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Section 6.3: 5, 7, 9, 15, 17, 29, 33

5. Consider $\frac{d^2y}{dt^2}+\omega^2 y=0$, $y(0)=1$, $y'(0)=0$

Let $y(t)=\cos(\omega t)$ be a solution.

$\mathcal{L}\left[\frac{d^2y}{dt^2}+\omega^2 y\right]=\mathcal{L}[0]$

$s^2\mathcal{L}[y]-sy(0)-\cancel{y'(0)}+\omega^2\mathcal{L}[y]=0$

$s^2\mathcal{L}[y]+\omega^2\mathcal{L}[y]-s=0$

$\mathcal{L}[y](s^2+\omega^2)=s$

$\therefore \mathcal{L}[y]=\frac{s}{s^2+\omega^2}$ if $y(t)=\cos(\omega t)$ is a solution, then

$\mathcal{L}[\cos(\omega t)]=\frac{s}{s^2+\omega^2}$ //

7. $f(t)=t\cos\omega t$

$\mathcal{L}[f(t)]=\mathcal{L}[t\cos\omega t]=\int_0^\infty t\cos\omega t e^{-st}\,dt$

Given that $\mathcal{L}[\cos(\omega t)]=\frac{s}{s^2+\omega^2}$

We know if $\mathcal{L}[f(t)]=F(s)$ then $\mathcal{L}[t^k f(t)]=-\frac{d^kF(s)}{ds^k}$

$\therefore \mathcal{L}[t\cos(\omega t)]=-\frac{d\mathcal{L}[\cos\omega t]}{ds}=-\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right)$

$=-\frac{s'(s^2+\omega^2)-(s^2+\omega^2)'s}{(s^2+\omega^2)^2}=-\frac{(s^2+\omega^2)-2s(s)}{(s^2+\omega^2)^2}$

$=\frac{2s^2-s^2-\omega^2}{(s^2+\omega^2)^2}$ $\therefore \mathcal{L}[t\cos\omega t]=\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$

9. $f(t)=t^2e^{at}$

$\mathcal{L}[f(t)]=\int_0^\infty t^2e^{at}e^{-st}\,dt=\int_0^\infty t^2e^{at-st}\,dt=\int_0^\infty t^2e^{-t(s-a)}\,dt$

D	I
t^2 (+)	$e^{-t(s-a)}$
$2t$ (−)	$-\frac{1}{s-a}e^{-t(s-a)}$
2 (+)	$\frac{1}{(s-a)^2}e^{-t(s-a)}$
0	$-\frac{1}{(s-a)^3}e^{-t(s-a)}$

$=-\frac{t^2}{s-a}e^{-t(s-a)}-\frac{2t}{(s-a)^2}e^{-t(s-a)}-\frac{2}{(s-a)^3}e^{-t(s-a)}\Big|_0^\infty$

$=0-\left(-\frac{0}{s-a}-\frac{0}{(s-a)^2}-\frac{2}{(s-a)^3}\right)$

$=\frac{2}{(s-a)^3}$ $\therefore \mathcal{L}[t^2e^{at}]=\frac{2}{(s-a)^3}$

15. $\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+10}\right]$

$\frac{1}{s^2+2s+10}=\frac{1}{s^2+2s+1+9}=\frac{1}{(s+1)^2+9}$

$\mathcal{L}[f]=\frac{1}{3}\left(\frac{3}{(s+1)^2+3^2}\right)$

$\mathcal{L}^{-1}[\mathcal{L}f]=\frac{1}{3}\mathcal{L}^{-1}\left[\frac{3}{(s+1)^2+3^2}\right]_{a=-1}$

$\frac{1}{3}e^{-t}\sin 3t$

17. $\mathcal{L}^{-1}\left[\frac{2s+3}{s^2+s+1}\right]$

$\frac{2s+3}{s^2+s+\frac{1}{4}+\frac{3}{4}}=\frac{2s+3}{(s+\frac{1}{2})^2+\frac{3}{4}}=\frac{2s+1}{(s+\frac{1}{2})^2+\frac{3}{4}}+\frac{2}{(s+\frac{1}{2})^2+\frac{3}{4}}$

$\to \mathcal{L}^{-1}\left[\frac{2s+1}{(s+\frac{1}{2})^2+(\frac{\sqrt3}{2})^2}\right]+2\left(\frac{2}{\sqrt3}\right)\mathcal{L}^{-1}\left[\frac{\frac{\sqrt3}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt3}{2})^2}\right]$

$=2\mathcal{L}^{-1}\left[\underbrace{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt3}{2})^2}}_{\text{cos shift, } a=-\frac{1}{2}}\right]+\frac{4}{\sqrt3}\mathcal{L}^{-1}\left[\underbrace{\frac{\frac{\sqrt3}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt3}{2})^2}}_{\text{sin shift, } a=-\frac{1}{2}}\right]$

$\omega=\frac{\sqrt3}{2}$

$\therefore \mathcal{L}^{-1}\left[\frac{2s+3}{s^2+s+1}\right]=2e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt3}{2}t\right)+\frac{4}{\sqrt3}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt3}{2}t\right)$

29. $\frac{d^2y}{dt^2}-4\frac{dy}{dt}+5y=2e^t$; $y(0)=3$, $y'(0)=1$

a) LHS: $\mathcal{L}[y''-4y'+5y]=s^2\mathcal{L}[y]-sy(0)-y'(0)-4s\mathcal{L}[y]+4y(0)+5\mathcal{L}[y]$ RHS: $\mathcal{L}[2e^t]=\frac{2}{s-1}$

$=\mathcal{L}[y](s^2-4s+5)-sy(0)+4y(0)-y'(0)$

b) $\mathcal{L}[y](s^2-4s+5)-3s+12-1=\frac{2}{s-1}$

$\mathcal{L}[y](s^2-4s+5)=\frac{2}{s-1}+3s-11$

$=\frac{2}{s-1}+\frac{(3s-11)(s-1)}{s-1}$

$=\frac{2}{s-1}+\frac{3s^2-3s-11s+11}{s-1}$

$\mathcal{L}[y](s^2-4s+5)=\frac{3s^2-14s+13}{s-1}$

$\mathcal{L}[y]=\frac{3s^2-14s+13}{(s-1)(s^2-4s+5)}$

$\frac{3s^2-14s+13}{(s-1)(s^2-4s+5)}=\frac{A}{s-1}+\frac{Bs+C}{s^2-4s+5}$

$3s^2-14s+13=As^2-4As+5A+Bs^2+Cs-Bs-C$

$\begin{cases}3=A+B\\-14=-4A+C-B\\13=5A-C\end{cases}$

$+\ 2=2A \quad A=1$

$A=1,\ B=2$
$C=-8$

$\mathcal{L}[y]=\frac{1}{s-1}+\frac{2s-8}{s^2-4s+5}$

c) $\mathcal{L}^{-1}[\mathcal{L}[y]]=\mathcal{L}^{-1}\left[\frac{1}{s-1}+\frac{2s-8}{s^2-4s+5}\right]$

$y(t)=\mathcal{L}^{-1}\left[\frac{1}{s-1}\right]+\mathcal{L}^{-1}\left[\frac{2s-8}{s^2-4s+5}\right]$

$=e^t+\mathcal{L}^{-1}\left[\frac{2s-8}{s^2-4s+5}\right]=e^t+\mathcal{L}^{-1}\left[\frac{2s-8}{s^2-4s+4+1}\right]$

$y(t)=e^t+\mathcal{L}^{-1}\left[\frac{2s-8}{(s-2)^2+1}\right]$

$\frac{2s-8}{(s-2)^2+1}=\frac{2s-4}{(s-2)^2+1}-\frac{4}{(s-2)^2+1}$

$=2\cdot\frac{s-2}{(s-2)^2+1}-4\cdot\frac{1}{(s-2)^2+1}$

$\mathcal{L}^{-1}\left[\frac{2s-8}{(s-2)^2+1}\right]=2\mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2+1}\right]-4\mathcal{L}^{-1}\left[\frac{1}{(s-2)^2+1}\right]$

$=2e^{2t}\cos t-4e^{2t}\sin t$

$\therefore y(t)=e^t+2e^{2t}\cos t-4e^{2t}\sin t$

33. $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+9y=20u_2(t)\sin(t-2)$; $y(0)=1$, $y'(0)=2$

a) **LHS:** $\mathcal{L}[y''+4y'+9y]=s^2\mathcal{L}[y]-sy(0)-y'(0)+4s\mathcal{L}[y]-4y(0)+9\mathcal{L}[y]$

$$=\mathcal{L}[y](s^2+4s+9)-y(0)(s+4)-y'(0)$$

RHS: $\mathcal{L}[20u_2(t)\sin(t-2)]=20e^{-2s}\left(\frac{1}{s^2+1}\right)$

b) $\mathcal{L}[y](s^2+4s+9)-s-4-2=\frac{20e^{-2s}}{s^2+1}$

$\mathcal{L}[y](s^2+4s+9)=\frac{20e^{-2s}}{s^2+1}+s+6$

$$\mathcal{L}[y]=\frac{20e^{-2s}}{(s^2+1)(s^2+4s+9)}+\frac{s+6}{s^2+4s+9}$$

c) $\mathcal{L}^{-1}[\mathcal{L}[y]]=20\,\mathcal{L}^{-1}\underbrace{\left[\frac{e^{-2s}}{(s^2+1)(s^2+4s+9)}\right]}_{①}+\mathcal{L}^{-1}\underbrace{\left[\frac{s+6}{s^2+4s+9}\right]}_{②}$

① $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{(s^2+1)(s^2+4s+9)}\right]$: $f(t)=\frac{1}{(s^2+1)(s^2+4s+9)}$; $\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4s+9}$

$$1=As^3+4As^2+9As+Bs^2+4Bs+9B+Cs^3+Cs+Ds^2+D$$

$$\begin{cases}1=9B+D\\0=9A+4B+C\\0=4A+B+D\\0=A+C\end{cases}\Rightarrow\begin{cases}1=0A+9B+0C+D\\0=9A+4B+C+0D\\0=4A+B+0C+D\\0=A+0B+C+0D\end{cases}\Rightarrow\begin{pmatrix}0&9&0&1\\9&4&1&0\\4&1&0&1\\1&0&1&0\end{pmatrix}\begin{pmatrix}A\\B\\C\\D\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \text{rref}\begin{pmatrix}0&9&0&1&1\\9&4&1&0&0\\4&1&0&1&0\\1&0&1&0&0\end{pmatrix};\ \begin{matrix}A=-\frac{1}{20}\\B=\frac{1}{10}\\C=\frac{1}{20}\\D=\frac{1}{10}\end{matrix}$$

$$f(t)=\frac{-\frac{1}{20}s+\frac{1}{10}}{s^2+1}+\frac{\frac{1}{20}s+\frac{1}{10}}{s^2+4s+9}=-\frac{1}{20}\left(\frac{s-2}{s^2+1}\right)+\frac{1}{20}\left(\frac{s+2}{s^2+4s+9}\right)=-\frac{1}{20}\left(\frac{s-2}{s^2+1}-\frac{s+2}{s^2+4s+9}\right)$$

$$\mathcal{L}^{-1}[f(t)]=-\frac{1}{20}\left(\mathcal{L}^{-1}\left[\frac{s}{s^2+1}-2\frac{1}{s^2+1}\right]-\mathcal{L}^{-1}\left[\frac{s+2}{s^2+4s+9}\right]\right)\rightarrow\frac{s+2}{(s+2)^2+5}=\frac{s+2}{(s+2)^2+\sqrt{5}^2}$$

$$=-\frac{1}{20}\left(\cos t-2\sin t-e^{-2t}\cos(\sqrt{5}t)\right)$$

e^{-2s} : $a=2$ | $f(t-2)=-\frac{1}{20}\left(\cos(t-2)-2\sin(t-2)-e^{-2(t-2)}\cos(\sqrt{5}(t-2))\right)$

① $\mathcal{L}^{-1}\left[\frac{20e^{-2s}}{(s^2+1)(s^2+4s+9)}\right]=-u_2(t)\left[\cos(t-2)-2\sin(t-2)-e^{-2(t-2)}\cos(\sqrt{5}(t-2))\right]$

$$=u_2(t)\left[e^{-2(t-2)}\cos(\sqrt{5}(t-2))-\cos(t-2)+2\sin(t-2)\right]$$

② $\mathcal{L}^{-1}\left[\frac{s+6}{s^2+4s+9}\right]=\mathcal{L}^{-1}\left[\frac{s+6}{(s+2)^2+5}\right]=\mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+\sqrt{5}^2}+\frac{4}{(s+2)^2+\sqrt{5}^2}\right]$

$$=e^{-2t}\cos(\sqrt{5}t)+\frac{4}{\sqrt{5}}\mathcal{L}^{-1}\left[\frac{\sqrt{5}}{(s+2)^2+\sqrt{5}^2}\right]$$

$$\mathcal{L}^{-1}\left[\frac{s+6}{s^2+4s+9}\right]=e^{-2t}\cos(\sqrt{5}t)+\frac{4}{\sqrt{5}}e^{-2t}\sin(\sqrt{5}t)$$

$$y(t)=e^{-2t}\cos(\sqrt{5}t)+\frac{4}{\sqrt{5}}e^{-2t}\sin(\sqrt{5}t)-u_2(t)\left[e^{-2(t-2)}\cos(\sqrt{5}(t-2))-\cos(t-2)+2\sin(t-2)\right]$$

Section 6.4: 1, 3, 5, 9

1. $$\lim_{\Delta t\to 0}\left(\frac{e^{s\Delta t}-e^{-s\Delta t}}{2\Delta t}\right)=\lim_{\Delta t\to 0}\left(\frac{se^{s\Delta t}+se^{-s\Delta t}}{2}\right)=\frac{se^{0}+se^{-s(0)}}{2}=\frac{2s}{2}\quad\therefore\ \lim_{\Delta t\to 0}\left(\frac{e^{s\Delta t}-e^{-s\Delta t}}{2\Delta t}\right)=s$$

$$\begin{cases} e^{s(0)}-e^{-s(0)}=1-1=0\\ 2(0)=0\end{cases}$$ L'Hopital works ✓ → Differentiate top/bottom w/ respect to Δt

3. $$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=\delta_3(t);\ y(0)=1,\ y'(0)=1$$

LHS: $$\mathcal{L}[y''+2y'+5y]=s^2\mathcal{L}[y]-sy(0)-y'(0)+2s\mathcal{L}[y]-2y(0)+5\mathcal{L}[y]$$
$$=s^2\mathcal{L}[y]+2s\mathcal{L}[y]+5\mathcal{L}[y]-s-1-2$$
$$=\mathcal{L}[y](s^2+2s+5)-s-3$$

RHS: $$\mathcal{L}[\delta_3(t)]=e^{-3s}$$
$$\mathcal{L}[y](s^2+2s+5)-s-3=e^{-3s}$$
$$\mathcal{L}[y]=\frac{e^{-3s}}{s^2+2s+5}+\frac{s+3}{s^2+2s+5}$$
$$\mathcal{L}^{-1}[\mathcal{L}[y]]=\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2+2s+5}\right]_{①}+\mathcal{L}^{-1}\left[\frac{s+3}{(s+1)^2+4}\right]_{②}$$

① $\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2+2s+5}\right]$: $$f(t)=\frac{1}{s^2+2s+5}=\frac{1}{(s+1)^2+4}=\frac{1}{2}\cdot\frac{2}{(s+1)^2+2^2}$$
$$\mathcal{L}^{-1}[f(t)]=\frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+2^2}\right]=\frac{1}{2}e^{-t}\sin(2t)$$
e^{-3s}: $a=3$ | $f(t-3)=\frac{1}{2}e^{-(t-3)}\sin(2(t-3))$
$$\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2+2s+5}\right]=\frac{1}{2}u_3(t)e^{-(t-3)}\sin(2(t-3))$$

② $$\mathcal{L}^{-1}\left[\frac{s+3}{(s+1)^2+2^2}\right]=\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right]+\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+2^2}\right]$$
$$=e^{-t}\cos 2t+e^{-t}\sin 2t$$
$$\mathcal{L}^{-1}\left[\frac{s+3}{(s+1)^2+2^2}\right]=e^{-t}(\cos 2t+\sin 2t)$$

$$y(t)=\frac{1}{2}u_3(t)e^{-(t-3)}\sin(2(t-3))+e^{-t}(\cos 2t+\sin 2t)$$

5. $$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+3y=\delta_1(t)-3\delta_4(t);\ y(0)=y'(0)=0$$

LHS: $$\mathcal{L}[y''+2y'+3y]=s^2\mathcal{L}[y]-sy(0)-y'(0)+2s\mathcal{L}[y]-y(0)+3\mathcal{L}[y]$$
$$=s^2\mathcal{L}[y]+2s\mathcal{L}[y]+3\mathcal{L}[y]$$
$$=\mathcal{L}[y](s^2+2s+3)$$

RHS: $$\mathcal{L}[\delta_1(t)-3\delta_4(t)]=e^{-s}-3e^{-4s}$$
$$\mathcal{L}[y](s^2+2s+3)=e^{-s}-3e^{-4s}$$
$$\mathcal{L}[y]=\frac{e^{-s}}{s^2+2s+3}-\frac{3e^{-4s}}{s^2+2s+3}$$
$$\mathcal{L}^{-1}[\mathcal{L}[y]]=\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2+2s+3}\right]_{①}-3\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^2+2s+3}\right]_{②}$$

① $\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2+2s+3}\right]$: $$f(t)=\frac{1}{s^2+2s+3}=\frac{1}{(s+1)^2+2}=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{(s+1)^2+\sqrt{2}^2}$$
$$\mathcal{L}^{-1}[f(t)]=\frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left[\frac{\sqrt{2}}{(s+1)^2+\sqrt{2}^2}\right]=\frac{1}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t)$$
e^{-s}: $a=1$ | $f(t-1)=\frac{1}{\sqrt{2}}e^{-(t-1)}\sin(\sqrt{2}(t-1))$
$$\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2+2s+3}\right]=\frac{1}{\sqrt{2}}u_1(t)e^{-(t-1)}\sin(\sqrt{2}(t-1))$$

② $\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^2+2s+3}\right]$: $$f(t)=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{(s+1)^2+\sqrt{2}^2}$$
$$\mathcal{L}^{-1}[f(t)]=\frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left[\frac{\sqrt{2}}{(s+1)^2+\sqrt{2}^2}\right]=\frac{1}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t)$$
e^{-4s}: $a=4$ | $f(t-4)=\frac{1}{\sqrt{2}}e^{-(t-4)}\sin(\sqrt{2}(t-4))$
$$\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^2+2s+3}\right]=\frac{1}{\sqrt{2}}u_4(t)e^{-(t-4)}\sin(\sqrt{2}(t-4))$$

$$y(t)=\frac{1}{\sqrt{2}}\left[u_1(t)e^{-(t-1)}\sin(\sqrt{2}(t-1))-3u_4(t)e^{-(t-4)}\sin(\sqrt{2}(t-4))\right]$$

9. $\frac{d^2y}{dt^2} + 2y = \sum_{n=1}^{\infty} \delta_n(t)\ ;\ y(0)=y'(0)=0$

a) LHS: $\mathcal{L}[y''+2y] = s^2\mathcal{L}[y] - sy(0) - y'(0) + 2\mathcal{L}[y]$

$= \mathcal{L}[y](s^2+2)$

RHS: $\mathcal{L}[\sum_{n=1}^{\infty} \delta_n(t)] = \sum_{n=1}^{\infty} e^{-ns}$

$\mathcal{L}[y](s^2+2) = \sum_{n=1}^{\infty} e^{-ns}$

$\mathcal{L}[y] = \frac{1}{s^2+2}\sum_{n=1}^{\infty} e^{-ns}$

b) $\mathcal{L}^{-1}[\mathcal{L}[y]] = \mathcal{L}^{-1}[\frac{1}{s^2+2}\sum_{n=1}^{\infty} e^{-ns}]$

$y(t) = \sum_{n=1}^{\infty} \mathcal{L}^{-1}[\frac{e^{-ns}}{s^2+2}]$

$f(t) = \frac{1}{s^2+2} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{s^2+\sqrt{2}^2} = \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$

e^{-ns}: $a=n$ | $f(t-n) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}(t-n))$

$\sum_{n=1}^{\infty} \mathcal{L}^{-1}[\frac{e^{-ns}}{s^2+2}] = \frac{1}{\sqrt{2}} u_n(t)\sin(\sqrt{2}(t-n))$

$y(t) = \frac{1}{\sqrt{2}}\sum_{n=1}^{\infty} u_n(t)\sin(\sqrt{2}(t-n))$

c) Given: $y(t) = \frac{1}{\sqrt{2}}\sum_{n=1}^{\infty} u_n(t)\sin(\sqrt{2}(t-n))$

The amplitude of the sine function is $\frac{1}{\sqrt{2}}$, and it is independent of both t and n. Sin is periodic, and $u_n(t)$ simply 'turns on' at $t \geq n$. This comes together to show that the behavior oscillates with constant period $\frac{2\pi}{\sqrt{2}}$ and amplitude of $\frac{1}{\sqrt{2}}$.