MATH 223—Worksheet 4

Due: 2023–10–20

1. Consider the function $f(x, y) = x + 2xy^2$.

   (a) Explain how you know f is differentiable everywhere.

   (b) Give an equation for the tangent plane to the surface $z = f(x, y)$ at $(a, b) = (1, -3)$.

2. Calculate the distance between $Q = (0, 0, \frac{3}{2})$ and the surface $z = f(x, y) = 2xy + 1$.

SKIP 3. How do we measure the "flatness" of a surface in 3d Euclidean space? Let's think about it a few different ways, using the function $f(x, y) = \cos(x)\sin(y) + \sin(x + y)$ as our motivating example. If we plot f, we see that "bumpy" is the best description of the surface. The question is how to best quantify the bumpiness of f?

   Consider the triangular domain R in the first quadrant of $\mathbb{R}^2$ trapped between the x-axis, the line $x = 2\pi$, and the line $y = \frac{1}{2}x$.

SKIP 3b, c, d

   (a) Let us start by bounding the values of f. How can we be sure f has a local minimum and local maximum over R?

   (b) Find all extrema of f over R. Show your work.

   (c) Classify the critical points you found in (b).

   (d) Based on (b), what is the range of f over R?

   (e) The range of f helps us understand the space the surface takes up. However, it does not really tell us much about the behaviour of f over R. Let us get an idea of where f is concentrated by measuring the mean value of f over R.
   N.B. $\int \cos^2(x) = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$ and $\int \sin^2(x) = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$.

   (f) We can use our answer to (e) to measure the squared deviation of f with respect to the mean, that is, $\iint_R (f - \text{Mean}_R f)^2\, dA$. For this problem, use software to evaluate the double integral; do not try to do it by hand!

4. The number we found for (f) does not mean much on its own, so let us compute the squared deviation of the function $g(x, y) = 2\sin(x^2)$ on the same domain R.

   (a) Find the range of g over R. (Hint: This should take next to no thinking.)

   (b) Compute the mean value of g over R.

   (c) Compute the squared deviation of g over R, using software, i.e., not by hand.

   (d) Compare the results from 1(f) and 2(c).

13.8.13

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SKIP QUESTION 3

Worksheet 4 - MATH223

1) $f(x,y) = x + 2xy^2$

a) $f_x = 1 + 2y^2$
$f_y = 4xy$
$\nabla f(x,y) = \langle 1+2y^2, 4xy \rangle$

Because x and y are cts. functions/variables, f is cts. for all x and y. The gradient of f is also cts. for all x and y for this same reason. Since the Gradient of f is cts. for all x and y inputs, f is differentiable everywhere.

b) $(a,b) = (1,-3)$, $\nabla f = \langle 1+2y^2, 4xy \rangle$, $f(x,y) = x + 2xy^2 \Rightarrow f(1,-3) = 19$
$\nabla f(1,-3) = \langle 19, -12 \rangle$ $0 = 19(x-1) - 12(y+3) - (z-19)$

2) $Q = (0, 0, \frac{3}{2})$, $f(x,y) = 2xy + 1$ Some point on f:
$\nabla f = \langle 2y, 2x \rangle \Rightarrow \vec{d}_n = \langle 2y, 2x, -1 \rangle$ $P = \langle x, y, 2xy+1 \rangle$

Given P and Q: $\vec{PQ} = \langle 0, 0, \frac{3}{2} \rangle - \langle x, y, 2xy+1 \rangle = \langle -x, -y, -2xy + \frac{1}{2} \rangle$

$\vec{PQ} \parallel \vec{d}$ $\therefore$ $\langle -x, -y, -2xy + \frac{1}{2} \rangle = K\langle 2y, 2x, -1 \rangle$
$-x = 2yK \Rightarrow x = -2yK$ | $-y = 2xK$ | $-2xy + \frac{1}{2} = -K$

$-y = 2(-2yK)K$
$-y = -4yK^2$
$K^2 = \frac{1}{4}$ $K = \pm \frac{1}{2}$

$\frac{-x}{2y} = K$ and $\frac{-y}{2x} = K$
$\frac{-x}{2y} = \frac{-y}{2x} \Rightarrow -2x^2 = -2y^2 \Rightarrow y = \pm x$

Plug each coordinate into $\langle -x, -y, -2xy + \frac{1}{2} \rangle$

$-2xy + \frac{1}{2} = -(\pm \frac{1}{2})$ choose neg. sign to get non-zero answers
$-2xy = -1$
$xy = \frac{1}{2}$, $y = \pm x$
$y^2 = \frac{1}{2} \Rightarrow y = \pm\sqrt{\frac{1}{2}} = \pm \frac{\sqrt{2}}{2}$
$x^2 = \frac{1}{2} \Rightarrow x = \pm\sqrt{\frac{1}{2}} = \pm \frac{\sqrt{2}}{2}$

Four possible Candidates
$P_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \therefore \vec{P_1Q} = \langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\frac{1}{2} \rangle$
$P_2 = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) \therefore \vec{P_2Q} = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{3}{2} \rangle$
$P_3 = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \therefore \vec{P_3Q} = \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{3}{2} \rangle$
$P_4 = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) \therefore \vec{P_4Q} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{1}{2} \rangle$

$\|\vec{PQ}\| = \sqrt{x^2 + y^2 + (-2xy + \frac{1}{2})^2}$, magnitude is dependent on only z coordinate
$\|\vec{P_1Q}\| = \|\vec{P_4Q}\|$, and $\|\vec{P_2Q}\| = \|\vec{P_3Q}\|$

Distance calculated by finding $\|\vec{P_1Q}\|$ and $\|\vec{P_2Q}\|$ and choosing smaller value

$\|\vec{P_1Q}\| = \sqrt{(\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2 + (-2(\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) + \frac{1}{2})^2} = \sqrt{\frac{1}{2} + \frac{1}{2} + \frac{1}{4}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$

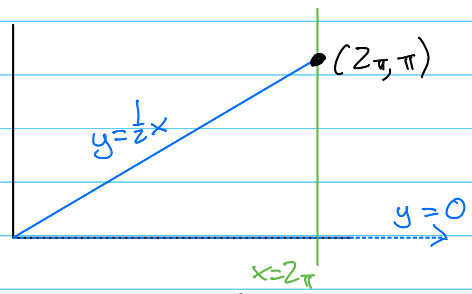
$\|\vec{P_2Q}\| = \sqrt{(-\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2 + (-2(-\frac{\sqrt{2}}{2})(\frac{\sqrt{2}}{2}) + \frac{1}{2})^2} = \sqrt{\frac{1}{2} + \frac{1}{2} + \frac{9}{4}} = \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2}$

$\frac{\sqrt{5}}{2} < \frac{\sqrt{13}}{2}$ $\therefore$ Dist from $(0, 0, \frac{3}{2})$ to $f(x,y) = 2xy + 1$ is $\frac{\sqrt{5}}{2}$

4) $g(x,y) = 2\sin(x^2)$ trapped btwn. x-axis, $x = 2\pi$, $y = \frac{1}{2}x$ ← Domain R

a) The range of g over R is $-2 \le g \le 2$, because sin cannot obtain values greater than one. A scalar of 2 amplifies the range by a factor of 2 as well.

b) Average (mean) Value of g over R: $\dfrac{\iint_R g(x,y)\,dA}{\iint_R dA}$



$$Mean = \frac{\int_0^{\pi}\int_{2y}^{2\pi} 2\sin(x^2)\,dx\,dy}{\int_0^{\pi}\int_{2y}^{2\pi} dx\,dy} = \frac{-\frac{1}{2}\left(\cos(4\pi^2) - 1\right)}{\pi^2}$$

$$= .0612$$

Work

$u = x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

numerator: $2\int_0^{\pi}\int_{2y}^{2\pi} \sin(x^2)\,dx\,dy = 2\int_0^{2\pi}\int_0^{x/2} \sin(x^2)\,dy\,dx = 2\int_0^{2\pi} \sin(x^2)\underbrace{\int_0^{x/2} dy}_{\frac{x}{2} - 0}\,dx = 2\int_0^{2\pi} \sin(x^2)\left(\frac{x}{2}\right)dx$

$$= \int_0^{2\pi} x\sin(x^2)\,dx = \frac{1}{2}\int_0^{4\pi^2} \sin(u)\,du = \frac{1}{2}\left(-\cos u\right)\Big|_0^{4\pi^2}$$

$$= -\frac{1}{2}\left(\cos(4\pi^2) - \cos(0)\right) = -\frac{1}{2}\left(\cos(4\pi^2) - 1\right)$$

denom: $\int_0^{\pi}\int_{2y}^{2\pi} dx\,dy = \int_0^{\pi}(2\pi - 2y)\,dy = 2\pi y - y^2\Big|_0^{\pi} = 2\pi(\pi) - \pi^2 = 2\pi^2 - \pi^2 = \pi^2$

c) Squared Deviation of g over R: $\int_0^{\pi}\int_{2y}^{2\pi}\left(2\sin(x^2) - Mean_R\, g\right)^2 dx\,dy$

Using Software:

$$\int_0^{\pi}\int_{2y}^{2\pi}\left(2\sin(x^2) - \frac{-\frac{1}{2}\left(\cos(4\pi^2) - 1\right)}{\pi^2}\right)dx\,dy$$

$$= \int_0^{\pi}\int_{2y}^{2\pi}\left(2\sin(x^2) - 0.0612\right)^2 dx\,dy = 19.804$$

d) Extension of Problem 3, cannot complete

13.8.18) $f(x,y) = 3y - 2x^2$, constrained by $y = x^2+x-1$ and $y = x$

Set Gradient to 0 to find extrema

$\nabla f = \langle -4x, 3 \rangle$ $\qquad \nabla f = \vec{0}$

$f_{xx} = -4,\ f_{yy} = 0$ $\qquad 0 = -4x \quad 0 = 3$

$f_{xy} = 0$

No global Extrema

Second Derivative Test

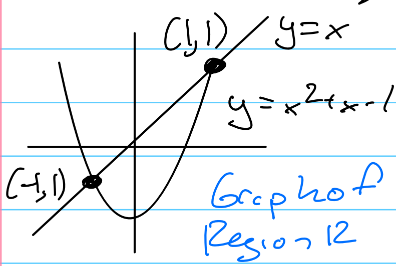
$D = f_{xx} f_{yy} - f_{xy}^2$

$D = 0 \therefore$ inconclusive

Test endpoints on x^2+x-1

$$f(x,y) = 3(x^2+x-1) - 2x^2$$
$$= 3x^2 + 3x - 3 - 2x^2$$
$$= x^2 + 3x - 3$$
$$\frac{df}{dx} = 2x + 3$$
$$0 = 2x + 3$$
$$x = -\frac{3}{2}$$
$$y = \left(-\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right) - 1$$
$$y = \frac{9}{4} - \frac{10}{4} = -\frac{1}{4}$$

One Canditate: $\left(-\frac{3}{2}, -\frac{1}{4}\right)$



Graph of Region R

Test endpoints on $y = x$

$$f(x,y) = 3x - 2x^2$$
$$\frac{df}{dx} = 3 - 4x$$
$$0 = 3 - 4x$$
$$x = \frac{3}{4} \qquad y = \frac{3}{4}$$

Second Candidate: $\left(\frac{3}{4}, \frac{3}{4}\right)$

Test Candidates

$$f\left(-\frac{3}{2}, -\frac{1}{4}\right) = 3\left(-\frac{1}{4}\right) - 2\left(-\frac{3}{2}\right)^2 = -\frac{21}{4}$$
$$f\left(\frac{3}{4}, \frac{3}{4}\right) = 3\left(\frac{3}{4}\right) - 2\left(\frac{3}{4}\right)^2 = \frac{9}{8}$$

Test Intersections of Region

$$f(1,1) = 3(1) - 2(1)^2 = 1$$
$$f(-1,-1) = 3(-1) - 2(-1)^2 = -5$$

Maximum at $\left(\frac{3}{4}, \frac{3}{4}\right)$ $z = \frac{9}{8}$

Minimum at $(-1,-1)$ $z = -5$

$f\left(-\frac{3}{2}, -\frac{1}{4}\right) > f(-1,-1)$, but $\left(-\frac{3}{2}, -\frac{1}{4}\right)$ is out of the domain