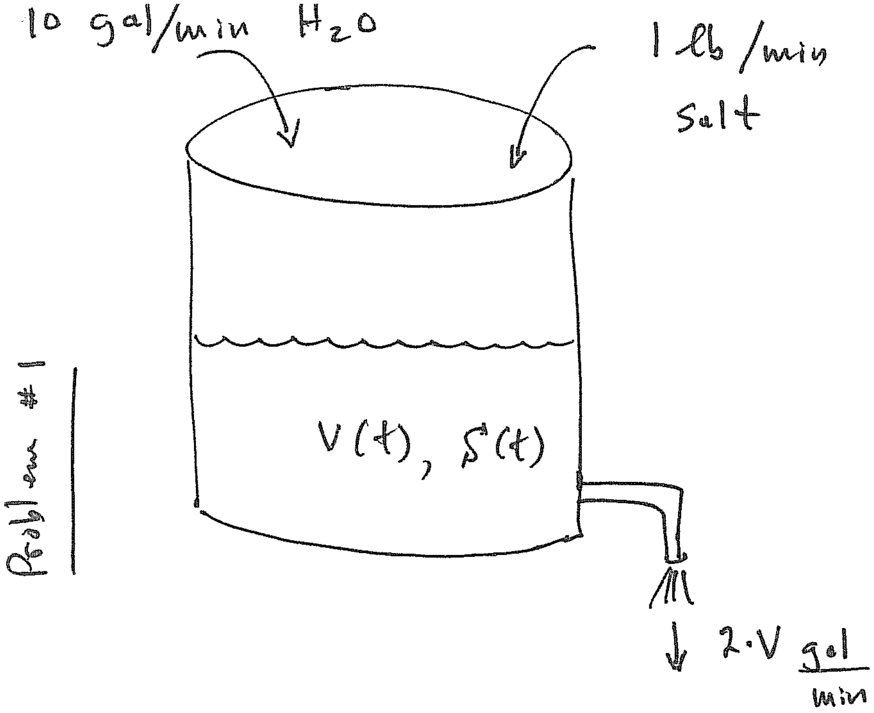
Problem #1



Concentration $C(t) = \dfrac{S(t)}{V(t)}$

(d) As $t \to \infty$, we see from (a–c) that

$V \to 5$ gallons water $= V_\infty$

$S \to \frac{1}{2}$ pound salt $= S_\infty$

$C \to \frac{1}{10} \frac{\text{pounds}}{\text{gal}} = \dfrac{S_\infty}{V_\infty}$.

(a) $\dfrac{dV}{dt} = 10 \frac{\text{gal}}{\text{min}}$ (influx) $- 2V \frac{\text{gal}}{\text{min}}$ (efflux)

so $\dfrac{dV}{dt} = 10 - 2V$

As $t \to \infty$, $V(t) \to 5$ gallons, the steady-state

(b) If $2V \frac{\text{gal}}{\text{min}}$ flow out, it takes $(2V \cdot C)$ lbs of salt with it. Therefore

$$\frac{dS}{dt} = -(2V)\frac{\text{gal}}{\text{min}} \cdot C\left(\frac{\text{lbs}}{\text{gal}}\right) + 1\left(\frac{\text{lbs}}{\text{min}}\right)$$

$$= -2VC + 1$$

Since $V \cdot C = S$, we have

$\dfrac{dS}{dt} = -2S + 1$. As $t \to \infty$, $S \to \frac{1}{2}$.

(c) Since $S = V \cdot C$, $\dfrac{dS}{dt} = \dfrac{dV}{dt} C + \dfrac{dC}{dt} V$.

Therefore,

$$\frac{dC}{dt} = \frac{1}{V}\left(\frac{dS}{dt} - C\frac{dV}{dt}\right) = \frac{1}{V}\left(-2VC + 1 - (10 - 2V)C\right)$$

$$= \frac{1}{V}\left(1 - 10C\right)$$

Regardless of V, as $t \to \infty$, $C \to \frac{1}{10}$.

Problem 2

$$y' + \frac{1}{2} y = \frac{1}{2} e^{t/3} \quad (*)$$

Multiply by a $\mu(t)$ such that $\frac{d\mu}{dt} = \frac{1}{2}\mu$, or $\mu = e^{t/2}$.

$$y' e^{t/2} + \frac{1}{2} e^{t/2} y = \frac{1}{2} e^{t/3} \cdot e^{t/2}$$

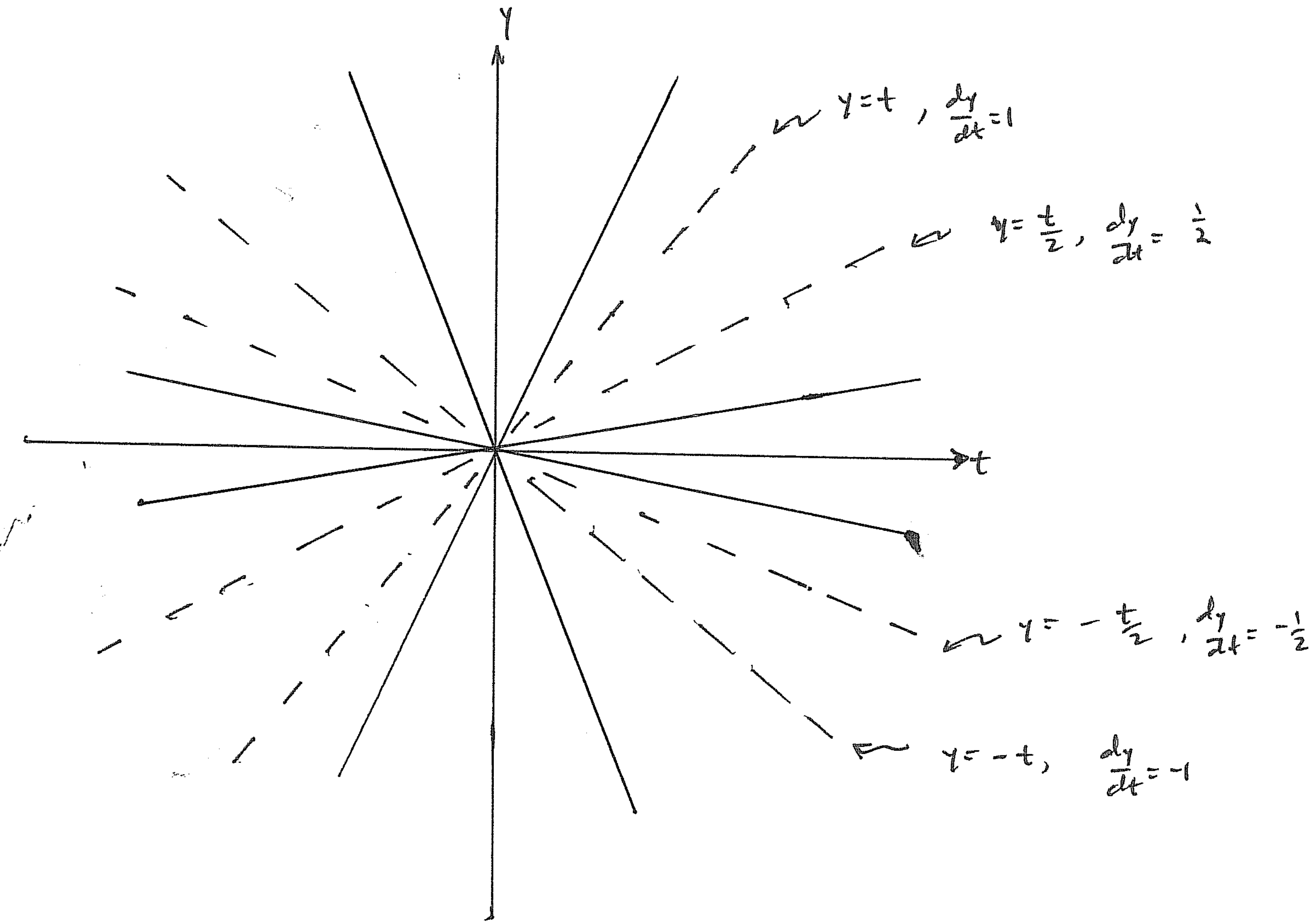
$$= \frac{d}{dt}\left[y e^{t/2} \right] = \frac{1}{2} e^{5t/6}.$$ Integrating gives

$$y(t) e^{t/2} = \frac{1}{2}\int e^{5t/6}\, dt = \frac{1}{2}\left[\frac{e^{5t/6}}{5/6} \right] + C = \frac{3}{5} e^{5t/6} + C$$

$$\boxed{y(t) = Ce^{-t/2} + \frac{3}{5} e^{t/3}}$$

This is the general solution to $(*)$.

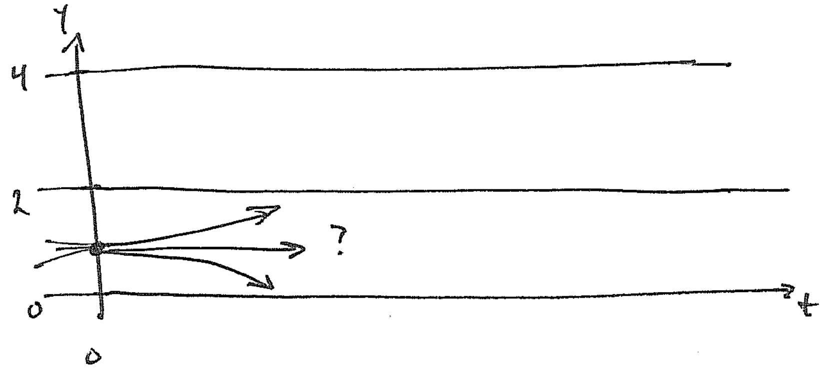
Problem 3



$\frac{dy}{dt} = \frac{y}{t}$ has solutions $\int \frac{dy}{y} = \int \frac{dt}{t}$, or $\ln|y| = \ln|t| + k$, or $y = ct$ for c arbitrary.

All solutions are straight lines that cross through the point $(0,0)$.

Problem 4



We know that $y(t)$ obeys $0 < y(t) < 2$ if $y(0) = 1$, since $y = 2$ and $y = 0$ are fixed points, and $y' = f(y)$ satisfies the hypotheses of the uniqueness theorem.

$\frac{dy}{dt} = 1 - \cos(y)$ has equilibria at $\cos y = 1$ or $0, \pm 2\pi, \pm 2k\pi$ $(k = 0, 1, 2, 3, \ldots)$

Everywhere else, $\frac{dy}{dt} > 0$, so every equilibrium is a node.