# Build Your Own Exam

MATH223 - Roland Baumann

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## Author Note

This document covers topics spanning the Calculus 3 curriculum, up to and *excluding* vector fields. This is by no means a comprehensive examination of your abilities as a student, and it is important to remember that some things I find important may not be stressed by your professor, and vice versa. A solution manual can be found at the end of this document, but it should only be referenced in the event that the exam has been fully completed. Best of luck in the rest of your calculus journey.

Each Question has an assigned point value, and partial credit is given in some parts. If point distribution is not specified, it is up to the student to decide what, and how many, points should be lost. The exam has **60 points** in total, and is **13 Questions** in Length.

**Build Your Own Exam 1:** *4 Questions - 20 Points*

1. **(4pts)** Given the two vectors $u = \langle 2, 5, -9 \rangle$ and $v = \langle 1, 2, 3 \rangle$, compute…
   a. The dot product

   b. The cross product

2. **(8pts)** Given the vector-valued function $r(t) = \langle e^{2t}, \cos(4t), 5t^3 \rangle$ the position function of a particle…
   a. Determine the particle's velocity and acceleration functions. **(3pts)**

   b. Determine the particle's displacement on the interval $[0,2\pi]$. **(2pts)**

   c. Determine the distance traveled of the particle on the interval $[0,2\pi]$. **(3pts)**

3. **(3pts)** Find the unit tangent vector for the function $r(t) = \langle 6\cos(2t), \sin(13t), 5\cos(t) \rangle$

4. **(5pts)** Given the lines $L_1(t) = \langle 3t-7, 2t-3, 4t+5 \rangle$ and $L_2(t) = \langle t-1, 3t-5, 2t+1 \rangle$
   a. Write the symmetric equations of each line, if possible. If this cannot be done, explain why. **(2pts)**

   b. Determine whether the lines intersect at a point, are parallel, skew, or the same line. **(3pts)**

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**Build Your Own Exam 2:** *4 Questions - 20 Points*

1. **(5pts)** Given the function $f(x) = yx^4 - x^3 + 2xy^2 - 4y$, compute…
   a. The Gradient $\nabla f$ **(2pts)**

   b. The Directional Derivative, $D_u f$, at point P = (1,3) in the direction $\langle 3, 4\rangle$. **(3pts)**

2. **(7pts)** Let $z = f(x, y) = x^2y - 2xy^2 - 8xy$, $v(s, t) = \langle 2s^2t, 3s^2 - t^2\rangle$,and $(s, t) = (2, 3)$ Compute $\frac{\delta z}{\delta s}$ and $\frac{\delta z}{\delta t}$ using the *Multivariable Chain Rule*.

3. **(3pts)** Let $5 = zexp(x^2) - 3y$ define z implicitly as a function of x and y. Compute $\frac{\delta z}{\delta x}$ and $\frac{\delta z}{\delta y}$ using *Implicit Differentiation*.

4. **(5pts)** Given the function $f(x) = 2xy - 3x^2 + xy^2 + 3$, find the critical points and classify them with the ***Second Derivative Test***.

**Build Your Own Exam 3:** *4 Questions - 20 Points*

1. Consider a region $R$ bounded by 2 circles and the x-axis. Let the inner radius be 3, and the outer radius be 4… **(6pts)**
   a. Graph the region $R$, with correct labels and axes, and *set up* a double integral to find the area of the region. **(2pts)**

   b. Let $R$ describe a lamina with density function $\delta(x, y) = x^2 + y^2$. Given this function, calculate the mass of this lamina using *Polar Coordinates*. **(4pts)**

2. Consider a region $R$, a rectangle with one corner at the origin and the opposite corner at the point (2, 3). Let this region describe a lamina of constant density, $\delta(x, y) = 3$. **(5pts)**
   a. Find the center of mass of this lamina using *Double Integration*. **(4pts)**

   b. Briefly explain why the center of mass you found makes sense for this lamina. **(1pt)**

**Build Your Own Exam 3:** *4 Questions - 20 Points*

3. Consider the function $f(x) = x^3 - 2x + 4y$, bounded by the curve $g(x) = x^3$ the curve $h(x) = -x^2 + 2$, and the lines $x = 0$ and $y = 0$. **(6pts)**
   a. Determine the Surface Area of the function $f(x)$ over this region. **(5pts)**

   b. Briefly Explain why you chose the order of integration that you did. **(1pt)**

4. Use Cylindrical Coordinates to prove that a Cylinder with radius *r* and height *h* has a volume of $V = \pi r^2 h$. **(2pts).**

5. In your own words, explain why it is necessary to put an extra term, such as r, in the integrand created when switching coordinate systems. **(1pt)**

Build Your Own Exam **Solutions**

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This ends the exam portion of this document. You will find solutions to the previous questions in the latter part of this document. Grade yourself fairly, and remember that mistakes on my end are possible, so do not treat this solution key as 100% correct.

**Build Your Own Exam 1:** *4 Questions - 20 Points* Solutions

1. **(4pts)** Given the two vectors u = ⟨2, 5, -9⟩and v = ⟨1, 2, 3⟩, compute…
   a. The dot product

   $\vec{u}\cdot\vec{v} = 2(1)+5(2)+(-9)3 = -15$

   b. The cross product

   $\vec{u}\times\vec{v} = \langle 2,5,-9\rangle \times \langle 1,2,3\rangle = \langle 15-(-18), -(6-(-9)), 4-5\rangle = \langle 33,-15,-1\rangle$

2. **(8pts)** Given the vector-valued function r(t) =⟨$e^{2t}$ ,cos(4t) ,$5t^3$⟩the position function of a particle…
   a. Determine the particle's velocity and acceleration functions. **(3pts)**

   $\vec{r}\,'(t) = \vec{v}(t) = \langle 2e^{2t}, 4\cos(4t), 15t^2\rangle$

   $\vec{r}\,''(t) = a(t) = \langle 4e^{2t}, 16\cos(4t), 30t\rangle$

   b. Determine the particle's displacement on the interval [0,2π]. **(2pts)**

   Distplacement $= \vec{r}(b)-\vec{r}(a) = \langle e^{2(2\pi)}, \cos(4\cdot 2\pi), 5(2\pi)^3\rangle - \langle e^{2(0)}, \cos(4\cdot 0), 5(0)^3\rangle$

   $= \langle e^{4\pi}-0, 1-1, 40\pi^3-0\rangle = \langle e^{4\pi}, 0, 40\pi^3\rangle$

   c. Determine the distance traveled of the particle on the interval [0,2π]. **(3pts)**

   Distance Traveled $= \int_a^b \|\vec{v}(t)\|\,dt = \int_0^{2\pi}\sqrt{(2e^{2t})^2+(4\cos(4t))^2+(15t^2)^2} = 286790.62$

3. **(3pts)** Find the unit tangent vector for the function r(t) =⟨6cos(2t), sin(13t), 5cos(t)⟩

   $\vec{r}\,'(t) = \langle -12\sin(2t), 13\cos(13t), -5\sin t\rangle$

   $\|\vec{r}\,'(t)\| = \sqrt{144\sin^2(2t)+169\cos^2(13t)+25\sin^2 t}$

   $\vec{T}(t) = \dfrac{\langle -12\sin 2t, 13\cos 13t, -5\sin t\rangle}{\sqrt{144\sin^2(2t)+25\sin^2 t+169\cos^2 13t}}$

   Cannot use trig identity b/c angles are not the same!

4. **(5pts)** Given the lines $L_1$(t) =⟨3t-7, 2t- 3, 4t+5⟩and $L_2$(t) = ⟨t-1,3t-5, 2t+1⟩
   a. Write the symmetric equations of each line, if possible. If this cannot be done, explain why. **(2pts)**

   $L_1$: $x=3t-7$, $y=2t-3$, $z=4t+5$ $\qquad \dfrac{x+7}{3}=\dfrac{y+3}{2}=\dfrac{z-5}{4}$

   $L_2$: $x=t-1$, $y=3t-5$, $z=2t+1$ $\qquad \dfrac{x+1}{1}=\dfrac{y+5}{3}=\dfrac{z-1}{2}$

   Symmetric Only DNE when lines have constants ending & vel-vector to y, or z.

   b. Determine whether the lines intersect at a point, are parallel, skew, or the same line. **(3pts)**

   Parallel NO!

   $\dfrac{\langle 3,2,4\rangle}{\sqrt{3^2+2^2+4^2}} = \langle .557, \ldots$

   $\dfrac{\langle 1,3,2\rangle}{\sqrt{1^2+3^2+2^2}} = \langle .267\ldots$

   First terms ≠ so you can stop

   This means they're not the same line either

   Intersect or Skew

   $3t-7=s-1$

   $2t-3=3s-5$

   $4t+5=2s+1$

   $s=3t-6$

   $2t-3=3(3t-6)-5$

   $2t-3=9t-18-5$

   $20=7t \quad t=\frac{20}{7}$

   $s=3(\frac{20}{7})-6=\frac{18}{7}$

   $t=\frac{20}{7}, \quad s=\frac{18}{7}$

   $L_1(\frac{20}{7}) = \langle \frac{11}{7}, \frac{19}{7}, \frac{115}{7}\rangle$

   $L_2(\frac{18}{7}) = \langle \frac{11}{7}, \frac{19}{7}, \frac{43}{7}\rangle$

   $\frac{115}{7} \neq \frac{43}{7} \therefore$ lines are skew

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**Build Your Own Exam 2:** *4 Questions - 20 Points*

1. **(5pts)** Given the function $f(x) = yx^4 - x^3 + 2xy^2 - 4y$, compute…
   a. The Gradient $\nabla f$ **(2pts)**

$\nabla f = \langle 4yx^3 - 3x^2 + 2y^2, \ x^4 + 4xy - 4 \rangle$

   b. The Directional Derivative, $D_u f$, at point P = (1,3) in the direction $\langle 3, 4\rangle$. **(3pts)**

$\nabla f(1,3) = \langle 4(3)(1)^3 - 3(1)^2 + 2(3)^2, \ 1^4 + 4(1)(3) - 4 \rangle$

$= \langle 12 - 3 + 18, \ 1 + 12 - 4 \rangle = \langle 27, 9 \rangle = 9\langle 3, 1 \rangle$

$\hat{u} = \frac{\langle 3, 4 \rangle}{\sqrt{3^2+4^2}} = \left\langle \frac{3}{\sqrt{25}}, \frac{4}{\sqrt{25}} \right\rangle \qquad D_{\hat{u}} f = 9\langle 3, 1\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = 9\left(\frac{9}{5} + \frac{4}{5}\right) = \frac{9(13)}{5}$

2. **(7pts)** Let $z = f(x, y) = x^2y - 2xy^2 - 8xy$, $v(s, t) = \langle 2s^2t, 3s^2 - t^2 \rangle$, and $(s, t) = (2, 3)$ Compute $\frac{\delta z}{\delta s}$ and $\frac{\delta z}{\delta t}$ using the *Multivariable Chain Rule*.

$\nabla f = \langle 2xy - 2y^2 - 8y, \ x^2 - 4xy - 8x \rangle$

$J_v = \left[ \frac{\partial v}{\partial s} \ \Big| \ \frac{\partial v}{\partial t} \right] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$

$(s,t) = (2,3) \Rightarrow v(2,3) = \langle 2(2)^2(3), \ 3(2)^2 - (3)^2 \rangle$

$= \langle 24, 3 \rangle$

$\therefore x = 24, \ y = 3$

$= \begin{bmatrix} 4st & 2s^2 \\ 6s & -2t \end{bmatrix}$

$\nabla f(72, 7) = \langle 2(24)(3) - 2(3)^2 - 8(3), \ 24^2 - 4(24)(3) - 8(24) \rangle$

$= \langle 102, 96 \rangle$

$J_v(2,3) = \begin{bmatrix} 24 & 8 \\ 12 & -6 \end{bmatrix}$

$\nabla f \circ v = \left\langle \frac{\partial z}{\partial s}, \frac{\partial z}{\partial t} \right\rangle = \nabla f \cdot J_v$

$\nabla f \circ v = \langle 102, 96 \rangle \begin{bmatrix} 24 & 8 \\ 12 & -6 \end{bmatrix} = \langle 3600, 240 \rangle$

$\therefore \frac{\partial z}{\partial s} = 3600, \quad \frac{\partial z}{\partial t} = 240$

3. **(3pts)** Let $5 = z exp(x^2) - 3y$ define z implicitly as a function of x and y. Compute $\frac{\delta z}{\delta x}$ and $\frac{\delta z}{\delta y}$ using *Implicit Differentiation*.

$5 = z\exp(x^2) - 3y, \quad \text{let } f = z\exp(x^2) - 3y - 5$

$\frac{\partial f}{\partial z} = \exp(x^2), \quad \frac{\partial f}{\partial x} = 2xz\exp(x^2), \quad \frac{\partial f}{\partial y} = -3$

$\frac{\partial z}{\partial x} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = \frac{-(2xz\exp(x^2))}{\exp(x^2)} = -2xz$

$\frac{\partial z}{\partial y} = \frac{-\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = \frac{-(-3)}{\exp(x^2)} = \frac{3}{\exp(x^2)}$

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4. **(5pts)** Given the function $f(x) = 2xy - 3x^2 + xy^2 + 3$, find the critical points and classify them with the *Second Derivative Test*.

$\nabla f = \langle 2y - 6x + y^2, 2x + 2xy \rangle$

$2y - 6x + y^2 = 0$

$2x + 2xy = 0$

$2x(1+y) = 0$

$x = 0, y = -1$

$x = 0$: $2y + y^2 = 0$

$y(2+y) = 0$

$y = 0, y = -2$

$y = -1$: $-2 - 6x + 1 = 0$

$-6x = 1$

$x = -\frac{1}{6}$

2nd Derivative Test / Hessian

$H = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$

$H = \begin{bmatrix} -6 & 2+2y \\ 2+2y & 2x \end{bmatrix}$

$|H| = D = -6(2x) - (2+2y)^2$

Test Critical Points

$D(0,0) = -4$

$D(0,-2) = -2$

$D(-\frac{1}{6},-1) = 2$, $f_{xx} = -6$

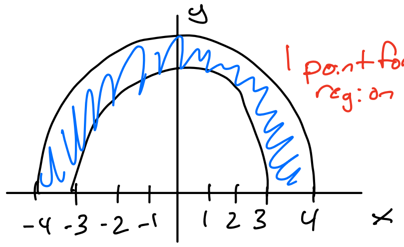
$(0,0)$ is a saddle point b/c $D<0$

$(0,-2)$ is a saddle point b/c $D<0$

$(-\frac{1}{6},-1)$ is a local max b/c $D>0$ and $f_{xx}<0$

**Build Your Own Exam 3:** *4 Questions - 20 Points*

1. Consider a region $R$ bounded by 2 circles and the x-axis. Let the inner radius be 3, and the outer radius be 4… **(6pts)**
   a. Graph the region $R$, with correct labels and axes, and *set up* a double integral to find the area of the region. **(2pts)**




1 point for region

$3 \le r \le 4$
$0 \le \theta \le \pi$

$A = \int_0^{\pi}\int_3^{4} r\,dr\,d\theta$ 1 point for integrand

   b. Let $R$ describe a lamina with density function $\delta(x, y) = x^2 + y^2$. Given this function, calculate the mass of this lamina using *Polar Coordinates*. **(4pts)**

$\text{Mass} = \int_0^{\pi}\int_3^{4} \delta(r,\theta)\,dr\,d\theta$

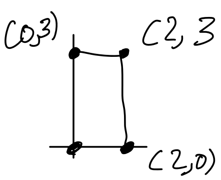
$\delta(r,\theta) = r^2\cos^2\theta + r^2\sin^2\theta = r^2$ 2 points for conversion to polar

$\text{Mass} = \int_0^{\pi}\int_3^{4} r^2(r)\,dr\,d\theta = \frac{1}{4}\int_0^{\pi} r^4\Big|_3^4\,d\theta = \frac{1}{4}\int_0^{\pi} 175\,d\theta = \frac{175\pi}{4}$

1 point for setting up correct integral

1 points for evaluating correctly

2. Consider a region $R$, a rectangle with one corner at the origin and the opposite corner at the point $(2, 3)$. Let this region describe a lamina of constant density, $\delta(x, y) = 3$. **(5pts)**
   a. Find the center of mass of this lamina using *Double Integration*. **(4pts)**




$\delta(x,y) = 3$

$\text{Mass} = \int_0^3\int_0^2 3\,dx\,dy = 3\int_0^3 2\,dy = 3(6) = 18$ 1 point for M

$\text{Mass}_x = \int_0^3\int_0^2 3x\,dx\,dy = \frac{3}{2}\int_0^3 x^2\Big|_0^2 dy = 6\int_0^3 dy = 6y\Big|_0^3 = 18$

$\text{Mass}_y = \int_0^3\int_0^2 3y\,dx\,dy = 3\int_0^3 xy\Big|_0^2 dy = 6\int_0^3 y\,dy = \frac{6}{2}(y^2)\Big|_0^3 = 3(3^2) = 27$

1 point for y-axis moment, 1 for x-axis

$(\bar{x}, \bar{y}) = \left(\frac{18}{18}, \frac{27}{18}\right) = (1, 1.5)$

1 point for find center of mass

   b. Briefly explain why the center of mass you found makes sense for this lamina. **(1pt)**

Because the region is a rectangle of constant density, its center of mass should be in the center of the region.

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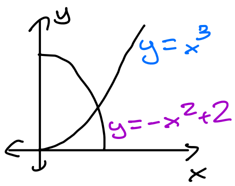
**Build Your Own Exam 3:** *4 Questions - 20 Points*

3. Consider the function $f(x) = x^3 - 2x + 4y$, bounded by the curve $g(x) = x^3$ the curve $h(x) = -x^2 + 2$, and the lines $x = 0$ and $y = 0$. **(6pts)**
   a. Determine the Surface Area of the function $f(x)$ over this region. **(5pts)**




$x^3 = -x^2+2$

$x^3+x^2-2=0$

$x=1$, $g(1)=1$

$f(x) = x^3 - 2x + 4y$

$\nabla f = \langle 3x^2-2, 4\rangle$

1 point for Gradient

1 point for bounds

1 point for integrand

1 point for evaluation

$$S = \int_0^1\int_{x^3}^{-x^2+2} \sqrt{1+(3x^2-2)^2+(4)^2}\,dy\,dx$$

$$= \int_0^1 y\sqrt{17+(3x^2-2)^2}\,\Big|_{x^3}^{-x^2+2} dx$$

$$= \int_0^1 \left((-x^2+2)\sqrt{17+(3x^2-2)^2} - (x^3)\sqrt{17+(3x^2-2)^2}\right) dx$$

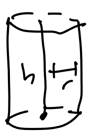
$$= 6.219$$

1 point for answer

   b. Briefly Explain why you chose the order of integration that you did. **(1pt)**

Because the gradient has only one variable in it, the first integral should be in terms of the other variable to make calculations as simple as possible

4. Use Cylindrical Coordinates to prove that a Cylinder with radius *r* and height *h* has a volume of $V = \pi r^2 h$. **(2pts).**




1 point for setup

$$V = \int_0^h\int_0^{2\pi}\int_0^r r\,dr\,d\theta\,dh = \frac{1}{2}\int_0^h\int_0^{2\pi} r^2\,d\theta\,dh$$

$$= \pi\int_0^h r^2\,dh = \pi r^2 h$$

1 point for evaluating

5. In your own words, explain why it is necessary to put an extra term, such as r, in the integrand created when switching coordinate systems. **(1pt)**

Terms, like $r^2 \sin\theta$ or $r$, need to be added when switching coordinate systems to account for space dilation.

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