Group 7 Trevor Swan

MATH 223 — Worksheet 1

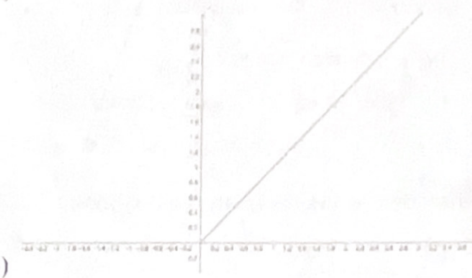
Due: 2023–09–08

1. For each of the following vector-valued functions, indicate the corresponding graph, and explain how you know.

- $r_1(t) = (9t, 9t)$

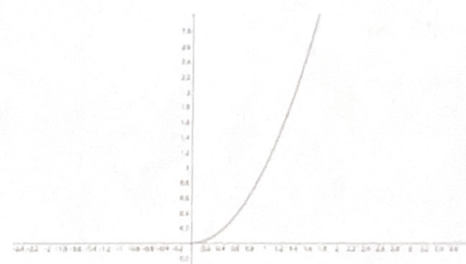
(a)

- $r_2(t) = (t^2, t^4)$



(b)

- $r_3(t) = 9(\cos^2 t, 1 - \sin^2 t)$



- $r_4(t) = (2\cos t, \sin t + \frac{1}{2}t)$

(c)

2. For each of the following maps $(x, y) = F(u, v)$, figure out what the map does to horizontal and vertical lines, and sketch an example. For example, the map $F(u, v) = (x(u, v), y(u, v)) = (u - v, 2u)$ sends the vertical line (u, b) to $(u - b, 2u)$ which corresponds to $y = 2(x + b.)$

   (a) (polar) $F(r, \theta) = (r\cos\theta, r\sin\theta)$.

   (b) (hyperbolic) $F(u, v) = (ve^u, ve^{-u})$, for $v \geq 0$.

   (c) (hyperbolic trig) $F(u, v) = (u\cosh v, u\sinh v)$.

   (d) $F(u, v) = (\frac{1}{2}(u + v), \frac{1}{2}(u - v))$.

3. Electrical power is measured by $P = VI$, where P is power in Watts (Joules per second), V is voltage in Volts, and I is current in Amps. Voltage is also related to current by the identity $V = IR$, where R is resistance in Ohms. (You can ignore the units entirely.)

   (a) Sketch the following curves in V-I-coordinates: $P = 1$; $P = 4$; $R = e$; $R = e^2$.

(b) The hyperbolic map takes (V, I) to $(u, v) = (\frac{1}{2}\ln\frac{V}{I}, \sqrt{VI}) = (\frac{1}{2}\ln R, \sqrt{P})$. Sketch the following curves in hyperbolic coordinates: $P = 1$; $P = 4$; $R = e$; $R = e^2$.

4. For each of the following domains D, sketch the domain D in cartesian coordinates. Choose either **polar** or **hyperbolic trig** coordinates, and sketch the domain in those coordinates.

   (a) D is the annulus centered at the origin with inner radius 1 and outer radius $\sqrt{2}$.

   (b) D is the region between $y = \frac{\sqrt{3}}{3}x$, $y = \sqrt{3}x$, $x^2 + y^2 = 4$, and $x^2 + y^2 = 9$.

   (c) D is the region between $y = \frac{3}{5}x$, $y = \frac{4}{5}x$, and $y^2 = x^2 - 4$.

5. Let $r_1(t) = \langle 3, 3, 4\rangle t + \langle -6, -5, -8\rangle$ and $r_2(t) = \langle 1, 0, 1\rangle t + \langle 0, 1, 0\rangle$ be two lines, and let $r(t) = r_1(t) - r_2(t)$ be their difference at time t.

   (a) Compute $r'(t)$.

   (b) For what t is $\|r(t)\|$ the smallest?

   (c) Calculate the distance between r_1 and r_2 as demonstrated in Chapter 11.5 of the textbook.

   (d) Did your answers to (b) and (c) match? Explain why or why not.

   (e) Find the area of the triangle formed by r, r_1, and r_2.

6. Here is a parametrized curve that traces out a helix: $r(t) = \langle 2\cos t, 2\sin t, 5t\rangle$.

   (a) Compute $r'(t)$ and $r''(t)$.

   (b) Give an equation for the tangent line to r at $t = \pi$.

   (c) Compute the arclength function $L(t)$ for the curve $r(t)$.

Recommeded problems from the textbook:

11.4 All.

11.6 7–19 and 27.

12.1 All.

1. $r_1(t) = (9t, 9t)$ b Linear Equation

$x = 9t \quad y = 9t \quad y = x$

$r_2(t) = (t^2, t^4)$ c Quadratic

$x = t^2 \quad y = t^4 \quad y = x^2$

$r_3(t) = 9(\cos^2 t, 1-\sin^2 t)$ b Linear Equation

$x = \cos^2 t \quad y = 1-\sin^2 t = \cos^2 t \quad y = x$

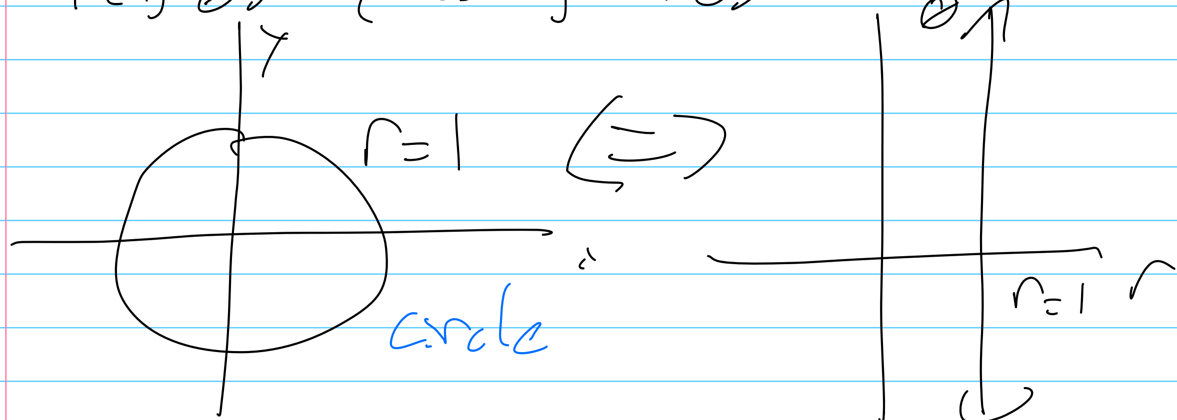
$r_4(t) = (2\cos t, \sin t + \frac{1}{2}t)$ a

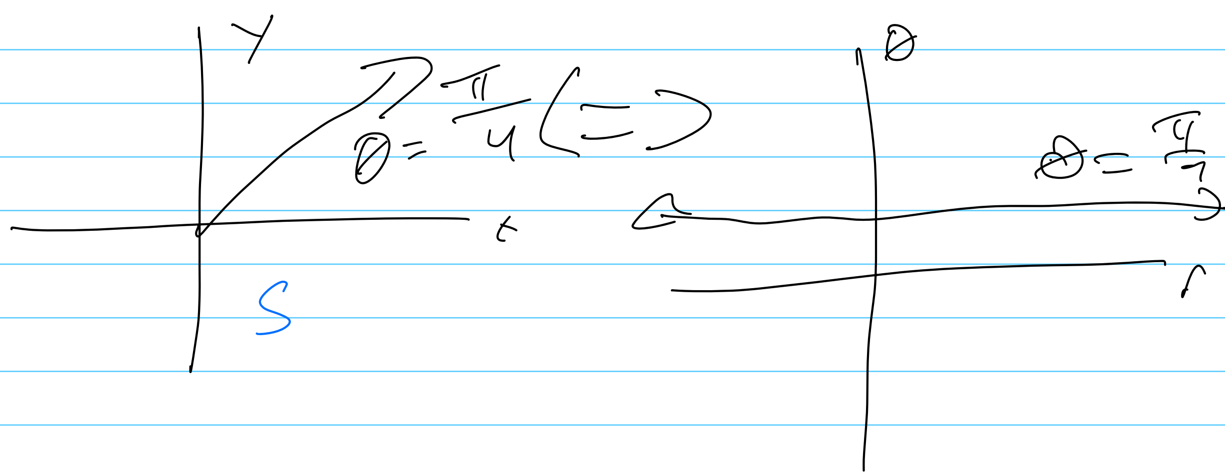
t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	2	0	-2	0	2
y	0	$1+\frac{\pi}{4}$	$\frac{\pi}{2}$	$-1+\frac{3\pi}{4}$	π

Plotted points

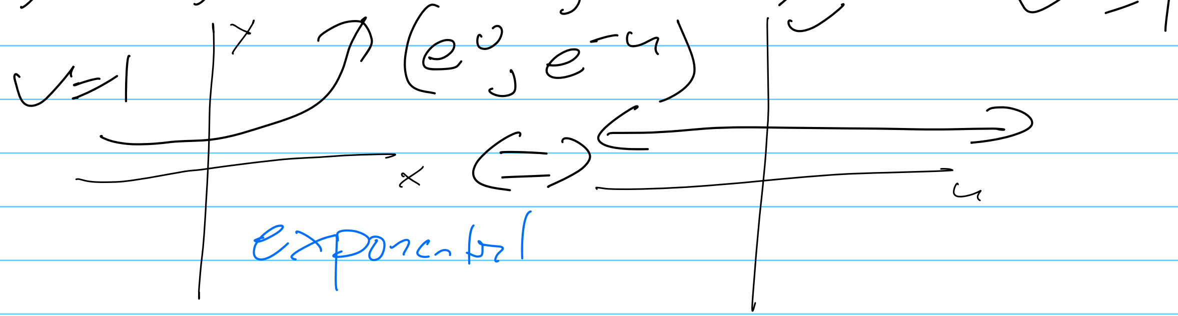
2. $F(r, \theta) = (r\cos\theta, r\sin\theta)$

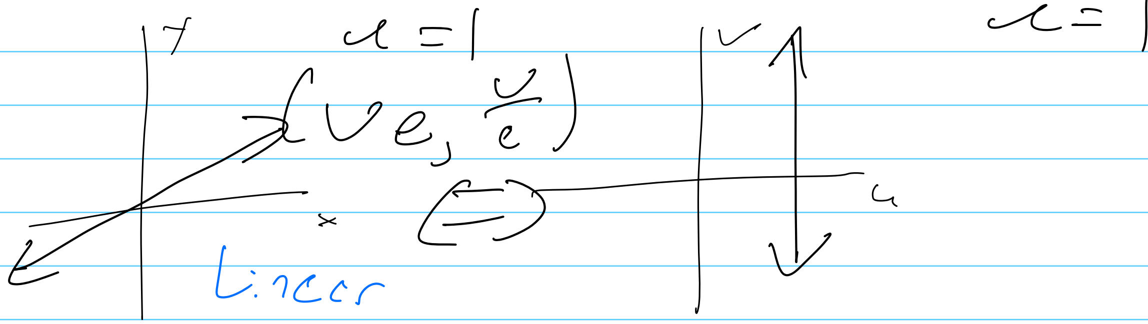
$F(1, \theta) = (\cos\theta, \sin\theta)$



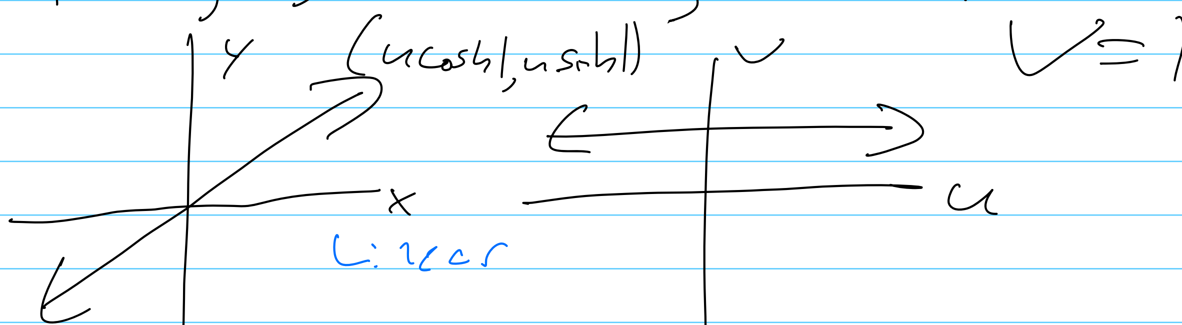


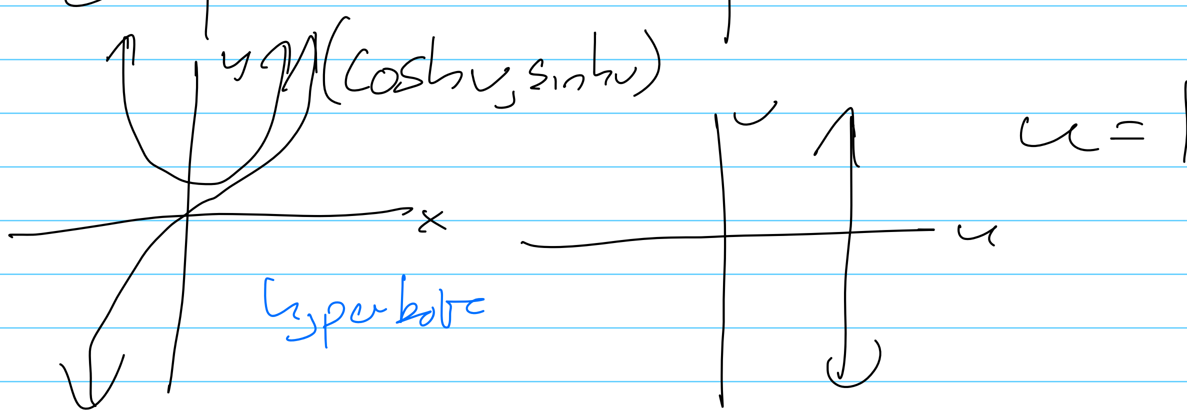
b) $F(u,v) = (ve^{u}, ve^{-u})$





c) $F(u,v) = (u\cosh v, u\sinh v)$

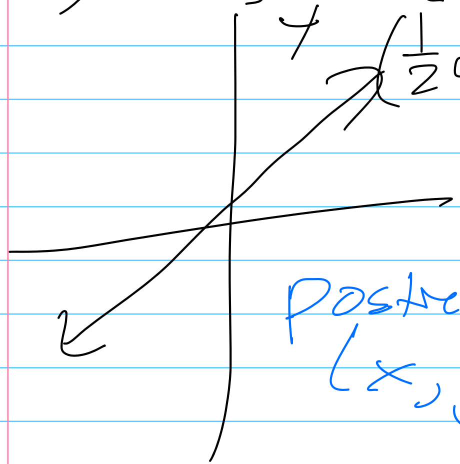


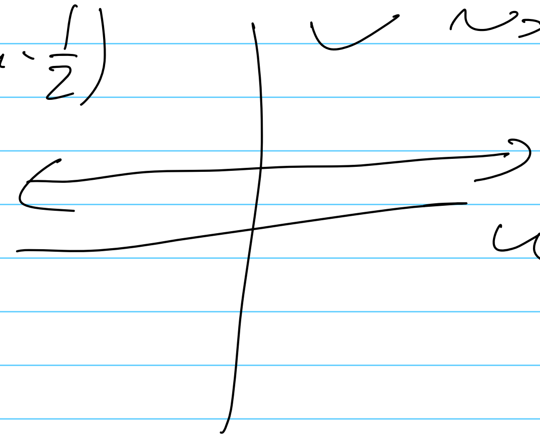


d) $F(u,v) = \left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right)$

$v = 1$

$\left(\frac{1}{2}u + \frac{1}{2}, \frac{1}{2}u - \frac{1}{2}\right)$



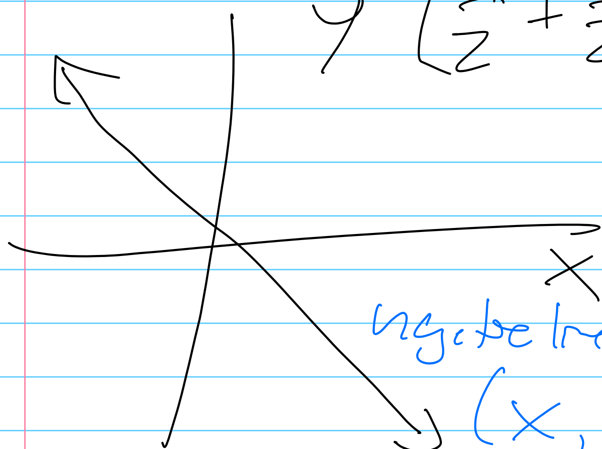


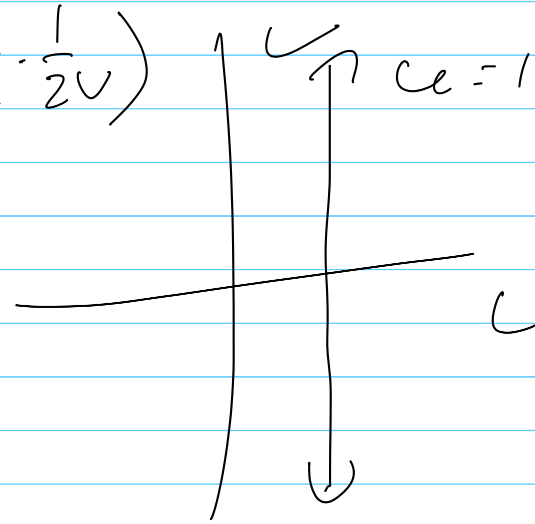
positive linear

(x, y)

$\left(\frac{1}{2} + \frac{1}{2}v, \frac{1}{2} - \frac{1}{2}v\right)$

$u = 1$



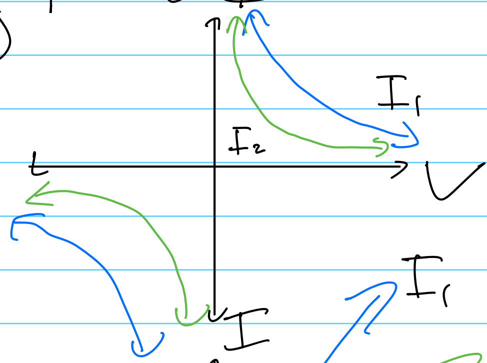


negative linear

$(x, -y)$

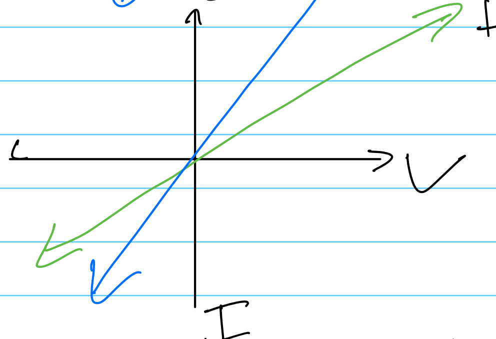
3. $P = VI$ and $V = IR$

a)



$I = \frac{P}{V}$, $P = 1$, $P = 4$

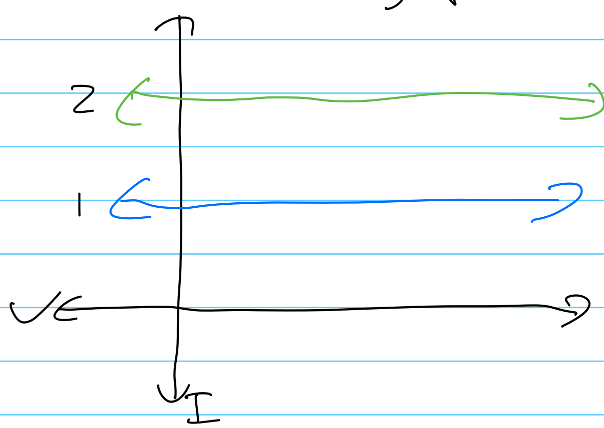
$I_1 = \frac{1}{V}$ $\quad$ $I_2 = \frac{4}{V}$



$I = \frac{V}{R}$, $R = e$, $R = e^2$

$I_1 = \frac{1}{e}V$ $\quad$ $I_2 = \frac{1}{e^2}V$

b) $(\frac{1}{2}\ln R, \sqrt{P})$

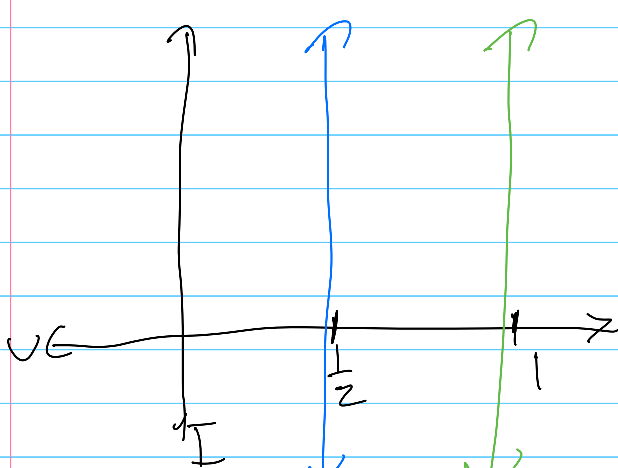


$\frac{1}{2}\ln R, \sqrt{1}$

$y = 1$, constant

$\frac{1}{2}\ln R, \sqrt{4}$

$y = 2$, constant



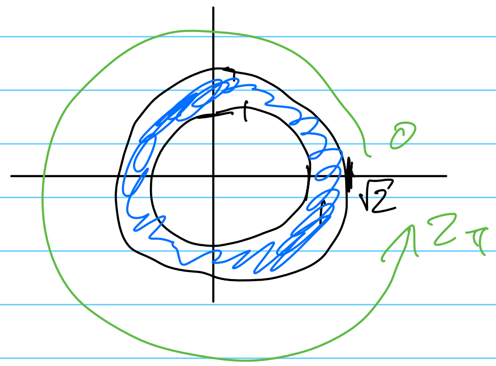
$\frac{1}{2}\ln e, \sqrt{P}$

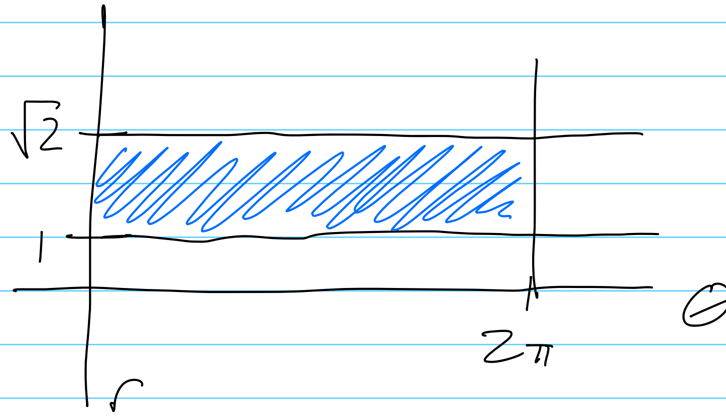
$x = \frac{1}{2}$ $\quad$ $y = \sqrt{P}$

$\frac{1}{2}\ln e^2, \sqrt{P}$

$x = 2$ $\quad$ $y = \sqrt{P}$

4. a) $r=1 \quad R=\sqrt{2}$

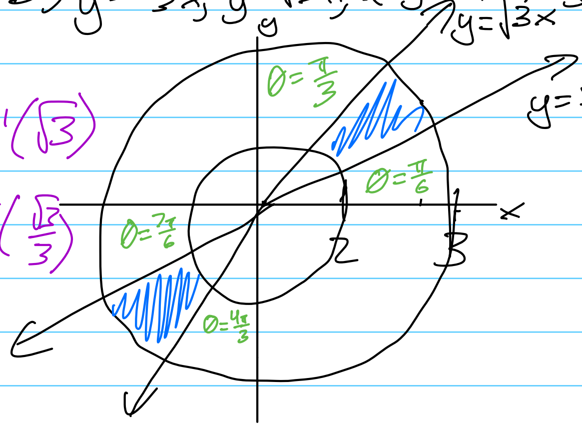


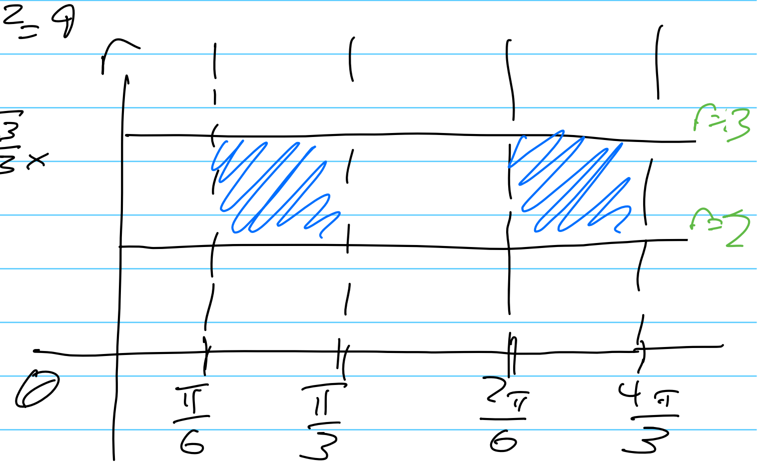


b) $y=\frac{\sqrt{3}}{3}x,\ y=\sqrt{3}x,\ x^2+y^2=4,\ x^2+y^2=9$

$\theta_1=\tan^{-1}(\sqrt{3})$

$\theta_2=\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$





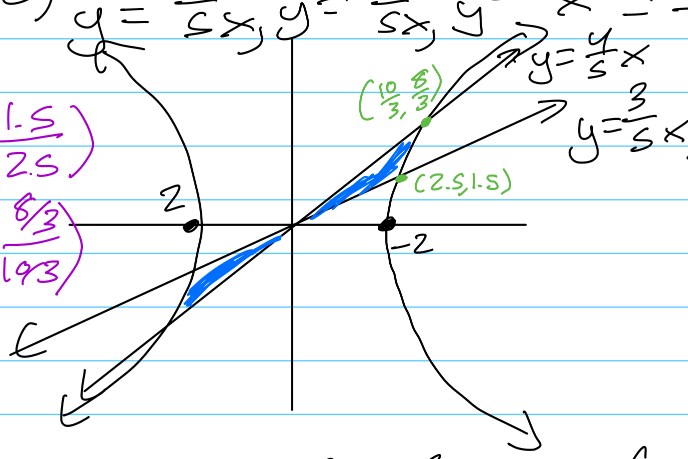
c) $y=\frac{3}{5}x,\ y=\frac{4}{5}x,\ y^2-x^2=-4$

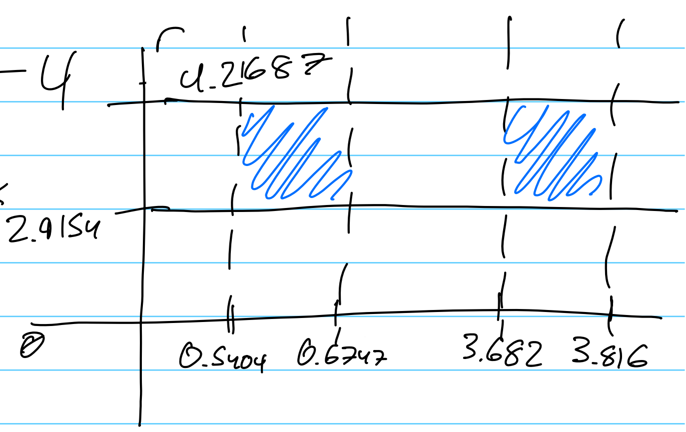
$\theta_1=\tan^{-1}\left(\frac{1.5}{2.5}\right)$

$\theta_2=\tan^{-1}\left(\frac{8/3}{10/3}\right)$

$\theta_3=\theta_1+\pi$

$\theta_4=\theta_2+\pi$





Intersections $\left(\pm\frac{10}{3}, \pm\frac{8}{3}\right)$ and $(\pm 2.5, \pm 1.5)$

$\theta_1=\tan\left(\frac{1.5}{2.5}\right)=0.5404$ and 3.682 $\quad d_1=\sqrt{\left(\frac{8}{3}\right)^2+\left(\frac{10}{3}\right)^2}=4.2687$

$\theta_2=\tan\left(\frac{8/3}{10/3}\right)=0.6747$ and 3.816 $\quad d_2=\sqrt{1.5^2+2.5^2}=2.9154$

5. $r_1(t) = \langle 3t-6, 3t-5, 4t-8 \rangle$

$r_2(t) = \langle t, 1, t \rangle$

a) $r(t) = r_1(t) - r_2(t)$

$= \langle 2t-6, 3t-6, 3t-8 \rangle$

$r'(t) = \langle 2, 3, 3 \rangle$

b) $\|r(t)\| = \sqrt{(2t-6)^2 + (3t-6)^2 + (3t-8)^2}$

$= \sqrt{4t^2 - 24t + 36 + 9t^2 - 36t + 36 + 9t^2 - 48t + 64}$

$= \sqrt{22t^2 - 108t + 136}$

Minimum $\|r(t)\|$ when $\|r'(t)\| = 0$

$\|r'(t)\| = \frac{1}{2}(22t^2 - 108t + 136)^{-1/2}(44t - 108)$

$\|r'(t)\| = 0$ when $44t - 108 = 0$

$t = \frac{27}{11}$

c) Let $t = s$ for $r_2(t)$

$\langle 3t-6, 3t-5, 4t-8 \rangle = \langle s, 1, s \rangle$

$3t - 5 = 1$ $\quad$ $3(2) - 6 = 0$ $\quad$ $s = 0$

$3t = 6$ $\quad t = 2$ $\quad$ $6 - 6 = 0 = s$ $\quad$ $t = 2$

The lines intersect at $(0, 1, 0)$, so the distance $= 0$

d) No, because the lines intersect at different times, b and c have different answers.

$t = \frac{27}{11}$ from b

$\|r(\frac{27}{11})\| = \sqrt{(2(\frac{27}{11}) - 6)^2 + (3(\frac{27}{11}) - 6)^2 + (3(\frac{27}{11}) - 8)^2}$

$= 2.6799 \neq 0 \leftarrow$ from part c

6. $r(t) = (2\cos t, 2\sin t, 5t)$

a) $r'(t) = (-2\sin t, 2\cos t, 5)$

$r''(t) = (-2\cos t, -2\sin t, 0)$

b) $r'(\pi) = (0, -2, 5)$

$r(\pi) = (-2, 0, 5\pi)$

$\ell(x) = (-2, 0, 5\pi) + (0, -2, 5)t$

$= (-2, 0, 5\pi) + (0, -2t, 5t)$

$\ell(t) = (-2, -2t, 5\pi + 5t)$

c) $L(t) = \int_0^t \|r'(t)\|\, dt$

$= \int_0^t \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 25}\, dt$

$= \int_0^t \sqrt{4\sin^2 t + 4\cos^2 t + 25}\, dt$

$= \int_0^t \sqrt{4(\sin^2 t + \cos^2 t) + 25}\, dt$

$= \int_0^t \sqrt{4 + 25}\, dt = \int_0^t \sqrt{29}\, dt$

$L(t) = \sqrt{29}\, t$