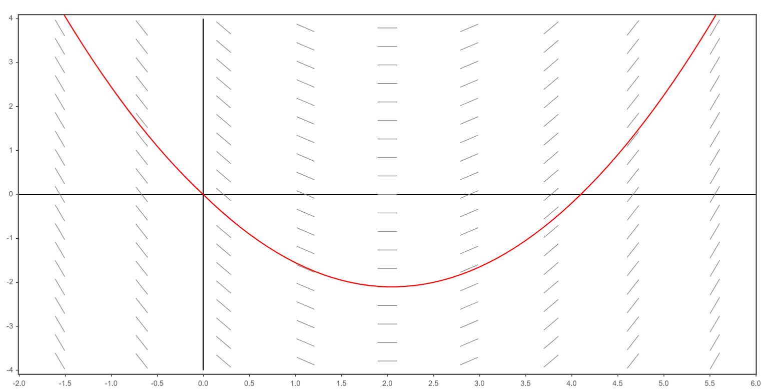
MATH 224: Practice Problems for Test 1

Slope Fields for Ordinary Differential Equations

Problem 1. Draw a slope field for each of the specified equations, sketch the solutions corresponding to the provided initial values, and briefly describe the behavior of these solutions. Ensure that you draw a sufficient number of slopes in each slope field to accurately depict the solution behaviors.

1. $\dfrac{dy}{dt} = t - 2$, $y(0) = 0$.

   Solution. Since $\frac{dy}{dt}$ depends only on t, slopes are equal along vertical lines.



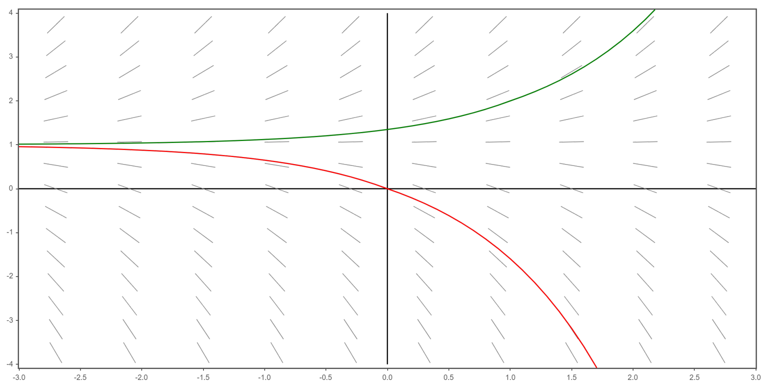
   - The particular solution has the minimum value $y(2) \approx 2$.
   - As $t \to \infty$, $y(t)$ approaches infinity.
   - As $t \to -\infty$, $y(t)$ approaches negative infinity.

2. $\dfrac{dy}{dt} = y - 1$.

   (a) $y(0) = 0$.

(b) $y(1) = 2$.

Solution. Since $\frac{dy}{dt}$ depends only on y, slopes are equal along horizontal lines.



(a) The particular solution $y(0) = 0$:

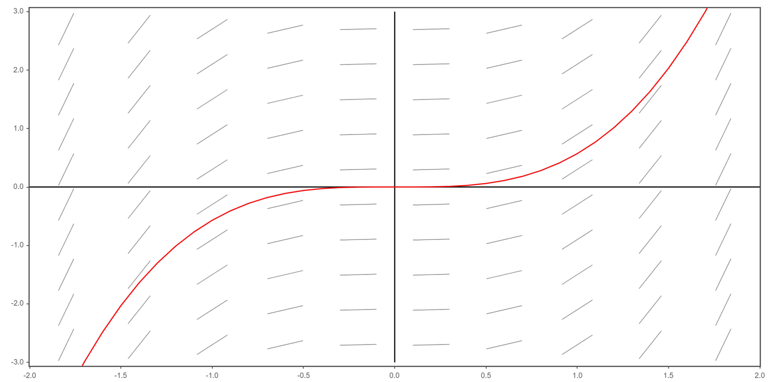
- As $t \to \infty$, $y(t)$ tends toward negative infinity.
- As $t \to -\infty$, $y(t)$ asymptotically approaches 1.

(b) The particular solution $y(1) = 2$:

- As $t \to \infty$, $y(t)$ tends toward infinity.
- As $t \to -\infty$, $y(t)$ asymptotically approaches 1.

3. $\dfrac{dy}{dt} = 2t^2$, $y(0) = 0$.

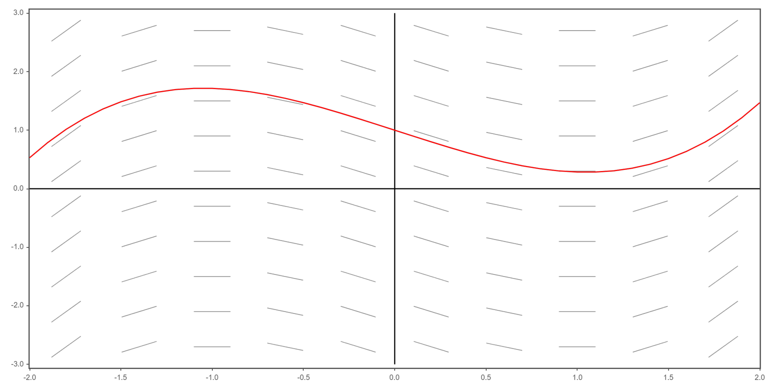
Solution. Since $\frac{dy}{dt}$ depends only on t, slopes are equal along vertical lines.



- The particular solution has the inflection point at $y(0) = 0$.
- As $t \to \infty$, $y(t)$ tends toward infinity.
- As $t \to -\infty$, $y(t)$ tends toward negative infinity.

4. $\dfrac{dy}{dx} = x^2 - 1$, $y(0) = 1$.

   Solution. Since $\frac{dy}{dx}$ depends only on x, slopes are equal along vertical lines.



- The particular solution has the local maximum value at $y(-1) \approx 1.5$.
- The particular solution has the local minimum value at $y(1) \approx 0.4$.

- As $t \to \infty$, $y(t)$ approaches infinity.
- As $t \to -\infty$, $y(t)$ approaches negative infinity.

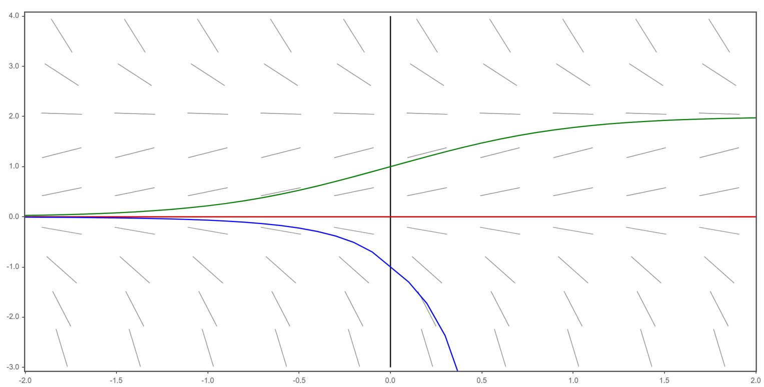
5. $\dfrac{dy}{dt} = 2y - y^2$.

   (a) $y(0) = 0$.

   (b) $y(0) = 1$.

   (c) $y(0) = -1$.

Solution. Since $\frac{dy}{dt}$ depends only on y, slopes are equal along horizontal lines.



(a) The particular solution $y(0) = 0$: The solution is identically zero for all values of x.

(b) The particular solution $y(0) = 1$:

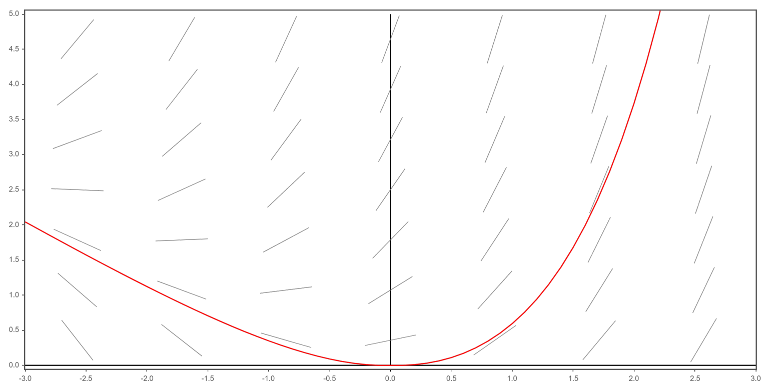
- As $t \to \infty$, $y(t)$ asymptotically approaches 2
- As $t \to -\infty$, $y(t)$ asymptotically approaches 0.

(c) The particular solution $y(0) = -1$:

- As $t \to \infty$, $y(t)$ tends toward negative infinity.
- As $t \to -\infty$, $y(t)$ asymptotically approaches 0.

6. $\dfrac{dy}{dx} = x + y$, $y(0) = 0$.

Solution.



- The particular solution has the local minimum value at $y(0) = 0$.
- As $t \to \infty$, $y(t)$ tends toward infinity.
- As $t \to -\infty$, $y(t)$ tends toward negative infinity.

7. $\dfrac{dy}{dx} = -\dfrac{y}{x}$.

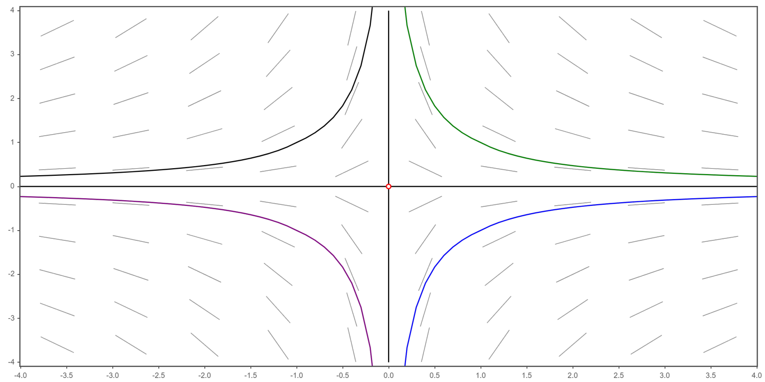
(a) $y(1) = 1$.

(b) $y(1) = -1$.

(c) $y(-1) = 1$.

(d) $y(-1) = -1$.

Solution.



(a) The particular solution $y(1) = 1$:

- As $t \to \infty$, $y(t)$ asymptotically approaches 0.
- As $t \to 0$, $y(t)$ tends toward infinity.

(b) The particular solution $y(1) = -1$:

- As $t \to \infty$, $y(t)$ asymptotically approaches 0.
- As $t \to 0$, $y(t)$ tends toward negative infinity.

(c) The particular solution $y(-1) = -1$:

- As $t \to 0$, $y(t)$ tends toward negative infinity.
- As $t \to -\infty$, $y(t)$ asymptotically approaches 0.

(d) The particular solution $y(-1) = 1$:

- As $t \to 0$, $y(t)$ tends toward infinity.
- As $t \to -\infty$, $y(t)$ asymptotically approaches 0.

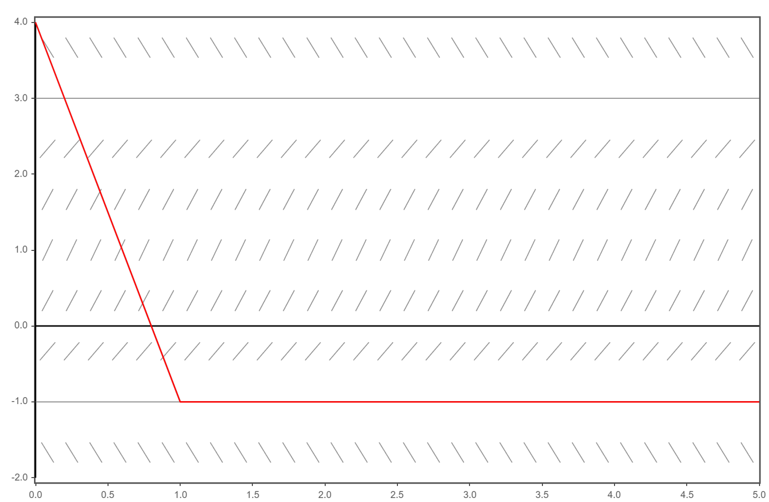
Euler's Methods

Problem 2. Use Euler's method to approximate the solution, then sketch the graph of $y(t)$ to visualize the results. Perform this calculation manually, without using any computational tools.

1. $\dfrac{dy}{dt} = (3-y)(y+1)$, $y(0) = 4$, $0 \le t \le 5$, and $\Delta t = 1$.

   Solution. $y_{k+1} = y_k + (3-y_k)(y_k+1)$

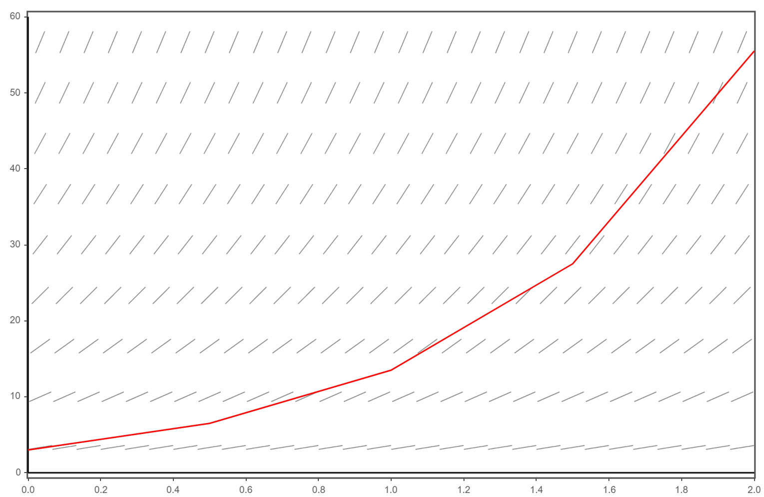
k	t_k	y_k
0	0	4
1	1	-1
2	2	-1
3	3	-1
4	4	-1
5	5	-1



2. $\dfrac{dy}{dt} = 2y + 1$, $y(0) = 3$, $0 \le t \le 2$, and $\Delta t = 0.5$.

Solution. $y_{k+1} = y_k + 0.5(2y_k + 1) = 2y_k + 0.5$.

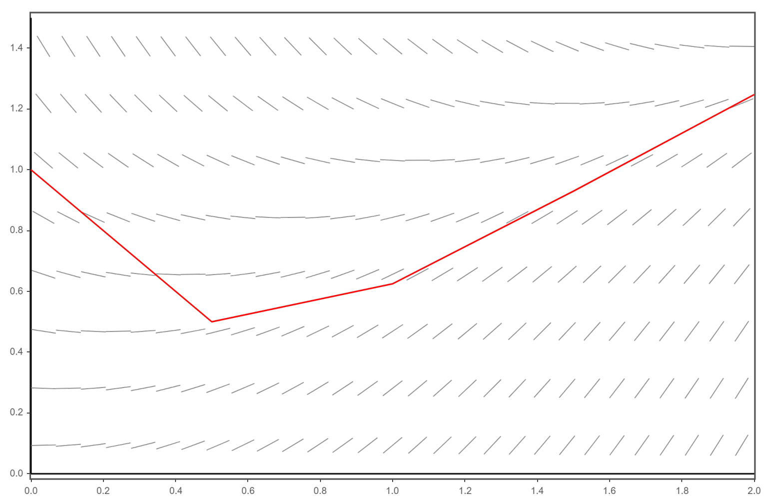
k	t_k	y_k
0	0	3
1	0.5	6.5
2	1	13.5
3	1.5	27.5
4	2	55.5



3. $\dfrac{dy}{dt} = t - y^2$, $y(0) = 1$, $0 \le t \le 1$, and $\Delta t = 0.25$.

Solution. $y_{k+1} = y_k + 0.5(t_k - y_k^2)$

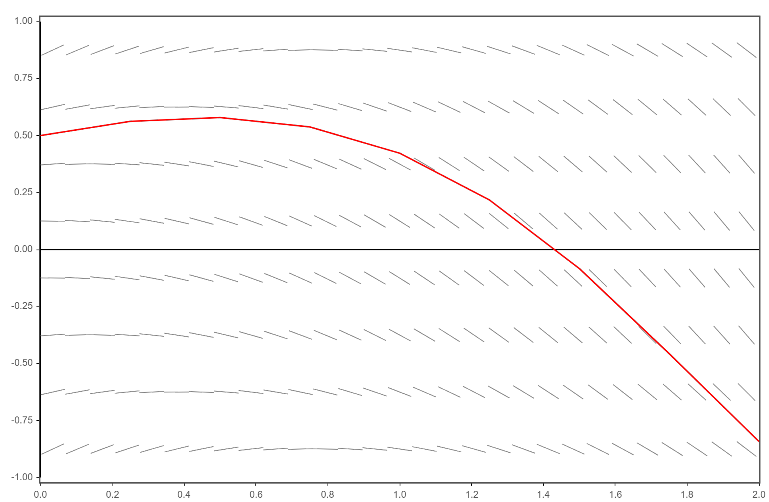
k	t_k	y_k
0	0	1
1	0.5	0.5
2	1	0.625
3	1.5	0.9297
4	2	1.2475



4. $\dfrac{dy}{dt} = y^2 - t$, $y(0) = 0.5$, $0 \le t \le 2$, and $\Delta t = 0.25$.

Solution. $y_{k+1} = y_k + 0.5(y_k^2 - t_k)$

k	t_k	y_k
0	0	0.5
1	0.25	0.5625
2	0.5	0.5791
3	0.75	0.5379
4	1	0.4228
5	1.25	0.2175
6	1.5	-0.0832
7	1.75	-0.4565
8	2	-0.8419



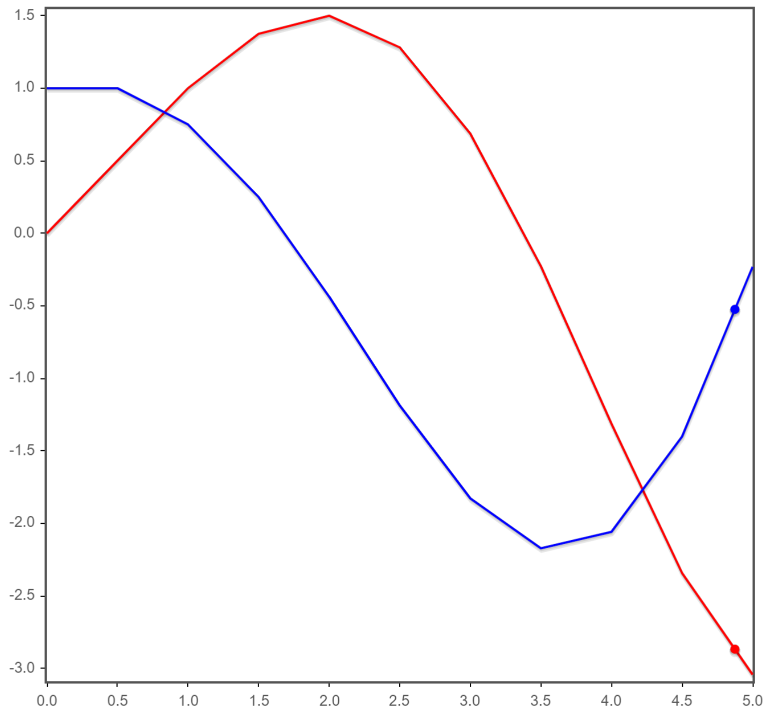
Problem 3. Use Euler's method to approximate the solution, then sketch the graphs of $x(t)$ and $y(t)$ to visualize the results. Perform this calculation manually, without using any computational tools.

1. Initial condition $(x(0), y(0)) = (0, 1)$, a step size of $\Delta t = 0.5$ and $0 \leq t \leq 5$:

$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = -x,$$

Solution. $x_{k+1} = x_k + 0.5y_k$, $y_{k+1} = y_k - 0.5x_k$

k	t_k	x_k	y_k
0	0	1	1
1	0.5	0.5	1
2	1	1	0.75
3	1.5	1.375	0.25
4	2	1.5	-0.4375
5	2.5	1.2823	-1.1875
6	3	0.6875	-1.8281
7	3.5	-0.2266	-2.1719
8	4	-1.3125	-2.0585
9	4.5	-2.3418	-1.4024
10	5	3.04297	-0.23145

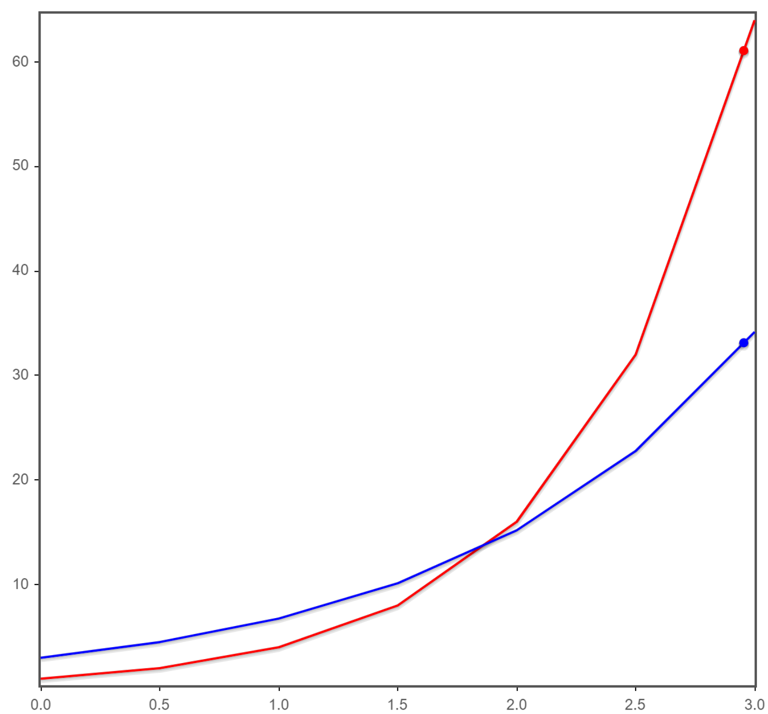


2. Initial condition $(x(0), y(0)) = (1, 3)$ and a step size of $\Delta t = 0.5$ and $0 \leq t \leq 5$:

$$\frac{dx}{dt} = 2x,$$
$$\frac{dy}{dt} = y,$$

Solution. $x_{k+1} = x_k + (0.5)(2x_k) = 2x_k$, $y_{k+1} = y_k + 0.5y_k = 1.5y_k$.

k	t_k	x_k	y_k
0	0	1	3
1	0.5	2	4.5
2	1	4	6.75
3	1.5	8	10.125
4	2	16	15.1875
5	2.5	32	22.78125
6	3	64	34.17188



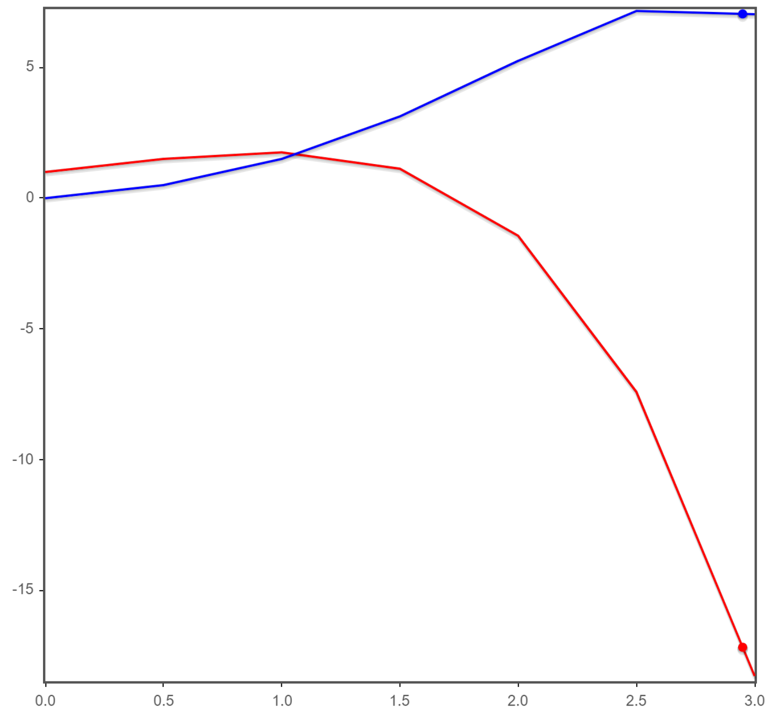
3. Initial condition $(x(0), y(0)) = (1, 0)$ and a step size of $\Delta t = 0.5$ and

$0 \leq t \leq 3$:

$$\frac{dx}{dt} = x - 2y$$
$$\frac{dy}{dt} = x + y$$

Solution. $x_{k+1} = x_k + (0.5)(x_k - 2y_k) = 1.5x_k - y_k$, $y_{k+1} = y_k + 0.5(x_k + y_k) = 0.5x_k + 1.5y_k$.

k	t_k	x_k	y_k
0	0	1	0
1	0.5	1.5	0.5
2	1	1.75	1.5
3	1.5	1.125	3.125
4	2	-1.4375	5.25
5	2.5	-7.40625	7.15625
6	3	-18.26563	7.03125

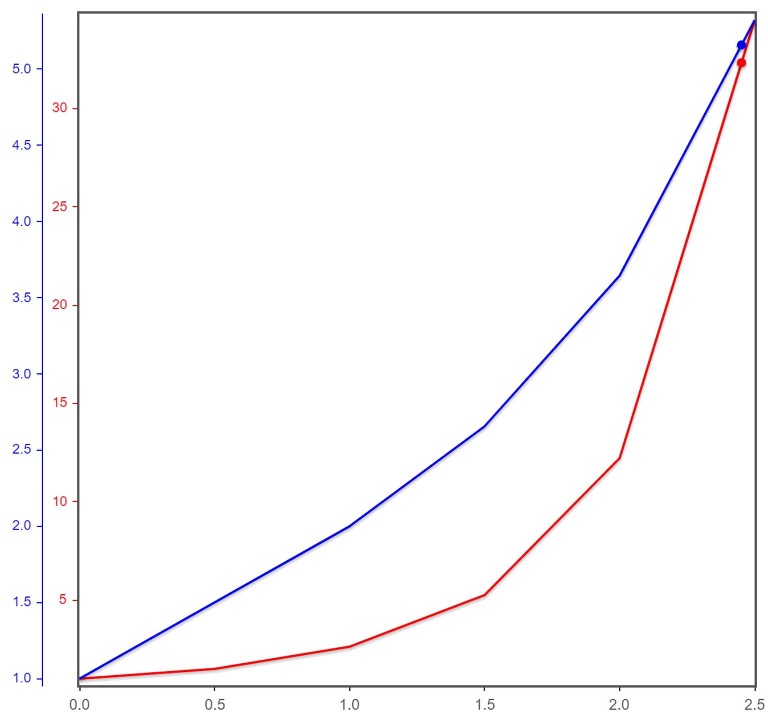


4. Initial condition $(x(0), y(0)) = (1, 1)$ and a step size of $\Delta t = 0.5$ and $0 \leq t \leq 2.5$:

$$\frac{dx}{dt} = xy$$
$$\frac{dy}{dt} = \frac{x}{y}$$

Solution. $x_{k+1} = x_k + (0.5)(x_k y_k)$, $y_{k+1} = y_k + (0.5)\frac{x_k}{y_k}$.

k	t_k	x_k	y_k
0	0	1	1
1	0.5	1.5	1.5
2	1	2.625	2
3	1.5	5.25	2.65625
4	2	12.2227	3.6445
5	2.5	34.4954	5.3214



Bifurcations

Problem 4. For each of the following one-parameter families

$$\frac{dy}{dt} = y^2 - 10y + a$$
$$\frac{dy}{dt} = y^2 - ay + 1$$
$$\frac{dy}{dt} = \alpha - |y|$$
$$\frac{dy}{dt} = (y^2 - \alpha)(y^2 - 4)$$
$$\frac{dy}{dt} = y^3 + ay^2 + y$$

1. Locate the bifurcation values.

2. Draw the bifurcation diagram. These diagrams should depict how the equilibrium points evolve as the parameter changes. Specifically, illustrate the phase lines for the system when the parameter is slightly less than, exactly equal to, and slightly more than the bifurcation values.

3. On your bifurcation diagrams, clearly indicate each equilibrium point and categorize them as sources, sinks, or nodes

Solution.

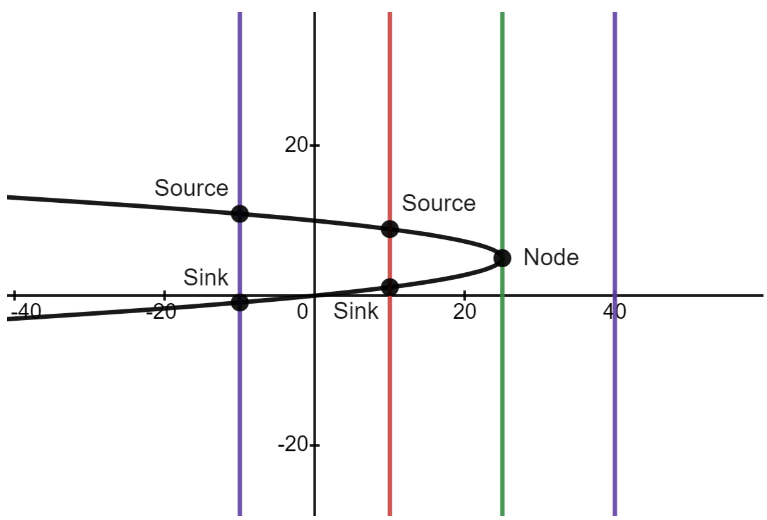
- $\dfrac{dy}{dt} = y^2 - 10y + a$: Solve the equilibria in terms of a.

$$y^2 - 10y + a = 0 \Rightarrow y = 5 \pm \sqrt{25 - a}$$

* ⋆ If $25 - a < 0$, meaning that $a < 25$, the DE has no equilibrium solutions.
* ⋆ If $25 - a = 0$, meaning that $a = 25$, the DE has one equilibrium solution $y = 5$.
* ⋆ If $25 - a > 0$, meaning that $a < 25$, the DE has two equilibrium solutions $y = 5 \pm \sqrt{25 - a}$.

Conclusion: The bifurcation values are $a = 15$.

Bifurcation Diagram: We draw the phase lines at $a = -10, 10, 25, 40$.



- $\dfrac{dy}{dt} = \alpha - |y|$: Solve the equilibria in terms of α.

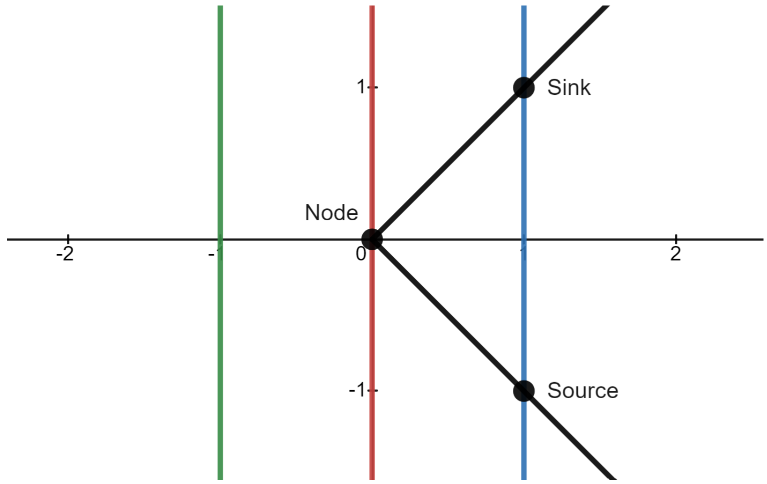
$$\alpha - |y| = 0 \Rightarrow |y| = \alpha$$

  - If $\alpha < 0$, the DE has no equilibrium solutions.
  - If $\alpha = 0$, the DE has one equilibrium solution $y = 0$.

– If $\alpha > 0$, the DE has two equilibrium solutions $y = \pm\alpha$.

Conclusion: The bifurcation value is $\alpha = 0$.

Bifurcation Diagram: We draw the phase lines at $\alpha = -1, 0, 1$.



- $\dfrac{dy}{dt} = y^2 - ay + 1$: Solve the equilibria in terms of a.

$$y^2 - ay + 1 = 0 \Rightarrow y = \frac{a \pm \sqrt{a^2 - 4}}{2}$$

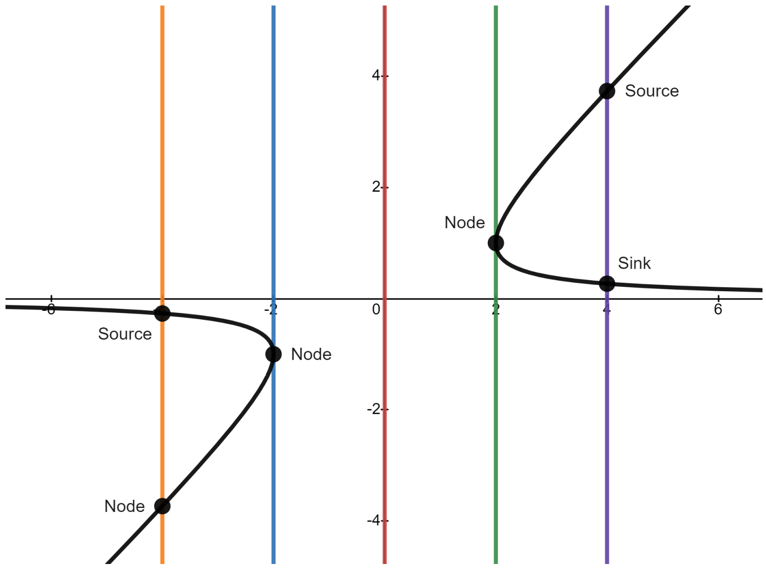
  ⋆ If $a^2 - 4 < 0$, meaning that $-2 < a < 2$, the DE has no equilibrium solutions.

  ⋆ If $a^2 - 4 = 0$, meaning that $a = \pm 2$, the DE has one equilibrium solution $y = \frac{a}{2}$.

  ⋆ If $a^2 - 4 > 0$, meaning that $a > 2$ or $a < -2$, the DE has two equilibrium solutions $y = \frac{a \pm \sqrt{a^2 - 4}}{2}$.

Conclusion: The bifurcation values are $\alpha = \pm 2$.

Bifurcation Diagram: We draw the phase lines at $\alpha = -4, -2, 0, 2, 4$.



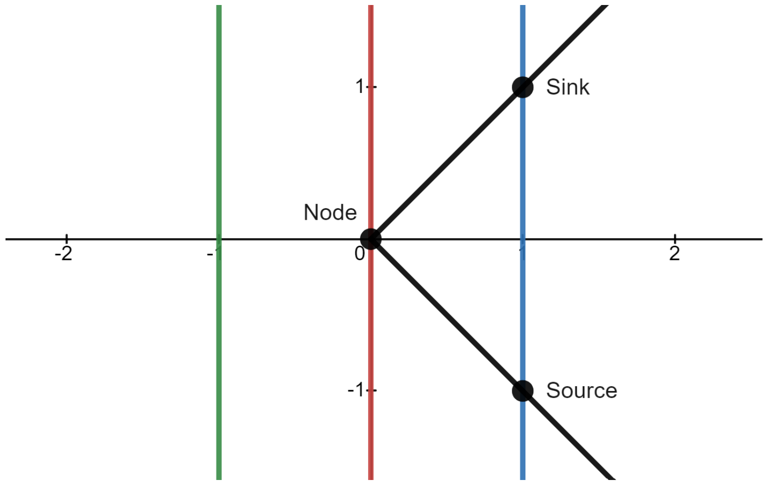
- $\dfrac{dy}{dt} = \alpha - |y|$: Solve the equilibria in terms of α.

$$\alpha - |y| = 0 \Rightarrow |y| = \alpha$$

  - If $\alpha < 0$, the DE has no equilibrium solutions.
  - If $\alpha = 0$, the DE has one equilibrium solution $y = 0$.
  - If $\alpha > 0$, the DE has two equilibrium solutions $y = \pm\alpha$.

Conclusion: The bifurcation value is $\alpha = 0$.

Bifurcation Diagram: We draw the phase lines at $\alpha = -1, 0, 1$.



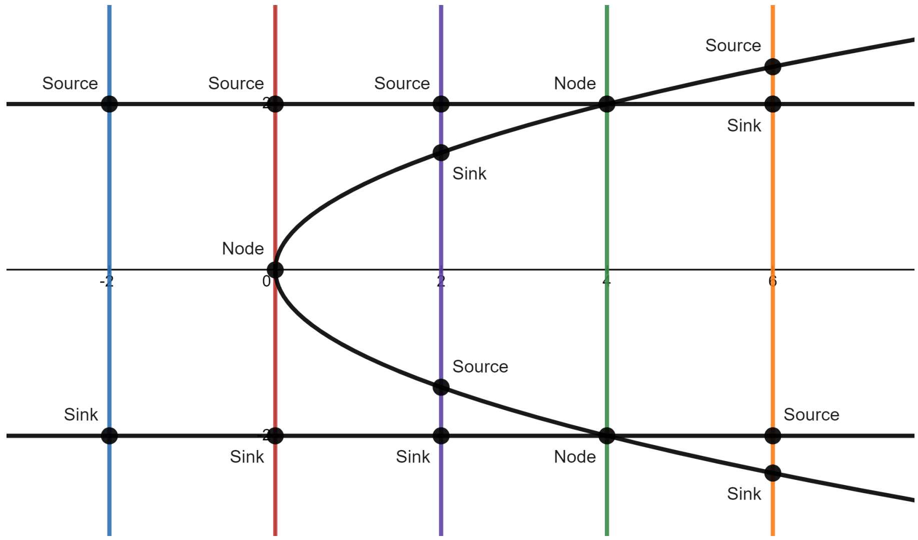
- $\dfrac{dy}{dt} = (y^2 - \alpha)(y^2 - 4)$: Solve the equilibria in terms of a.

$$(y^2 - \alpha)(y^2 - 4) = 0 \Rightarrow y = \pm\sqrt{\alpha}, \pm 2$$

  - If $\alpha < 0$, the DE has two equilibrium solutions $y = \pm 2$.
  - If $\alpha = 0$, the DE has three equilibrium solutions $y = 0, \pm 2$.
  - If $0 < \alpha < 4$, the DE has four equilibrium solutions $y = \pm\sqrt{\alpha}, \pm 2$.
  - If $\alpha = 4$, the DE has two equilibrium solutions $y = \pm 2$.
  - If $4 < \alpha$, the DE has four equilibrium solutions $y = \pm\sqrt{\alpha}, \pm 2$.

Conclusion: The bifurcation values are $\alpha = 0, 4$.

Bifurcation Diagram: We draw the phase lines at $\alpha = -2, 0, 2, 4, 6$.



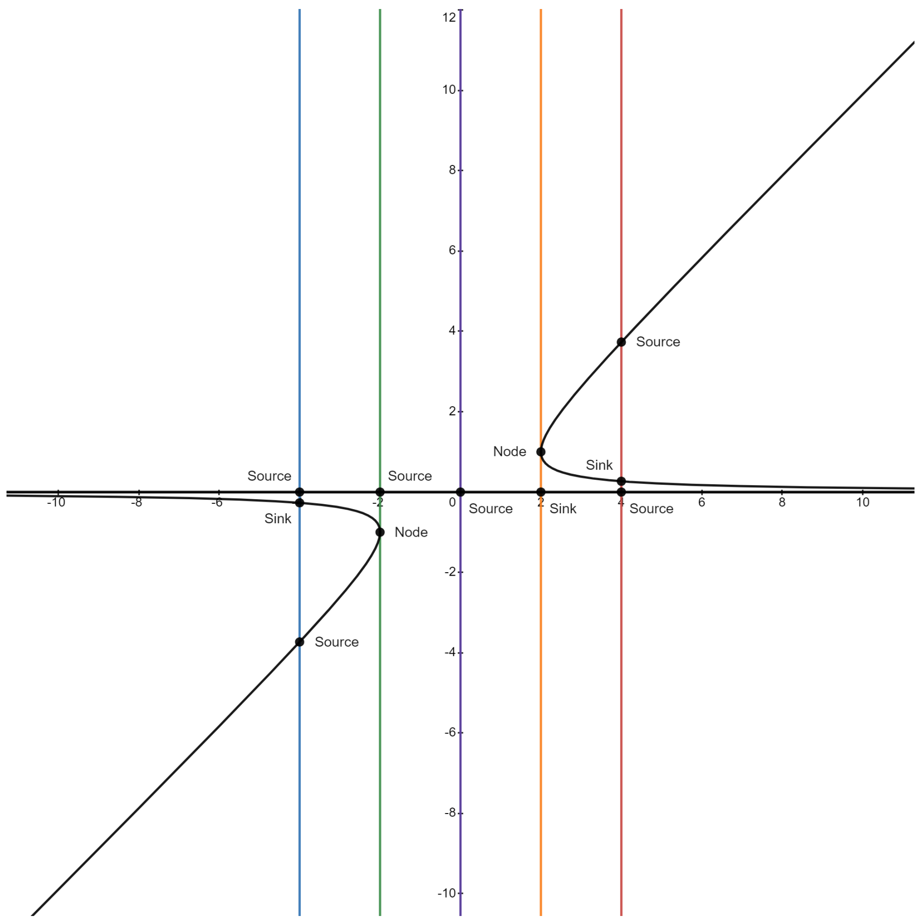
- $\dfrac{dy}{dt} = y^3 + ay^2 + y$: Solve the equilibria in terms of a.

$$y^3 + ay^2 + y = 0 \Rightarrow y(y^2 + ay + 1) = 0$$
$$\Rightarrow y = 0, \frac{a \pm \sqrt{a^2 - 4}}{2}$$

  - If $a^2 - 4 < 0$, meaning that $-2 < a < 2$ the DE has one equilibrium solutions $y = 0$.
  - If $a^2 - 4 = 0$, meaning that ± 2, the DE has two solutions.
    * If $a = 2$, $y = 0, 1$.
    * If $a = -2$, $y = 0, -1$.
  - If $a^2 - 4 > 0$, meaning that $a > 2$ or $a < -2$, the DE has three equilibrium solutions $y = 0, \dfrac{a \pm \sqrt{a^2 - 4}}{2}$.

Conclusion: The bifurcation values are $a = \pm 2$.

Bifurcation Diagram: We draw the phase lines at $\alpha = -4, -2, 0, 2, 4$.



Analytic Methods for First-Order Differential Equations

Problem 5. (Method of Undermined Coefficients) By solving $y_h(t)$ and $y_p(t)$, determine the general solution of the differential equations (DEs) or the specific solutions for the initial value problems (IVPs) provided.:

1. $\dfrac{dy}{dt} = -4y + 3e^{-t}$.

   Solution.

   - **Homogeneous Equation:**

   $$\frac{dy}{dt} = -4y \Rightarrow y_h(t) = Ce^{-4t}$$

   - **Guess:** $b(t) = 3e^{-t}$. Since the exponential of $y_h(t)$ and $b(t)$ are different, we guess $y_p(t) = ce^{-t}$. Substitute it into the DE to solve for c:

   $$\begin{aligned}
   \frac{dy}{dt} &= -4y + 3e^{-t} \\
   \frac{d}{dt}(ce^{-t}) &= (-4)(ce^{-t}) + 3e^{-t} \\
   -ce^{-t} &= -4ce^{-t} + 3e^{-t} \\
   3ce^{-t} &= 3e^{-t} \\
   c &= 1.
   \end{aligned}$$

   Hence, $y_p(t) = e^{-t}$.

- **Combine:** The general solution is $y(t) = Ce^{-4t} + e^{-t}$.

2. $\dfrac{dy}{dt} = 2y + \sin(2t)$.

   Solution.

   - **Homogeneous Equation:**

   $$\frac{dy}{dt} = 2y \Rightarrow y_h(t) = Ce^{2t}$$

   - **Guess:** $b(t) = \sin(2t)$. We guess $y_p(t) = A\sin(2t) + B\cos(2t)$. Substitute it into the DE to solve for A and B:

   $$\begin{aligned} \frac{dy}{dt} &= 2y + \sin(2t) \\ \frac{d}{dt}(A\sin(2t) + B\cos(2t)) &= 2(A\sin(2t) + B\cos(2t)) + \sin(2t) \\ 2A\cos(2t) - 2B\sin(2t) &= (2A+1)\sin(2t) + 2B\cos(2t). \end{aligned}$$

   Identifying $\cos(2t)$ and $\sin(2t)$ on both sides of the equation,

   $$\begin{array}{rcl} 2A & = & 2B \\ 2A+1 & = & -2B \end{array} \Rightarrow A = B = -\frac{1}{4}.$$

   Therefore, $y_p(t) = -\dfrac{1}{4}\sin(2t) - \dfrac{1}{4}\cos(2t)$.

   - **Combine:** The general solution is $y(t) = Ce^{2t} - \frac{1}{4}\sin(2t) - \frac{1}{4}\cos(2t)$.

3. $\dfrac{dy}{dt} + 2y = 3t^2 + 2t - 1.$

Solution.

- **Homogeneous Equation:**

$$\frac{dy}{dt} = -2y \Rightarrow y_h(t) = Ce^{-2t}$$

- **Guess:** $b(t) = 3t^2 + 2t - 1$. We guess $y_p(t) = At^2 + Bt + c$. Substitute it into the DE to solve for A and B:

$$\begin{aligned} \frac{dy}{dt} + 2y &= 3t^2 + 2t - 1 \\ \left[\frac{d}{dt}(At^2 + Bt + C)\right] + 2(At^2 + Bt + C) &= 3t^2 + 2t - 1 \\ 2At^2 + (2A + B)t + (B + 2C) &= 3t^2 + 2t - 1. \end{aligned}$$

Identifying t^2, t, and the constant terms on both sides of the equation,

$$\begin{array}{rclcrcl} 2A &=& 3 & & A &=& \frac{3}{2} \\ 2A + 2B &=& 2 & \Rightarrow & B &=& \frac{-1}{2} \\ B + 2C &=& -1 & & C &=& \frac{1}{4} \end{array}$$

Therefore, $y_p(t) = \dfrac{3}{2}t^2 - \dfrac{1}{2}t + \dfrac{1}{4}$.

- **Combine:** The general solution is $y(t) = Ce^{-2t} + \frac{3}{2}t^2 - \frac{1}{2}t + \frac{1}{4}$.

4. $\dfrac{dy}{dt} + y = \cos(2t),\ y(0) = 5.$

Solution.

- **Homogeneous Equation:**

$$\frac{dy}{dt} = -y \Rightarrow y_h(t) = Ce^{-t}$$

- **Guess:** $b(t) = \cos(2t)$. We guess $y_p(t) = A\sin(2t) + B\cos(2t)$. Substitute it into the DE to solve for A and B:

$$\begin{aligned}\frac{dy}{dt} &= -y + \cos(2t)\\ \frac{d}{dt}(A\sin(2t) + B\cos(2t)) &= -(A\sin(2t) + B\cos(2t)) + \cos(2t)\\ 2A\cos(2t) - 2B\sin(2t) &= -A\sin(2t) + (-B+1)\cos(2t).\end{aligned}$$

Identifying $\cos(2t)$ and $\sin(2t)$ on both sides of the equation,

$$\begin{aligned}-2B &= -A\\ 2A &= -B+1\end{aligned} \Rightarrow A = \frac{2}{5}, B = \frac{1}{5}$$

Therefore, $y_p(t) = \dfrac{2}{5}\sin(2t) + \dfrac{1}{5}\cos(2t)$.

- **Combine:** The general solution is $y(t) = Ce^{-t} + \frac{2}{5}\sin(2t) + \frac{1}{5}\cos(2t)$.

- **Initial Value:**

$$
\begin{aligned}
y(0) = 5 \Rightarrow C + \frac{1}{5} &= 5 \\
\Rightarrow C &= \frac{24}{5}
\end{aligned}
$$

The solution for the IVP is $y(t) = \frac{24}{5}e^{-t} + \frac{2}{5}\sin(2t) + \frac{1}{5}\cos(2t)$.

5. $\dfrac{dy}{dt} - 2y = 7e^{2t},\ y(0) = 3.$

Solution.

- **Homogeneous Equation:**

$$\frac{dy}{dt} = 2y \Rightarrow y_h(t) = Ce^{2t}$$

- **Guess:** $b(t) = 7e^{2t}$. Since the exponential of $y_h(t)$ and $b(t)$ are the same as e^{2t}, we guess $y_p(t) = cte^{2t}$. Substitute it into the DE to solve for c:

$$
\begin{aligned}
\frac{dy}{dt} &= 2y + 7e^{2t} \\
\frac{d}{dt}(cte^{2t}) &= (2)(cte^{2t}) + 7e^{2t} \\
ce^{2t} + 2cte^{2t} &= 2cte^{2t} + 7e^{2t} \\
ce^{2t} &= 7e^{2t} \\
c &= 7.
\end{aligned}
$$

Hence, $y_p(t) = 7te^{2t}$.

- **Combine:** The general solution is $y(t) = Ce^{2t} + 7te^{2t}$.

- **Initial Value:**

$$y(0) = 3 \Rightarrow C = 3.$$

The solution for the IVP is $y(t) = 3e^{2t} + 7te^{2t}$.

6. $\dfrac{dy}{dt} - 3y = 4\cos(t)e^{2t},\ y(0) = 4.$

Solution.

- **Homogeneous Equation:**

$$\frac{dy}{dt} = 3y \Rightarrow y_h(t) = Ce^{3t}$$

- **Guess:** $b(t) = 4\cos(t)e^{2t}$. We guess $y_p(t) = e^{2t}(A\cos(t) + B\sin(t))$.

  Substitute it into the DE to solve for A and B:

$$\begin{aligned}
&\frac{dy}{dt} = 3y + 4\cos(t)e^{2t} \\
\Rightarrow &\frac{d}{dt}(e^{2t}(A\cos(t) + B\sin(t))) = (3)(e^{2t}(A\cos(t) + B\sin(t))) + 4\cos(t)e^{2t} \\
\Rightarrow &(2A + B)e^{2t}\cos(t) + (-A + 2B)e^{2t}\sin(t) = (3A + 4)e^{2t}\cos(t) + 3Be^{2t}\sin(t)
\end{aligned}$$

$$\begin{aligned} 2A + B &= 3A + 4 \\ -A + 2B &= 3B \end{aligned} \Rightarrow A = -2,\ B = 2$$

Hence, $y_p(t) = -2e^{2t}\cos(t) + 2e^{2t}\sin(t)$.

- **Combine:** The general solution is $y(t) = Ce^{3t} - 2e^{2t}\cos(t) + 2e^{2t}\sin(t)$.

- **Initial Value:**

$$y(0) = 4 \Rightarrow C = 6.$$

The solution for the IVP is $y(t) = 6e^{3t} - 2e^{2t}\cos(t) + 2e^{2t}\sin(t)$.

7. $\dfrac{dy}{dt} + y = \cos(2t) + 3\sin(2t) + e^{-t},\ y(0) = 0.$
Solution.

- **Homogeneous Equation:**

$$\frac{dy}{dt} = -y \Rightarrow y_h(t) = Ce^{-t}$$

- **Guess:** $b(t) = \cos(2t) + 3\sin(2t) + e^{-t}$. We guess $y_p(t) = A\sin(2t) + B\cos(2t) + Cte^{-t}$. Substitute it into the DE to solve for A, B, and c:

$$
\begin{aligned}
\frac{dy}{dt} &= -y + \cos(2t) \\
\frac{d}{dt}(A\sin(2t) + B\cos(2t) + Cte^{-t}) &= -(A\sin(2t) + B\cos(2t) + Cte^{-t}) \\
&\quad + \cos(2t) + 3\sin(2t) + e^{-t} \\
2A\cos(2t) - 2B\sin(2t) + Ce^{-t} - Cte^{-t} &= (-A+3)\sin(2t) \\
&\quad + (-B+1)\cos(2t) - Cte^{-t} + e^{-t}.
\end{aligned}
$$

Identifying $\cos(2t)$, $\sin(2t)$, and e^{-t} on both sides of the equation,

$$
\begin{aligned}
2A &= -B+1 \\
-2B &= -A+3 \quad \Rightarrow A = \frac{2}{5},\ B = \frac{1}{5},\ C = 1 \\
C &= 1
\end{aligned}
$$

Therefore, $y_p(t) = \dfrac{2}{5}\sin(2t) + \dfrac{1}{5}\cos(2t) + te^{-t}$.

- **Combine:** The general solution is $y(t) = Ce^{-t} + \frac{2}{5}\sin(2t) + \frac{1}{5}\cos(2t) + te^{-t}$.
- **Initial Value:**

$$
\begin{aligned}
y(0) = 0 &\Rightarrow C + \frac{1}{5} = 0 \\
&\Rightarrow C = -\frac{1}{5}
\end{aligned}
$$

The solution for the IVP is $y(t) = -\frac{1}{5}e^{-t} + \frac{2}{5}\sin(2t) + \frac{1}{5}\cos(2t) + te^{-t}$.

Problem 6. (Method of Integrating Factors) Using the method of integrating factors, Determine the general solution of the differential equations (DEs) or the specific solutions for the initial value problems (IVPs) provided.

1. $\dfrac{dy}{dt} = -\dfrac{y}{1+t} + t^2$.

   Solution.

   - **Standard form:** $\dfrac{dy}{dt} + \dfrac{y}{1+t} = t^2$. Hence, $g(t) = \frac{1}{1+t}$, and $b(t) = t^2$.

   - **Integrating Factor:**

$$\begin{aligned} \mu(t) &= e^{\int \frac{1}{1+t}\,dt} \\ &= e^{\ln(1+t)} \\ &= 1+t \end{aligned}$$

   - **The Integrand:**

$$\begin{aligned} \int \mu(t)b(t)\,dt &= \int (1+t)t^2\,dt \\ &= \int (t^2+t^3)\,dt \\ &= \frac{t^3}{3} + \frac{t^4}{4} + C \end{aligned}$$

- **The General Solution:**

$$\begin{aligned}
y(t) &= \frac{1}{\mu(t)} \int \mu(t) b(t)\, dt \\
&= \frac{1}{t+1}\left(\frac{t^3}{3} + \frac{t^4}{4} + C\right) \\
y(t) &= \frac{t^3}{3(t+1)} + \frac{t^4}{4(t+1)} + \frac{C}{t+1}
\end{aligned}$$

2. $\dfrac{dy}{dt} + 3t^2 y = e^t(3t^2+1)$.

   Solution.

   - **Standard form:** $\dfrac{dy}{dt} + 3t^2 y = e^t(3t^2+1)$. Hence, $g(t) = 3t^2$, and $b(t) = e^t(3t^2+1)$.

   - **Integrating Factor:**

   $$\begin{aligned}
   \mu(t) &= e^{\int 3t^2\, dt} \\
   &= e^{t^3}
   \end{aligned}$$

   - **The Integrand:**

   $$\begin{aligned}
   \int \mu(t) b(t)\, dt &= \int e^{t^3} e^t (3t^2+1)\, dt \\
   &= \int e^{t^3+t}(3t^2+1)\, dt
   \end{aligned}$$

Substitution: $u = t^3 + t$. Hence,

$$\int \mu(t)b(t)\,dt = e^{t^3+t} + C$$

- **The General Solution:**

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)} \int \mu(t)b(t)\,dt \\ &= \frac{1}{e^{t^3}}\left(e^{t^3+t} + C\right) \\ y(t) &= e^t + Ce^{-t^3}. \end{aligned}$$

3. $\dfrac{dy}{dt} + \dfrac{2}{t}y = \dfrac{\cos(t)}{t}$.

   Solution.

   - **Standard form:** $\dfrac{dy}{dt} + \dfrac{2}{t}y = \dfrac{\cos(t)}{t}$. Hence, $g(t) = \frac{2}{t}$, and $b(t) = \frac{\cos(t)}{t}$.

   - **Integrating Factor:**

$$\begin{aligned} \mu(t) &= e^{\int \frac{2}{t}\,dt} \\ &= t^2 \end{aligned}$$

- **The Integrand:**

$$\int \mu(t)b(t)\,dt = \int (t^2)\left(\frac{\cos(t)}{t}\right)\,dt$$
$$= \int t\cos(t)\,dt$$

Integration by Parts: $u = t$, $dv = \cos(t)\,dt$, $du = dt$, and $v = \sin(t)$. Hence,

$$\int \mu(t)b(t)\,dt = t\sin(t) - \int \sin(t)\,dt = t\sin(t) + \cos(t) + C$$

- **The General Solution:**

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)b(t)\,dt$$
$$= \frac{1}{t^2}\left(t\sin(t) + \cos(t) + C\right)$$
$$y(t) = \frac{\sin(t)}{t} + \frac{\cos(t)}{t^2} + \frac{C}{t^2}.$$

4. $2\dfrac{dy}{dt} + y = 3t^2.$

Solution.

- **Standard form:** $\dfrac{dy}{dt} + \dfrac{1}{2}y = 3t^2$. Hence, $g(t) = \frac{1}{2}$, and $b(t) = 3t^2$.

- **Integrating Factor:**

$$\mu(t) = e^{\int \frac{1}{2}\,dt}$$
$$= e^{t/2}$$

- **The Integrand:** Using Integration by Parts twice

$$\begin{aligned}
\int \mu(t)b(t)\,dt &= \int 2t^2e^{t/2}\,dt \\
&= 4t^2e^{t/2} - 8\int te^{t/2}\,dt \\
&= 4t^2e^{t/2} - 8\left(2te^{t/2} - 2\int e^{t/2}\,dt\right) \\
&= 4t^2e^{t/2} - 16te^{t/2} + 32e^{t/2} + C
\end{aligned}$$

- **The General Solution:**

$$\begin{aligned}
y(t) &= \frac{1}{\mu(t)}\int \mu(t)b(t)\,dt \\
&= \frac{1}{e^{t/2}}\left(4t^2e^{t/2} - 16te^{t/2} + 32e^{t/2} + C\right) \\
y(t) &= 4t^2 - 16t + 32 + Ce^{-t/2}.
\end{aligned}$$

5. $\dfrac{dy}{dt} = \dfrac{1}{1+t}y + 4t^2 + 4t,\ y(1) = 10.$

Solution.

- **Standard form:** $\dfrac{dy}{dt} - \dfrac{1}{1+t}y = 4t^2 + 4t$. Hence, $g(t) = -\frac{1}{1+t}$, and $b(t) = 4t^2 + 4t$.

- **Integrating Factor:**

$$\begin{aligned} \mu(t) &= e^{\int -\frac{1}{1+t}\,dt} \\ &= e^{-\ln(1+t)} \\ &= \frac{1}{1+t} \end{aligned}$$

- **The Integrand:**

$$\begin{aligned} \int \mu(t)b(t)\,dt &= \int \frac{1}{1+t}(4t^2+4t)\,dt \\ &= \int 4t\,dt \\ &= 2t^2 + C \end{aligned}$$

- **The General Solution:**

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)} \int \mu(t)b(t)\,dt \\ &= (t+1)\left(2t^2 + C\right) \end{aligned}$$

- **Initial Value:**

$$y(1) = 10 \Rightarrow 2(C+2) = 10$$
$$\Rightarrow C = 3$$

The solution for the IVP is $y(t) = (t+1)\left(2t^2+3\right)$.

6. $\dfrac{dy}{dt} - \dfrac{2}{t}y = 2t^2,\ y(-2) = 4.$

Solution.

- **Standard form:** $\dfrac{dy}{dt} - \dfrac{2}{t}y = 2t^2$. Hence, $g(t) = -\frac{2}{t}$, and $b(t) = 2t^2$.

- **Integrating Factor:**

$$\begin{aligned}\mu(t) &= e^{\int -\frac{2}{t}\,dt} \\ &= e^{-2\ln(t)} \\ &= \frac{1}{t^2}\end{aligned}$$

- **The Integrand:**

$$\begin{aligned}\int \mu(t)b(t)\,dt &= \int \frac{1}{t^2}(2t^2)\,dt \\ &= \int 2\,dt \\ &= 2t + C\end{aligned}$$

- **The General Solution:**

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)} \int \mu(t) b(t)\, dt \\ &= t^2 \left(2t + C\right) \\ &= 2t^3 + Ct^2 \end{aligned}$$

- **Initial Value:**

$$\begin{aligned} y(-2) = 4 &\Rightarrow -16 + 4C = 4 \\ &\Rightarrow C = 5 \end{aligned}$$

The solution for the IVP is $y(t) = 2t^3 + 5t^2$.

7. $\dfrac{dy}{dt} - y = 2te^{2t},\ y(0) = 1.$

Solution.

- **Standard form:** $\dfrac{dy}{dt} - y = 2te^{2t}$. Hence, $g(t) = -1$, and $b(t) = 2te^{2t}$.

- **Integrating Factor:**

$$\begin{aligned} \mu(t) &= e^{\int (-1)\, dt} \\ &= e^{-t} \end{aligned}$$

- **The Integrand:** Using Integration by Parts:

$$
\begin{aligned}
\int \mu(t)b(t)\,dt &= \int e^{-t}(2te^{2t})\,dt \\
&= \int 2te^{t}\,dt \\
&= 2te^{t} - 2\int e^{t}\,dt \\
&= 2te^{t} - e^{t} + C
\end{aligned}
$$

- **The General Solution:**

$$
\begin{aligned}
y(t) &= \frac{1}{\mu(t)} \int \mu(t)b(t)\,dt \\
&= \frac{1}{e^{-t}}\left(2te^{t} - e^{t} + C\right) \\
y(t) &= te^{2t} - e^{2t} + Ce^{t}.
\end{aligned}
$$

- **Initial Value:**

$$
\begin{aligned}
y(0) = 1 &\Rightarrow C - 1 = 1 \\
&\Rightarrow C = 2
\end{aligned}
$$

The solution for the IVP is $y(t) = te^{2t} - e^{2t} + 2e^{t}$.

8. $t\dfrac{dy}{dt} + (t+1)y = 2te^{-t},\ y(0) = 0.$

Solution.

- **Standard form:** $\dfrac{dy}{dt} + (1 + \dfrac{1}{t})y = 2e^{-t}$. Hence, $g(t) = 1 + \frac{1}{t}$, and $b(t) = 2e^{-t}$.

- **Integrating Factor:**

$$\begin{aligned}\mu(t) &= e^{\int 1+\frac{1}{t}\,dt} \\ &= e^{t+\ln(t)} \\ &= te^{t}\end{aligned}$$

- **The Integrand:**

$$\begin{aligned}\int \mu(t)b(t)\,dt &= \int (te^{t})(2te^{-t})\,dt \\ &= \int 2t\,dt \\ &= t^2 + C\end{aligned}$$

- **The General Solution:**

$$\begin{aligned}y(t) &= \frac{1}{\mu(t)} \int \mu(t)b(t)\,dt \\ &= \frac{1}{te^{t}} \left(t^2 + C\right) \\ &= te^{-t} + \frac{Ce^{-t}}{t}\end{aligned}$$

- **Initial Value:**

$$y(0) = 0 \Rightarrow C = 0$$

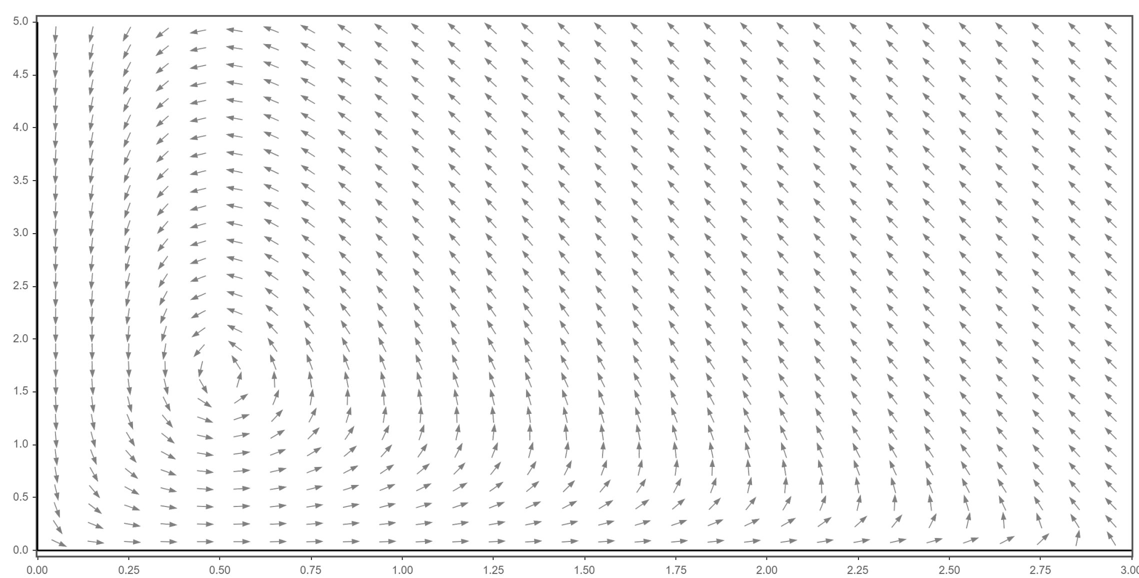
The solution for the IVP is $y(t) = te^{-t}$.

More on Systems of Differential Equations

Problem 7. Analyze the fates for the prey (R) and predator (F) populations in each of the following predator-prey systems given their respective initial conditions and diretion fields.

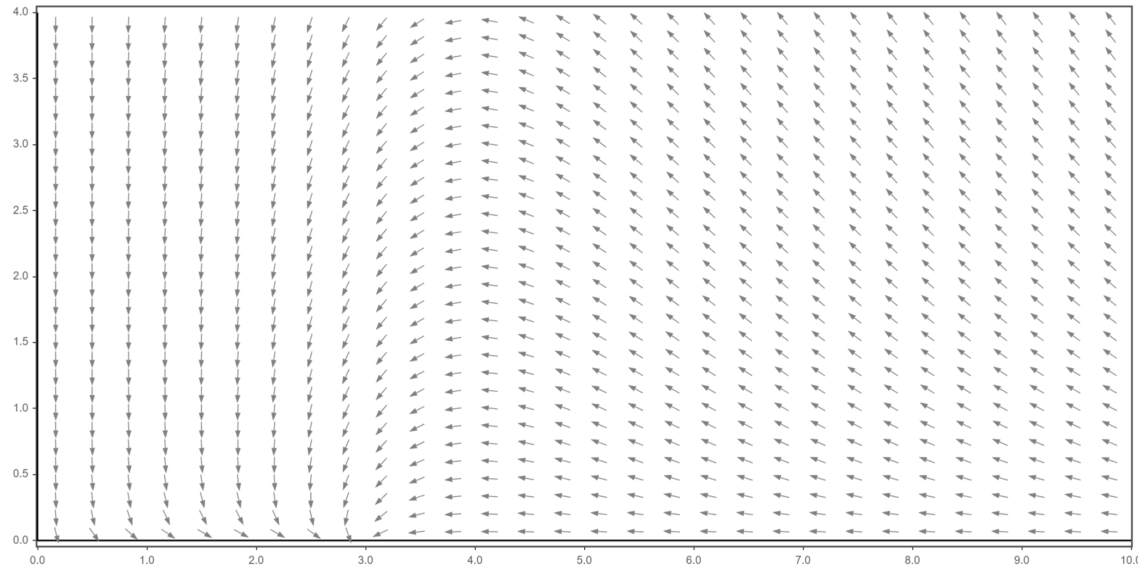
1. Initial value $(R(0), F(0)) = (1, 0.3)$ and the system is

$$\frac{dR}{dt} = 2(1 - \frac{R}{3})R - RF$$
$$\frac{dF}{dt} = -2F + 4RF$$



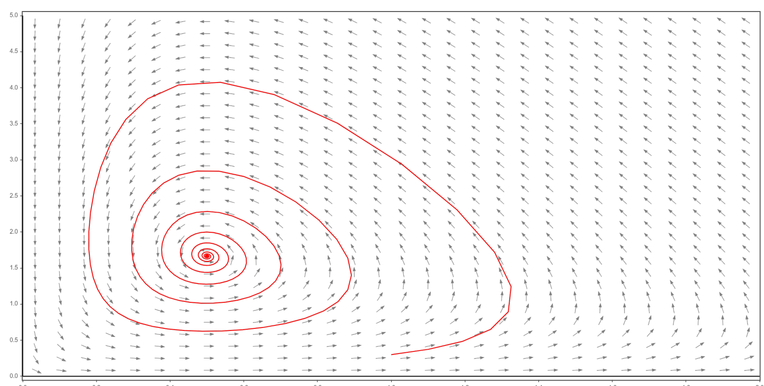
2. Initial value $(R(0), F(0)) = (8, 1)$ and the system is

$$\frac{dR}{dt} = 2(1 - \frac{R}{3})R - RF$$
$$\frac{dF}{dt} = -16F + 4RF$$



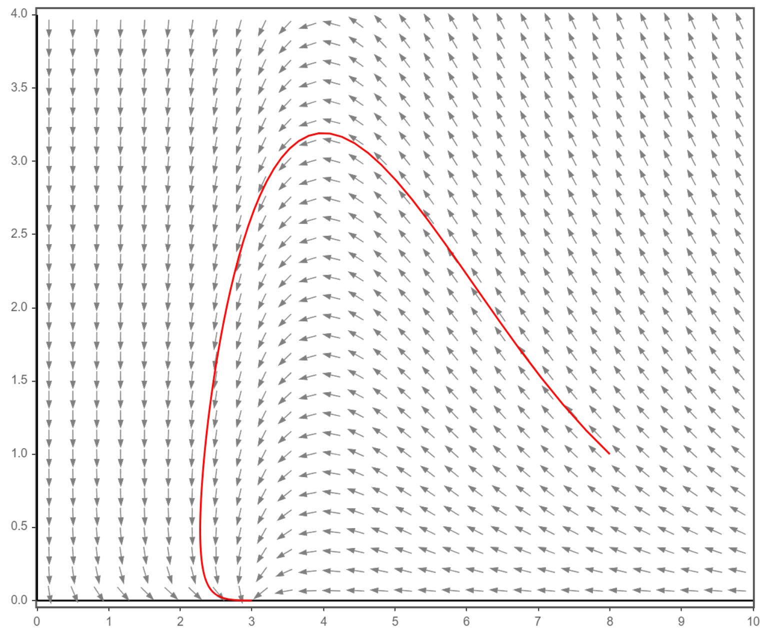
Solution.

1. Using the slope field applet, the solution curve for the initial value $(R(0), F(0)) = (1, 0.3)$ is obtained



Conclusion: Two species coexist and approach the balanced state $(R, F) \approx (0.5, 1.7)$.

2. Using the slope field applet, the solution curve for the initial value $(R(0), F(0)) = (1, 0.3)$ is obtained



Conclusion: Two species cannot coexist. Specifically, the predator will become extinct, while the prey will continue to exist and thrive.

Problem 8. For each of the following first order systems:

$$\begin{aligned} \frac{dx}{dt} &= 1 \\ \frac{dy}{dt} &= 0 \end{aligned} \qquad\qquad \begin{aligned} \frac{dx}{dt} &= x \\ \frac{dy}{dt} &= 2y \end{aligned}$$

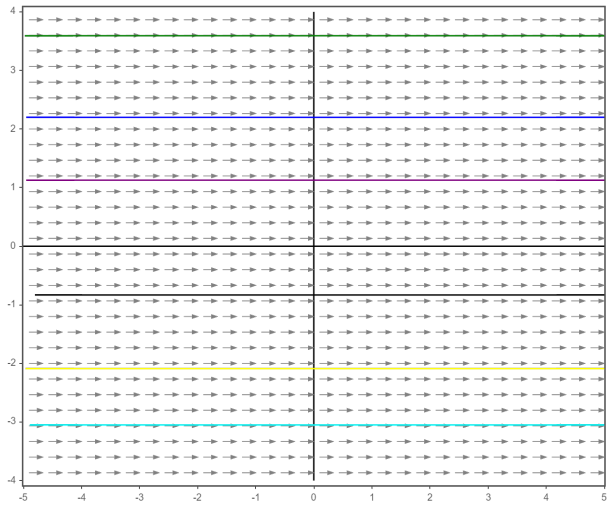
1. Hand-draw the direction field for the system on the phase plane. Include sufficient vectors to clearly illustrate its geometric pattern.

2. Draw a phase portrait of the system, including at least six unique solution curves to capture the dynamics comprehensively.

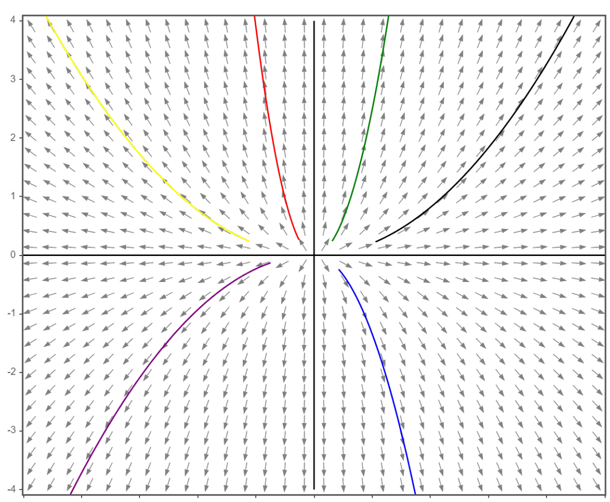
3. Briefly describe the behavior of the solutions.

Solution

- For the system $\begin{aligned} \frac{dx}{dt} &= 1 \\ \frac{dy}{dt} &= 0 \end{aligned}$: All solutions curves are horizontal.



- For the system $\begin{aligned} \frac{dx}{dt} &= x \\ \frac{dy}{dt} &= 2y \end{aligned}$: Each solution remains in the same quadrant as the initial point and approaches either positive or negative infinity.



Problem 9. (Patially Decoupled Systems) Solve the Initial Value Problems (IVPs) for the Given Partially Decoupled Systems with Specified Initial Conditions:

1. Initial Value $(x(0), y(0)) = (-1, 3)$ and

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = -y$$

Solution. We solve the partiallly decoupled system as follows:

- Solving $\dfrac{dy}{dt} = -y$, $y(t) = Ce^{-t}$.
- Initial value $y(0) = 3$: $C = 3$. Hence, $y(t) = 3e^{-t}$.
- Substitute $y(t)$ into the first equation:

$$\frac{dx}{dt} = 2x + 3e^{-t}$$

- (Exercise) Using either "Guess Method" or "Integrating Factor Method", $x(t) = Ce^{2t} - e^{-t}$.
- Initial Value $x(0) = -1$: $C = 0$. Hence, $x(t) = -e^{-t}$.
- **Conclusion.** The solution is $(x(t), y(t)) = (-e^{-t}, 3e^{-t})$.

2. Initial Value $(x(0), y(0)) = (0, 1)$ and

$$\frac{dx}{dt} = 2x - 8y^2$$
$$\frac{dy}{dt} = -3y$$

Solution. We solve the partiallly decoupled system as follows:

- Solving $\frac{dy}{dt} = -3y$, $y(t) = Ce^{-3t}$.
- Initial value $y(0) = 1$: $C = 1$. Hence, $y(t) = e^{-3t}$.
- Substitute $y(t)$ into the first equation:

$$\frac{dx}{dt} = 2x - 8e^{-6t}$$

- (Exercise) Using either "Guess Method" or "Integrating Factor Method", $x(t) = Ce^{2t} + e^{-6t}$.
- Initial Value $x(0) = 0$: $C = -1$. Hence, $x(t) = -e^{2t} + e^{-6t}$.
- **Conclusion.** The solution is $(x(t), y(t)) = (-e^{2t} + e^{-6t}, e^{-3t})$.

3. Initial Value $(x(0), y(0)) = (0, 1)$ and

$$\frac{dx}{dt} = xy$$
$$\frac{dy}{dt} = y + 1,$$

Solution. We solve the partiallly decoupled system as follows:

- (Exercise) Solve $\dfrac{dy}{dt} = y + 1$ using either "Guess Method" or "Integrating Factor Method",

$$y(t) = Ce^t - 1$$

- Initial value $y(0) = 0$: $C = 1$. Hence, $y(t) = e^t - 1$.
- Substitute $y(t)$ into the first equation:

$$\frac{dx}{dt} = x(e^t - 1)$$

- Use the Method of Seperation of Variables:

$$\begin{aligned}
\frac{dx}{dt} = x(e^t - 1) \Rightarrow \int \frac{1}{x}\,dx &= \int (e^t - 1)\,dt \\
&\Rightarrow \ln|x| = e^t - t + C \\
&\Rightarrow |x(t)| = e^{e^t - t + C} \\
&\Rightarrow x(t) = ke^{e^t + t}, \text{ for constants } k
\end{aligned}$$

- Initial Value $x(0) = 1$: $ke = 1$; thus $k = e^{-1}$. Hence,

$$x(t) = e^{-1}e^{e^t + t} = e^{e^t + t - 1}.$$

- **Conclusion.** The solution is $(x(t), y(t)) = (e^{e^t+t-1}, e^t - 1)$.

4. Initial Value $(x(0), y(0)) = (0, 2)$ and

$$
\begin{aligned}
\frac{dx}{dt} &= \sin(t)\sqrt{x} \\
\frac{dy}{dt} &= 3y + 10\sqrt{x}
\end{aligned}
$$

Solution. We solve the partiallly decoupled system as follows:

- Solve $\frac{dx}{dt} = \sin(t)\sqrt{x}$ using Separation of Variables:

$$
\begin{aligned}
\frac{dx}{dt} &= \sin(t)\sqrt{x} \\
\int \frac{1}{\sqrt{x}}\,dx &= \int \sin(t)\,dt \\
2\sqrt{x} &= -\cos(t) + C \\
x(t) &= \left(-\frac{\cos(t)}{2} + \frac{C}{2}\right)^2
\end{aligned}
$$

- Initial value $x(0) = 0$: $C = 1$. Hence, $x(t) = \left(-\frac{\cos(t)}{2} + \frac{1}{2}\right)^2$.
- Substitute $x(t)$ into the second equation:

$$
\frac{dy}{dt} = 2y - 5\cos(t) + 5
$$

- Use either "Integrating Factor" or "Guess Method":

$$y(t) = Ce^{2t} - \sin(t) + 2\cos(t) - \frac{5}{2}$$

- Initial Value $y(0) = 2$: $C + 2 - \frac{5}{2} = 2$; thus $C = \frac{5}{2}$. Hence,

$$y(t) = \frac{5}{2}e^{2t} - \sin(t) + 2\cos(t) - \frac{5}{2}.$$

- **Conclusion.** The solution is

$$(x(t), y(t)) = \left(\left(-\frac{\cos(t)}{2} + \frac{1}{2}\right)^2, \frac{5}{2}e^{2t} - \sin(t) + 2\cos(t) - \frac{5}{2}\right)$$

5. Initial Value $(x(0), y(0)) = (0, 1)$ and

$$\begin{aligned}\frac{dx}{dt} =& (2t+1)e^{-x} \\ \frac{dy}{dt} =& 2y - 2e^{x}\end{aligned}$$

Solution. We solve the partiallly decoupled system as follows:

- Solve $\dfrac{dx}{dt} = \sin(t)\sqrt{x}$ using Separation of Variables:

$$\begin{aligned}
\frac{dx}{dt} &= (2t+1)e^{-x} \\
\int e^x \, dx &= \int (2t+1)\, dt \\
e^x &= t^2 + t + C \\
x(t) &= \ln(t^2 + t + C)
\end{aligned}$$

- Initial value $x(0) = 0$: $C = 1$. Hence, $x(t) = \ln(t^2 + t + 1)$.
- Substitute $x(t)$ into the second equation:

$$\frac{dy}{dt} = 2y - 2(t^2 + t + 1)$$

- Use either "Integrating Factor" or **"Guess Method"**:

$$y(t) = Ce^{2t} + t^2 + 2t + 2$$

- Initial Value $y(0) = 1$: $C + 2 = 1$; thus $C = -1$. Hence,

$$y(t) = -e^{2t} + t^2 + 2t + 2.$$

- **Conclusion.** The solution is

$$(x(t), y(t)) = \big(\ln(t^2 + t + 1), -e^{2t} + t^2 + 2t + 2\big)$$