#1

$$A = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \qquad \vec{Y} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(a) and (b) are best solved together. From the example in the supplemental problem, we can guess $\xi_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector. Indeed $A\xi_1 = -3\xi_1$, so $\lambda_1 = -3$ is an eigenvalue.

As in the supplement, $\lambda_2 = \lambda_3 = 0$ is a repeated eigenvalue. Two linearly independent eigenvectors are $\xi_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\xi_3 = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$.

(c) The general solution is $\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$.

(d) For $\vec{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ we have to solve

$$c_1 + c_2 - c_3 = 1 \qquad (Eq\,1)$$
$$c_1 \qquad + 2c_3 = 2 \qquad (Eq\,2)$$
$$c_1 - c_2 - c_3 = 3 \qquad (Eq\,3)$$

$(Eq\,1) - (Eq\,3)$ gives $2c_2 = -2$, or $c_2 = -1$

$(Eq\,1) + (Eq\,2) + (Eq\,3)$ gives $3c_1 = 6$, or $c_1 = 2$

$(Eq\,2)$ gives $2c_3 = 2 - c_1 = 0$, or $c_3 = 0$.

So the solution is

$$\boxed{\vec{x}(t) = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

#2 (a) $B = \begin{pmatrix} -2 & -1 \\ -2 & -3 \end{pmatrix}$ $Tr(B) = -5$ $Det(B) = (-2)(-3) - (-2)(-1)$

$= 6 - 2 = 4$

$Q = Tr^2 - 4 Det = 25 - 16 = 9 > 0$ so real distinct roots.

$\lambda_\pm = \frac{1}{2}\{ Tr \pm \sqrt{Q} \} = \frac{1}{2}\{-5 \pm \sqrt{9}\} = -1, -4$

$\lambda_+ = -1$ $(B - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2-(-1) & -1 \\ -2 & -3-(-1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} = ($

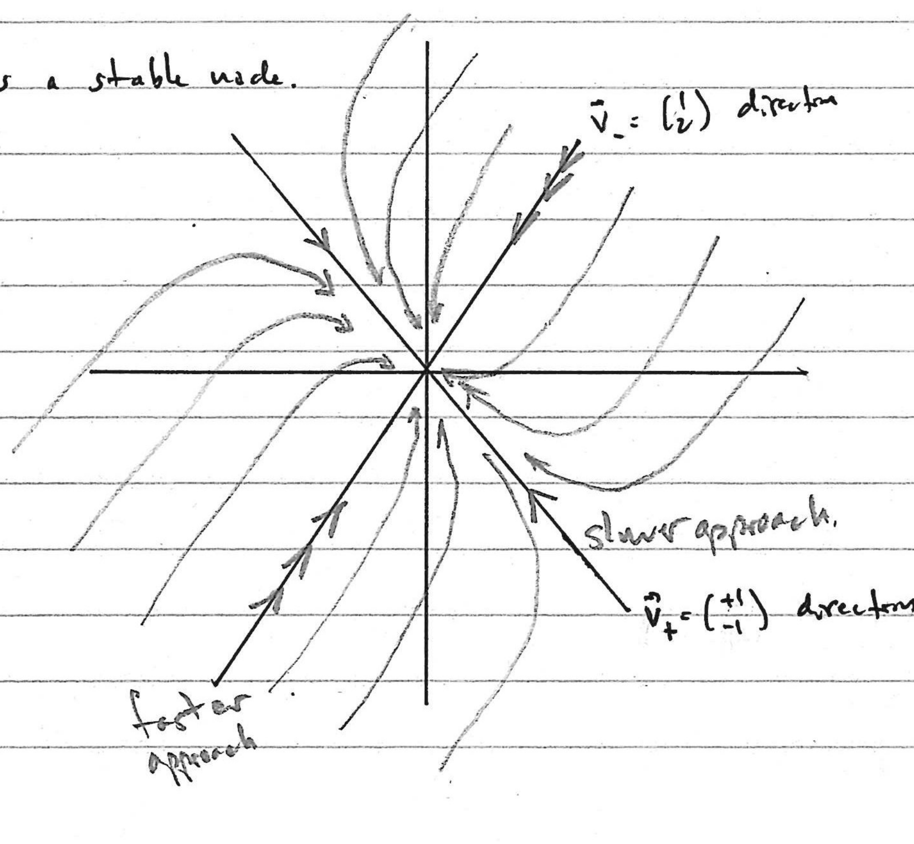
$\vec{v}_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for $\lambda_+ = -1$

$\lambda_- = -4$ $(B - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2-(-4) & -1 \\ -2 & -3-(-4) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{v}_- = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for $\lambda_- = -4$

(b) $\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the general solution.

(c) The system is a stable node.



$3-(-4)=7$

#3 $C = \begin{pmatrix} -1 & 2 \\ -2 & -3 \end{pmatrix}$ $\qquad Tr = -4 \quad Det = 7 \quad Q = Tr^2 - 4Det = 16 - 28 = -12$

$$\lambda_\pm = \tfrac{1}{2}\{Tr \pm \sqrt{Q}\}$$
$$= \tfrac{1}{2}\{-4 \pm \sqrt{-12}\}$$
$$= \tfrac{1}{2}\{-4 \pm 2i\sqrt{3}\}$$
$$= -2 \pm i\sqrt{3}$$

$$e^{-2t}\left(A\cos(\sqrt{3}\,t) + B\sin(\sqrt{3}\,t)\right)$$

so $k = 2$ and $\omega = \sqrt{3}$.

#4 $D = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ $\qquad \lambda = 2, 2$ (repeated)

$$\boxed{\begin{aligned} x &= e^{+2t} x_0 \\ y &= y_0 e^{2t} + t x_0 e^{2t} \end{aligned}}$$

$x' = 2x$

$y' = x + 2y$

check: $y' = 2y + x_0 e^{2t}$

Vector form $\vec{Y}(t) = e^{2t}\vec{Y}_0 + te^{2t}\vec{Y}_1$ where

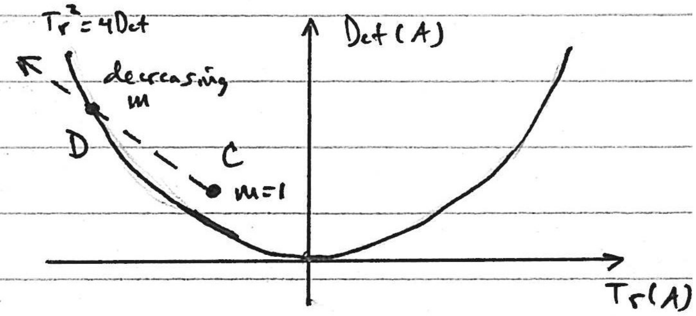
$$\vec{Y}_1 = (D - \lambda I)\vec{Y}_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ x_0 \end{pmatrix}$$

#5 (a) The linear system is $\frac{d\vec{Y}}{dt} = A\vec{Y}$ with $\vec{Y} = \begin{pmatrix} y \\ dy/dt \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ -\frac{b}{m} & -\frac{k}{m} \end{pmatrix}$

$\mathrm{Tr}(A) = -\frac{k}{m}$ and $\mathrm{Det}(A) = \frac{b}{m}$. The eigenvalues are

$$\lambda_\pm = \frac{1}{2}\left\{ \mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}^2(A) - 4\,\mathrm{Det}(A)} \right\} = \frac{1}{2}\left\{ -\frac{k}{m} \pm \sqrt{\frac{k^2}{m^2} - 4\frac{b}{m}} \right\}$$

$$= \frac{1}{2m}\left\{ -k \pm \sqrt{k^2 - 4bm} \right\}$$



(b) If $k = b = 1$ then

$\mathrm{Tr} = -\frac{1}{m}$, $\mathrm{Det} = \frac{1}{m}$

For $m = 1$, $\mathrm{Tr}^2 - 4\,\mathrm{Det} = -3 < 0$

so we are above the curve $\mathrm{Tr}^2 = 4\,\mathrm{Det}$.

(c) At point C ($m = 1$, $\mathrm{Tr} = -1$, $\mathrm{Det} = +1$) the eigenvalues are complex and the system shows underdamped oscillations.

(d) At point D $\mathrm{Tr}^2 = 4\,\mathrm{Det}$, or $\left(\frac{-1}{m_*}\right)^2 = 4\left(\frac{1}{m_*}\right)$, or $m_* = \frac{1}{4}$.

As m decreases through $m_* = \frac{1}{4}$, the eigenvalues change from imaginary (complex) to real through a repeated root. At $m = \frac{1}{4}$ the system is critically damped.

(e) As m decreases further, the eigenvalues become more negative, and the system becomes over damped.

Extra credit for this answer:

$$\boxed{\lim_{m \to 0} (\lambda_-) = -\infty, \quad \lim_{m \to 0} (\lambda_+) = -\frac{b}{k}}$$