Differential Equations (MATH 224) – Fall 2014 – Thomas Section
First Midterm

Your name:____.

Note: this is a closed-book, closed-notes exam. The time allotted is 50 minutes. You may have one 8.5×11 inch sheet of notes (front and back) for reference. Please turn in your notes with your exam. Calculators are not allowed. You may use a straight-edge or ruler. Each problem is worth 25 points. Your grade will be based both on the correctness of your answers and on the clarity of your writing.

BEFORE YOU BEGIN, copy the following integrity agreement on your paper below, and sign it.

I agree to complete this exam without giving or receiving aid.

Integrity agreement:

____.

Signature & Date:

____.

Number	Grade
1	
2	
3	
4	
Total	

1. Consider the partially decoupled system

$$\frac{dx}{dt} = xy, \qquad \frac{dy}{dt} = y + 1.$$

(a) Derive the general solution.

$$\int \frac{dy}{y+1} = \ln(y+1) = \int dt = t + \text{Const}$$

$$\boxed{y(t) = A e^{t} - 1}$$

$$\frac{dx}{dt} = x \cdot (A e^{t} - 1) \quad \text{so} \quad \int \frac{dx}{x} = \ln x = \int (A e^{t} - 1)\, dt$$

$$= A e^{t} - t + \text{const.}$$

So

$$\boxed{x(t) = B \exp[A e^{t} - t]}$$

(b) Find the equilibrium points of the system.

$\frac{dy}{dt} = 0$ only when $y = -1$. $\frac{dx}{dt} = 0$ for $y = -1$ only when $x = 0$.

$\boxed{(x = 0, y = -1) \text{ is the only fixed point.}}$

(c) Find the solution that satisfies the initial condition $(x_0, y_0) = (1, 0)$.

Take $t_0 = 0$. $x(0) = B \exp[A - 0]$ and $y(0) = A - 1$.

For $y(0) = 0$, take $A = 1$.

For $x(0) = 1$, take $B = e^{-1}$.

$$\boxed{\begin{aligned} x(t) &= e^{-1} \exp[e^{t} - t] = \exp[e^{t} - t - 1] \\ y(t) &= e^{t} - 1 \end{aligned}}$$

2. Consider the differential equation $dy/dt = -2ty^2$.

(a) Calculate its general solution. (Please indicate if you use the back for scratch work.)

$$\int \frac{dy}{y^2} = \frac{y^{-1}}{-1} = \int -2t\,dt = -t^2 + \text{constant} \quad \text{so}$$

$$\boxed{y = \frac{1}{t^2 + c} \quad \text{or} \quad y = 0} \quad \text{is the general solution.}$$

(b) Find all values of y_0 such that the solution to the initial-value problem

$$\frac{dy}{dt} = -2ty^2, \qquad y(-1) = y_0,$$

does not blow up (or blow down) in finite time. In other words, find all y_0 such that the solution is defined for all real t.

$y_0 = y(-1) = \frac{1}{1+c}$. Evidently (from (a)) y remains finite for all t provided either $y = 0$ or $c > 0$. Since

$c = \frac{1}{y_0} - 1$, $c > 0$ means $y_0 < 1$.

Solutions $y(t)$ exist for all t if $y(-1) = y_0 < 1$ (including $y_0 = 0$).

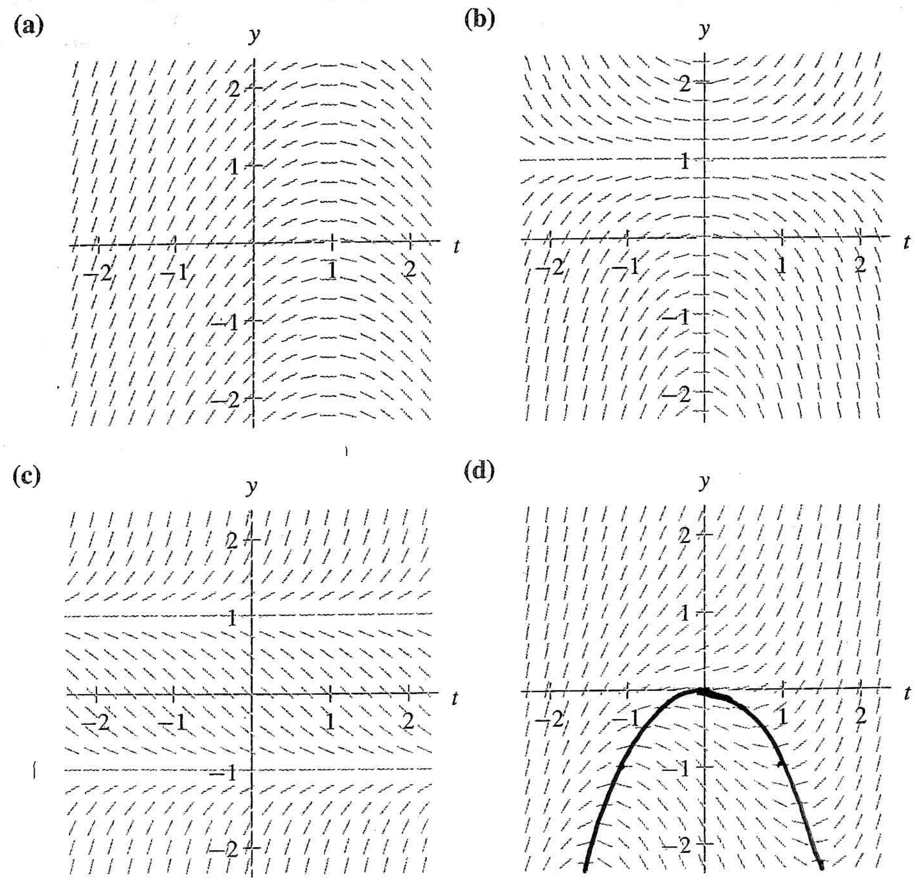
3. Eight differential equations and four slope fields are given below. Determine the equation that corresponds to each slope field and state briefly how you know your choice is correct.

(a) matches [4] because: $y' = 0$ for $t = 1$ and all y in eq 1, 4 only. $y' < 0$ for $t > 1$ in eq 4 but not eq 1.

(b) matches [7] because: $y' = ty - t = t \cdot (y-1) = 0$ for $y = 1$, and all t, and for $t = 0$, and all y.

(c) matches [8] because: $y' = 0$ for $y = \pm 1$ for all t in eq 2, 8 only. $y' > 0$ for $y > 1$ in eq 8 but not eq 2.

(d) matches [6] because: $y' = 0$ along the parabola $y = -t^2$ (see sketch) only for eq 6.



$$\frac{dy}{dt} = t - 1 \quad (1)$$
$$\frac{dy}{dt} = 1 - y^2 \quad (2)$$
$$\frac{dy}{dt} = y - t^2 \quad (3)$$
$$\frac{dy}{dt} = 1 - t \quad (4)$$
$$\frac{dy}{dt} = 1 - y \quad (5)$$
$$\frac{dy}{dt} = y + t^2 \quad (6)$$
$$\frac{dy}{dt} = ty - t \quad (7)$$
$$\frac{dy}{dt} = y^2 - 1 \quad (8)$$

4. The air in a small room 20 ft by 5 ft by 10 ft is 3% carbon monoxide (CO). Starting at time $t = 0$, air containing 1% CO is blown into the room at the rate of 100ft^3 per hour, and well mixed air flows out through a vent at the same rate.

(a) Write an initial-value problem for the amount of carbon monoxide in the room over time.

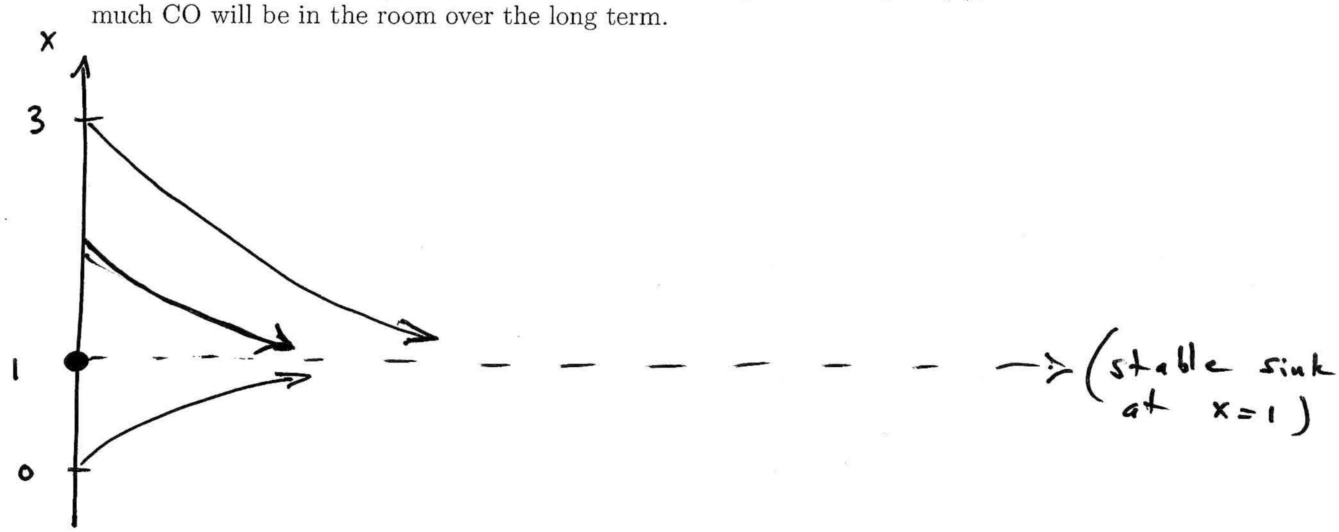
Let x be the % CO. The room volume is $20 \times 5 \times 10 = 1000 \text{ ft}^3$. The initial value is $x(0) = 3$. The influx is 1 % CO and the efflux is x % [redacted] CO. The inflow brings in $.01 \cdot 100 \frac{\text{ft}^3}{\text{hr}}$ of CO and the outflow removes $\frac{x}{100} \cdot 100 \frac{\text{ft}^3}{\text{hr}}$. The percent CO obeys

$$\frac{d}{dt}\left(\frac{x}{100} \cdot 1000\right) = -\frac{x}{100} \cdot 100 + .01 \cdot 100 = 1 - x$$

so the IVP is

$$\boxed{\frac{dx}{dt} = \frac{1-x}{10}, \qquad x(0) = 3}$$

(b) Sketch the phase line corresponding to the initial-value problem in part (a), and determine how much CO will be in the room over the long term.



(c) Estimate the time at which the air in the room will be reduced to 2% carbon monoxide.

$$(x(t) - 1) = e^{-t/10}(x(0) - 1) = 2e^{-t/10} \quad \text{so}$$

$$x(t) = 1 + 2e^{-t/10}.$$

So $x(t) = 2$ when $e^{-t/10} = \frac{1}{2}$, or $e^{t/10} = 2$,

or $t = 10 \ln(2) \approx 6.93$ hours