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tcs94

Section 6.2: 3, 5, 7, 11, 13, 17

3. $\mathcal{L}[g_a(t)]$ $\quad g_a(t) = \begin{cases} t/a, & \text{for } t<a \\ 1, & t \ge a \end{cases}$

$a \ge 0$

$$\mathcal{L}[g_a(t)] = \int_0^\infty g_a(t)e^{-st}\,dt$$
$$= \frac{1}{a}\int_0^a te^{-st}\,dt + \int_a^\infty e^{-st}$$

D	I
t	e^{-st}
1	$-\frac{1}{s}e^{-st}$
0	$\frac{1}{s^2}e^{-st}$

$$= \frac{1}{a}\left(-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right)\Big|_0^a - \frac{1}{s}e^{-st}\Big|_a^\infty$$
$$= \frac{1}{a}\left(-\frac{a}{s}e^{-as} - \frac{1}{s^2}e^{-as} - \left(0 - \frac{1}{s^2}\right)\right) + \frac{1}{s}e^{-as}$$
$$= \frac{1}{a}\left(-\frac{a^2 s e^{-as}}{as^2} - \frac{ae^{-as}}{as^2} + \frac{a}{as^2}\right) + \frac{1}{s}e^{-as}$$
$$= \frac{1}{a}\left(\frac{e^{-as}(-a^2 s - a) + a}{as^2}\right) + \frac{1}{s}e^{-as}$$
$$= \frac{1}{\cancel{a}}\left(\frac{\cancel{a}(e^{-as}(-as-1)+1)}{as^2}\right) + \frac{1}{s}e^{-as}$$
$$= \frac{\cancel{-ase^{-as}} - e^{-as} + 1 + \cancel{ase^{-as}}}{as^2} = \boxed{\frac{1-e^{-as}}{as^2}}$$

5. $\mathcal{L}^{-1}\left[\frac{e^{-3s}}{(s-1)(s-2)}\right]$ $\quad \frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$

① $\mathcal{L}\left[\frac{1}{(s-1)(s-2)}\right]$ $\quad 1 = A(s-2) + B(s-1)$

$1 = As - 2A + Bs - B$

$\begin{cases} 0 = A + B \\ 1 = -2A - B \end{cases}$ $\quad A = -B$, $\; 1 = -2(-B) - B$ $\quad B = 1$, $\; A = -1$

$\frac{1}{s-2} - \frac{1}{s-1}$ $\quad \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s-2}\right] - \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s-1}\right]$

$= e^{2t} - e^{t}$ $\quad \mathcal{L}[u_a(t)f(t-a)] = e^{-as}\mathcal{L}[f(t)]$

② $\longrightarrow$ $y(t) = u_3(t)e^{2(t-3)} - u_3(t)e^{t-3}$

7. $\mathcal{L}^{-1}\left[\frac{14e^{-s}}{(3s+2)(s-4)}\right]$ $\quad \frac{14}{(3s+2)(s-4)} = \frac{A}{3s+2} + \frac{B}{s-4}$

① $\mathcal{L}\left[\frac{14}{(3s+2)(s-4)}\right]$ $\quad 14 = A(s-4) + B(3s+2)$

$14 = As - 4A + 3Bs + 2B$

$\begin{cases} 0 = A + 3B \\ 14 = 2B - 4A \end{cases}$ $\quad A = -3B$, $\; 14 = 2B + 12B$ $\quad B = 1$, $\; A = -3$ $\quad \frac{-3}{3s+2} = \frac{-1}{s+\frac{2}{3}}$

$$\mathcal{L}^{-1}\left[\frac{-1}{s+\frac{2}{3}} + \frac{1}{s-4}\right] = -e^{-2t/3} + e^{4t}$$

$\therefore$ $y(t) = u_1(t)\left(e^{4(t-1)} - e^{\frac{-2(t-1)}{3}}\right)$

11. $\frac{dy}{dt} = -y + u_2(t)e^{-2(t-2)}$, $y(0) = 1$

LHS $\mathcal{L}[y'] = s\mathcal{L}[y] - y(0) = s\mathcal{L}[y] - 1$

RHS $\mathcal{L}[-y + u_2(t)e^{-2(t-2)}] = -\mathcal{L}[y] + e^{-2s}\left(\frac{1}{s+2}\right)$

$$s\mathcal{L}[y] - 1 = -\mathcal{L}[y] + \frac{e^{-2s}}{s+2}$$
$$\mathcal{L}[y](s+1) = \frac{e^{-2s}}{s+2} + 1 \to \mathcal{L}[y] = \frac{e^{-2s}}{(s+2)(s+1)} + \frac{1}{s+1}$$
$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{(s+2)(s+1)}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$\frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$ $\quad 1 = A(s+1) + B(s+2)$; $\; 1 = As + A + Bs + 2B$

$\begin{cases} 1 = A + 2B \\ 0 = A + B \end{cases}$ $\quad A = -B$, $\; 1 = -B + 2B$ $\quad B = 1$, $\; A = 1$

$$f(t) = \mathcal{L}^{-1}\left[\frac{-1}{s+2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = -e^{-2t} + e^{-t}$$

e^{-2s}: $a = 2$, $f(t-2) = -e^{-2(t-2)} + e^{-(t-2)}$

$$y(t) = u_2(t)\left(-e^{-2(t-2)} + e^{-(t-2)}\right) + e^{-t}$$

$\therefore$ $y(t) = u_2(t)\left(e^{2-t} - e^{4-2t}\right) + e^{-t}$

13. $\frac{dy}{dt} = -y + u_1(t)(t-1)$, $y(0) = 2$

LHS $\mathcal{L}[y'] = s\mathcal{L}[y] - y(0) = s\mathcal{L}[y] - 2$ $\quad a = 1$

RHS $\mathcal{L}[-y + u_1(t)(t-1)] = -\mathcal{L}[y] + \mathcal{L}[u_1(t)(t-1)]$

$$s\mathcal{L}[y] - 2 = -\mathcal{L}[y] + \frac{e^{-s}}{s^2}$$
$$\mathcal{L}[y](s+1) = \frac{e^{-s}}{s^2} + 2 \to \mathcal{L}[y] = \frac{e^{-s}}{s^2(s+1)} + \frac{2}{s+1}$$
$$\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2(s+1)}\right] + \mathcal{L}^{-1}\left[\frac{2}{s+1}\right]$$

$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$ $\quad 1 = As(s+1) + B(s+1) + Cs^2$; $\; 1 = As^2 + As + Bs + B + Cs^2$

$\begin{cases} 1 = B \\ 0 = A + B \\ 0 = A + C \end{cases}$ $\quad A = -1$, $B = 1$, $C = 1$ $\; \to \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$

$$f(t) = \mathcal{L}^{-1}\left[-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right] = -1 + t + e^{-t}$$

e^{-s}: $a = 1$, $f(t-1) = -1 + (t-1) + e^{-(t-1)}$

$$y(t) = u_1(t)\left(-1 + t - 1 + e^{1-t}\right) + 2e^{-t}$$

$\therefore$ $y(t) = u_1(t)\left(t - 2 + e^{1-t}\right) + 2e^{-t}$

16. A periodic Proof just for fun

Suppose $f(t+T)=f(t)\ \forall T$, so $f(t)$ is periodic

Show $\mathcal{L}[f(t)] = \frac{1}{1-e^{-Ts}}\int_0^T f(t)e^{-st}dt$

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \int_{2T}^{3T} f(t)e^{-st}dt + \dots$$

$$= \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} f(t)e^{-st}dt \qquad u = t-nT,\quad du = dt$$

$$= \int_0^T f(u+nT)e^{-s(u+nT)}du \qquad \text{periodic so } f(u+nT)=f(u)$$

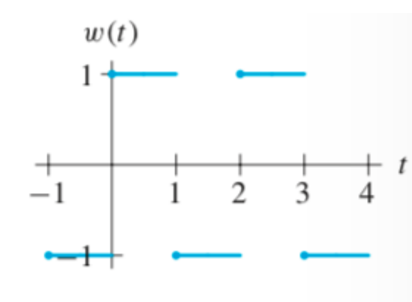
$$= \int_0^T f(u)\,e^{-su}e^{-snT}du = \sum_{n=0}^{\infty} e^{-snT}\int_0^T f(u)e^{-su}du$$

$$= \int_0^T e^{-su}f(u)du\left(\sum_{n=0}^{\infty} e^{-snT}\right) \qquad \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}\ |r|<1 \quad e^{-sT}<0 \text{ for } s>0\ ✓$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-su}f(u)du = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

as $u=t-nT=t$ since $f(t)$ is periodic

17.



A square wave with period 2.

For some $n \in \mathbb{Z}$,

$$w(t) = \begin{cases} 1 & \text{if } 2n \le t < 2n+1 \\ -1 & \text{if } 2n+1 \le t < 2n+2 \end{cases}$$

Compute $\mathcal{L}[w(t)]$

Consider various 'n' values

$n=0$: $w(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ -1 & \text{if } 1 \le t < 2 \end{cases}$

$n=1$: $w(t) = \begin{cases} 1 & \text{if } 2 \le t < 3 \\ -1 & \text{if } 3 \le t < 4 \end{cases}$

$n=2$: $w(t) = \begin{cases} 1 & \text{if } 4 \le t < 5 \\ -1 & \text{if } 5 \le t < 6 \end{cases}$

Periodic with $T=2$ as given and shown

From #16, a periodic function w/ Period t $f(t+T)=f(t)$ for all t

$$\mathcal{L}[f] = \frac{1}{1-e^{-Ts}}\int_0^T f(t)e^{-st}dt$$

Let $f(t)=w(t)$, and $T=2$

$$\mathcal{L}[w(t)] = \frac{1}{1-e^{-2s}}\int_0^2 w(t)e^{-st}dt$$

$$= \frac{1}{1-e^{-2s}}\left(\int_0^1 e^{-st}dt + \int_1^2 -e^{-st}dt\right)$$

$$= \frac{1}{1-e^{-2s}}\left(-\frac{1}{s}e^{-st}\Big|_0^1 + \frac{1}{s}e^{-st}\Big|_1^2\right)$$

$$= \frac{1}{1-e^{-2s}}\left(-\frac{1}{s}e^{-s} + \frac{1}{s} + \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-s}\right)$$

$$= \frac{1}{s(1-e^{-2s})}\left(-e^{-s} + 1 + e^{-2s} - e^{-s}\right)$$

$$\therefore \mathcal{L}[w(t)] = \frac{e^{-2s}-2e^{-s}+1}{s(1-e^{-2s})}$$