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Section 3.1: 3, 5, 15, 17, 25

3. $\frac{dx}{dt} = ax + by$, $\frac{dy}{dt} = cx + dy$

$a = 1,\ b = 0,\ c = 2,\ d = 1$

Since a is positive and b is 0, Paul (x(t))'s profit (value) increases as profits increase and are not impacted at all by Bob's profits. Bob's profits increase with both Bob's and Paul's profits. That being said, Bob's profits increase two times as quickly with respect to Paul's profits.

5. $\begin{cases} \frac{dx}{dt} = 2x + y \\ \frac{dy}{dt} = x + y \end{cases}$ $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

15. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\vec{Y}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$\frac{d\vec{Y}}{dt} = 0$ @ eq. $\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Suppose $\det(A) = 0 \therefore ad - bc = 0$

$\begin{cases} ax_0 + by_0 = 0 \\ cx_0 + dy_0 = 0 \end{cases}$ Let $a \neq 0$

$\therefore d = \frac{bc}{a}$ is valid

and $x_0 = -\frac{b}{a} y_0$

Plug into system: $c\left(-\frac{b}{a} y_0\right) + \left(\frac{bc}{a}\right) y_0 = 0$

$0 = 0$ ✓

Because $0=0$, $x_0 = -\frac{b}{a} y_0$ is a valid solution

Let $b \neq 0$ and $a = 0$

$\therefore c = \frac{ad}{b}$ is valid and $c = 0$

Plug into system: $0x_0 + by_0 = 0 \Rightarrow y_0 = 0$

$0x_0 + dy_0 = 0 \Rightarrow y_0 = 0$

Straight line solution on x-axis

Let $a = 0$ and $b = 0$

$\therefore cx_0 + dy_0 = 0$ is the only set of eq points

17. $\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$

a) Let $q = 0$ and $p \neq 0$

$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} + pv = 0 \end{cases} \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -p & 0 \end{pmatrix}\begin{pmatrix} v \\ y \end{pmatrix}$

$\therefore v = 0$ and $-pv = 0$

$v = 0$

All eq points, therefore, lie on the line $v=0$

b) Let $q = p = 0$

$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = 0 \end{cases} \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v \\ y \end{pmatrix}$

$\therefore v = 0$ and $0 = 0$

$v = 0$ is where all eq. points lie

25. $\frac{dY}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} Y$

a) $Y(t) = \begin{pmatrix} te^{2t} \\ -(t+1)e^{2t} \end{pmatrix} = \begin{pmatrix} te^{2t} \\ -(te^{2t} + e^{2t}) \end{pmatrix}$

$\therefore Y'(t) = \begin{pmatrix} 2te^{2t} + e^{2t} \\ -(2te^{2t} + e^{2t} + 2e^{2t}) \end{pmatrix}$

$\begin{pmatrix} 2te^{2t} + e^{2t} \\ -2te^{2t} - 3e^{2t} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} te^{2t} \\ -(te^{2t} + e^{2t}) \end{pmatrix}$

(i) $2te^{2t} + e^{2t} = te^{2t} + te^{2t} + e^{2t}$

$2te^{2t} + e^{2t} = 2te^{2t} + e^{2t}$

$0 = 0$ ✓

(ii) $-2te^{2t} - 3e^{2t} = te^{2t} - 3te^{2t} - 3e^{2t}$

$-2te^{2t} - 3e^{2t} = -2te^{2t} - 3e^{2t}$

$0 = 0$ ✓

b) Use Linearity Principle

$\vec{Y} = \begin{pmatrix} te^{2t} \\ -(t+1)e^{2t} \end{pmatrix}$ is a solution from (a)

$\vec{Y}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $k\vec{Y} = \vec{Y}_1$, $\vec{Y}_1(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$\therefore \begin{pmatrix} 0 \\ -1 \end{pmatrix} = k\begin{pmatrix} 0 \\ 2 \end{pmatrix} \rightarrow k = -2$ ← Plug into $\vec{Y}$

$\therefore$ Solution is $\vec{Y} = \begin{pmatrix} -2te^{2t} \\ (2t+2)e^{2t} \end{pmatrix}$

Section 3.2: 3, 5, 13, 15, 17, 19, 21

3. $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -5 & -2 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

a) $p(\lambda) = \det(A - I\lambda)$ $A = \begin{pmatrix} -5 & -2 \\ -1 & -4 \end{pmatrix}$

$A - I\lambda = \begin{pmatrix} -5-\lambda & -2 \\ -1 & -4-\lambda \end{pmatrix}$

$p(\lambda) = (-5-\lambda)(-4-\lambda) - 2$

$= 20 + 9\lambda + \lambda^2 - 2$

$= \lambda^2 + 9\lambda + 18$

$= (\lambda+6)(\lambda+3) \Rightarrow \boxed{\lambda_1 = -6,\ \lambda_2 = -3}$

b) $\lambda_1 = -6$: $\begin{pmatrix} -5 & -2 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -6\begin{pmatrix} x \\ y \end{pmatrix}$

$-5x - 2y = -6x \Rightarrow y = \frac{1}{2}x$

$-x - 4y = -6y \Rightarrow y = \frac{1}{2}x$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ for $y_0 = \frac{1}{2}x_0$

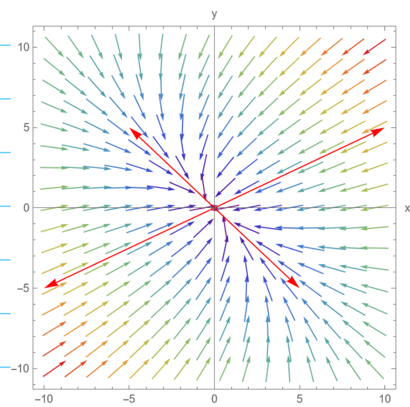
$\lambda_2 = -3$: $\begin{pmatrix} -5 & -2 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$

$-5x - 2y = -3x \Rightarrow y = -x$

$-x - 4y = -3y \Rightarrow -x = y$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ for $y = -x$

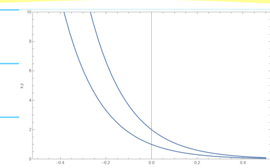
c)



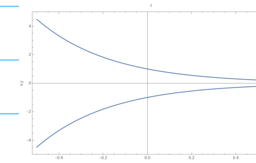
d) $Y_1(t) = e^{-6t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$Y_2(t) = e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Y_1:



Y_2:



e) $\vec{Y}(t) = k_1 e^{-6t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

5. $\frac{dx}{dt} = -\frac{x}{2}$, $\frac{dy}{dt} = x - \frac{y}{2}$

a) $A = \begin{pmatrix} -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix} \Rightarrow A - I\lambda = \begin{pmatrix} -\frac{1}{2}-\lambda & 0 \\ 1 & -\frac{1}{2}-\lambda \end{pmatrix}$

$p(\lambda) = (-\frac{1}{2}-\lambda)(-\frac{1}{2}-\lambda)$ $\boxed{\lambda_1 = \lambda_2 = \frac{1}{2}}$

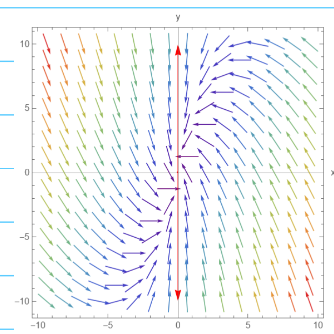
b) $\lambda = -\frac{1}{2}$: $\begin{pmatrix} -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}$

$-\frac{1}{2}x = -\frac{1}{2}x \Rightarrow x \in \mathbb{R}$

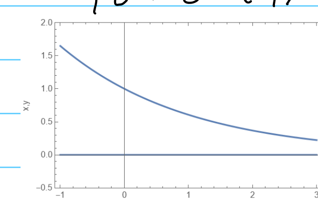
$x - \frac{1}{2}y = -\frac{1}{2}y \Rightarrow x = 0$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ y_1 \end{pmatrix}$ for $y_1 \in \mathbb{R}$

c)



d) $\vec{Y}(t) = e^{-\frac{1}{2}t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



e) This problem does not meet the requirement specified of "two real and distinct eigenvalues. A general solution cannot be made!

13. $\frac{d\vec{Y}}{dt} = \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$p(\lambda) = (-4-\lambda)(-3-\lambda) - 2$

$= 12 + 7\lambda + \lambda^2 - 2$

$= \lambda^2 + 7\lambda + 10$

$= (\lambda+5)(\lambda+2) \Rightarrow \lambda_1 = -5,\ \lambda_2 = -2$

$\lambda_1 = -5$: $\begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix}$

$\left.\begin{matrix} -4x + y = -5x \Rightarrow y = -x \\ 2x - 3y = -5y \Rightarrow 2x = -2y \end{matrix}\right\} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = -2$: $\begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$

$\left.\begin{matrix} -4x + y = -2x \Rightarrow y = 2x \\ 2x - 3y = -2y \Rightarrow x = \frac{1}{2}y \end{matrix}\right\} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\therefore \vec{Y}(t) = k_1 e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

a) $Y(0) = (1, 0)$ $\quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} = k_1 e^{-5(0)}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-2(0)}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{Y}(t) = \frac{2}{3}e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{3}e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$1 = k_1 + k_2$ $\qquad 0 = -k_1 + 2k_2$

$k_1 = 1 - k_2$ $\qquad 0 = k_2 - 1 + 2k_2 \Rightarrow k_2 = \frac{1}{3},\ k_1 = \frac{2}{3}$

b) $Y(0) = (2, 1)$ $\quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} = k_1 e^{-5(0)}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-2(0)}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{Y}(t) = e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$2 = k_1 + k_2$ $\qquad 1 = -k_1 + 2k_2$

$k_1 = 2 - k_2$ $\qquad 1 = k_2 - 2 + 2k_2 \Rightarrow k_2 = 1,\ k_1 = 1$

c) $Y(0) = (-1, -2)$ $\quad \begin{pmatrix} -1 \\ -2 \end{pmatrix} = k_1 e^{-5(0)}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-2(0)}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{Y}(t) = -e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$-1 = k_1 + k_2$ $\qquad -2 = -k_1 + 2k_2$

$k_1 = -1 - k_2$ $\qquad -2 = 1 + k_2 + 2k_2 \Rightarrow k_2 = -1,\ k_1 = 0$

15. $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $\qquad \lambda = a$: $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = a\begin{pmatrix} x \\ y \end{pmatrix}$

$p(\lambda) = (a-\lambda)(a-\lambda)$

$\lambda_1 = \lambda_2 = a$

$\therefore \lambda = a$ is the only eigenvalue

$ax = ax \quad x \in \mathbb{R}$

$ay = ay \quad y \in \mathbb{R}$

$\therefore$ Any vector $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ that is nonzero is an eigenvector for the given matrix

17. $B = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$

$p(\lambda) = (a-\lambda)(d-\lambda) - b^2$

$= ad - a\lambda - d\lambda + \lambda^2 - b^2$

$= \lambda^2 - (a+d)\lambda + ad - b^2$

$D = (a-d)^2 - 4b^2$

$p(\lambda) = \lambda^2 - (a+d)\lambda + ad - b^2$

$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-b^2)}}{2}$

$\therefore D = a^2 + 2ad + d^2 - 4ad + 4b^2$

$= a^2 - 2ad + d^2 + 4b^2$

$D = (a-d)^2 + 4b^2$

$D \geq$ for all a, b, d..

so B has at least one real eigenvalue

let $a = d$ and $b \neq 0$:

$D = (0)^2 + 4b^2 = 4b^2$

$\Rightarrow \lambda = \frac{2d \pm \sqrt{4b^2}}{2}$ has 2 distinct real roots

let $a \neq d$ and $b \neq 0$

$\lambda = \frac{a+d \pm \sqrt{(a-d)^2 + 4b^2}}{2}$ has 2 real distinct roots

for $b \neq 0$, all choices of a and d result two real eigenvalues for B

19. $\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$; $p, q \in \mathbb{R}^+$

a) $\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} + pv + qy = 0 \end{cases}$ $\quad \frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}$

b) $p(\lambda) = \det(A - I\lambda)$

$A - I\lambda = \begin{pmatrix} -\lambda & 1 \\ -q & -p-\lambda \end{pmatrix}$ $\quad p(\lambda) = (-\lambda)(-p-\lambda) + q$

$= p\lambda + \lambda^2 + q$

$\therefore\ p(\lambda) = \lambda^2 + p\lambda + q$

c) $\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$

d) $D = p^2 - 4q$ must be positive

$0 < p^2 - 4q \rightarrow p^2 > 4q$

e) $p^2 - 4q$ is less than p^2

$-p \pm \sqrt{p^2 - 4q} \leftarrow$ evaluates to less than p

Numerator always negative, so $\lambda < 0$

21. $\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0$

a) $\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} + 7v + 10y = 0 \end{cases}$ $\quad \frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -10 & -7 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}$

b) $p(\lambda) = (-\lambda)(-7-\lambda) + 10$

$= 7\lambda + \lambda^2 + 10$

$= (\lambda+5)(\lambda+2) \Rightarrow \lambda_1 = -5,\ \lambda_2 = -2$

$\lambda_1 = -5$: $\begin{pmatrix} 0 & 1 \\ -10 & -7 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix} = -5\begin{pmatrix} y \\ v \end{pmatrix}$

$v = -5y$

$-10y - 7v = -5v \quad v = -5y$ $\quad \begin{pmatrix} 1 \\ -5 \end{pmatrix}$ or $\begin{pmatrix} y \\ -5y \end{pmatrix}$

$\lambda_2 = -2$: $\begin{pmatrix} 0 & 1 \\ -10 & -7 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix} = -2\begin{pmatrix} y \\ v \end{pmatrix}$

$v = -2y$

$-10y - 7v = -2v \quad v = -2y$ $\quad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ or $\begin{pmatrix} y \\ -2y \end{pmatrix}$

c) $\lambda_1 = -5$: $\vec{Y}_1(t) = k_1 e^{-5t}\begin{pmatrix} 1 \\ -5 \end{pmatrix}$

$\vec{Y}_1(t) = \begin{pmatrix} k_1 e^{-5t} \\ -5k_1 e^{-5t} \end{pmatrix}$

$\lambda_2 = -2$: $\vec{Y}(t) = k_2 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\vec{Y}(t) = \begin{pmatrix} k_2 e^{-2t} \\ -2k_2 e^{-2t} \end{pmatrix}$

d) Solution from 2.3)

b) Let $y(t) = \alpha e^{st}$

$\alpha s^2 e^{st} + 7\alpha s e^{st} + 10\alpha e^{st} = 0$

$s^2 + 7s + 10 = 0$

$(s+5)(s+2) = 0 \rightarrow s = -5$ and $s = -2$

α is arbitrary here

$\therefore\ y_1(t) = e^{-5t}$ and $y_2(t) = e^{-2t}$

$5e^{-5t}$ $\quad Y_1 = (e^{-5t}, -5e^{-5t})$

$2e^{-2t}$ $\quad Y_2 = (e^{-2t}, -2e^{-2t})$

Multiples of a solution are also solutions, so the two different methods are equivalent