PROBLEM	POINTS	GRADE
1	25	
2	25	
3	25	
4	25	
5	10	

1. For the equation $\frac{dy}{dt} = y^5 t$,

   (a) Find the general solution. (15 points)

   (b) Find the solution with the following initial conditions: $y(0) = 1$. (10 points)

a) $\int \frac{dy}{y^5} = \int t\,dt \Rightarrow -\frac{1}{4} y^{-4} = \frac{1}{2}t^2 + C$

So, $y(t) = (-2t^2 - 4C)^{-1/4}$

Don't forget about $y(t) \equiv 0$!

b) Substituting $y(0) = 1$ into the general solution gives $C = -1/4$. Thus, the specific solution is $y(t) = (-2t^2+1)^{-1/4}$

2. Consider the equation $\frac{dy}{dt} = -y^2 - 2y + 2$ $= f(y)$) autonomus

(a) Sketch the phase line for the equation. Classify all equilibrium points (sinks, sources, nodes). (15 points)

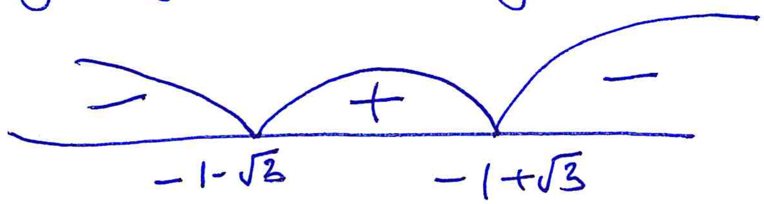
Find equilibrium points by setting $\frac{dy}{dt} = 0$:

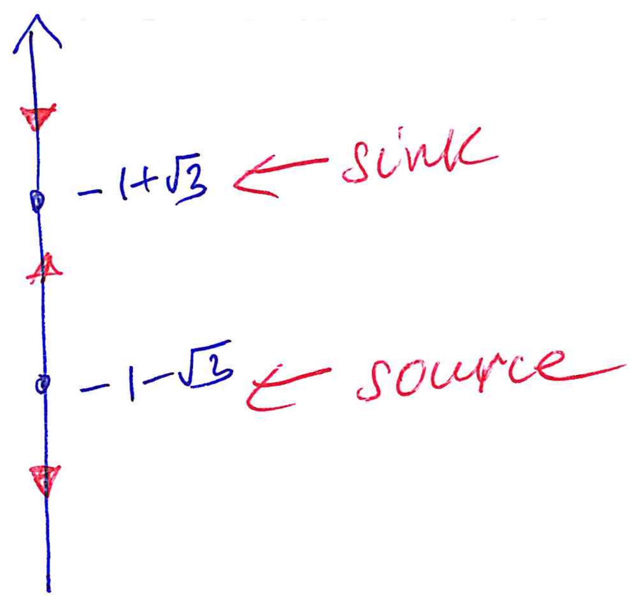
$-y^2 - 2y + 2 = 0 \Rightarrow$

$y^2 + 2y - 2 = 0 \qquad y = \frac{-2 \pm \sqrt{4+8}}{2}$

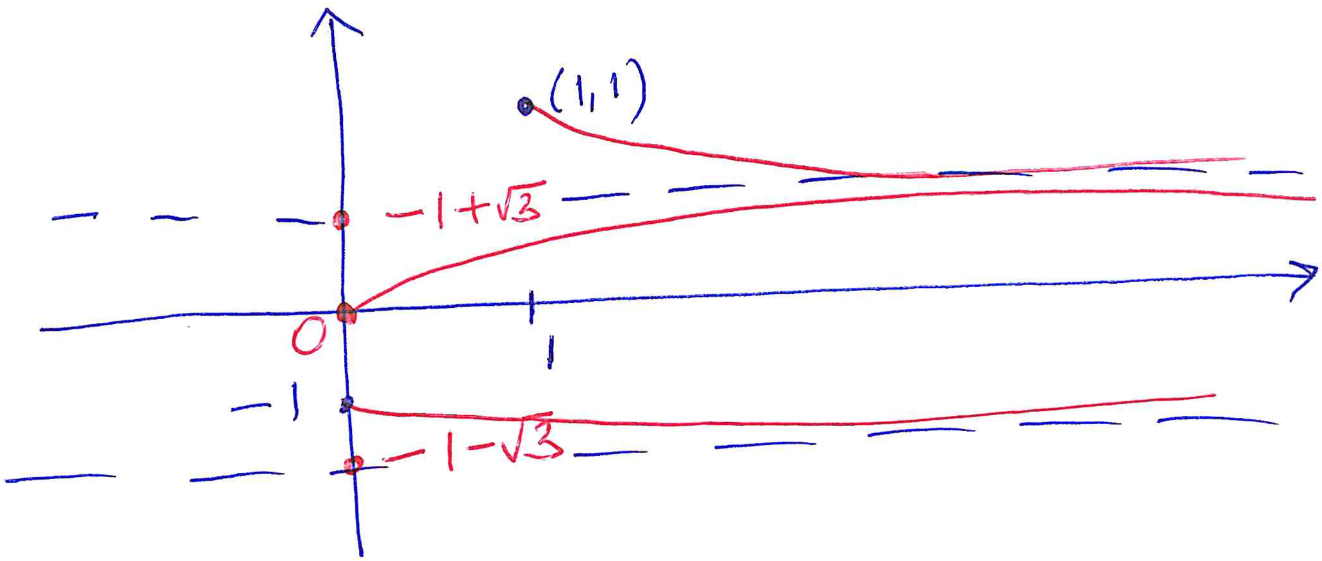
$y_1 = -1-\sqrt{3}$

$y_2 = -1+\sqrt{3}$





(b) Determine the long-term behavior of solutions with each of the following initial conditions: $y(0) = 0$, $y(1) = 1$, $y(0) = -1$. (10 points)



3. (a) Show that $y_1(t) = \frac{1}{t-1}$ and $y_2(t) = \frac{1}{t-2}$ are solutions to $\frac{dy}{dt} = -y^2$. (10 points)

(b) What can you say about solutions of $\frac{dy}{dt} = -y^2$ for which the initial condition $-1 < y(0) < -1/2$? (15 points)

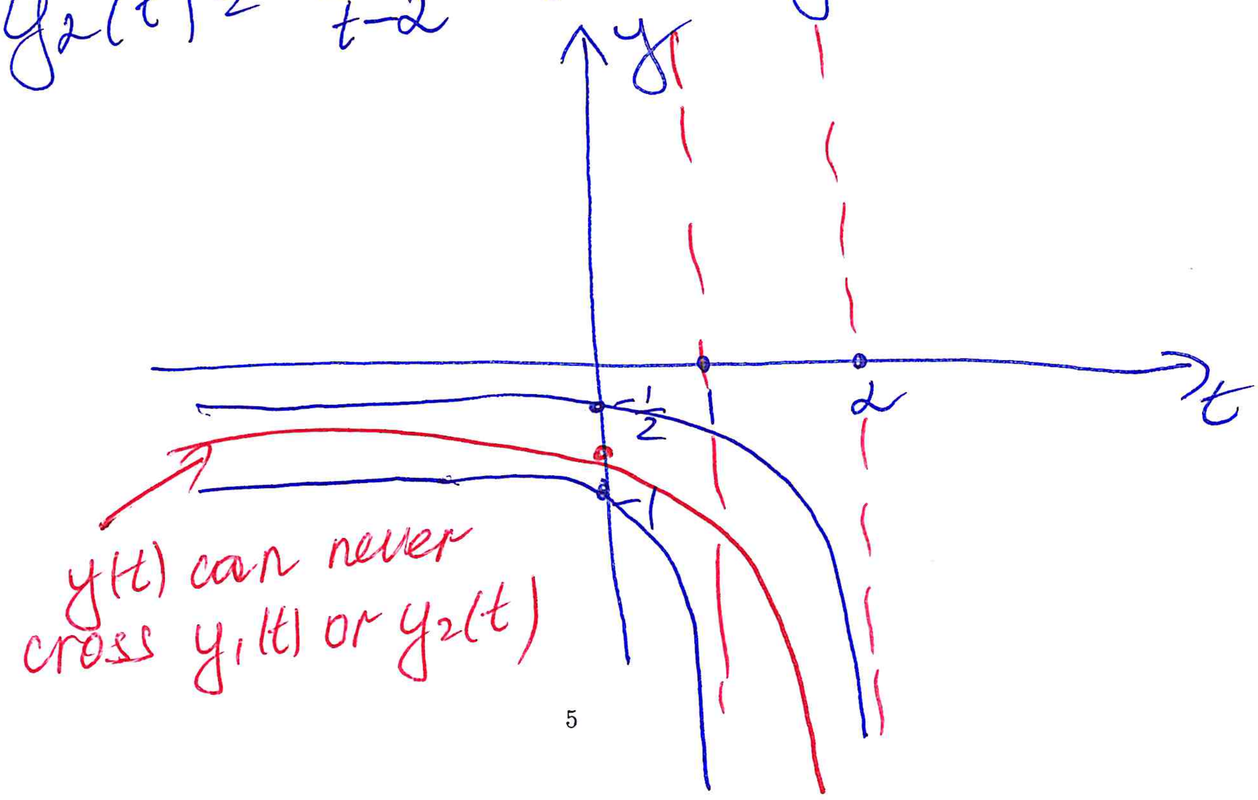
a) $y_1 = \frac{1}{t-1}$, $\frac{dy_1}{dt} = -\frac{1}{(t-1)^2} = -y_1^2$

$y_2 = \frac{1}{t-2}$ $\frac{dy_2}{dt} = -\frac{1}{(t-2)^2} = -y_2^2$

b) Equation $\frac{dy}{dt} = -y^2 = f(y)$ is autonomous and $\frac{\partial f}{\partial y} = -2y$ is continuous for any point (t,y). So any point (t_0, y_0) in ty-space can be crossed only by single curve $\Rightarrow$ none of two curves can intersect.

Solution $y_1(t) = \frac{1}{t-1}$ satisfies $y(0) = -1$

$y_2(t) = \frac{1}{t-2}$ satisfies $y(0) = -\frac{1}{2}$



4. (a) For the equation $\frac{dy}{dt} = -2y + \sin t + e^t$

i. Find the homogeneous part of the solution. (5 points)

ii. Find the general solution. (10 points)

i) Homogeneous part: $\frac{dy}{dt} = -2y$

$y(t) = Ke^{-2t}$

ii) We propose particular solution of the form: $y_p(t) = A\cos t + B\sin t + De^t$

Substitute $y_p(t)$ into the differential equation:

$\frac{d}{dt}(A\cos t + B\sin t + De^t) = -2(A\cos t + B\sin t + De^t) + \sin t + e^t$ ~~$-A\sin t + B\cos t + De^t =$~~

$= -2A\cos t - 2B\sin t - 2De^t + \sin t + e^t$

For $\cos t$: $B = -2A \Rightarrow B = 2(-2B+1)$

For $\sin t$: $-A = -2B + 1$ $\quad B = \frac{2}{5}$,

For e^t: $D = 1 - 2D$ $\quad A = -\frac{1}{5}$

$D = 1/3$

General solution $y(t) = Ke^{-2t} - \frac{1}{5}\cos t + \frac{2}{5}\sin t + e^t/3$

iii. Find the solution's curve $y(t)$ that satisfies the initial condition $y(0) = \frac{1}{5}$. Is it unique? Explain your answer using the Existence and Uniqueness Theorem. (10 points)

$y(0) = K - \frac{1}{5} + \frac{1}{3} = \frac{1}{5}$

$K = \frac{2}{5} - \frac{1}{3} = \frac{6-5}{15} = 1/15$

From Uniqueness Theorem solution is unique as $f(y,t) = -2y + \ln t + e^t$ is continuously differentiable.

$\frac{\partial f}{\partial y}$ is continuous.

(3) $V = 480 \, (km^3)$

$r = 350 \, (km^3)$

c - pollutant concentration of Huron

$x(0) = 5cV = x_0$

$x(t)$ - amount of pollutant in Erie

$\frac{dx}{dt} = rc - \frac{r}{V} x$, let $p = \frac{r}{V}$, $q = rc$

$\frac{dx}{dt} + \frac{r}{V} x = rc = \frac{dx}{dt} + px = q$,

Integrating factor $\rho = e^{pt}$

$x(t) = e^{-pt}\left[x_0 + \frac{q}{p}(e^{pt} - 1)\right] =$

$= e^{-rt/V}\left[5cV + \frac{rc}{r/V}(e^{rt/V} - 1)\right]$:

$x(t) = cV + 4cVe^{-rt/V}$

To find when $x(t) = 2cV$:

$cV + 4cVe^{-rt/V} = 2cV$

$t = \frac{V}{r} \ln 4 \approx 1.9$ years