Section 3.1 - Properties of Linear Systems

Consider $\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases}$, a, b, c, d are constants called coefficients of the system

- This system is called a 2D linear system w/ constant coefficients
  - ↳ It is also known as a planar system
- Autonomous system (no t dependence)

Uniqueness Thm: $\begin{cases} \frac{dx}{dt} = ax + by = f \\ \frac{dy}{dt} = cx + dy = g \end{cases}$ $\quad \frac{\partial f}{\partial x} = a, \ \frac{\partial f}{\partial y} = b, \quad \frac{\partial g}{\partial x} = c, \ \frac{\partial g}{\partial y} = d$

Constants exists. so uniqueness and existence thm work here

Let A be a 2x2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and let y be the vector $\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\therefore A\vec{Y} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \Rightarrow$ Our system can be written as: $\frac{d\vec{Y}}{dt} = A\vec{Y}$.

A is called the coefficient matrix

Basic matrix multiplication!

Example 1: Write the system below in matrix form.

$\begin{cases} \frac{dy}{dt} = -5x \\ \frac{dx}{dt} = 2x - 3y \end{cases}$ $\quad \frac{d\vec{Y}}{dt} = \begin{pmatrix} -5 & 0 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Determinant of a 2x2

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det(A) = ad - bc$

Matrices with a zero determinant are called singular or degenerate

Be comfortable taking determinants of matrices

Equilibrium Points

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{cases} ax_0 + by_0 = 0 \\ cx_0 + dy_0 = 0 \end{cases}$$

$\vec{Y}(x_0, y_0) = (0, 0)$ is a trivial solution

Thm: If A is a matrix with $\det(A) \neq 0$, then the only equilibrium point for the system is the origin

$\det(A) \neq 0 \Rightarrow$ matrix A is nonsingular $\rightarrow$ columns of A are linearly independent vectors

$A = \left(\begin{array}{c|c} a & b \\ c & d \end{array}\right)$ Columns: $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$

Recall: $\vec{u} = \lambda\vec{w}$ for some constant λ if $\vec{u}$ and $\vec{w}$ are linearly dependent

$\det(A) = 0 \Rightarrow$ matrix A is singular

The Linearity Principle

Suppose $\frac{d\vec{Y}}{dt} = A\vec{Y}$ is a linear system of DEs

Infinitely many solutions when $KY_1 + KY_2$

① If $\vec{Y}(t)$ is a solution of the system and K is a constant, $K\vec{Y}(t)$ is also a solution

Proof: Let $\vec{Y}(t)$ be a solution of $\frac{d\vec{Y}}{dt}$

$$\text{If } \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad K\vec{Y} = \begin{pmatrix} Kx \\ Ky \end{pmatrix} \Rightarrow \frac{d(K\vec{Y})}{dt} = \begin{pmatrix} K\frac{dx}{dt} \\ K\frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} Kx \\ Ky \end{pmatrix}$$

$$\text{Then } \begin{pmatrix} aKx + bKy \\ cKx + dKy \end{pmatrix} = \begin{pmatrix} K(ax + by) \\ K(cx + dy) \end{pmatrix} = K(\vec{Y}) = K\vec{Y}$$

Yay!

② If $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are two solutions of the system, $Y_1(t) + Y_2(t)$ is also a solution

Proof: Let $Y_1(t)$ and $Y_2(t)$ be solutions of $\frac{d\vec{y}}{dt}$

$$\vec{y} = \vec{y}_1 + \vec{y}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \text{ where } \vec{y}_1 = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \text{ and } \vec{y}_2 = \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$$

$$\frac{d(\vec{y}_1 + \vec{y}_2)}{dt} = \begin{pmatrix} \frac{dx_1}{dt} + \frac{dx_2}{dt} \\ \frac{dy_1}{dt} + \frac{dy_2}{dt} \end{pmatrix} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dy_1}{dt} \end{pmatrix} + \begin{pmatrix} \frac{dx_2}{dt} \\ \frac{dy_2}{dt} \end{pmatrix} = \frac{d\vec{y}_1}{dt} + \frac{d\vec{y}_2}{dt} \longrightarrow \frac{dy_1}{dt} + \frac{dy_2}{dt} = A\vec{Y}_1 + A\vec{Y}_2$$

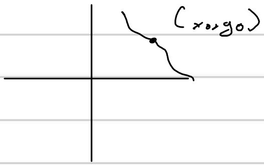
$$A(\vec{y}_1 + \vec{y}_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} a(x_1 + x_2) + b(y_1 + y_2) \\ c(x_1 + x_2) + d(y_1 + y_2) \end{pmatrix} = \begin{pmatrix} ax_1 + by_1 \\ cx_1 + dy_1 \end{pmatrix} + \begin{pmatrix} ax_2 + by_2 \\ cx_2 + dy_2 \end{pmatrix} = A\vec{Y}_1 + A\vec{Y}_2$$

Initial Value Problems

Given two solutions $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$, we know $K_1\vec{Y}_1 + K_2\vec{Y}_2$ is also a solution. Are all of these solutions?

Will the solution to any IVP be in this form? ← or $\vec{y}_3(t) \neq K_1\vec{y}_1 + K_2\vec{y}_2$ but $\vec{y}_3(t)$ is a solution

Proof:
$$\begin{cases} \frac{dx}{dt} = ax+by \\ \frac{dy}{dt} = cx+dy \\ x(t_0)=x_0 \\ y(t_0)=y_0 \end{cases}$$



(x_0,y_0) is an arbitrary solution. then $\vec{y}_3 = K_1\vec{y}_1+K_2\vec{y}_2$ as $K_1\vec{y}_1+K_2\vec{y}_2$ describes all solutions by Uniqueness thm.

$\therefore \vec{y}(t) = K_1\vec{y}_1 + K_2\vec{y}_2$

$\frac{\partial f}{\partial x}=a, \frac{\partial f}{\partial y}=b, \frac{\partial g}{\partial x}=c, \frac{dg}{dy}=d$

Linear Independence

← Vectors that don't lie on the same line, or vectors that are not constant multiples of each other

Suppose (x_1,y_1) and (x_2,y_2) are two linearly independent vectors in the plane. For any given vector (x_0,y_0), there are constants K_1 and K_2 s.t. $K_1\begin{pmatrix}x_1\\y_1\end{pmatrix} + K_2\begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}x_0\\y_0\end{pmatrix}$

$\forall \vec{v}$, if $K_1, K_2 \in \mathbb{R}$, then $\vec{v} = K_1\vec{y}_1 + K_2\vec{y}_2$ for linearly independent vectors y_1 and y_2

You can express any vector in $\mathbb{R}^2$ space using multiples of linearly independent vectors.

Suppose $\vec{y}_1$ and $\vec{y}_2$ are solutions of a linear system $\frac{d\vec{Y}}{dt} = A\vec{Y}$. If $\vec{Y}_1(0)$ and $\vec{Y}_2(0)$ are linearly independent, then for any initial condition $\vec{Y}(0) = (x_0, y_0)$, we can find constants K_1 and K_2 so that $K_1\vec{Y}_1 + K_2\vec{Y}_2$ is the solution to the IVP

Section 3.2 - Straight Line Solutions

Example to Introduce Topic

Consider $\frac{d\vec{Y}}{dt} = A\vec{Y} = \begin{pmatrix}4&2\\1&3\end{pmatrix}\vec{Y} \iff \begin{cases}\frac{dx}{dt} = 4x+2y\\ \frac{dy}{dt} = x+3y\end{cases}$

$A=\begin{pmatrix}4&2\\1&3\end{pmatrix}$ ∴ $\det(A) = 12\ 2 = 10 \neq 0$ → linearly independent solutions

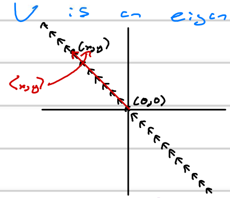
Eigenvectors and Eigenvalues ← It is central to this course to understand this

Along a straight-line solution, the vector field must point either directly towards or away from the origin

↳ At a point (x,y) on a straight-line solution, $\vec{F}(x,y)$ must be some multiple of (x,y). We seek vectors (x,y) s.t. $A\vec{Y} = \lambda\vec{Y} \iff \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \lambda\begin{pmatrix}x\\y\end{pmatrix}$

Given A, a scalar λ is an eigenvalue for A if $\exists\, \vec{v} \neq 0$ s.t. $A\vec{v} = \lambda\vec{v}$. The vector $\vec{v}$ is called an eigenvector corresponding to λ ← eigen = characteristic

If $\vec{v}$ is an eigenvector then any multiple of $\vec{v}$ also is.



$\begin{cases}\frac{dx}{dt} = ax+by\\ \frac{dy}{dt} = cx+dy\end{cases}$

Vector $\langle x,y\rangle$ is tangent to $\frac{d\vec{Y}}{dt} = \begin{pmatrix}ax+by\\cx+dy\end{pmatrix} = \lambda\begin{pmatrix}x\\y\end{pmatrix}$ ← Can change the length of the vector and still remain on straight-line solution

Basically a fancy way of saying any constant multiple of a vector that is on a straight line solution is also a solution

Let $\begin{cases}\vec{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix}\\ A\begin{pmatrix}a&b\\c&d\end{pmatrix}\end{cases}$ $A\vec{v} = \begin{pmatrix}av_1+bv_2\\cv_1+dv_2\end{pmatrix} = \lambda\vec{v} = \begin{pmatrix}\lambda v_1\\ \lambda v_2\end{pmatrix}$

System: $\begin{matrix}av_1+bv_2=\lambda v_1\\ cv_1+dv_2=\lambda v_2\end{matrix} \Rightarrow \begin{cases}(a-\lambda)v_1+bv_2=0\\ cv_1+(d-\lambda)v_2=0\end{cases}$ Vectors $\langle a-\lambda, b\rangle$ and $\langle c, d-\lambda\rangle$ must be linearly dependent

$\begin{pmatrix}a-\lambda & b\\ c & d-\lambda\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ To be linearly dependent,

$$\det\begin{pmatrix}a-\lambda & b\\ c & d-\lambda\end{pmatrix} = 0$$

$$(a-\lambda)(d-\lambda)-cb=0$$

$$ad-\lambda(a+b)+\lambda^2-cb=0$$

$$\lambda^2-\lambda(a+b)+(ad-bc)=0$$

Characteristic Equation or Polynomial

roots are eigenvalues

Complex roots covered in 3.4/3.5

Eigenvalue
↳ characteristic value

Eigenvector
↳ characteristic vector

Example 1

Consider $\frac{d\vec{Y}}{dt} = A\vec{Y} = \begin{pmatrix}4&2\\1&3\end{pmatrix}\begin{pmatrix}\vec{Y}\end{pmatrix} \iff \begin{cases}\frac{dx}{dt} = 4x+2y\\ \frac{dy}{dt} = x+3y\end{cases}$ Find straight-line solutions

$\begin{pmatrix}4&2\\1&3\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \lambda\begin{pmatrix}v_1\\v_2\end{pmatrix}$ $\det\begin{pmatrix}4-\lambda & 2\\ 1 & 3-\lambda\end{pmatrix} = 0$

$$(4-\lambda)(3-\lambda)-2=0$$

$$\lambda^2-7\lambda+10=0$$ Factor or quad formula

$$(\lambda-5)(\lambda-2)=0$$

$\lambda_1 = 5, \lambda_2 = 2$ ← Different eigenvalues have different eigenvectors

λ_2: $\begin{pmatrix}4&2\\1&3\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = 2\begin{pmatrix}v_1\\v_2\end{pmatrix}$

$\begin{cases}4v_1+2v_2=2v_1\\ v_1+3v_2=2v_2\end{cases} \Rightarrow \begin{cases}2v_1+2v_2=0\\ v_1+v_2=0\end{cases} \rightarrow v_1=-v_2$ (linearly dependent)

choose easy solution ↓

$\begin{pmatrix}v_1\\v_2\end{pmatrix} = C\begin{pmatrix}1\\-1\end{pmatrix} \Rightarrow \begin{pmatrix}C\\-C\end{pmatrix}$ General Vector — Infinitely Many

λ_1: $\begin{pmatrix}4&2\\1&3\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = 5\begin{pmatrix}v_1\\v_2\end{pmatrix}$

$\begin{cases}4v_1+2v_2=5v_1\\ v_1+3v_2=5v_2\end{cases} \Rightarrow \begin{cases}-v_1+2v_2=0\\ v_1-2v_2=0\end{cases} \rightarrow v_1=2v_2$ (linearly dependent)

$\begin{pmatrix}v_1\\2v_2\end{pmatrix} = C\begin{pmatrix}1\\2\end{pmatrix} \Rightarrow \begin{pmatrix}C\\2C\end{pmatrix}$

Theorem

Suppose the matrix A has a real eigenvalue λ associated with eigenvector $\vec{v}$. Then the linear system $\frac{d\vec{Y}}{dt} = A\vec{Y}$ has the straight line solution $\vec{Y}(t) = e^{\lambda t}\vec{v}$

Link back to example 1:

$$\lambda_1 = 2: \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \lambda_2 = 5: \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightarrow \begin{matrix} \vec{y}_1 = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} \\ \vec{y}_2 = e^{5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{5t} \\ 2e^{5t} \end{pmatrix} \end{matrix}$$

Two linearly independent solutions

Linearity principle: $y(t)$ is a solution where $y_1(t) + y_2(t) = y(t)$, if y_1 and y_2 are also solutions

General Solution: $\vec{y}(t) = K_1\begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} + K_2\begin{pmatrix} e^{5t} \\ 2e^{5t} \end{pmatrix}$

$$\vec{y}(t) = \begin{pmatrix} K_1e^{2t} + K_2e^{5t} \\ -K_1e^{2t} + 2K_2e^{5t} \end{pmatrix}$$

Distinct eigenvalues correspond to linearly independent eigenvectors

Theorem

If λ_1 and λ_2 are real distinct eigenvalues with eigenvectors $\vec{v}_1$ and $\vec{v}_2$, then the two solutions are linearly independent and

$\vec{Y}(t) = K_1e^{\lambda_1 t}\vec{v}_1 + K_2e^{\lambda_2 t}\vec{v}_2$ is the general solution to the system.

Example 2

Characteristic Equation

$$p(\lambda) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

Solve the IVP $\begin{cases} \frac{dx}{dt} = 3x \\ \frac{dy}{dt} = x - 2y \end{cases}$ $(x(0), y(0)) = (5, 0)$

$A = \begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix}$ $\quad p(\lambda) = \begin{pmatrix} 3-\lambda & 0 \\ 1 & -2-\lambda \end{pmatrix} = (3-\lambda)(-2-\lambda) - 0 = 0$

Characteristic Polynomial

$\lambda_1 = 3, \lambda_2 = -2$

$\lambda_1 = 3: \begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$

$3x = 3x$

$x - 2y = 3y \quad x = 5y$

$v_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ for $\lambda_1 = 3$

$\lambda_2 = -2: \begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$

$3x = -2x \quad x = 0$

$x - 2y = -2y \quad y = C$

$\lambda = -2 \rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ← must be 0 ← y can be anything, choose easiest

$\therefore \vec{Y}(t) = K_1e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + K_2e^{3t}\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ ← use thm. to get solution vector

$t = 0$: $\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5K_2 \\ K_1 + K_2 \end{pmatrix}$ $\quad 5K_2 = 5 \quad K_2 = 1 \quad K_1 + K_2 = 0 \quad K_1 = -1$

$$\boxed{\begin{matrix} \vec{Y}(t) = -e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} \\ \text{or } \vec{Y}(t) = \begin{pmatrix} 5e^{3t} \\ -e^{-2t} + e^{3t} \end{pmatrix} \end{matrix}}$$

Section 3.3 - Phase Portraits for Real Eigenvalues

Consider $\frac{d\vec{Y}}{dt} = A\vec{Y}$ with two distinct real eigenvalues $\lambda_1 < \lambda_2$

The equilibrium point at the origin is called:

- Sink if $\lambda_1 < \lambda_2 < 0$ ← both negative
- Source if $0 < \lambda_1 < \lambda_2$ ← both positive
- Saddle if $\lambda_1 < 0$ and $\lambda_2 > 0$ ← One each

Solution Curves tend towards the dominant eigenvector

Example: A sink

$$A = \begin{pmatrix} -1 & -2 \\ 1 & -4 \end{pmatrix} \longrightarrow \lambda_1 = -3 \quad \text{and} \quad \lambda_2 = -2$$

do some work.. $\vec{Y}(t) = \begin{pmatrix} k_1 e^{-3t} + 2k_2 e^{-2t} \\ k_1 e^{-3t} + k_2 e^{-2t} \end{pmatrix}$

$e^{-\lambda t} \to 0$ as $t \to \infty$

Regardless of k_1 and k_2, the solution approaches 0 as $t \to \infty$

Example: A source

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

Characteristic Equation: $p(\lambda) = \det(A - I\lambda) = \det\begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$$= (4-\lambda)(3-\lambda) - 2 = 0$$
$$\lambda^2 - 7\lambda + 12 - 2 = 0$$
$$\lambda^2 - 7\lambda + 10 = 0$$
$$(\lambda - 5)(\lambda - 2) = 0$$
$$\lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 5$$

$$\lambda = \frac{7 \pm \sqrt{49 - 4(10)}}{2} = \frac{7 \pm 3}{2} = 5, 2$$

do some work.. $\vec{Y}(t) = \begin{pmatrix} -k_1 e^{2t} + 2k_2 e^{5t} \\ k_1 e^{2t} + k_2 e^{5t} \end{pmatrix}$

$e^{\lambda t} \to \infty$ as $t \to \infty$

Regardless of k_1 and k_2, the solution approaches ∞ as $t \to \infty$

Example: A saddle

$$A = \begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix} \longrightarrow \lambda_1 = -2 \quad \text{and} \quad \lambda_2 = 3$$

do some work.. $\vec{Y}(t) = \begin{pmatrix} 5k_2 e^{3t} \\ k_1 e^{-2t} + k_2 e^{3t} \end{pmatrix}$

The choice of k_1 and k_2 will result in different behavior around the origin

Section 3.4 - Complex Eigenvalues

Example 1: No apparent straight line solutions

$$\frac{dx}{dt} = 2x+2y, \quad \frac{dy}{dt} = -4x+6y$$

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix} \qquad p(\lambda) = (2-\lambda)(6-\lambda)+8$$

$$0 = \lambda^2 - 8\lambda + 20$$

$$\lambda = \frac{8 \pm \sqrt{64-80}}{2} = \frac{8 \pm 4i}{2} = 4 \pm 2i$$

we want corresponding eigenvectors:

$$\lambda_1 = 4-2i: \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (4-2i)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Solve like normal

Set $v_1 = 1$ then solve for v_2

$$\begin{matrix} 2v_1 + 2v_2 = (4-2i)v_1 \\ -4v_1 + 6v_2 = (4-2i)v_2 \end{matrix} \Rightarrow \begin{cases} (-2+2i)v_1 + 2v_2 = 0 \\ -4v_1 + (2+2i)v_2 = 0 \end{cases} \Rightarrow \begin{matrix} v_1 = 1 \\ v_2 = 1-i \end{matrix}$$

$$\vec{v} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$$\lambda_2 = 4+2i: \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (4+2i)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{matrix} 2v_1 + 2v_2 = (4+2i)v_1 \\ -4v_1 + 6v_2 = (4+2i)v_2 \end{matrix} \Rightarrow \begin{cases} (-2-2i)v_1 + 2v_2 = 0 \\ -4v_1 + (2-2i)v_2 = 0 \end{cases} \Rightarrow \begin{matrix} v_1 = 1 \\ v_2 = 1+i \end{matrix}$$

$$\vec{v} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$v_1 = 1: -2-2i + 2v_2 = 0$$

$$2v_2 = 2+2i \rightarrow v_2 = 1+i$$

Pick easy value and solve for other variable to set =

$$\vec{Y}(t) = e^{\lambda t}\vec{v} = e^{(4+2i)t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix} = \begin{pmatrix} e^{(4+2i)t} \\ (1+i)e^{(4+2i)t} \end{pmatrix}$$

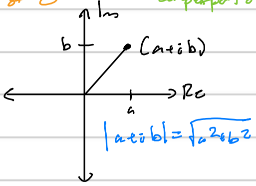
For Conjugates Im part becomes negative. Unnecessary to compute both Eigenvectors and solutions

Complex Numbers

- Complex Conjugates: $a+ib$, $a-ib$
- Any Complex Number C can be written the standard form $C = a \pm ib$
  - ↳ $Re(C) = a$, $Im(C) = b$

real part of C

complexpartof C



$|a+ib| = \sqrt{a^2+b^2}$

- Cannot compare Complex numbers directly but Imagnitude can be found. Its the distance to (0,0)

$$\boxed{mag = \sqrt{a^2+b^2}}$$ Pythagorean

- Conjugates: $(a+ib)(a-ib)$

$$= a^2 - iab + iab + b^2$$

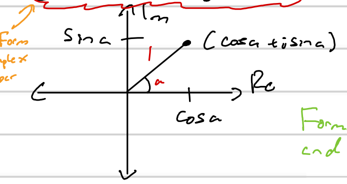
$$= \boxed{a^2 + b^2}$$

Euler's Formula

$$e^{i\pi} = -1$$

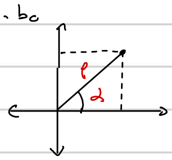
↳ $$\boxed{e^{ia} = \cos a + i \sin a}$$

Polar Form of Complex Number



A complex C can be

$$C = \rho e^{i\alpha}$$



Formula helps seperate real and complex parts of complex numbers

Useful for Multiplication

$$C_1 = \rho_1 e^{i\alpha_1}, \quad C_2 = \rho_2 e^{i\alpha_2}$$

$$C_1 C_2 = \rho_1 \rho_2 e^{i(\alpha_1+\alpha_2)}$$

$$\therefore \quad e^{a+ib} = e^a \cdot e^{ib} = e^a(\cos b + i \sin b)$$ ← Proof uses power series

$\therefore$ for $\lambda = 4+2i$, $e^{(4+2i)t} = e^{4t} \cdot e^{2it} = e^{4t}(\cos(2t) + i\sin(2t))$

$$\vec{Y}(t) = \begin{pmatrix} e^{(4+2i)t} \\ (1+i)e^{(4+2i)t} \end{pmatrix} = \begin{pmatrix} e^{4t}(\cos 2t + i \sin 2t) \\ e^{4t}(1+i)(\cos 2t + i \sin 2t) \end{pmatrix}$$

$$= \begin{pmatrix} e^{4t}(\cos 2t + i \sin 2t \\ e^{4t}(\cos 2t + i \sin 2t + i \cos 2t - \sin 2t \end{pmatrix}$$

$$= \begin{pmatrix} e^{4t}\cos 2t + e^{4t} i \sin 2t \\ e^{4t}(\cos 2t - \sin 2t) + e^{4t} i(\cos 2t + \sin 2t) \end{pmatrix} = \begin{pmatrix} e^{4t}\cos 2t \\ e^{4t}\cos 2t - e^{4t}\sin 2t \end{pmatrix} + i\begin{pmatrix} e^{4t}\sin 2t \\ e^{4t}\cos 2t + e^{4t}\sin 2t \end{pmatrix}$$

Use Eulers Formula to Separate Real and Imaginary parts of solutions

$$\vec{Y}_{re}(t) \quad + \quad i \quad \vec{Y}_{im}(t)$$

Complex Conjugate

Properties of Complex conjugates

For λ_1, $\vec{Y}_{re} + i\vec{Y}_{im}$

For λ_2, $\vec{Y}_{re} - i\vec{Y}_{im}$

Sums/Differences of Solutions are solutions
Scalar multiples of Solutions are also Solutions } Proves the below

$\therefore$ linearity Principle → $\vec{Y}_{re}(t)$ and $\vec{Y}_{im}$ are linearly independent solutions

Theorem

Suppose $\vec{Y}(t)$ is a complex-valued system to a linear system $\frac{d\vec{Y}}{dt} = A\vec{Y}$, where A has real coefficients.

Suppose $\vec{Y}(t) = \vec{Y}_{re}(t) + i\vec{Y}_{im}(t)$, where $\vec{Y}_{re}(t)$ and $\vec{Y}_{im}(t)$ are real valued functions of t.

$\therefore$ The real, $\vec{Y}_{re}(t)$, and imaginary, $\vec{Y}_{im}(t)$, are both solutions to the original system $\frac{d\vec{Y}}{dt} = A\vec{Y}$.

Proof in slides if you're interested ☺ ← Page 7

Complex Eigenvalues

Suppose $\frac{d\vec{Y}}{dt} = A\vec{Y}$ is a linear system with complex eigenvalues $\lambda_{1,2} = \alpha \pm i\beta$ for $\beta \neq 0$. Then the complex solutions have the form $\vec{Y}(t) = e^{(\alpha + i\beta)t}\vec{V}$ where $\vec{V}$ is a complex eigenvector. This can be written as:

$$\vec{Y}(t) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))\vec{V}$$

- $\cos(\beta t)$ and $\sin(\beta t)$ oscillate with period $2\pi/\beta$. This is the natural period
- The natural frequency is the number of cycles that solutions make in one unit of time $\left(\frac{\beta}{2\pi}\right)$
- β is the angular frequency in terms of radians per unit time

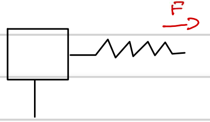
Behavior:

α value

$\alpha < 0 \rightarrow$ solutions spiral to origin — spiral sink
$\alpha > 0 \rightarrow$ solutions spiral from origin — spiral source
$\alpha = 0 \rightarrow$ solutions are periodic — center

Classification of Origin

Example: Springs w/o friction



Frictionless

$\vec{F} = my'' = -ky$

$\therefore my'' + ky = 0$

$$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -\left(\frac{k}{m}\right)y \end{cases}$$

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}$$

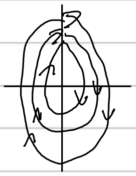
$p(\lambda) = (-\lambda)^2 + \frac{k}{m}$

$0 = \lambda^2 + \frac{k}{m}$

$0 = \left(\lambda - i\sqrt{\frac{k}{m}}\right)\left(\lambda + i\sqrt{\frac{k}{m}}\right)$

$\lambda_1 = i\sqrt{\frac{k}{m}}$ and $\lambda_2 = -i\sqrt{\frac{k}{m}}$

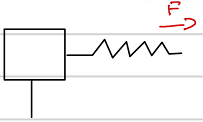
$\alpha = 0$, so origin is a center



To get direction of curves, check one point. All other curves will follow that same direction.

↑ either Cw or ccw

Example: Springs w/ friction



$\vec{F} = my'' = -by' - ky$

$\therefore my'' + by' + ky = 0$

$$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -\left(\frac{k}{m}\right)y + bv \end{cases}$$

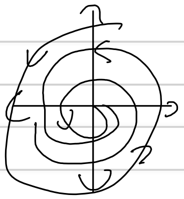
Example Sink

$$\begin{cases} \frac{dx}{dt} = -3x - 5y \\ \frac{dy}{dt} = 3x - y \end{cases}$$

$p(\lambda) = (-3-\lambda)(-1-\lambda) + 15$

$0 = \lambda^2 + 4\lambda + 18$

$\lambda = -2 \pm i\sqrt{14}$



Section 3.5 - Repeated Eigenvalues

Suppose D, the discriminant of $p(\lambda)$ is 0, so $\lambda_1 = \lambda_2 = \vec{v}$. Then $\vec{v}$ is a repeated eigenvalue

Example 1

$$A = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \qquad \begin{aligned} p(\lambda) &= (-2-\lambda)(-4-\lambda)+1 \\ &= \lambda^2 + 6\lambda + 9 \\ 0 &= (\lambda+3)^2 \Rightarrow \lambda = -3 \end{aligned}$$

$$\lambda = -3: \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} -2x - y = -3x \\ x - 4y = -3y \end{cases} \Rightarrow \begin{cases} x - y = 0 \\ x - y = 0 \end{cases} \qquad \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \vec{y}_1(t) = e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot K_1$$

General Solutions for Repeated Eigenvalues

$\vec{Y}(t) = e^{\lambda t} V_0 + t e^{\lambda t} \vec{V}_1$ where $\vec{V}_0 = (x_0, y_0)$ is an arbitrary initial condition and $\vec{V}_1$ is determined from $\vec{V}_0$ as $V_1 = (A - \lambda I) V_0$.

Recall: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- If $\vec{V}_1 = \vec{0}$, then $\vec{V}_0$ is an eigenvector and $\vec{Y}(t)$ is a straight line solution
- Otherwise $\vec{V}_1$ is an eigenvector

$$\left(\vec{y}(t) = e^{\lambda t}\vec{V}_0 + t e^{\lambda t}\vec{V}_1\right) A$$

$$A\vec{y}(t) = e^{\lambda t} A\vec{V}_0 + t e^{\lambda t} A\vec{V}_1 \leftarrow \text{vector}$$

$$\frac{d\vec{y}}{dt} = e^{\lambda t}(\lambda\vec{V}_0 + \vec{V}_1) + t e^{\lambda t}(\lambda\vec{V}_1) = e^{\lambda t}(A\vec{V}_0) + t e^{\lambda t}(AV_1)$$

$$\begin{cases} \lambda\vec{V}_0 + \vec{V}_1 = A\vec{V}_0 \\ A\vec{V}_1 = \lambda\vec{V}_1 \end{cases}$$

V_1 is eigenvector of A

λ is eigenvector of A

Find V_0 from System $\Rightarrow \vec{V}_1 = (A - \lambda I)\vec{V}_0$

Example 2

$$\frac{dY}{dt} = A\vec{Y}, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}, \quad \vec{Y}(0) = (1, 0), \quad \lambda = 3$$

$$V_1 = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \left(\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\vec{V}_0$$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

$-x_0 + y_0 = K_1$

$-x_0 + y_0 = K_2$

$K_1 = K_2 = y_0 - x_0$

General Solution

$$\vec{Y}(t) = e^{3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^{3t}\begin{pmatrix} y_0 - x_0 \\ y_0 - x_0 \end{pmatrix}$$

Lemma

Two Solutions Y_1 and Y_2

- $\vec{Y}_1(t) = V_0 e^{\lambda t} + V_1 t e^{\lambda t}$
- $\vec{Y}_2(t) = W_0 e^{\lambda t} + W_1 t e^{\lambda t}$

are equal for all t iff $V_0 = W_0$ and $V_1 = W_1$

Question

Is vector $\vec{y}_2 = e^{\lambda t}\vec{V}_0 + t e^{\lambda t}\vec{V}_1$ linearly independent with $\vec{y}_1(t) = e^{\lambda t} V_1$

Recall; Linearly Independent if $\forall \alpha \in \mathbb{R}$ $\vec{y}_1 \neq \alpha\vec{y}_2$

Yes they are linearly Independent

Section 3.6 - Summary

Second Order Linear equations

$ay'' + by' + cy = 0$

① Find Characteristic Polynomial Roots

- $as^2 + bs + c = 0$

② Determine steps to take based on roots

Value of s	General Solution
$s_1 \neq s_2 \in \mathbb{R}$	$y(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$
$s = \alpha \pm \beta i$ in $\mathbb{C}$	$y(t) = K_1 e^{\alpha t}\cos(\beta t) + K_2 e^{\alpha t}\sin(\beta t)$
s in $\mathbb{R}$ repeated	$y(t) = K_1 e^{st} + K_2 t e^{st}$

Using Matrices

① Setup matrix from System

$y'' + py' + qy = 0 \quad \begin{cases} y' = v \\ v' = -pv - qy \end{cases}$

$$\vec{v}(t) = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix} \longrightarrow \frac{d\vec{v}}{dt} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\vec{v}$$

② Determine eigenvalues and eigenvectors from characteristic Polynomial

Unforced Harmonic Oscillators

Standard Form: $my'' + by' + ky = 0$

- Damped when $b > 0$
- Undamped when $b = 0$
- Not an Oscillator when $b < 0$

Case 2: Underdamped

- $b^2 - 4km < 0$
- $s = \alpha \pm \beta i$ with $\alpha > 0$
- Solutions Oscillate with period $2\pi/\beta$ and tend towards $y = 0$

Case 3a) Overdamped

- Eigenvalues $\in \mathbb{R}^-$, s_1 and s_2

Case 3b) Critically Damped

- One negative Eigenvalue s
- $y(t) = K_1 e^{st} + K_2 t e^{st}$
- No oscillation and tend to $y = 0$

Section 3.7 - Trace-Determinant Plane

- A way of summarizing everything learned so far

Consider $\dfrac{d\vec{Y}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\vec{Y}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Definition

$\text{Trace}(A) = \text{Tr}(A) = a + d$

Sum of its Diagonal Elements

Characteristic Polynomial: $p(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc)$. We let:

- $T = a + d$, the trace of the matrix A
- $D = ad - bc$, the determinant of the matrix

$p(\lambda) = \lambda^2 - \text{Trace}(A) + \det(A)$

We can rewrite $p(\lambda)$ as $p(\lambda) = \lambda^2 - T\lambda + D$ with roots

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} \longrightarrow \lambda = \frac{T}{2} \pm \frac{\sqrt{T^2 - 4D}}{2} = \frac{T}{2} \pm \sqrt{\frac{T^2}{4} - D}$$

Case A: Complex Eigenvalues $(D > T^2/4)$

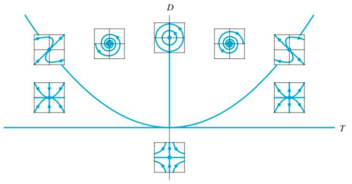
- Eigenvalues are complex with $\alpha = T/2$
  - ① $T < 0$: Origin is a spiral sink
  - ② $T > 0$: Origin is a spiral source
  - ③ $T = 0$: Origin is a center

Case B: Repeated Eigenvalue $(D = T^2/4)$

① $T < 0$: Origin is repeated root sink

② $T > 0$: Origin is repeated root source

End Result



Harmonic Oscillator

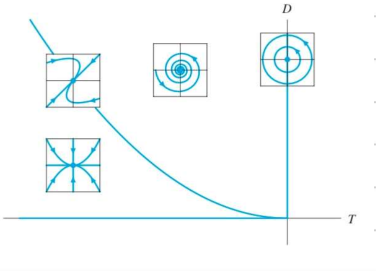
$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \longrightarrow \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}$$

$m > 0$, $k > 0$, and $b \geq 0$

mass

spring constant

damping coefficient



Changing **Parameters in a linear system**

- Different values of D and T can drastically change behavior as seen in the summary products
- Bifurcation Behavior

Example: Bifurcations

Consider $\frac{d\vec{v}}{dt} = \begin{pmatrix} a & a \\ 1 & 0 \end{pmatrix} \vec{v}$, and find bifurcation value(s)

$A = \begin{pmatrix} a & a \\ 1 & 0 \end{pmatrix}$ $\quad$ $\text{Tr}(A) = a$, $\quad$ $\det(A) = -a$

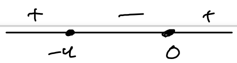
Recall: $\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} \Rightarrow \lambda = \frac{a \pm \sqrt{a^2 + 4a}}{2}$

Bifurcation points:

$a^2 + 4a = 0$

$a(a+4) = 0$

$a = 0, \; a = -4$



$a < -4$: Two real roots, sink $\rightarrow \frac{a - \sqrt{a^2+4a}}{2} < 0$ and $\frac{a + \sqrt{a^2+4a}}{2} < 0$

First Bifurcation Point $\boxed{a = -4}$:

$-4 < a < 0$: Two complex roots, spiral sink

Second Bifurcation Point $\boxed{a = 0}$:

$a > 0$: Two real roots, node (saddle) $\rightarrow \frac{a - \sqrt{a^2+4a}}{2} < 0$ and $\frac{a + \sqrt{a^2+4a}}{2}$

Example: A different system

Consider $\frac{d\vec{y}}{dt} = \begin{pmatrix} -2 & a \\ -2 & 0 \end{pmatrix} \vec{y}$, find equilibrium / bifurcation points

$\text{Trace}(A) = -2$

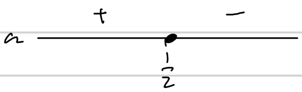
$\det(A = 2a$

$\lambda = \frac{-2 \pm \sqrt{4 - 8a}}{2} = -1 \pm \sqrt{1 - 2a}$

$4 - 8a = 0$

$a = \frac{1}{2}$

$4 - 8a$



$a < \frac{1}{2}$: Two real eigenvalues

$\sqrt{1 - 2a} > 1$

Section 4.1 - Forced Harmonic Oscillators

Undamped: $my'' = -ky - by'$

A damping force adds a function of time to the RHS

Forced: $my'' = ky - by' + f(t)$

$\Rightarrow y'' + py' + qy = g(t)$

where $p = \frac{b}{m}$, $q = \frac{k}{m}$, $g(t) = \frac{f(t)}{m}$

$g(t)$ is the forcing function

Second order, linear, nonhomogeneous, non autonomous differential equation

Extended Linearity Principle

Use this to solve Forced Harmonic Oscillators

(NH): $y'' + p_0 y' + qy = g(t)$

(H): $y'' + p_0 y' + qy = 0$

① If $y_h(t)$ is a sln to (H) and $y_p(t)$ is a sln to (NH), $y_h(t) + y_p(t)$ is also a sln to (NH)

② If $y_p(t)$ and $y_q(t)$ are both slns to (NH), then $y_p(t) - y_q(t)$ is a sln to (H)

If $K_1 y_1(t) + K_2 y_2(t)$ is the general sln to (H), then $y(t) = K_1 y_1(t) + K_2 y_2(t) + y_p(t)$ is the general sln to (NH)

③ General Solution of (H): $K_1 y_1(t) + K_2 y_2(t)$ for some linearly independent functions $y_1(t)$ and $y_2(t)$

Example: $y'' + 4y' + 13y = -2e^{-2t}$

(NH): $y'' + 4y' + 13y = -2e^{-2t}$

(H): $y'' + 4y' + 13y = 0$

Guess Sln to (H): $y(t) = we^{st}$, plug into (H)

$ws^2e^{st} + 4wse^{st} + 13we^{st} = 0$

$\cancel{we^{st}}(s^2 + 4s + 13) = 0$ characteristic equation

$s = \frac{-4 \pm \sqrt{16 - 4(13)}}{2} \Rightarrow s = -2 \pm 3i$

$y(t) = e^{(2-3i)t} = e^{-2t} \cdot e^{-3it}$

$= e^{-2t}(\cos(3t) - i\sin(3t))$

$y_1(t) = e^{-2t}\cos(3t)$

$y_2(t) = e^{-2t}\sin(3t)$

$\therefore y_h(t) = K_1 e^{-2t}\cos(3t) + K_2 e^{-2t}\sin(3t)$

Guess Sln to (NH): $y_p(t) = Ae^{-2t}$

$\cancel{e^{-2t}}(4A - 8A + 13A) = -2\cancel{e^{-2t}}$

$9A = -2$

$A = -\frac{2}{9}$

$y_p(t) = -\frac{2}{9}e^{-2t}$

$$\boxed{\therefore y(t) = K_1 e^{-2t}\cos(3t) + K_2 e^{-2t}\sin(3t) - \frac{2}{9}e^{-2t}}$$

Some Vocabulary

For large t values, initial conditions do not affect behavior.

① $y_p(t)$ is called the forced response, or steady state response

② $y_h(t)$ is called the natural response, or free response

Behavior is more complicated when Damping Coefficient is 0.

For large t values...

$y(t) = K_1 y_1(t) + K_2 y_2(t) + y_p(t) \approx y_p(t)$

External Force ↓

Example: $y'' + 2y' + y = e^{-t}$

Can predict behavior will decay as $t \to \infty$

(NH): $y'' + 2y' + y = e^{-t}$

(H): $y'' + 2y' + y = 0$

Guess sln to (H): $y_h(t) = e^{st}$

$\cancel{e^{st}}(s^2 + 2s + 1) = 0$

$s^2 + 2s + 1 = 0$

oops! $(s+1)^2 = 0 \to s_1 = s_2 = -1$

Just add t factor!

$y_h(t) = K_1 e^{-t} + K_2 te^{-t}$

Question

Suppose $\tilde{y}(t)$ is also a solution to (NH)

$y(t) - \tilde{y}(t)$ is a solution of

$y'' + 2y' + y = 0$ ← (H) part

Guess sln to (NH): $y_p(t) = Ae^{-t}$

$Ae^{-t} - 2Ae^{-t} + Ae^{-t} = e^{-t}$

oops! $0 = e^{-t}$ Cannot be true ever !!

Guess again w/ t factor: $y_p(t) = Ate^{-t}$ ← Don't even need to consider as its part of $y_h(t)$

$y' = Ae^{-t} - Ate^{-t}$

$y'' = -Ae^{-t} - A(e^{-t} - te^{-t})$

$\underbrace{-Ae^{-t} - Ae^{-t}}_{-2Ae^{-t}} + \underbrace{Ate^{-t}}_{\text{Cancels}} + \underbrace{2Ae^{-t}}_{\text{Cancels}} - \underbrace{2Ate^{-t} + Ate^{-t}}_{-Ate^{-t}} = e^{-t}$

Guess again w/ t^2: $y_p(t) = At^2e^{-t}$

$y' = A(2te^{-t} - t^2e^{-t}) = 2Ate^{-t} - At^2e^{-t}$

$y'' = A(2e^{-t} - 2te^{-t} - 2te^{-t} + t^2e^{-t})$

$= A(2e^{-t} - 4te^{-t} + t^2e^{-t}) = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$

$t^2e^{-t}(A - 2A + A) + te^{-t}(-2A + 2A) + e^{-t}(2A) = e^{-t}$

$e^{-t}2A = e^{-t}$

$2A = 1 \to A = \frac{1}{2}$

$$\boxed{\therefore y(t) = \frac{1}{2}t^2e^{-t} + K_1 e^{-t} + K_2 te^{-t}}$$

Choose a guess that is not a part of $y_h(t)$ for any values of K_1, K_2

$\lim_{t \to \infty} y(t) = 0$

Example: $y''+5y'+6y=2$ $\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$

Solve the IVP

(NH): $y''+5y'+6y=2$

(H): $y''+5y'+6y=0$

$p(\lambda)=\lambda^2+5\lambda+6=0$; $\lambda_1=-2, \lambda_2=-3$

$y_h(t)=K_1e^{-3t}+K_2e^{-2t}$

Guess: $y_p(t)=A$

$6A=2$

$A=\frac{1}{3}$

$y(t)=\frac{1}{3}+K_1e^{-3t}+K_2e^{-2t}$

Have to find Particular Soln

from initial condition

IVP: $y(0)=\frac{1}{3}+K_1+K_2=0$

$y'(0)=-3K_1-2K_2=0$

$K_2=-\frac{3}{2}K_1$ $K_1=\frac{2}{3}, K_2=-1$

$$\boxed{y(t)=\frac{1}{3}+\frac{2}{3}e^{-3t}-e^{-2t}}$$

Section 4.2 - Sinusoidal Functions

We will consider $y''+py'+qy=g(t)$ where $g(t)$ is a sine or cosine function

Definition: Here $g(t+T)=g(t)$ is true for the forcing period T

This is called Periodic Forcing

Algorithm

① Find general Solution to (H)

② Find particular solution to (NH)

Guess: $y_p(t)=\alpha\cos(t)+\beta\sin t$

multiply t by ω if (NH) has $e^{\omega t}$

There is another method using complex numbers

Example: $y''+3y'+2y=5\cos t$

(NH): $y''+3y'+2y=5\cos t$

(H): $y''+3y'+2y=0$

$s^2+3s+2=0$

$(s+2)(s+1)=0$; $s_1=-2, s_2=-1$

$y_h(t)=K_1e^{-2t}+K_2e^{-t}$

$$\boxed{y(t)=K_1e^{-2t}+K_2e^{-t}+\frac{1}{2}\cos t+\frac{3}{2}\sin t.}$$

Guess: $y_p(t)=\alpha\cos t+\beta\sin t$

$y'=-\alpha\sin t+\beta\cos t$

$y''=-\alpha\cos t-\beta\sin t$

$\underbrace{-\alpha\cos t}\,\underbrace{-\beta\sin t}\,\underbrace{-3\alpha\sin t}+\underbrace{3\beta\cos t}+\underbrace{2\alpha\cos t}+\underbrace{2\beta\sin t}=5\cos t$

$\cos t(-2\alpha+3\beta+\alpha)+\sin t(2\beta-3\alpha-\beta)=5\cos t+0\sin t$

$-\alpha+3\beta+2\alpha=5$

$-\beta-3\alpha+2\beta=0$

$\begin{cases} 3\beta+\alpha=5 \\ \beta-3\alpha=0 \end{cases}$

$\alpha=\frac{1}{3}\beta$ $3\beta+\frac{1}{3}\beta=5$

$\alpha=\frac{1}{2}$ $\frac{10}{3}\beta=5$

$\beta=\frac{3}{2}$

$y_p(t)=\frac{1}{2}\cos t+\frac{3}{2}\sin t$

Prove $A\sin t+B\cos t=C\sin t+D\cos t$

iff $A=C$ and $B=D$

$t=0$: $B=D$ ✓

$t=\frac{\pi}{2}$: $A-B=C-D$

$B=D\rightarrow A=C$ ✓

As $t\to\infty$, all solutions will tend to $y_p(t)=\frac{1}{2}\cos t+\frac{3}{2}\sin t$

Another Method of Solution

Given $y''+py'+qy=g(t)$ where $g(t)=a\sin(\omega t)$ or $g(t)=a\cos(\omega t)$

① Consider $y''+py'+qy=ae^{i\omega t}$

② Guess a complex solution: $y_c(t)=\alpha e^{i\omega t}$

$y_c(t)=\alpha e^{i\omega t}$

$=\alpha(\cos(\omega t)+i\sin(\omega t))$

$=y_{re}(t)+i\,y_{im}(t)$ ⇐ After Multiplying α through eq.

③ If original $g(t)$ was $a\sin(\omega t)$, then $y_p(t)=y_{im}(t)$

If original $g(t)$ was $a\cos(\omega t)$, then $y_p(t)=y_{re}(t)$

Example 2: $y''+4y'+20y=-3\sin(2t)$

(NH) $y''+4y'+20y=-3\sin(2t)$

(H) $y''+4y'+20y=0$

$s^2+4s+20=0$

$s=\frac{-4\pm\sqrt{16-80}}{2}$

$s=-2\pm4i$

$y_h(t)=K_1e^{-2t}\cos4t+K_2e^{-2t}\sin4t$

Solve NH: $y''+4y'+20y=-3e^{i2t}$

$y_p(t)=\alpha e^{i2t}$ i is just a constant

$y_p'(t)=2\alpha ie^{i2t}$

$y_p''(t)=-4\alpha e^{i2t}$

$\cancel{e^{i2t}}(-4\alpha+8\alpha i+20\alpha)=-3\cancel{e^{i2t}}$

$\alpha(16+8i)=-3$

$\alpha=\frac{-3}{16+8i}\frac{(16-8i)}{(16-8i)}=\frac{-48+24i}{320}$

Could solve IVP given y, y'

$y_p(t)=\frac{-48+24i}{320}e^{2it}$

$=\left(\frac{-48}{320}+i\frac{24}{320}\right)(\cos2t+i\sin2t)$

$=-\frac{48}{320}\cos2t-\frac{24}{320}\sin2t+i\underbrace{\left(-\frac{48}{320}\sin2t+\frac{24}{320}\cos2t\right)}$

Original DE has sin, so we only want the imaginary part

$\therefore y_p(t)=\frac{3}{40}\cos2t-\frac{3}{20}\sin2t$

$$\boxed{y(t)=K_1e^{-2t}\cos4t+K_2e^{-2t}\sin4t+\frac{3}{40}\cos2t-\frac{3}{20}\sin2t}$$

All solutions converge to $y_p(t)$

Section 4.3 - Undamped Forcing and Resonance

We will consider undamped harmonic oscillators with sinusoidal forcing in the form:

(NH): $y''+qy=g(t)$ where $g(t)=a\sin\omega t$ or $a\cos\omega t$

① First, solve (H): $y''+qy=0$ with general solution

$y_h(t)=K_1\cos\beta t+K_2\sin\beta t$ where $\beta=\sqrt{q}$ y_h oscillates with period $2\pi/\beta$ and amplitude $A=\sqrt{K_1^2+K_2^2}$

Deriving general Solution: $y''+qy=0$

$s^2+q=0$

$s=\pm i\sqrt{q}$ $\rightarrow e^{i\sqrt{q}}=e^{0t}(\cos(\sqrt{q}\,t)+i\sin(\sqrt{q}\,t))$

Let $\beta=\sqrt{q}$

$=\cos\beta t+i\sin\beta t$ $\rightarrow$ $\therefore \boxed{y_h(t)=K_1\cos(\sqrt{q}\,t)+K_2\sin(\sqrt{q}\,t)}$

② Now, find particular solution to NH. Guess $y_p(t)=a\sin\omega t$ or $y_p(t)=a\cos\omega t$ based on $g(t)$ in (NH)

Two Cases: $q\neq\omega^2$ and our guess works $q=\omega^2$ our guess doesn't work

$y(t)=K_1\cos\beta t+K_2\sin\beta t+\frac{a}{q-\omega^2}\sin\omega t$ $\uparrow$ or cos

↳ This is called the Resonant Case

If q and ω^2 are very close, then we're called near resonant

A good example of resonance is a bridge being disrupted and wobbling because of the wind

Exercise 1: Simplify $y_h(t)$

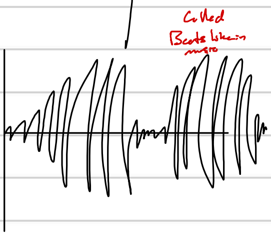
We can say: $K_1\cos\beta t+K_2\sin\beta t=K\cos(\beta t-\varphi)$, where $K_1=K\cos\varphi$ and $K_2=K\sin\varphi$

Proof: $\cos(\beta t-\varphi)=K\cos\beta t\cos\varphi+K\sin\beta t\sin\varphi$

$K=\sqrt{K_1^2+K_2^2}$

$=\sqrt{(K\cos\varphi)^2+(K\sin\varphi)^2}$

$K=\sqrt{K^2}$ trig identity



Example: No Resonance

(NH): $y''+4y=\sin 3t$ $\uparrow$ Freq = 3 No resonance

(H): $y''+4y=0$ Freq = 2

$s^2+4=0$ $y_h(t)=K_1\cos 2t+K_2\sin 2t$

$S=\pm 2i$

Guess $y_p(t)$: $y_p(t)=\alpha e^{i3t}$

$y_p'(t)=3i\alpha e^{i3t}$

$y_p''(t)=-9\alpha e^{i3t}$

Plug in: $e^{i3t}(-9\alpha+4\alpha)=5e^{i3t}$

$\therefore \alpha=-1$

$y_p(t)=-e^{i3t}=-\cos 3t-\underbrace{i\sin 3t}$ $\rightarrow$ $y_p(t)=-\sin 3t$

$\therefore \boxed{y(t)=K_1\cos 2t+K_2\sin 2t-\sin 3t}$

Example: Resonant Case

(NH): $y''+4y=\sin 2t$

(H): $y''+4y=0$

$s^2+4=0;\ s=\pm 2i$

$y_h(t)=K_1\cos 2t+K_2\sin 2t$

Note external force lies in the solution to (H) ⇐ resonant case

Let $y''+4y=e^{2it}$ sin is original, look for Y_{im}

Guess: $y_p=Ae^{i2t}$ doesn't work!

$y_p''(t)=-4Ae^{i2t}$ $\therefore \cancel{-4Ae^{i2t}}+\cancel{4Ae^{i2t}}=e^{2it}$

oops! $0=e^{2it}$

Add t Factor!

Guess: $y_p=Ate^{i2t}$

$y_p'(t)=Ae^{i2t}+2iAte^{i2t}$

$y_p''(t)=2iAe^{i2t}+2i(Ae^{i2t}+2iAte^{i2t})$

$=Ae^{i2t}(4i-4t)$

$\therefore Ae^{i2t}(4i-4t)+4Ate^{i2t}=e^{i2t}$

$A\cancel{e^{i2t}}(4i-4t+4t)=\cancel{e^{i2t}}\rightarrow A=\frac{1}{4i}\frac{(-4i)}{(-4i)}$

Multiply A by conjugate

$A=\frac{-4i}{16}=-\frac{1}{4}i$ $\therefore y_p(t)=-\frac{1}{4}ite^{i2t}$

$y_p(t)=-\frac{1}{4}it(\cos 2t+i\sin 2t)$

$=\underbrace{-\frac{1}{4}it\cos 2t}-\frac{1}{4}t\sin 2t$

$y(t)=K_1\cos 2t+K_2\sin 2t-\frac{1}{4}t\cos 2t$

Plot $y(t)$

Amplitude grows to ∞