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tcs94

Section 4.1: 3, 7, 11, 21, 37

3. $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 5e^{3t}$

(H): $y'' - y' - 2y = 0$

$s^2 - s - 2 = 0$

$(s-2)(s+1) = 0$; $s_1 = 2, s_2 = -1$

$\boxed{y_h(t) = K_1e^{2t} + K_2e^{-t}}$

Guess: $y_p(t) = \alpha e^{3t}$

$e^{3t}(9\alpha - 3\alpha - 2\alpha) = 5e^{3t}$

$4\alpha = 5$

$\alpha = \frac{5}{4}$

$y_p(t) = \frac{5}{4}e^{3t}$

$\boxed{y(t) = K_1e^{2t} + K_2e^{-t} + \frac{5}{4}e^{3t}}$

7. $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 4y = e^{4t}$

(H): $y'' - 5y' + 4y = 0$

$s^2 - 5s + 4 = 0$

$(s-1)(s-4) = 0$; $s_1 = 1, s_2 = 4$

$\boxed{y_h(t) = K_1e^{t} + K_2e^{4t}}$

Guess: $y_p(t) = \alpha te^{4t}$ bc e^{4t} is in $y_h(t)$

$y_p'(t) = \alpha e^{4t} + 4\alpha te^{4t}$

$y_p''(t) = 4\alpha e^{4t} + 4\alpha e^{4t} + 16\alpha te^{4t}$

$= 8\alpha e^{4t} + 16\alpha te^{4t}$

$\cancel{e^{4t}}(8\alpha + 16\alpha t - 5\alpha - 20\alpha t + 4\alpha t) = \cancel{e^{4t}}$

$3\alpha = 1 \quad \alpha = \frac{1}{3} \Rightarrow y_p(t) = \frac{1}{3}te^{4t}$

$\boxed{y(t) = K_1e^{t} + K_2e^{4t} + \frac{1}{3}te^{4t}}$

11. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = -3e^{-2t} \quad y(0) = y'(0) = 0$

(H): $y'' + 4y' + 13y = 0$

$s^2 + 4s + 13 = 0$

$s = \frac{-4 \pm \sqrt{16 - 4(13)}}{2} = -2 \pm 3i$

$\boxed{y_h(t) = K_1e^{-2t}\cos(3t) + K_2e^{-2t}\sin(3t)}$

Guess: $y_p(t) = \alpha e^{-2t}$

$\cancel{e^{-2t}}(4\alpha - 8\alpha + 13\alpha) = -3\cancel{e^{-2t}}$

$9\alpha = -3 \quad \alpha = -\frac{1}{3}$

$y_p(t) = -\frac{1}{3}e^{-2t}$

$\boxed{y(t) = K_1e^{-2t}\cos(3t) + K_2e^{-2t}\sin(3t) - \frac{1}{3}e^{-2t}}$

$y(0)$: $y(0) = K_1\cos(0) + K_2\sin(0) - \frac{1}{3}$

$0 = K_1 - \frac{1}{3} \quad K_1 = \frac{1}{3}$

$y'(0)$: $y'(t) = K_1(-2e^{-2t}\cos 3t - 3e^{-2t}\sin 3t) + K_2(-2e^{-2t}\sin 3t + 3e^{-2t}\cos 3t) + \frac{2}{3}e^{-2t}$

$y'(0) = K_1(-2-0) + K_2(0+3) + \frac{2}{3}$

$0 = -2K_1 + 3K_2 + \frac{2}{3}$

$0 = -\frac{2}{3} + 3K_2 + \frac{2}{3} \quad K_2 = 0$

$\boxed{y(t) = \frac{1}{3}e^{-2t}\cos(3t) - \frac{1}{3}e^{-2t}}$

21. $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 4y = 5$

a) (H): $y'' - 5y' + 4y = 0$

$s^2 - 5s + 4 = 0$

$(s-4)(s-1) = 0;\ s_1 = 4,\ s_2 = 1$

$y_h(t) = K_1 e^{4t} + K_2 e^{t}$

Guess: $y_p(t) = \alpha$

$0 - 0 + 4\alpha = 5$

$4\alpha = 5$

$\alpha = \frac{5}{4}$

$y(t) = K_1 e^{4t} + K_2 e^{t} + \frac{5}{4}$

b) $y(0) = y'(0) = 0$

$y(0)$: $y(0) = K_1 + K_2 + \frac{5}{4}$

$0 = K_1 + K_2 + \frac{5}{4}$

$y'(0)$: $y'(t) = 4K_1 e^{4t} + K_2 e^{t}$

$y'(0) = 4K_1 + K_2$

$0 = 4K_1 + K_2$

$\begin{cases} K_1 + K_2 = -\frac{5}{4} \\ 4K_1 + K_2 = 0 \end{cases}$

$K_2 = -4K_1$

$K_1 - 4K_1 = -\frac{5}{4}$

$-3K_1 = -\frac{5}{4}$

$K_2 = -\frac{5}{3} \leftarrow K_1 = \frac{5}{12}$

$y(t) = \frac{5}{12} e^{4t} - \frac{5}{3} e^{t} + \frac{5}{4}$

37. $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{-t} + 4$

a) (H): $y'' + 5y' + 6y = 0$

$s^2 + 5s + 6 = 0$

$(s+3)(s+2) = 0;\ s_1 = -3,\ s_2 = -2$

$y_h(t) = K_1 e^{-3t} + K_2 e^{-2t}$

Guess: $y_p(t) = \alpha e^{-t} + \beta$

$\alpha e^{-t} - 5\alpha e^{-t} + 6\alpha e^{-t} + 6\beta = e^{-t} + 4$

$\begin{cases} \alpha - 5\alpha + 6\alpha = 1 \\ 6\beta = 4 \end{cases}$ $\quad 2\alpha = 1 \quad \alpha = \frac{1}{2}$ $\quad \beta = \frac{2}{3}$

$y(t) = K_1 e^{-3t} + K_2 e^{-2t} + \frac{1}{2} e^{-t} + \frac{2}{3}$

b) $y(0) = y'(0) = 0$

$y(0)$: $y(0) = K_1 + K_2 + \frac{1}{2} + \frac{2}{3}$

$0 = K_1 + K_2 + \frac{7}{6}$

$y'(0)$: $y'(t) = -3K_1 e^{-3t} - 2K_2 e^{-2t} - \frac{1}{2} e^{-t}$

$y'(0) = -3K_1 - 2K_2 - \frac{1}{2}$

$0 = -3K_1 - 2K_2 - \frac{1}{2}$

$\begin{cases} K_1 + K_2 = -\frac{7}{6} \\ -3K_1 - 2K_2 = \frac{1}{2} \end{cases}$

$K_1 = -\frac{7}{6} - K_2$

$-3(-\frac{7}{6} - K_2) - 2K_2 = \frac{1}{2}$

$\frac{7}{2} + 3K_2 - 2K_2 = \frac{1}{2}$

$K_2 = -\frac{6}{2} = -3$

$K_1 = -\frac{7}{6} - (-3) = \frac{11}{6}$

$y(t) = \frac{11}{6} e^{-3t} - 3e^{-2t} + \frac{1}{2} e^{-t} + \frac{2}{3}$

c) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{11}{6} e^{-3t} - 3e^{-2t} + \frac{1}{2} e^{-t} + \frac{2}{3} = \frac{2}{3}$

As $t \to \infty$ the long term behavior of $y(t)$, the solution tends to $\frac{2}{3}$

Section 4.2: 1, 3, 11, 17, 19

1. $$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=\cos t$$

(H): $y''+3y'+2y=0$

$s^2+3s+2=0$

$(s+2)(s+1)=0$; $s_1=-2$, $s_2=-1$

$$y_h(t)=K_1e^{-2t}+K_2e^{-t}$$

Guess: $y_p(t)=\alpha\cos t+\beta\sin t$

$y_p'(t)=-\alpha\sin t+\beta\cos t$

$y_p''(t)=-\alpha\cos t-\beta\sin t$

$$\underbrace{-\alpha\cos t}\;\underbrace{-\beta\sin t}\;\underbrace{-3\alpha\sin t}\;\underbrace{+3\beta\cos t}\;\underbrace{+2\alpha\cos t}\;\underbrace{+2\beta\sin t}=\underbrace{\cos t}$$

$$\begin{cases}-\alpha+3\beta+2\alpha=1 & -\alpha+9\alpha+2\alpha=1 & \alpha=\frac{1}{10}\\ -\beta-3\alpha+2\beta=0 & \beta=3\alpha & \beta=\frac{3}{10}\end{cases}$$

$$y(t)=K_1e^{-2t}+K_2e^{-t}+\frac{1}{10}\cos t+\frac{3}{10}\sin t$$

3. $$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=\sin t$$

(H): $y''+3y'+2y=0$

$s^2+3s+2=0$

$(s+2)(s+1)=0$; $s_1=-2$, $s_2=-1$

$$y_h(t)=K_1e^{-2t}+K_2e^{-t}$$

Guess: $y_p(t)=\alpha\cos t+\beta\sin t$

$y_p'(t)=-\alpha\sin t+\beta\cos t$

$y_p''(t)=-\alpha\cos t-\beta\sin t$

$$\underbrace{-\alpha\cos t}\;\underbrace{-\beta\sin t}\;\underbrace{-3\alpha\sin t}\;\underbrace{+3\beta\cos t}\;\underbrace{+2\alpha\cos t}\;\underbrace{+2\beta\sin t}=\underbrace{\sin t}$$

$$\begin{cases}-\alpha+3\beta+2\alpha=0 & \alpha=-3\beta & \alpha=-\frac{3}{10}\\ -\beta-3\alpha+2\beta=1 & -\beta+9\beta+2\beta=1 & \beta=\frac{1}{10}\end{cases}$$

$$y(t)=K_1e^{-2t}+K_2e^{-t}-\frac{3}{10}\cos t+\frac{1}{10}\sin t$$

11. $$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+8y=\cos t \qquad y(0)=y'(0)=0$$

(H): $y''+6y'+8y=0$

$s^2+6s+8=0$

$(s+4)(s+2)=0$; $s_1=-4$, $s_2=-2$

$$y_h(t)=K_1e^{-4t}+K_2e^{-2t}$$

Guess: $y_p(t)=\alpha e^{it}$

$\cancel{e^{it}}(-\alpha+6\alpha i+8\alpha)=\cancel{e^{it}}$

$7\alpha+6\alpha i=1$

$\alpha=\frac{1}{7+6i}\,\frac{(7-6i)}{(7-6i)}$

$\alpha=\frac{7-6i}{85}$

$y_p(t)=\frac{7-6i}{85}e^{it}$

$=\left(\frac{7}{85}-\frac{6}{85}i\right)(\cos t+i\sin t)$

$=\frac{7}{85}\cos t+\frac{6}{85}\sin t+\underbrace{\text{im part}}$

Only need y_{re} as original was $\cos t$

$$y(t)=K_1e^{-4t}+K_2e^{-2t}+\frac{7}{85}\cos t+\frac{6}{85}\sin t$$

$y(0)$: $y(0)=K_1+K_2+\frac{7}{85}$

$0=K_1+K_2+\frac{7}{85}$

$y'(0)$: $y'(t)=-4K_1e^{-4t}-2K_2e^{-2t}-\frac{7}{85}\sin t+\frac{6}{85}\cos t$

$y'(0)=-4K_1-2K_2+\frac{6}{85}$

$0=-4K_1-2K_2+\frac{6}{85}$

$$\begin{cases}K_1+K_2=-\frac{7}{85}\\ -4K_1-2K_2=-\frac{6}{85}\end{cases}$$

$K_1=-\frac{7}{85}-K_2$

$-4\left(-\frac{7}{85}-K_2\right)-2K_2=-\frac{6}{85}$

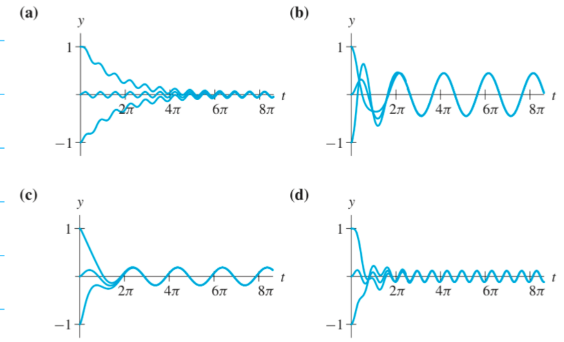
$\frac{28}{85}+4K_2-2K_2=-\frac{6}{85}$

$2K_2=-\frac{34}{85}$ $\quad K_2=-\frac{17}{85}=-\frac{1}{5}$

$K_1=-\frac{7}{85}-\left(-\frac{1}{5}\right)=\frac{10}{85}=\frac{2}{17}$

$$y(t)=\frac{2}{17}e^{-4t}-\frac{1}{5}e^{-2t}+\frac{7}{85}\cos t+\frac{6}{85}\sin t$$

17. $\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \cos \omega t$



(i) $p=5, q=3, \omega=1$ (iv) $p=1, q=3, \omega=2$

(ii) $p=1, q=3, \omega=1$ (v) $p=5, q=1, \omega=3$

(iii) $p=5, q=2, \omega=2$ (vi) $p=1, q=1, \omega=3$

Characteristic Equations Period $= \frac{2\pi}{\omega}$

(i) $s^2+5s+3=0$, $\omega=1$

$s = \frac{-5 \pm \sqrt{25-4(3)}}{2}$; $s_1=-4.3$, $s_2=-0.7$, Period $= \frac{2\pi}{1} = 2\pi$

(iv) $s^2+s+3=0$, $\omega=2$

Same as (ii) $s = -\frac{1}{2} \pm i\frac{\sqrt{11}}{2}$, Period $= \frac{2\pi}{2} = \pi$

(ii) $s^2+s+3=0$, $\omega=1$

$s = \frac{-1 \pm \sqrt{1-4(3)}}{2}$; $s = -\frac{1}{2} \pm i\frac{\sqrt{11}}{2}$, Period $= \frac{2\pi}{1} = 2\pi$

(v) $s^2+5s+1=0$, $\omega=3$

$s = \frac{-5 \pm \sqrt{25-4}}{2}$; $s_1=-4.8$, $s_2=-0.2$, Period $= \frac{2\pi}{3}$

(iii) $s^2+5s+2=0$, $\omega=2$

$s = \frac{-5 \pm \sqrt{25-4(2)}}{2}$; $s_1=-4.5$, $s_2=-0.5$, Period $= \frac{2\pi}{2} = \pi$

(vi) $s^2+s+1=0$, $\omega=3$

$s = \frac{-1 \pm \sqrt{1-4}}{2}$; $s = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, Period $= \frac{2\pi}{3}$

Matching Graphs

a) Period Approx. $\frac{2\pi}{3}$, must be (v) or (vi). (v) as Identical Period can't have complex!

b) Period Approx. 2π, must be (i) or (ii). (ii) as Different Periods @ $t=0$ → complex

c) Period Approx. 2π, must be (i) or (ii). (i) as Identical Periods over time, no varying!

d) Period Approx. $\frac{2\pi}{3}$, must be (v) or (vi). (vi) as Different Period @ $t=0$ need complex terms

19. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = e^{-t}\cos t$

a) (H): $y'' + 4y' + 20y = 0$

$s^2 + 4s + 20 = 0$

$s = \frac{-4 \pm \sqrt{16-4(20)}}{2}$; $s = -2 \pm 4i$

$y_h(t) = K_1 e^{-2t}\cos 4t + K_2 e^{-2t}\sin 4t$

$e^{-t}\cos t = e^{-t}e^{it} = e^{(i-1)t}$

Guess: $y_p(t) = \alpha e^{(i-1)t}$

$y_p'(t) = \alpha(i-1)e^{(i-1)t}$

$y_p''(t) = \alpha(i-1)^2 e^{(i-1)t} = \alpha(-1-2i+1)e^{(i-1)t} = -2\alpha i e^{(i-1)t}$

$\cancel{e^{(i-1)t}}(-2\alpha i + 4\alpha(i-1) + 20\alpha) = \cancel{e^{(i-1)t}}$

$\alpha(-2i + 4i - 4 + 20) = 1$ $\alpha = \frac{1}{16+2i}$

$\alpha = \frac{1}{16+2i}\cdot\frac{(16-2i)}{(16-2i)} = \frac{16-2i}{260} = \frac{8-i}{130} = \frac{8}{130} - i\frac{1}{130} = \frac{4}{65} - i\frac{1}{130}$

$y_p(t) = \left(\frac{4}{65} - i\frac{1}{130}\right)\left(e^{(i-1)t}\right) = \left(\frac{4}{65} - i\frac{1}{130}\right)e^{-t}(\cos t + i\sin t)$

$y_p(t) = \frac{4}{65}e^{-t}\cos t + \frac{1}{130}e^{-t}\sin t + \text{Im}$ cos t in original, only need Re!

$y(t) = K_1 e^{-2t}\cos 4t + K_2 e^{-2t}\sin 4t + \frac{4}{65}e^{-t}\cos t + \frac{1}{130}e^{-t}\sin t$

b) $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} K_1 e^{-2t}\cos 4t + K_2 e^{-2t}\sin 4t + \frac{4}{65}e^{-t}\cos t + \frac{1}{130}e^{-t}\sin t = 0$

The long term behavior is found by computing $\lim_{t\to\infty} y(t)$. Since all terms have an exponential, All solutions tend to 0 as $t\to\infty$.