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Group 5

MATH 223—Worksheet 3

Due: 2023–10–13

1. Use implicit differentiation to compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the quadric surface equation
$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - x^2.$$

2. My friend's tendency to procrastinate can be modeled as follows:
   - Let $f(x, y)$ be a function representating procrastination, where x and y are my friend's energy and stress levels, respectively.
   I've noticed that my friend procastinates most when they are very tired or very stressed or when there is a big difference between energy and stress levels. So I model
   $$f(x, y) = \frac{1}{x^2} + y^2 + (y - x)^2.$$
   - Based on careful observation, I have determined that x and y are themselves a function of time of day and sleep. So, let t represent time of day in hours (where 0 is midnight), let s represent amount of sleep the night before in hours, and let $(x, y) = v(t, s)$ be the vector-valued function that determines energy and stress based on time of day and sleep.
   - Here is the v I came up with:
   $$v(t, s) = \langle 10\exp(-0.3(t - 9)^2 - 0.1(s - 8)^2), (t - s)^2\rangle.$$

   (a) As best as you can, explain why I chose to model v the way I did.

   (b) Compute the gradient ∇f.

   (c) If it is 10 AM and my friend has slept 5 hours the night before, what are their current energy and stress levels. Round to the nearest integer.

   (d) Use your previous answers to compute $\nabla(f \circ v)$.

   (e) How will my friend's procrastination level change? Describe its sensitivity to time of day and sleep at the chosen point $(t, s) = (10, 5)$.

3. The total differential of a function $f(x, y)$ can be expressed as $df = \nabla f \cdot \langle dx, dy\rangle$. For (x, y) close to (a, b), we can approximate
$$f(x, y) \approx \nabla f(a, b) \cdot (\langle x, y\rangle - \langle a, b\rangle) + f(a, b).$$
Suppose we wanted to estimate the value of $(\frac{\pi}{3})^e$. Let $f(x, y) = x^y$.

   (a) Compute ∇f. (Hint: Write $f(x, y) = x^y = \exp(y \ln x)$.)

   (b) Use $(a, b) = (1, 3)$ in the formula above to approximate $(\frac{\pi}{3})^e$. Don't use a calculator for this part.

   (c) Use a calculator to check the error between $(\frac{\pi}{3})^e$ and the approximation you found in part (b).

4. Consider the function $f(x, y) = x^2 + 2xy + y^2$. Partials

   (a) Compute all the first derivatives and second derivatives of f.

(b) Draw a contour plot for z-values 0, 1, 2, and 3. Mark the point $A : (0.5, 0.5)$.

(c) Define the following vectors $v_1 = \langle -2, 2 \rangle$, $v_2 = \langle 0, 1 \rangle$, $v_3 = \langle -1, 0 \rangle$, and $v_4 = \langle 1, 1 \rangle$.

  i. Based **only** on the contour plot, along which vector is the directional derivative of f at A the most positive? most negative? zero?

  ii. Check your answer by computing the directional derivative of f at A along each vector.

5. **Challenge Problem (Optional).** Let us prove the formulae for implicit differentiation: given an equation $f(x, y, z) = c$ for constant $c \in \mathbb{R}$,

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{f_y}{f_z}.$$

(a) Treat z as a function of x and y, and let $v(x, y, \cancel{z}) = (x, y, z(x, y))$ control the inputs to f. Compute the Jacobian J_v.

(b) Compute $\nabla(f \circ v)$ as a function of x and y, using the chain rule.

(c) Since $f \circ v$ is constant, set $\nabla(f \circ v) = 0$ and solve for $\left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle$.

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Worksheet 3 - MATH223

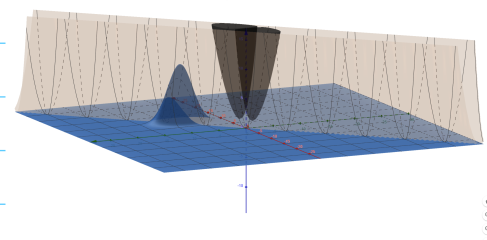
1) $\dfrac{y^2}{9} + \dfrac{z^2}{4} = 1 - x^2$ $\qquad 2z\dfrac{\partial z}{\partial x} = -8x \quad \dfrac{\partial z}{\partial x} = \dfrac{-8x}{2z} \quad \dfrac{\partial z}{\partial x} = \dfrac{-4x}{z}$

$z = 4\left(1 - x^2 - \dfrac{y^2}{9}\right)$ $\qquad 2z\dfrac{\partial z}{\partial y} = -\dfrac{8y}{9} \quad \dfrac{\partial z}{\partial y} = \dfrac{-8y}{18} \quad \dfrac{\partial z}{\partial y} = \dfrac{-4y}{9}$

$z^2 = 4 - 4x^2 - \dfrac{4y^2}{9}$

$f(x,y) = \dfrac{1}{x^2} + y^2 + (y-x)^2$ $\qquad v(t,s) = \langle 10\exp(-0.3(t-9)^2 - 0.1(s-8)^2), (t-s)^2 \rangle$

2) a) The first component of v corresponds to the x input of f, while the second component of v corresponds to y. x is represented by an equation in the form e^{-x^2}, where graphs the normal distribution, or a curve with a constant area. This means that regardless of how much sleep or stress the friend has, the relationship will still be proportional and continue to be the same. The scalars and constants in the exponent exist to position and scale the bump graphed. The y term is squared so that the y term is never negative. This makes sense in the context of the problem because stress levels cannot be negative. The graph is included here.



The blue graphs the x component of v, orange is the y, black is $f(x,y)$

b) $\nabla f = \langle f_x, f_y \rangle$, $f(x,y) = x^{-2} + y^2 + (y-x)^2$

$f_x = -2x^{-3} + 2(y-x)(-1)$

$\quad = -2x^{-3} - 2(y-x)$ $\qquad \nabla f(x,y) = \left\langle \dfrac{-2}{x^3} - 2(y-x),\ 2y + 2(y-x) \right\rangle$

$f_y = 2y + 2(y-x)$

c) $v(10,5) = \langle 10\exp(-0.3(10-9)^2 - 0.1(5-8)^2), (10-5)^2 \rangle$

$= \langle 10\exp(-0.3 - 0.1(9)), 25 \rangle = \langle 10\exp(-0.3-0.9), 25 \rangle$

$= \langle 10\exp(-1.2), 25 \rangle = \langle 3.01, 25 \rangle$

To the nearest integer: $\langle 3, 25 \rangle$ $\qquad$ energy level $= 3$

stress level $= 25$

d) $\nabla(f \circ v) = \nabla f \cdot J_v$ $\quad J_v = \left[\frac{\partial v}{\partial t} \,\Big|\, \frac{\partial v}{\partial s} \right]$

$\nabla f = \langle \frac{-2}{x^3} - 2(y-x),\ 2y + 2(y-x) \rangle$

$\nabla f(3, 25) = \langle -44.07, 94 \rangle$

$$J_v = \begin{bmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \end{bmatrix} \Rightarrow \begin{aligned} \frac{\partial x}{\partial t} &= (10\exp(-0.3(t-9)^2 - 0.1(s-8)^2))(-0.6(t-9)) \\ \frac{\partial y}{\partial t} &= 2(t-s) \\ \frac{\partial x}{\partial s} &= (10\exp(-0.3(t-9)^2 - 0.1(s-8)^2))(-0.2(s-8)) \\ \frac{\partial y}{\partial s} &= 2(t-s)(-1) = -2(t-s) \end{aligned}$$

$$J_v(10,5) = \begin{bmatrix} \frac{\partial x}{\partial t}(10,5) & \frac{\partial x}{\partial s}(10,5) \\ \frac{\partial y}{\partial t}(10,5) & \frac{\partial y}{\partial s}(10,5) \end{bmatrix} \quad \begin{aligned} \frac{\partial x}{\partial t}(10,5) &= -1.8 \\ \frac{\partial y}{\partial t}(10,5) &= 10 \\ \frac{\partial x}{\partial s}(10,5) &= 1.8 \\ \frac{\partial y}{\partial s}(10,5) &= -10 \end{aligned}$$

$J_v(10,5) = \begin{bmatrix} -1.8 & 1.8 \\ 10 & -10 \end{bmatrix}$ and $\nabla f(v(10,5)) = \langle -44.07, 94 \rangle$

$$\nabla(f \circ v) = \nabla f(v(t,s)) \cdot J_v = \langle -44.07, 94 \rangle \cdot \begin{bmatrix} -1.8 & 1.8 \\ 10 & -10 \end{bmatrix}$$

$$= \langle 1019.326, -1019.326 \rangle$$

$$\nabla(f \circ v) = \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \rangle = \langle 1019.326, -1019.326 \rangle$$

e) The friend is equally affected by the time of day and sleep. The two values in the gradient have equal absolute values, but the friend's procrastination is impacted differently at t=10 and s=5. They will procrastinate more with a change of time of day, and will procrastinate less with more hours of sleep. Their procrastination will be affected by an amount of equal magnitude, in opposite ways.

3) a) $f(x,y) = x^y = e^{y\ln x}$

$f_x = e^{y\ln x}\left(\frac{y}{x}\right) = x^y\left(\frac{y}{x}\right) = x^y y x^{-1} = yx^{y-1}$ ← power rule $\quad \nabla f = \langle yx^{y-1}, x^y \ln x \rangle$

$f_y = e^{y\ln x}(\ln x) = x^y(\ln x)$ ← $\frac{d}{dx}(a^x)$ derivative

b) $f(x,y) \approx \nabla f(a,b) \cdot (\langle x,y \rangle - \langle a,b \rangle) + f(a,b)$, given $(a,b) = (1,3)$, approx. $\left(\frac{\pi}{3}\right)^e$

$f(1,3) = 1^3 = 1$, $\nabla f(1,3) = \langle 3(1)^2, 1\ln 1 \rangle = \langle 3, 0 \rangle$

$f(\frac{\pi}{3}, e) = ?$ $\quad f(\frac{\pi}{3}, e) \approx \nabla f(1,3) \cdot (\langle \frac{\pi}{3}, e \rangle - \langle 1,3 \rangle) + f(1,3)$ $\quad f(\frac{\pi}{3}, e) \approx \pi - 2$

$= \langle 3, 0 \rangle \cdot \langle \frac{\pi}{3} - 1, e - 3 \rangle + 1$

$= (\pi - 3) + 0 + 1 = \pi - 2$

c) via calculator: $f(\frac{\pi}{3}, e) = (\frac{\pi}{3})^e = 1.1336$

approximation of $f(\frac{\pi}{3}, e) = 1.14159$

The error (|difference|) is about 0.00804, very low!

4) $f(x,y) = x^2 + 2xy + y^2$

a) find $f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$

$f_x = 2x + 2y$ $\qquad f_{xx} = 2$

$f_y = 2x + 2y$ $\qquad f_{yy} = 2$

$f_{xy} = 2$ $\qquad f_{yx} = 2$

b) $z = 0, 1, 2, 3$, $A: (0.5, 0.5)$ $\qquad z = x^2 + 2xy + y^2$

$0 = x^2 + 2xy + y^2$

$$x = \frac{-2y \pm \sqrt{4y^2 - 4(1)(y^2)}}{2} = \frac{-2y \pm \sqrt{0}}{2}$$

$x = -y$, $y = -x$

$1 = x^2 + 2xy + y^2$

$$x = \frac{-2y \pm \sqrt{4y^2 - 4(1)(y^2 - 1)}}{2} = \frac{-2y \pm \sqrt{4}}{2}$$

$x = -y \pm 1$, $y = -x \pm 1$

$2 = x^2 + 2xy + y^2$

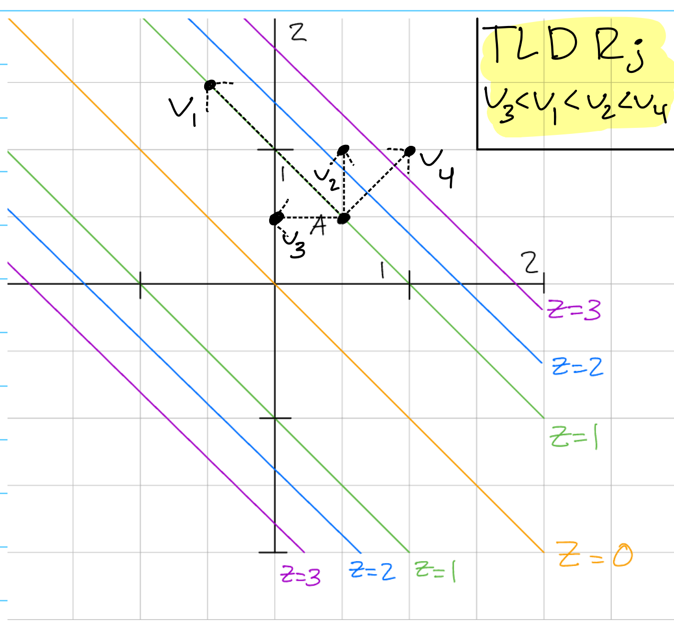
$$x = \frac{-2y \pm \sqrt{4y^2 - 4(1)(y^2 - 2)}}{2} = \frac{-2y \pm \sqrt{8}}{2}$$

$x = -y \pm \sqrt{2}$, $y = -x \pm \sqrt{2}$

$3 = x^2 + 2xy + y^2$

$$x = \frac{-2y \pm \sqrt{4y^2 - 4(1)(y^2 - 3)}}{2} = \frac{-2y \pm \sqrt{12}}{2}$$

$x = -y \pm \sqrt{3}$ $\qquad y = -x \pm \sqrt{3}$



TLDR; $v_3 < v_1 < v_2 < v_4$

c) $v_1 = \langle -2, 2 \rangle$, $v_2 = \langle 0, 1 \rangle$, $v_3 = \langle -1, 0 \rangle$, $v_4 = \langle 1, 1 \rangle$

i) As seen by the plot to the left, v_1 points in the direction of the contour line $z=1$. z is constant along this direction, so this directional derivative should equal 0. v_2 and v_4 both point in directions where the z value is increasing, as seen where the z value goes from 1 to 2 for v_2 and 1 to 3+ for v_4. Due to the contour values increasing more at a more rapid pace for v_4, it is where the Directional derivative is most positive. v_3 is in the direction of $z=0$, a lower contour line, so this is where the directional derivative is most negative.

ii) Let $\vec{u}_n$ be the unit vectors of Defined vectors, $\vec{v}_n$.

$$\vec{u}_1 = \frac{\langle -2, 2\rangle}{\sqrt{4+4}} = \frac{\langle -2, 2\rangle}{\sqrt{8}} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$$

$\vec{u}_2 = \langle 0, 1\rangle$
$\vec{u}_3 = \langle -1, 0\rangle$
These are already unit vectors

$$\vec{u}_4 = \frac{\langle 1, 1\rangle}{\sqrt{1+1}} = \frac{\langle 1, 1\rangle}{\sqrt{2}} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$$

$\nabla f = \langle 2x+2y, 2y+2x\rangle$ using part A

$\nabla f(A) = \nabla f(0.5, 0.5)$

$= \langle 2(0.5)+2(0.5), 2(0.5), 2(0.5)\rangle$

$\nabla f(0.5, 0.5) = \langle 2, 2\rangle$

$D_{\vec{u}_1} f = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle \cdot \langle 2, 2\rangle$

$= -\sqrt{2} + \sqrt{2} = 0$

$D_{\vec{u}_2} f = \langle 0, 1\rangle \cdot \langle 2, 2\rangle = 2$

$D_{\vec{u}_3} f = \langle -1, 0\rangle \cdot \langle 2, 2\rangle = -2$

$D_{\vec{u}_4} f = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle \cdot \langle 2, 2\rangle$

$= \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

$D_{\vec{u}_3} f < D_{\vec{u}_1} f < D_{\vec{u}_2} f < D_{\vec{u}_4} f$

Because unit vectors $\vec{u}_n$ correspond to those defined in 4c, my assumptions of the directional derivatives is true.

5) Challenge Problem

$f(x,y,z) = c$ for $c \in \mathbb{R}$, Prove $\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}$ and $\frac{\partial z}{\partial y} = -\frac{f_y}{f_z}$

a) Let $z = f(x,y)$ and $v(x,y) = (x, y, z(x,y))$

$$J_v = \left[\frac{\partial v}{\partial x} \;\middle|\; \frac{\partial v}{\partial y} \right] = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix}$$

b) $\nabla(f \circ v) = \nabla f \cdot J_v$

$\nabla f = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$

$$\nabla(f \circ v) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix}$$

$$\nabla(f \circ v) = \left\langle f_x(x,y,z) + f_z(x,y,z)\frac{\partial z}{\partial x},\; f_y(x,y,z) + f_z(x,y,z)\frac{\partial z}{\partial y}\right\rangle$$

c) It is known that $f(x,y,z) = c$, a constant, so $f \circ v$ is also constant as v controls the inputs to f. We can now set $\nabla(f \circ v) = 0$ and solve for $\left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right\rangle$.

$\langle 0, 0\rangle = \left\langle f_x + f_z \frac{\partial z}{\partial x}, f_y + f_z \frac{\partial z}{\partial y}\right\rangle$

Note: Partials from B have been condensed to save space

$0 = f_x + f_z \frac{\partial z}{\partial x}$ $\quad$ $0 = f_y + f_z \frac{\partial z}{\partial y}$

$-f_x = f_z \frac{\partial z}{\partial x}$ $\quad$ $-f_y = f_z \frac{\partial z}{\partial y}$

$\frac{\partial z}{\partial x} = \frac{-f_x}{f_z}$ $\quad$ $\frac{\partial z}{\partial y} = \frac{-f_y}{f_z}$

Solving for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ proves the shortcut for implicit differentiation.