Spring 2024 Math 224 Homework 7 (Keys)

Instructions: show all steps to get full credits.

Problem 1.

1. Write the each expression in the form of $(s-a)^2+\omega^2$.

   (a) $s^2+2s+10=(s+1)^2+3^2$.

   (b) $s^2-4s+5=(s-2)^2+1^2$.

   (c) $s^2+s+1=\left(s+\dfrac{1}{2}\right)^2+\left(\dfrac{\sqrt{3}}{2}\right)^2$.

   (d) $s^2+6s+10=(s+3)^2+1^2$

2. Using the result in (1), compute the inverse Laplace transform of the given function.

   (a) $\dfrac{1}{s^2+2s+10}$.

   Solution.

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{1}{s^2+2s+10}\right] &= \mathscr{L}^{-1}\left[\frac{1}{(s+1)^2+3^2}\right]\\ &= \frac{1}{3}\mathscr{L}^{-1}\left[\frac{3}{(s+1)^2+3^2}\right]\\ &= \frac{1}{3}e^{-t}\sin(3t)\end{aligned}$$

   (b) $\dfrac{s}{s^2-4s+5}$.

Solution.

$$
\begin{aligned}
\mathscr{L}^{-1}\left[\frac{s}{s^2-4s+5}\right] &= \mathscr{L}^{-1}\left[\frac{s}{(s-2)^2+1^2}\right] \\
&= \mathscr{L}^{-1}\left[\frac{s-2}{(s-2)^2+1^2}\right] + \mathscr{L}^{-1}\left[\frac{2}{(s-2)^2+1^2}\right] \\
&= \mathscr{L}^{-1}\left[\frac{s-2}{(s-2)^2+1^2}\right] + 2\mathscr{L}^{-1}\left[\frac{1}{(s-2)^2+1^2}\right] \\
&= e^{2t}\cos(t) + 2e^{2t}\sin(t)
\end{aligned}
$$

(c) $\dfrac{2s+3}{s^2+s+1}$.

Solution.

$$
\begin{aligned}
\mathscr{L}^{-1}\left[\frac{2s+3}{s^2+s+1}\right] &= \mathscr{L}^{-1}\left[\frac{2s+3}{(s+1/2)^2+(\sqrt{3}/2)^2}\right] \\
&= 2\mathscr{L}^{-1}\left[\frac{s+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right] \\
&+ \frac{4}{\sqrt{3}}\mathscr{L}^{-1}\left[\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right] \\
&= 2e^{-t/2}\cos(\sqrt{3}t/2) + \frac{4}{\sqrt{3}}e^{-t/2}\sin(\sqrt{3}t/2)
\end{aligned}
$$

(d) $\dfrac{s+1}{s^2+6s+10}$.

Solution.

$$
\begin{aligned}
\mathscr{L}^{-1}\left[\frac{s+1}{s^2+6s+10}\right] &= \mathscr{L}^{-1}\left[\frac{s+1}{(s+3)^2+1^2}\right] \\
&= \mathscr{L}^{-1}\left[\frac{s+3}{(s+3)^2+1^2}\right] - 2\mathscr{L}^{-1}\left[\frac{1}{(s-2)^2+1^2}\right] \\
&= e^{-3t}\cos(t) - 2e^{-3t}\sin(t)
\end{aligned}
$$

Problem 2. Solve the initial value problems (IVPs):

1. $\dfrac{d^2y}{dt^2} - y = e^{2t}$, $y(0) = 1$, $y'(0) = -1$.
   Solution.

   - **Laplace Transform:**

$$\begin{aligned}
\mathscr{L}\left[\frac{d^2y}{dt^2} - y\right] &= \mathscr{L}\left[e^{2t}\right] \\
(s^2\mathscr{L}[y] - sy(0) - y'(0)) - \mathscr{L}[y] &= \mathscr{L}\left[e^{2t}\right] \\
(s^2\mathscr{L}[y] - s + 1) - \mathscr{L}[y] &= \frac{1}{s-2} \\
\mathscr{L}[y] &= \frac{s-1}{s^2-1} + \frac{1}{(s-2)s^2-1} \\
\mathscr{L}[y] &= \frac{1}{s+1} + \frac{e^{2s}}{(s-2)(s-1)(s+1)}
\end{aligned}$$

   - **Partial Fraction Decomposition:** (Exercise)

$$\frac{1}{s+1} + \frac{1}{(s-2)(s-1)(s+1)} = -\frac{1}{2}\cdot\frac{1}{s-1} + \frac{7}{6}\cdot\frac{1}{s+1} + \frac{1}{3}\cdot\frac{1}{s-2}$$

   - **Inverse Laplace Transform:**

$$\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left[-\frac{1}{2}\cdot\frac{1}{s-1} + \frac{7}{6}\cdot\frac{1}{s+1} + \frac{1}{3}\cdot\frac{1}{s-2}\right] \\
&= -\frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{s-1}\right] + \frac{7}{6}\mathscr{L}^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{3}\mathscr{L}^{-1}\left[\frac{1}{s-2}\right] \\
y(t) &= -\frac{1}{2}e^{t} + \frac{7}{6}e^{-t} + \frac{1}{3}e^{2t}
\end{aligned}$$

2. $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 5y = 2e^{t}$, $y(0) = 3$, $y'(0) = 1$.
   Solution.

- **Laplace Transform:**

$$
\begin{aligned}
\mathscr{L}\left[\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y\right] &= \mathscr{L}\left[2e^t\right] \\
(s^2\mathscr{L}[y] - sy(0) - y'(0)) - 2(s\mathscr{L}[y] - y(0)) + 5\mathscr{L}[y] &= \frac{2}{s-1} \\
(s^2\mathscr{L}[y] - 3s - 1) - 2(s\mathscr{L}[y] - 3) + 5\mathscr{L}[y] &= \frac{2}{s-1} \\
(s^2 - 2s + 5)\mathscr{L}[y] &= 3s - 5 + \frac{2}{s-1}
\end{aligned}
$$

Hence,

$$
\mathscr{L}[y] = \frac{3s-5}{s^2-2s+5} + \frac{2}{(s-1)(s^2-2s+5)} = \frac{3s^2-8s+7}{(s-1)(s^2-2s+5)}
$$

- **Partial Fraction Decomposition:**

$$
\begin{aligned}
\frac{3s^2-8s+7}{(s-1)(s^2-2s+5)} &= \frac{A}{s-1} + \frac{Bs+C}{s^2-2s+5} \\
&= \frac{(A+B)s^2 + (-2A-B+C)s + (5A-C)}{(s-1)(s^2-2s+5)}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
A + B &= 3 \\
-2A - B + C &= -8 \\
5A - C &= 7
\end{aligned}
$$

Solving the equations, $A = 1/2$, $B = 5/2$, $C = -9/2$. Therefore,

$$
\frac{3s^2-8s+7}{(s-1)(s^2-2s+5)} = \frac{1}{2}\cdot\frac{1}{s-1} + \frac{1}{2}\cdot\frac{5s-9}{s^2-2s+5}
$$

- **Inverse Laplace Transform:**

$$\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left[\frac{3s^2-8s+7}{(s-1)(s^2-2s+5)}\right] \\
&= \mathscr{L}^{-1}\left[\frac{1}{2}\cdot\frac{1}{s-1}+\frac{1}{2}\cdot\frac{5s-9}{s^2-2s+5}\right] \\
&= \frac{1}{2}\cdot\mathscr{L}^{-1}\left[\frac{1}{s-1}\right]+\frac{1}{2}\cdot\mathscr{L}^{-1}\left[\frac{5s-9}{s^2-2s+5}\right] \\
y(t) &= \frac{1}{2}e^t+\frac{1}{2}\cdot\mathscr{L}^{-1}\left[\frac{5s-9}{s^2-2s+5}\right]
\end{aligned}$$

  - Compute $\mathscr{L}^{-1}\left[\dfrac{5s-9}{s^2-2s+5}\right]$:

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{5s-9}{s^2-2s+5}\right] &= \mathscr{L}^{-1}\left[\frac{5s-9}{(s-1)^2+2^2}\right] \\
&= 5\mathscr{L}^{-1}\left[\frac{s-1}{(s-1)^2+2^2}\right]-2\mathscr{L}^{-1}\left[\frac{2}{(s-1)^2+2^2}\right] \\
&= 5e^t\cos(2t)-2e^t\sin(2t)
\end{aligned}$$

  - The solution is

$$y(t)=\frac{1}{2}e^t+\frac{5}{2}e^t\cos(2t)-e^t\sin(2t)$$

3. $\dfrac{d^2y}{dt^2}+6\dfrac{dy}{dt}+13y=13u_4(t)$, $y(0)=3$, $y'(0)=1$.
Solution.

- **Laplace Transform:**

$$\begin{aligned}
\mathscr{L}\left[\frac{d^2y}{dt^2}+6\frac{dy}{dt}+13y\right] &= \mathscr{L}\left[13u_4(t)\right] \\
(s^2\mathscr{L}[y]-sy(0)-y'(0))+6(s\mathscr{L}[y]-y(0))+13\mathscr{L}[y] &= 13\frac{e^{4s}}{s} \\
(s^2\mathscr{L}[y]-3s-1)+6(s\mathscr{L}[y]-3)+13\mathscr{L}[y] &= 13\frac{e^{4s}}{s} \\
(s^2+6s+13)\mathscr{L}[y] &= 3s+19+13\frac{e^{4s}}{s}
\end{aligned}$$

Hence,

$$\mathscr{L}[y] = \frac{3s+19}{s^2+6s+13} + \frac{13e^{4s}}{s(s^2+6s+13)}$$

- **Partial Fraction Decomposition:**

$$\begin{aligned}\frac{13}{s(s^2+6s+13)} &= \frac{A}{s} + \frac{Bs+C}{s^2+6s+13} \\ &= \frac{(A+B)s^2+(6A+C)s+13A}{s(s^2+6s+13)}\end{aligned}$$

Hence,

$$\begin{aligned}A+B &= 0 \\ 6A+C &= 0 \\ 13A &= 13\end{aligned}$$

Solving the equations, $A = 1$, $B = -1$, $C = -6$. Therefore,

$$\frac{13}{s(s^2+6s+13)} = \frac{1}{s} - \frac{s+6}{s^2+6s+13}$$

- **Inverse Laplace Transform:**

$$\begin{aligned}y(t) &= \mathscr{L}^{-1}\left[\frac{3s+19}{s^2+6s+13} + \frac{13e^{4s}}{s(s^2+6s+13)}\right] \\ &= \mathscr{L}^{-1}\left[\frac{3s+19}{s^2+6s+13} + e^{4s}\cdot\frac{1}{s} - e^{4s}\cdot\frac{s+6}{s^2+6s+13}\right] \\ &= \mathscr{L}^{-1}\left[\frac{3s+19}{s^2+6s+13}\right] + \mathscr{L}^{-1}\left[e^{4s}\cdot\frac{1}{s}\right] - \mathscr{L}^{-1}\left[e^{4s}\cdot\frac{s+6}{s^2+6s+13}\right]\end{aligned}$$

  - Compute $\mathscr{L}^{-1}\left[e^{4s}\cdot\frac{1}{s}\right] = u_4(t)$.
  - Compute $\mathscr{L}^{-1}\left[\frac{3s+19}{s^2+6s+13}\right]$:

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{3s+19}{s^2+6s+13}\right] &= \mathscr{L}^{-1}\left[\frac{3s+19}{(s+3)^2+2^2}\right] \\ &= 3\mathscr{L}^{-1}\left[\frac{s+3}{(s+3)^2+2^2}\right] + 5\mathscr{L}^{-1}\left[\frac{2}{(s+3)^2+2^2}\right] \\ &= 3e^{-3t}\cos(2t) + 5e^{-3t}\sin(2t)\end{aligned}$$

– Compute $\mathscr{L}^{-1}\left[\dfrac{3s+19}{s^2+6s+13}\right]$:

$$\begin{aligned}
\mathscr{L}^{-1}\left[e^{4s}\cdot\frac{s+6}{s^2+6s+13}\right] &= \mathscr{L}^{-1}\left[e^{4s}\cdot\frac{s+6}{(s+3)^2+2^2}\right]\\
&= \mathscr{L}^{-1}\left[e^{4s}\cdot\frac{s+3}{(s+3)^2+2^2}\right]\\
&+ \frac{3}{2}\mathscr{L}^{-1}\left[e^{4s}\cdot\frac{2}{(s+3)^2+2^2}\right]\\
&= u_4(t)e^{-3(t-4)}\cos(2(t-4))\\
&+ \frac{3}{2}u_4(t)e^{-3(t-4)}\sin(2(t-4))
\end{aligned}$$

– The solution is

$$\begin{aligned}
y(t) &= 3e^{-3t}\cos(2t)+5e^{-3t}\sin(2t)+u_4(t)\\
&\quad - u_4(t)e^{-3(t-4)}\cos(2(t-4)) - \frac{3}{2}u_4(t)e^{-3(t-4)}\sin(2(t-4))
\end{aligned}$$

4. $\dfrac{d^2y}{dt^2}+4\dfrac{dy}{dt}+9y = 20u_2(t)\sin(t-2)$, $y(0)=1$, $y'(0)=2$.

Solution.

- **Laplace Transform:**

$$\begin{aligned}
\mathscr{L}\left[\frac{d^2y}{dt^2}+4\frac{dy}{dt}+9y\right] &= \mathscr{L}\left[20u_2(t)\sin(t-2)\right]\\
(s^2\mathscr{L}[y]-sy(0)-y'(0))+4(s\mathscr{L}[y]-y(0))+9\mathscr{L}[y] &= 20e^{-2s}\frac{1}{s^2+1}\\
(s^2\mathscr{L}[y]-s-2)+4(s\mathscr{L}[y]-1)+9\mathscr{L}[y] &= 20e^{-2s}\frac{1}{s^2+1}\\
(s^2+4s+9)\mathscr{L}[y] &= s+6+20e^{-2s}\frac{1}{s^2+1}
\end{aligned}$$

Hence,

$$\mathscr{L}[y] = \frac{s+6}{s^2+4s+9} + e^{-2s}\frac{20}{(s^2+1)(s^2+4s+9)}$$

- **Partial Fraction Decomposition:**

$$\frac{20}{(s^2+1)(s^2+4s+9)} = \frac{A+Bs}{s^2+1} + \frac{C+Ds}{s^2+4s+9}$$
$$= \frac{(B+D)s^3 + (A+4B+C)s^2 + (4A+9B+D)s + (9A+C)}{(s^2+1)(s^2+4s+9)}$$

Hence,

$$\begin{aligned} B+D &= 0 \\ A+4B+C &= 0 \\ 4A+9B+D &= 0 \\ 9A+C &= 20 \end{aligned}$$

Solving the equations, $A = 2$, $B = -1$, $C = 2$, $D = 1$. Therefore,

$$\frac{20}{(s^2+1)(s^2+4s+9)} = \frac{2-s}{s^2+1} + \frac{2+s}{s^2+4s+9}$$

- **Inverse Laplace Transform:**

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left[\frac{s+6}{s^2+4s+9} + e^{-2s}\frac{20}{(s^2+1)(s^2+4s+9)}\right] \\ &= \mathscr{L}^{-1}\left[\frac{s+6}{s^2+4s+9} + e^{-2s}\cdot\frac{s+2}{s^2+4s+9} - e^{-2s}\cdot\frac{s-2}{s^2+1}\right] \\ &= \mathscr{L}^{-1}\left[\frac{s+6}{s^2+4s+9}\right] + \mathscr{L}^{-1}\left[e^{-2s}\cdot\frac{s+2}{s^2+4s+9}\right] \\ &\quad - \mathscr{L}^{-1}\left[e^{-2s}\cdot\frac{s-2}{s^2+1}\right] \end{aligned}$$

  - Compute $\mathscr{L}^{-1}\left[\frac{s+6}{s^2+4s+9}\right]$.

$$\begin{aligned} \mathscr{L}^{-1}\left[\frac{s+6}{s^2+4s+9}\right] &= \mathscr{L}^{-1}\left[\frac{s+6}{(s+2)^2+(\sqrt{5})^2}\right] \\ &= \mathscr{L}^{-1}\left[\frac{s+2}{(s+2)^2+(\sqrt{5})^2}\right] + \frac{4}{\sqrt{5}}\mathscr{L}^{-1}\left[\frac{\sqrt{5}}{(s+2)^2+(\sqrt{5})^2}\right] \\ &= e^{-2t}\cos(\sqrt{5}t) + \frac{4}{\sqrt{5}}e^{-2t}\sin(\sqrt{5}t) \end{aligned}$$

– Compute $\mathscr{L}^{-1}\left[e^{-2s}\cdot\dfrac{s+2}{s^2+4s+9}\right]$.

$$\begin{aligned}\mathscr{L}^{-1}\left[e^{-2s}\cdot\frac{s+2}{s^2+4s+9}\right] &= \mathscr{L}^{-1}\left[e^{-2s}\cdot\frac{s+2}{(s+2)^2+(\sqrt{5})^2}\right]\\ &= u_2(t)e^{-2(t-2)}\cos(\sqrt{5}(t-2))\end{aligned}$$

– Compute $\mathscr{L}^{-1}\left[e^{-2s}\cdot\dfrac{s-2}{s^2+1}\right]$:

$$\begin{aligned}\mathscr{L}^{-1}\left[e^{-2s}\cdot\frac{s-2}{s^2+1}\right] &= \mathscr{L}^{-1}\left[e^{-2s}\cdot\frac{s}{s^2+1}\right] - 2\mathscr{L}^{-1}\left[e^{-2s}\cdot\frac{1}{s^2+1}\right]\\ &= u_2(t)\cos(t-2) - 2u_2(t)\sin(t-2)\end{aligned}$$

– The solution is

$$\begin{aligned}y(t) &= e^{-2t}\cos(\sqrt{5}t) + \frac{4}{\sqrt{5}}e^{-2t}\sin(\sqrt{5}t) + u_2(t)e^{-2(t-2)}\cos(\sqrt{5}(t-2))\\ &\quad - u_2(t)\cos(t-2) + 2u_2(t)\sin(t-2)\end{aligned}$$

5. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 3y = \delta_4(t)$, $y(0)=1$, $y'(0)=0$.

Solution.

- **Laplace Transform:**

$$\begin{aligned}\mathscr{L}\left[\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y\right] &= \mathscr{L}\left[\delta_4(t)\right]\\ (s^2\mathscr{L}[y] - sy(0) - y'(0)) + 2(s\mathscr{L}[y] - y(0)) + 3\mathscr{L}\left[y\right] &= e^{-4s}\\ (s^2\mathscr{L}[y] - s) + 2(s\mathscr{L}[y] - 1) + 3\mathscr{L}\left[y\right] &= e^{-4s}\\ (s^2+2s+3)\mathscr{L}[y] &= s+2+e^{-4s}\end{aligned}$$

Hence,

$$\mathscr{L}[y] = \frac{s+2}{s^2+2s+3} + e^{-4s}\frac{1}{s^2+2s+3}$$

- **Inverse Laplace Transform:**

$$
\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left[\frac{s+2}{s^2+2s+3} + e^{-4s}\frac{1}{s^2+2s+3}\right] \\
&= \mathscr{L}^{-1}\left[\frac{s+1}{s^2+2s+3}\right] + \mathscr{L}^{-1}\left[e^{-4s}\frac{1}{s^2+2s+3}\right] \\
&= \mathscr{L}^{-1}\left[\frac{s+1}{(s+1)^2+(\sqrt{2})^2}\right] + \mathscr{L}^{-1}\left[\frac{1}{(s+1)^2+(\sqrt{2})^2}\right] \\
&+ \mathscr{L}^{-1}\left[e^{-4s}\frac{1}{(s+1)^2+(\sqrt{2})^2}\right] \\
&= \mathscr{L}^{-1}\left[\frac{s+1}{(s+1)^2+(\sqrt{2})^2}\right] + \frac{1}{\sqrt{2}}\mathscr{L}^{-1}\left[\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}\right] \\
&+ \frac{1}{\sqrt{2}}\mathscr{L}^{-1}\left[e^{-4s}\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}\right] \\
y(t) &= e^{-t}\cos(\sqrt{2}t) + \frac{1}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t) + \frac{1}{\sqrt{2}}u_4(t)e^{-(t-4)}\sin(\sqrt{2}(t-4))
\end{aligned}
$$

6. $\dfrac{d^2y}{dt^2} + 10\dfrac{dy}{dt} + 27y = \delta_1(t) - 3\delta_4$, $y(0) = 0$, $y'(0) = 0$.

Solution.

- **Laplace Transform:**

$$
\begin{aligned}
\mathscr{L}\left[\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 27y\right] &= \mathscr{L}\left[\delta_1(t) - 3\delta_4\right] \\
(s^2\mathscr{L}[y] - sy(0) - y'(0)) + 10(s\mathscr{L}[y] - y(0)) + 27\mathscr{L}[y] &= e^{-s} - 3e^{-4s} \\
s^2\mathscr{L}[y] + 10s\mathscr{L}[y] + 27\mathscr{L}[y] &= e^{-s} - 3e^{-4s} \\
(s^2 + 10s + 27)\mathscr{L}[y] &= e^{-s} - 3e^{-4s}
\end{aligned}
$$

Hence,

$$
\mathscr{L}[y] = e^{-s}\frac{1}{s^2+10s+27} - 3e^{-4s}\frac{1}{s^2+10s+27}
$$

- **Inverse Laplace Transform:**

$$
\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left[e^{-s}\frac{1}{s^2+10s+27} - 3e^{-4s}\frac{1}{s^2+10s+27}\right]\\
&= \mathscr{L}^{-1}\left[e^{-s}\frac{1}{s^2+10s+27}\right] - 3\mathscr{L}^{-1}\left[e^{-4s}\frac{1}{s^2+10s+27}\right]\\
&= \mathscr{L}^{-1}\left[e^{-s}\frac{1}{(s+5)^2+(\sqrt{2})^2}\right] - 3\mathscr{L}^{-1}\left[e^{-4s}\frac{1}{(s+5)^2+(\sqrt{2})^2}\right]\\
&= \frac{1}{\sqrt{2}}\mathscr{L}^{-1}\left[e^{-s}\frac{\sqrt{2}}{(s+5)^2+(\sqrt{2})^2}\right] - \frac{3}{\sqrt{2}}\mathscr{L}^{-1}\left[e^{-4s}\frac{\sqrt{2}}{(s+5)^2+(\sqrt{2})^2}\right]\\
y(t) &= \frac{1}{\sqrt{2}}u_1(t)e^{-5(t-1)}\sin(\sqrt{2}(t-1)) - \frac{3}{\sqrt{2}}u_4(t)e^{-5(t-4)}\sin(\sqrt{2}(t-4))
\end{aligned}
$$