Section 1.1 - Modeling Via Differential Equations

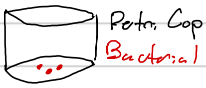
To build a Model:

- Clearly state all assumptions being made
- Completely describe all vars and parameters
  - dependent variable
  - independent variable (usually time)
  - Parameters: do not change with time but may be adjusted on a per model basis

Assumptions allow equations to be derived from vars and parameters

Example: Unlimited Population Growth

Assumptions needed to create more accurate models



Predict the population of bacteria in the cup.

Variables: t = time (independent)

$S(t)$ = Population of bacteria

'rate of change' synonymous with derivative

Assumption: Number of new bacteria is proportional to the population itself

rate of change $= \frac{dS}{dt} = KS$

Two Equations: $\frac{dS}{dt} = KS$ ← differential Eq.

$S(0) = S_0$ ← Initial Condition

We need to find a function! $S(t)$

Guess: $S(t) = S_0 \cdot e^{Kt}$, so $\frac{dS}{dt} = K \cdot S_0 e^{Kt}$ ← Works for short time period but space/resources limiting!

Terminology

- A constant solution to a DE is called an equilibrium solution. A solution $P(t)$ st $\frac{dP}{dt} = 0$ for all t is an equilibrium solution
- The pair of equations $\frac{dP}{dt} = KP$ and $P(0) = P_0$ is called an initial value problem
- Ordinary DEs lack Partial Derivatives

Example: Limited Resources and the Logistic Population Model

Variables: t = time

P = Population

K = growth rate coef.

N = Carrying capacity

Model: $\frac{dP}{dt} = K\left(1 - \frac{P}{N}\right)P$

Called the Logistic Population Model, using defined vars

Assumptions: small population → rate of growth proportional to P

Population too large → rate of growth decreases

Our model is a first order DE, and it is nonlinear because of the P^2 term.

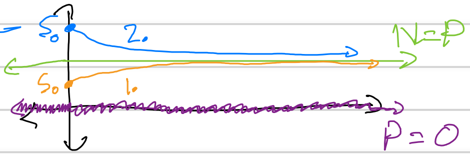
- A DE is nonlinear if the term involving the dependent variable is nonlinear
- The order of a DE is determined by the order of the derivative of the equation

What are the equilibrium solutions to our model?

We need the rate of change to be 0!

$$\frac{dP}{dt} = KP\left(1 - \frac{P}{N}\right)$$

$\frac{dP}{dt} = 0$ if $P = 0$ or $N = P$



1. If initial value lies btwn $P=0$ and $N=P$, Model approaches N like an asymptote
2. If initial value is larger than N, the derivative is negative, and approaches N

Example: Consider $\frac{dy}{dt} = y^4 + y^3 - 2y^2$

(a) Equilibrium Solutions, $\frac{dy}{dt} = 0$

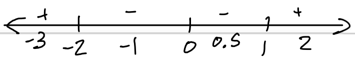
$y^4 + y^3 - 2y^2 = 0$

$y^2(y^2 + y - 2) = 0$

$y^2(y-1)(y+2) = 0$

$y = 0, y = 1, y = -2$

(b) values of y where $y(t)$ inc. and dec. $\frac{dy}{dt} = y^2(y-1)(y+2)$



Sign Diagrams helpful

Model Graph is derived from increasing and decreasing regions above

Predator - Prey Systems

Let: R = prey and F = predator (ex. rabbits and foxes)

Assumptions:
- If no foxes → rabbits reproduce proportional to population size
- If no rabbits → fox population decreases at rate prop. to pop. size
- Rate at which rabbits are being eaten by foxes is proportional to their interaction rate
- Fox births are proportional to the number of rabbits eaten

Parameters: (Assume all ≥ 0)

α: growth rate coef. of rabbits (alpha)

γ: death rate coef. of foxes (gamma)

β: Constant that measures # of fox-rabbit interactions where the rabbit is eaten (beta)

δ: constant that measures the benefit to fox population of an eaten rabbit (delta)

Parameters do not change with time

Modeling:

$$\begin{cases} \frac{dR}{dt} = \alpha R - \beta RF \\ \frac{dF}{dt} = -\gamma F + \delta RF \end{cases}$$

This can be applied to other population types as well, not just rabbit-fox populations. Its just convenient for variable names

Definitions

① Analytic methods: search for explicit formulas for solutions

② Qualitative methods: Use of graphical models

③ Numerical methods: Use of technology

will appear in text. Numerical is used a lot bc models are complicated

Section 1.2 - Separation of Variables

Standard form of first order DE: (*) $\frac{dy}{dt} = f(t,y)$

Example: Consider $\frac{dy}{dt} = y^2(2t+3)$

Is $y(t) = -\frac{1}{t^2+3t}$ a solution? What about $y(t) = 3t+2$

First eq: $y(t) = -\frac{1}{t^2+3t}$ in $\frac{dy}{dt} = y^2(2t+3)$

$\frac{dy}{dt} = \frac{2t+3}{(t^2+3t)^2}$ bc $y = -\frac{1}{t^2+3t}$ $\frac{dy}{dt} = y^2(2t+3)$, yes

Second eq: $y(t) = 3t+2$ in $\frac{dy}{dt} = y^2(2t+3)$

$\frac{dy}{dt} = 3 \neq (3t+2)^2(2t+3)$

Special Cases

① $\frac{dy}{dt} = g(t)$, can be solved by integrating

② $\frac{dy}{dt} = h(y)$, called an autonomous DE

Steps: ① Take derivative of equation
② Plug eq. into DE
③ Check Equivalence

Separable Equations work in the form $\frac{dy}{dt} = g(y) \cdot h(t)$

Example: Solve the IV problem $\frac{dy}{dt} = \frac{t^3}{y^2}$, $y(0) = 2$.

① Separate: $y^2\,dy = t^3\,dt$

② Integrate: $\int y^2\,dy = \int t^3\,dt$ > Don't forget constant

$\frac{1}{3}y^3 + C_1 = \frac{1}{4}t^4 + C_2$ Constants can be combined to make a general constant

③ Simplify: $y^3 = \frac{3}{4}t^4 + C$

$y(t) = (\frac{3}{4}t^4 + C)^{1/3}$

④ Solve for C: $y(0) = C^{1/3} \to C^{1/3} = 2 \to C = 8$

⑤ Solution: $y(t) = (\frac{3}{4}t^4 + 8)^{1/3}$

You can check your solution by differentiating and plugging in, using the steps outlined in example 1.

Warning! there may be additional solutions corresponding to $h(y) = 0$

Example: Solve the IV problem $\frac{dy}{dt} = t^2 y^3$, $y(0) = 0$

① Separate: $\frac{dy}{y^3} = t^2\,dt$ Dividing by y^3 eliminates the $y=0$ possibility, so be careful!

② Integrate: $\int y^{-3}\,dy = \int t^2\,dt$

$-\frac{1}{2}y^{-2} + C_1 = \frac{1}{3}t^3 + C_2$ Always check for division by 0 when separating variables

$y^{-2} = -\frac{2}{3}t^3 + C$

③ Simplify: $y = \sqrt{\frac{1}{-\frac{2}{3}t^2 + C}}$

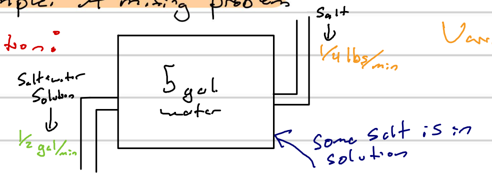
④ Solve for C: $y(0) = \sqrt{\frac{1}{C}}$ but $c = 0$ makes the function undefined!

⑤ Solution: $y(t) = 0$ is the only solution Changing the initial condition completely changes the solution

$\frac{d}{dt}(0) = 0$ and $\frac{dy}{dt}(0) = t^2(0)^3 = 0$

Example: A mixing problem

Situation:



Variables: t = time independent

$S(t)$ = amt. of salt dependent

$W = 5$ gal

DE Model: $\frac{dS}{dt} = \frac{1}{4} - \frac{1}{2}\left(\frac{S}{5}\right)$

rate of change of salt

$$\begin{cases} \frac{dS}{dt} = \frac{1}{4} - \frac{S}{10} \\ S(0) = 0 \end{cases}$$

Density of the mixture $= \frac{\text{amount of salt}}{\text{Volume}}$

Units: $\frac{1\text{ gal}}{2\text{ min}}\left(\frac{S(t)\text{ salt}}{\text{gal}}\right) \to \frac{\text{salt}}{\text{min}}$ ✓

Solving: $\frac{dS}{dt} = \frac{1}{4} - \frac{S}{10} = f(S)$ Autonomous Equation

$\frac{dS}{\frac{1}{4} - \frac{S}{10}} = dt$ if you divide by vars check for ×/0 errors!

Consider $\frac{1}{4} - \frac{S}{10} = 0$ $S = \frac{10}{4} = \frac{5}{2}$ $S(t) = \frac{5}{2}$

$\int \frac{dS}{\frac{1}{4} - \frac{S}{10}} = \int dt$

$-10 \ln\left|\frac{1}{4} - \frac{S}{10}\right| + C_1 = t + C_2$

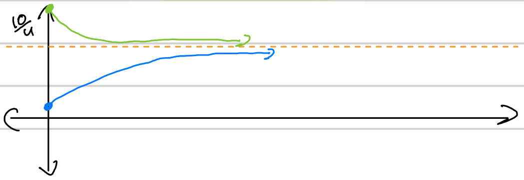
$\ln\left|\frac{1}{4} - \frac{S}{10}\right| = -\frac{1}{10}t + C$

$K = \pm e^C$

$\frac{1}{4} - \frac{S}{10} = e^{-\frac{1}{10}t + C}$ $-\frac{S}{10} = e^{-\frac{1}{10}t + C} - \frac{1}{4}$ $S(0) = 0$

$S = Ke^{-\frac{1}{10}t} + \frac{5}{2}$ $0 = Ce^0 + \frac{5}{2}$

$S(t) = -\frac{5}{2}e^{-\frac{1}{10}t} + \frac{5}{2}$ $C = -\frac{5}{2}$

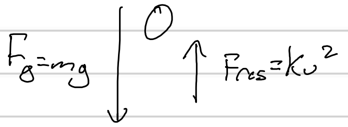


Example: Free falling object

Assumptions: Force of Air Resistance $\propto v^2$ ← Experimentally tested

Second Newton's Law: $\vec{F} = ma = m\frac{dv}{dt}$

$\frac{dv}{dt} = g - \frac{Kv^2}{m}$ => $\vec{F} = mg - Kv^2$



$F = mg - Kv^2$

Variables: t = time $v(t)$ = velocity

Model: $\frac{dv}{dt} = g - \frac{Kv^2}{m} = g\left(1 - \frac{Kv^2}{mg}\right)$ let $\alpha = \sqrt{\frac{K}{mg}}$

$\frac{dv}{dt} = g(1 - v^2\alpha^2)$

$\alpha = \sqrt{\frac{K}{mg}}$ to simplify integration → diff. of squares

Solving: $\int \frac{dv}{1 - v^2\alpha^2} = \int g\,dt$ Partial Fraction Decomposition $\frac{1}{1 - v^2\alpha^2} = \frac{\frac{1}{2}}{1 + \alpha v} + \frac{\frac{1}{2}}{1 - \alpha v}$

$\int \frac{dv}{1 + \alpha v} + \int \frac{dv}{1 - \alpha v} = 2gt + C$ $\frac{1 + \alpha v}{1 - \alpha v} = Ke^{2\alpha gt}$ $K = \pm e^C$ bc $\ln|x|$ can be $-\infty < x < \infty$

$\frac{1}{\alpha}\ln|1 + \alpha v| - \frac{1}{\alpha}\ln|1 - \alpha v| = 2gt + C$

$\ln\left|\frac{1 + \alpha v}{1 - \alpha v}\right| = 2\alpha gt + C$ $v(t)\ \frac{1}{\alpha}\frac{Ke^{2\alpha gt} - 1}{Ke^{2\alpha gt} + 1}$ where $\alpha = \sqrt{\frac{K}{mg}}$

Terminal Velocity: $\lim_{t \to \infty} \frac{1}{\alpha}\frac{K - e^{-2\alpha gt}}{K + e^{-2\alpha gt}} = \frac{1}{\alpha} = \frac{1}{\sqrt{\frac{K}{mg}}} = \sqrt{\frac{mg}{K}}$

Derivation: Newton's law of cooling

Assumptions: Decrease in Temp $\propto$ the diff. btwn objects and environments temps

Variables: t = time, $T(t)$ = Temperature

Model: $T_0 = T(0) = 30$ F →

$$\begin{cases} \frac{dT}{dt} = -K(T - T_0) \\ T_0 = T_1 \end{cases}$$

Section 1.3 - Slope Fields

Suppose we have models:

① $\frac{dy}{dt} = \sin(t^2)$

$y(t) = \int \sin(t^2)\,dt$

② $\frac{dy}{dt} = \frac{1}{e^{y^2}}$

$t = \int e^{y^2}\,dy$

These are unsolvable by elementary methods so we need to use so-called numerical methods to solve

Slope Fields

Let $\begin{cases} \frac{dy}{dt} = f(t, y) \\ y(t_1) = y_1 \end{cases}$

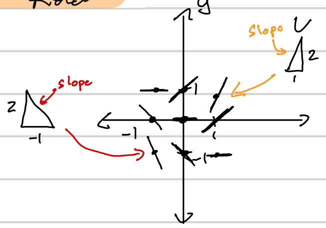
At the point (t, y), $f(t, y)$ is the slope of the tangent line @ (t, y)

The values on the right side of the equation yield the slopes of the tangents at all points on the graph of a solution $y(t)$

Example: Constructing a slope field

Consider $y' = t + y$

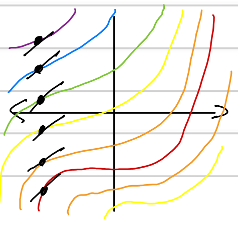
t	y	slope, y'
-1	0	-1
-1	-1	-2
-1	1	0
0	0	0
0	-1	-1
0	1	1
1	0	1
1	-1	0
1	1	2



Given an initial condition, we can sketch a potential solution curve. Graphing takes a while by hand, so technology is very useful here.

Special Case 1

① $\frac{dy}{dt} = f(t)$, where slopes only depend on t. Slopes are parallel along vertical lines.



Poorly Drawn, but slope doesn't depend on the y-coordinate!

Special Case 2 (autonomous)

$\frac{dy}{dt} = f(y)$. Slopes only depend on y, slopes parallel along horizontal lines. Slope doesn't depend on the t coordinate!

Section 1.4 - Euler's Method

Consider $\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$

Approximations made by starting at (t_0, y_0), drawing a tangent line with slope $f(t_0,y_0)$ ending at point (t_1, y_1) where $t_1 = t_0 + \Delta t$. Repeatable, leads to y_n approximations.

Note: Smaller Δt values result in more accurate tangent line approximations

Recursive Formulas:

$$t_{k+1} = t_k + \Delta t$$

$$y_{k+1} = y_k + f(t_k, y_k)\Delta t \leftarrow \text{slope at } (t_k, y_k) \text{ times } \Delta t$$

> Simple, yet powerful

Example: Euler's Method by hand

$y' = 2y+1$, $y(0)=3$. Compute $0 \le t \le 2$ with $\Delta t = 0.5$

① $t_0 = 0$, $t_1 = 0.5$

$y_1 = y_0 + \Delta t(2y_0+1)$

$= 3 + 0.5(7) = 6.5$

② $t_0 = 0$, $t_{-1} = -0.5$

$y_{-1} = 3 + -0.5(2y_0+1)$

$= 3 - 0.5(7) = -0.5$

③ $t_1 = 0.5$ $t_2 = 1$

$y_2 = 6.5 + 0.5(2 \cdot 6.5 + 1)$

$= 6.5 + 7 = 13.5$

④ $t_2 = 1$ $t_3 = 1.5$

$y_3 = 13.5 + 0.5(2 \cdot 13.5 + 1)$

$= 13.5 + 14 = 27.5$

⑤ $t_3 = 1.5$ $t_4 = 2$

$y_4 = 27.5 + 0.5(2 \cdot 27.5 + 1)$

$= 27.5 + 28 = 55.5$

For Comparison: $\frac{dy}{dt} = 2y+1$

$\int \frac{dy}{2y+1} = \int dt$ but $y = -\frac{1}{2}$ is lost in the sauce"

$\frac{1}{2}\ln|2y+1| = t + C$

$\ln|2y+1| = 2t + C$

$2y+1 = Ke^{2t}$

$y = Ke^{2t} - \frac{1}{2}$, $y(0) = 3$

$3 = K - \frac{1}{2} \rightarrow K = 3.5$

$y = 3.5e^{2t} - 0.5$

Generally, approximations are worse with rapidly growing functions or with unbounded models

Example: Eulers Method by hand

$y' = (3-y)(y+1)$, $y(0) = 4$, $\Delta t = 1$ with $0 \le t \le 5$

① $y(0) = 4$

② $t = 1$

$y = 4 + (-5) = -1$

③ $t = 2$

$y = -1 + (0) = -1$

$y_{k+1} = y_k + f(t_k, y_k)\Delta t$

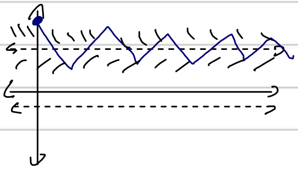
④ $t = 3$

$y = -1 + 0 = -1$

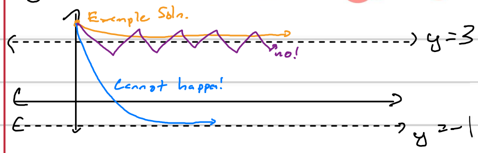
-1 is a general soln. but not one for this initial condition

The step size is so big that we get a false solution.

A step size of 0.5 doesn't work either, as it oscillates around $y=3$



↑ Due to slopes around $y=3$, shown above



Seeing how your graph should look, such as knowing $y(0)=4$ will approach, but never = or cross 3, tells us which solutions to accept.

Section 1.5 - Existence and Uniqueness of Solutions

Suppose an equation $f(x)=x^5+5x-10$. Solve $f(x)=0$

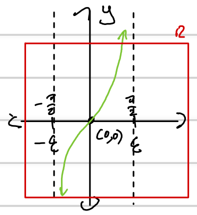
$f(x)=x^5+5x-10$

Intermediate Value Thm.

$f(1)=-4$ and $f(2)=32$ $\therefore$ $f(x)=0$ when $1<x<2$, now we can use numerical methods

Existence Thm: Suppose $f(t,y)$ is a cts. function on a open rectangle R in the ty-plane. If (t_0,y_0) is a point in R, then there exists an $\varepsilon>0$ and a function $y(t)$ defined for $t_0-\varepsilon<t<t_0+\varepsilon$ that solves the initial value problem

Example 1 $\frac{dy}{dt}=1+y^2$, $y(0)=0$, $t_0=0$ and $y_0=0$



A sln exists in some rectangle R

$$\int \frac{dy}{1+y^2}=\int dt$$

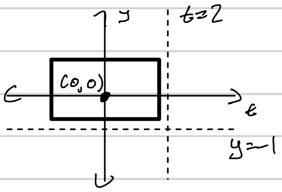
$\arctan y = t+C \Big|_{(0,0)} \therefore C=0$

$y=\tan t$

$\varepsilon=\frac{\pi}{2}$, tangent defined over $-\frac{\pi}{2}<t<\frac{\pi}{2}$

Example 2: Restricted Domain

$\frac{dy}{dt}=\frac{1}{(y+1)(t-2)}$, $y(0)=0$, not cts !!



Can be solved through integration, takes a while so omitted

Uniqueness Thm: Suppose $f(t,y)$ and $\frac{\partial f}{\partial y}$ are cts. functions on an open rectangle R in the ty-plane that contains (t_0,y_0). Then the solutions cannot cross equilibrium slns.

If two solutions are ever in the same place at the same time, then they are the same function (for all t for which they are both defined.

Example 3: Uniqueness fails

Consider $y'=3y^{2/3}$ with $y(0)=0$)

Let $f(t,y)=3y^{2/3}$ cts.

$\frac{\partial f}{\partial y}=2y^{-1/3}$ not cts. @ $y=0$

$\therefore$ Uniqueness Thm. does not work here !!

Can be solved numerically or by integrating

$$\int \frac{dy}{3y^{2/3}}=\int dt$$

Example 4: Role of Equilibrium Solutions

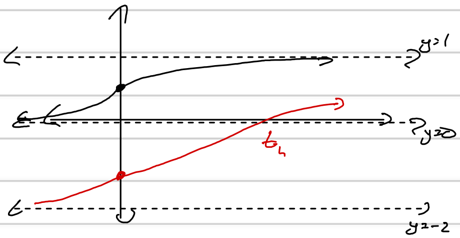
Consider $\frac{dy}{dt}=y^4+y^3-2y^2$ cts., $y(t_0)=y_0$

$\frac{\partial f}{\partial y}=4y^3+3y^2-4y$ cts. as well

Uniqueness Thm. says slns. cross

$f(y)$ is cts for all points in the ty-plane

$y'=y^2(y^2+y-2)=y^2(y+2)(y-1) \Rightarrow y=0, y=-2, y=1$



Predicting initial condit.

if $0<y(t_0)<1$

then $0<y(t)<1$

entire curve squeezed b/twn solutions

A solution can never cross an equilibrium solution because of uniqueness theorem

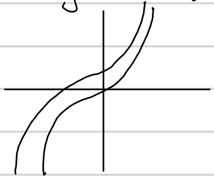
also guarantees solutions will not cross

↳ If a line $h(t)$ is drawn and crosses $y=0$, there would be two solutions at t_n

Example: Uniqueness Thm

Suppose $f(y,t)=y'$ where $y=t^3$ and $y=t^3+1$ are both solutions. What can be said about solutions

$$\begin{cases} y'=f(y,t) \\ f, \frac{\partial f}{\partial y} \text{ are cts in ty} \\ y(0)=a, \ 0\le a<1 \end{cases}$$



$t^3<y(t)<t^3+1$ as it is impossible for solutions to cross the equilibrium solutions

Section 1.6 - Equilibria

We will consider autonomous DE in this section, in the form $\frac{dy}{dt} = f(y)$

Recall: Slopes are parallel along horizontal lines in the ty plane

↳ we can describe an entire field if we know the slopes along one vertical line only

This line is called the phase line, and contains compressed slope info for an autonomous DE

Recall: Equilibrium points are the y-values when $f(y)=0$.

↖ Note: There can be holes in a phase line

Theorem

↙ So Existence and Uniqueness apply

Consider a function $\frac{dy}{dt} = f(y)$ where $f(y)$ are cts and differentiable. Suppose $y(t)$ is a solution

① If $f(y(0))=0$, then $y(0)$ is an equilibrium point and $y(t)=y(0)$ for all t

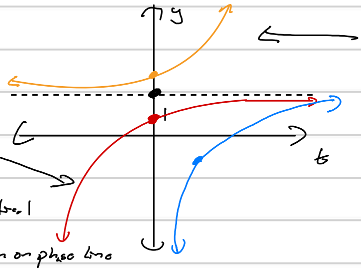
② If $f(y(0))>0$, then $y(t)$ is increasing for all t and either $y(t)\to\infty$, or $y(t)\to$ larger eq. point

③ If $f(y(0))<0$, then $y(t)$ is decreasing for all t and either $y(t)\to-\infty$, or $y(t)\to$ smaller eq. point

Example: Phase Lines

Draw the phase line for $\begin{cases} y' = (y-1)^2 \\ y(0) = \frac{1}{2} \end{cases}$

$\lim_{t\to\infty} y(t) = 1$



⟵ This doesn't always approach inf, as there could be a vertical asymptote

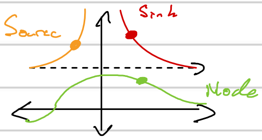
Actual sln of $y = 1 + \frac{1}{C-t}$, w. the vertical asymptote @ $t=C$, which cannot be seen on phase line

Example: Stinky ass

Draw the phase line for $\begin{cases} y' = \frac{1}{y+1} \\ y(0) = x \end{cases}$

undefined @ $y=-1$!!

No eq. points



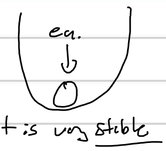
Classification of Eq. points let y_0 be an initial condition

① A sink is where y_0 asymptotically approaches y_0 as $t\to\infty$

② A source is where y_0 asymptotically approaches y_0 as $t\to-\infty$

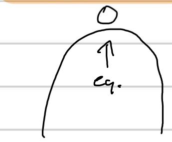
③ A node is neither a source nor a sink

Think of sinks as



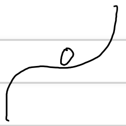
it is very stable

Think of sources as



unstable, loses eq. easily

Think of nodes



half stable

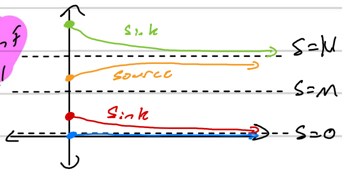
Modified Logistic Population Model

$$\boxed{\frac{dS}{dt} = kS\left(1-\frac{S}{N}\right)\left(\frac{S}{M}-1\right)}$$

N is carrying capacity, M is sparsity

- $S(t)$ is population and k is growth rate coef

Population can't be negative!



Section 1.7 - Bifurcations

In this section we are interested in $\frac{dy}{dt} = f_\mu(y)$ where μ is the parameter of interest.

A parameter where there is a significant change in behavior is a bifurcation

Example: Find Equilibrium points for $y' = y^2 + 3y + \mu$

$f_\mu(y) = 0$

Equilibrium points: $y = \frac{-3 \pm \sqrt{9 - 4\mu}}{2}$

$$\left.\begin{array}{l} \text{if } \mu = \frac{9}{4} \Rightarrow \text{one solution} \\ \text{if } \mu > \frac{9}{4} \Rightarrow \text{No real solution} \\ \text{if } \mu < \frac{9}{4} \Rightarrow 2 \text{ solutions} \end{array}\right\}$$

$\frac{9}{4}$ is a location of large change.

$\mu = \frac{9}{4}$ is a bifurcations point

Bifurcation Plot - A graph in the μy plane

① Plot parameter μ along horizontal axis

② Draw phase line with μ eq. points

Example: bifurcation plot

Draw bifurcation diagram for $y' = y^3 - \alpha y$

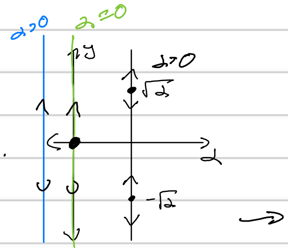
$y' = f_\alpha(y) = y^3 - \alpha y$

$0 = y^3 - \alpha y$

$0 = y(y^2 - \alpha)$

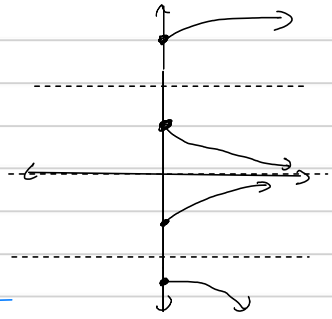
$0 = y(y - \sqrt{\alpha})(y + \sqrt{\alpha})$

$$\left\{\begin{array}{l} \text{if } \alpha > 0\text{: } 3 \text{ eq sols.} \\ \text{if } \alpha < 0\text{: } 1 \text{ eq sol} \\ \text{if } \alpha = 0\text{: } 1 \text{ eq. sols} \end{array}\right.$$

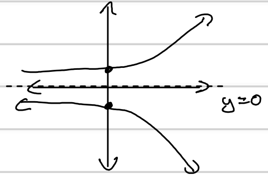


The behavior is the same regardless of α choice if in range

for $\alpha > 0$



for $\alpha < 0$



Notice the Dramatic Difference in graph.

This is why its a bifurcation

$f_C(P) = kP(1 - \frac{P}{N}) - C$ where C is a parameter of fish being caught.

autonomous, f does not depend on t

Example: Application to Sustainability. Use above eqn

$$\frac{dP}{dt} = kP(1 - \frac{P}{N}) - C$$

Solve: $f_C(P) = kP(1 - \frac{P}{N})$

$0 = kP - \frac{K}{N}P^2 - C$

C is just a constant

$* (-\frac{N}{K})$

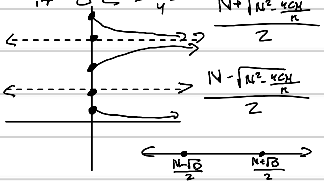
$0 = P^2 - NP + \frac{CN}{K}$

$$P = \frac{N \pm \sqrt{N^2 - 4(\frac{CN}{K})}}{2}$$

Solve the discriminant

$N^2 = \frac{4CN}{K} \Rightarrow C = \frac{NK}{4}$

if $C < \frac{NK}{4}$



$$\left\{\begin{array}{ll} \text{if } C < \frac{NK}{4} & 2 \text{ solutions} \\ \text{if } c > \frac{NK}{4} & \text{no eq.} \\ \text{if } c = \frac{NK}{4} & \text{one eq.} \end{array}\right.$$

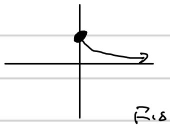
$F_C(P) = P^2 + kP - C = 0$

$$P_1 = \frac{N - \sqrt{P^2 - \frac{4CN}{K}}}{2}$$

$$P_2 = \frac{N + \sqrt{P2 - \frac{4CN}{K}}}{2}$$

Plug into general factor to get behavior

if $C > \frac{NK}{4}$



$f(t) < 0$ for all t

Fish die!!

Section 1.8 - Linear Equations

Linear DEs

$\frac{dy}{dt} = a(t)y + b(t)$, where a and b are arbitrary functions of t

Examples: $y' = e^{2t}y - \sin t$ ✓ $y' = ty^2$ ✓

$y' = 3y + 4$ ✓ $\frac{dP}{dt} = kP(1 - \frac{P}{N})$ ✗

$y' = t^2 y$ ✓

notice how the function of t doesn't matter. Only care about dependent variable

- If $b(t) = 0$ for all t, the equation is called homogeneous (or unforced)
  - ↳ otherwise called nonhomogeneous (or forced)
- If $a(t)$ is constant, the linear equation is called a constant-coefficient equation.

The linearity Principle (Homogeneous Case)

If $y_h(t)$ is a solution of the homogeneous linear equation $\frac{dy}{dt} = a(t)y$, then any constant multiple of $y_h(t)$ is a solution.

Given: $y'(t) = a(t)y$

$y_h(t)$ - is a solution

Prove: $k \cdot y_h(t)$ - is also a solution

$$(k \cdot y_h(t))' = k \cdot y_h'(t) = a(t) \cdot (k \cdot y_h(t))$$

General Solution to Homogeneous Equation: $y_h(t) = ke^{\int a(t)dt}$

Suppose: $\frac{dy}{dt} = a(t) \cdot y$

$\int \frac{dy}{y} = \int a(t)dt$

$\ln|y| = \int a(t)dt$

$|y| = e^{\int a(t)dt} \cdot e^{c}$

$y(t) = k \cdot e^{\int a(t)dt}$ where $k = \pm e^{c}$

but $y(t) = 0$ is a solution

$\therefore y(t) = k \cdot e^{\int a(t)dt}$, $k \in \mathbb{R}$

The extended linearity principle

(NH) $\frac{dy}{dt} = a(t)y + b(t)$

(H) $\frac{dy}{dt} = a(t)y$

The general solution to (NH) is $y = y_h + y_p$, where y_h is the general solution to (H) and y_p is a particular solution to (NH)

Given: $y_p(t)$ is particular solution to (NH)

$y_h(t)$ is general solution to (H)

Prove: $y(t) = y_p(t) + y_h(t)$ is also a solution to (NH)

$$\frac{dy}{dt} = (y_p(t) + y_h(t))' = y_p'(t) + y_h'(t) = \underline{a(t)y_p(t) + b(t)} + \underline{a(t)y_h(t)}$$

$$= a(t)(y_p(t) + y_h(t)) + b(t)$$

$\therefore (y_p(t) + y_h(t))$ is a solution to NH

Two solutions of NH, their difference is a solution to the corresponding H.

Algorithm for solving linear equations

① Find the general solution $y_h(t)$ of H

② Find one particular solution to NH by guessing for now

③ Find the general solution by adding general solution of H to particular solution of NH

$y = y_h(t) + y_p(t)$

Example 1: Solve the IVP $y' = 3y - e^{4t}$, $y(0) = 5$

(NH): $y' = 3y - e^{4t}$

(H): $y' = 3y$

$a(t) = 3$

$b(t) = -e^{4t}$

$\therefore y_h(t) = k \cdot e^{\int 3\,dt} = k \cdot e^{3t}$

Guess for NH: $y(t) = -e^{4t}$, $y' = -4e^{4t}$

$-4e^{4t} = -3e^{4t} - e^{4t}$ ✓

$y(t) = -e^{4t} + ke^{3t}$

$y(0) = -1 + k(1) = 5$

$k = 6$

$\therefore y(t) = -e^{4t} + 6e^{3t}$

Example 2: Find solution to $y' = 3y + \sin 2t$

no initial condition, just general!

(NH): $y' = 3y + \sin 2t$

(H): $y' = 3y$

$y_h = ke^{3t}$

$a(t) = 3$

$b(t) = \sin 2t$

Guess: $y_p(t)$ of $y' = 3y + \sin(t)$

Try ⇒ $y_p(t) = A\cos 2t + B\sin 2t$

$\therefore y'_p(t) = -2A\sin 2t + 2B\cos 2t \overset{\text{eq NH}}{=} 3(A\cos 2t + B\sin 2t) + \sin 2t$

$\underline{-2A}\sin 2t + \underline{2B}\cos 2t = \underline{3A}\cos 2t + \underline{3B}\sin 2t + \underline{\sin 2t}$

$\sin 2t$: $-2A = 3B + 1$

$\cos 2t$: $2B = 3A \quad \therefore B = \frac{3A}{2}$

$-2A = \frac{9A}{2} + 1$

$-\frac{13A}{2} = 1$

$\therefore A = -\frac{2}{13}, \ B = -\frac{3}{13}$

$y_p(t) = -\frac{2}{13}\cos 2t - \frac{3}{13}\sin 2t$

general solution:

$$y(t) = ke^{3t} - \frac{2}{13}\cos 2t - \frac{3}{13}\sin 2t$$

Example 3: Sometimes guessing not be successful

(NH): $y' = 4y - e^{4t}$

(H): $y' = 4y$

$y = ke^{4t}$

$a(t) = 4$

$b(t) = -e^{4t}$

Guess: $y_p(t) = C\cdot e^{4t}$

$4Ce^{4t} = 4Ce^{4t} - e^{4t}$

~~$-e^{4t} = 0$~~ ← can never happen

If this happens, add t factor

$\therefore y_p(t) = C\cdot t\cdot e^{4t}$

$y'_p(t) = Ce^{4t} + 4Cte^{4t}$

$\Rightarrow C\cdot e^{4t} + C\cdot 4te^{4t} = 4Cte^{4t} - e^{4t}$

$C = -1$

$\therefore y_p(t) = -te^{4t}$

Section 1.9 - Integrating Factors

Using Integrating Factors to solve 1st order NHDE

① Rewrite NH DE in the form:

$$(NH)\ \frac{dy}{dt} + g(t)y = b(t)$$ ← Standard form

② Multiply Both sides by some function $\mu(t)$

$$\mu(t)\frac{dy}{dt} + \mu(t)g(t)y = \mu(t)b(t)$$

$\mu(t)$ is called an integrating factor

③ Simplify using product rule $\frac{d\mu}{dt} = \mu g$

$$\frac{d}{dt}\left(\mu(t)y(t)\right) = \frac{d\mu}{dt}y + \frac{dy}{dt}\mu$$

④ Simplify

$$\frac{d\mu}{dt}y = \mu(t)g(t)y(t)$$

⑤ Solve for μ

$$\int\frac{d\mu}{\mu} = \int g \Rightarrow \ln|\mu| = \int g(t)dt \Rightarrow \mu = ke^{\int g(t)dt}$$

Condensed Steps

① Make sure its in the standard form

② Compute the integrating factor $\mu(t) = e^{\int g(t)dt}$

③ Multiply both sides by $\mu(t)$

④ Integrate both sides with respect to t

⑤ Solve for $y(t)$ by dividing by $\mu(t)$

TL;DR:

Dont Forget abt constant when evaluating

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)b(t)\,dt, \text{ where } \mu(t) = e^{\int g(t)dt}$$

Example 1: $\frac{dy}{dt} = \frac{3}{t}y + t^5$ for $t > 0$

① $\frac{dy}{dt}\underbrace{\boxed{-\frac{3}{t}}}_{g(t)}y = t^5,\ t > 0$ Standard form

② $\mu(t) = e^{\int -\frac{3}{t}dt}$ ← Dont need a constant here

$\mu(t) = e^{-3\ln(t)}$ $t > 0$, so no $|t|$

$\mu(t) = t^{-3}$

③ $\underbrace{y' t^{-3} - 3t^{-4}y} = t^2$

Conveniently product rule $(y\cdot t^{-3})' = t^2$

④ $\int(y\cdot t^{-3})' = \int t^2$

$y\cdot t^{-3} = \frac{t^3}{3} + C$

⑤ $y = t^3C + \frac{t^6}{3}$

Example 2: $\frac{dy}{dt} = (\sin t)y + 4$

① $\frac{dy}{dt} - (\sin t)y = 4$

② $\mu(t) = e^{\int(-\sin t)dt} = e^{\cos t}$

③ $\frac{dy}{dt}e^{\cos t} - e^{\cos t}\sin t\, y = 4$

$(y\cdot e^{\cos t})' = 4e^{\cos t}$

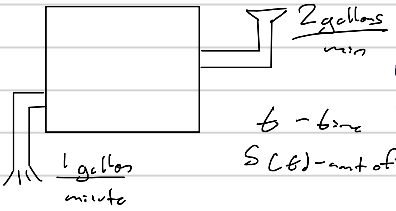
④ $\int(y\cdot e^{\cos t})'dt = \int 4e^{\cos t}dt$

$y\cdot e^{\cos t} = 4\int e^{\cos t}dt$

⑤ $y(t) = 4e^{-\cos t}\cdot\int e^{\cos t}dt$

If the integral becomes too challenging, you can only leave it in the integral form.

Example 3: Salt in a tank... again



Volume = 15 + t

density = $\frac{St}{\text{Volume}} = \frac{S(t)}{15+t}$

t - time

S(t) - amt of salt

$$\frac{dS}{dt} = 1 - \frac{S(t)}{15+t},\ S(0) = 6 \qquad S(15) = ?$$

① $\frac{dS}{dt} + \frac{S}{15+t} = 1$

② $\mu(t) = e^{\int\frac{1}{15+t}dt}$

$\mu(t) = e^{\ln(15+t)} = 15+t$

③ $\frac{dS}{dt}(15+t) + S = 15+t$

$(S\cdot(15+t))' = 15+t$

④ $\int(S\cdot(15+t))'dt = \int(15+t)dt$

$S(t)\cdot(15+t) = 15t + \frac{1}{2}t^2 + C$

⑤ $S(t) = \frac{15t + \frac{1}{2}t^2 + C}{15+t}$

IVP: $S(t) = \frac{15t + \frac{1}{2}t^2 + C}{15+t}$, $S(0) = 6$

$6 = \frac{C}{15}$ ∴ $C = 90$ ⇒ $S(t) = \frac{15t + \frac{1}{2}t^2 + 90}{15+t}$

Section 2.1 - Modeling Via Systems

Recall: Predator - Prey Model ← common model

Parameters: α = growth rate of rabbits

γ = death rate of foxes

β = Number of rabbit deaths, roughly

δ = benefit to fox population from rabbit

$$\left.\begin{aligned}\frac{dR}{dt} &= \alpha R - \beta RF \\ \frac{dF}{dt} &= -\gamma F + \delta RF\end{aligned}\right\}$$

This is a first order system of DEs

A solution to this is a pair of functions that satisfy both equations!

Also given a pair of IVs as well

Equilibrium solutions (R,F) are solutions if they make $\frac{dR}{dt}$ and $\frac{dF}{dt} = 0$

Example 1: parameters given for predator-prey model

Given: $\left.\begin{aligned}\alpha &= 2 \\ \beta &= 1.2 \\ \gamma &= 1 \\ \delta &= 0.9\end{aligned}\right\}$ $\quad \frac{dR}{dt} = 2R - 1.2RF \qquad \frac{dF}{dt} = -F + 0.9RF$

Equilibrium Solutions: have to get creative

$$\begin{aligned}2R - 1.2RF &= 0 \\ -F + 0.9RF &= 0\end{aligned} \;\Big> \; \begin{aligned}R(2 - 1.2F) &= 0 \\ F(0.9R - 1) &= 0\end{aligned} \;\Big\} \; \begin{aligned}&R=0,\ F=0, \\ &R = \tfrac{1}{0.9},\ F = \tfrac{2}{1.2}\end{aligned}$$

Plotting rabbits and foxes shows a graph independent of time, called a phase portrait

Two equilibrium solutions found: $(R,F) = (0,0)$ $\quad (1.11, 1.67)$

The RF-plane is called the phase plane, analogous to a phase line

Phase Portraits: A graph in the RF-plane of solutions to multiple initial value problems for a system

The models above aren't fully accurate, don't account for carrying capacity. We look to logistic!

Logistic Modification

$$\frac{dR}{dt} = \alpha R\left(1 - \frac{R}{N}\right) - \beta RF \quad \text{and} \quad \frac{dF}{dt} = -\gamma F + \delta RF$$

Example 2: Examining Logistic Model

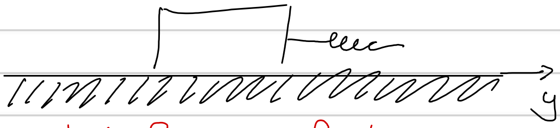
$(0,0)$ is called a trivial solution for these models

$$\begin{cases} 2R\left(1 - \frac{R}{2}\right) - 1.2FR = 0 \\ -F - 0.9RF = 0 \end{cases}$$

A phase portrait of these curves shows they all approach equilibrium

↖ Could solve by factoring or numerical methods, but it is omitted

Undamped Mass-Spring System



N2L

$$F_{net} = ma = m\frac{d^2y}{dt^2}$$

Hooke's Law

$$F_{spring} = -ky$$

$$\left.\right\} \; m\frac{d^2y}{dt^2} = -ky \;\Downarrow\; \frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

Second Order DE

IV of $y(0) = 0$ for this problem

Logically this should be a sinusoid solution

$\frac{d^2y}{dt^2} - \frac{k}{m}y = 0$ is called the simple undamped harmonic oscillator

We can turn second order / higher order DEs to lower ones with other variables

$v = \frac{dy}{dt}$ and $a = \frac{dv}{dt}$, giving us $\frac{dv}{dt} = -\frac{k}{m}y$ where $\frac{dy}{dt} = v$

add this because we used the relationship

This gives us a first order system which is better for analysis

Example: Guessing

Consider $\begin{cases} \frac{d^2y}{dt^2} + \frac{k}{m}y = 0. \\ y(t) = \cos(\beta t) \end{cases}$ what β is $y(t) = \cos(\beta t)$ a solution

plug in

$$-\beta^2\cos(\beta t) + \frac{k}{m}\cos(\beta t) = 0 \quad \therefore \beta = \pm\sqrt{\frac{k}{m}}$$

Cosine is even so ± doesn't matter

$$y(t) = \cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

Section 2.2 - Geometry of Systems

Consider Predator-Prey Model Example:

$$\frac{dR}{dt} = 2R - 1.2RF$$

$$\frac{dF}{dt} = -F + 0.9RF$$

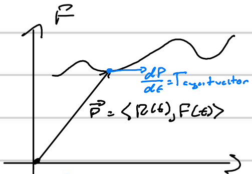
Solution vector

$$\vec{P}(t) = \begin{pmatrix} R(t) \\ F(t) \end{pmatrix}$$

vector valued function

Then, Calc 3 vector derivative rules

$$\frac{d\vec{P}}{dt} = \begin{pmatrix} \frac{dR}{dt} \\ \frac{dF}{dt} \end{pmatrix} = \begin{pmatrix} 2R - 1.2RF \\ -F + 0.9RF \end{pmatrix}$$

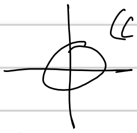


Slope Fields" in R

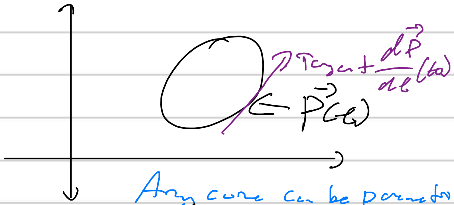
Remember that any curve can be parametrized

Instead of a slope field we get a vector field

Recall Circle Parametrization:



Circle: $(\cos t, \sin t)$
$0 \le t \le 2\pi$



Any curve can be parametrized by vector functions

When all vectors on the vector plot of a system are the same length, it is called a direction field

Example: $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x + x^3 = 0$

a) convert to a system

$\frac{dx}{dt} = y$, $\frac{d^2x}{dt^2} = \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = -2y + 3x - x^3$

System, 2 functions →

$$\begin{cases} \frac{dx}{dt} = y(t) \\ \frac{dy}{dt} = -2y + 3x - x^3 \end{cases}$$

This technique can be applied to any order!

b) $\vec{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $\frac{d\vec{y}}{dt} = \begin{pmatrix} y(t) \\ -2y + 3x - x^3 \end{pmatrix}$ → vector field

Equilibrium when $\frac{d\vec{y}}{dt} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{cases} y = 0 \\ -2y + 3x - x^3 = 0 \end{cases} \Rightarrow \begin{matrix} y = 0 \\ 3x - x^3 = 0 \\ x(3 - x^2) = 0 \end{matrix}$

$x = 0$, $x = +\sqrt{3}$ and $x = -\sqrt{3}$

$(0, 0)$ ← Trivial
$(\sqrt{3}, 0)$
$(-\sqrt{3}, 0)$

Section 2.3 - Damped Harmonic Oscillators

In this model, assume the strength of the damping force is proportional to velocity

The damping force is: $-b\dfrac{dy}{dt}$ where $b>0$ is the damping constant

$y(t)$ - position $\qquad \dfrac{dy}{dt}$ - velocity $\qquad \dfrac{d^2y}{dt^2}$ - acceleration

N2L yields: $$m\frac{d^2y}{dt^2} = \underbrace{-b\frac{dy}{dt}}_{\text{Damping force}}\underbrace{-ky}_{\text{Spring}} \Leftrightarrow m\frac{d^2y}{dt^2}+b\frac{dy}{dt}+ky=0 \Leftrightarrow my''+by'+ky=0$$

F=ma

Since mass ≠ 0: $$\frac{d^2y}{dt^2}+\frac{b}{m}\left(\frac{dy}{dt}\right)+\frac{k}{m}y=0$$

for simplicity we say $\frac{b}{m}=p$ and $\frac{k}{m}=q$ $\Rightarrow y''+py'+qy=0$

Transform $y''+py'+qy=0$ to a System...

$\dfrac{dy}{dt}=v \qquad \dfrac{d^2y}{dt^2}=\dfrac{dv}{dt}$ $$\begin{cases}\dfrac{dv}{dt}=-pv-qy\\ \dfrac{dy}{dt}=v\end{cases}$$

We can observe Behaviors of solutions for this by making use of "VectorPlot" in Mathematica

$$\therefore \vec{y}=\begin{pmatrix}v\\y\end{pmatrix}=\begin{pmatrix}-pv-qy\\v\end{pmatrix}$$ (idea from Section 2.2)

Example: Guess-and-test method

Let $$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=0$$

Guess some exp function

← It should be either a trig function or exponential

Let $y(t)=\alpha e^{st}$

$y'=\alpha s e^{st}$ and $y''=\alpha s^2 e^{st}$

$$\therefore \cancel{\alpha}s^2e^{st}+5\cancel{\alpha}se^{st}+6\cancel{\alpha}e^{st}=0$$

$$e^{st}(s^2+5s+6)=0$$

$e^{st}\neq 0$ so division works

This is called a characteristic equation $\Rightarrow s^2+5s+6=0$

$$s=\frac{-5\pm\sqrt{25-24}}{2},\quad s=-3,\ s=-2$$

→ $y_1(t)=e^{-2t}$, $y_2(t)=e^{-3t}$

y_1+y_2 is also a solution

$(y_1+y_2)''+5(y_1+y_2)'+6(y_1+y_2)=0$

$\therefore$ General Solution: $y(t)=\alpha e^{-2t}+\beta e^{-3t}$

for $\alpha,\beta\in\mathbb{R}$

Consider $$\lim_{t\to\infty}y(t)=\lim_{t\to\infty}\alpha e^{-2t}+\beta e^{-3t}=0$$

← Motion, or oscillation, will stop over time

Section 2.4 - Decoupled Systems

A system of DEs is decoupled if the rate of change of at least one of the dependent variables depends only on its own value.

Example: A completely decoupled system

$$\begin{cases} \frac{dx}{dt} = 3x \\ \frac{dy}{dt} = -2y \end{cases}$$

Separable!

1. $\frac{dx}{x} = 3dt$

$x(t) = K_1 e^{3t}$

2. $\frac{dy}{y} = 2dt$

$y(t) = K_2 e^{-2t}$

Vector Notation

If $\vec{y} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ Then, $\frac{dy}{dt} = \begin{pmatrix} 3x \\ -2y \end{pmatrix}$

Plotting this reveals $(0,0)$ is a source

Example: Partially Decoupled System

$$\begin{cases} \frac{dx}{dt} = -5x \\ \frac{dy}{dt} = 2x - 3y \end{cases}$$

1. $x(t) = K_1 e^{-5t}$

$\therefore \frac{dy}{dt} = 2K_1 e^{-5t} - 3y$ linear eq!

$\frac{dy}{dt} + 3y = 2K_1 e^{-5t}$

Can either solve by doing NH/H or by integrating factors

Solve linear: $y' + 3y = 2K_1 e^{-5t}$ (NH)

$y' + 3y = 0$ (H)

$y_h(t) = K_2 e^{-3t}$

Guess: $y_p(t) = Ce^{-5t}$, $y_p'(t) = -5Ce^{-5t}$

$\therefore -5Ce^{-5t} + 3Ce^{-5t} = 2K_1 e^{-5t}$

$-2C = 2K_1 \Rightarrow C = -K_1$

general solution $y(t) = y_h(t) + y_p(t)$

$y(t) = K_2 e^{-3t} - K_1 e^{-5t}$

$$\begin{cases} x(t) = K_1 e^{-5t} \\ y(t) = K_2 e^{-3t} - K_1 e^{-5t} \end{cases}$$

Section 2.5 - Euler's Method for Systems

Consider the system

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases}$$

A solution curve $\vec{Y}(t) = (x(t), y(t))$ whose tangent vector agrees w/ the vector field:

$\vec{F}(\vec{Y}) = (f(x,y), g(x,y))$

A vector of derivatives, "tangent vector"

with Initial Condition (x_0, y_0)

Algorithm for Euler's Method

① Select a step size Δt, starting at (x_0, y_0). Use $\Delta t \vec{F}(x_0, y_0)$ to make the first step

$(x_1, y_1) = (x_0, y_0) + \Delta t \vec{F}(x_0, y_0)$

② Compute $\vec{F}(x_1, y_1)$ and do the following

$(x_2, y_2) = (x_1, y_1) + \Delta t \vec{F}(x_1, y_1)$

③ Repeat until you don't need to anymore

Very similar to Euler's method from Chapter 1, just with coordinates instead.

$\vec{F}(x_0, y_0)$ is equivalent to:

$\left(\frac{dx}{dt}(x_0, y_0), \frac{dy}{dt}(x_0, y_0)\right)$

↑ Derivatives evaluated at point

Section 2.6 - Existence and Uniqueness for Systems

Non autonomous Systems

$$\frac{d\vec{P}}{dt} = \vec{F}(t,\vec{P}) = \begin{pmatrix} f(t,x,y) \\ g(t,x,y) \end{pmatrix}$$

A vector field $\vec{F}(t,\vec{P})$ is continuously differentiable if

$$\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}$$ are cts.

Connection to Singular DE

If a vector field $\vec{F}$ is cts and differentiable, then a solution exists around an initial value and it is unique on the interval.

Geometrically { Two solutions cannot start at the same place at the same time

Same can be said for autonomous systems, but we don't need to check $\frac{\delta f}{\partial t}$ or $\frac{\delta g}{\delta t}$

It is wise to check these things before applying numerical methods

Example 1 : Show that $x(t) \to \infty$ when $t \to \infty$

$$\begin{cases} \frac{dx}{dt} = x^2+y = f \\ \frac{dy}{dt} = xy = g \end{cases} \qquad \vec{F} = \begin{pmatrix} x^2+y \\ xy \end{pmatrix} \text{ is cts. differentiable}$$

$(x_0, y_0) = (0,1)$ $\qquad \frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 1 \Big| \frac{\partial g}{\partial x} = y, \frac{\partial g}{\partial y} = x$

$\frac{dx}{dt} \geq 0$ $\quad \forall t > 0$ $\quad$ x and y are increasing functions

$\frac{dy}{dt} \geq 0$

You can deduce this from the initial condition (0,1)

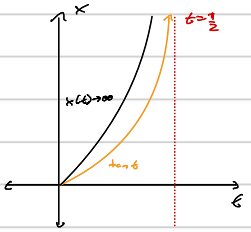
Consider $\frac{dx}{dt} = x^2+1$, $x(0) = 0$

$$\int \frac{dx}{x^2+1} = \int dt$$

$\arctan(x) = t$

$x(t) = \tan(t)$

$\lim_{t \to \frac{\pi}{2}^-} \tan(t) = \infty$



Example 2

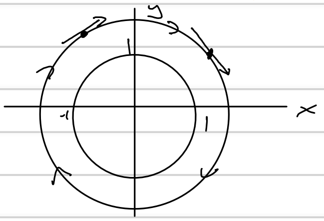
$$\begin{cases} \frac{dx}{dt} = y(x^2+y^2-1) \\ \frac{dy}{dt} = -x(x^2+y^2-1) \end{cases}$$

Equilibriums : (0,0)

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = -\frac{x}{y}$$

Solve $\begin{cases} y\,dy = -x\,dx \\ x^2+y^2 = C \end{cases}$

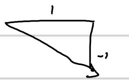
Construct Phase Portrait



Curves are all circles with radius C centered @ (0,0)

Vector @ (1,1)

$\langle 1, -1 \rangle$



Vector @ (-1,1)

$\langle 1, 1 \rangle$

Direction determined by vectors at points!

Test 1 - Friday 3/1

- Covers chapter 1 and 2
- Practice exams on Canvas
  - ↳ Solutions provided down the line
- Extra office hours
-

Section 3.1 - Properties of Linear Systems

Consider $\begin{cases} \frac{dx}{dt} = ax+by \\ \frac{dy}{dt} = cx+dy \end{cases}$, a, b, c, d are constants called coefficients of the system

- This system is called a 2D linear system w/ constant coefficients
  - ↳ It is also known as a planar system
- Autonomous system (no t dependence)

Uniqueness Thm: $\begin{cases} \frac{dx}{dt} = ax+by = f \\ \frac{dy}{dt} = cx+dy = g \end{cases}$ $\frac{\partial f}{\partial x} = a$, $\frac{\partial f}{\partial y} = b$, $\frac{\partial g}{\partial x} = c$, $\frac{\partial g}{\partial y} = d$

Constants are cts. so uniqueness and existence thm work here

Let A be a 2x2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and let y be the vector $\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\therefore A\vec{Y} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

⇒ Our system can be written as: $\frac{d\vec{Y}}{dt} = A\vec{Y}$. A is called the coefficient matrix

Basic matrix multiplication!

Example 1: Write the system below in matrix form.

$$\begin{cases} \frac{dy}{dt} = -5x \\ \frac{dx}{dt} = 2x-3y \end{cases} \qquad \frac{d\vec{Y}}{dt} = \begin{pmatrix} -5 & 0 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Determinant of a 2x2

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\det(A) = ad - bc$

Matrices with a zero determinant are called singular or degenerate

Be comfortable taking determinants of matrices

Equilibrium Points

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{cases} ax_0 + by_0 = 0 \\ cx_0 + dy_0 = 0 \end{cases}$$

$\vec{Y}(x_0, y_0) = (0, 0)$ is a trivial solution

Thm: If A is a matrix with $\det(A) \neq 0$, then the only equilibrium point for the system is the origin

$\det(A) \neq 0$ ⇒ matrix A is nonsingular → columns of A are linearly independent vectors

$$A = \left(\begin{array}{c|c} a & b \\ c & d \end{array}\right) \quad \text{Colums: } \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$$

Recall: $\vec{u} = \lambda\vec{w}$ for some constant λ if $\vec{u}$ and $\vec{w}$ are linearly dependent

$\det(A) = 0$ ⇒ matrix A is singular

The Linearity Principle

Suppose $\frac{d\vec{Y}}{dt} = A\vec{Y}$ is a linear system of DEs

Infinitely many solutions when $kY_1 + kY_2$

① If $\vec{Y}(t)$ is a solution of the system and k is a constant, $k\vec{Y}(t)$ is also a solution

Proof: Let $\vec{Y}(t)$ be a solution of $\frac{d\vec{y}}{dt}$

$$\text{If } \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad k\vec{Y} = \begin{pmatrix} kx \\ ky \end{pmatrix} \Rightarrow \frac{d(k\vec{Y})}{dt} = \begin{pmatrix} k\frac{dx}{dt} \\ k\frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} kx \\ ky \end{pmatrix}$$

$$\text{Then } \begin{pmatrix} akx + bky \\ ckx + dky \end{pmatrix} = \begin{pmatrix} k(ax+by) \\ k(cx+dy) \end{pmatrix} = k(\vec{Y}) = k\vec{Y}$$ Yay!

② If $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are two solutions of the system, $Y_1(t) + Y_2(t)$ is also a solution

Proof: Let $Y_1(t)$ and $Y_2(t)$ be solutions of $\frac{d\vec{y}}{dt}$

$$\vec{y} = \vec{y}_1 + \vec{y}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \text{ where } \vec{y}_1 = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \text{ and } \vec{y}_2 = \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$$

$$\frac{d(\vec{y}_1 + \vec{y}_2)}{dt} = \begin{pmatrix} \frac{dx_1}{dt} + \frac{dx_2}{dt} \\ \frac{dy_1}{dt} + \frac{dy_2}{dt} \end{pmatrix} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dy_1}{dt} \end{pmatrix} + \begin{pmatrix} \frac{dx_2}{dt} \\ \frac{dy_2}{dt} \end{pmatrix} = \frac{d\vec{y}_1}{dt} + \frac{d\vec{y}_2}{dt} \longrightarrow \frac{dy_1}{dt} + \frac{dy_2}{dt} = A\vec{Y}_1 + A\vec{Y}_2$$

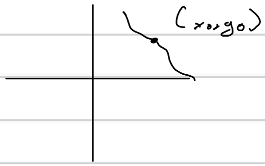
$$A(\vec{y}_1 + \vec{y}_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} a(x_1 + x_2) + b(y_1 + y_2) \\ c(x_1 + x_2) + d(y_1 + y_2) \end{pmatrix} = \begin{pmatrix} ax_1 + by_1 \\ cx_1 + dy_1 \end{pmatrix} + \begin{pmatrix} ax_2 + by_2 \\ cx_2 + dy_2 \end{pmatrix} = A\vec{Y}_1 + A\vec{Y}_2$$

Initial Value Problems

Given two solutions $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$, we know $K_1\vec{Y}_1 + K_2\vec{Y}_2$ is also a solution. Are all of these solutions?

Will the solution to any IVP be in this form? ← or $\vec{y}_3(t) \neq K_1\vec{y}_1 + K_2\vec{y}_2$ but $\vec{y}_3(t)$ is a solution

Proof:
$$\begin{cases} \frac{dx}{dt} = ax+by \\ \frac{dy}{dt} = cx+dy \\ x(t_0)=x_0 \\ y(t_0)=y_0 \end{cases}$$



(x_0,y_0) is an arbitrary solution.
then $\vec{y}_3 = K_1\vec{y}_1 + K_2\vec{y}_2$ as
$K_1\vec{y}_1 + K_2\vec{y}_2$ describes all solutions
by Uniqueness thm.

$\therefore \vec{y}(t) = K_1\vec{y}_1 + K_2\vec{y}_2$

$\frac{\partial f}{\partial x}=a, \frac{\partial f}{\partial y}=b, \frac{\partial g}{\partial x}=c, \frac{dg}{dy}=d$

Linear Independence

← Vectors that don't lie on the same line, or vectors that are not constant multiples of each other

Suppose (x_1,y_1) and (x_2,y_2) are two linearly independent vectors in the plane. For any given vector (x_0,y_0), there are constants K_1 and K_2 s.t. $K_1\begin{pmatrix}x_1\\y_1\end{pmatrix} + K_2\begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}x_0\\y_0\end{pmatrix}$

$\forall \vec{v}, \exists K_1, K_2 \in \mathbb{R}$, then $\vec{v} = K_1\vec{y}_1 + K_2\vec{y}_2$ for linearly independent vectors y_1 and y_2

You can express any vector in $\mathbb{R}^2$ space using multiples of linearly independent vectors.

Suppose $\vec{y}_1$ and $\vec{y}_2$ are solutions of a linear system $\frac{d\vec{Y}}{dt} = A\vec{Y}$. If $\vec{Y}_1(0)$ and $\vec{Y}_2(0)$ are linearly independent, then for any initial condition $\vec{Y}(0) = (x_0,y_0)$, we can find constants K_1 and K_2 so that $K_1\vec{Y}_1 + K_2\vec{Y}_2$ is the solution to the IVP

Section 3.2 - Straight Line Solutions

Example to Introduce Topic

Consider $\frac{d\vec{Y}}{dt} = A\vec{Y} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\vec{Y} \iff \begin{cases} \frac{dx}{dt} = 4x+2y \\ \frac{dy}{dt} = x+3y \end{cases}$

$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ $\therefore$ $\det(A) = 12\,2 = 10 \neq 0$ → linearly independent solutions

Eigenvectors and Eigenvalues

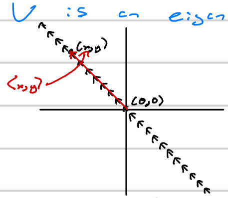
← It is critical to this course to understand this

Along a straight-line solution, the vector field must point either directly towards or away from the origin

↳ At a point (x,y) on a straight-line solution, $\vec{F}(x,y)$ must be some multiple of (x,y). we seek vectors (x,y) s.t. $A\vec{Y} = \lambda\vec{Y} \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$

Given A, a scalar λ is an eigenvalue for A if $\exists\, \vec{v} \neq 0$ s.t. $A\vec{v} = \lambda\vec{v}$. The vector $\vec{v}$ is called an eigenvector corresponding to λ ← eigen = characteristic

If $\vec{v}$ is an eigenvector, then any multiple of $\vec{v}$ also is.



$\begin{cases} \frac{dx}{dt} = ax+by \\ \frac{dy}{dt} = cx+dy \end{cases}$

Vector $\langle x,y\rangle$ is tangent to $\frac{d\vec{Y}}{dt} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$

Can change the length of the vector and still a straight-line solution

Basically a fancy way of saying any constant multiple of a vector that is on a straight line solution is also a solution

Let $\begin{cases} \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ A\begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{cases}$ $A\vec{v} = \begin{pmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{pmatrix} = \lambda\vec{v} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$

System: $\begin{matrix} av_1 + bv_2 = \lambda v_1 \\ cv_1 + dv_2 = \lambda v_2 \end{matrix} \Rightarrow \begin{cases} (a-\lambda)v_1 + bv_2 = 0 \\ cv_1 + (d-\lambda)v_2 = 0 \end{cases}$

Vectors $\langle a-\lambda, b\rangle$ and $\langle c, d-\lambda\rangle$ must be linearly dependent

$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ To be linearly dependent,

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - cb = 0$$

$$ad - \lambda(a+b) + \lambda^2 - cb = 0$$

$$\lambda^2 - \lambda(a+b) + (ad-bc) = 0$$

Eigenvalue
↳ characteristic value

Eigenvector
↳ characteristic vector

Characteristic Equation or Polynomial

roots are eigenvalues

Complex roots covered in 3.4/3.5

Example 1

Consider $\frac{d\vec{Y}}{dt} = A\vec{Y} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}(\vec{Y}) \iff \begin{cases} \frac{dx}{dt} = 4x+2y \\ \frac{dy}{dt} = x+3y \end{cases}$ Find straight line solutions

$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\det\begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix} = 0$

$$(4-\lambda)(3-\lambda) - 2 = 0$$

$$\lambda^2 - 7\lambda + 10 = 0$$ Factor or quad formula

$$(\lambda-5)(\lambda-2) = 0 \qquad \lambda_1 = 5, \quad \lambda_2 = 2$$

Different eigenvalues have different eigenvectors

λ_2: $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

linearly dependent

$\begin{cases} 4v_1 + 2v_2 = 2v_1 \\ v_1 + 3v_2 = 2v_2 \end{cases} \Rightarrow \begin{cases} 2v_1 + 2v_2 = 0 \\ v_1 + v_2 = 0 \end{cases}$ $v_1 = -v_2$

choose easy solution

Infinitely Many

General Vector

$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = C\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} C \\ -C \end{pmatrix}$

λ_1: $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 5\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

linearly dependent

$\begin{cases} 4v_1 + 2v_2 = 5v_1 \\ v_1 + 3v_2 = 5v_2 \end{cases} \Rightarrow \begin{cases} -v_1 + 2v_2 = 0 \\ v_1 - 2v_2 = 0 \end{cases}$ $v_1 = 2v_2$

↓

$\begin{pmatrix} v_1 \\ 2v_2 \end{pmatrix} = C\begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} C \\ 2C \end{pmatrix}$

Theorem

Suppose the matrix A has a real eigenvalue λ associated with eigenvector $\vec{v}$. Then the linear system $\frac{d\vec{Y}}{dt} = A\vec{Y}$ has the straight-line solution $\vec{Y}(t) = e^{\lambda t}\vec{v}$

Link back to example 1: $\lambda_1 = 2: \begin{pmatrix}1\\-1\end{pmatrix}$, $\lambda_2 = 5: \begin{pmatrix}1\\2\end{pmatrix}$ $\rightarrow$ $\vec{y}_1 = e^{2t}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}e^{2t}\\-e^{2t}\end{pmatrix}$, $\vec{y}_2 = e^{5t}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}e^{5t}\\2e^{5t}\end{pmatrix}$ Two linearly independent solutions

Linearity principle: $y(t)$ is a solution where $y_1(t) + y_2(t) = y(t)$, if y_1 and y_2 are also solutions

General Solution: $\vec{y}(t) = K_1\begin{pmatrix}e^{2t}\\-e^{2t}\end{pmatrix} + K_2\begin{pmatrix}e^{5t}\\2e^{5t}\end{pmatrix}$

$$\vec{y}(t) = \begin{pmatrix}K_1e^{2t} + K_2e^{5t}\\ -K_1e^{2t} + 2K_2e^{5t}\end{pmatrix}$$

Distinct eigenvalues correspond to linearly independent eigenvectors

Theorem

If λ_1 and λ_2 are real distinct eigenvalues with eigenvectors $\vec{v}_1$ and $\vec{v}_2$, then the two solutions are linearly independent and

$$\vec{Y}(t) = K_1e^{\lambda_1 t}\vec{v}_1 + K_2e^{\lambda_2 t}\vec{v}_2$$

is the general solution to the system.

Example 2

Characteristic Equation

$$p(\lambda) = \det\begin{pmatrix}a-\lambda & b\\ c & d-\lambda\end{pmatrix}$$

Solve the IVP $\begin{cases}\frac{dx}{dt} = 3x\\ \frac{dy}{dt} = x - 2y\end{cases}$ $(x(0), y(0)) = (5, 0)$

$A = \begin{pmatrix}3 & 0\\ 1 & -2\end{pmatrix}$ $\quad p(\lambda) = \begin{pmatrix}3-\lambda & 0\\ 1 & -2-\lambda\end{pmatrix} = (3-\lambda)(-2-\lambda) - 0 = 0$

Characteristic Polynomial

$\lambda_1 = 3, \lambda_2 = -2$

$\lambda_1 = 3: \begin{pmatrix}3 & 0\\ 1 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 3\begin{pmatrix}x\\y\end{pmatrix}$

$3x = 3x$

$x - 2y = 3y \qquad x = 5y$

$v_1 = \begin{pmatrix}5\\1\end{pmatrix}$ for $\lambda_1 = 3$

$\lambda_2 = -2: \begin{pmatrix}3 & 0\\ 1 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = -2\begin{pmatrix}x\\y\end{pmatrix}$

$3x = -2x \qquad x = 0$

$x - 2y = -2y \qquad y = C$

$\lambda = -2 \rightarrow v_2 = \begin{pmatrix}0\\1\end{pmatrix}$ ← must be 0 ← y can be anything, choose easiest

$\therefore \vec{Y}(t) = K_1e^{-2t}\begin{pmatrix}0\\1\end{pmatrix} + K_2e^{3t}\begin{pmatrix}5\\1\end{pmatrix}$ ← use thm. to get solution vector

$t = 0$: $\begin{pmatrix}5\\0\end{pmatrix} = \begin{pmatrix}5K_2\\K_1 + K_2\end{pmatrix}$ $\quad 5K_2 = 5$, $K_2 = 1$ $\quad K_1 + K_2 = 0$, $K_1 = -1$

$$\boxed{\begin{gathered}\vec{Y}(t) = -e^{-2t}\begin{pmatrix}0\\1\end{pmatrix} + e^{3t}\begin{pmatrix}5\\1\end{pmatrix}\\ \text{or } \vec{Y}(t) = \begin{pmatrix}5e^{3t}\\ -e^{-2t} + e^{3t}\end{pmatrix}\end{gathered}}$$

Section 3.3 - Phase Portraits for Real Eigenvalues

Consider $\frac{d\vec{Y}}{dt} = A\vec{Y}$ with two distinct real eigenvalues $\lambda_1 < \lambda_2$

The equilibrium point at the origin is called:

- Sink if $\lambda_1 < \lambda_2 < 0$ ← both negative
- Source if $0 < \lambda_1 < \lambda_2$ ← both positive
- Saddle if $\lambda_1 < 0$ and $\lambda_2 > 0$ ← One each

Solution Curves tend towards the dominant eigenvector

Example: A sink

$$A = \begin{pmatrix} -1 & -2 \\ 1 & -4 \end{pmatrix} \rightarrow \lambda_1 = -3 \quad \text{and} \quad \lambda_2 = -2$$

do some work.. $\vec{Y}(t) = \begin{pmatrix} K_1 e^{-3t} + 2K_2 e^{-2t} \\ K_1 e^{-3t} + K_2 e^{-2t} \end{pmatrix}$ $\quad e^{-\lambda t} \to 0$ as $t \to \infty$

Regardless of K_1 and K_2, the solution approaches 0 as $t \to \infty$

Example: A source

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

Characteristic Equation: $p(\lambda) = \det(A - I\lambda) = \det\begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$$= (4-\lambda)(3-\lambda) - 2 = 0$$
$$\lambda^2 - 7\lambda + 12 - 2 = 0$$
$$\lambda^2 - 7\lambda + 10 = 0$$
$$(\lambda - 5)(\lambda - 2) = 0$$

$$\lambda = \frac{7 \pm \sqrt{49 - 4(10)}}{2} = \frac{7 \pm 3}{2} = 5, 2$$

$$\lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 5$$

do some work.. $\vec{Y}(t) = \begin{pmatrix} -K_1 e^{2t} + 2K_2 e^{5t} \\ K_1 e^{2t} + K_2 e^{5t} \end{pmatrix}$ $\quad e^{\lambda t} \to \infty$ as $t \to \infty$

Regardless of K_1 and K_2, the solution approaches ∞ as $t \to \infty$

Example: A saddle

$$A = \begin{pmatrix} 3 & 0 \\ 1 & -2 \end{pmatrix} \longrightarrow \lambda_1 = -2 \quad \text{and} \quad \lambda_2 = 3$$

do some work.. $\vec{Y}(t) = \begin{pmatrix} 5K_2 e^{3t} \\ K_1 e^{-2t} + K_2 e^{3t} \end{pmatrix}$

The choice of K_1 and K_2 will result in different behavior around the origin

Section 3.4 - Complex Eigenvalues

Example 1: No apparent straight line solutions

$$\frac{dx}{dt} = 2x + 2y, \quad \frac{dy}{dt} = -4x + 6y$$

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix} \quad p(\lambda) = (2-\lambda)(6-\lambda) + 8$$

$$0 = \lambda^2 - 8\lambda + 20 \rightarrow \lambda = \frac{8 \pm \sqrt{64-80}}{2} = \frac{8 \pm 4i}{2} = 4 \pm 2i$$

we want corresponding eigenvectors:

$$\lambda_1 = 4-2i: \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (4-2i)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Solve like normal

Set $v_1 = 1$ then solve for v_2

$$\begin{matrix} 2v_1 + 2v_2 = (4-2i)v_1 \\ -4v_1 + 6v_2 = (4-2i)v_2 \end{matrix} \Rightarrow \begin{cases} (-2+2i)v_1 + 2v_2 = 0 \\ -4v_1 + (2+2i)v_2 = 0 \end{cases} \Rightarrow \begin{matrix} v_1 = 1 \\ v_2 = 1-i \end{matrix} \quad \vec{v} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$$\lambda_2 = 4+2i: \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (4+2i)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{matrix} 2v_1 + 2v_2 = (4+2i)v_1 \\ -4v_1 + 6v_2 = (4+2i)v_2 \end{matrix} \Rightarrow \begin{cases} (-2-2i)v_1 + 2v_2 = 0 \\ -4v_1 + (2-2i)v_2 = 0 \end{cases} \Rightarrow \begin{matrix} v_1 = 1 \\ v_2 = 1+i \end{matrix} \quad \vec{v} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$v_1 = 1: -2-2i + 2v_2 = 0$$

$$2v_2 = 2+2i \rightarrow v_2 = 1+i$$

Pick easy value and solve for other variable to get $\vec{v}$

$$\vec{Y}(t) = e^{\lambda t}\vec{v} = e^{(4+2i)t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix} = \begin{pmatrix} e^{(4+2i)t} \\ (1+i)e^{(4+2i)t} \end{pmatrix}$$

For conjugate Im part becomes negative. Unnecessary to compute both eigenvectors and solutions

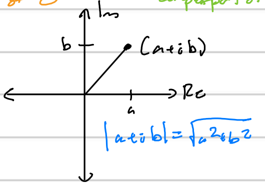
Complex Numbers

- Complex Conjugates: $a+ib$, $a-ib$
- Any Complex Number C can be written the standard form $C = a \pm ib$

↳ $Re(C) = a$, $Im(C) = b$

real part of C

complex part of C



$|a+ib| = \sqrt{a^2+b^2}$

- Cannot compare Complex numbers directly but magnitude can be found. Its the distance to (0,0)

$$mag = \sqrt{a^2+b^2}$$

Pythagorean

- Conjugates: $(a+ib)(a-ib)$

$$= a^2 - iab + iab + b^2$$

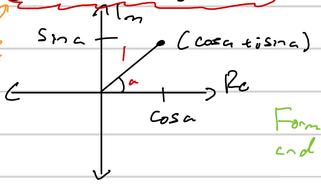
$$= a^2 + b^2$$

Euler's Formula

$$e^{i\pi} = -1$$

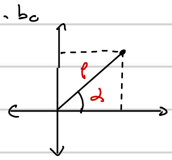
↳ $e^{ia} = \cos a + i\sin a$

Polar Form of Complex Number



A complex C can be

$$C = \rho e^{i\alpha}$$



Formula helps separate real and complex parts of complex numbers

Useful for Multiplication

$$C_1 = \rho_1 e^{i\alpha_1}, \quad C_2 = \rho_2 e^{i\alpha_2}$$

$$C_1C_2 = \rho_1\rho_2 e^{i(\alpha_1+\alpha_2)}$$

$$\therefore e^{a+ib} = e^a \cdot e^{ib} = e^a(\cos b + i\sin b)$$

← Proof uses power series

$$\therefore \text{ for } \lambda = 4+2i, \quad e^{(4+2i)t} = e^{4t} \cdot e^{2it} = e^{4t}(\cos(2t) + i\sin(2t))$$

$$\vec{Y}(t) = \begin{pmatrix} e^{(4+2i)t} \\ (1+i)e^{(4+2i)t} \end{pmatrix} = \begin{pmatrix} e^{4t}(\cos 2t + i\sin 2t) \\ e^{4t}(1+i)(\cos 2t + i\sin 2t) \end{pmatrix}$$

$$= \begin{pmatrix} e^{4t}(\cos 2t + i\sin 2t \\ e^{4t}(\cos 2t + i\sin 2t + i\cos 2t - \sin 2t \end{pmatrix}$$

$$= \begin{pmatrix} e^{4t}\cos 2t + e^{4t} i\sin 2t \\ e^{4t}(\cos 2t - \sin 2t) + e^{4t} i(\cos 2t + \sin 2t) \end{pmatrix} = \begin{pmatrix} e^{4t}\cos 2t \\ e^{4t}\cos 2t - e^{4t}\sin 2t \end{pmatrix} + i\begin{pmatrix} e^{4t}\sin 2t \\ e^{4t}\cos 2t + e^{4t}\sin 2t \end{pmatrix}$$

Use Eulers Formula to Separate Real and Imaginary parts of solutions

$$\vec{Y}_{re}(t) \qquad + \quad i \quad \vec{Y}_{im}(t)$$

Complex Conjugate

Properties of Complex conjugates

$$\left\{ \begin{matrix} \text{For } \lambda_1, & \text{For } \lambda_2, \\ \vec{Y}_{re} + i\vec{Y}_{im} & \vec{Y}_{re} - i\vec{Y}_{im} \end{matrix} \right.$$

Sums/Differences of Solutions are solutions
Scalar multiples of Solutions are also Solutions } Proves the below

$\therefore$ linearity Principle → $\vec{Y}_{re}(t)$ and $\vec{Y}_{im}$ are linearly independent solutions

Theorem

Suppose $\vec{Y}(t)$ is a complex-valued system to a linear system $\frac{d\vec{Y}}{dt} = A\vec{Y}$, where A has real coefficients.

Suppose $\vec{Y}(t) = \vec{Y}_{re}(t) + i\vec{Y}_{im}(t)$, where $\vec{Y}_{re}(t)$ and $\vec{Y}_{im}(t)$ are real valued functions of t.

$\therefore$ The real, $\vec{Y}_{re}(t)$, and imaginary, $\vec{Y}_{im}(t)$, are both solutions to the original system $\frac{d\vec{Y}}{dt} = A\vec{Y}$.

Proof on slides if you're interested ☺ ← Page 7

Complex Eigenvalues

Suppose $\frac{d\vec{Y}}{dt} = A\vec{Y}$ is a linear system with complex eigenvalues $\lambda_{1,2} = \alpha \pm i\beta$ for $\beta \neq 0$. Then the complex solutions have the form $\vec{Y}(t) = e^{(\alpha + i\beta)t}\vec{V}$ where $\vec{V}$ is a complex eigenvector. This can be written as:

$$\vec{Y}(t) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))\vec{V}$$

- $\cos(\beta t)$ and $\sin(\beta t)$ oscillate with period $2\pi/\beta$. This is the natural period
- The natural frequency is the number of cycles that solutions make in one unit of time $\left(\frac{\beta}{2\pi}\right)$
- β is the angular frequency in terms of radians per unit time

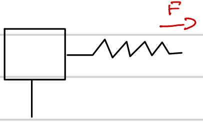
Behavior:

α value

- $\alpha < 0 \rightarrow$ solutions spiral to origin — spiral sink
- $\alpha > 0 \rightarrow$ solutions spiral from origin — spiral source
- $\alpha = 0 \rightarrow$ solutions are periodic — center

Classification of Origin

Example: Springs w/o friction



Frictionless

$\vec{F} = my'' = -ky$

$\therefore my'' + ky = 0$

$$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -\left(\frac{k}{m}\right)y \end{cases}$$

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}$$

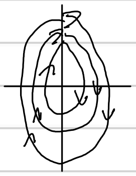
$p(\lambda) = (-\lambda)^2 + \frac{k}{m}$

$0 = \lambda^2 + \frac{k}{m}$

$0 = \left(\lambda - i\sqrt{\frac{k}{m}}\right)\left(\lambda + i\sqrt{\frac{k}{m}}\right)$

$\lambda_1 = i\sqrt{\frac{k}{m}}$ and $\lambda_2 = -i\sqrt{\frac{k}{m}}$

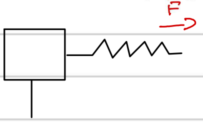
$\alpha = 0$, so origin is a center



To get direction of curves, check one point. All other curves will follow that same direction.

↑ either cw or ccw

Example: Springs w/ friction



$\vec{F} = my'' = -by' - ky$

$\therefore my'' + by' + ky = 0$

$$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -\left(\frac{k}{m}\right)y + bv \end{cases}$$

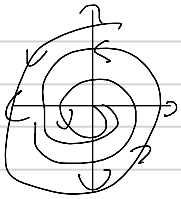
Example Sink

$$\begin{cases} \frac{dx}{dt} = -3x - 5y \\ \frac{dy}{dt} = 3x - y \end{cases}$$

$p(\lambda) = (-3-\lambda)(-1-\lambda) + 15$

$0 = \lambda^2 + 4\lambda + 18$

$\lambda = -2 \pm i\sqrt{14}$



Section 3.5 - Repeated Eigenvalues

Suppose D, the discriminant, of $p(\lambda)$ is 0, so $\lambda_1 = \lambda_2 = \vec{v}$. Then $\vec{v}$ is a repeated eigenvalue

Example 1

$$A = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}$$

$$p(\lambda) = (-2-\lambda)(-4-\lambda)+1$$
$$= \lambda^2 + 6\lambda + 9$$
$$0 = (\lambda+3)^2 \Rightarrow \lambda = -3$$

$$\lambda = -3: \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} -2x - y = -3x \\ x - 4y = -3y \end{cases} \Rightarrow \begin{cases} x - y = 0 \\ x - y = 0 \end{cases} \quad \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \vec{y}_1(t) = e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot K_1$$

General Solutions for Repeated Eigenvalues

$\vec{Y}(t) = e^{\lambda t} V_0 + t e^{\lambda t} \vec{V}_1$ where $\vec{V}_0 = (x_0, y_0)$ is an arbitrary initial condition and $\vec{V}_1$ is determined from $\vec{V}_0$ as $V_1 = (A - \lambda I) V_0$.

Recall: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- If $\vec{V}_1 = 0$, then $\vec{V}_0$ is an eigenvector and $Y(t)$ is a straight line solution
- Otherwise $\vec{V}_1$ is an eigenvector

$$(\vec{y}(t) = e^{\lambda t}\vec{V}_0 + t e^{\lambda}\vec{V}_1) A$$

$$A\vec{y}(t) = e^{\lambda t} A\vec{V}_0 + t e^{\lambda} A\vec{V}_1 \leftarrow \text{vector}$$

$$\frac{d\vec{y}}{dt} = e^{\lambda t}(\lambda\vec{V}_0 + \vec{V}_1) + t e^{\lambda t}(\lambda\vec{V}_1) = e^{\lambda t}(A\vec{V}_0) + t e^{\lambda t}(AV_1)$$

$$\begin{cases} \lambda\vec{V}_0 + \vec{V}_1 = A\vec{V}_0 \\ A\vec{V}_1 = \lambda\vec{V}_1 \end{cases}$$

V_1 is eigenvector of A

λ is eigenvector of A

Find V_0 from system $\Rightarrow \vec{V}_1 = (A - \lambda I)\vec{V}_0$

Example 2

$$\frac{dY}{dt} = A\vec{Y}, \quad A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}, \quad \vec{Y}(0) = (1, 0), \quad \lambda = 3$$

$$V_1 = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \left(\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\vec{V}_0$$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$$

$$-x_0 + y_0 = K_1$$
$$-x_0 + y_0 = K_2$$

$$K_1 = K_2 = y_0 - x_0$$

Lemma

Two Solutions Y_1 and Y_2

- $\vec{Y}_1(t) = V_0 e^{\lambda t} + V_1 t e^{\lambda t}$
- $\vec{Y}_2(t) = W_0 e^{\lambda t} + W_1 t e^{\lambda t}$

are equal for all t iff $V_0 = W_0$ and $V_1 = W_1$

Question

Is vector $\vec{y}_2 = e^{\lambda t}\vec{V}_0 + t e^{\lambda t}\vec{V}_1$ linearly independent with $\vec{y}_1(t) = e^{\lambda t}V_1$

Recall; Linearly Independent if $\forall \alpha \in \mathbb{R}$ $\vec{y}_1 \neq \alpha\vec{y}_2$

Yes they are linearly independent

General Solution

$$\vec{Y}(t) = e^{3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^{3t}\begin{pmatrix} y_0 - x_0 \\ y_0 - x_0 \end{pmatrix}$$

Section 3.6 - Summary

Second Order Linear equations

$ay'' + by' + cy = 0$

① Find Characteristic Polynomial Roots

- $as^2 + bs + c = 0$

② Determine steps to take based on roots

Value of s	General Solution
$s_1 \neq s_2 \in \mathbb{R}$	$y(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$
$s = a \pm \beta i$ in $\mathbb{C}$	$y(t) = K_1 e^{\alpha t}\cos(\beta t) + K_2 e^{\alpha t}\sin(\beta t)$
s in $\mathbb{R}$ repeated	$y(t) = K_1 e^{st} + K_2 t e^{st}$

Using Matrices

① Setup matrix from System

$y'' + py' + qy = 0 \quad \begin{cases} y' = v \\ v' = -pv - qy \end{cases}$

$$\vec{v}(t) = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix} \longrightarrow \frac{d\hat{v}}{dt} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\vec{V}$$

② Determine eigenvalues and eigenvectors from characteristic Polynomial

Unforced Harmonic Oscillators

Standard Form: $my'' + by' + ky = 0$

$\begin{cases} \text{Damped when } b > 0 \\ \text{Undamped when } b = 0 \\ \text{Not an Oscillator when } b < 0 \end{cases}$

Case 2: Underdamped

- $b^2 - 4km < 0$
- $s = \alpha \pm \beta t$ with $\alpha > 0$
- Solutions Oscillate with period $2\pi/\beta$ and tend towards $y = 0$

Case 3a) Overdamped

- Eigenvalues $\in \mathbb{R}^-$, s_1 and s_2

Case 3b) Critically Damped

- One negative Eigenvalue s
- $y(t) = K_1 e^{st} + K_2 t e^{st}$
- No oscillation and tend to $y = 0$

Section 3.7 - Trace-Determinant Plane

- A way of summarizing everything learned so far

Consider $\frac{d\vec{Y}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\vec{Y}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Characteristic Polynomial: $p(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc)$. We let:

- $T = a + d$, the trace of the matrix A
- $D = ad - bc$, the determinant of the matrix

$p(\lambda) = \lambda^2 - \text{Trace}(A) + \det(A)$

We can rewrite $p(\lambda)$ as $p(\lambda) = \lambda^2 - T\lambda + D$ with roots

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} \longrightarrow \lambda = \frac{T}{2} \pm \frac{\sqrt{T^2 - 4D}}{2} = \frac{T}{2} \pm \sqrt{\frac{T^2}{4} - D}$$

Definition

$\text{Trace}(A) = \text{Tr}(A) = a + d$

Sum of its Diagonal Elements

Case A: Complex Eigenvalues $(D > T^2/4)$

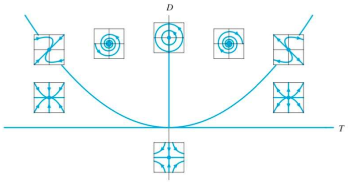
- Eigenvalues are complex with $\alpha = T/2$
  - ① $T < 0$: Origin is a spiral sink
  - ② $T > 0$: Origin is a spiral source
  - ③ $T = 0$: Origin is a center

Case B: Repeated Eigenvalue $(D = T^2/4)$

① $T < 0$: Origin is repeated root sink

② $T > 0$: Origin is repeated root source

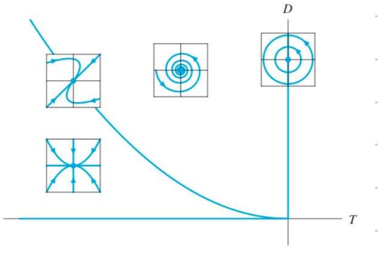
End Result



Harmonic Oscillator

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \longrightarrow \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}$$

$\underset{\text{mass}}{m > 0}$, $\underset{\text{Spring constant}}{k > 0}$, and $\underset{\text{damping coefficient}}{b \geq 0}$



Changing Parameters in a linear system

- Different values of D and T can drastically change behavior as seen in the summary portraits
- Bifurcation Behavior

Example: Bifurcations

Consider $\frac{d\vec{x}}{dt} = \begin{pmatrix} a & a \\ 1 & 0 \end{pmatrix}\vec{x}$, and find bifurcation value(s)

$A = \begin{pmatrix} a & a \\ 1 & 0 \end{pmatrix}$ $\quad Tr(A) = a$, $\det(A) = -a$

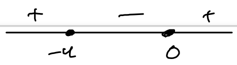
Recall: $\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} \Rightarrow \lambda = \frac{a \pm \sqrt{a^2 + 4a}}{2}$

Bifurcation points:

$a^2 + 4a = 0$

$a(a+4) = 0$

$a = 0,\ a = -4$



$a < -4$: Two real roots, sink $\to \frac{a - \sqrt{a^2+4a}}{2} < 0$ and $\frac{a + \sqrt{a^2+4a}}{2} < 0$

First Bifurcation Point $\boxed{a = -4}$:

$-4 < a < 0$: Two complex roots, spiral sink

Second Bifurcation Point $\boxed{a = 0}$:

$a > 0$: Two real roots, node (saddle) $\to \frac{a - \sqrt{a^2+4a}}{2} < 0$ and $\frac{a + \sqrt{a^2+4a}}{2}$

Example: A different system

Consider $\frac{d\vec{y}}{dt} = \begin{pmatrix} -2 & a \\ -2 & 0 \end{pmatrix}\vec{y}$, find equilibrium / bifurcation points

$Trace(A) = -2$

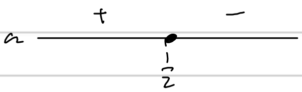
$\det(A = 2a$

$\lambda = \frac{-2 \pm \sqrt{4 - 8a}}{2} = -1 \pm \sqrt{1 - 2a}$

$4 - 8a = 0$

$a = \frac{1}{2}$

$4 - 8a$



$a < \frac{1}{2}$: Two real eigenvalues

$\sqrt{1-2a} > 1$

Section 4.1 - Forced Harmonic Oscillators

Undamped: $my'' = -ky - by'$

A damping force adds a function of time to the RHS

Forced: $my'' = Ky - by' + f(t)$

$\Rightarrow y'' + py' + qy = g(t)$

where $p = \frac{b}{m}$, $q = \frac{K}{m}$, $g(t) = \frac{f(t)}{m}$

$g(t)$ is the forcing function

Second order, linear, nonhomogeneous, non autonomous differential equation

Extended Linearity Principle

Use this to solve Forced Harmonic Oscillators

(NH): $y'' + p_0 y' + qy = g(t)$

(H): $y'' + p_0 y' + qy = 0$

① If $y_h(t)$ is a sln to (H) and $y_p(t)$ is a sln to (NH), $y_h(t) + y_p(t)$ is also a sln to (NH)

② If $y_p(t)$ and $y_q(t)$ are both slns to (NH), then $y_p(t) - y_q(t)$ is a sln to (H)

If $K_1 y_1(t) + K_2 y_2(t)$ is the general sln to (H), then $y(t) = K_1 y_1(t) + K_2 y_2(t) + y_p(t)$ is the general sln to (NH)

③ General Solution of (H): $K_1 y_1(t) + K_2 y_2(t)$ for some linearly independent functions $y_1(t)$ and $y_2(t)$

Example: $y'' + 4y' + 13y = -2e^{-2t}$

(NH): $y'' + 4y' + 13y = -2e^{-2t}$

(H): $y'' + 4y' + 13y = 0$

Guess Sln to (H): $y(t) = we^{st}$, plug into (H)

$$ws^2e^{st} + 4wse^{st} + 13we^{st} = 0$$

$$\cancel{we^{st}}(s^2 + 4s + 13) = 0$$

characteristic equation

$$s = \frac{-4 \pm \sqrt{16 - 4(13)}}{2} \Rightarrow s = -2 \pm 3i$$

$$y(t) = e^{(-2-3i)t} = e^{-2t} \cdot e^{-3it}$$

$$= e^{-2t}(\cos(3t) - i\sin(3t))$$

$$y_1(t) = e^{-2t}\cos(3t)$$

$$y_2(t) = e^{-2t}\sin(3t)$$

$$\therefore y_h(t) = K_1 e^{-2t}\cos(3t) + K_2 e^{-2t}\sin(3t)$$

Guess Sln to (NH): $y_p(t) = Ae^{-2t}$

$$\cancel{e^{-2t}}(4A - 8A + 13A) = -2\cancel{e^{-2t}}$$

$$9A = -2$$

$$A = -\frac{2}{9}$$

$$y_p(t) = -\frac{2}{9}e^{-2t}$$

$$\boxed{\therefore y(t) = K_1 e^{-2t}\cos(3t) + K_2 e^{-2t}\sin(3t) - \frac{2}{9}e^{-2t}}$$

Some Vocabulary

For Large t values, initial conditions do not effect behavior.

① $y_p(t)$ is called the final response, or steady state response

② $y_h(t)$ is called the natural response, or free response

Behavior is more complicated when Damping Coefficient is 0

For Large t values...

$$\boxed{y(t) = K_1 y_1(t) + K_2 y_2(t) + y_p(t) \approx y_p(t)}$$

External Force

Example: $y'' + 2y' + y = e^{-t}$

Can predict behavior will decay as $t \to \infty$

(NH): $y'' + 2y' + y = e^{-t}$

(H): $y'' + 2y' + y = 0$

Guess sln to (H): $y_h(t) = e^{st}$

$$\cancel{e^{st}}(s^2 + 2s + 1) = 0$$

$$s^2 + 2s + 1 = 0$$

oops! $(s+1)^2 = 0 \rightarrow s_1 = s_2 = -1$

Just add t factor! $y_h(t) = K_1 e^{-t} + K_2 te^{-t}$

Question

Suppose $\tilde{y}(t)$ is also a solution to (NH)

$y(t) - \tilde{y}(t)$ is a solution of

$y'' + 2y' + y = 0$ ← (H) part

Guess sln to (NH): $y_p(t) = Ae^{-t}$

$$Ae^{-t} - 2Ae^{-t} + Ae^{-t} = e^{-t}$$

oops! $0 = e^{-t}$ Cannot be the even !!

Guess again w/ t factor: $y_p(t) = Ate^{-t}$ ← Don't even need to consider as its part of $y_h(t)$

$$\left.\begin{aligned} y' &= Ae^{-t} - Ate^{-t} \\ y'' &= -Ae^{-t} - A(e^{-t} - te^{-t}) \end{aligned}\right\} \underbrace{-Ae^{-t} - Ae^{-t}}_{-2Ae^{-t}} \underbrace{+ Ate^{-t}}_{\text{Cancels}} \underbrace{+ 2Ae^{-t}}_{\text{Cancels}} \underbrace{- 2Ate^{-t} + Ate^{-t}}_{-Ate^{-t}} = e^{-t}$$

Guess again w/ t^2: $y_p(t) = \underline{At^2e^{-t}}$

$$y' = A(2te^{-t} - t^2e^{-t}) = \underline{2Ate^{-t}} - \underline{At^2e^{-t}}$$

$$y'' = A(2e^{-t} - 2te^{-t} - 2te^{-t} + t^2e^{-t})$$

$$= A(2e^{-t} - 4te^{-t} + t^2e^{-t}) = \underline{2Ae^{-t}} - \underline{4Ate^{-t}} + \underline{At^2e^{-t}}$$

$$t^2e^{-t}(A - 2A + A) + te^{-t}(-2A + 2A) + e^{-t}(2A) = e^{-t}$$

$$e^{-t}2A = e^{-t}$$

$$2A = 1 \rightarrow A = \frac{1}{2}$$

$$\boxed{\therefore y(t) = \frac{1}{2}t^2e^{-t} + K_1 e^{-t} + K_2 te^{-t}}$$

Choose a guess that is not a part of $y_h(t)$ for any values of K_1, K_2

$$\lim_{t \to \infty} y(t) = 0$$

Example: $y''+5y'+6y=2$ $\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$

Solve the IVP

(NH): $y''+5y'+6y=2$

(H): $y''+5y'+6y=0$

$p(\lambda)=\lambda^2+5\lambda+6=0$; $\lambda_1=-2, \lambda_2=-3$

$y_h(t)=K_1e^{-3t}+K_2e^{-2t}$

Guess: $y_p(t)=A$

$6A=2$

$A=\frac{1}{3}$

$y(t)=\frac{1}{3}+K_1e^{-3t}+K_2e^{-2t}$

Have to find particular soln from initial condition

IVP: $y(0)=\frac{1}{3}+K_1+K_2=0$

$y'(0)=-3K_1-2K_2=0$

$K_2=-\frac{3}{2}K_1$ $K_1=\frac{2}{3}, K_2=-1$

$$\boxed{y(t)=\frac{1}{3}+\frac{2}{3}e^{-3t}-e^{-2t}}$$

Section 4.2 - Sinusoidal Functions

We will consider $y''+py'+qy=g(t)$ where $g(t)$ is a sine or cosine function

Definition: Here $g(t+T)=g(t)$ is true for the forcing period T

This is called Periodic Forcing

Algorithm

① Find general solution to (H)

② Find particular solution to (NH)

Guess: $y_p(t)=\alpha\cos(t)+\beta\sin t$

multiply t by ω if (NH) has $e^{\omega t}$

There is another method using complex numbers

Example: $y''+3y'+2y=5\cos t$

(NH): $y''+3y'+2y=5\cos t$

(H): $y''+3y'+2y=0$

$s^2+3s+2=0$

$(s+2)(s+1)=0$; $s_1=-2, s_2=-1$

$y_h(t)=K_1e^{-2t}+K_2e^{-t}$

$$\boxed{y(t)=K_1e^{-2t}+K_2e^{-t}+\frac{1}{2}\cos t+\frac{3}{2}\sin t.}$$

Guess: $y_p(t)=\alpha\cos t+\beta\sin t$

$y'=-\alpha\sin t+\beta\cos t$

$y''=-\alpha\cos t-\beta\sin t$

$\underbrace{-\alpha\cos t}\ \underbrace{-\beta\sin t}\ \underbrace{-3\alpha\sin t}\ \underbrace{+3\beta\cos t}\ \underbrace{+2\alpha\cos t}\ \underbrace{+2\beta\sin t}=5\cos t$

$\cos t(-2\alpha+3\beta+\alpha)+\sin t(2\beta-3\alpha-\beta)=5\cos t+0\sin t$

$-\alpha+3\beta+2\alpha=5$

$-\beta-3\alpha+2\beta=0$

$\begin{cases} 3\beta+\alpha=5 \\ \beta-3\alpha=0 \end{cases}$

$\alpha=\frac{1}{3}\beta$ $3\beta+\frac{1}{3}\beta=5$

$\alpha=\frac{1}{2}$ $\frac{10}{3}\beta=5$

$\beta=\frac{3}{2}$

$y_p(t)=\frac{1}{2}\cos t+\frac{3}{2}\sin t$

Prove $A\sin t+B\cos t=C\sin t+D\cos t$ iff $A=C$ and $B=D$

$t=0$: $B=D$ ✓

$t=\frac{\pi}{2}$: $A-B=C-D$

$B=D\to A=C$ ✓

As $t\to\infty$, all solutions will tend to $y_p(t)=\frac{1}{2}\cos t+\frac{3}{2}\sin t$

Another Method of Solution

Given $y''+py'+qy=g(t)$ where $g(t)=a\sin(\omega t)$ or $g(t)=a\cos(\omega t)$

① Consider $y''+py'+qy=ae^{i\omega t}$

② Guess a complex solution: $y_c(t)=\alpha e^{i\omega t}$

$y_c(t)=\alpha e^{i\omega t}$

$=\alpha(\cos(\omega t)+i\sin(\omega t))$

$=y_{re}(t)+i\,y_{im}(t)$ ← After Multiplying α through eq.

③ If original $g(t)$ was $a\sin(\omega t)$, then $y_p(t)=y_{im}(t)$

If original $g(t)$ was $a\cos(\omega t)$, then $y_p(t)=y_{re}(t)$

Example 2: $y''+4y'+20y=-3\sin(2t)$

(NH) $y''+4y'+20y=-3\sin(2t)$

(H) $y''+4y'+20y=0$

$s^2+4s+20=0$

$s=\dfrac{-4\pm\sqrt{16-80}}{2}$

$s=-2\pm4i$

$y_h(t)=K_1e^{-2t}\cos4t+K_2e^{-2t}\sin4t$

Solve NH: $y''+4y'+20y=-3e^{i2t}$

$y_p(t)=\alpha e^{i2t}$ i is just a constant

$y'_p(t)=2\alpha ie^{i2t}$

$y''_p(t)=-4\alpha e^{i2t}$

$\cancel{e^{i2t}}(-4\alpha+8\alpha i+20\alpha)=-3\cancel{e^{i2t}}$

$\alpha(16+8i)=-3$

$\alpha=\dfrac{-3}{16+8i}\dfrac{(16-8i)}{(16-8i)}=\dfrac{-48+24i}{320}$

$y_p(t)=\dfrac{-48+24i}{320}e^{2it}$

$=\left(\dfrac{-48}{320}+i\dfrac{24}{320}\right)(\cos2t+i\sin2t)$

$=-\frac{48}{320}\cos2t-\frac{24}{320}\sin2t+i\underbrace{\left(-\frac{48}{320}\sin2t+\frac{24}{320}\cos2t\right)}$

Original DE has sin, so we only want the imaginary part

$\therefore y_p(t)=\frac{3}{40}\cos2t-\frac{3}{20}\sin2t$

$$\boxed{y(t)=K_1e^{-2t}\cos4t+K_2e^{-2t}\sin4t+\frac{3}{40}\cos2t-\frac{3}{20}\sin2t}$$

Could solve IVP given y_0, y'_0

All solutions converge to $y_p(t)$

Section 4.3 - Undamped Forcing and Resonance

We will consider undamped harmonic oscillators with sinusoidal forcing in the form:

(NH): $y''+qy=g(t)$ where $g(t)=a\sin\omega t$ or $a\cos\omega t$

① First, solve (H): $y''+qy=0$ with general solution

$y_h(t)=K_1\cos\beta t+K_2\sin\beta t$ where $\beta=\sqrt{q}$ y_h oscillates with period $2\pi/\beta$ and amplitude $A=\sqrt{K_1^2+K_2^2}$

Deriving general Solution: $y''+qy=0$

$s^2+q=0$

$s=\pm i\sqrt{q}$

$e^{i\sqrt{q}}=e^{0t}(\cos(\sqrt{q}\,t)+i\sin(\sqrt{q}\,t))$

Let $\beta=\sqrt{q}$

$=\cos\beta t+i\sin\beta t$

$\therefore \boxed{y_h(t)=K_1\cos(\sqrt{q}\,t)+K_2\sin(\sqrt{q}\,t)}$

② Now, find particular solution to NH. Guess $y_p(t)=a\sin\omega t$ or $y_p(t)=a\cos\omega t$ based on $g(t)$ in (NH)

Two Cases: $q\neq\omega^2$ and our guess works

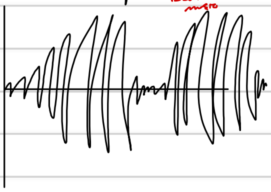
$y(t)=K_1\cos\beta t+K_2\sin\beta t+\dfrac{a}{q-\omega^2}\sin\omega t$ ↑ or cos

$q=\omega^2$ our guess doesn't work

↳ This is called the Resonant Case

If q and ω^2 are very close, they are called near resonant

A good example of resonance is a bridge being disrupted and wobbling because of the wind



Exercise 1: Simplifying $y_h(t)$

We can say: $K_1\cos\beta t+K_2\sin\beta t=K\cos(\beta t-\varphi)$, where $K_1=K\cos\varphi$ and $K_2=K\sin\varphi$

Proof: $\cos(\beta t-\varphi)=K\cos\beta t\cos\varphi+K\sin\beta t\sin\varphi$

$K=\sqrt{K_1^2+K_2^2}$

$=\sqrt{(K\cos\varphi)^2+(K\sin\varphi)^2}$

$K=\sqrt{K^2}$ trig identity

Example: No Resonance

(NH): $y''+4y=\sin 3t$ No resonance

(H): $y''+4y=0$ Freq=3 Freq=2

$s^2+4=0$

$s=\pm 2i$

$y_h(t)=K_1\cos 2t+K_2\sin 2t$

Guess $y_p(t)$: $y_p(t)=\alpha e^{i3t}$

$y'_p(t)=3i\alpha e^{i3t}$

$y''_p(t)=-9\alpha e^{i3t}$

Plug in: $e^{i3t}(-9\alpha+4\alpha)=5e^{i3t}$

$\therefore\ \alpha=-1$

$y_p(t)=-e^{i3t}=-\cos 3t-\underbrace{i\sin 3t}$

$y_p(t)=-\sin 3t$

$\therefore \boxed{y(t)=K_1\cos 2t+K_2\sin 2t-\sin 3t}$

Example: Resonant Case

(NH): $y''+4y=\sin 2t$

(H): $y''+4y=0$

$s^2+4=0$; $s=\pm 2i$

$y_h(t)=K_1\cos 2t+K_2\sin 2t$

Note external force lies in the solution to (H) ← resonant case

Let $y''+4y=e^{2it}$ sin in original, look for Y_{im}

Guess: $y_p=Ae^{i2t}$ doesn't work!

$y''_p(t)=-4Ae^{i2t}$ $\therefore \cancel{-4Ae^{i2t}+4Ae^{i2t}}=e^{2it}$

oops! $0=e^{2it}$

Add t factor:

Guess: $y_p=Ate^{i2t}$

$y'_p(t)=Ae^{i2t}+2iAte^{i2t}$

$y''_p(t)=2iAe^{i2t}+2i(Ae^{i2t}+2iAte^{i2t})$

$=Ae^{i2t}(4i-4t)$

$\therefore Ae^{i2t}(4i-4t)+4Ate^{i2t}=e^{i2t}$

$A\cancel{e^{i2t}}(4i-4t+4t)=\cancel{e^{i2t}}\rightarrow A=\dfrac{1}{4i}\dfrac{(-4i)}{(-4i)}$

Multiply A by conjugate

$A=\dfrac{-4i}{16}=-\dfrac{1}{4}i$ $\therefore\ y_p(t)=-\frac{1}{4}ite^{i2t}$

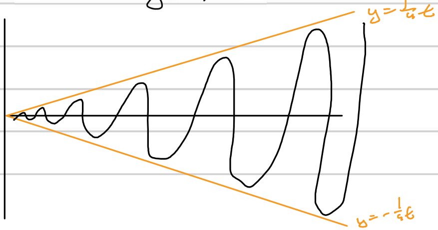
$y_p(t)=-\frac{1}{4}it(\cos 2t+i\sin 2t)$

$=\underbrace{-\frac{1}{4}it\cos 2t}-\frac{1}{4}t\sin 2t$

$y(t)=K_1\cos 2t+K_2\sin 2t-\frac{1}{4}t\cos 2t$

Plot $y(t)$

Amplitude grows to ∞



Section 6.1 - Laplace Transforms

The Laplace Transform of a function $y(t)$ uses integration to 'compare' $y(t)$ to the exponential function e^{st} for various values of s.

Laplace Transform of $y(t)$ is $Y(s)$: $Y(s) = \int_0^\infty y(t)e^{-st}\,dt$

$Y(s)$ called image of $y(t)$

for all s where this improper integral converges

Denoted by $\mathcal{L}[y]$

Recall: $$\int_0^\infty y(t)e^{-st}\,dt = \lim_{b\to\infty}\int_0^b y(t)e^{-st}\,dt$$ if limit exists, integral converges; diverges otherwise

Example: Compute $\mathcal{L}[e^t]$

$e^t \to Y(s)$

$$\int_0^\infty e^t \cdot e^{-st}\,dt = \lim_{b\to\infty}\int_0^b e^{t-st}\,dt = \lim_{b\to\infty}\int_0^b e^{t(1-s)} = \lim_{b\to\infty} \frac{1}{1-s}e^{t(1-s)}\Big|_0^b$$

$$\lim_{b\to\infty} \frac{1}{1-s}e^{t(1-s)}\Big|_0^b = \lim_{b\to\infty}\frac{1}{1-s}e^{b(1-s)} - \left(\frac{1}{1-s}e^0\right) = \lim_{b\to\infty}\frac{1}{1-s}\left(\underbrace{e^{b(1-s)}}_{\to 0 \text{ for } s>1} - 1\right)$$

$s>1$: $\lim_{b\to\infty} \frac{1}{1-s}e^{b(1-s)} = 0$

$s<1$: DNE

$s=1$: $\mathcal{L}[y] = 0$

if $s>1$ $\mathcal{L}[y] = \frac{1}{s-1}$

Laplace Transform Table

① $\mathcal{L}[e^{at}] = \frac{1}{s-a}$, $s>a$

② $\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$

③ $\mathcal{L}[t] = \frac{1}{s^2}$

④ $\mathcal{L}[a] = \frac{a}{s}$

⑤ $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

Sufficient Conditions

① $y(t)$ is piecewise-cts for $t \geq 0$

② $y(t)$ is of exponential order: if positive C, M, T such that

$|y(t)| \leq Me^{ct}$ $\forall\, t \geq T$ Bound

Example: $\mathcal{L}[e^{at}]$ for arbitrary a

$|e^{at}| \leq Me^{ct}$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at}e^{-st}\,dt = \lim_{b\to\infty}\int_0^b e^{t(a-s)}\,dt = \lim_{b\to\infty}\frac{1}{a-s}e^{t(a-s)}\Big|_0^b$$

Converges iff $(a-s) \leq 0$ or $s \geq a$

Properties of the Laplace Transform

$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$

Derivation: $$\mathcal{L}[y'] = \lim_{b\to\infty}\int_0^b \frac{dy}{dt}e^{-st} = \lim_{b\to\infty}\left[y\cdot e^{-st}\Big|_0^b + s\int_0^b y\,e^{-st}\,dt\right.$$

$$= \lim_{b\to\infty} \underbrace{y(b)e^{-sb}}_{0} \underbrace{- y(0)e^0}_{-y(0)} + \underbrace{s\int_0^b ye^{-st}}_{\mathcal{L}[y]}dt$$

func+func

Linear Operator: $\mathcal{L}[f+g] = \mathcal{L}[f] + \mathcal{L}[g]$

constant func. $\mathcal{L}[cg] = c\mathcal{L}[g]$

] Integration Rules/Properties

Example: $\frac{dy}{dt} + 4y = 6, \quad y(0) = 0$

$$\mathcal{L}[y' + 4y] = \mathcal{L}[y'] + 4\mathcal{L}[y] = -\overset{\nearrow 0}{y(0)} + s\mathcal{L}[y] + 4\mathcal{L}[y]$$

$$s\mathcal{L}[y] + 4\mathcal{L}[y] = \mathcal{L}[6] \leftarrow 6\,\mathcal{L}[1]$$

$$(4+s)\mathcal{L}[y] = \frac{6}{s}$$

$$\therefore \mathcal{L}[y] = \frac{6}{s(s+4)}$$

Must Find $\mathcal{L}^{-1}\left[\frac{6}{s(s+4)}\right]$

$$\frac{6}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4} = \frac{A(s+4) + Bs}{s(s+4)}$$

$$6 = A(s+4) + Bs \qquad A + B = 0 \qquad A = \frac{3}{2}$$

$$\rightarrow 6 = s(A+B) + 4A \qquad 4A = 6 \qquad B = -\frac{3}{2}$$

$$\mathcal{L}^{-1}\left[\mathcal{L}[y] = \frac{3}{2}\underbrace{\left(\frac{1}{s}\right)}_{\mathcal{L}[1]} - \frac{3}{2}\underbrace{\left(\frac{1}{s+4}\right)}_{\mathcal{L}[e^{-4t}]}\right]$$

$$y(t) = \frac{3}{2} - \frac{3}{2}e^{-4t} \quad \therefore$$

Inverse Laplace Transform

$$\mathcal{L}^{-1}[F] = f \quad \text{iff} \quad \mathcal{L}[f] = F$$

$$\mathcal{L}^{-1}[\mathcal{L}[f(t)]] = f(t)$$

Properties of Inverse Image (Laplace Trans called Image)

① Uniqueness: If f is cts with $\mathcal{L}[f] = F$, then f is the only such cts. Function.

② Inverse is also a Linear operator: $\mathcal{L}^{-1}[f+g] = \mathcal{L}^{-1}[f] + \mathcal{L}^{-1}[g]$

$$\mathcal{L}^{-1}[cf] = c\,\mathcal{L}^{-1}[f]$$

Example: Compute $\mathcal{L}[t]$

$$\mathcal{L}[t] = \lim_{b\to\infty} \int_0^b t \cdot e^{-st}\,dt$$

$$= \lim_{b\to\infty} \left[\frac{1}{-s}\int_0^b t\,de^{-st}\right]$$

$$= \lim_{b\to\infty} \left[\frac{1}{-s}\left(\cancelto{0}{te^{-st}\Big|_0^b} - \int_0^b e^{-st}\,dt\right)\right] = \frac{1}{s^2}$$

$\frac{dy}{dt} = -y + e^{-2t}, \quad y(0) = 2$

$$\text{LHS: } \mathcal{L}[y'] = s\mathcal{L}[y] - y(0) = s\mathcal{L}[y] - 2$$

$$\text{RHS: } \mathcal{L}[-y + e^{-2t}] = -\mathcal{L}[y] + \frac{1}{s+2}$$

$$s\mathcal{L}[y] - 2 = -\mathcal{L}[y] + \frac{1}{s+2}$$

$$\mathcal{L}[y](s+1) = \frac{1}{s+2} + 2$$

$$\mathcal{L}[y] = \frac{1}{(s+2)(s+1)} + \frac{2}{s+1}$$

$$\frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$1 = A(s+1) + B(s+2)$$

$$\begin{cases} 0 = A + B \\ 1 = A + 2B \end{cases} \qquad \begin{aligned} 1 &= A + 2(-A) \\ 1 &= -A \quad A = -1 \quad B = 1 \end{aligned}$$

$$\mathcal{L}[y] = \frac{-1}{s+2} + \frac{1}{s+1} + \frac{2}{s+1}$$

$$\mathcal{L}^{-1}[\mathcal{L}[y]] = -\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$$y(t) = -e^{-2t} + e^{-t} + 2e^{-t}$$

$$y(t) = 3e^{-t} - e^{-2t}$$

Section 6.2 - Discontinuous Functions

Some functions have external forcing functions that turn on @ a certain time

Heaviside Function for $a \geq 0$ is

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

Discontinuous at Point a

Solving DEs w/ Laplace

① Take $\mathcal{L}$ of both sides
② Solve for $\mathcal{L}$
③ Compute $\mathcal{L}^{-1}$ for RHS ← usually Partial Fractions
④ Determine DE solution

Laplace Transform Table

① $\mathcal{L}[e^{at}] = \frac{1}{s-a},\ s > a$
② $\mathcal{L}[\frac{dy}{dt}] = s\mathcal{L}[y] - y(0)$
③ $\mathcal{L}[t] = \frac{1}{s^2}$
④ $\mathcal{L}[a] = \frac{a}{s}$
⑤ $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$
⑥ $\mathcal{L}[u_a(t)] = \frac{e^{-as}}{s}$
⑦ $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}\mathcal{L}[f(t)]$

Compute $\mathcal{L}[u_a(t)]$

$$\mathcal{L}[u_a(t)] = \int_0^\infty u_a(t)e^{-st}\,dt$$

$$= \underbrace{\int_0^a u_a(t)e^{-st}\,dt}_{u_a = 0 \text{ for } t<a} + \underbrace{\int_a^\infty u_a(t)e^{-st}\,dt}_{u_a = 1 \text{ for } t>a}$$

$$= 0 + \lim_{b\to\infty}\int_a^b e^{-st}\,dt = \lim_{b\to\infty} \frac{1}{-s}\left(\overset{0}{\cancel{e^{-bs}}} - e^{-as}\right) = \frac{e^{-as}}{s}$$

Cts. for $s \geq 0$

Example: $y' + 7y = u_2(t),\ y(0) = 3$

LHS: $\mathcal{L}[y' + 7y] = \mathcal{L}[y'] + 7\mathcal{L}[y] = -y(0) + s\mathcal{L}[y] + 7\mathcal{L}[y]$

RHS: $\mathcal{L}[u_2(t)] = \frac{e^{-2s}}{s}$

Set. Eq. $-3 + s\mathcal{L}[y] + 7\mathcal{L}[y] = \frac{e^{-2s}}{s}$

$\mathcal{L}[y](s+7) = \frac{e^{-2s}}{s} + 3$

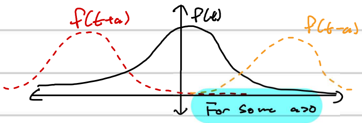
$\left(\mathcal{L}[y] = \frac{e^{-2s}}{s(s+7)} + \frac{3}{s+7}\right)\mathcal{L}^{-1}$

$\rightarrow y = \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s(s+7)}\right] + \mathcal{L}^{-1}\left[\frac{3}{s+7}\right]$

$\rightarrow y(t) = \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s(s+7)}\right] + 3e^{-7t}$

Horizontal Shift $a > 0$

Given $f(t)$, $f(t-a)$ shifts the function horizontally by 'a' units



Horizontal Shifts with $u_a(t)$

Suppose $g(t) = u_a(t)f(t-a)$, Compute $\mathcal{L}[g(t)]$

$$\mathcal{L}[u_a(t)f(t-a)] = \int_a^\infty f(t-a)e^{-st}\,dt$$

Heavisides just here

U sub $u = t-a$, $\frac{du}{dt} = 1$

$u_a = 0$ for $t<a$, omit

$$= \int_0^\infty f(u)e^{-s(u+a)}\,du$$

$$= e^{-as}\lim_{b\to\infty}\int_0^b f(u)e^{-su}\,du = e^{-as}\mathcal{L}[f(t)]$$

$\mathcal{L}[f \text{ shifted by } a] = e^{-as}\mathcal{L}[f]$

$\mathcal{L}^{-1}[e^{-as}\mathcal{L}[f]] = f$ shifted by a

Finish example: $y(t) = \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s(s+7)}\right] + 3e^{-7t}$

$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s(s+7)}\right] = \mathcal{L}^{-1}[u_2(t)f(t-2)]$

$\mathcal{L}[f] = \frac{1}{s(s+7)} = \frac{A}{s} + \frac{B}{s+7}$

$1 = As + 7A + Bs$

$A = \frac{1}{7},\ B = -\frac{1}{7}$

$\rightarrow \mathcal{L}^{-1}\left[\frac{1}{7}\cdot\frac{1}{s} - \frac{1}{7}\cdot\frac{1}{s+7}\right]$

$= \frac{1}{7} - \frac{1}{7}e^{-7t}$

$\therefore \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s(s+7)}\right] = u_2(t)\left(\frac{1}{7} - \frac{1}{7}e^{-7(t-2)}\right)$

$$\rightarrow \boxed{y(t) = u_2(t)\left(\frac{1}{7} - \frac{1}{7}e^{-7(t-2)}\right) + 3e^{-7t}}$$

Whats going on: $\frac{e^{-2s}}{s(s+7)}$ is an image of function $u_2(t)\cdot f(t-2)$

where $\mathcal{L}[f] = \frac{1}{s(s+7)}$

- $f = \mathcal{L}^{-1}\left[\frac{1}{s(s+7)}\right] = \frac{1}{7} - \frac{1}{7}e^{-7t}$ ← without shift

$\rightarrow \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s(s+7)}\right] = u_2(t)\left(\frac{1}{7} - \frac{1}{7}e^{-7(t-2)}\right)$ with shift now

Example: $y' = -y + u_1(t)(t-1)$, $y(0) = 2$

Look for $f(t)$ non-shifted

$e^{-s}\mathcal{L}[f(t)]$

$f(t) = t$

LHS $\mathcal{L}[y'] = -y(0) + s\mathcal{L}[y]$

RHS $\mathcal{L}[-y + u_1(t)(t-1)] = -\mathcal{L}[y] + \mathcal{L}[u_1(t)(t-1)]$

$= -\mathcal{L}[y] + \frac{e^{-s}}{s^2}$

Solve for $\mathcal{L}[y]$
$$\begin{cases} -2 + s\mathcal{L}[y] = -\mathcal{L}[y] + \frac{e^{-s}}{s^2} \\ \mathcal{L}[y](s+1) = \frac{e^{-s}}{s^2} + 2 \end{cases} \longrightarrow \mathcal{L}[y] = \frac{2}{s+1} + \frac{e^{-s}}{s^2(s+1)}$$

$\rightarrow y = \mathcal{L}^{-1}\left[\frac{2}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2(s+1)}\right]$

$= 2e^{-t} +$

$\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2(s+1)}\right]$ is an image of a horizontally shifted by 1.

$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)}\right]$: $\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$

$1 = As(s+1) + B(s+1) + Cs^2$

$1 = As^2 + As + Bs + B + Cs^2$

$$\begin{cases} 0 = A + C \\ 0 = A + B \\ 1 = B \end{cases}$$

$B = 1$, $A = -1$, $C = 1$

$\frac{1}{s^2(s+1)} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$

$$\left.\begin{aligned} \mathcal{L}\left[-\tfrac{1}{s}\right] &= -1 \\ \mathcal{L}\left[\tfrac{1}{s^2}\right] &= t \\ \mathcal{L}\left[\tfrac{1}{s+1}\right] &= e^{-t} \end{aligned}\right\} \mathcal{L}\left[\frac{1}{s^2(s+1)}\right] = (t-1) + e^{-t}$$

Shift to right one, subtract one from t

$((t-1)-1) + e^{-(t-1)} = t - 2 + e^{-t+1}$

Shift $f(t) = (t-1) + e^{-t}$ to the right by 1, and multiply by $u_1(t)$ to get $\mathcal{L}^{-1}$

$u_1(t)[f(t-1)] = u_1(t)[(t-2) + e^{-t+1}]$

$$\boxed{\therefore y(t) = 2e^{-t} + u_1(t)[(t-2) + e^{-t+1}]}$$

Laplace Transform is needed for discts. forcing functions like $u_c(t)$

$\mathcal{L}[y] = \frac{e^{-3s}}{(s-1)(s-2)}$ $a = 3$

$\mathcal{L}^{-1}[\mathcal{L}[y]] = \mathcal{L}^{-1}\left[\frac{e^{-3s}}{(s-1)(s-2)}\right]$

$\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} = \frac{-1}{s-1} + \frac{1}{s-2}$

$1 = A(s-2) + B(s-1)$

$$\begin{cases} 0 = A + B \\ 1 = -2A - B \end{cases}$$

$1 = -2A + A$

$A = -1$

$-\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-2}\right]$

$f(t) = -e^{t} + e^{2t}$

$f(t-a) = -e^{(t-3)} + e^{2(t-3)}$

$y(t) = u_3(t)\left(e^{2(t-3)} - e^{(t-3)}\right)$

$\frac{dy}{dt} = -y + u_2(t)e^{-2(t-2)}$, $y(0) = 1$

$f(t) = e^{-2t}$

LHS: $\mathcal{L}[y'] = s\mathcal{L}[y] - y(0) = s\mathcal{L}[y] - 1$

RHS: $\mathcal{L}[-y + u_2(t)e^{-2(t-2)}] = -\mathcal{L}[y] + e^{-2s}\left(\frac{1}{s+2}\right)$

$s\mathcal{L}[y] - 1 = -\mathcal{L}[y] + \frac{e^{-2s}}{s+2}$

$\mathcal{L}[y](s+1) = \frac{e^{-2s}}{s+2} + 1$

$\mathcal{L}[y] = \frac{e^{-2s}}{(s+2)(s+1)} + \frac{1}{s+1}$

$\mathcal{L}^{-1}[\mathcal{L}[y]] = \mathcal{L}^{-1}\left[\frac{e^{-2s}}{(s+2)(s+1)} + \frac{1}{s+1}\right]$

$\frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$

$1 = A(s+1) + B(s+2)$

$$\begin{cases} 0 = A + B \\ 1 = A + 2B \end{cases}$$

$B = -A$

$1 = A - 2A$

$A = -1$ $B = 1$

$f(t) = \frac{-1}{s+2} + \frac{1}{s+1}$

$\mathcal{L}^{-1}[f(t)] = -e^{-2t} + e^{-t}$

$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{-e^{-2t} - e^{-t}} + \frac{1}{s+1}\right]$

$f(t) = e^{-t} - e^{-2t}$ $a = 2$

$f(t-a) = e^{-(t-2)} - e^{-2(t-2)}$

$y(t) = u_2(t)\left(e^{2-t} - e^{-2(t-2)}\right) + e^{-t}$

Section 6.3 - Second Order Equations

We will study $y'' + py' + qy = f(t)$ with forcing functions $\sin(\omega t)$, $\cos(\omega t)$, $e^{at}\sin(\omega t)$, and $e^{at}\cos(\omega t)$

$$\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0)$$

Proof for $\mathcal{L}[y'']$

$$\mathcal{L}[y''] = \int_0^\infty y'' e^{-st}\,dt = y' e^{-st}\Big|_0^\infty - s\int_0^\infty y' e^{-st}\,dt$$

$$\begin{array}{cc} \underline{D} & \underline{I} \\ e^{-st} & y'' \\ -se^{-st} & y' \end{array} \qquad \begin{array}{l} -y'(0) - s(y(0) - s\mathcal{L}[y]) \end{array}$$

$+$ $-$

$$\therefore\ \mathcal{L}[y''] = -y'(0) - sy(0) + s^2\mathcal{L}[y] \quad //$$

Proof for $\mathcal{L}[\sin \omega t]$

$$\mathcal{L}[\sin \omega t] = \int_0^\infty \sin(\omega t)\, e^{-st}\,dt$$

$$\begin{array}{cc} \underline{D} & \underline{I} \\ \sin(\omega t) & e^{-st} \\ \omega\cos(\omega t) & -\frac{1}{s}e^{-st} \end{array}$$

$+$ $-$

$$-\frac{\sin(\omega t)}{s}e^{-st}\Big|_0^\infty + \frac{\omega}{s}\int_0^\infty \cos(\omega t)e^{-st}\,dt$$

$$= \frac{\omega}{s}\mathcal{L}[\cos(\omega t)]$$

$$\mathcal{L}[\cos(\omega t)] = \int_0^\infty \cos(\omega t)e^{-st}\,dt$$

$$\begin{array}{cc} \underline{D} & \underline{I} \\ \cos \omega t & e^{-st} \\ -\omega\sin(\omega t) & -\frac{1}{s}e^{-st} \end{array}$$

$+$

$$-\frac{1}{s}\cos(\omega t)e^{-st}\Big|_0^\infty - \frac{\omega}{s}\int_0^\infty \sin(\omega t)e^{-st}\,dt$$

$$= \frac{1}{s} - \frac{\omega}{s}\mathcal{L}[\sin(\omega t)]$$

Solve for $\mathcal{L}[\sin \omega t]$ $\quad \mathcal{L}[\sin \omega t] = \frac{\omega}{s}\left(\frac{1}{s} - \frac{\omega}{s}\mathcal{L}[\sin(\omega t)]\right)$

$$= \frac{\omega}{s^2} - \frac{\omega^2}{s^2}\mathcal{L}[\sin(\omega t)]$$

$$\mathcal{L}[\sin(\omega t)]\left(1 + \frac{\omega^2}{s^2}\right) = \frac{\omega}{s^2}$$

$$= \frac{\frac{\omega}{s^2}}{1 + \frac{\omega^2}{s^2}} = \frac{\omega}{s^2\left(\frac{s^2+\omega^2}{s^2}\right)}$$

$$\therefore\ \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2} \quad //$$

Shifting origin on s-axis

Given $f(t)$ with $\mathcal{L}[f] = F(s)$ known

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{at}f(t)e^{-st}\,dt$$

$$= \int_0^\infty f(t)e^{-(s-a)t}\,dt$$

$$= F(s-a)$$

$$\therefore\ \mathcal{L}[e^{at}f(t)] = F(s-a) \quad \text{for } F(s) = \mathcal{L}[f(t)]$$

Recall: $\mathcal{L}[e^{-as}f(t)] = u_a(t)f(t-a)$

Laplace Transform Table

① $\mathcal{L}[e^{at}] = \frac{1}{s-a}$, $s > a$

② $\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$

③ $\mathcal{L}[t] = \frac{1}{s^2}$

④ $\mathcal{L}[a] = \frac{a}{s}$

⑤ $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

⑥ $\mathcal{L}[u_a(t)] = \frac{e^{-as}}{s}$

⑦ $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}\mathcal{L}[f(t)]$

⑧ $\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0)$

⑨ $\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$

⑩ $\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$

⑪ $\mathcal{L}[e^{at}\sin(\omega t)] = \frac{\omega}{(s-a)^2+\omega^2}$

⑫ $\mathcal{L}[e^{at}\cos(\omega t)] = \frac{s-a}{(s-a)^2+\omega^2}$

Trig $\mathcal{L}^{-1}$

- $\mathcal{L}^{-1}\left[\frac{\omega}{s^2+\omega^2}\right] = \sin(\omega t)$
- $\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right] = \cos(\omega t)$

Trig Product $\mathcal{L}^{-1}$

- $\mathcal{L}^{-1}\left[\frac{\omega}{(s-a)^2+\omega^2}\right] = \sin(\omega t)$
- $\mathcal{L}^{-1}\left[\frac{s-a}{(s-a)^2+\omega^2}\right] = \cos(\omega t)$

Extra Point Problem (+1)

Derive $\mathcal{L}[y^{(n)}]$ Using Mathematical Induction

↳ Laplace image for any order function

Two different Shifts

- e^{as}: shift in t-axis
  ↳ $u_a(t)f(t-a)$
- e^{at}: shift in s-axis
  ↳ $F(s-a)$ for $F(s) = \mathcal{L}[f(t)]$

'Proving' $\mathcal{L}[e^{at}\sin(\omega t)]$ and cos

$$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

↳ $\mathcal{L}[e^{at}\sin(\omega t)]$

shift in s-axis

$$\equiv F(s-a) \text{ where } F(s) = \mathcal{L}[\sin \omega t]$$

$$\therefore\ \mathcal{L}[e^{at}\sin(\omega t)] = \frac{\omega}{(s-a)^2+\omega^2}$$

$$\mathcal{L}[e^{at}\cos(\omega t)] \xrightarrow[\text{s-axis}]{\text{shift in}} \frac{(s-a)}{(s-a)^2+\omega^2}$$

Example: Compute: $\mathcal{L}^{-1}\left[\overset{①}{\frac{s+3}{s^2+6s+10}}\right]$ and $\mathcal{L}^{-1}\left[\overset{②}{\frac{s+1}{s^2+6s+10}}\right]$ Pattern: Looks like $\frac{s-a}{(s-a)^2+\omega^2}$

① $\mathcal{L}^{-1}\left[\frac{s+3}{s^2+6s+10}\right]=\mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2+1^2}\right]$ where $\begin{cases} a=-3 \\ \omega=1 \end{cases}$ so $\mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2+1^2}\right]=e^{-3t}\cos t$

② $\mathcal{L}^{-1}\left[\frac{s+1}{s^2+6s+10}\right]=\mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2+1^2}-\frac{2}{(s+3)^2+1^2}\right]=e^{-3t}\cos t-2\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2+1^2}\right]$ where $\begin{cases} a=-3 \\ \omega=1 \end{cases}$

Need to transform problem into something that we know how to contend with

so $\mathcal{L}^{-1}\left[\frac{s+1}{(s+3)^2+1^2}\right]=e^{-3t}-2e^{-3t}\sin t$

Example: Use Laplace Transform to solve the IVP

$y''+4y'+20y=\sin 2t,\ y(0)=0,\ y'(0)=4$

LHS: $\mathcal{L}[y''+4y'+20y]=\underline{s^2\mathcal{L}[y]-sy(0)-y'(0)}+\underline{4s\mathcal{L}[y]-4y(0)}+\underline{20\mathcal{L}[y]}$

$=s^2\mathcal{L}[y]-0-4+4s\mathcal{L}[y]-0+20\mathcal{L}[y]$

$=s^2\mathcal{L}[y]+4s\mathcal{L}[y]+20\mathcal{L}[y]-4$

RHS: $\mathcal{L}[\sin 2t]=\frac{2}{s^2+4}$

$\frac{2}{s^2+4}=\mathcal{L}[y](s^2+4s+20)-4 \longrightarrow \mathcal{L}[y]=\frac{2}{(s^2+4s+20)(s^2+4)}+\frac{4}{s^2+4s+20}$

$\mathcal{L}^{-1}[\mathcal{L}[y]]=\mathcal{L}^{-1}\left[\overset{②}{\frac{2}{(s^2+4s+20)(s^2+4)}}\right]+\mathcal{L}^{-1}_{①}\left[\frac{4}{s^2+4s+20}\right]$

① $\mathcal{L}^{-1}\left[\frac{4}{s^2+4s+20}\right]=\mathcal{L}^{-1}\left[\frac{4}{(s+2)^2+4^2}\right]=e^{-2t}\sin 4t$

$\omega=4,\ a=-2$

② $\mathcal{L}^{-1}\left[\frac{2}{(s^2+4s+20)(s^2+4)}\right]$ PFD $\frac{2}{(s^2+4s+20)(s^2+4)}=\frac{As+B}{s^2+4s+20}+\frac{Cs+D}{s^2+4}$

$2=(As+B)(s^2+4)+(Cs+D)(s^2+4s+20)$

$2=As^3+4As+Bs^2+4B+Cs^3+4Cs^2+20Cs+Ds^2+4Ds+20D$

s^0: $2=4B+20D$

s^1: $0=4A+20C+4D$

s^2: $0=B+4C+D$

s^3: $0=A+C$

$A=-C$

$0=-4C+20C+4D$

$0=16C+4D$

$0=4(4C+D) \rightarrow 4C+D=0$

$\therefore 0=B+0 \quad B=0$

$2=4(0)+20D$

$D=\frac{1}{10}$

$4C+D=0$

$C=-\frac{1}{40}$

$\begin{cases} A=\frac{1}{40} \\ B=0 \\ C=-\frac{1}{40} \\ D=\frac{1}{10} \end{cases}$

Example: $\mathcal{L}[t\cdot\cos\omega t]=\int_0^\infty t\cdot\cos\omega t\cdot e^{-st}\,dt$ #7 in book

$\int_0^\infty te^{st}\cos\omega t\,dt$ can be done using by parts, but it's very time consuming

Instead: $y''+\omega^2y=f(t)$ $y(0)=0,\ y'(0)=1$

$y(t)=t\cos\omega t$ is a sol to $y''+\omega^2y=-2\omega\sin\omega t$

LHS: $\mathcal{L}[y''+\omega^2y]=s^2\mathcal{L}[y]-sy(0)-y'(0)+\omega^2\mathcal{L}[y]$

RHS: $\mathcal{L}[-2\omega\sin\omega t]=-2\omega\left(\frac{\omega}{s^2+\omega^2}\right)$

$s^2\mathcal{L}[y]-1+\omega^2\mathcal{L}[y]=-2\left(\frac{\omega^2}{s^2+\omega^2}\right)$

$\mathcal{L}[y](s^2+\omega^2)=1-\frac{2\omega^2}{s^2+\omega^2}$

$\mathcal{L}[y]=\frac{1}{s^2+\omega^2}-\frac{2\omega^2}{s^2+\omega^2}$

$\therefore \mathcal{L}[y]=\frac{s^2-\omega^2}{s^2+\omega^2}$

Simpler Method

$\mathcal{L}[\cos(\omega t)]=\frac{s}{s^2+\omega^2}=\int_0^\infty \cos\omega t e^{-st}\,dt$

$\frac{d}{ds}\left(\int_0^\infty \cos\omega t e^{-st}\,dt\right)=\int_0^\infty \frac{d}{ds}\left(\cos\omega t e^{-st}\right)dt$

$=-\int_0^\infty t\cos\omega t e^{-st}\,dt$

Using this relationship we can say that:

$\mathcal{L}[t^kf(t)]=-\frac{d^k}{ds^k}F(s)$

where $F(s)=\mathcal{L}[f(t)]$

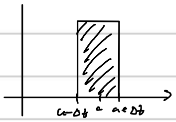
Section 6.4 - Delta Functions and Impulse Forcing

δ function also called Dirac Function or δ-Dirac Function

Suppose we have $y''+py'+qy=g(t)$ where $g(t)=\begin{cases}0, & t\neq a\\ \infty, & t=a\end{cases}$

We need to turn this into a usable formula, so let Δt represent a small increment of time so

$$g_{\Delta t}(t)=\begin{cases}0 & \text{if } 0\le t<a-\Delta t\\ K & \text{if } a-\Delta t\le t\le a+\Delta t\\ 0 & \text{if } t>a+\Delta t\end{cases}$$

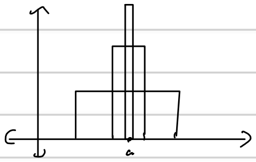


Delta Dirac Function

just lim as Δt goes to 0

$$\delta_a(t)=\lim_{\Delta t\to 0} g_{\Delta t}(t)$$

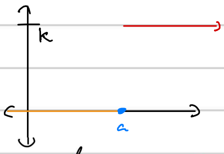
as $\Delta t\to 0$, force $\to\infty$ to preserve area of rectangle



Consider $u_a(t)$ – Heaviside Function

Suppose $K*u_a(t)$

instantaneous jump to K @ $t=a$.



You can think of $\delta_a(t)$ as $\frac{d}{dt}(u_a(t))$

Laplace Transform Table

① $\mathcal{L}[e^{at}]=\frac{1}{s-a},\ s>a$

② $\mathcal{L}[\frac{dy}{dt}]=s\mathcal{L}[y]-y(0)$

③ $\mathcal{L}[t]=\frac{1}{s^2}$

④ $\mathcal{L}[a]=\frac{a}{s}$

⑤ $\mathcal{L}[t^n]=\frac{n!}{s^{n+1}}$

⑥ $\mathcal{L}[u_a(t)]=\frac{e^{-as}}{s}$

⑦ $\mathcal{L}[u_a(t)f(t-a)]=e^{-as}\mathcal{L}[f(t)]$

⑧ $\mathcal{L}[y'']=s^2\mathcal{L}[y]-sy(0)-y'(0)$

⑨ $\mathcal{L}[\sin(\omega t)]=\frac{\omega}{s^2+\omega^2}$

⑩ $\mathcal{L}[\cos(\omega t)]=\frac{s}{s^2+\omega^2}$

⑪ $\mathcal{L}[e^{at}\sin(\omega t)]=\frac{\omega}{(s-a)^2+\omega^2}$

⑫ $\mathcal{L}[e^{at}\cos(\omega t)]=\frac{s-a}{(s-a)^2+\omega^2}$

⑬ $\mathcal{L}[\delta_a(t)]=e^{-as}$

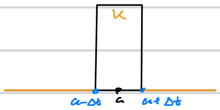
⑭ $\mathcal{L}[t^K f(t)]=-\frac{d^K}{ds^K}(\mathcal{L}[f(t)])$

δ-Dirac $\mathcal{L}^{-1}$

$$\mathcal{L}^{-1}[e^{-as}]=\delta_a(t)$$

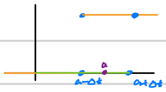
Laplace Image of δ-Dirac

Given $g_{\Delta t}(t)=\begin{cases}0, & 0\le t<a-\Delta t\\ K, & a-\Delta t\le t\le a+\Delta t\\ 0, & t>a+\Delta t\end{cases}$



☆ $K*u_{a-\Delta t}$

☆ $K*u_{a+\Delta t}$



So we can say $g_{\Delta t}(t)=K(u_{a-\Delta t}+u_{a+\Delta t})$

$$\delta_a(t)=\lim_{\Delta t\to 0} g_{\Delta t}(t)=\lim_{\Delta t\to 0}\frac{1}{2\Delta t}(u_{a-\Delta t}+u_{a+\Delta t})$$ Let $\Delta t=h$

$$\mathcal{L}[g_h(t)]=\frac{1}{2h}\left(\mathcal{L}[u_{a-h}(t)]-\mathcal{L}[u_{a+h}(t)]\right)$$

$$=\frac{e^{-as}}{2s}\left(\frac{e^{hs}-e^{-hs}}{h}\right)$$

$$\delta_a(t)=\lim_{h\to 0}\frac{e^{-as}}{2s}\left(\frac{e^{hs}-e^{-hs}}{h}\right)=e^{-as}\lim_{h\to 0}\left(\frac{he^{hs}+he^{-hs}}{1}\right)=e^{-as}$$

$\therefore\ \mathcal{L}[\delta_a(t)]=e^{-as}$

Notation: $\delta(t)$ means $\delta_0(t)$

so $\mathcal{L}[\delta(t)]=1$ here

Example: $y''+y=4\delta_{2\pi}(t)$ (a) $y(0)=0,\ y'(0)=0$ (b) $y(0)=1,\ y'(0)=0$

LHS: $\mathcal{L}[y''+y]=s^2\mathcal{L}[y]-sy(0)-y'(0)+\mathcal{L}[y]$

(a) $=\mathcal{L}[y](s^2+1)-s(0)-0$

RHS: $\mathcal{L}[4\delta_{2\pi}(t)]=4e^{-2\pi s}$

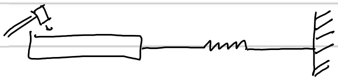
$$\mathcal{L}[y]=4\cdot\frac{e^{-2\pi s}}{s^2+1}$$

$$y(t)=4\mathcal{L}^{-1}\left[\frac{e^{-2\pi s}\cdot 1}{s^2+1}\right]$$ sin t

$$y(t)=4u_{2\pi}(t)\sin(t-2\pi)$$

Sin is periodic $y(t)=4u_{2\pi}(t)\sin(t)$

Example

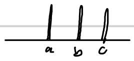


Oscillator periodically hit with hammer with constant force (1J)

Homogeneous Part: $y''+2y=0$ ← no friction! Make a model and draw solution

Need to use $\delta_a(t)$, but it only happens once, while this is periodic.

$\delta_a(t)+\delta_b(t)+\delta_c(t)+\ldots$



External force $f(t)=?$

Should be infinite sum of periodic forcing

Define $f(t)$: Periodic Forcing: $f(t)=\sum_{i=0}^{\infty}\delta_i(t)$

Solve eq: (NH) $y''+2y=\sum_{i=0}^{\infty}\delta_i(t)$, Initial Condition: $y(0)=y'(0)=0$

LHS: $\mathcal{L}[y''+2y]=s^2\mathcal{L}[y]-sy(0)-y'(0)+2\mathcal{L}[y]$

$=\mathcal{L}[y](s^2+2)$

RHS: $\mathcal{L}\left[\sum_{i=0}^{\infty}\delta_i(t)\right]=\sum_{i=0}^{\infty}e^{-is}$ Looks like $\mathcal{L}[\sin\omega t]=\frac{\omega}{s^2+\omega^2}$

$\rightarrow \mathcal{L}[y]=\frac{1}{s^2+2}\sum_{i=0}^{\infty}e^{-is}=\sum_{i=0}^{\infty}\frac{e^{-is}}{s^2+2}$

$y(t)=\mathcal{L}^{-1}\left[\sum_{i=0}^{\infty}\frac{e^{-is}}{s^2+2}\right]$ Linear operator!

$y(t)=\sum_{i=0}^{\infty}\mathcal{L}^{-1}\left[\frac{e^{-is}}{s^2+2}\right]$ image of function shifted right by i

$\rightarrow y(t)=\sum_{i=0}^{\infty}\mathcal{L}^{-1}\left[\frac{e^{-is}}{s^2+2}\right]$

$f(t)$: $\mathcal{L}^{-1}\left[\frac{1}{s^2+2}\right]=\frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left[\frac{\sqrt{2}}{s^2+2}\right]$

$f(t)=\frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$

e^{-is}: $a=i$ → $f(t-i)=\frac{1}{\sqrt{2}}\sin(\sqrt{2}(t-i))$

$$\boxed{y(t)=\frac{1}{\sqrt{2}}\sum_{i=0}^{\infty}u_i(t)\left[\sin(\sqrt{2}(t-i))\right]}$$

correct answer, but how can we predict behavior?

Estimate long term behavior of solution:

$y(t)=\frac{1}{\sqrt{2}}\sum_{i=0}^{\infty}u_i(t)\sin(\sqrt{2}(t-i))$

Recall: if $f(t)=f(t+T)$ ← periodic function

Then $\mathcal{L}[f(t)]=\frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}\,dt$

This conclusion will be derived in HW 12 ← Problem #9 from Section 6.4

Section 5.1 - Equilibrium Point Analysis

Example: A competing species model

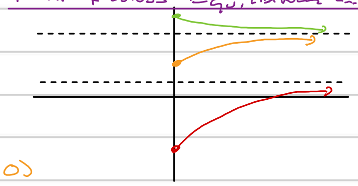
$$\begin{cases} \frac{dx}{dt} = 2x(1-\frac{x}{2}) - xy \\ \frac{dy}{dt} = 3y(1-\frac{y}{3}) - 2xy \end{cases}$$

- Increase in either species population has adverse effects on other species
- We cannot solve this system with our methods, must use numerical methods

$x(t)$ and $y(t)$ represent competing species battling for resources

Before using numerical methods, we should try to get rough idea about $x(t)$ and $y(t)$

Recall Previous Equilibrium analysis



Equilibrium analysis done in $x(t)-y(t)$ plane

We can turn this into a general system

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases}$$ if (x_0, y_0) is a solution, then we can move it to the origin: $\begin{cases} u = x - x_0 \\ v = y - y_0 \end{cases}$

Find Eq points $(x' = y' = 0)$

$$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = 0 \end{cases} > \begin{cases} 0 = 2x - x^2 - xy \\ 0 = 3y - y^2 - 2xy \end{cases}$$

$0 = x(2 - x - y)$ $(0,0)$ ← trivial solution

$0 = y(3 - y - 2x)$ Let $x = 0$: $(0,3)$

Let $y = 0$: $(2,0)$

or: $(1,1)$

Can make a phase portrait from this

This yields $$\begin{cases} \frac{du}{dt} = f(x_0 + u, y_0 + v) \\ \frac{dv}{dt} = g(x_0 + u, y_0 + v) \end{cases}$$ > Substitute this

Recall Tangent plane: $z = f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

We can approximate $f(x,y)$ as $f(x_0+u, y_0+v) \approx f(x_0,y_0) + f_x(x_0,y_0)u + f_y(x_0,y_0)v$

Same thing do it for $g(x_0, y_0)$ too

This yields: $$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} \approx \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Jacobian matrix

Can analyze eq points with the Jacobian matrix if points are close to eq

Example: $$\begin{cases} \frac{dx}{dt} = 2x(1-\frac{x}{2}) - xy \\ \frac{dy}{dt} = 3y(1-\frac{y}{3}) - 2xy \end{cases}$$

Analyze @ (1,1)

Uniqueness and uniqueness says solutions exist as 2 are all pts. endpoint

① Compute Jacobian

$$J_s = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 2 - 2x - y & -x \\ -2y & 3 - 2y - 2x \end{pmatrix}$$

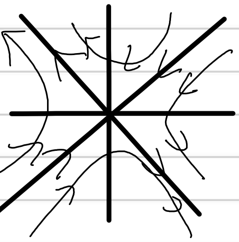
② Evaluate Jacobian @ eq point

$$J_s(1,1) = \begin{pmatrix} 2-2-1 & -1 \\ -2 & 3-2-2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}$$

③ Solve System

$$\frac{d\vec{y}}{dt} = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix} \vec{y}, \quad \vec{y} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{cases} \lambda_1 < 0 \\ \lambda_2 > 0 \end{cases}$$ So eq is a saddle

$$p(\lambda) = \lambda^2 - (-2)\lambda + (-1)$$
$$= \lambda^2 + 2\lambda - 1$$
$$\lambda = \frac{-2 \pm \sqrt{4 - 4(-1)}}{2} = \frac{-2 \pm \sqrt{8}}{2}$$
$$\lambda_{1,2} = -1 \pm \sqrt{2}$$



Because eq is saddle, we can see that $(1,1)$ is very unstable and small changes in population lead to drastic population systems

④ Apply Jacobian analysis to other eq points

Analyze @ (0,0)

$$J_p = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$p(\lambda) = \lambda^2 - 5\lambda + 6$

$\lambda_1 = 2$ $\lambda_2 > \lambda_1 > 0$ So source

$\lambda_2 = 3$

Analyze @ (2,0)

$$J_p = \begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix}$$

$p(\lambda) = \lambda^2 + 3\lambda + 2$

$\lambda_1 = -1$ $0 > \lambda_1 > \lambda_2$ So sink

$\lambda_2 = -2$

Analyze @ (0,3)

$$J_p = \begin{pmatrix} -1 & 0 \\ -6 & -3 \end{pmatrix}$$

$p(\lambda) = \lambda^2 + 4\lambda + 3$

$\lambda_1 = -3$ $0 > \lambda_1 > \lambda_2$ So sink

$\lambda_2 = -1$

This analysis helps construct accurate phase portraits

For systems

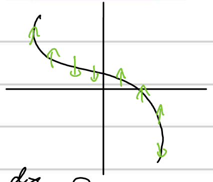
$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases}$$

- x-nullcline is the set of points (x,y) where $f(x,y)=0$, the zero level set/curve of $f(x,y)$.
- y-nullcline is the set of points (x,y) where $g(x,y)=0$

On x-nullcline, vector fields point up/down. They point left/right on y-nullcline

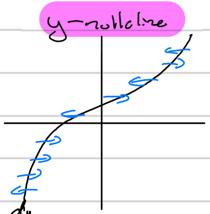
✰ Equilibrium points occur when x and y nullclines intersect ✰

x-nullcline



$\frac{dx}{dt} = f(x,y) = 0$

y-nullcline



$\frac{dy}{dt} = g(x,y) = 0$

Compute nullclines for $\begin{cases} \frac{dx}{dt} = 2x(1-\frac{x}{2}) - xy \\ \frac{dy}{dt} = 3y(1-\frac{y}{3}) - 2xy \end{cases}$

x-nullcline $\frac{dx}{dt} = 2x - x^2 - xy$

$0 = x(2-x-y)$

$\therefore$ $x=0$

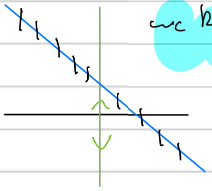
$y=-x+2$

y-nullclines $\frac{dy}{dt} = 3y - y^2 - 2xy$

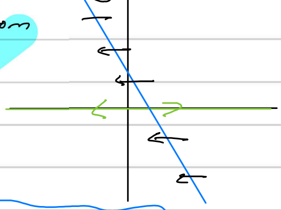
$0 = y(3-y-2x)$

$\therefore$ $y=0$

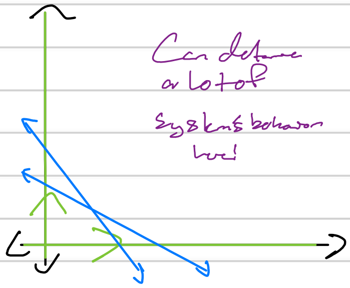
$y=-2x+3$



we know (0,0) is a source from previous example



We can then draw the nullclines on one plane



Can determine a lot of system's behavior here!

4 quadrants formed here, If a solution enters one of the triangular regions, it can never leave

Final Exam - 5/2

- Thursday May 2nd
- 3:30-6:30 pm ← more than enough time to finish
- Degrace 312
- Bring ID

Practice Tests on Canvas, also Extra Office Hours 11am-5pm on T, W

- Laplace Transforms and basic Identities provided
- Front and back handwritten formula sheet allowed

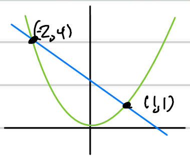
Visit office to pickup any quizzes!

Review Lecture

5.1 Review: Use Linearization to Solve and Analyze with Nullclines

$$\begin{cases} \frac{dx}{dt} = 2-x-y \\ \frac{dy}{dt} = y-x^2 \end{cases}$$

$$\begin{cases} 0 = 2-x-y \rightarrow y = 2-x \\ 0 = y-x^2 \rightarrow y = x^2 \end{cases}$$



2 equilibriums
① $(1,1)$
② $(-2,4)$

> Need Jacobian to Analyze

Let $\begin{cases} f(x,y) = 2-x-y \\ g(x,y) = y-x^2 \end{cases}$ $\therefore$ $J_s = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -2x & 1 \end{pmatrix}$

① $J_s(1,1) = \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}$

$p(\lambda) = \lambda^2 - (0)\lambda + -1-2$

$0 = \lambda^2 - 3 \rightarrow \lambda = \pm\sqrt{3}$

Both Positive and Negative Eigenvalues so $(1,1)$ is saddle

② $J_s(-2,4) = \begin{pmatrix} -1 & -1 \\ 4 & 1 \end{pmatrix}$

$p(\lambda) = \lambda^2 - (0)\lambda + -1+4$

$0 = \lambda^2 + 3 \rightarrow \lambda = \pm i\sqrt{3}$

$\alpha = 0$, spiral source at $(-2,4)$

Cover all cases:
- repeated
- Complex
  ↳ $\alpha = 0, \alpha \neq 0$
- Behavior

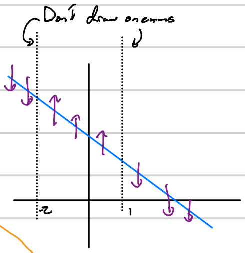
Nullclines

x-nullcline: $\frac{dx}{dt} = 0$

$2-x-y = 0$

$y = 2-x$

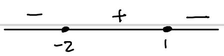
Direction of x-nullcline determined from $\frac{dy}{dt}$



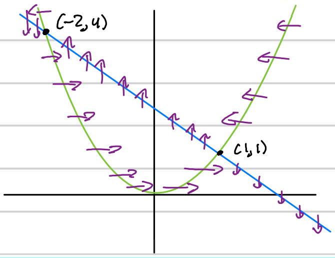
Don't draw on exams

$\frac{dy}{dt} = y-x^2 = 2-x-x^2 = -(x^2+x-2)$

$= -(x+2)(x-1)$



Total Graph:



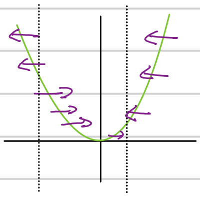
y-nullcline: $\frac{dy}{dt} = 0$

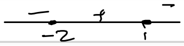
$y - x^2 = 0$

$y = x^2$

$\frac{dx}{dt} = 2-x-y = 2-x-x^2$

Same equation as above





Expect One Problem of this on Final

Chapter 6: Find Function whose Laplace Image is:

$$F(s) = \frac{2e^{5s} + e^{5s}}{s^2-3s+2}$$

$u_5(t)f(t-5)$

$$\mathcal{L}^{-1}[F(s)] = 2\cdot\mathcal{L}^{-1}\left[\frac{e^{5s}}{s^2-3s+2}\right] + \mathcal{L}^{-1}\left[\frac{e^{5s}}{s^2-3s+2}\right]$$

① $f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2-3s+2}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s-1)(s-2)}\right] \rightarrow \frac{A}{s-1} + \frac{B}{s-2}$

Redundant Process I'm not writing this ‼

Chapter 3/4: Linear System, solution

$$\begin{cases} \frac{dx}{dt} = -3x-3y \\ \frac{dy}{dt} = y-5x \end{cases} \qquad A = \begin{pmatrix} -3 & -3 \\ -5 & 1 \end{pmatrix}$$

$p(\lambda) = \lambda^2 - (-2)\lambda + -3-15$

$0 = \lambda^2 + 2\lambda - 18$

$(-3-\lambda)(1-\lambda) - 15$