x \geq 0$, be a curve in the xz-plane. The surface formed by revolving this curve about the z-axis has the equation

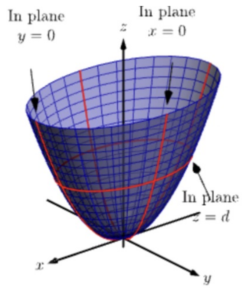
$$z = f\left(\sqrt{x^2+y^2}\right)$$

Quadric Surface - the graph of the general second-degree equation in 3 variables:

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J=0$$

when D, E, or F are not zero, the basic shapes are rotated in space.

There are 6 basic quadric surfaces - elliptic paraboloid, elliptic cone, ellipsoid, hyperboloid of one sheet, hyperboloid of 2 sheets, and the hyperbolic paraboloid.



Consider the elliptic paraboloid $z = \frac{1}{4}x^2 + y^2$

If we intersect this shape with the plane $z = d$ we have the equation:

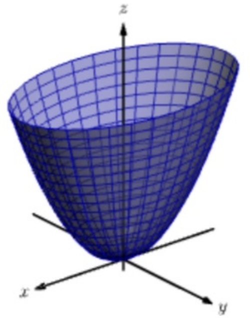
$$d = \frac{1}{4}x^2 + y^2$$ ellipse

Divide both sides by d:
$$1 = \frac{x^2}{4d} + \frac{y^2}{d}$$

Consider cross-sections parallel to the xz-plane, letting $y=0$ yields a parabola - $z = \frac{1}{4}x^2$

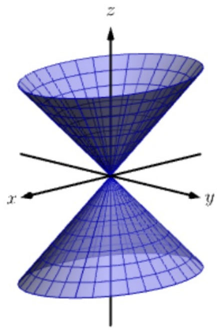
Some cross sections are parabolas while others are ellipses, hence the name.

The Six basic Quadric Surfaces



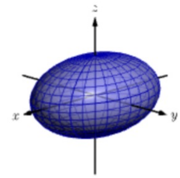
Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



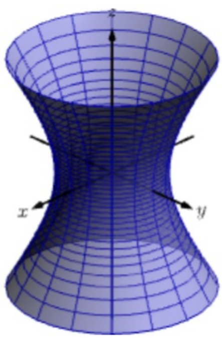
Elliptic Cone

$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



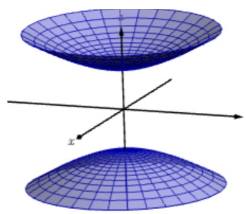
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



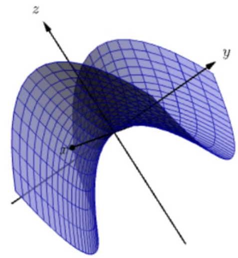
Hyperboloid of One sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



Hyperboloid of Two sheets

$$\frac{z^2}{c^2} - \frac{x^2}{c^2} - \frac{y^2}{b^2} = 1$$



Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Quadric Surface Identification Guide

$$\frac{x^2}{a^2} \square \frac{y^2}{b^2} \square \frac{z^n}{c^2} = 1$$

- z
  - $+\,-$ → Elliptic Paraboloid
  - $-\,-$ → Hyperbolic Paraboloid
- z^2
  - $+\,-$
    - $c=1$ → Elliptic Cone
    - $c\neq1$ → Hyperboloid in One Sheet
  - $-\,-$ → Hyperboloid in 2 sheets
  - $+\,+$ → Ellipsoid

Chapter 11.2 Notes

Vectors - convey both magnitude and direction information. A directed line segment.

Given points P and Q (in a plane or space), we denote with $\overrightarrow{PQ}$ the vector from P to Q. P is the initial point. Q is terminal.

The magnitude, length, or norm of $\overrightarrow{PQ}$ is the length of the line segment $\overline{PQ}$

Two vectors are equal if they have the same mag. and direction

We use: $\mathbb{R}^2$ (r two) to represent all the vectors in the plane
$\mathbb{R}^3$ (r three) to represent all the vectors in space

A vector whose x-displacement is "a" and whose y-displacement is "b" will have the terminal point (a, b), when the initial point is the origin, (0, 0).

Component Form

- ... of a vector in $\mathbb{R}^2$ whose terminal point is (a, b) when its initial point is (0, 0), is $\langle a, b \rangle$.
- ... of a vector in $\mathbb{R}^3$ whose terminal point is (a, b, c) when its initial point is (0, 0, 0) is $\langle a, b, c \rangle$.

When $P = (x_1, y_1)$ and $Q = (x_2, y_2)$

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

When $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Head to tail Rule

The addition/sum of the vectors $\vec{u}$ and $\vec{v}$ is

$$\vec{u}+\vec{v} = \langle u_1+v_1, u_2+v_2 \rangle \quad \text{plane}$$

$$= \langle u_1+v_1, u_2+v_2, u_3+v_3 \rangle \quad \text{space}$$

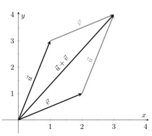
In words, starting at an initial point, go out $\vec{u}$, then go out $\vec{v}$.

The multiplication of a scalar c and $\vec{v}$ is

$$c\vec{v} = c\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle \quad \text{plane}$$

$$= c\langle v_1, v_2, v_3 \rangle = \langle cv_1, cv_2, cv_3 \rangle \quad \text{space}$$

If vectors $\vec{u}$ and $\vec{v}$ represent forces acting on a body, the sum $\vec{u}+\vec{v}$ gives the resulting force. Because of the physical applications of vector addition, the sum is often referred to as the resultant vector, or just the "resultant"



The head to tail rule is also known as the parallelogram law, which holds true in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

Zero vector - the vector whose initial point is also its terminal point. In $\mathbb{R}^2$, it's $(0,0)$, In $\mathbb{R}^3$, it's $(0,0,0)$. Usually denoted by $\vec{0}$.

Following are true for all scalars and vectors...

1. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
2. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
3. $\vec{v}+\vec{0}=\vec{v}$ (additive id)
4. $(cd)\vec{v}=c(d\vec{v})$
5. $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$ or $(c+d)\vec{v}=c\vec{v}+d\vec{v}$
6. $0\vec{v}=0$
7. $\|c\vec{v}\| = |c| \cdot \|\vec{v}\|$
8. $\|\vec{u}\|=0$ if, and only if, $\vec{u}=\vec{0}$

Unit Vector - A vector w/ magnitude 1 $||\vec{u}|| = 1$

If we divide a vector by its magnitude, it becomes a vector of length one, or a unit vector.

Example 11.2.21. Using Unit Vectors. Let $\vec{v} = \langle 3, 1\rangle$ and let $\vec{w} = \langle 1, 2, 2\rangle$.

1. Find the unit vector in the direction of $\vec{v}$.
2. Find the unit vector in the direction of $\vec{w}$.
3. Find the vector in the direction of $\vec{v}$ with magnitude 5.

Unit vector in direction of a nonzero vector:

$\vec{u} = \frac{1}{||\vec{v}||}\vec{v}$

1. $||\vec{v}|| = \sqrt{3^2+1^2} = \sqrt{10}$ $\quad \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle$

2. $||\vec{w}|| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$ $\quad \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$

3. Let $\vec{u} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle$ $\quad 5\vec{u} = \left\langle \frac{15}{\sqrt{10}}, \frac{5}{\sqrt{10}} \right\rangle$

$\vec{v} = ||\vec{v}|| \frac{1}{||\vec{v}||}\vec{v}$ or $\vec{v} = \underbrace{||\vec{v}||}_{\text{mag.}} \cdot \underbrace{\left(\frac{1}{||\vec{v}||}\vec{v}\right)}_{\text{direction}}$

Parallel Vectors

- Unit vectors u_1 and u_2 are parallel if $\vec{u}_1 = \pm\vec{u}_2$
- Nonzero vectors $\vec{v}_1$ and $\vec{v}_2$ are parallel if their respective unit vectors are parallel
- Vectors $\vec{v}_1$ and $\vec{v}_2$ are parallel if there is a scalar $c \neq 0$ such that $\vec{v}_1 = c\vec{v}_2$

The component form of all unit vectors in $\mathbb{R}^2$ is $\langle \cos\theta, \sin\theta \rangle$ for some angle θ

A vector $\vec{u}_3$ in $\mathbb{R}^3$ is a unit vector if, and only if, its component form is:

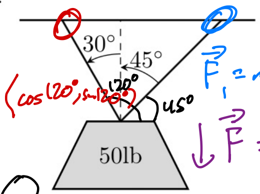
$\langle \sin\theta\cos(\varphi), \sin\theta\sin(\varphi), \cos\theta \rangle$ for some angles θ and φ

Standard Unit Vectors

- In $\mathbb{R}^2$, the standard unit vectors are
  $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$
- In $\mathbb{R}^3$, the standard unit vectors are
  $\vec{i} = \langle 1, 0, 0 \rangle$ and $\vec{j} = \langle 0, 1, 0 \rangle$
  and $\vec{k} = \langle 0, 0, 1 \rangle$

Example 11.2.26. Finding Component Forces. Consider a weight of 50lb hanging from two chains, as shown in Figure 11.2.27. One chain makes an angle of 30° with the vertical, and the other an angle of 45°. Find the force applied to each chain.

$\vec{F}_2 = m_2 \langle \cos 120^\circ, \sin 120^\circ \rangle$



$\vec{F}_1 = m_1 \langle \cos 45^\circ, \sin 45^\circ \rangle$

$\downarrow \vec{F} = \langle 0, -50 \rangle$

$\vec{F}_1 + \vec{F}_2 + \vec{F} = 0$

$\langle 0, -50 \rangle + \langle m_1 \cos 45^\circ, m_1 \sin 45^\circ \rangle + \langle m_2 \cos 120^\circ, m_2 \sin 120^\circ \rangle = \vec{0}$

$m_1 \cos 45^\circ + m_2 \cos 120^\circ = 0$ $\quad (25.88, 36.6)$

$m_1 \sin 45^\circ + m_2 \sin 120^\circ = 50$ $\quad m_1 = 25.88$

$m_2 = 36.6$

Example 11.2.31. Finding Component Force. A weight of 25lb is suspended from a chain of length 2ft while a wind pushes the weight to the right with constant force of 5lb as shown in Figure 11.2.32. What angle will the chain make with the vertical as a result of the wind's pushing? How much higher will the weight be?

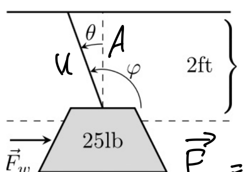
$5 + m \cos \varphi = 0$

$-25 + m \sin \varphi = 0$

$m \cos \varphi = -5$

$m \sin \varphi = 25$

$\vec{F}_c = m \langle \cos \varphi, \sin \varphi \rangle$



$\downarrow \vec{F}_g = -25\vec{j}$

$\vec{F}_w = 5\vec{i}$

$\vec{F}_c + \vec{F}_w + \vec{F}_g = \vec{0}$

$\vec{F}_c = \langle -5, 25 \rangle$

$\|\vec{F}_c\| = \sqrt{5^2 + 25^2}$

$5\sqrt{26} = 25.5\text{lb}$

$5 + 5\sqrt{26} \cos \varphi = 0$ $\quad 11.31^\circ$ w/ vertical

$\varphi = 101.31^\circ$ $\quad \cos = \frac{A}{H}$ $\quad 2 \cos 11.31 = 1.96\text{ft}$

Chapter 11.3 Notes

Dot Product - yields a scalar, not a vector

Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ in $\mathbb{R}^2$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Properties of the dot product

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
4. $\vec{0} \cdot \vec{v} = 0$
5. $\vec{v} \cdot \vec{v} = ||\vec{v}||^2$

The Dot product and angles -

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$

Then, $\vec{u} \cdot \vec{v} = ||\vec{u}|| \; ||\vec{v}|| \cos\theta$

where θ, $0 \leq \theta \leq \pi$, is the angle btwn. $\vec{u}$ and $\vec{v}$

Can be rewritten as $\frac{\vec{u}}{||\vec{u}||} \cdot \frac{\vec{v}}{||\vec{v}||} = \cos\theta$,

the dot product of 2 unit vectors.

The dot product of two directions gives the cosine of the angle between them

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \; ||\vec{v}||} \quad / \quad \theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \; ||\vec{v}||}\right)$$

Orthogonal - Vectors are orthogonal, or perpendicular, if their product is 0. This is true for all nonzero vectors

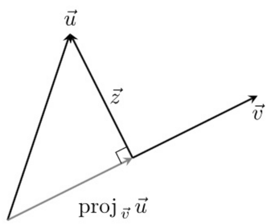
Orthogonal Projection - Let nonzero vectors $\vec{u}$ and $\vec{v}$ be given. The orthogonal projection of $\vec{u}$ onto $\vec{v}$, denoted $\text{proj}_{\vec{v}}\vec{u}$:

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \left(\vec{u}\cdot\frac{\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|}$$

A special case of this occurs when $\vec{v}$ is a unit vector. Here, the formula reduces to just:

$$\text{proj}_{\vec{v}}\vec{u} = (\vec{u}\cdot\vec{v})\vec{v}, \text{ as } \vec{v}\cdot\vec{v} = 1$$

When $\vec{v}$ is a unit vector, essentially providing only direction information, the dot product of $\vec{u}$ and $\vec{v}$ gives how much $\vec{u}$ is in the direction of $\vec{v}$



$$\vec{u} = \text{proj}_{\vec{v}}\vec{u} + \vec{z}$$

Silly? no

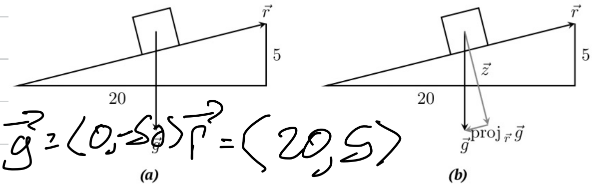
$$\vec{z} = \vec{u} - \text{proj}_{\vec{v}}\vec{u}$$

$$\vec{u} = \text{proj}_{\vec{v}}\vec{u} + (\vec{u} - \text{proj}_{\vec{v}}\vec{u})$$

Orthogonal Decomposition of vectors - Let nonzero vectors $\vec{u}$ and $\vec{v}$ be given. Then $\vec{u}$ can be written as the sum of two vectors, one of which is parallel to $\vec{v}$, and one is orthogonal to $\vec{v}$!

$$\vec{u} = \underbrace{\text{proj}_{\vec{v}}\vec{u}}_{\parallel \vec{v}} + \underbrace{(\vec{u} - \text{proj}_{\vec{v}}\vec{u})}_{\perp \vec{v}}$$

Example 11.3.25. Orthogonally decomposing a force vector. Consider Figure 11.3.26.(a), showing a box weighing 50lb on a ramp that rises 5ft over a span of 20ft. Find the components of force, and their magnitudes, acting on the box (as sketched in Figure 11.3.26.(b)):



$\vec{g} = \langle 0, -50\rangle \quad \vec{r} = \langle 20, 5\rangle$

Figure 11.3.26. Sketching the ramp and box in Example 11.3.25. Note: The vectors are not drawn to scale.

1. in the direction of the ramp, and
2. orthogonal to the ramp.

1. $\text{proj}_{\vec{r}}\,\vec{g} = \dfrac{\vec{g}\cdot\vec{r}}{\vec{r}\cdot\vec{r}}\,\vec{r}$

$= \dfrac{-250}{425}\langle 20, 5\rangle$

$= \langle -11.76, -2.94\rangle$

mag $= 12.13$ lb

2. $\langle 0, -50\rangle - \langle -11.76, -2.94\rangle$

$= \langle 11.76, -47.06\rangle$

mag $= 48.81$ lb

Vector application to work -

The application of a force "F" to move an object in a straight line a distance "d" produces work. The amt. of work "W" is:

$W = Fd$ (where F is in the direction of travel)

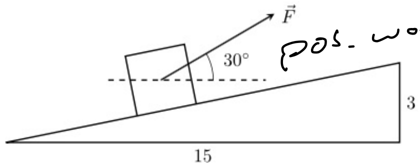
Let $\vec{F}$ be a constant force that moves an object in a straight line from point "P" to point "Q". Let $\vec{d} = \vec{PQ}$. The work "W" done by $\vec{F}$ along $\vec{d}$ is:

$$W = \vec{F}\cdot\vec{d}$$

If the angle btwn $\vec{F}$ and $\vec{d}$ is acute the expression will be positive. When the angle is obtuse, the force is causing motion in the opposite direction of $\vec{d}$, resulting in negative work.

the expression too

Example 11.3.29. Computing work. A man slides a box along a ramp that rises 3ft over a distance of 15ft by applying 50lb of force as shown in Figure 11.3.30. Compute the work done.



pos. work

$$W = \vec{F} \cdot \vec{d}$$

$$\vec{F} = 50\langle \cos 30, \sin 30 \rangle = \langle 43.3, 25 \rangle$$

$$\vec{d} = \langle 15, 3 \rangle$$

$$W = \langle 43.3, 25 \rangle \cdot \langle 15, 3 \rangle$$

$$= 43.3(15) + 25(3) = 724.5$$

Chapter 11.4 Notes

Cross product - Creates a vector $\vec{w}$ that is perpendicular to both non-parallel, nonzero vectors $\vec{u}$ and $\vec{v}$ in space.

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in $\mathbb{R}^3$. The cross product of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \times \vec{v}$, is the vector:

$$\vec{u} \times \vec{v} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$$

U components in blue, V components in red

Convenient way to do cross product

Let $\vec{u} = \langle 2, -1, 4 \rangle$ and $\vec{v} = \langle 3, 2, 5 \rangle$...

Start with a 3x3 matrix, with the first row being the unit vectors $\vec{i}, \vec{j}, \vec{k}$

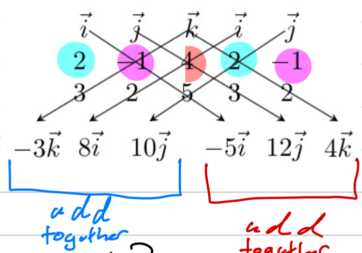
$$\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 3 & 2 & 4 \end{matrix}$$

$\vec{u}$ is highlighted throughout

Repeat the first 2 columns after the original 3

$$\begin{matrix} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 2 & -1 & 4 & 2 & -1 \\ 3 & 2 & 4 & 3 & 2 \end{matrix}$$

Now draw diagonals from each unit vector, and compute the products. Subtract the products on the left from the products on the right.



$$\vec{u} \times \vec{v} = (-5\vec{i} + 12\vec{j} + 4\vec{k}) - (-3\vec{k} + 8\vec{i} + 10\vec{j})$$

$$= -13\vec{i} + 2\vec{j} + 7\vec{k} = \langle -13, 2, 7 \rangle$$

Properties of the cross Product-

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ Anticommunicative
2. a) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

   b) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
3. $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
4. a) $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$ Orthogonality

   $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
5. $\vec{u} \times \vec{u} = \vec{0}$ or $\vec{u} \times \vec{0} = \vec{0}$
6. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

The cross product and angles

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^3$, then

$$||\vec{u} \times \vec{v}|| = ||\vec{u}||\,||\vec{v}|| \sin\theta$$

where θ, $0 \le \theta \le \pi$ is the angle btwn $\vec{u}$ and $\vec{v}$

At $\theta = 0, \pi$ the vectors are parallel, and the magnitude of their cross product is 0. The only vector w/ a magnitude of 0 is $\vec{0}$, so the cross product of parallel vectors is $\vec{0}$.

Example 11.4.10. The cross product and angles. Let $\vec{u} = \langle 1, 3, 6\rangle$ and $\vec{v} = \langle -1, 2, 1\rangle$ as in Example 11.4.5. Verify Theorem 11.4.8 by finding θ, the angle between $\vec{u}$ and $\vec{v}$, and the magnitude of $\vec{u} \times \vec{v}$.

$||\vec{u}|| = \sqrt{1+4+36} = \sqrt{46}$

$||\vec{v}|| = \sqrt{1+4+1} = \sqrt{6}$

$u_2v_3 - u_3v_2$

$-(u_1v_3 - u_3v_1)$

$u_1v_2 - u_2v_1$

$3(1) - 6(2) = 3 - 12 = -9$

$-(1(1) - 6(-1)) = -(1+6) = -7$

$1(2) - 3(-1) = 2 + 3 = 5$

$\vec{u} \times \vec{v} = \langle -9, -7, 5\rangle$

$||\vec{u} \times \vec{v}|| = \sqrt{81 + 49 + 25}$

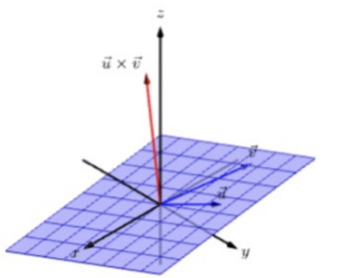
$= \sqrt{155}$

$\theta = \sin^{-1}\left(\frac{\sqrt{155}}{(\sqrt{46})(\sqrt{6})}\right)$

$\theta = \cos^{-1}\left(\frac{11}{\sqrt{46}\sqrt{6}}\right)$

Right Hand Rule for cross products

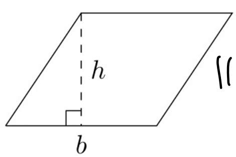
Given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ with the same initial point, point the index finger of your right hand in the direction of $\vec{u}$ and let your middle finger point in the direction of $\vec{v}$. The natural direction of the thumb is the direction of the cross product of $\vec{u}$ and $\vec{v}$.



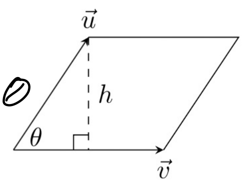
Applications of the cross product

Area of a Parallelogram:

$$A = \|\vec{u}\|\,\|\vec{v}\| \sin\theta = \|\vec{u} \times \vec{v}\|$$



$h = \|\vec{u}\| \sin\theta$



A 3-dimentional Parallelogram is called a Parallelepiped, where each face is parallel to the opposite face

The volume of a parallelepiped defined by $\vec{u}, \vec{v}, \vec{w}$

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| = |(\vec{u} \times \vec{v}) \cdot \vec{w}| \ldots$$

Torque - the measure of the turning force applied to an obj.
Usually represented by the greek letter τ, tau

When a force $\vec{F}$ is applied to a lever arm $\vec{\ell}$, the resulting torque is:

$$\vec{\tau} = \vec{\ell} \times \vec{F}$$

Example 11.4.15. Area of a triangle. Find the area of the triangle with vertices $A = (1,2)$, $B = (2,3)$ and $C = (3,1)$, as pictured in Figure 11.4.16.

$\vec{AB} = \langle 1, 1 \rangle$

$\vec{AC} = \langle 2, -1 \rangle$

Cross product is only $\mathbb{R}^3$, rewrite with $z=0$

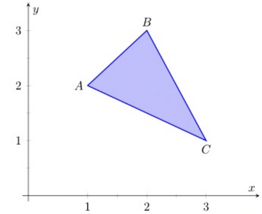


Figure 11.4.16. Finding the area of a triangle in Example 11.4.15

$A = \frac{1}{2}bh = \frac{1}{2}(\text{parallelogram}) = \frac{1}{2}\|\vec{u} \times \vec{v}\|$

$\vec{AB} = \langle 1, 1, 0 \rangle$ $\quad A = \frac{1}{2}\langle (0-0), -(0-0), (-1-2) \rangle$

$\vec{AC} = \langle 2, -1, 0 \rangle$ $\quad = \frac{1}{2}\langle 0, 0, -3 \rangle = \langle 0, 0, -\frac{3}{2} \rangle$

$A = \sqrt{(-\frac{3}{2})^2} = \frac{3}{2}$

Example 11.4.18. Finding the volume of parallelepiped. Find the volume of the parallelepiped defined by the vectors $\vec{u} = \langle 1,1,0 \rangle$, $\vec{v} = \langle -1,1,0 \rangle$ and $\vec{w} = \langle 0,1,1 \rangle$.

$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$

$\vec{v} \times \vec{w} = \langle 1, 1, -1 \rangle$, $\langle 1, 1, 0 \rangle \cdot \langle 1, 1, -1 \rangle$

$V = 1 + 1 = 2 \text{ units}^3$

Example 11.4.20. Computing torque. A lever of length 2ft makes an angle with the horizontal of $45°$. Find the resulting torque when a force of 10lb is applied to the end of the level where:

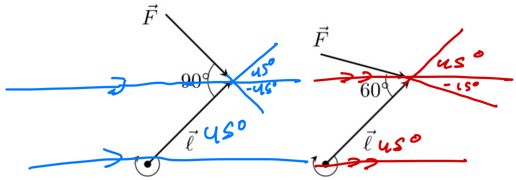


Figure 11.4.21. Showing a force being applied to a lever in Example 11.4.20

1. the force is perpendicular to the lever, and
2. the force makes an angle of $60°$ with the lever, as shown in

1. $\vec{\ell} = 2\langle \cos 45°, \sin 45°\rangle$
$= \langle \sqrt{2}, \sqrt{2} \rangle$
$\vec{F} = 10\langle \cos -45, \sin -45 \rangle$
$= \langle 5\sqrt{2}, -5\sqrt{2} \rangle$

$\langle \sqrt{2}, \sqrt{2}, 0 \rangle$

$\langle 5\sqrt{2}, -5\sqrt{2}, 0 \rangle$

$$\langle \sqrt{2}, \sqrt{2}, 0 \rangle \times \langle 5\sqrt{2}, -5\sqrt{2}, 0 \rangle = \langle 0, 0, -10-10 \rangle$$

$\vec{T} = \langle 0, 0, -20 \rangle$, magnitude $= 20$

2. $\vec{\ell} = \langle \sqrt{2}, \sqrt{2} \rangle$ $\quad \vec{F} = 10\langle \cos -15, \sin -15 \rangle$
$= \langle 9.659, -2.588 \rangle$

Usually, don't round until the end.

$$\langle \sqrt{2}, \sqrt{2}, 0 \rangle \times \langle 9.659, -2.588, 0 \rangle$$

$\vec{T} = \langle 0, 0, -17.321 \rangle$, magnitude $= 17.321$

Chapter 11.5 Notes

Lines - A point and a slope is needed to find the equation of a line. The slope conveys direction information.

To define a line one needs a point on the line and the direction of the line.

This holds true in space and on planes.

A line can be represented traditionally as:

$$\vec{l}(t) = \vec{p} + t\vec{d}$$

note: this is very similar to $y = mx + b$

The vector equation of the line through P in the direction of $\vec{d}$

Another way to represent a line -

Let $P = (x_0, y_0, z_0)$, $\vec{p} = \langle x_0, y_0, z_0 \rangle$ and let $\vec{d} = \langle a, b, c \rangle$. The equation of the line through P in the direction of $\vec{d}$ is:

$$\vec{l} = \vec{p} + t\vec{d} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

vector-valued equation $= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

The three components here can be solved for t and are the parametric equations of the line through $\vec{p}$ in the direction of $\vec{d}$.

$x = x_0 + at$ $\quad t = \dfrac{x - x_0}{a}$

$y = y_0 + bt$ $\quad t = \dfrac{y - y_0}{b}$

$z = z_0 + ct$ $\quad t = \dfrac{z - z_0}{c}$

Each highlighted form has its pros/cons based on context

Symmetric equations of the line through $\vec{p}$ in the direction of $\vec{d}$:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$\langle a, b, c\rangle = \langle \vec{d}_1, \vec{d}_2, \vec{d}_3 \rangle$
$\langle x_0, y_0, z_0\rangle = \langle \vec{P}_1, \vec{P}_2, \vec{P}_3 \rangle$] Variable Relations

$\vec{l} = \vec{P} + t\vec{d}$

Example 11.5.5. Finding the equation of a line. Give all three equations, as given in Definition 11.5.4, of the line through $P = (2, 3, 1)$ in the direction of $\vec{d} = \langle -1, 1, 2\rangle$. Does the point $Q = (-1, 6, 6)$ lie on this line?

$\vec{l} = \langle 2, 3, 1\rangle + t\langle -1, 1, 2\rangle$ NO!

$x = 2 - t \quad y = 3 + t \quad z = 1 + 2t$

$t = \frac{x-2}{-1}$ a $\quad t = \frac{y-3}{1}$ b $\quad t = \frac{z-1}{2}$ c

$t = \frac{-1-2}{-1} = 3 \quad t = \frac{6-3}{1} = 3 \quad t = \frac{6-1}{2} = 2.5$

Example 11.5.7. Finding the equation of a line through two points. Find the parametric equations of the line through the points $P = (2, -1, 2)$ and $Q = (1, 3, -1)$.

$\vec{PQ} = \langle -1, 4, -3\rangle \quad \vec{l} = \langle 2, -1, 2\rangle + t\langle \vec{PQ}\rangle$

$x = 2 - t \quad y = -1 + 4t \quad z = 2 - 3t$

$t = \frac{x-2}{-1} \quad t = \frac{y+1}{4} \quad t = \frac{z-2}{-3}$

In a plane, two distinct lines can either be parallel or will intersect at exactly one point.
In space, given equations of two lines, it can be difficult to tell if they are distinct or not. Given $\vec{l}_1 = \vec{P}_1 + t\vec{d}_1$ and $\vec{l}_2 = \vec{P}_2 + t\vec{d}_2$, we have four possibilities:

- The same line: They share all points
- Intersecting lines: They share only one points
- Parallel lines: $\vec{d}_1 \parallel \vec{d}_2$, no points in common
- Skew lines: $\vec{d}_1 \nparallel \vec{d}_2$, no points in common

Example 11.5.9. Comparing lines. Consider lines ℓ_1 and ℓ_2, given in parametric equation form:

$$\ell_1 : \begin{array}{rcl} x &=& 1+3t \\ y &=& 2-t \\ z &=& t \end{array} \qquad \ell_2 : \begin{array}{rcl} x &=& -2+4s \\ y &=& 3+s \\ z &=& 5+2s \end{array}.$$

Not ∥ or same line

Determine whether ℓ_1 and ℓ_2 are the same line, intersect, are parallel, or skew.

$\vec{d}_1 = \langle 3, -1, 1 \rangle \quad \vec{d}_2 = \langle 4, 1, 2 \rangle$

$1 + 3t = -2 + 4s$

$2 - t = 3 + s$

$t = 5 + 2s$

$2 - (5 + 2s) = 3 + s$

$-3 - 2s = 3 + s$

$-6 = 3s \quad s = -2$

$t = 5 + 2(-2) = 1$

$t = 1, \ S = -2$

↑ check to see if this satisfies all 3 eqs.

$1 + 3(1) = -2 + 4(-2)$

$4 \neq -10 \ \therefore$ Skew

Example 11.5.11. Comparing lines. Consider lines ℓ_1 and ℓ_2, given in parametric equation form:

$$\ell_1 : \begin{array}{rcl} x &=& -0.7+1.6t \\ y &=& 4.2+2.72t \\ z &=& 2.3-3.36t \end{array} \qquad \ell_2 : \begin{array}{rcl} x &=& 2.8-2.9s \\ y &=& 10.15-4.93s \\ z &=& -5.05+6.09s. \end{array}$$

If it is hard to tell if the $\vec{d}$ vectors are the same, convert to unit vectors!

Determine whether ℓ_1 and ℓ_2 are the same line, intersect, are parallel, or skew.

$\vec{d}_1 = \langle 1.6, 2.72, -3.36 \rangle \quad \vec{u}_1 = \dfrac{\vec{d}_1}{\|\vec{d}_1\|} = \langle 0.3471, .5901, -.7289 \rangle$

$\vec{d}_2 = \langle -2.9, -4.93, 6.09 \rangle \quad \vec{u}_2 = \dfrac{\vec{d}_2}{\|\vec{d}_2\|} = \langle -0.3471, -.5901, .7289 \rangle$

$\vec{u}_1 = -\vec{u}_2$, they are parallel

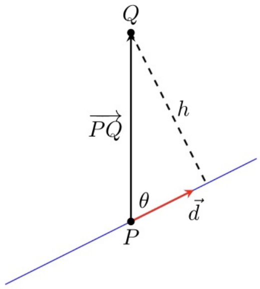
Need to check if they are the same line

$P_1 = (-.7, 4.2, 2.3) \quad \vec{\ell}_2 = \dfrac{x - 2.8}{2.9} = \dfrac{y - 10.15}{-4.93} = \dfrac{z + 5.05}{6.09}$

Plug into $\vec{\ell}_2$ eqs. to the point

$1.2069 = 1.2069 = 1.2069$ Same line

Shortest line is always the perpendicular line!



Distance – the length of the shortest line segment from the point to the line.

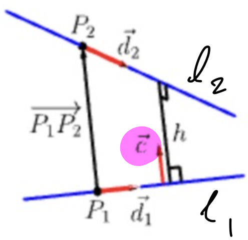
From This: $h = \|\vec{PQ}\| \sin\theta$

Using the cross Product:

$$\|\vec{PQ} \times \vec{d}\| = \|\vec{PQ}\| \, \|\vec{d}\| \sin\theta$$

Dividing both sides by $\|\vec{d}\|$ obtains the equation:

$$h = \frac{\|\vec{PQ} \times \vec{d}\|}{\|\vec{d}\|}$$



Determining the shortest segment between lines –

Let $\vec{l}_1 = \vec{P}_1 + t\vec{d}_1$

$\vec{l}_2 = \vec{P}_2 + t\vec{d}_2$

To find the direction orthogonal to both $\vec{d}_1$ and $\vec{d}_2$ take the cross product

$$\vec{c} = \vec{d}_1 \times \vec{d}_2$$

$\vec{P_1P_2}$ connects two points on the lines together

we need the magnitude of the orthogonal projection onto $\vec{c}$:

$$h = \|\operatorname{proj}_{\vec{c}} \vec{P_1P_2}\|$$

$$= \left\| \frac{\vec{P_1P_2} \cdot \vec{c}}{\vec{c} \cdot \vec{c}} \vec{c} \right\|$$

$$= \frac{|\vec{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|^2} \|\vec{c}\|$$

$$h = \frac{|\vec{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|}$$

Key Idea - Distance to Lines

- Let P be a point on the line "ℓ" that is parallel to $\vec{d}$. The distance h from a point Q to the line ℓ is:

$$h = \frac{\|\vec{PQ} \times \vec{d}\|}{\|\vec{d}\|}$$

- Let P_1 be a point on line ℓ_1 that is parallel to $\vec{d}_1$, and let P_2 be a point on ℓ_2 parallel to $\vec{d}_2$, and let $\vec{c} = \vec{d}_1 \times \vec{d}_2$, where lines ℓ_1 and ℓ_2 are not parallel. The distance h between the two lines is:

$$h = \frac{|\vec{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|}$$

where $\vec{c} = \vec{d}_1 \times \vec{d}_2$

Example 11.5.18. Finding the distance from a point to a line. Find the distance from the point $Q = (1,1,3)$ to the line $\vec{\ell}(t) = \langle 1,-1,1\rangle + t\,\langle 2,3,1\rangle$.

$\vec{PQ} = \langle 0, 2, 2\rangle$ $\quad \vec{d} = \langle 2, 3, 1\rangle$

$\vec{PQ} \times \vec{d} = \langle 2-6, -(0-4), 0-4\rangle$

$= \|\langle -4, 4, -4\rangle\| = \sqrt{3(16)} = \sqrt{48}$

$\|\vec{d}\| = \sqrt{14}$ $\quad h = \frac{\sqrt{48}}{\sqrt{14}} = 1.8516$

Example 11.5.19. Finding the distance between lines. Find the distance between the lines

$$\ell_1 : \begin{array}{rcl} x &=& 1+3t \\ y &=& 2-t \\ z &=& t \end{array} \qquad \ell_2 : \begin{array}{rcl} x &=& -2+4s \\ y &=& 3+s \\ z &=& 5+2s. \end{array}$$

$\vec{P_1P_2} = \langle -3, 1, 5\rangle$

$\vec{d}_1 = \langle 3, -1, 1\rangle$

$\vec{d}_2 = \langle 4, 1, 2\rangle$

$\vec{c} = \vec{d}_1 \times \vec{d}_2 = \langle -2-1, -(6-4), 3+4\rangle = \langle -3, -2, 7\rangle$

$\|\vec{c}\| = \sqrt{62}$ $\quad h = \frac{\langle -3,1,5\rangle \cdot \langle -3,-2,7\rangle}{\sqrt{62}} = \frac{9+(-2)+35}{\sqrt{62}} = \frac{42}{\sqrt{62}}$

Chapter 11.6 Notes

Plane - a plane in space can be defined given a point on the plane and the direction the plane "faces". The direction of the plane will be supplied by an orthogonal vector to the plane. This is often called a normal vector.

Let two points "P" and "Q" be in the plane. A vector "$\vec{n}$" is orthogonal to the plane if $\vec{n}$ is perpendicular to $\overrightarrow{PQ}$ for all "P" and all "Q".

$$\vec{n} \cdot \overrightarrow{PQ} = 0 \text{, for all } P \text{ and } Q$$

Equations of a plane in Standard and General Forms

Let $P = (x_0, y_0, z_0)$ be a point in the plane and let $\vec{n} = \langle a, b, c \rangle$ be a normal vector to the plane. A point $Q = (x, y, z)$ lies in this plane if, and only if $\overrightarrow{PQ}$ is orthogonal to $\vec{n}$. $\overrightarrow{PQ} = \langle x - x_0, y - y_0, z - z_0 \rangle$ ...

$$\overrightarrow{PQ} \cdot \vec{n} = 0$$

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$$

Standard Form

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = d$$ ← General Form

Example 11.6.4. Finding the equation of a plane. Write the equation of the plane that passes through the points $P = (1, 1, 0)$, $Q = (1, 2, -1)$ and $R = (0, 1, 2)$ in standard form.

$\overrightarrow{PQ}$ and $\overrightarrow{PR}$ in plane

$\overrightarrow{PQ} = \langle 0, 1, -1 \rangle$ $\overrightarrow{PR} = \langle -1, 0, 2 \rangle$

$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2, 1, 1 \rangle$ $2(x-1) + 1(y-1) + z = 0$

Example 11.6.6. Finding the equation of a plane. Verify that lines ℓ_1 and ℓ_2, whose parametric equations are given below, intersect, then give the equation of the plane that contains these two lines in general form.

The lines intersect at $P = (-1, 3, 0)$

$$\ell_1: \begin{array}{rcl} x &=& -5+2s \\ y &=& 1+s \\ z &=& -4+2s \end{array} \Bigg| \begin{array}{l} =-1 \\ =3 \\ =0 \end{array} \quad s=2 \qquad \ell_2: \begin{array}{rcl} x &=& 2+3t \\ y &=& 1-2t \\ z &=& 1+t \end{array} \Bigg| \begin{array}{l} =-1 \\ =3 \\ =0 \end{array} \quad t=-1$$

$-5+2s = 2+3t \qquad s = -2t$

$1+s = 1-2t \qquad -5+2(-2t) = 2+3t \qquad s=2$

$-4+2s = 1+t \qquad -5+-4t = 2+3t \qquad t=-1$

$\vec{d}_1 = \langle 2, 1, 2 \rangle \quad \vec{d}_2 = \langle 3, -2, 1 \rangle$

$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \langle 1-(-4), -(2-6), -4-3 \rangle$

$\vec{n} = \langle 5, 4, -7 \rangle$

$5(x+1) + 4(y-3) - 7z = 0 \qquad 5x + 4y - 7z = 7$

$5x + 5 + 4y - 12 - 7z = 0$

Example 11.6.8. Finding the equation of a plane. Give the equation, in standard form, of the plane that passes through the point $P = (-1, 0, 1)$ and is orthogonal to the line with vector equation $\vec{\ell}(t) = \langle -1, 0, 1 \rangle + t \langle 1, 2, 2 \rangle$.

$\vec{d} = \langle 1, 2, 2 \rangle = \vec{n}$

$1(x+1) + 2(y-0) + 2(z-1) = 0$

$(x+1) + 2y + 2(z-1)$

Example 11.6.10. Finding the intersection of two planes. Give the parametric equations of the line that is the intersection of the planes p_1 and p_2, where:

$$p_1: x - (y-2) + (z-1) = 0$$
$$p_2: -2(x-2) + (y+1) + (z-3) = 0$$

$p_1: x - y + 2 + z - 1 \qquad x - y + 1 = -z \qquad z = y - x - 1$

$p_2: -2x + 4 + y + 1 + z - 3 = 0 \qquad 2 - 2x + y = -z \qquad z = 2x - y - 2$

$y - x - 1 = 2x - y - 2 \qquad x=3, \; y=4, \; z=0 \qquad \vec{n}_1 = \langle 1, -1, 1 \rangle$

$2y = 3x - 1 \qquad P = (3, 4, 0) \qquad \vec{n}_2 = \langle -2, 1, 1 \rangle$

$y = \frac{1}{2}(3x-1) \qquad \vec{d} = \vec{n}_1 \times \vec{n}_2 = \langle -1-1, -(1+2), 1-2 \rangle$

$\vec{d} = \langle -2, -3, -1 \rangle$

$x = -2t + 3, \; y = -3t + 4, \; z = -t$

Infinite eqs.

Example 11.6.13. Finding the intersection of a plane and a line. Find the point of intersection, if any, of the line $\ell(t) = \langle 3, -3, -1 \rangle + t \langle -1, 2, 1 \rangle$ and the plane with equation in general form $2x + y + z = 4$.

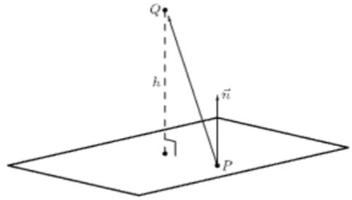
plane $\vec{n} = \langle 2, 1, 1 \rangle$
line $\vec{d} = \langle -1, 2, 1 \rangle$
$\vec{n} \neq \vec{d}$, lines are not $\parallel$ or $\perp$

$\ell(t)$, $x = 3 - t$
$y = -3 + 2t$
$z = -1 + t$

$2x + y + z = 4 \Rightarrow 2(3-t) + (-3+2t) + (-1+t) = 4$
$6 - 2t - 3 + 2t - 1 + t = 4$
$t = 2$

$(1, 1, 1)$

Distances - For example, when a plane with a normal vector is sketched containing a point P while a point Q is not on the plane, we get the distance from point Q to the plane by measuring the length of the projection of $\vec{PQ}$ on $\vec{n}$:



$$\|\text{proj}_{\vec{n}} \vec{PQ}\| = \left\| \frac{\vec{n} \cdot \vec{PQ}}{\|\vec{n}\|^2} \vec{n} \right\| = \frac{|\vec{n} \cdot \vec{PQ}|}{\|\vec{n}\|}$$

Distance from a point to a Plane -
Let a plane with normal vector $\vec{n}$ be given, and let Q be a point. The distance "h" from Q to the plane is:

$$h = \frac{|\vec{n} \cdot \vec{PQ}|}{\|\vec{n}\|}$$, where P is any point on the plane

Example 11.6.18. Distance between a point and a plane. Find the distance between the point $Q = (2, 1, 4)$ and the plane with equation $2x - 5y + 6z = 9$.

$\vec{n} = \langle 2, -5, 6 \rangle$ $\quad 2(0) - 5(0) + 6(z) = 9$ $\quad z = 1.5$ $\quad P = (0, 0, 1.5)$
$\vec{PQ} = \langle 2, 1, 2.5 \rangle$ $\quad h = \dfrac{|4 + (-5) + 15|}{\sqrt{65}} = \dfrac{14}{\sqrt{65}}$

Chapter 12

Chapter 12.1 Notes

A vector valued function is a function with the form:

$$\vec{r}(t) = \langle f(t), g(t) \rangle \text{ or } \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

where f, g, and h are real valued functions

The domain of $\vec{r}$ is the set of all values of t for which $\vec{r}(t)$ is defined. The range is the set of all possible output vectors

The graph of a vector valued function is the set of all terminal points of $\vec{r}$ where the initial point of each vector is always the origin

Operations on vector-valued functions -

Let $\vec{r}_1(t) = \langle f_1(t), g_1(t) \rangle$ and $\vec{r}_2(t) = \langle f_2(t), g_2(t) \rangle$ be vector valued functions in $\mathbb{R}^2$, with c be a scalar.

- $\vec{r}_1(t) \pm \vec{r}_2(t) = \langle f_1(t) \pm f_2(t), g_1(t) \pm g_2(t) \rangle$
- $c\vec{r}_1(t) = \langle cf_1(t), cg_1(t) \rangle$

Displacement - the difference of two locations $\vec{r}(t_1) - \vec{r}(t_0)$

Let $\vec{r}(t)$ be a vector valued function with $t_0 < t_1$ being values in the domain. The displacement $\vec{d}$ of $\vec{r}$, from $t = t_0$ to $t = t_1$ is

$$\vec{d} = \vec{r}(t_1) - \vec{r}(t_0)$$

Average Rate of Change - The average rate of change of $\vec{r}$ on $[t_0, t_1]$, is: $Avg = \dfrac{\vec{r}(t_1) - \vec{r}(t_0)}{t_1 - t_0}$

- A helix is a circle with a direction

Chapter 12.2 Notes

The Limit of $\vec{r}$, as it approaches c, is $\vec{L}$ expressed:

$$\lim_{t \to c} \vec{r}(t) = L$$

In $\mathbb{R}^2$, Let $\vec{r}(t) = \langle f(t), g(t) \rangle$, $\lim_{t \to c} \vec{r}(t) = \langle \lim_{t \to c} f(t), \lim_{t \to c} g(t) \rangle$

In $\mathbb{R}^3$ Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $\lim_{t \to c} \vec{r}(t) = \langle \lim_{t \to c} f(t), \lim_{t \to c} g(t), \lim_{t \to c} h(t) \rangle$

Continuity of vector-valued functions

- $\vec{r}(t)$ is cts. @ c if $\lim_{t \to c} \vec{r}(t) = r(c)$
- If $\vec{r}(t)$ is cts. @ all c in I, then $\vec{r}(t)$ is cts. on I

Let $\vec{r}(t)$ be a vector-valued function defined on an open interval I containing c. Then $\vec{r}(t)$ is cts. @ c if, and only if, each of its components cts. @ c.

Derivative of a vector valued function

- The derivative of $\vec{r}$ at $t = c$ is: $\vec{r}\,'(c) = \lim_{h \to 0} \frac{\vec{r}(c+h) - \vec{r}(c)}{h}$
- The derivative of $\vec{r}$ is: $\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$

If a vector valued function has a derivative for all c in an open interval I, it is differentiable on I.

The derivative of $\vec{r}(t)$, $\vec{r}\,'(t)$ is equal to the derivative of the components of $\vec{r}(t)$.

Tangent Vectors / Tangent Lines - I is open interval w/ c and $\vec{r}\,'(c) \neq \vec{0}$.

- A vector $\vec{v}$ is tangent to $\vec{r}(t)$ at $t = c$ if $\vec{v}$ is parallel to $\vec{r}\,'(c)$
- An equation of the tangent line is $\vec{l}(t) = \vec{r}(c) + t\vec{r}\,'(c)$

Example 12.2.19. Finding tangent lines to curves in space. Let $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ on $[-1.5, 1.5]$. Find the vector equation of the line tangent to the graph of $\vec{r}$ at $t = -1$.

$\vec{r}(-1) = \langle -1, 1, -1 \rangle$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle \big|_{t=-1} = \langle 1, -2, 3 \rangle$

$\vec{\ell}(t) = \langle -1, 1, -1 \rangle + t\langle 1, -2, 3 \rangle$

If $\vec{r}\,'(0) = \langle 0, 0 \rangle$ or $\langle 0, 0, 0 \rangle$ or $\vec{0}$, this means the tangent line has no direction, and we cannot find the equation of the tangent line

Smooth Vector Valued Functions

Let $\vec{r}(t)$ be a differentiable vector valued function on an open interval I where $\vec{r}\,'(t)$ is cts. on I. $\vec{r}(t)$ is smooth on I if $\vec{r}\,'(t) \neq \vec{0}$ on I.

Properties of vector valued functions

Let $\vec{r}(t)$ and $\vec{s}(t)$ be vector valued functions and let f be a real valued function, $\vec{r}(t)$, $\vec{s}(t)$ and $f(t)$ are diff.

1. $\frac{d}{dt}\left(\vec{r}(t) \pm \vec{s}(t)\right) = \vec{r}\,'(t) \pm \vec{s}\,'(t)$

2. $\frac{d}{dt}\left(c\vec{r}(t)\right) = c\vec{r}\,'(t)$

Product Rule

3. $\frac{d}{dt}\left(f(t)\vec{r}(t)\right) = f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$

4. $\frac{d}{dt}\left(\vec{r}(t) \cdot \vec{s}(t)\right) = \vec{r}\,'(t) \cdot \vec{s}(t) + \vec{r}(t) \cdot \vec{s}\,'(t)$

5. $\frac{d}{dt}\left(\vec{r}(t) \times \vec{s}(t)\right) = \vec{r}\,'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}\,'(t)$

6. $\frac{d}{dt}\left(\vec{r}(f(t))\right) = \vec{r}\,'(f(t))\, f'(t)$

Chain Rule

Vector valued functions of constant length

Let $\vec{r}(t)$ be a vector valued function of constant length that is differentiable on an open interval I. That is, $\|\vec{r}(t)\| = c$ for all t in I; $\vec{r}(t) \cdot \vec{r}(t) = c^2$ for all t in I. Then $\vec{r}(t) \cdot \vec{r}'(t) = 0$ for all t in I.

Constant length is a vector that traces out part of a circle.

Integration of vector valued functions

Let $\vec{r}(t)$ be cts. on $[a,b]$. An antiderivative of $\vec{r}(t)$ is a function $\vec{R}(t)$ where $\vec{R}'(t) = \vec{r}(t)$.

The set of all antiderivatives of $\vec{r}(t)$ is the indefinite integral of $\vec{r}(t)$ • $\int \vec{r}(t)\,dt$

The definite integral of $\vec{r}(t)$ on $[a,b]$: $\int_a^b \vec{r}(t)\,dt = \lim_{\|\Delta t\| \to 0} \sum_{i=1}^{n} \vec{r}(c_i)\Delta t_i$

Keep in mind: $\int_a^b \vec{r}'(t)\,dt = \vec{r}(b) - \vec{r}(a)$

Let $\vec{r}(t) = \langle f(t), g(t) \rangle$ in $\mathbb{R}^2$ that is cts on $[a,b]$

- $\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt \right\rangle$
- $\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt \right\rangle$

Holds true in $\mathbb{R}^3$ too

Arc length $= \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt$ in parametric where $x = f(t)$ and $y = g(t)$

Arc length of Vector Valued Functions

Let $\vec{r}(t)$ be a vector valued function where $\vec{r}'(t)$ is cts, on $[a,b]$. The arc length L of the graph of $\vec{r}(t)$ is

$$L = \int_a^b \|\vec{r}'(t)\|\,dt$$

Typically $a = 0$ and $b = t$ for problems

Chapter 12.3 Notes

Velocity, speed, and acceleration

Let $\vec{r}(t)$ be a position function in $\mathbb{R}^2$ or $\mathbb{R}^3$:

Velocity: The instantaneous rate of position change:

$$\vec{v}(t) = \vec{r}\,'(t)$$

Speed: The magnitude of velocity: $\|\vec{v}(t)\|$

acceleration: The instantaneous rate of velocity change:

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$

Objects with a constant speed

If an object moves with a constant speed, then its velocity and acceleration vectors are orthogonal: $\vec{v}(t) \cdot \vec{a}(t) = 0$

If acceleration is not perpendicular to velocity, then there is some acceleration in the direction of travel, influencing speed. If speed is constant, then $\vec{v}(t) \perp \vec{a}(t)$

Projectile motion: The motion of objects under only the influence of gravity

It is customary to write initial position $\vec{r}(0) = \langle x_0, y_0 \rangle$ and initial velocity $\vec{v}(0) = \langle v_x, v_y \rangle$. It is also customary to write velocity in terms of its speed and direction.

$\vec{v}(0) = v_0 \langle \cos\theta, \sin\theta \rangle$, where θ is a measure counter-clockwise of the x-axis

The position function of a projectile propelled from an initial position $\vec{r}_0 = \langle x_0, y_0 \rangle$, with the initial speed v_0 and angle of elevation θ, neglecting all acceleration but gravity is:

$$\vec{r}(t) = \left\langle (v_0 \cos\theta)t + x_0, -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + y_0 \right\rangle$$

Letting $\vec{v}_0 = v_0 \langle \cos\theta, \sin\theta \rangle$, $\vec{r}(t) = \left\langle 0, -\frac{1}{2}gt^2 \right\rangle + \vec{v}_0 t + \vec{r}_0$

Example 12.3.14. Projectile Motion. Sydney shoots her Red Ryder® bb gun across level ground from an elevation of 4 ft, where the barrel of the gun makes a 5° angle with the horizontal. Find how far the bb travels before landing, assuming the bb is fired at the advertised rate of 350 ft/s and ignoring air resistance.

$g = 9.8\ m/s$
$= 32\ ft/s$

$\vec{r}(t) = \langle 350\cos 5\, t, -16t^2 + 350\sin 5\, t + 4\rangle$
$= \langle 346.67t, -16t^2 + 30.50t + 4\rangle$

$-16t^2 + 30.50t + 4 = 0 \quad t = 2.03$ Quadratic formula

$x = 346.67(2.03) = 703.74\ ft$

Distance Travelled

Let $\vec{v}(t)$ be a velocity function for a moving object. The distance traveled for that object on $[a,b]$:

$$\text{Distance Traveled} = \int_a^b \|\vec{v}(t)\|\,dt$$

This is identical to the arc length formula

Recall: Average value of a function $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

Average speed and velocity

Average speed: $\dfrac{\text{distance traveled}}{\text{travel time}} = \dfrac{\int_a^b \|\vec{r}'(t)\|dt}{b-a}$

$$\text{Avg. Speed} = \frac{1}{b-a}\int_a^b \|\vec{v}(t)\|\,dt$$

Average Velocity: $\dfrac{\text{displacement}}{\text{travel time}} = \dfrac{\int_a^b \vec{r}'(t)\,dt}{b-a}$

$$\text{Avg. Velocity} = \frac{1}{b-a}\int_a^b \vec{r}'(t)\,dt$$

Chapter 12.4 Notes

Unit Tangent Vector

Let $\vec{r}(t)$ be a smooth function on an open interval I. The Unit vector $\vec{T}(t)$ is:

$$\vec{T}(t) = \frac{1}{\|\vec{r}'(t)\|}\vec{r}'(t)$$

Unit Normal Vector

Let $\vec{r}(t)$ be a vector-valued function where the unit tangent vector, $\vec{T}(t)$, is smooth on an open interval I. The unit normal vector $\vec{N}$ is:

$$\vec{N}(t) = \frac{1}{\|\vec{T}'(t)\|}\vec{T}'(t)$$

If $\vec{u} = \langle u_1, u_2\rangle$ is a unit vector in $\mathbb{R}^2$, then the only unit vectors orthogonal to $\vec{u}$ are $\langle -u_2, u_1\rangle$ and $\langle u_2, -u_1\rangle$. Given $\vec{T}(t)$, we can quickly determine $\vec{N}(t)$ if we know which term to multiply by -1.

Unit Normal Vectors in $\mathbb{R}^2$

Let $\vec{r}(t)$ be a vector valued function in $\mathbb{R}^2$ where $\vec{T}'(t)$ is smooth on an open interval I. Let t_0 be in I and $\vec{T}(t_0) = \langle t_1, t_2\rangle$, then $\vec{N}(t_0)$ is either:

$$\vec{N}(t_0) = \langle -t_2, t_1\rangle \quad \text{or} \quad \vec{N}(t_0) = \langle t_2, -t_1\rangle$$

whichever is the vector that points to the concave side of the graph of $\vec{r}$.

Applications to Acceleration

Let $\vec{r}(t)$ be a position function. Acceleration, $\vec{a}(t)$, lies in the plane defined by $\vec{T}$ and $\vec{N}$. That is, there are scalar functions $a_T(t)$ and $a_N(t)$ s.t.

$$\vec{a}(t) = a_T(t)\vec{T}(t) + a_N(t)\vec{N}(t)$$

$$\text{proj}_{\vec{T}(t)} \vec{a}(t) = \frac{\vec{a}(t)\cdot\vec{T}(t)}{\vec{T}(t)\cdot\vec{T}(t)}\vec{T}(t) = \underbrace{(\vec{a}(t)\cdot\vec{T}(t))}_{a_T}\vec{T}(t)$$

Because $\vec{T}(t)$ is a unit vector, $\vec{T}(t)\cdot\vec{T}(t) = 1$

Acceleration in the Plane Defined by $\vec{T}$ and $\vec{N}$

Let $\vec{r}(t)$ be a position function with acceleration $\vec{a}(t)$ and unit tangent and normal vectors $\vec{T}(t)$ and $\vec{N}(t)$. Then $\vec{a}(t)$ lies in the plane defined by $\vec{T}(t)$ and $\vec{N}(t)$; that is, there exist scalars a_T and a_N s.t.

$$\vec{a}(t) = a_T\vec{T}(t) + a_N\vec{N}(t)$$

$$a_T = \vec{a}(t)\cdot\vec{T}(t) = \frac{d}{dt}\left(\|\vec{v}(t)\|\right)$$

$$a_N = \vec{a}(t)\cdot\vec{N}(t) = \sqrt{\|\vec{a}(t)\|^2 - a_T^2}$$

Measures Rate of change of speed, the amount of acceleration in the direction of travel

$$= \frac{\|\vec{a}(t)\times\vec{v}(t)\|}{\|\vec{v}(t)\|} = \|\vec{v}(t)\|\|\vec{T}'(t)\|$$

$$a_T = \frac{d}{dt}\left(\|\vec{v}(t)\|\right), \quad a_N = \|\vec{v}(t)\|\,\|\vec{T}'(t)\|$$

Chapter 12.5 Notes

How to find the arc length

$$\text{arc Length} = \int_0^t \|\vec{r}\,'(u)\|\,du$$

Arc Length Parameter

Let $\vec{r}(s)$ be a vector-valued function. The parameter s is the arc length if, and only if, $\|\vec{r}\,'(s)\| = 1$.

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt} \qquad \vec{r}\,'(s) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \vec{T}(t)$$

$$\vec{r}\,'(t) = \vec{r}\,'(s) \cdot \|\vec{r}\,'(t)\|$$

Curvature

Let $\vec{r}(s)$ be a vector-valued function where s is the arc length parameter. The curvature K of the graph of $\vec{r}$ is:

$$K = \left\| \frac{d\vec{T}}{ds} \right\| = \|\vec{T}\,'(s)\|$$

If $\vec{r}(s)$ is parametrized by the arc length...

$$\vec{T}(s) = \frac{\vec{r}\,'(s)}{\|\vec{r}\,'(s)\|} \text{ and } \vec{N}(s) = \frac{\vec{T}\,'(s)}{\|\vec{T}\,'(s)\|} \Rightarrow \vec{T}\,'(s) = K\vec{N}(s)$$

Formulas for Curvature

Let C be a smooth curve in the plane or space

1. If C is defined by $y = f(x)$, then

$$K = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

2. If C is defined as a vector-valued function in the plane, $\vec{r}(t) = \langle x(t), y(t) \rangle$, then

$$K = \frac{|x'y'' - x''y'|}{((x')^2 + (y')^2)^{3/2}}$$

3. If C is defined in space by a vector-valued function $\vec{r}(t)$, then

$$K = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{\vec{a}(t) \cdot \vec{N}(t)}{\|\vec{v}(t)\|^2}$$

Curvature and Motion

$$a_T = \frac{d}{dt}(\|\vec{v}(t)\|) = \frac{d}{dt}(s'(t)) = s''(t)$$

$$a_N = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|} \text{ and } K = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|^3}$$

so $a_N = K\|\vec{v}(t)\|^2 = K(s'(t))^2$

Using this we can redefine the old $\vec{a}(t)$:

$$\vec{a}(t) = s''(t)\vec{T}(t) + K\|\vec{v}(t)\|^2 \vec{N}(t)$$

$$\text{Radius} = \frac{1}{K}$$

Chapter 13

Chapter 13.1 Notes

Function of Two Variables

Let D be a subset of $\mathbb{R}^2$. A function f of two variables is a rule that assigns each pair (x, y) in D a value $z = f(x, y)$ in $\mathbb{R}$. D is the domain of f, R is the range.

Contour lines, or level curves, show how much or how quickly the elevation is changing

Functions of Three Variables are the same as those with Two variables, but in $\mathbb{R}^3$ and $f(x, y, z)$

A function of three variables is a hypersurface drawn in 4 dimensions

Chapter 13.2 Notes

Open Disk, Boundary and Interior Points, Open and Closed Sets, Bounded Sets

- An **open disk** B in $\mathbb{R}^2$ centered at (x_0, y_0) with radius r is the set of all points (x, y) such that $\sqrt{(x-x_0)^2 + (y-y_0)^2} < r$
- Let S be a set of points in $\mathbb{R}^2$. A point P is a **boundary point** of S if all open disks centered at P contain both points in S and points not in S
- A point P in S is an **interior point** of S if there is an open disk centered at P that contains only points in S
- A set S is **open** if every point in S is an interior point
- A set S is **closed** if it contains all of its boundary points
- A set S is **bounded** if there is an $M > 0$ such that the open disk, centered at the origin with radius M, contains S. A set that is not bounded is **unbounded**

Limits of Functions of Two Variables

Definition 13.2.7. Limit of a Function of Two Variables. Let S be a set containing $P = (x_0, y_0)$ where every open disk centered at P contains points in S other than P, let f be a function of two variables defined on S, except possibly at P, and let L be a real number. The ***limit*** of $f(x, y)$ as (x, y) approaches (x_0, y_0) is L, denoted

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all (x, y) in S, where $(x, y) \neq (x_0, y_0)$, if (x, y) is in the open disk centered at (x_0, y_0) with radius δ, then $|f(x, y) - L| < \varepsilon$.

ε δ (epsilon-delta) definition of a limit

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

Basic Limit Properties

Let $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = K$

- Constants $\lim_{(x,y)\to(x_0,y_0)} b = b$
- Identity $\lim_{(x,y)\to(x_0,y_0)} x = x_0$ $\quad \lim_{(x,y)\to(x_0,y_0)} y = y_0$
- Sums/Differences $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \pm g(x,y)) = L \pm K$
- Scalars $\lim_{(x,y)\to(x_0,y_0)} b \cdot f(x,y) = bL$
- Products $\lim_{(x,y)\to(x_0,y_0)} f(x,y) \cdot g(x,y) = LK$
- Quotients $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = L/K$, $K \neq 0$
- Powers $\lim_{(x,y)\to(x_0,y_0)} f(x,y)^n = L^n$

Where don't limits exist?

If it is possible to arrive at different limiting values by approaching (x_0, y_0) along different paths, then the limit does not exist.

Continuity

Let a function $f(x,y)$ be defined on a set S containing the point (x_0, y_0)

- f is cts. at (x_0, y_0) if $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$
- f is cts. on S if f is cts. at all points in S. If f is cts. at all points in $\mathbb{R}^2$, we say f is cts. everywhere

Properties of cts. Functions

Let f and g be cts. on a set S, let c be a real number, and n be a positive integer. If these conditions are met, the following are cts. on S.

- Sums/Differences: $f \pm g$
- Constant: $c \cdot f$
- Products: $f \cdot g$
- Quotients: f/g if $g \neq 0$ on S
- Powers: f^n
- Roots: $\sqrt[n]{f}$ n=even $\Rightarrow f \geq 0$ on S, n=odd $\Rightarrow$ f cts for all

- Compositions: Adjust f and g's defs.. Let f be cts on S, where the range of f on S is J. Let G be a single variable function cts. on J. Then, $g \circ f = g(f(x,y))$ is cts. on S

Functions of 3 or more Variables: Open Balls, Limits, Continuity

- An open ball in $\mathbb{R}^3$ centered at (x_0, y_0, z_0) with radius r is the set of all points (x,y,z) such that

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = r$$

- Let D be a set in $\mathbb{R}^3$ containing (x_0, y_0, z_0) where every open ball centered at (x_0, y_0, z_0) contains points of D other than (x_0, y_0, z_0) and let $f(x,y,z)$ be a function of 3 variables defined on D, except possibly at (x_0, y_0, z_0). The limit of $f(x,y,z)$ as (x,y,z) approaches (x_0,y_0,z_0) is L, denoted:

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L$$

- Let $f(x,y,z)$ be defined on a set D containing (x_0, y_0, z_0). f is cts. at (x_0, y_0, z) if

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0, y_0 z_0)$$

If f is cts. for all points on D, f is cts. on D

Chapter 13.3 Notes

Partial Derivatives are used to determine how z changes with respect to x or y.

- Just as $\frac{d}{dx}(5x^2) = 10x$, $\frac{\delta}{\delta x}(x^2 y) = 2xy$ (y is a coef.)
- Just as $\frac{d}{dx}(5^3) = 0$, $\frac{\delta}{\delta x}(y^3) = 0$ (y is a constant)

f_x is the way to write $\frac{\delta f}{\delta x}$, partial derivative of f with respect to x

↳ The same notation can be used for other variables

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Example 13.3.5. Finding partial derivatives. Find $f_x(x,y)$ and $f_y(x,y)$ in each of the following.

1. $f(x,y) = x^3y^2 + 5y^2 - x + 7$
2. $f(x,y) = \cos(xy^2) + \sin(x)$

To take the Partial derivative of a function with respect to some variable x, treat all others as constants or coefficients

1. $f_x(x,y) = 3x^2y^2 - 1$
   $f_y(x,y) = 2yx^3 + 10y$
2. $f_x(x,y) = -y^2\sin(xy^2) + \cos x$
   $f_y(x,y) = -2xy\sin(xy^2)$

Tangent Planes

Let $f(x,y)$ be a function whose first order partial derivatives exist at (a,b). The tangent plane to the surface $z = f(x,y)$ at $(a, b, f(a,b))$ is:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Example 13.3.11. Finding a tangent plane equation. Find the equation of tangent plane to the surface $z = x^2 + 3y^2$ at $(x,y) = (1,-1)$.

$f(x,y) = x^2 + 3y^2$, $(x,y) = (1,-1)$ $\qquad f(1,-1) = 1 + 3 = 4$

$f_x(x,y) = 2x \Big|_{\substack{x=1 \\ y=-1}} = f_x(1,-1) = 2$

$f_y(x,y) = 6y \Big|_{\substack{x=1 \\ y=-1}} = f_y(1,-1) = -6$

$$z = 4 + 2(x-1) - 6(y+1)$$

Second Order Partial Derivatives

Let $z = f(x,y)$ be cts on a set S

- The second partial derivative of f with respect to x, then x is:
  $$\frac{\delta}{\delta x}\left(\frac{\delta f}{\delta x}\right) = \frac{\delta^2 f}{\delta x^2} = (f_x)_x = f_{xx}$$
- The second partial derivative of f with respect to x, then y is:
  $$\frac{\delta}{\delta y}\left(\frac{\delta f}{\delta x}\right) = \frac{\delta^2 f}{\delta y \delta x} = (f_x)_y = f_{xy}$$

Similar for Definitions $\frac{\delta^2 f}{\delta y^2} = f_{yy}$ and $\frac{\delta^2 f}{\delta x \delta y} = f_{yx}$

Example 13.3.13. Second partial derivatives. For each of the following, find all six first and second partial derivatives. That is, find

$$f_x, f_y, f_{xx}, f_{yy}, f_{xy} \text{ and } f_{yx}.$$

$$f(x,y) = x^3y^2 + 2xy^3 + \cos(x)$$

$f_x = 3x^2y^2 + 2y^3 - \sin x$, $f_{xx} = 6xy^2 - \cos x$

$f_y = 2yx^3 + 6xy^2$, $f_{yy} = 2x^3 + 12xy$

$f_{xy} = 6x^2y + 6y^2$, $f_{yx} = 6yx^2 + 6y^2$

If f is cts, $f_{xy} = f_{yx}$, order does not matter

First order partial derivatives tell us how much f changes with respect to a variable, while second order tells us the concavity of f in a direction.

Chapter 13.4 Notes

Total Differential

Let $z = f(x, y)$ be cts. Let dx and dy represent changes in x and y respectively. Partial Derivatives f_x and f_y exist. The total differential of z is:

$$dz = f_x(x, y)dx + f_y(x, y)dy$$

Differentiability

At a given point (x_0, y_0), let E_x and E_y describe functions of dx and dy such that $E_x dx$ and $E_y dy$ describes the error associated with approximating Δz with dz

$$\Delta z = dz + E_x dx + E_y dy$$

- f is differentiable at (x_0, y_0) if as dx and dy go to 0, so do E_x and E_y.
- f is differentiable on S if f is differentiable at every point on S. If f is differentiable on $\mathbb{R}^2$, it is differentiable everywhere

$$\Delta z = f(x_0 + dx, y_0 + dy) - f(x_0, y_0)$$

Continuity and Differentiability

Let $z = f(x, y)$ be defined on a set S

- If f is differentiable at (x_0, y_0), f is cts. at (x_0, y_0)
- If f_x and f_y are both cts. on S, then f is cts. on S

Approximations

Example 13.4.9. Approximating with the total differential. Let $z = \sqrt{x}\sin(y)$. Approximate $f(4.1, 0.8)$.

Let $x_0 = 4$

Let $y_0 = \frac{\pi}{4}$

$dz \approx f(4.1, 0.8) - f(4, \frac{\pi}{4})$

$dx = 4.1 - 4 = 0.1$

$f_x = \frac{1}{2}x^{-1/2}(\sin y)$

$= \frac{\sin y}{2\sqrt{x}}$

$f_y = \sqrt{x}\cos y$

$f_y(4, \frac{\pi}{4}) = \sqrt{2}$

$dy = 0.8 - \frac{\pi}{4} = 0.015$

$f(x_0, y_0) = f(4, \frac{\pi}{4}) = \sqrt{2}$

$f_x(4, \frac{\pi}{4}) = \sqrt{2}/8$

$dz = f_x(4, \frac{\pi}{4})dx + f_y(4, \frac{\pi}{4})dy$

$= \frac{\sqrt{2}}{8}(0.1) + \sqrt{2}(0.015) \approx 0.039$

$\Delta z + f(x_0, f_y) = f(x_0 + dx, y_0 + dy) \qquad \Delta z \approx dz$

$f(4.1, 0.8) \approx 0.039 + \sqrt{2} = 1.4531$

New position = old position + amount of change, so

New position $\approx$ old position + approximate amount of change

Example 13.4.10. Approximating an unknown function. Given that $f(2,-3)=6$, $f_x(2,-3)=1.3$ and $f_y(2,-3)=-0.6$, approximate $f(2.1,-3.03)$.

$f(2.1,-3.03)$ $\quad d_x = 0.1 \quad d_y = -0.03 \quad f(2,-3)=6$

$f(2.1,-3.03) \approx (f_x d_x + f_y d_y) + 6$

$f_x d_x = 1.3(0.1) = 0.13 \qquad f_y d_y = (-0.6)(-.03) = .018$

$f(2.1,-3.03) \approx (0.13 + .018) + 6 = 6.148$

Tangent Plane Approximation

Recall: $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

If we assume (x,y) is close to (a,b) and set $d_x = x-a$ and $d_y = y-b$

$dz = f_x(a,b)d_x + f_y(a,b)d_y = f_x(a,b)(x-a) + f_y(a,b)(y-b)$

If $\ell(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ then

$\ell(x,y) = f(a,b) + dz$ or $dz = \ell(x,y) - f(a,b)$

Sensitivity Analysis

Example 13.4.11. Sensitivity analysis. A cylindrical steel storage tank is to be built that is 10ft tall and 4ft across in diameter. It is known that the steel will expand/contract with temperature changes; is the overall volume of the tank more sensitive to changes in the diameter or in the height of the tank?

Cylindrical Solid Volume

$V = \pi r^2 h$

Total Differential

$dV = (2\pi r h)dr + (\pi r^2)dh$

$$\frac{\partial V}{\partial r} = V_r(r,h) = 2\pi r h \Big|_{(2,10)} = 40\pi$$

$$\frac{\delta V}{\delta h} = V_h(r,h) = \pi r^2 \Big|_{(2,10)} = 4\pi$$

A small change in radius will be multiplied by 40π, while only 4π for small height changes.

Volume is more sensitive to radius

Multivariable Differentiability

Let $w = f(x,y,z)$, $dw = f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z)dz$

This function, dw, is the total differential

$\Delta w = dw + E_x dx + E_y dy + E_z dz$

Very similar to 2 variable functions

$$\Delta w = dw + E_x dx + E_y dy + E_z dz.$$

1. We say f is ***differentiable at*** (x_0,y_0,z_0) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $\|\langle dx,dy,dz\rangle\| < \delta$, then $\|\langle E_x,E_y,E_z\rangle\| < \varepsilon$.
2. We say f is ***differentiable on*** B if f is differentiable at every point in B. If f is differentiable on $\mathbb{R}^3$, we say that f is ***differentiable everywhere***.

Continuity and Differentiability

- If f is differentiable at (x_0,y_0,z_0), then f is cts at (x_0,y_0,z_0)
- If f_x, f_y, and f_z are cts on B, then f is differentiable on B

Chapter 13.5 Notes

Multivariable Chain Rule

Let $z = f(x,y)$, $x = g(t)$, and $y = h(t)$, where f, g, and h are differentiable functions. Then $z = f(x,y) = f(g(t), h(t))$

$$\frac{dz}{dt} = \frac{df}{dt} = f_x(x,y)\frac{dx}{dt} + f_y(x,y)\frac{dy}{dt}$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \langle f_x, f_y\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle$$

$= \nabla f \cdot \langle x', y'\rangle$ where x' and y' are with respect to t

Example 13.5.5. Using the Multivariable Chain Rule. Let $z = x^2y + x$, where $x = \sin(t)$ and $y = e^{5t}$. Find $\frac{dz}{dt}$ using the Chain Rule.

$z = x^2y + x$

$x = \sin t$, $y = e^{5t}$ ← Substitute this one

$\nabla f = \langle 2xy + 1, x^2\rangle$

$x' = \cos t$ $\quad y' = 5e^{5t}$

$$\frac{dz}{dt} = \nabla f \cdot \langle x', y'\rangle = \cos t(2xy+1) + 5e^{5t}x^2$$

Multivariable Chain Rule

To find partial of f with respect to s or t

$z = f(x,y)$, $v(s,t) = \langle x, y\rangle$ $\qquad \nabla f(v(s,t)) = \nabla f(x,y) \cdot J_v$

$$J_v = \left[\frac{\partial v}{\partial s} \;\Big|\; \frac{\partial v}{\partial t}\right] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

This can be applied to higher dims.

Example 13.5.10. Using the Multivariable Chain Rule, Part II. Let $z = x^2y + x$, $x = s^2 + 3t$ and $y = 2s - t$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, and evaluate each when $s = 1$ and $t = 2$.

$\nabla f = \langle 2xy + 1, x^2\rangle$

$z = x^2y + x \qquad v(s,t) = \langle s^2 + 3t, 2s - t\rangle$

$$J_v = \left[\frac{\partial v}{\partial s} \;\Big|\; \frac{\partial v}{\partial t}\right] = \begin{bmatrix} 2s & 3 \\ 2 & -1 \end{bmatrix}$$

$\nabla(f \circ v) = \langle 2xy+1, x^2\rangle \cdot J_v = \langle 2s(2xy+1) + 2x^2, 3(2xy+1) - x^2\rangle$

$\frac{\partial z}{\partial s} = 2s(2xy+1) + 2x^2 \qquad s = 1, t = 2 \qquad \frac{\partial z}{\partial s}(1,2) = 100$

$v(1,2) = \langle 7, 0\rangle$

$\frac{\partial z}{\partial t} = 3(2xy+1) - x^2 \qquad x = 7, 0 \qquad \frac{\partial z}{\partial t}(1,2) = -46$

Implicit Differentiation

$$\frac{dy}{dx} = -\frac{f_x(x,y)}{f_y(x,y)}$$

3D+

$$\frac{\partial z}{\partial x} = -\frac{f_x(x,y,z)}{f_z(x,y,z)} \qquad \frac{\partial z}{\partial y} = -\frac{f_y(x,y,z)}{f_z(x,y,z)}$$

Chapter 13.6 Notes

Directional Derivatives

let $z = f(x,y)$ be differentiable and $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector. The directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

$\nabla f = \langle f_x, f_y \rangle$

$$\nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos\theta = \|\nabla f\| \cos\theta$$

- Maximized when $\cos\theta = 1$, gradient has same direction as $\vec{u}$
- Minimized when $\cos\theta = -1$, gradient has opposite direction as $\vec{u}$
- Equal to 0 when $\cos\theta = 0$, gradient and $\vec{u}$ are orthogonal

The Gradient and Directional Derivatives

- Max value of $D_{\vec{u}} f(x_0, y_0)$ is $\|\nabla f(x_0, y_0)\|$; the direction of maximal z increase is $\nabla f(x_0, y_0)$
- Min value of $D_{\vec{u}} f(x_0, y_0)$ is $-\|\nabla f(x_0, y_0)\|$; the direction of minimal z increase is $-\nabla f(x_0, y_0)$
- Let $P = (x_0, y_0)$. At P, $\nabla f(x_0, y_0)$ is orthogonal to the level curve passing through (x_0, y_0)

The Gradient and Directional Derivatives with Three variables

- Max val of $D_{\vec{u}} F$ is $\|\nabla F\|$, where the angle btwn. ∇F and $\vec{u}$ is 0. The direction of max increase is ∇F
- Min val of $D_{\vec{u}} F$ is $-\|\nabla F\|$, where the angle btwn. ∇F and $\vec{u}$ is π. The direction of min increase is $-\nabla F$
- $D_{\vec{u}} F = 0$ when ∇F and $\vec{u}$ are orthogonal. The gradient is orthogonal to the level surfaces

Quick Concepts

- $D_{\vec{u}} f = f_x$ when $\vec{u} = \langle 1, 0, 0 \rangle$, or i
- $D_{\vec{u}} f = f_y$ when $\vec{u} = \langle 0, 1, 0 \rangle$, or j
- $D_{\vec{u}} f = f_z$ when $\vec{u} = \langle 0, 0, 1 \rangle$ or k
- The gradient is orthogonal to level curves

Chapter 13.7 Notes

Directional Tangent Lines

Let $z=f(x,y)$ be differentiable and $\vec{u}=\langle u_1,u_2\rangle$ be a unit vector

- The line l_x through $(x_0,y_0,f(x_0,y_0))$ parallel to $\langle 1,0,f_x(x_0,y_0)\rangle$ is the tangent line to f in the direction of x at (x_0,y_0)
- The line l_y through $(x_0,y_0,f(x_0,y_0))$ parallel to $\langle 0,1,f_y(x_0,y_0)\rangle$ is the tangent line to f in the direction of y at (x_0,y_0)
- The line $l_{\vec{u}}$ through $(x_0,y_0,f(x_0,y_0))$ parallel to $\langle u_1,u_2,D_{\vec{u}}f(x_0,y_0)\rangle$ is the tangent line to f in the direction of $\vec{u}$ at (x_0,y_0)

$$l_x(t)=\begin{cases} x=x_0+t \\ y=y_0 \\ z=z_0+f_x(x_0,y_0)t \end{cases} \qquad l_y(t)=\begin{cases} x=x_0 \\ y=y_0+t \\ z=z_0+f_y(x_0,y_0)t \end{cases} \qquad l_{\vec{u}}(t)=\begin{cases} x=x_0+u_1t \\ y=y_0+u_2t \\ z=z_0+D_{\vec{u}}ft \end{cases}$$

Normal Line

Let $a=f_x(x_0,y_0)$ and $b=f_y(x_0,y_0)$

1. A nonzero vector parallel to $\vec{n}=\langle a,b,-1\rangle$ is orthogonal to f at $P=(x_0,y_0,f(x_0,y_0))$
2. The line l_n through P with direction parallel to $\vec{n}$ is the normal line to f at P

The set of parametric equations of the normal line to a surface $z=f(x,y)$ at $(x_0,y_0,f(x_0,y_0))$ is:

$$l_n(t)=\begin{cases} x=x_0+at \\ y=y_0+bt \\ z=f(x_0,y_0)-t \end{cases}$$

Tangent

$a=f_x(x_0,y_0)$, $b=f_y(x_0,y_0)$, $\vec{n}=\langle a,b,-1\rangle$, $P=(x_0,y_0,f(x_0,y_0))$

The plane through P with normal vector $\vec{n}$ is the tangent plane to f at P. The standard form of the plane is:

$$a(x-x_0)+b(y-y_0)-(z-f(x_0,y_0))=0$$

The Gradient and Level Surfaces

Let $w=F(x,y,z)$ be differentiable on a set D containing (x_0,y_0,z_0) with gradient ∇F, where $F(x_0,y_0,z_0)=c$.

The vector $\nabla F(x_0,y_0,z_0)$ is orthogonal to the level surface $F(x,y,z)=c$ at (x_0,y_0,z_0)

Example 13.7.19. Using the gradient to find a tangent plane. Find the equation of the plane tangent to the ellipsoid $\frac{x^2}{12}+\frac{y^2}{6}+\frac{z^2}{4}=1$ at $P=(1,2,1)$.

$\nabla F(x,y,z) = \langle F_x, F_y, F_z \rangle = \langle \frac{x}{6}, \frac{y}{3}, \frac{z}{2} \rangle$

$\nabla F(1,2,1) = \langle \frac{1}{6}, \frac{2}{3}, \frac{1}{2} \rangle$

The tangent plane at $(1,2,1)$ is $\frac{1}{6}(x-1)+\frac{2}{3}(y-2)+\frac{1}{2}(z-1)=0$

This can be simplified to be:

$f_x(a,b,c)(x-a)+f_y(a,b,c)(y-b)+f_z(a,b,c)(z-c)=0$

which is the equation of a plane with normal vector $\nabla F(a,b,c)$

Example 13.7.21. Finding the tangent plane of a level surface. Determine the equation of the tangent plane to the level surface $x^2yz^3-\sin(x-3z)+4xy^2-3yz=0$ at the point $(3,0,1)$. (Note that this is the same problem as Example 13.5.15.)

$f_x = 2xyz^3 - \cos(x-3z)+4y^2$ $\quad f_x(3,0,1) = -1$

$f_y = x^2z^3+8xy-3z$ $\quad f_y(3,0,1)=6$

$f_z = 3x^2yz^2+3\cos(x-3z)-3y$ $\quad f_z(3,0,1)=3$

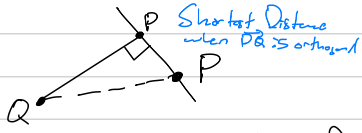
$-1(x-3)+6(y-0)+3(z-1)=0$

Distance Between Point and Surface

P on $f(x,y)=z$ $\quad g(x,y,z)=f(x,y)-z$

Q not on surface $\quad \nabla g = \langle f_x, f_y, -1\rangle$

$\nabla g \perp$ surface



$\overrightarrow{PQ} = k\nabla g = k\langle f_x, f_y, -1\rangle$ ← Solve this

Chapter 13.8 Exercises

Relative and absolute Extrema — Let $f(x,y)$ be defined at a point $P=(x_0,y_0)$

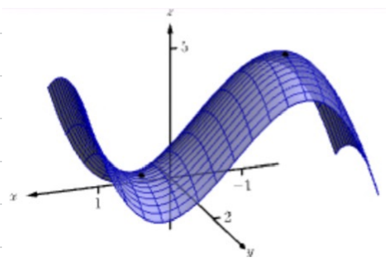
1. If $f(x_0,y_0) \geq f(x,y)$ for all (x,y), f has an absolute maximum at P.
   If $f(x_0,y_0) \leq f(x,y)$ for all (x,y), f has an absolute min at P
2. If a circle can be drawn at P where $f(x_0,y_0) \geq f(x,y)$ for all (x,y) in the circle, there is a relative max at P. If this exists for P but only if $f(x_0,y_0) \leq f(x,y)$, there is a relative min at P.
3. Absolute mins/maxs are designated absolute extremum. Relative mins/maxs are designated relative extremum.

Critical Points — $P=(x_0,y_0)$ is a critical point if

- $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$ or
- $f_x(x_0,y_0)$ and/or $f_y(x_0,y_0)$ is undefined

If f has a relative extreme at P, then P is a critical point of f

To find Critical Points, compute Partials and solve system of equations



There is clearly a relative max at $(-1,2)$ but neither at $(1,2)$.

- walking parallel to $(1,2)$ along the y-axis suggests it's a relative max.
- walking parallel to $(1,2)$ along the x-axis suggests it's a relative min

This is called a saddle point

Second Derivative Test — Let $P=(x_0,y_0)$ be a critical point in f which has all first and second partials defined

$$D = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - (f_{xy}(x_0,y_0))^2$$

1. If $D>0$ and $f_{xx}(x_0,y_0)>0$, f has a relative minimum at P
2. If $D>0$ and $f_{xx}(x_0,y_0)<0$, f has a relative maximum at P
3. If $D<0$, then f has a saddle point at P
4. If $D=0$, the test is inconclusive

Extreme Value Theorem — If $f=z(x,y)$ is cts on a closed, bounded set S, f has a max and min value on S. Check Endpoints!

Chapter 14

Chapter 14.1 Notes

Iterated Integration

the process of repeatedly integrating the results of previous integration. Let a, b, c, and d be numbers and let $g_1(x), g_2(x)$, $h_1(y)$, and $h_2(y)$ be functions of x and y, respectively.

1. $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left(\int_{h_1}^{h_2} f(x,y)\,dx \right) dy$

2. $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left(\int_{g_1}^{g_2} f(x,y)\,dy \right) dx$

Notice how the outer integral is evaluated with constants, not functions

Area of a Plane Region

Let a region be bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$

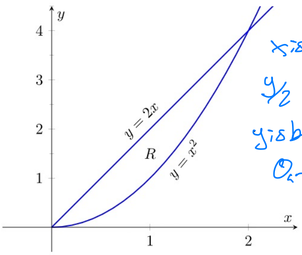
$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$$

$dy\,dx$ is often rewritten as dA

Let a region be bounded by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\,dy$$

Example 14.1.12. Area of a plane region. Find the area of the region enclosed by $y = 2x$ and $y = x^2$, as shown in Figure 14.1.13.



x is bounded by $y/2$ and $\sqrt{y}$

y is bounded by 0 and 4

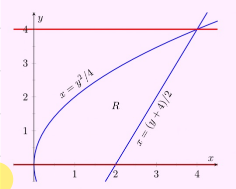
$$\int_0^2 \int_{x^2}^{2x} dy\,dx = \int_0^2 (2x - x^2)\,dx$$

$$= x^2 - \tfrac{1}{3}x^3 \Big|_0^2 = 4 - \tfrac{8}{3}$$

$$\int_0^4 \int_{y/2}^{\sqrt{y}} dx\,dy = \int_0^4 \left(\sqrt{y} - \tfrac{y}{2}\right) dy$$

$$= \tfrac{2}{3} y^{3/2} - \tfrac{1}{4} y^2 \Big|_0^4 = \tfrac{16}{3} - 4$$

Example 14.1.16. Changing the order of integration. Change the order of integration of $\displaystyle\int_0^4 \int_{y^2/4}^{(y+4)/2} 1\,dx\,dy$.



Note how function bounds change for area

$x = \dfrac{y+4}{2}$

$2x = y + 4$

$y = 2x - 4$

$x = y^2/4$

$4x = y^2$

$y = \pm 2\sqrt{x}$

$$\int_0^2 \int_0^{2\sqrt{x}} dy\,dx + \int_2^4 \int_{2x-4}^{2\sqrt{x}} dy\,dx$$

Graphing can be very useful

Chapter 14.2 Notes

Double Integrals and Signed Volume

Let $z = f(x,y)$ be a cts. function defined by a region R in the xy plane. The signed volume V under f over R is

$$V = \iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy\,dx$$

This can be written as a Riemann Sum: $\lim_{\|\Delta A\| \to 0} \sum_{i=1}^{n} f(x_i,y_i)\Delta A_i$

Fubini's Thm

If R is bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1}^{g_2} f(x,y)\,dy\,dx$$

If R is bounded by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$

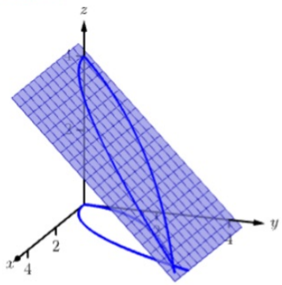
$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1}^{h_2} f(x,y)\,dx\,dy$$

Average Value of f on R

Let $z = f(x,y)$ be a cts. function bounded by R in xy plane. The average value of f on R is:

$$\frac{\iint_R f(x,y)\,dA}{\iint_R dA}$$

Example 14.2.21. Evaluating a double integral. Evaluate $\iint_R (4-y)\,dA$, where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$, graphed in Figure 14.2.22.



$y^2 = 4x \qquad x^2 = 4y$

$y = x^2/4$

$\left(\frac{x^2}{4}\right)^2 = 4x$

$\frac{x^4}{16} = 4x \qquad x^4 - 64x = 0$

$x(x^3 - 64) = 0$

$x = 0, 4$

$$V = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} (4-y)\,dx\,dy = \int_0^4 \left(2\sqrt{y} - \frac{y^2}{4}\right)(4-y)\,dy = 11.73$$

Chapter 14.3

Evaluating Double Integrals with Polar Coordinates

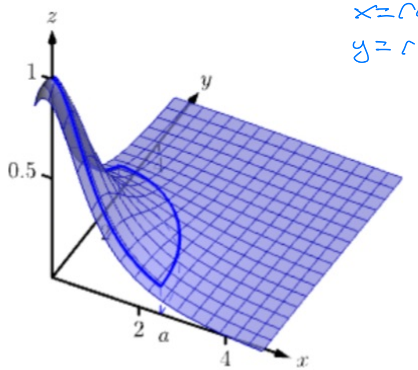
Let $z=f(x,y)$ be a cts. function defined over a closed, bounded region R in the xy-plane, where R is bounded by $\alpha \le \theta \le \beta$ and $g_1 \le r \le g_2$

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Example 14.3.9. Evaluating a double integral with polar coordinates.

Find the volume under the surface given by the graph of $f(x,y) = \dfrac{1}{x^2+y^2+1}$ over the sector of the circle with radius a centered at the origin in the first quadrant, as shown in Figure 14.3.10.

$x = r\cos\theta$
$y = r\sin\theta$



$$\frac{1}{x^2+y^2+1} \Rightarrow \frac{1}{r^2\cos^2\theta + r^2\cos^2\theta + 1} = \frac{1}{r^2+1}$$

$$\iint_R f(x,y)\,dA = \int_0^{\pi/2}\int_0^a \left(\frac{1}{r^2+1}\right) r\,dr\,d\theta = \int_0^{\pi/2}\int_0^a \frac{r}{r^2+1}\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}(\ln|r^2+1|)\Big|_0^a\,d\theta = \frac{1}{2}\int_0^{\pi/2} \ln(a^2+1)\,d\theta$$

$$= \frac{1}{2}\ln(a^2+1)\,\theta\Big|_0^{\pi/2} = \frac{\pi}{4}\ln(a^2+1)$$

Chapter 14.4 Notes

Mass of a Lamina with Variable Density

Let $\delta(x,y)$ be a cts. density function of a lamina corresponding to a closed, bounded plane region R.

$$\text{mass } M = \iint_R dm = \iint_R \delta(x,y)\, dA$$

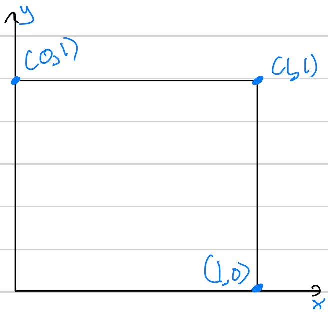
Center of Mass of a planar Lamina

Let a planar lamina be represented by a closed region R

1. mass: $M = \iint_R \delta(x,y)\, dA$
2. moment about the x-axis $M_y = \iint_R y\, \delta(x,y)\, dA$
3. moment about the y-axis $M_x = \iint_R x\, \delta(x,y)\, dA$
4. The center of mass of the lamina is

$$(\bar{x}, \bar{y}) = \left(\frac{M_x}{M}, \frac{M_y}{M}\right)$$

Example 14.4.19. Finding the center of mass of a lamina. Find the center of mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see Figure 14.4.18), with variable density $\delta(x,y) = (x+y+2)$ g/cm^2. (Note: this is the lamina from Example 14.4.5.)



$$\text{mass} = \int_0^1\int_0^1 (x+y+2)\, dx\, dy = 3g$$

$$M_x = \int_0^1\int_0^1 y\,(x+y+2)\, dx\, dy = \frac{19}{12} g$$

$$M_y = \int_0^1\int_0^1 x\,(x+y+2)\, dx\, dy = \frac{19}{12} g$$

$$(\bar{x}, \bar{y}) = \left(\frac{19/12}{3}, \frac{19/12}{3}\right) = \left(\frac{19}{36}, \frac{19}{36}\right)$$

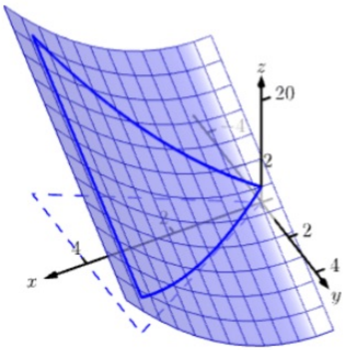
Chapter 14.5 Notes

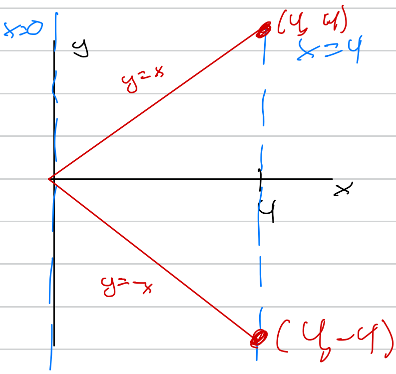
Surface Area

let $z = f(x,y)$ where ∇f is cts over a region R.

$$S = \iint_R dS = \iint_R \sqrt{1+f_x^2+f_y^2}\, dA$$

Example 14.5.8. Finding surface area over a region. Find the area of the surface $f(x,y) = x^2 - 3y + 3$ over the region R bounded by $-x \le y \le x$, $0 \le x \le 4$, as pictured in Figure 14.5.9.





$$f(x,y) = x^2 - 3y + 3$$

$$\nabla f = \langle 2x, -3 \rangle$$

$$S = \int_0^4 \int_{-x}^{x} \sqrt{1+4x^2+9}\, dy\, dx$$

$$= \int_0^4 y\sqrt{4x^2+10}\, \Big|_{-x}^{x} dx$$

$$= \int_0^4 \left(x\sqrt{4x^2+10} - -x\sqrt{4x^2+10}\right) dx$$

$$= \int_0^4 2x\sqrt{4x^2+10}\, dx$$

$u = 4x^2+10$

$\frac{du}{dx} = 8x \quad dx = \frac{du}{8x}$

$$= \int_{10}^{74} 2x\sqrt{u}\, du\left(\frac{1}{8x}\right) = \frac{1}{4}\int_{10}^{74} \sqrt{u}\, du$$

$$= \frac{1}{4}\left(\frac{2}{3}u^{3/2}\right)\Big|_{10}^{74} = \frac{1}{6}\left(74^{3/2} - 10^{3/2}\right)$$

$$= 100.825 \text{ units}^2$$

Chapter 14.6 Notes

Triple Integrals and Iterated Integration

Volume 3D

$$\text{Volume} = \iiint_D dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\,dy\,dx$$

Mass, Center of Mass of Solids

$$\text{mass} = \iiint_D dm = \iiint_D \delta(x,y,z)\,dV$$

$$\text{moment about } yz \text{ plane} = M_{yz} = M_x = \iiint_D x\,\delta(x,y,z)\,dV$$

$$\text{moment about } xz \text{ plane} = M_{xz} = M_y = \iiint_D y\,\delta(x,y,z)\,dV$$

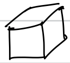
$$\text{moment about } xy \text{ plane} = M_{xy} = M_z = \iiint_D z\,\delta(x,y,z)\,dV$$

$$\text{Center of mass: } (\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_x}{m}, \frac{M_y}{m}, \frac{M_z}{m}\right)$$

Textbook Conversion: $(\rho, \varphi, \theta) \mapsto (\rho, \theta, \frac{\pi}{2} - \varphi)$
(roland) (textbook)

Chapter 14.7 Notes/November 3 Lecture

Cartesian



(x, y, z)

we don't really need these

Cartesian → Cylindrical

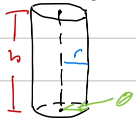
$(x, y, z) \mapsto (\sqrt{x^2+y^2}, \arctan(\frac{y}{x}), z)$

Cartesian → Spherical

$(x, y, z) \mapsto (\sqrt{x^2+y^2+z^2}, \arctan(\frac{\sqrt{x^2+y^2}}{z}), \arctan(\frac{y}{x}))$

Cylindrical

(r, θ, h)



$0 \le r < \infty$
$0 \le \theta \le 2\pi$
$-\infty < h < \infty$

Polar with a z coordinate

Cylindrical → Cartesian

$(r, \theta, h) \mapsto (r\cos\theta, r\sin\theta, h)$

Cylindrical → Spherical

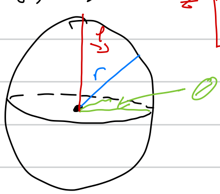
$(r, \theta, h) \mapsto (r\cos\theta, r\sin\theta, h)$
$\mapsto (\sqrt{r^2+h^2}, \arctan\frac{r}{h}, \theta)$

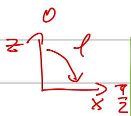
Using known formulae to convert

but useful either in this class

Spherical

(ρ, φ, θ)





$0 \le \rho < \infty$
$0 \le \varphi \le \pi$
$0 \le \theta \le 2\pi$

φ starts at north pole ($\varphi = 0$) and goes to south pole ($\varphi = \pi$)

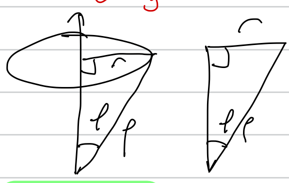
Spherical → Cartesian

$(\rho, \varphi, h) \mapsto (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$

Spherical → Cylindrical

$(\rho, \varphi, h) \mapsto (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$
$\mapsto (\rho\sin\varphi, \theta, \rho\cos\varphi)$

Using known formulae to convert



$r = \sin\varphi$

Conversion Map

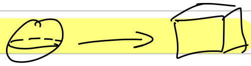
Cylindrical		Cartesian		Spherical
$(\sqrt{x^2+y^2}, \arctan\frac{y}{x}, z)$	←	(x, y, z)	→	$(\sqrt{x^2+y^2+z^2}, \arctan\frac{\sqrt{x^2+y^2}}{z}, \arctan\frac{y}{x})$
(r, θ, h)	→	$(r\cos\theta, r\sin\theta, h)$	→	$(\sqrt{r^2+h^2}, \arctan\frac{r}{h}, \theta)$
$(\rho\sin\varphi, \theta, \rho\cos\varphi)$	←	$(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$	←	(ρ, φ, θ)

This column is most common

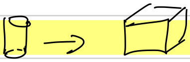
Jacobians and Coordinate Systems

det J accounts for space dialation when changing coordinate systems

$$\iiint_D F\,dV = \iiint f\cdot|\det J_v|\,dV$$

F:  $\det J_F = \rho^2\sin(\varphi)$

$$\iiint_D F\,dV = \iiint f\cdot\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

F:  $\det J_F = r$ Same for Polar

$$\iiint_D F\,dV = \iiint F\cdot r\,dr\,d\theta\,dh$$

Textbook → Roland has J_F of I ∴ No difference

Example Problem

$0 \le z \le \sqrt{4-x^2-y^2}+3$ 

$x^2+y^2 \le 4$

$\mapsto$ Plug in Conversions

$0 \le h \le \sqrt{4-r^2\cos^2\theta - r^2\sin^2\theta}+3$

$= \sqrt{4-r^2}+3$

$r^2 \le 4 \qquad r \le 2$

r must be positive

$(r,\theta,h) \to (r\cos\theta, r\sin\theta, h)$
$\quad\quad\quad\quad\quad\quad\quad \shortparallel$
$\quad\quad\quad\quad\quad\quad (x, y, z)$

Then setup Integral with conversions. It can be faster than taking cross-sections

$$V = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{4-r^2}+3} r\,dh\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2\int_0^{\sqrt{4-r^2}+3} r\,dh\,dr$$

This term doesn't effect the other integrals so you can take it out

$$= 2\pi\int_0^2\int_0^{\sqrt{4-r^2}+3} r\,dh\,dr$$

Much cleaner

Center of Mass

$$\frac{1}{m}\langle M_x, M_y, M_z\rangle$$

$\text{Mass} = \iiint_R \delta\,dV \qquad M_x = \iiint_R \delta x\,dV$

Examples

1) Volume of 

$0 \le x^2+y^2+z^2 \le 1$

$\Rightarrow 0 \le \rho \le 1$

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\pi}\int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi$$

$$= 2\pi \int_0^1 \rho\, d\rho \int_0^{\pi} \sin\phi\, d\phi$$

$$= \frac{2\pi}{3}\rho^3\Big|_0^1 \cdot (-\cos\phi)\Big|_0^{\pi}$$

$$= \frac{2\pi}{3}(-\cos\pi - (-\cos 0)) = \frac{4\pi}{3}$$

2) Center $(0, 0, 1)$ 

$0 \le x^2+y^2+(z-1)^2 \le 1$

$0 \le r^2 + (h-1)^2 \le 1$

Look @ $r^2+(h-1)^2=1$

$r = \sqrt{1-(h-1)^2}$

$$V = \int_0^{2\pi}\int_0^{2}\int_0^{\sqrt{1-(h-1)^2}} r\, dr\, dh\, d\theta$$

$$= 2\pi \int_0^2 \int_0^{\sqrt{1-(h-1)^2}} r\, dr\, dh$$

$$= \pi \int_0^2 r^2\Big|_0^{\sqrt{1-(h-1)^2}} dh$$

$$= \pi \int_0^2 1-(h-1)^2\, dh$$

$$= \pi\left(h - \frac{1}{3}(h-1)^3\right)\Big|_0^2$$

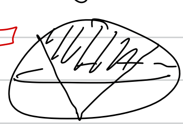
$$= \pi\left(2 - \frac{1}{3} - \left(0+\frac{1}{3}\right)\right)$$

$$= \pi\left(2-\frac{2}{3}\right) = \frac{4\pi}{3}$$

3) $R: x^2+y^2+z^2 \le 4z,\ z \ge \sqrt{x^2+y^2}$ ← Cone

$x^2+y^2+z^2-4z+4 \le 4$ Complete $\square$

$x^2+y^2+(z-2)^2 \le 2^2$



Region bounded by a cone and a sphere

(a) $\rho^2 - 4\rho\cos\phi = 0$

$\rho(\rho - 4\cos\phi) = 0$

$\rho = 0$ or $\rho = 4\cos\phi$

(b) $0 = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} - \rho\cos\phi$

$= \rho(\sin\phi - \cos\phi)$

$\rho = 0$ or $\sin\phi = \cos\phi$ $\quad \phi = \frac{\pi}{4}$

θ is dropped btwn 0 and π

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{4\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\frac{\pi}{4}} \rho^3\Big|_0^{4\cos\phi} \sin\phi\, d\phi$$

$$= \frac{2\pi}{3}\int_0^{\frac{\pi}{4}} 2^6 \cos^3\phi \sin\phi\, d\phi$$

$$= \frac{2\pi}{3}\cdot 2^6 (\cos^4\phi)\Big|_0^{\frac{\pi}{4}} = 8\pi$$

Chapter 15.1 Notes

Line Integrals over a Scalar Field

Let C be a smooth curve parametrized by s, the arc length parameter, and let f be a cts. function s. A line integral is an integral in the form

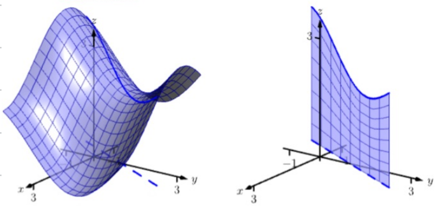
$$\int_C f(s)\,ds$$ when C is closed (starts and ends at the same point): $$\oint_C f(s)\,ds$$

Recall the arc length parameter $s(t) = \int_0^t ||\vec{r}\,'(u)||\,du$

Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t), k(t)\rangle$ for $a \le t \le b$

$$\int_C f(s)\,ds = \underbrace{\int_a^b f(g(t), h(t))\,||\vec{r}\,'(t)||\,dt}_{\text{In 2D}},\quad = \underbrace{\int_a^b f(g(t), h(t), k(t))\,||\vec{r}\,'(t)||\,dt}_{\text{In 3D}}$$

Example 15.1.5. Evaluating a line integral: area under a surface over a curve. Find the area under the surface $f(x, y) = \cos(x) + \sin(y) + 2$ over the curve C, which is the segment of the line $y = 2x + 1$ on $-1 \le x \le 1$, as shown in Figure 15.1.6.



$y = 2x+1$: $\vec{r}(t) = \langle t, 2t+1\rangle$

$||\vec{r}\,'(t)|| = \sqrt{1+2^2} = \sqrt{5}$

$$\int_C f(s)\,ds = \int_{-1}^{1} (\cos(t) + \sin(2t+1) + 2)\sqrt{5}\,dt$$

$$= \sqrt{5}\left(\sin(t) - \tfrac{1}{2}\cos(2t+1) + 2t\right)\Big|_{-1}^{1}$$

$= 14.418$ units^2

Line Integral Properties are the same as all other integrations

Mass, Center of Mass of Thin wire

Let a thin wire lie along a smooth curve C with cts. density function $\delta(s)$, where s is the arc length parameter

Mass of the thin wire $M = \int_C \delta(s)\,ds$

Moment about the x-axis is $M_x = \int_C x\,\delta(s)\,ds$

Moment about the y-axis is $M_y = \int_C y\,\delta(s)\,ds$

Moment about the z-axis is $M_z = \int_C z\,\delta(s)\,ds$

Center of Mass is $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_x}{M}, \frac{M_y}{M}, \frac{M_z}{M}\right)$

Chapter 15.2 Notes

Recoll: $\vec{u}\times\vec{v} = \langle u_2v_3-u_3v_2, -(u_1v_3-u_3v_1), u_1v_2-u_2v_1\rangle$

Vector Fields

1. A vector field in the plane is a function $\vec{F}(x,y)$ whose domain is a subset of $\mathbb{R}^2$ and whose output is a 2 Dimensional vector

$$\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle$$

2. A vector field in space is a function $\vec{F}(x,y,z)$ whose domain is a subset of $\mathbb{R}^3$ and whose output is a 3 Dimensional vector

$$\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$$

For vector fields, the point (x,y) or (x,y,z) corresponds to the tail of a vector, whose components are controlled by a function $\vec{F}$

Exact Magnitudes are not as important as relative magnitudes

Short Hand

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$, $\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle \Rightarrow \vec{F} = \langle M, N\rangle$

Same applies in 3 Dimensions

Divergence of a vector field

$$\text{div } \vec{F} = \nabla f \cdot \vec{F}$$

- In the plane, $\text{div } \vec{F} = M_x + N_y$
- In Space, $\text{div } \vec{F} = M_x + N_y + P_z$

Curl of a vector field

- let $\vec{F} = \langle M, N\rangle$ be a vector field in the plane. The curl of $\vec{F}$ is

$$\text{curl } \vec{F} = N_x - M_y$$

- let $\vec{F} = \langle M, N, P\rangle$ be a vector field in space. The curl of $\vec{F}$ is

$$\text{curl } \vec{F} = \nabla\times\vec{F} = \langle P_y - N_z, M_z - P_x, N_x - M_y\rangle$$

Example 15.2.16. Computing divergence and curl of vector fields in space. Compute the divergence and curl of each of the following vector fields.

1. $\vec{F} = \langle x^2 + y + z, -x - z, x + y\rangle$

$$\vec{F}(x,y,z) = \langle M, N, P\rangle$$

$\text{div } \vec{F} = \nabla\cdot\vec{F} = M_x + N_y + P_z = 2x + 0 + 0 = 2x$

$\text{curl } \vec{F} = \nabla\times\vec{F} = \langle P_y - N_z, M_z - P_x, N_x - M_y\rangle$

$= \langle 1-(-1), 1-1, -1-1\rangle = \langle 2, 0, -2\rangle$

The Jacobian and Vector fields

2D

$F(x,y) = \langle M, N \rangle$

curl $F = N_x - M_y$

div $F = M_x + N_y$

$$J_F = \begin{bmatrix} M_x & M_y \\ N_x & N_y \end{bmatrix}$$

Subtract from left to right to find curl

Add for divergence

$F(x,y,z) = \langle M, N, P \rangle$

$$J_F = \begin{bmatrix} M_x & M_y & M_z \\ N_x & N_y & N_z \\ P_x & P_y & P_z \end{bmatrix}$$

minus minus minus

div $F = M_x + N_y + P_z$

curl $F = \langle P_y - N_z, -(P_x - M_z), N_x - M_y \rangle$ ✓

Add for div $\vec{F}$

Subtract components on left and right, starting at P_y and moving towards N_x.

Chapter 15.3 Notes

Recall: $\vec{T} = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

Line Integral over a vector field

Let $\vec{F}$ be a vector field with cts. components defined on a smooth curve C, parametrized by $\vec{r}(t)$, and let $\vec{T}$ be the unit Tangent vector of $\vec{r}(t)$. The line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\, ds$$

If c is parametrized by a cts. and differentiable $\vec{r}(t)$...

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \left(\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\right) dt$$

These integrals have identical properties to those previously studied

Conservative Field, Path Independent

Let $\vec{F}$ be a vector field on an open, connected domain D in the plane or in space containing points A and B. If the line integral $\int_C \vec{F} \cdot d\vec{r}$ has the same value for all choices of path C starting at A and ending at B, then

- $\vec{F}$ is a conservative field
- The line integral $\int_C \vec{F} \cdot d\vec{r}$ is path independent and can be written as

$$\int_C \vec{F} \cdot d\vec{r} = \int_A^B \vec{F} \cdot d\vec{r}$$

- $\vec{F}$ is conservative if and only if there exists a differentiable function f such that $\vec{F} = \nabla f$
- If $\vec{F}$ is conservative, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

Recall and Relate

FTC

Potential Function

Let f be a differentiable function in the plane or space. If $\vec{F} = \nabla f$, the gradient of f, then f is a potential function of $\vec{F}$.

Curl of a Conservative Field

Computing Curl can prove conservative

Let $\vec{F}$, a vector field, have cts. partials on a connected domain D in the plane or space. Then $\vec{F}$ is conservative if and only if curl $\vec{F} = 0$ or $\vec{0}$, in 2D or 3D, respectively

Chapter 15.4 Notes

Flow, Flux

Let $\vec{F} = \langle M, N \rangle$ be a vector field with cts. components defined on a smooth curve C, parametrized by $\vec{r}(t) = \langle f(t), g(t) \rangle$, let $\vec{T}$ be the unit tangent vector of $\vec{r}(t)$, and $\vec{n}$ be the clockwise $90°$ rotation of $\vec{T}$.

- The flow of $\vec{F}$ along C is: $\int_C \vec{F} \cdot \vec{T} \, ds = \int_C \vec{F} \cdot d\vec{r}$
- The flux of $\vec{F}$ across C is: $\int_C \vec{F} \cdot \vec{n} \, ds = \int_C M\,dy - N\,dx = \int_C (Mg'(t) - Nf'(t))\,dt$

Green's Theorem

Let R be a closed, bounded region of the plane whose boundary C is composed of finitely many smooth curves, let $\vec{r}(t)$ be a counterclockwise parametrization of C, and let $\vec{F} = \langle M, N \rangle$ where N_x and M_y are cts. over R. Then,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \operatorname{curl} \vec{F} \, dA$$

The divergence Theorem (In the plane)

Let R be a closed, bounded region of the plane whose boundary C is composed of finitely many smooth curves, let $\vec{r}(t)$ be a counterclockwise parametrization of C, and let $\vec{F} = \langle M, N \rangle$ where M_x and N_y are cts. over R

$$\oint_C \vec{F} \cdot \vec{n} \, ds = \iint_R \operatorname{div} \vec{F} \, dA$$

Chapter 15.5 Notes

Parametrized Surface

Let $\vec{r}(u,v) = \langle f(u,v), g(u,v), h(u,v) \rangle$ be a cts. vector-valued function that is one-to-one on the interior of its domain R in the u-v plane. The set of all terminal points of $\vec{r}$, or the range of $\vec{r}$, is the surface S, and $\vec{r}$ along with its domain R form a parametrization of S

This parametrization is smooth on R: if $\vec{r}_u$ and $\vec{r}_v$ are cts. and $\vec{r}_u \times \vec{r}_v$ is never $\vec{0}$ on the interior of R

Example 15.5.4. Parametrizing a surface over a rectangle. Parametrize the surface $z = x^2 + 2y^2$ over the rectangular region R defined by $-3 \le x \le 3$, $-1 \le y \le 1$.

Cartesian coordinates: $x = u$ $y = v$

$\vec{r}(u,v) = \langle u, v, u^2 + 2v^2 \rangle$

Example 15.5.6. Parametrizing a surface over a circular disk. Parametrize the surface $z = x^2 + 2y^2$ over the circular region R enclosed by the circle of radius 2 that is centered at the origin.

circular region: $x = r v\cos u$ $y = r v \sin u$

$\vec{r}(u,v) = \langle 2v\cos u, 2v\sin u, 4v^2\cos^2 u + 8v^2\sin^2 u$

Let a surface S be a graph of a function $f(x,y)$, where the domain of f is a closed, bounded region R in the xy-plane. Let R be bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where $\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$

let $h(u,v) = g_1(u) + v(g_2(u) - g_1(u))$

S can be parametrized as: $\vec{r}(u,v) = \langle u, h(u,v), f(u, h(u,v)) \rangle$, $a \le u \le b$, $0 \le v \le 1$

Surface Area of Parametrically Defined Surfaces

Let $\vec{r}(u,v)$ be a smooth parametrization of a surface S over a closed, bounded region R of the uv-plane.

- The surface area differential dS is: $dS = \|\vec{r}_u \times \vec{r}_v\|\, dA$
- The surface area of S is

$$S = \iint_S dS = \iint_R \|\vec{r}_u \times \vec{r}_v\|\, dA$$

Partial derivative notation

Technology will most likely need to be used as these parametrizations do not end up pretty when the magnitude is calculated