MATH 223—Worksheet 5

Due: 2023–11–17

Part I

1. For each of the following vector-valued functions, compute the determinant of the Jacobian, $|J_F|$.

   Polar Coordinates. $F(r, \theta) = (r\cos\theta, r\sin\theta)$.

   Cylindrical Coordinates. $F(r, \theta, h) = (r\cos\theta, r\sin\theta, h)$.

   Spherical Coordinates. $F(r, \varphi, \theta) = (r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)$.

2. Consider a pop-up fidget toy in the shape of an upper hemisphere centered about the origin with inner radius of 2 and outer radius of 3. Suppose the fidget toy is not precision engineered and has a density function given by $\delta(x, y, z) = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$. Find the center of mass with respect to δ. (Hint: Look for shortcuts to make the computation significantly faster.) Is there an efficient way to measure the surface area of the toy? If so, measure it.

3. Use cylindrical and spherical coordinates to measure the volume of a cored apple: take a sphere of radius 2 centred about the origin and remove a vertical cylinder of radius 1 from the sphere, then measure the volume.

4. **Challenge:** Find the volume under the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ in the first octant for $a, b, c > 0$, using triple integration.

5. Textbook problems: ~~(a) 14.7.38; (b) 14.7.40~~.
   14.7.42

Part II

1. For each of the following vector fields, sketch the field, then find the divergence and curl.

   (a) The Volterra-Lodka system is the predator-prey model given by $F(x, y) = \langle 0.4y - 0.4x, -0.1y + 0.2xy \rangle$.

   (b) Here is a vaguely heart-shaped vector field: $F(x, y) = \langle y - x - xy, -x - y + x^2 \rangle$.

2. The gradient of a function determines a vector field $F(x, y) = \nabla f(x, y)$. In such a situation, we call f a **potential function** for the vector field F.

   (a) What can you say about the curl and divergence of a vector field that has a potential function?

   (b) Use your observations to show $G(x, y) = \langle y - \sin x, x^3 + \cos(2y) \rangle$ does **not** have a potential function.

   (c) Compute $\int_\alpha G$, for $\alpha(t) = \langle t, 0.5t \rangle$ with $t \in [0, 2\pi]$.

3. Consider the vector field

$$H(x, y) = \left\langle -\frac{2x}{(x^2 + y^2)^2}, -\frac{2y}{(x^2 + y^2)^2} \right\rangle.$$

   (a) Find a potential function for H.

   (b) Compute the integral of H along the segment from $(0, 0)$ to $(0, 1)$.

   (c) Use the previous part to compute the integral of H along the clockwise semi-circular path from $(0, 1)$ to $(0, 0)$, with the least work possible.

4. Without integrating anything, prove that $\oint_S F \cdot dr = 0$ for S the unit circle and

$$F = \langle -y^2 \sin(xy), \cos(xy) - xy \sin(xy) \rangle.$$

Trevor Swan

Worksheet 5 Part 1 - MATH223

1) Polar Coordinates $F(r,\theta)=\langle r\cos\theta, r\sin\theta\rangle$

$$J_F=\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix} \quad \det(J_F)=r\cos^2\theta+r\sin^2\theta=r$$

Cylindrical Coordinates $F(r,\theta,h)=\langle r\cos\theta, r\sin\theta, h\rangle$

$$J_F=\begin{bmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} \quad \det(J_F)=\cos\theta(r\cos\theta-0)-(-r\sin\theta)(\sin\theta-0)+0(0-0)$$

$$=r\cos^2\theta+r\sin^2\theta=r$$

Spherical Coordinates $F(\rho,\varphi,\theta)=\langle \rho\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi\rangle$

$$J_F=\begin{bmatrix}\sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta\\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta\\ \cos\varphi & -r\sin\varphi & 0\end{bmatrix}$$

$$\det(J_F)=\sin\varphi\cos\theta(0-(-\rho\sin\varphi)(\rho\sin\varphi\cos\theta))=(\sin\varphi\cos\theta)(\rho^2\sin^2\varphi\cos\theta)$$

$$-\rho\cos\varphi\cos\theta(0-\rho\sin\varphi\cos\theta\cos\varphi)=\rho^2\sin\varphi\cos^2\theta\cos^2\varphi$$

$$-\rho\sin\varphi\sin\theta(-\rho\sin^2\varphi\sin\theta-\rho\cos^2\varphi\sin\theta)=\rho^2\sin^3\varphi\sin^2\theta+\rho^2\sin\varphi\cos^2\varphi\sin^2\theta$$

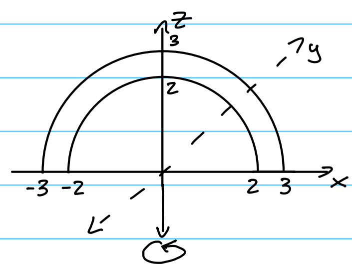
$$=\rho^2\sin^3\varphi\cos^2\theta+\rho^2\sin\varphi\cos^2\theta\cos^2\varphi+\rho^2\sin^3\varphi\sin^2\theta+\rho^2\sin\varphi\cos^2\varphi\sin^2\theta$$

$$=\rho^2\sin^3\varphi\cos^2\theta+\rho^2\sin^3\varphi\sin^2\theta=\rho^2\sin^3\varphi$$

$$+\rho^2\sin\varphi\cos^2\theta\cos^2\varphi+\rho^2\sin\varphi\cos^2\varphi\sin^2\theta=\rho^2\sin\varphi\cos^2\varphi(\cos^2\theta+\sin^2\theta)=\rho^2\sin\varphi\cos^2\varphi$$

$$=\rho^2\sin^3\varphi+\rho^2\sin\varphi\cos^2\varphi=\rho^2\sin\varphi(\sin^2\varphi+\cos^2\varphi)=\rho^2\sin\varphi$$

2)



$\delta(x,y,z)=(x^2+y^2+z^2)^{-1/2} \longrightarrow$ spherical

$x=\rho\sin\varphi\cos\theta, \quad y=\rho\sin\varphi\sin\theta, \quad z=\rho\cos\varphi$

Conversion: $\sqrt{x^2+y^2+z^2}^{-1} \Rightarrow \sqrt{\rho^2}^{-1} \Rightarrow \pm\rho^{-1} \Rightarrow \rho^{-1}$ ρ is only positive

$\therefore\ \delta(\rho,\varphi,\theta)=\rho^{-1}$

Bounds: $2\le\rho\le3, \quad 0\le\varphi\le\frac{\pi}{2}, \quad 0\le\theta\le2\pi$

$$\text{Mass}=\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_2^3(\rho^2\sin\varphi)\frac{1}{\rho}\,d\rho\,d\theta\,d\varphi=2\pi\int_0^{\frac{\pi}{2}}\int_2^3\rho\sin\varphi\,d\rho\,d\varphi$$

θ plays no role in integral

$$=2\pi\int_0^{\frac{\pi}{2}}\frac{1}{2}\rho^2\sin\varphi\Big|_2^3 d\varphi=\pi\int_0^{\frac{\pi}{2}}5\sin\varphi\,d\varphi=5\pi(-\cos\varphi)\Big|_0^{\frac{\pi}{2}}=5\pi$$

$$M_{yz}=M_x=\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_2^3\rho\sin\varphi\cos\theta\,\rho\sin\varphi\,d\rho\,d\theta\,d\varphi=\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_2^3\rho^2\sin^2\varphi\cos\theta\,d\rho\,d\theta\,d\varphi$$

$$=\frac{1}{3}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\rho^3\sin^2\varphi\cos\theta\Big|_2^3 d\theta\,d\varphi=\frac{1}{3}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}27\sin^2\varphi\cos\theta-8\sin^2\varphi\cos\theta\,d\theta\,d\varphi$$

$$=\frac{19}{3}\int_0^{\frac{\pi}{2}}\sin^2\varphi\sin\theta\Big|_0^{2\pi}d\varphi=\frac{19}{3}\int_0^{\frac{\pi}{2}}0\,d\varphi=0 \quad \therefore\ M_x=0$$

$M_{xz}=M_y=\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_2^3(\rho\sin\varphi\sin\theta)(\rho\sin\varphi)\,d\rho\,d\theta\,d\varphi=\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_2^3\rho^2\sin^2\varphi\sin\theta\,d\rho\,d\theta\,d\varphi$

$=\frac{1}{3}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\rho^3\sin^2\varphi\sin\theta\Big|_2^3 d\theta\,d\varphi=\frac{19}{3}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\sin^2\varphi\sin\theta\,d\theta\,d\varphi=-\frac{19}{3}\int_0^{\frac{\pi}{2}}\sin^2\varphi\cos\theta\Big|_0^{2\pi}d\varphi$

$=-\frac{19}{3}\int_0^{\frac{\pi}{2}}\sin^2\varphi(\cos 2\pi-\cos 0)\,d\varphi=-\frac{19}{3}\int_0^{\frac{\pi}{2}}0\,d\varphi=0$ $\therefore$ $M_y=0$

$M_{xy}=M_z=\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_2^3(\rho\cos\varphi)(\rho\sin\varphi)\,d\rho\,d\theta\,d\varphi=2\pi\int_0^{\frac{\pi}{2}}\int_2^3\rho^2\sin\varphi\cos\varphi\,d\rho\,d\varphi$ θ plays no role in int.

$=\frac{2\pi}{3}\int_0^{\frac{\pi}{2}}\rho^3\sin\varphi\cos\varphi\Big|_2^3 d\varphi=\frac{38\pi}{3}\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi\,d\varphi$ $u=\sin\varphi$ $u(\frac{\pi}{2})=1$ $\frac{du}{d\varphi}=\cos\varphi$ $u(0)=0$

$=\frac{38\pi}{3}\int_0^1 u\,du=\frac{19\pi}{3}(u^2)\Big|_0^1=\frac{19\pi}{3}$ $\therefore$ $M_z=\frac{19\pi}{3}$

Center of Mass: $(\bar{x},\bar{y},\bar{z})=\left(\frac{M_x}{m},\frac{M_y}{m},\frac{M_z}{m}\right)=\left(0,0,\frac{\frac{19\pi}{3}}{5\pi}\right)=\left(0,0,\frac{19}{15}\right)$

There is an efficient way to calculate surface area, it involves known surface area equations for spheres and circles. Integration is not necessary, as shown below

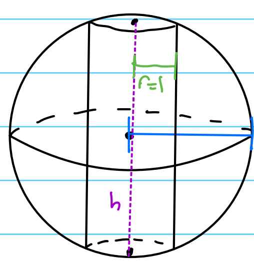
Surface Area: $S_{sphere}=4\pi r^2$ $\quad$ $Area_{Circle}=\pi r^2$

Dome: $S_B=\underbrace{(4\pi(3)^2}_{\text{Large}}+\underbrace{4\pi(2)^2}_{\text{Small}})\frac{1}{2}$
$=\frac{1}{2}(52\pi)=26\pi$

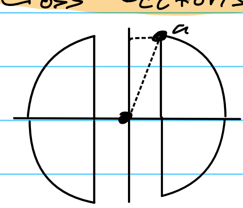
Base: $S_B=\underbrace{\pi(3)^2}_{\text{Large}}-\underbrace{\pi(2)^2}_{\text{Small}}$
$=5\pi$

$\therefore$ Surface Area: $S_o=26\pi+5\pi=31\pi$ units^2

3)

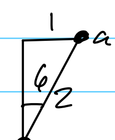


Cross Sections



Symmetric Region About Horizon
$0\le\theta\le 2\pi$

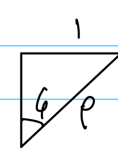
φ Bounds:



$\sin\varphi=\frac{1}{2}\Rightarrow\varphi=\frac{\pi}{6},\frac{5\pi}{6}$

$\frac{\pi}{6}\le\varphi\le\frac{5\pi}{6}$

ρ Bounds:



$\sin\varphi=\frac{1}{\rho}$
$\rho=\csc\varphi$

$\csc\varphi\le\rho\le 2$

$V=\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_{\csc\varphi}^{2}\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta=2\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_{\csc\varphi}^{2}\rho^2\sin(\varphi)\,d\rho\,d\varphi=\frac{2\pi}{3}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\rho^3\sin\varphi\Big|_{\csc\varphi}^{2}d\varphi$

$=\frac{2\pi}{3}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}8\sin\varphi-\csc^3\varphi\sin\varphi\,d\varphi=\frac{16\pi}{3}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\sin\varphi\,d\varphi-\frac{2\pi}{3}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\csc^2\varphi\,d\varphi$

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$$= -\frac{16\pi}{3}\cos\varphi\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \frac{2\pi}{3}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\csc^2\varphi\, d\varphi = -\frac{16\pi}{3}\left(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right) + \frac{2\pi}{3}(\cot\varphi)\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= \frac{16\pi}{3}(\sqrt{3}) + \frac{2\pi}{3}(-\sqrt{3}-\sqrt{3}) = \frac{16\sqrt{3}\pi}{3} - \frac{4\sqrt{3}\pi}{3} = \frac{12\sqrt{3}\pi}{3} = 4\sqrt{3}\pi$$

This can be confirmed by Other Methods not included to Save space

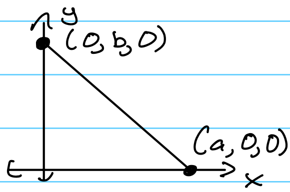
4) I approached this challenge by bounding z in terms of x and y, y in terms of x, and x in terms of the constant a. z and y both have constants in their upper bounds as well. This is a proof for the volume of a triangular pyramid. Cool!

Given: $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$; $a, b, c > 0$ ∴ First Octant Only

$z = c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

z upper bound

$x = a$ x upper bound



In the xy-plane, $z=0$ everywhere

∴ $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow y = b\left(1-\frac{x}{a}\right)$

y upper bound

The following will become relevant in the future: $\left(b-\frac{bx}{a}\right)^2 = b^2 - \frac{2b^2x}{a} + \frac{b^2x^2}{a^2} = b\left(b-\frac{2bx}{a}+\frac{bx^2}{a^2}\right)$

This expression is indicated by the orange Highlight Below

$$V = \int_0^a\int_0^{b-\frac{bx}{a}}\int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\,dy\,dx = \int_0^a\int_0^{b-\frac{bx}{a}} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$= c\int_0^a\int_0^{b-\frac{bx}{a}}\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx = c\int_0^a\left(y-\frac{xy}{a}-\frac{y^2}{2b}\right)\Big|_0^{b-\frac{bx}{a}} dx$$

$$= c\int_0^a b\left(1-\frac{x}{a}\right) - b\left(1-\frac{x}{a}\right)\left(\frac{x}{a}\right) - b\left(b-\frac{2bx}{a}+\frac{bx^2}{a^2}\right)\left(\frac{1}{2b}\right)dx$$

$$= bc\int_0^a 1 \underbrace{-\frac{x}{a}-\frac{x}{a}} + \frac{x^2}{a^2} - \left(b-\frac{2bx}{a}+\frac{bx^2}{a^2}\right)\left(\frac{1}{2b}\right)dx = bc\int_0^a 1-\frac{2x}{a}+\frac{x^2}{a^2}-\left(\frac{1}{2}-\frac{x}{a}+\frac{x^2}{2a^2}\right)dx$$

$$= bc\int_0^a\left(\underbrace{1}-\underbrace{\frac{2x}{a}}+\underbrace{\frac{x^2}{a^2}}-\underbrace{\frac{1}{2}}+\underbrace{\frac{x}{a}}-\underbrace{\frac{x^2}{2a^2}}\right)dx = bc\int_0^a\left(\frac{1}{2}-\frac{x}{a}+\frac{x^2}{2a^2}\right)dx$$

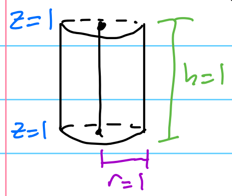
$$= bc\left(\frac{x}{2}-\frac{x^2}{2a}+\frac{x^3}{6a^2}\right)\Big|_0^a = bc\left(\frac{a}{2}-\frac{a^2}{2a}+\frac{a^3}{6a^2}\right) = bc\left(\underbrace{\frac{a}{2}}-\underbrace{\frac{a}{2}}+\frac{a}{6}\right)$$

$$= bc\left(\frac{a}{6}\right) = \frac{1}{6}abc \quad \therefore \quad \text{Volume} = \frac{1}{6}abc$$

This solution is confirmed by the known equation to find the volume of a triangular Pyramid with sides a, b, c. $V = \frac{1}{6}abc$

14.7.42) $x^2+y^2=1$ bounded by $z=0$ and $z=1$ conversion: $r^2\cos^2\theta + r^2\sin^2\theta = 1$

$r^2=1$ $r=\pm 1$ r is only positive $\therefore$ $r=1$



$$V=\int_0^1\int_0^{2\pi}\int_0^1 r\,dr\,dh\,d\theta = 2\pi\int_0^1\int_0^1 r\,dr\,dh$$

$$= \pi\int_0^1 r^2\Big|_0^1 dh = \pi\int_0^1 dh = \pi$$ $\therefore$ Volume = π

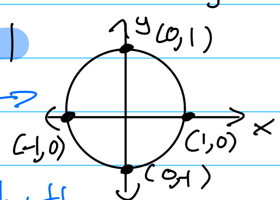
Cylindrical: $\int_0^1\int_0^{2\pi}\int_0^1 r\,dr\,dh\,d\theta$

This integral is by far the easiest to setup/evaluate

This conversion is simple, and is proved by the equation $V=\pi r^2 h$. Setting up integrals with rectangular and spherical coordinates proves to be more challenging...

$0\le z\le 1$

Cylinder Base →



$-1\le y\le 1$

$x^2+y^2=1$

$x^2=1-y^2$

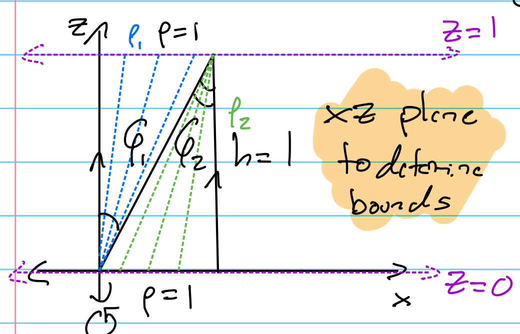
$x=\pm\sqrt{1-y^2}$

Rectangular: $\int_0^1\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy\,dz$

Not too hard to evaluate

This takes slightly longer to set up, but isn't too difficult $-\sqrt{1-y^2}\le x\le\sqrt{1-y^2}$

Converting to spherical proved to be most challenging, and is by far the least favorable way to approach a problem like this...



xz plane to determine bounds

$\varphi_1 = \arctan\left(\frac{1}{1}\right)=\frac{\pi}{4}$ $\quad 0\le\varphi_1\le\frac{\pi}{4}$

$\varphi_2 = \arctan\left(\frac{1}{1}\right)=\frac{\pi}{4}$ $\quad \frac{\pi}{4}\le\varphi_2\le\frac{\pi}{2}$

$\varphi = \frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$

max ρ at $\varphi=\frac{\pi}{4}$: $\cos\left(\frac{\pi}{4}\right)=\frac{1}{\rho} \Rightarrow \rho=\sqrt{2}$

ρ_1: $\cos(\varphi_1)=\frac{1}{\rho_1}\Rightarrow\rho_1=\sec(\varphi_1)$

ρ_2: $\sin(\varphi_2)=\frac{1}{\rho_2}\Rightarrow \rho_2=\csc(\varphi_2)$

Full Cylinder: $0\le\theta\le 2\pi$

$0\le\rho_1\le\sec(\varphi_1)$ and $0\le\rho_2\le\csc(\varphi_2)$

Spherical: $$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sec(\varphi)} \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta + \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\csc(\varphi)} \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

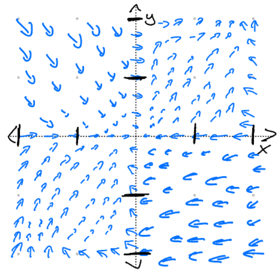
Obviously, this integral would prove most challenging to evaluate due to two integrals being present, inverse trig functions in the bounds, and the effort required to set up the integral.

Putting these 3 integrals in software confirms the expected π units^3 for the volume

Worksheet 5 Part 2 - MATH223

1) a) $F(x,y) = \langle 0.4y - 0.4x, -0.1y + 0.2xy \rangle$

$$J_F = \begin{bmatrix} -0.4 & 0.4 \\ 0.2y & -0.1+0.2x \end{bmatrix}$$



$\operatorname{div} \vec{F} = -0.1 + 0.2x + -0.4$

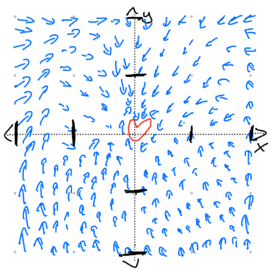
$= 0.2x - 0.5$

$\operatorname{curl} \vec{F} = 0.2y - 0.4$

The magnitudes here are greatest at $(-2, 2)$ and $(2, -2)$. The first and third quadrants have smaller, positive vectors in magnitude and direction, respectively

b) $F(x,y) = \langle y - x - xy, -x - y + x^2 \rangle$

$$J_F = \begin{bmatrix} -1-y & 1-x \\ -1+2x & -1 \end{bmatrix}$$



$\operatorname{div} \vec{F} = -1 - y + -1$

$= -2 - y$

$\operatorname{curl} \vec{F} = -1 + 2x - (1 - x)$

$= -2 + 3x$

This sketch is very poor, but there is a clear heart outlined, indicated by the red heart centered at $(0,0)$. Magnitudes are the greatest on the left, smallest at $(0,0)$ and medium-sized at $(2, \pm 2)$

2. a) Let $F(x,y) = \langle f_x, f_y \rangle$ and $\nabla f(x,y) = F(x,y)$

$$J_F = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$\operatorname{div} \vec{F} = f_{xx} + f_{yy}$

$\operatorname{curl} \vec{F} = 0$

For conservative vector fields

$f_{xy} = f_{yx} \therefore f_{xy} - f_{yx} = 0$

b) $G(x,y) = \langle y - \sin x, x^3 + \cos(2y) \rangle$

$$J_G = \begin{bmatrix} -\cos x & 1 \\ 3x^2 & -2\sin 2y \end{bmatrix} \Rightarrow \operatorname{Curl} \vec{G} = 3x^2 - 1$$

$\operatorname{curl} \vec{G} = 3x^2 - 1 \therefore \operatorname{curl} \vec{G} \neq 0$ and G does not have a potential function.

c) $\int_a G$ for $a(t) = \langle t, 0.5t \rangle$ with $t \in [0, 2\pi]$

$a'(t) = \langle 1, 0.5 \rangle$ $\quad G(a(t)) = \langle 0.5t - \sin t, t^3 + \cos(2(0.5t)) \rangle$

$$\int_0^{2\pi} \langle 0.5t - \sin t, t^3 + \cos t \rangle \cdot \langle 1, 0.5 \rangle \, dt = \int_0^{2\pi} (0.5t - \sin t + 0.5t^3 + 0.5\cos t)\,dt$$

$$= \left(\tfrac{1}{4}t^2 + \cos t + \tfrac{1}{8}t^4 + 0.5\sin t\right)\Big|_0^{2\pi} = \tfrac{1}{4}(2\pi)^2 + 1 + \tfrac{1}{8}(2\pi)^4 + 0 - (1 + 0)$$

$$= \tfrac{1}{4}(4\pi^2) + \tfrac{1}{8}(16\pi^4) = \pi^2 + 2\pi^4$$

$u = x^2+y^2$
$\frac{\partial u}{\partial x} = 2x \quad \partial x = \frac{\partial u}{2x}$

$u = x^2+y^2$
$\frac{\partial u}{\partial y} = 2y \quad \partial y = \frac{\partial u}{2y}$

3. $H(x,y) = \left\langle -\frac{2x}{(x^2+y^2)^2}, -\frac{2y}{(x^2+y^2)^2} \right\rangle$

a) Find a function $h(x,y)$ s.t. $\nabla h(x,y) = H(x,y)$

$$\int \frac{-2x}{(x^2+y^2)^2}\,dx = \int \frac{-1}{u^2}\,du = u^{-1} + C_1 \Rightarrow \frac{1}{x^2+y^2} + C_1$$

$$\int \frac{-2y}{(x^2+y^2)^2}\,dx = \int \frac{-1}{u^2}\,du = u^{-1} + C_2 \Rightarrow \frac{1}{x^2+y^2} + C_2$$

$$\frac{1}{x^2+y^2} + C_1 = \frac{1}{x^2+y^2} + C_2 \therefore C_1 = C_2 \qquad h(x,y) = \frac{1}{x^2+y^2}$$

b) Integrate H along $(0,1)$ to $(0,2)$

$\vec{r}(t) = \langle 0, t \rangle$ on $1 \le t \le 2$ $\qquad \vec{r}\,'(t) = \langle 0, 1 \rangle$

$$H(\vec{r}(t)) = H(0,t) = \left\langle \frac{-2(0)}{(0^2+t^2)^2}, \frac{-2t}{(0^2+t^2)^2} \right\rangle = \left\langle 0, \frac{-2}{t^3} \right\rangle$$

$$\int_1^2 \left\langle 0, \frac{-2}{t^3} \right\rangle \cdot \langle 0, 1 \rangle\,dt = \int_1^2 \frac{-2}{t^3}\,dt = t^{-2}\Big|_1^2 = \frac{1}{4} - 1 = -\frac{3}{4}$$

c) Because $H(x,y)$ has a potential function $h(x,y)$, H is a conservative function and therefore is path independent. Because H is being measured from the same two points, the line integral's magnitude will be equal. In this case, the path is traveled across in the opposite direction ($(0,2) \to (0,1)$ instead of $(0,1) \to (0,2)$), so the sign should be flipped. The integral of H in this case is $\frac{3}{4}$

4. $F = \langle -y^2\sin(xy), \cos(xy) - xy\sin(xy) \rangle$

$$J_F = \begin{bmatrix} -y^3\cos(xy) & -y^2x\cos(xy) - 2y\sin(xy) \\ -y\sin(xy) - (y\sin(xy) + xy^2\cos(xy)) & -x\sin(xy) - (x\sin(xy) + x^2y\cos(xy)) \end{bmatrix}$$

$$\text{curl}\,\vec{F} = -2y\sin(xy) - xy^2\cos(xy) - (-xy^2\cos(xy) - 2y\sin(xy))$$
$$= \underbrace{-2y\sin(xy)}\,\underbrace{-xy^2\cos(xy)}\,\underbrace{+xy^2\cos(xy)}\,\underbrace{+2y\sin(xy)} = 0$$

According to Green's Thm. $\oint_S \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\,dA$ for closed, bounded regions. The unit circle is obviously closed, so this thm. applies. The integral of 0 is 0, so $\oint_S \vec{F}\cdot d\vec{r} = \iint_{R^*} 0\,dA = 0$ for the unit circle

*The bounds do not matter, but $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ and $-1 \le x \le 1$