Spring 2024 Math 224 Homework 8

Instructions: show all steps to get full credits.

Problem 1. $m = 1$, $k = 8$, $b = 0$, with initial values $y(0) = 1$, $v(0) = 4$.

a) Write the second-order differential equation and convert it to the corresponding first-order linear system.
Answer.

- The second-order differential equation: $\dfrac{d^2y}{dt^2} + 8y = 0$.
- The corresponding system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -8y$$

b) Solve the initial value problem with respect to the corresponding linear system using the eigenvalues-eigenvectors method.
Answer.

- The coefficient matrix $A = \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}$.
- Solve for the eigenvalues:

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -8 & -\lambda \end{bmatrix} = \lambda^2 + 8$$

$\lambda_1 = \sqrt{8}i, \lambda_2 = -\sqrt{8}i$.

- Find an eigenvector for $\lambda_1 = \sqrt{8}i$:

$$\begin{bmatrix} -\sqrt{8}i & 1 \\ -8 & -\sqrt{8}i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \beta = \sqrt{8}\, i\alpha$$

Choose $\alpha = 1$, $\vec{v}_1 = \begin{bmatrix} 1 \\ \sqrt{8}\, i \end{bmatrix}$.

- Decompose $\mathbf{Y}_1(t) = e^{\lambda_1 t}\vec{v}_1$:

$$\begin{aligned}
\mathbf{Y}_1(t) &= e^{\sqrt{8}it}\begin{bmatrix} 1 \\ \sqrt{8}\,i \end{bmatrix} \\
&= (\cos(\sqrt{8}t) + i\sin(\sqrt{8}t))\begin{bmatrix} 1 \\ \sqrt{8}\,i \end{bmatrix} \\
&= \begin{bmatrix} \cos(\sqrt{8}t) \\ -\sqrt{8}\sin(\sqrt{8}t) \end{bmatrix} + i\begin{bmatrix} \sin(\sqrt{8}t) \\ \sqrt{8}\,\cos(\sqrt{8}\,t) \end{bmatrix}
\end{aligned}$$

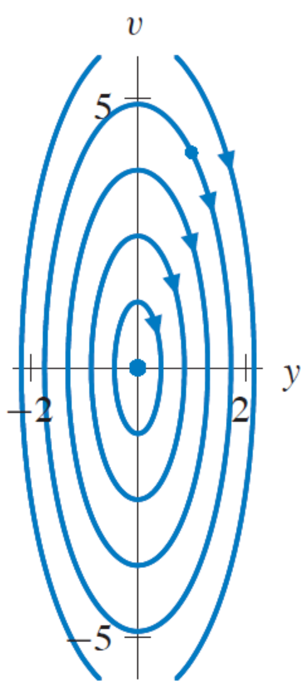
- The general solution:

$$\begin{aligned}
\mathbf{Y}(t) &= k_1\begin{bmatrix} \cos(\sqrt{8}t) \\ -\sqrt{8}\sin(\sqrt{8}t) \end{bmatrix} + k_2\begin{bmatrix} \sin(\sqrt{8}t) \\ \sqrt{8}\,\cos(\sqrt{8}\,t) \end{bmatrix} \\
&= \begin{bmatrix} k_1\cos(\sqrt{8}t) + k_2\sin(\sqrt{8}t) \\ -\sqrt{8}k_1\sin(\sqrt{8}t) + \sqrt{8}k_2\,\cos(\sqrt{8}\,t) \end{bmatrix}
\end{aligned}$$

- Initial values $y(0) = 1$, $v(0) = 4$: $k_1 = 1$, $k_2 = \sqrt{2}$.

$$\mathbf{Y}(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \cos(\sqrt{8}t) + \sqrt{2}\sin(\sqrt{8}t) \\ -\sqrt{8}\sin(\sqrt{8}t) + 4\,\cos(\sqrt{8}\,t) \end{bmatrix}$$

c) Sketch the phase portrait of the corresponding linear system.
Solution. The solutions are periodic and oscillate in the clockwise direction.



d) Classify the oscillator (as underdamped, overdamped, critically damped, or undamped) and, when appropriate, give the natural frenquency and the natural period.
Answer.

- The origin is a center. Hence, the system is underdamped.
- The natural period is $2\pi/\sqrt{8} = \pi/\sqrt{2}$, and the natural frequency is $\sqrt{2}/\pi$.

e) Use the"simplified method" in Lecture Note 3.6 to solve for $y(t)$ and $v(t)$.
Answer.

- The quadratic equation $\lambda^2 + 8 = 0$, so $\lambda = \pm i\sqrt{8}$.
- $y(t) = k_1 \cos(\sqrt{8}t) + k_2 \sin(\sqrt{8}t)$.
- $v(t) = y'(t) = -\sqrt{8}\, k_1 \sin(\sqrt{8}\, t) + \sqrt{8}k_2 \cos(\sqrt{8}\, t)$.
- Initial values Initial values $y(0) = 1$, $v(0) = 4$: $k_1 = 1$, $k_2 = \sqrt{2}$.
- The solutions:

$$\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \cos(\sqrt{8}t) + \sqrt{2}\sin(\sqrt{8}t) \\ -\sqrt{8}\sin(\sqrt{8}t) + 4\,\cos(\sqrt{8}\,t) \end{bmatrix}$$

Problem 2. $m = 9$, $k = 1$, $b = 6$, with initial values $y(0) = v(0) = 1$.

a) Write the second-order differential equation and convert it to the corresponding first-order linear system.
Answer.

- The second-order differential equation: $9\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + y = 0$.
- The corresponding system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{1}{9}y - \frac{2}{3}v. \end{aligned}$$

b) Solve the initial value problem with respect to the corresponding linear system using the eigenvalues-eigenvectors method.
Answer.

- The coefficient matrix $A = \begin{bmatrix} 0 & 1 \\ -1/9 & -2/3 \end{bmatrix}$.

- Solve for the eigenvalues:

$$\det(A-\lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -1/9 & -2/3-\lambda \end{bmatrix} = \lambda^2 + \frac{2}{3}\lambda + \frac{1}{9}$$

  $\lambda = -1/3$ is the repeated eigenvalue.

- Compute $\vec{v}_1$:

$$\vec{v}_1 = (A-\lambda I)\vec{v}_0 = \begin{bmatrix} 1/3 & 1 \\ -1/9 & -1/3 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0/3 + y_0 \\ -x_0/9 - y_0/3 \end{bmatrix}$$

- The general solution:

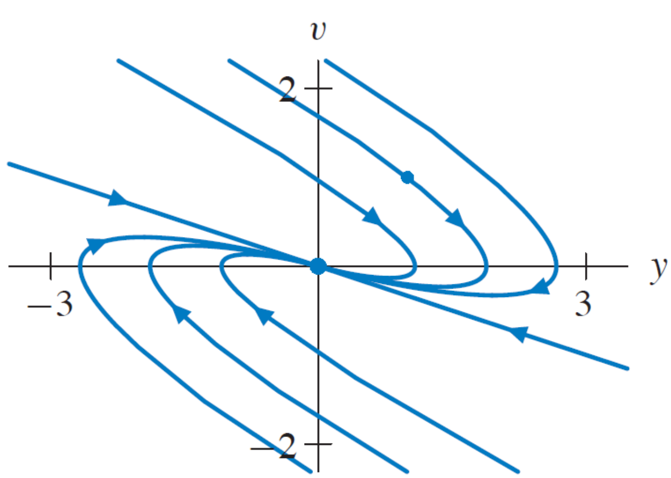
$$\begin{aligned} \mathbf{Y}(t) &= e^{\lambda t}\vec{v}_0 + te^{\lambda t}\,\vec{v}_1 \\ &= e^{-t/3}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + te^{-t/3}\begin{bmatrix} x_0/3 + y_0 \\ -x_0/9 - y_0/3 \end{bmatrix} \end{aligned}$$

- Initial values $y(0) = v(0) = 1$:

$$\mathbf{Y}(t) = e^{-t/3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + te^{-t/3}\begin{bmatrix} 4/3 \\ -4/9 \end{bmatrix}$$

$$\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} e^{-t/3} + \frac{4}{3}te^{-t/3} \\ e^{-t/3} - \frac{4}{9}te^{-t/3} \end{bmatrix}$$

c) Sketch the phase portrait of the corresponding linear system.
Solution.



d) Classify the oscillator (as underdamped, overdamped, critically damped, or undamped) and, when appropriate, give the natural frenquency and the natural period.
Answer.

- The origin is a sink with the repeated value. Hence, the system is critically damped.

e) Use the“simplified method” in Lecture Note 3.6 to solve for $y(t)$ and $v(t)$.
Answer.

- The quadratic equation $9\lambda^2 + 6\lambda + 1 = 0$, so $\lambda = -1/3$.
- $y(t) = k_1 e^{-t/3} + k_2 t e^{-t/3}$.
- $v'(t) = y'(t) = (-k_1/3 + k_2)e^{-t/3} + (-k_2/3)te^{-t/3}$.
- Initial values $y(0) = 1$, $v(0) = 1$: $k_1 = 1$, $k_2 = 4/3$.
- The solutions:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{-t/3} + \frac{4}{3}te^{-t/3} \\ e^{-t/3} - \frac{4}{9}te^{-t/3} \end{bmatrix}$$

Problem 3. $m = 1$, $k = 8$, $b = 6$, with initial values $y(0) = 1$, $v(0) = 0$.

a) Write the second-order differential equation and convert it to the corresponding first-order linear system.
Answer.

- The second-order differential equation: $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 8y = 0$.
- The corresponding system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -8y - 6v \end{aligned}$$

b) Solve the initial value problem with respect to the corresponding linear system using the eigenvalues-eigenvectors method.
Answer.

- The coefficient matrix $A = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$.
- Solve for the eigenvalues:

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -8 & -\lambda - 6 \end{bmatrix} = \lambda^2 + 6\lambda + 8$$

$\lambda_1 = -2, \lambda_2 = -4$.

- Find an eigenvector for $\lambda_1 = -2$:

$$\begin{bmatrix} 2 & 1 \\ -8 & -4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \beta = -2\alpha$$

  Choose $\alpha = 1$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

- Find an eigenvector for $\lambda_1 = -4$:

$$\begin{bmatrix} 4 & 1 \\ -8 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \beta = -4\alpha$$

  Choose $\alpha = 1$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

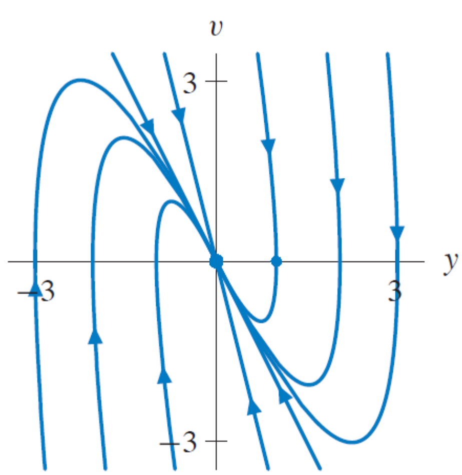
- The general solution:

$$\begin{aligned} \mathbf{Y}(t) &= k_1 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + k_2 e^{-4t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} k_1 e^{-2t} + k_2 e^{-4t} \\ -2k_1 e^{-2t} - 4k_2 e^{-4t} \end{bmatrix} \end{aligned}$$

- Initial values $y(0) = 1$, $v(0) = 0$: $k_1 = 2$, $k_2 = -1$.

$$\mathbf{Y}(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 2e^{-2t} - e^{-4t} \\ -4e^{-2t} + 4e^{-4t} \end{bmatrix}$$

c) Sketch the phase portrait of the corresponding linear system.
Solution. The solutions are periodic and oscillate in the clockwise direction.



d) Classify the oscillator (as underdamped, overdamped, critically damped, or undamped) and, when appropriate, give the natural frenquency and the natural period.
Answer.

- The origin is a sink. Hence, the system is overdamped.

e) Use the"simplified method" in Lecture Note 3.6 to solve for $y(t)$ and $v(t)$.
Answer.

- The quadratic equation $\lambda^2 + 6\lambda + 8 = 0$, so $\lambda = -2, -4$.
- $y(t) = k_1 e^{-2t}) + k_2 e^{-4t}$.
- $v(t) = y'(t) = -2k_1 e^{-2t}) - 4k_2 e^{-4t}$.
- Initial values $y(0) = 1$, $v(0) = 0$: $k_1 = 2$, $k_2 = -1$.
- The solutions:

$$\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 2e^{-2t} - e^{-4t} \\ -4e^{-2t} + 4e^{-4t} \end{bmatrix}$$

Problem 4. $m = 1$, $k = 5$, $b = 4$, with initial values $y(0) = 1$, $v(0) = 0$.

a) Write the second-order differential equation and convert it to the corresponding first-order linear system.
Answer.

- The second-order differential equation: $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 5y = 0$.
- The corresponding system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -5y - 4v \end{aligned}$$

b) Solve the initial value problem with respect to the corresponding linear system using the eigenvalues-eigenvectors method.
Answer.

- The coefficient matrix $A = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix}$.

- Solve for the eigenvalues:

$$\det(A-\lambda I)=\det\begin{bmatrix}-\lambda & 1\\ -5 & -\lambda-4\end{bmatrix}=\lambda^2+4\lambda+5$$

$\lambda_1=-2+i, \lambda_2=-2-i.$

- Find an eigenvector for $\lambda_1=-2+i$:

$$\begin{bmatrix}2-i & 1\\ -5 & -2-i\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\Rightarrow \beta=-(2-i)\alpha$$

Choose $\alpha=1$, $\vec{v}_1=\begin{bmatrix}1\\ -2+i\end{bmatrix}$.

- Decompose $\mathbf{Y}_1(t)=e^{\lambda_1 t}\vec{v}_1$:

$$\begin{aligned}\mathbf{Y}_1(t)&=e^{(-2+i)t}\begin{bmatrix}1\\ -2+i\end{bmatrix}\\ &=(e^{-2t}\cos(t)+ie^{-2t}\sin(t))\begin{bmatrix}1\\ -2+i\end{bmatrix}\\ &=\begin{bmatrix}e^{-2t}\cos(t)\\ -2e^{-2t}\cos(t)-e^{-2t}\sin(t)\end{bmatrix}+i\begin{bmatrix}e^{-2t}\sin(t)\\ e^{-2t}\cos(t)-2e^{-2t}\sin(t)\end{bmatrix}\end{aligned}$$

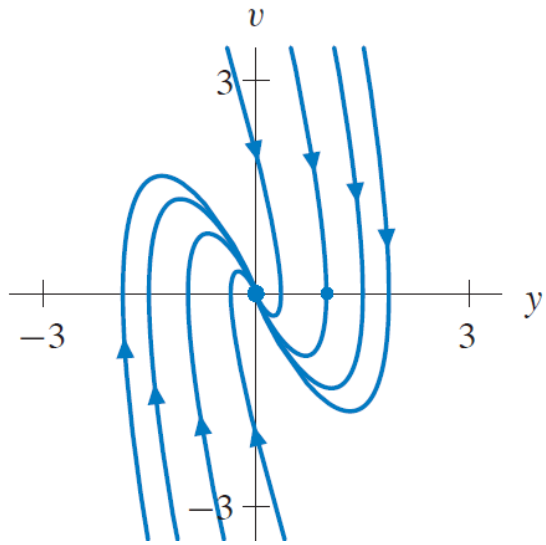
- The general solution:

$$\begin{aligned}\mathbf{Y}(t)&=k_1\begin{bmatrix}e^{-2t}\cos(t)\\ -2e^{-2t}\cos(t)-e^{-2t}\sin(t)\end{bmatrix}+k_2\begin{bmatrix}e^{-2t}\sin(t)\\ e^{-2t}\cos(t)-2e^{-2t}\sin(t)\end{bmatrix}\\ &=\begin{bmatrix}k_1e^{-2t}\cos(t)+k_2e^{-2t}\sin(t)\\ (-2k_1+k_2)e^{-2t}\cos(t)+(-k_1-2k_2)e^{-2t}\sin(t)\end{bmatrix}\end{aligned}$$

- Initial values $y(0)=1$, $v(0)=0$: $k_1=1$, $k_2=2$.

$$\mathbf{Y}(t)=\begin{bmatrix}y(t)\\ v(t)\end{bmatrix}=\begin{bmatrix}e^{-2t}\cos(t)+2e^{-2t}\sin(t)\\ -5e^{-2t}\sin(t)\end{bmatrix}$$

c) Sketch the phase portrait of the corresponding linear system.
Solution. The solutions are periodic and oscillate in the clockwise direction.



d) Classify the oscillator (as underdamped, overdamped, critically damped, or undamped) and, when appropriate, give the natural frenquency and the natural period.

Answer.

- The origin is a spiral sink. Hence, the system is underdamped.
- The natural period is 2π, and the natural frequency is $1/2\pi$.

e) Use the"simplified method" in Lecture Note 3.6 to solve for $y(t)$ and $v(t)$.

Answer.

- The quadratic equation $\lambda^2 + 4\lambda + 5 = 0$, so $\lambda = -2 \pm i$.
- $y(t) = k_1 e^{-2t}\cos(t) + k_2 e^{-2t}\sin(t)$.
- $v(t) = y'(t) = (-2k_1 + k_2)e^{-2t}\cos(t) + (-k_1 - 2k_2)e^{-2t}\sin(t)$.
- Initial values Initial values $y(0) = 1$, $v(0) = 4$: $k_1 = 1$, $k_2 = 2$.
- The solutions:

$$\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} e^{-2t}\cos(t) + 2e^{-2t}\sin(t) \\ -5e^{-2t}\sin(t) \end{bmatrix}$$

Problem 5. Use the Method of Undetermined Coefficients to solve the following initial value problems:

a) $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 3y = 2e^{2t}$, $y(0) = 0$, $y'(0) = 0$.

Solution.

- **Homogeneous equation:** $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 3y = 0.$

$$\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda = 1, 3$$

  Hence, $y_h(t) = k_1 e^t + k_2 e^{3t}$.

- **Guess:** $g(t) = 2e^{2t}$, then we guess $y_p(t) = Ce^{2t}$.
  - (Exercise) Solve for C: $C = -1$.
  - $y_p(t) = -e^{2t}$.
- The general solution:

$$y(t) = k_1 e^t + k_2 e^{3t} - e^{2t}$$

- (Exercise) The particular solution for the initial values $y(0) = 0$, $y'(0) = 0$: $k_1 = 1/2$, $k_2 = 1/2$.

$$y(t) = \frac{1}{2}e^t + \frac{1}{2}e^{3t} - e^{2t}$$

b) $\dfrac{d^2y}{dt^2} + 7\dfrac{dy}{dt} + 10y = e^{-2t}$, $y(0) = 0$, $y'(0) = 0$.

Solution.

- **Homogeneous equation:** $\dfrac{d^2y}{dt^2} + 7\dfrac{dy}{dt} + 10y = 0.$

$$\lambda^2 + 7\lambda + 10 = 0 \Rightarrow \lambda = -2, -5$$

  Hence, $y_h(t) = k_1 e^{-2t} + k_2 e^{-5t}$.

- **Guess:** $g(t) = e^{-2t}$, then we guess $y_p(t) = Cte^{-2t}$. The extra t is the result of e^{-2t} also appearing in $y_h(t)$.
  - (Exercise) Solve for C: $C = 1/3$.
  - $y_p(t) = \frac{1}{3}te^{-2t}$.
- The general solution:

$$y(t) = k_1 e^{-2t} + k_2 e^{-5t} + \frac{1}{3}te^{-2t}$$

- (Exercise) The particular solution for the initial values $y(0) = 0$, $y'(0) = 0$: $k_1 = -1/9$, $k_2 = 1/9$.
$$y(t) = -\frac{1}{9}e^{-2t} + \frac{1}{9}e^{-5t} + \frac{1}{3}te^{-2t}$$

c) $\dfrac{d^2y}{dt^2} + 10\dfrac{dy}{dt} + 25y = 2e^{-5t}$, $y(0) = 0$, $y'(0) = 1$.
Solution.

- **Homogeneous equation:** $\dfrac{d^2y}{dt^2} + 10\dfrac{dy}{dt} + 25y = 0$.
$$\lambda^2 + 10\lambda + 25 = 0 \Rightarrow \lambda = -5,$$
which is the repeated eigenvalue. Hence, $y_h(t) = k_1e^{-5t} + k_2te^{-5t}$.
- **Guess:** $g(t) = 2e^{-5t}$, then we guess $y_p(t) = Ct^2e^{-5t}$. The extra t^2 is the result of e^{-5t} and te^{-5t} both appearing in $y_h(t)$.
  - (Exercise) Solve for C: $C = 1$.
  - $y_p(t) = t^2e^{-5t}$.
- The general solution:
$$y(t) = k_1e^{-5t} + k_2te^{-5t} + t^2e^{-5t}$$
- (Exercise) The particular solution for the initial values $y(0) = 0$, $y'(0) = 1$: $k_1 = 0$, $k_2 = 1$.
$$y(t) = te^{-5t} + t^2e^{-5t}$$

d) $\dfrac{d^2y}{dt^2} + 4y = -3t^2 + 2t + 3$, $y(0) = 2$, $y'(0) = 0$.
Solution.

- **Homogeneous equation:** $\dfrac{d^2y}{dt^2} + 4y = 0$.
$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2\,i$$
Hence, $y_h(t) = k_1\cos(2t) + k_2\sin(2t)$.
- **Guess:** $g(t) = -3t^2+2t+3$, then we guess $y_p(t) = At^2+Bt+C$.
  - (Exercise) Solve for A, B, C: Substitute the guess and compare:
$$\begin{aligned} 4A &= -3 \\ 4B &= 2 \\ 2A + 4C &= 3 \end{aligned}$$
    Hence, $A = -3/4, B = 1/2, C = 9/8$.

- $y_p(t) = -\frac{3}{4}t^2 + \frac{1}{2}t + \frac{9}{8}$.

- The general solution:

$$y(t) = k_1 \cos(2t) + k_2 \sin(2t) - \frac{3}{4}t^2 + \frac{1}{2}t + \frac{9}{8}$$

- (Exercise) The particular solution for the initial values $y(0) = 2$, $y'(0) = 0$: $k_1 = 7/8$, $k_2 = -1/4$.

$$y(t) = \frac{7}{8}\cos(2t) - \frac{1}{4}\sin(2t) - \frac{3}{4}t^2 + \frac{1}{2}t + \frac{9}{8}$$

e) $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2t + e^{-2t}$, $y(0) = 3$, $y'(0) = 1$.

Solution.

- **Homogeneous equation:** $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 0$.

$$\lambda^2 + 6\lambda + 8 = 0 \Rightarrow \lambda = \pm - 2, -4$$

  Hence, $y_h(t) = k_1 e^{-2t} + k_2 e^{-4t}$.

- **Guess:** $g(t) = 2t + e^{-2t}$, then we guess $y_p(t) = At + B + Cte^{-2t}$.
  - (Exercise) Solve for A, B, C: Substitute the guess and compare:

$$\begin{aligned} 6A + 8B &= 0 \\ 8A &= 2 \\ 2C &= 1 \end{aligned}$$

    Hence, $A = 1/4, B = -3/16, C = 1/2$.
  - $y_p(t) = \frac{1}{4}t - \frac{3}{16} + \frac{1}{2}te^{-2t}$.

- The general solution:

$$y(t) = k_1 e^{-2t} + k_2 e^{-4t} + \frac{1}{4}t - \frac{3}{16} + \frac{1}{2}te^{-2t}$$

- (Exercise) The particular solution for the initial values $y(0) = 3$, $y'(0) = 1$: $k_1 = 13/2$, $k_2 = -53/16$.

$$y(t) = \frac{13}{2}e^{-2t} - \frac{53}{16}e^{-4t} + \frac{1}{4}t - \frac{3}{16} + \frac{1}{2}te^{-2t}$$

f) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 20y = e^{-t}\cos(t)$, $y(0) = 3$, $y'(0) = 1$.
Solution.

- **Homogeneous equation:** $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 20y = 0$.

$$\lambda^2 + 4\lambda + 20 = 0 \Rightarrow \lambda = -2 \pm 4i$$

  Hence, $y_h(t) = k_1 e^{-2t}\cos(4t) + k_2 e^{-2t}\sin(4t)$.

- **Guess:** $g(t) = e^{-t}\cos(t)$, then we guess $y_p(t) = e^{-t}(A\cos(t) + B\sin(t))$.
  - (Exercise) Solve for A, B: Substitute the guess and compare:

$$\begin{aligned} 16A + 2B &= 1 \\ -2A + 16B &= 0 \end{aligned}$$

    Hence, $A = 4/65, B = 1/130$.
  - $y_p(t) = e^{-t}(\frac{4}{65}\cos(t) + \frac{1}{130}\sin(t))$.

- The general solution:

$$y(t) = k_1 e^{-2t}\cos(4t) + k_2 e^{-2t}\sin(4t) + e^{-t}(\frac{4}{65}\cos(t) + \frac{1}{130}\sin(t))$$

- (Exercise) The particular solution for the initial values $y(0) = 3$, $y'(0) = 1$: $k_1 = 191/65$, $k_2 = 901/520$.

$$y(t) = \frac{191}{65}e^{-2t}\cos(4t) + \frac{901}{520}e^{-2t}\sin(4t) + e^{-t}(\frac{4}{65}\cos(t) + \frac{1}{130}\sin(t))$$

Problem 6. Using the complexifying method, compute the solution of the given initial-value problem:

a) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 2\cos(2t)$, $y(0) = y'(0) = 0$.
Solution.

- **Homogeneous equation:** $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 0$.

$$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = -1,$$

  which is the repeated eigenvalue. Hence, $y_h(t) = k_1 e^{-t} + k_2 t e^{-t}$.

- **Complexifying:**

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2e^{i2t}$$

- **Guess** $y_c(t)$**:** It is the non-resonant case. We guess $y_c(t) = Ce^{i2t}$.
  - (Exercise) Solve for C: $C = \dfrac{2}{-3+4i} = \dfrac{-6-8i}{25}$ by multiplying the conjugation $-3-4i$ to the numerator and denominator.
- **Decompose** $y_c(t)$

$$\begin{aligned} y_c(t) &= \frac{-6-8i}{25} e^{i2t} \\ &= \frac{-6-8i}{25}(\cos(2t) + i\sin(2t)) \\ &= \left(-\frac{6}{25}\cos(2t) + \frac{8}{25}\sin(2t)\right) + i\left(-\frac{8}{25}\cos(2t) - \frac{6}{25}\sin(2t)\right) \end{aligned}$$

- **Choose** $y_p(t)$**:** Since $g(t) = 2\cos(2t)$, we choose $y_p(t) = -\dfrac{6}{25}\cos(2t) + \dfrac{8}{25}\sin(2t)$ the real part of $y_c(t)$.
- The general solution:

$$y(t) = k_1 e^{-t} + k_2 t e^{-t} - \frac{6}{25}\cos(2t) + \frac{8}{25}\sin(2t)$$

- (Exercise) The particular solution for the initial values $y(0) = 0$, $y'(0) = 0$: $k_1 = 6/25$, $k_2 = -2/5$.

$$y(t) = \frac{6}{25}e^{-t} - \frac{2}{5}te^{-t} - \frac{6}{25}\cos(2t) + \frac{8}{25}\sin(2t)$$

b) $\dfrac{d^2y}{dt^2} + 4y = \sin(3t)$, $y(0) = 2, y'(0) = 0$.

Solution.

- **Homogeneous equation:** $\dfrac{d^2y}{dt^2} + 4y = 0$.

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i.$$

  Hence, $y_h(t) = k_1\cos(2t) + k_2\sin(2t)$.
- **Complexifying:**

$$\frac{d^2y}{dt^2} + 4y = e^{i3t}$$

- **Guess** $y_c(t)$**:** It is the non-resonant case. We guess $y_c(t) = Ce^{i3t}$.

- (Exercise) Solve for C: $C = -\frac{1}{5}$.

- **Decompose** $y_c(t)$

$$\begin{aligned} y_c(t) &= -\frac{1}{5}e^{i3t} \\ &= -\frac{1}{5}(\cos(3t) + i\sin(3t)) \\ &= -\frac{1}{5}\cos(3t) + i\left(-\frac{1}{5}\sin(3t)\right) \end{aligned}$$

- **Choose** $y_p(t)$**:** Since $g(t) = \sin(3t)$, we choose $y_p(t) = -\frac{1}{5}\sin(3t)$ the imaginary part of $y_c(t)$.
- The general solution:

$$y(t) = k_1\cos(2t) + k_2\sin(2t) - \frac{1}{5}\sin(3t)$$

- (Exercise) The particular solution for the initial values $y(0) = 2$, $y'(0) = 0$: $k_1 = 2$, $k_2 = 3/10$.

$$y(t) = 2\cos(2t) + \frac{3}{10}\sin(2t) - \frac{1}{5}\sin(3t)$$

c) $\frac{d^2y}{dt^2} + 4y = 3\cos(2t)$, $y(0) = y'(0) = 0$.

Solution.

- **Homogeneous equation:** $\frac{d^2y}{dt^2} + 4y = 0$.

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i.$$

Hence, $y_h(t) = k_1\cos(2t) + k_2\sin(2t)$.

- **Complexifying:**

$$\frac{d^2y}{dt^2} + 4y = 3e^{i2t}$$

- **Guess** $y_c(t)$**:** It is the resonant case. We guess $y_c(t) = Cte^{i2t}$.
  - (Exercise) Solve for C: $C = -\frac{3i}{4}$.

- **Decompose** $y_c(t)$

$$\begin{aligned} y_c(t) &= -\frac{3i}{4}te^{i2t} \\ &= -\frac{3i}{4}t(\cos(2t) + i\sin(2t)) \\ &= \frac{3}{4}t\sin(2t) + i\left(-\frac{3}{4}t\cos(2t)\right) \end{aligned}$$

- **Choose** $y_p(t)$**:** Since $g(t) = 3\cos(2t)$, we choose $y_p(t) = \frac{3}{4}t\sin(2t)$ the real part of $y_c(t)$.
- The general solution:

$$y(t) = k_1\cos(2t) + k_2\sin(2t) + \frac{3}{4}t\sin(2t)$$

- (Exercise) The particular solution for the initial values $y(0) = 0$, $y'(0) = 0$: $k_1 = 0$, $k_2 = 0$.

$$y(t) = \frac{3}{4}t\sin(2t)$$

d) $\dfrac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 1$, $y'(0) = -1$.

Solution.

- **Homogeneous equation:** $\dfrac{d^2y}{dt^2} + 9y = 0$.

$$\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i.$$

Hence, $y_h(t) = k_1\cos(3t) + k_2\sin(3t)$.

- **Complexifying:**

$$\frac{d^2y}{dt^2} + 9y = 3e^{i3t}$$

- **Guess** $y_c(t)$**:** It is the resonant case. We guess $y_c(t) = Cte^{i3t}$.
  - (Exercise) Solve for C: $C = -\dfrac{i}{6}$.
- **Decompose** $y_c(t)$

$$\begin{aligned} y_c(t) &= -\frac{i}{6}te^{i3t} \\ &= -\frac{i}{6}t(\cos(3t) + i\sin(3t)) \\ &= \frac{1}{6}t\sin(3t) + i\left(-\frac{1}{6}t\cos(3t)\right) \end{aligned}$$

- **Choose $y_p(t)$:** Since $g(t) = \sin(3t)$, we choose $y_p(t) = -\frac{1}{6}t\cos(3t)$ the imaginary part of $y_c(t)$.
- The general solution:

$$y(t) = k_1\cos(3t) + k_2\sin(3t) - \frac{1}{6}t\cos(3t)$$

- (Exercise) The particular solution for the initial values $y(0) = 1$, $y'(0) = -1$: $k_1 = 1$, $k_2 = -5/18$.

$$y(t) = \cos(3t) - \frac{5}{18}\sin(3t) - \frac{1}{6}t\cos(3t)$$