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Section 4.3: 1, 7, 9, 13, 19, 21, 23

1. $\frac{d^2y}{dt^2} + 9y = \cos t$

(H) $y'' + 9y = 0$

$s^2 + 9 = 0$

$s = \pm 3i$

$y_h(t) = K_1 \cos 3t + K_2 \sin 3t$

Guess NH: $y_p(t) = a e^{it}$

$y_p''(t) = -a e^{it}$

$-ae^{it} + 9ae^{it} = e^{it}$

$-a + 9a = 1 \rightarrow a = \frac{1}{8}$

$y_p(t) = \frac{1}{8} e^{it}$

$y_p(t) = \frac{1}{8}(\cos t + i \sin t)$

cos in original → only Re

$y_p(t) = \underbrace{\frac{1}{8}\cos t}_{} + \frac{1}{8} i \sin t$

$y(t) = K_1 \cos 3t + K_2 \sin 3t + \frac{1}{8}\cos t$

7. $\frac{d^2y}{dt^2} + 3y = \cos 3t$

(H): $y'' + 3y = 0$

$s^2 + 3 = 0$

$s = \pm\sqrt{3}\, i$

$y_h(t) = K_1 \cos\sqrt{3}t + K_2 \sin\sqrt{3}t$

Guess NH: $y_p(t) = a e^{3it}$

$y_p''(t) = -9a e^{3it}$

$e^{3it}(-9a + 3a) = e^{3it}$

$-9a + 3a = 1 \rightarrow a = -\frac{1}{6}$

$y_p(t) = -\frac{1}{6} e^{i3t}$

$y_p(t) = -\frac{1}{6}(\cos 3t + i \sin 3t)$

cos in original → only Re

$y_p(t) = \underbrace{-\frac{1}{6}(\cos 3t}_{} + i \sin 3t)$

$y(t) = K_1 \cos\sqrt{3}t + K_2 \sin\sqrt{3}t - \frac{1}{6}\cos 3t$

9. $\frac{d^2y}{dt^2} + 9y = \cos t \qquad y(0) = y'(0) = 0$

General Solution: $y(t) = K_1 \cos 3t + K_2 \sin 3t + \frac{1}{8}\cos t$ from (1)

$y(0) = K_1(1) + K_2(0) + \frac{1}{8}(1)$

$0 = K_1 + \frac{1}{8}$

$$\begin{cases} 0 = K_1 + \frac{1}{8} \\ 0 = 3K_2 \end{cases} \rightarrow \quad \begin{matrix} K_1 = -\frac{1}{8} \\ K_2 = 0 \end{matrix}$$

$y'(t) = -3K_1 \sin 3t + 3K_2 \cos 3t - \frac{1}{8}\sin t$

$y'(0) = -3K_1(0) + 3K_2(1) - \frac{1}{8}(0) \rightarrow 0 = 3K_2$

$y(t) = -\frac{1}{8}\cos 3t + \frac{1}{8}\cos t$

13. $\frac{d^2y}{dt^2} + 9y = 2\cos 3t \qquad y(0) = 2, \quad y'(0) = -9$

(H) $y'' + 9y = 0$

from (1)

$y_h(t) = K_1 \cos 3t + K_2 \sin 3t$

same freq as y_h, add t factor

Guess NH: $y_p(t) = a t e^{i3t}$

$y_p'(t) = ae^{i3t} + 3i a t e^{i3t}$

$y_p''(t) = 3iae^{i3t} + 3iae^{i3t} - 9ate^{i3t}$

$= 3ae^{i3t}(2i - 3t)$

$3ae^{i3t}(2i - 3t) + 9ate^{i3t} = 2e^{i3t}$

$6ai - 9at + 9at = 2 \rightarrow a = \frac{2}{6i} = \frac{1}{3i}\frac{(-i)}{(-i)} = \frac{-i}{3}$

$\therefore y_p(t) = -\frac{i}{3} t e^{i3t} = -\frac{i}{3} t \underbrace{(\cos 3t + i \sin 3t)}_{}$ want Re as cos in orig

$\hookrightarrow y_p(t) = \frac{1}{3} t \sin 3t$

$y(t) = K_1 \cos 3t + K_2 \sin 3t + \frac{1}{3} t \sin 3t$

$y'(t) = -3K_1 \sin 3t + 3K_2 \cos 3t + \frac{1}{3}\sin 3t + t\cos 3t$

$y(0) = K_1(1) + K_2(0) + 0$

$y'(0) = -3K_1(0) + 3K_2(1) + \frac{1}{3}(0) + 0$

$$\begin{cases} 2 = K_1 \\ -9 = 3K_2 \end{cases} \quad \begin{matrix} K_1 = 2 \\ K_2 = -3 \end{matrix}$$

$\therefore y(t) = 2\cos 3t - 3\sin 3t + \frac{1}{3} t \sin 3t$

19. $\frac{d^2y}{dt^2}+12y=3\cos 4t+2\sin t \qquad y(0)=y'(0)=0$

Define 2 NH: ① $y''+12y=3\cos 4t$ ② $y''+12y=2\sin t$

a) (H): $y''+12y=0$

$s^2+12=0$

$s=\pm\sqrt{12}\,i$

$y_h(t)=K_1\cos\sqrt{12}t+K_2\sin\sqrt{12}t$

Guess NH 1: $y_p(t)=\alpha_1 e^{i4t}$

$y_p''(t)=-16\alpha_1 e^{i4t}$

$-16\alpha_1 e^{i4t}+12\alpha_1 e^{i4t}=3e^{i4t}$

$-16\alpha_1+12\alpha_1=3 \quad \alpha_1=-\frac{3}{4}$

$y_p(t)=-\frac{3}{4}e^{i4t}$

$=-\frac{3}{4}(\cos 4t+i\sin 4t)$

$y_p(t)=-\frac{3}{4}\cos 4t$

Guess NH 2: $y_p(t)=\alpha_2 e^{it}$

$y_p''(t)=-\alpha_2 e^{it}$

$-\alpha_2 e^{it}+12\alpha_2 e^{it}=2e^{it}$

$-\alpha_2+12\alpha_2=2 \quad \alpha_2=\frac{2}{11}$

$y_p(t)=\frac{2}{11}e^{it}$

$=\frac{2}{11}(\cos t+i\sin t)$

$y_p(t)=\frac{2}{11}\sin t$

Sum is also a solution

$\therefore\ y(t)=K_1\cos\sqrt{12}t+K_2\sin\sqrt{12}t-\frac{3}{4}\cos 4t+\frac{2}{11}\sin t$

b) $y(0)=K_1(1)+K_2(0)-\frac{3}{4}(1)+\frac{2}{11}(0)$

$0=K_1-\frac{3}{4}$

$y'(t)=-\sqrt{12}K_1\sin t+\sqrt{12}K_2\cos t+3\sin t+\frac{2}{11}\cos t$

$y'(0)=0+\sqrt{12}K_2+0+\frac{2}{11}$

$0=\sqrt{12}K_2+\frac{2}{11}$

$\begin{cases}0=K_1-\frac{3}{4}\\ 0=\sqrt{12}K_2+\frac{2}{11}\end{cases} \qquad K_1=\frac{3}{4} \qquad K_2=-\frac{2}{11\sqrt{12}}$

$y(t)=\frac{3}{4}\cos\sqrt{12}t-\frac{2}{11\sqrt{12}}\sin\sqrt{12}t-\frac{3}{4}\cos 4t+\frac{2}{11}\sin t$

c)

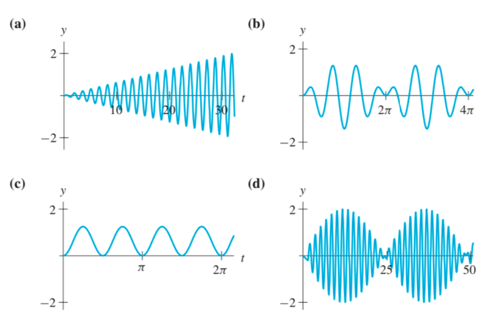
d) Because the frequencies of our solution are close to the natural frequencies given $\sqrt{12}\approx 4$, $4=4$, beats occurs. We could've predicted beats because the frequencies don't depend on any IVP.

21.

(i) $\frac{d^2y}{dt^2}+16y=10$ (ii) $\frac{d^2y}{dt^2}+16y=-10$

(iii) $\frac{d^2y}{dt^2}+16y=5\cos 3t$ (iv) $\frac{d^2y}{dt^2}+14y=2\cos 4t$

(v) $\frac{d^2y}{dt^2}+16y=\frac{1}{2}\cos 4t$ (vi) $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+16y=\cos 4t$



i) $s^2+16=0$, $s=\pm 4i$

ii) $s=\pm 4i$

iii) $s=\pm 4i$

iv) $s^2+14=0$, $s=\pm\sqrt{14}\,i$

v) $s=\pm 4i$

vi) Damped

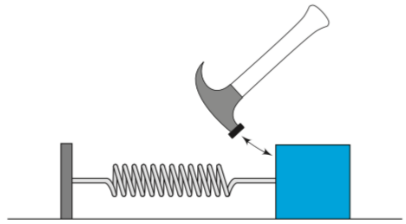
a) v, natural and resonant frequencies equal. Resonant Case

b) iii, near resonant but not as close as c

c) Positive non-periodic forcing, so it must be i

d) Heavy beats, very near resonant. iv has $s=\pm\sqrt{14}\,i\approx\pm 3.74i$, very close to 4 frequency for natural frequency. Must be iv to fit this behavior.

23.



Given

- Unit mass on frictionless surface
- Spring constant $k=16$
- Lightly tapped by hammer every T seconds
- First tap @ $t=0$

Model $\frac{d^2y}{dt^2}+16y=\cos\omega t$ where $\omega=\frac{1}{\text{Period}}$ ← T

Most likely begins in all cases below!

a) $T=1$

Forcing $=\frac{1}{1}=1$ Natural $=\frac{\sqrt{k/m}}{2\pi}=\frac{\sqrt{16}}{2\pi}=\frac{2}{\pi}=0.6366$

$1-0.6366=0.3634$

0.3634 is relatively large, so motion has a small amplitude response

b) $T=3/2$

Forcing $=\frac{1}{3/2}=\frac{2}{3}$ Natural $=0.6366$

$\frac{2}{3}-0.6366=0.03$

0.03 is small, so motion will have a very large amplitude response.

c) $T=2$

Forcing $=\frac{1}{2}=0.5$ Natural $=0.6366$

$|0.5-0.6366|=0.1366$

0.1366 is relatively small, so the motion will have a moderately large amplitude response.

d) $T=5/2$

Forcing $=\frac{1}{5/2}=\frac{2}{5}=0.4$ Natural $=0.6366$

$|0.4-0.6366|=0.2366$

0.2366 is relatively large, with a moderately small amplitude response for motion.

e) $T=3$

Forcing $=\frac{1}{3}=0.333$ Natural $=0.6366$

$|0.3-0.6366|=0.3033$

0.3033 is large, there will be a noticeably small amplitude response!

Section 6.1: 1, 3, 5, 11, 13, 15, 23

1. $f(t)=3$

$\mathcal{L}[3]=\int_0^\infty 3e^{-st}\,dt = \lim_{b\to\infty} 3\int_0^b e^{-st}\,dt$

$= \lim_{b\to\infty} -\frac{3}{s}e^{-st}\Big|_0^b$

$= \lim_{b\to\infty} -\frac{3}{s}\left(e^{-bs}-1\right)$

$= \frac{3}{s}$

$\mathcal{L}[3]=\frac{3}{s}$

3. $h(t)=-5t^2$

$\mathcal{L}[-5t^2]=\int_0^\infty -5t^2e^{-st} = \lim_{b\to\infty} -5\int_0^b t^2e^{-st}\,dt$

D	I
t^2 +	e^{-st}
$2t$ −	$-\frac{1}{s}e^{-st}$
2 +	$\frac{1}{s^2}e^{-st}$
0	$-\frac{1}{s^3}e^{-st}$

$= \lim_{b\to\infty} -5\left(-\frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\right)\Big|_0^b$

$= -5\left(0-0-0-\left(0-0-\frac{2}{s^3}\right)\right)$

$= -5\left(\frac{2}{s^3}\right) = \frac{-10}{s^3}$

$\mathcal{L}[5t^2]=\frac{-10}{s^3}$

5. Mathematical Induction to show $\mathcal{L}[t^n]=\frac{n!}{s^{n+1}}$, $s>0$

Basis Step: Show $p(0)$ is true

$\mathcal{L}[t^0]=\frac{0!}{s^{0+1}}=\frac{1}{s}$?

$\mathcal{L}[1]=\int_0^\infty 1e^{-st}\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\,dt$

$= \lim_{b\to\infty} -\frac{1}{s}e^{-st}\Big|_0^b$

$= -\frac{1}{s}e^{-bs}-\left(-\frac{1}{s}\right)$

$= \frac{1}{s}$

$\frac{1}{s}=\frac{1}{s}$ ✓

Inductive Step: Assume $p(k)$ is true. $p(k)$: $\mathcal{L}[t^k]=\frac{k!}{s^{k+1}}$ $s>0$

WTS: Show that $p(k+1)$ is true

$\mathcal{L}[t^{k+1}]=\int_0^\infty t^{k+1}e^{-st}\,dt = \lim_{b\to\infty}\int_0^b t^{k+1}e^{-st}\,dt$

(by parts) $= \lim_{b\to\infty} t^{k+1}\cdot -\frac{1}{s}e^{-st}\Big|_0^b - \int_0^b -\frac{1}{s}e^{-st}(k+1)t^k\,dt$

$= \lim_{b\to\infty} \frac{t^{k+1}e^{-st}}{-s}\Big|_0^b + \frac{k+1}{s}\int_0^b t^k e^{-st}\,dt$

$= \frac{k+1}{s}\mathcal{L}[t^k]$ Use $p(k)$ to rewrite

$= \frac{k+1}{s}\left(\frac{k!}{s^{k+1}}\right)$ for $s>0$

$= \frac{(k+1)!}{s^{k+2}}$ Simplify

$p(k+1)$: $\mathcal{L}[t^{k+1}]=\frac{(k+1)!}{s^{k+2}}$, $s>0$

By Mathematical Induction, we can therefore conclude that $\mathcal{L}[t^n]=\frac{n!}{s^{n+1}}$ is true for $s>0$

11. $\mathcal{L}[?]=\frac{4}{s(s+3)}$

$\frac{4}{s(s+3)}=\frac{A}{s}+\frac{B}{s+3}$

$4=A(s+3)+B(s)$

$4=As+3A+Bs$

$\begin{cases} 3A=4 & A=\frac{4}{3} \\ A+B=0 & B=-\frac{4}{3} \end{cases}$

$\frac{4}{s(s+3)}=\frac{4}{3s}-\frac{4}{3(s+3)}$

$\mathcal{L}^{-1}\left[\frac{4}{s(s+3)}\right]=\mathcal{L}^{-1}\left[\frac{4}{3s}-\frac{4}{3(s+3)}\right]$

$=\mathcal{L}^{-1}\left[\frac{4}{3s}\right]+\mathcal{L}^{-1}\left[\frac{-4}{3(s+3)}\right]$

$=\frac{4}{3}\mathcal{L}^{-1}\left[\frac{1}{s}\right]-\frac{4}{3}\mathcal{L}^{-1}\left[\frac{1}{s+3}\right]$

$=\frac{1}{3}(4)-\frac{4}{3}\left(e^{-3t}\right)$

$\frac{4}{3}-\frac{4}{3}e^{-3t}$

13. $\frac{2s+1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$ $A=-3$ $1=-4+2B-B$

$B=5$

$2s+1 = A(s-2) + B(s-1)$

$2s+1 = As - 2A + Bs - B$

$\begin{cases} 2 = A + B & A = 2-B \\ 1 = -2A - B & 1 = -2(2-B) - B \end{cases}$

$\mathcal{L}^{-1}\left[\frac{2s+1}{(s-1)(s-2)}\right] = -3\,\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + 5\,\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]$

$= -3e^{t} + 5e^{2t}$

15. $\frac{dy}{dt} = -y + e^{-2t}$, $y(0)=2$

a) $\mathcal{L}[y'] = \mathcal{L}[-y + e^{-2t}]$

LHS $\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$

RHS $\mathcal{L}[-y+e^{-2t}] = \mathcal{L}[-y] + \mathcal{L}[e^{-2t}]$

$\mathcal{L}[-y+e^{-2t}] = -\mathcal{L}[y] + \frac{1}{s+2}$

b) $s\mathcal{L}[y] - y(0) = -\mathcal{L}[y] + \frac{1}{s+2}$

$s\mathcal{L}[y] + \mathcal{L}[y] = \frac{1}{s+2} + 2$

$\mathcal{L}[y](s+1) = \frac{1}{s+2} + 2$

$\mathcal{L}[y] = \frac{1}{(s+2)(s+1)} + \frac{2}{s+1}$

$\mathcal{L}[y] = \frac{2(s+2)+1}{(s+2)(s+1)} \longrightarrow \mathcal{L}[y] = \frac{2s+5}{(s+2)(s+1)}$

c) $\frac{2s+5}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$ $\begin{cases} 2 = A + B & A = 2-B \\ 5 = 2A + B & 5 = 2(2-B) + B \end{cases}$

$2s+5 = As + 2A + Bs + B$ $5 = 4 - 2B + B$ $B=-1$ $A=3$

$\frac{2s+5}{(s+1)(s+2)} = 3\left(\frac{1}{s+1}\right) - \left(\frac{1}{s+2}\right)$ $\mathcal{L}^{-1}\left[\frac{2s+5}{(s+1)(s+2)}\right] = 3\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}\left[\frac{1}{s+2}\right]$ $y(t) = 3e^{-t} - e^{-2t}$

23. $\frac{dy}{dt} = -y + t^2$, $y(0)=1$

a) $\mathcal{L}[y'] = \mathcal{L}[-y+t^2]$

LHS $\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$

RHS $\mathcal{L}[-y+t^2] = \mathcal{L}[-y] + \mathcal{L}[t^2]$

$\mathcal{L}[-y+t^2] = -\mathcal{L}[y] + \frac{2}{s^3}$

b) $s\mathcal{L}[y] - y(0) = -\mathcal{L}[y] + \frac{2}{s^3}$

$s\mathcal{L}[y] + \mathcal{L}[y] = \frac{2}{s^3} + 1$

$\mathcal{L}[y](s+1) = \frac{2}{s^3} + 1$

$\mathcal{L}[y] = \frac{2}{s^3(s+1)} + \frac{1}{s+1}$

$\mathcal{L}[y] = \frac{2+s^3}{s^3(s+1)}$

c) $\frac{2+s^3}{s^3(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1}$ $\mathcal{L}^{-1}\left[\frac{2+s^3}{s^3(s+1)}\right] = 2\mathcal{L}^{-1}\left[\frac{1}{s}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{s^3}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$

$2+s^3 = As^2(s+1) + Bs(s+1) + C(s+1) + Ds^3$ $= 2 - 2t + t^2 - e^{-t}$

$2+s^3 = As^3 + As^2 + Bs^2 + Bs + Cs + C + Ds^3$

$\begin{cases} 2 = C & B = -C \\ 1 = A + D & 0 = A - C \\ 0 = A + B & A = C \\ 0 = B + C & D = 1 - A \end{cases}$ $\begin{cases} A = 2, B = -2 \\ C = 2, D = -1 \end{cases}$

$y(t) = 2 - 2t + t^2 - e^{-t}$