1. (a) Which of the following vectors is an eigenvector of the matrix $A = \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix}$?

i. $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad A\mathbf{u} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

ii. $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad A\mathbf{v} = \begin{pmatrix} -7 \\ 1 \end{pmatrix} \neq \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

iii. $\mathbf{w} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ $\quad A\mathbf{w} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = -2\begin{pmatrix} -1 \\ -2 \end{pmatrix} \quad \lambda = -2$

(b) Solve the initial-value problem

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

$$(-4-\lambda)(-3-\lambda) - 2 = \lambda^2 + 7\lambda + 12 - 2 = \lambda^2 + 7\lambda + 10 = (\lambda+5)(\lambda+2)$$

$$\lambda_1 = -5, \quad \vec{w}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = -2, \quad \vec{w}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -4+5 & 1 \\ 2 & -3+5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\vec{Y}(t) = e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} c_1 + e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} c_2$$

$$\vec{Y}(0) = \begin{pmatrix} c_1 + c_2 \\ -c_1 + 2c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

$$\begin{aligned} c_1 + c_2 &= -1 \\ -c_1 + 2c_2 &= -2 \\ \hline 3c_2 &= -3 \end{aligned} \qquad \begin{aligned} c_2 &= -1 \\ c_1 &= 0 \end{aligned}$$

$$\boxed{\vec{Y}(t) = -e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ solves the IVP}}$$

2. Convert the third-order differential equation

$$\frac{d^3y}{dt^3} + p\frac{d^2y}{dt^2} + q\frac{dy}{dt} + ry = 0$$

where p, q and r are constants, to a three-dimensional linear system written in matrix form.

Let $v = \frac{dy}{dt}$ and let $w = \frac{dv}{dt}$. Then we have

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = w$$

$$\frac{dw}{dt} = \frac{d^3y}{dt^3} = -ry - q\frac{dy}{dt} - p\frac{d^2y}{dt^2} = -ry - qv - pw$$

Writing $\vec{Y} = \begin{pmatrix} y \\ v \\ w \end{pmatrix}$ we have, in matrix form

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r & -q & -p \end{pmatrix} \vec{Y}$$

3. Consider the partially decoupled system

$$\frac{dx}{dt} = 2x - 8y^2$$
$$\frac{dy}{dt} = -3y.$$

(a) Derive the general solution.

$y(t) = y_0 e^{-3t}$ satisfies the second equation, regardless of $x(t)$.

Using $y(t)$ as a forcing term, we have $\frac{dx}{dt} = 2x - 8y_0^2 e^{-6t}$.

For a particular solution, take $x(t) = a e^{-6t}$. Equating $x' = -6a e^{-6t}$

with $2x - 8y^2 = 2ae^{-6t} - 8y_0^2 e^{-6t}$ gives $y_0^2 = a$.

The associated homogeneous solutions are $x(t) = k e^{+2t}$.

The general solutions $x(t) = ke^{2t} + y_0^2 e^{-6t}$, $y(t) = y_0 e^{-3t}$.

(b) Find the solution that satisfies the initial condition $x_0 = 0, y_0 = 1$.

At $t=0$ the general solution reads $x(0) = k + y_0^2$, $y(0) = y_0$.

Therefore $y_0 = 1$ and $k = -1$. The solution is

$x(t) = -e^{2t} + e^{-6t}$

$y(t) = e^{-3t}$

(i) $\dfrac{dx}{dt} = -x$, $\quad \dfrac{dy}{dt} = y - 1$

(ii) $\dfrac{dx}{dt} = x^2 - 1$, $\quad \dfrac{dy}{dt} = y$

(iii) $\dfrac{dx}{dt} = x + 2y$, $\quad \dfrac{dy}{dt} = -y$

(iv) $\dfrac{dx}{dt} = 2x$, $\quad \dfrac{dy}{dt} = y$

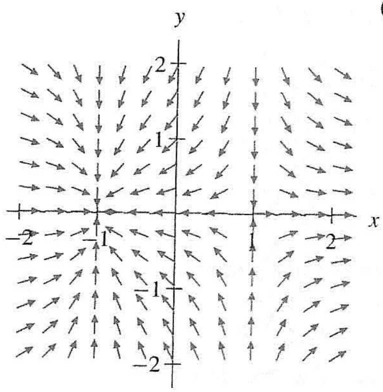
(v) $\dfrac{dx}{dt} = x$, $\quad \dfrac{dy}{dt} = 2y$

(vi) $\dfrac{dx}{dt} = x - 1$, $\quad \dfrac{dy}{dt} = -y$

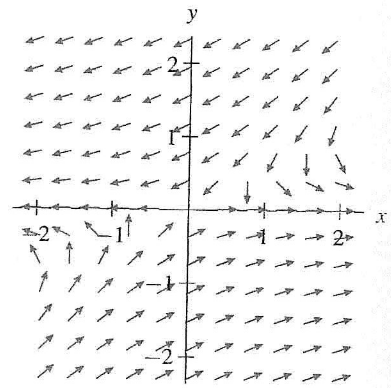
(vii) $\dfrac{dx}{dt} = x^2 - 1$, $\quad \dfrac{dy}{dt} = -y$

(viii) $\dfrac{dx}{dt} = x - 2y$, $\quad \dfrac{dy}{dt} = -y$

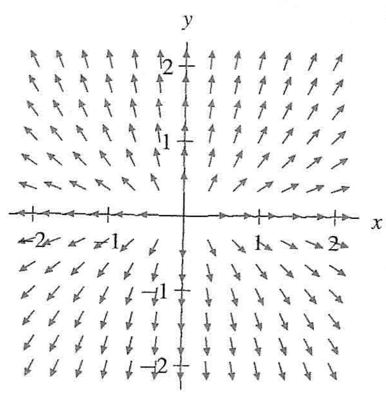
(a)



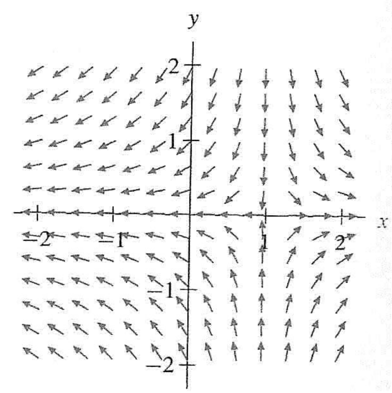
(b)



(c)



(d)



4. The figure (next page) shows eight systems of differential equations and four direction fields. For each direction field, determine which system of equations it corresponds to. Explain your reasoning carefully.

(a) Direction field (a) corresponds to system Vii because: There are two equilibria, so the system is nonlinear, indicating (ii) or (vii).

If $x = 1$ and $y > 0$ then $y' < 0$, inconsistent with (ii), but consistent with (vii).

(b) Direction field (b) corresponds to system Viii because: There is one saddle equilibrium. If $y = 0$, then x' is proportional to x and $y' = 0$, consistent only with (iii) or (viii).

At $x = 1, y = 1$ we have $x' = 3, y' = -1$ for (iii), which doesn't match

At $x = 1, y = 1$ we have $x' = -1, y' = -1$ for (viii), which does match.

(c) Direction field (c) corresponds to system V because: There is one source fixed point, consistent with either (v) or (iv).

The y coordinate grows faster than x, indicating (v).

(d) Direction field (d) corresponds to system Vi because: There is one saddle equilibrium at $x = +1$, $y = 0$. The displacement $x - 1$ grows while y shrinks, matching (vi). ~~[illegible]~~