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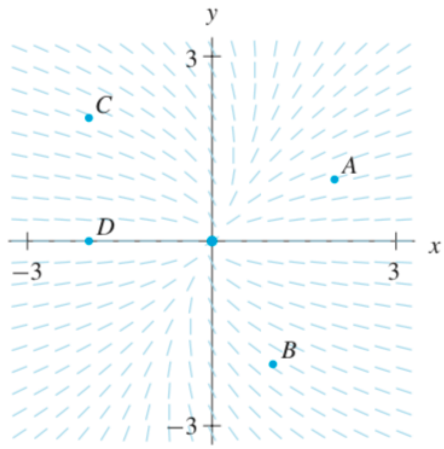
Section 3.3: 19, 23, 25, 27

1a) $\begin{cases} \frac{dx}{dt} = -2x + \frac{1}{2}y \\ \frac{dy}{dt} = -y \end{cases}$

a)

Quadrant		Value/Dir	
I	(1,2)	$\begin{pmatrix}-1\\-2\end{pmatrix}$	↙
II	(−1,2)	$\begin{pmatrix}3\\-2\end{pmatrix}$	↘
III	(−1,−2)	$\begin{pmatrix}1\\2\end{pmatrix}$	↗
IV	(1,−2)	$\begin{pmatrix}-3\\2\end{pmatrix}$	↖

The direction vectors in the quadrants point towards the origin, so its a sink



b) $A = \begin{pmatrix} -2 & \frac{1}{2} \\ 0 & -1 \end{pmatrix}$

$p(\lambda) = (-2-\lambda)(-1-\lambda)$
$= 2 + 2\lambda + \lambda + \lambda^2$
$0 = \lambda^2 + 3\lambda + 2$
$0 = (\lambda+2)(\lambda+1)$
$\lambda_1 = -2$ and $\lambda_2 = -1$

$\lambda_1 = -2$: $\begin{pmatrix} -2 & \frac{1}{2} \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$

$-2x + \frac{1}{2}y = -2x \Rightarrow y = 0$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$-y = -2y \Rightarrow y = 0$

$\lambda_2 = -1$: $\begin{pmatrix} -2 & \frac{1}{2} \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$

$-2x + \frac{1}{2}y = -x \Rightarrow y = 2x$ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
$-y = -y \Rightarrow x \in \mathbb{R}$

Solution to (b)

$\vec{Y}(t) = K_1 e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + K_2 e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

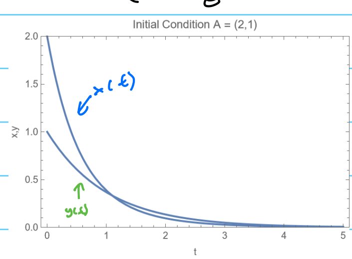
$\vec{Y}(t) = \begin{pmatrix} K_1 e^{-2t} + K_2 e^{-t} \\ 2K_2 e^{-t} \end{pmatrix}$

I will make the plots for (c) in mathematica, but the equation calculations are done by hand!

c) $A(0) = (2,1)$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} K_1 + K_2 \\ 2K_2 \end{pmatrix}$ $K_2 = \frac{1}{2}$, $K_1 = \frac{3}{2}$

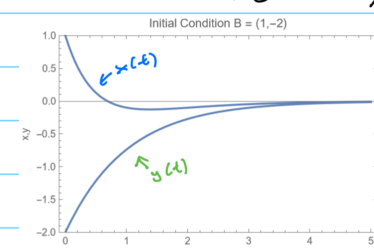
$\therefore \vec{Y}(t) = \begin{pmatrix} \frac{3}{2}e^{-2t} + \frac{1}{2}e^{-t} \\ e^{-t} \end{pmatrix}$



$B(0) = (1,-2)$

$\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} K_1 + K_2 \\ 2K_2 \end{pmatrix}$ $K_2 = -1$, $K_1 = 2$

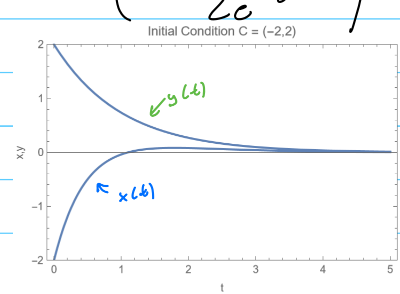
$\therefore \vec{Y}(t) = \begin{pmatrix} 2e^{-2t} - e^{-t} \\ -2e^{-t} \end{pmatrix}$



$C(0) = (-2,2)$

$\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} K_1 + K_2 \\ 2K_2 \end{pmatrix}$ $K_2 = 1$, $K_1 = -3$

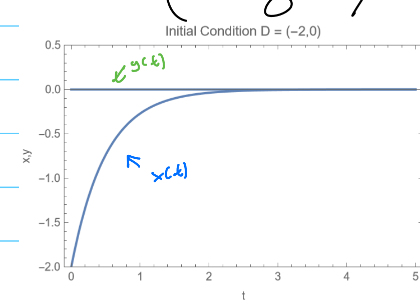
$\therefore \vec{Y}(t) = \begin{pmatrix} -3e^{-2t} + e^{-t} \\ 2e^{-t} \end{pmatrix}$



$D = (-2,0)$

$\begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} K_1 + K_2 \\ 2K_2 \end{pmatrix}$ $K_2 = 0$, $K_1 = -2$

$\therefore \vec{Y}(t) = \begin{pmatrix} -2e^{-2t} \\ 0 \end{pmatrix}$



23. $\frac{dY}{dt} = \begin{pmatrix} -0.2 & -0.1 \\ 0 & -0.1 \end{pmatrix} Y$

$\lambda_1 = -0.2$: $\begin{pmatrix} -0.2 & -0.1 \\ 0 & -0.1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -0.2\begin{pmatrix} x \\ y \end{pmatrix}$

$-0.2x - 0.1y = -0.2x \quad y=0$

$-0.1y = -0.2y \quad y=0$

$\lambda_1 = 0.2$: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = -0.1$: $\begin{pmatrix} -0.2 & -0.1 \\ 0 & -0.1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -0.1\begin{pmatrix} x \\ y \end{pmatrix}$

$-0.2x - 0.1y = -0.1x \quad y = -x$

$-0.1y = -0.1y \quad y = \mathbb{R}$

$\lambda_2 = -0.1$: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

a) $\begin{cases} \frac{dx}{dt} = -0.2x - 0.1y \\ \frac{dy}{dt} = -0.1y \end{cases}$

$p(\lambda) = (-0.2 - \lambda)(-0.1 - \lambda)$

$\lambda_1 = -0.2$ and $\lambda_2 = -0.1$

The presence of the new species 'y' has a negative effect on 'x', the native species. The new species is unaffected by the presence of the native fish.

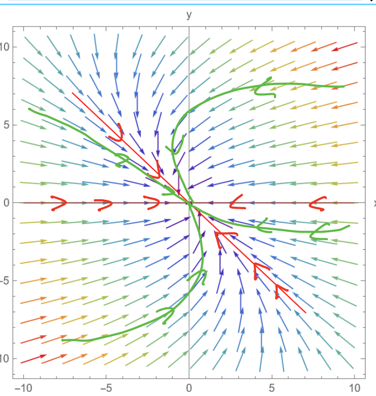
$\vec{Y}(t) = K_1 e^{-0.2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + K_2 e^{-0.1t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$,

$\vec{Y}(t) = \begin{pmatrix} K_1 e^{-0.2t} - K_2 e^{-t} \\ K_2 e^{-0.1t} \end{pmatrix}$

b) $x(0) = I$ and $y(0) = 0$ $\begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} K_1 - K_2 \\ K_2 \end{pmatrix}$

$K_2 = 0 \Rightarrow \vec{Y}(t) = \begin{pmatrix} Ie^{-0.2t} \\ 0 \end{pmatrix}$

c) Plot from Mathematica



$\lim_{t\to\infty} Ie^{-0.2t} = 0$. This aligns with the model as the population tends to its equilibrium value, 0.

c) As seen to the left, all solutions tend toward the origin. Straight line solutions do this as well, and lie on $y=-x$ and x-axis. The origin is a sink.

d) If the new species is added, the native species will drop below equilibrium, but will shortly go back up as the new species begins to die out.

25. $\frac{dY}{dt} = \begin{pmatrix} -0.2 & 0.1 \\ 0 & -0.1 \end{pmatrix} Y$

$\lambda_1 = -0.2$: $\begin{pmatrix} -0.2 & 0.1 \\ 0 & -0.1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -0.2\begin{pmatrix} x \\ y \end{pmatrix}$ $\quad$ $\lambda_2 = -0.1$: $\begin{pmatrix} -0.2 & 0.1 \\ 0 & -0.1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -0.1\begin{pmatrix} x \\ y \end{pmatrix}$

Same $p(\lambda)$ as 23 $\quad \lambda_1 = -0.2, \lambda_2 = -0.1$

$-0.2x + 0.1y = -0.2x \quad x = \mathbb{R}$; $-0.1y = -0.2y \quad y = 0$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$-0.2x + 0.1y = -0.1x \quad y = x$; $-0.1y = -0.1y \quad y = \mathbb{R}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

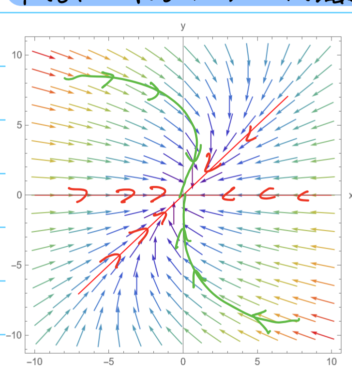
a) The native fish population is increased by the new fish, but the new fish are not impacted by the natives.

$\vec{Y}(t) = K_1 e^{-0.2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + K_2 e^{-0.1t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} K_1 e^{-0.2t} + K_2 e^{-0.1t} \\ K_2 e^{-0.1t} \end{pmatrix}$

b) $x(0) = I$ and $y(0) = 0$

$\begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} K_1 + K_2 \\ K_2 \end{pmatrix}$ $\quad K_2 = 0, \; K_1 = I \Rightarrow \vec{Y}(t) = \begin{pmatrix} Ie^{-0.2t} \\ 0 \end{pmatrix}$

c) Plot from Mathematica



$\lim_{t\to\infty} Ie^{-0.2t} = 0$ which aligns with the model.

c) The origin is a sink. Straight line solutions on $y=x$ and the x-axis. All other solutions tend to the origin.

d) Population of natives increases initially from equilibrium then returns to equilibrium as new fish die out.

27. $\frac{dY}{dt} = \begin{pmatrix}-2 & 1\\ 0 & 2\end{pmatrix} Y$

a) $(0,0)$ is a saddle if $\lambda_1 < 0$ and $\lambda_2 > 0$

$p(\lambda) = (-2-\lambda)(2-\lambda) \Rightarrow \lambda_1 = -2,\ \lambda_2 = 2$

Because $\lambda_1 = -2 < 0$ and $\lambda_2 = 2 > 0$ the origin $(0,0)$ is a saddle point.

b) $\lambda_1 = -2$: $\begin{pmatrix}-2 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = -2\begin{pmatrix}x\\ y\end{pmatrix}$

$-2x + y = -2x \quad x = \mathbb{R}$ if $y = 0$ $\quad \begin{pmatrix}1\\ 0\end{pmatrix}$

$2y = -2y \quad y = 0$

$\lambda_2 = 2$: $\begin{pmatrix}-2 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 2\begin{pmatrix}x\\ y\end{pmatrix}$

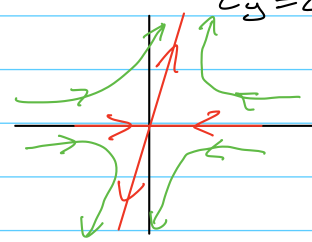
$-2x + y = 2x \quad y = 4x \quad \begin{pmatrix}1\\ 4\end{pmatrix}$

$2y = 2y \quad x = \mathbb{R}$

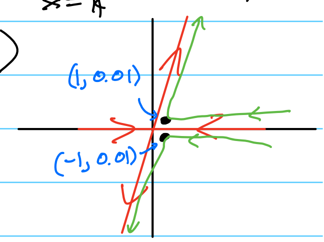
$\vec{Y}(t) = K_1 e^{-2t}\begin{pmatrix}1\\ 0\end{pmatrix} + K_2 e^{2t}\begin{pmatrix}1\\ 4\end{pmatrix}$

$= \begin{pmatrix} K_1 e^{-2t} + K_2 e^{2t}\\ 4K_2 e^{2t}\end{pmatrix}$

eigenvectors lie on x-axis and $y = 4x$



c)



d) $A_1 = (1, 0.01)$: $\begin{pmatrix}1\\ 0.01\end{pmatrix} = \begin{pmatrix}K_1 + K_2\\ 4K_2\end{pmatrix}$ $\quad K_2 = 0.0025$, $K_1 = 0.9975$

$\vec{Y}_1 = \begin{pmatrix}0.9975e^{-2t} + 0.0025e^{2t}\\ 0.01e^{2t}\end{pmatrix} \Rightarrow$ $x_1(t) = 0.9975e^{-2t} + 0.0025e^{2t}$, $y_1(t) = 0.01e^{2t}$

$A_2 = (1, -0.01)$: $\begin{pmatrix}1\\ -0.01\end{pmatrix} = \begin{pmatrix}K_1 + K_2\\ 4K_2\end{pmatrix}$ $\quad K_2 = -0.0025$, $K_1 = 1.0025$

$\vec{Y}_2 = \begin{pmatrix}1.0025e^{-2t} - 0.0025e^{2t}\\ -0.01e^{2t}\end{pmatrix}$ $\quad x_2(t) = 1.0025e^{-2t} - 0.0025e^{2t}$, $y_2(t) = -0.01e^{2t}$

Distance Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, $\boxed{\text{Find } t \text{ when } d = 1}$

$(x_2 - x_1)^2 = (1.0025e^{-2t} - 0.0025e^{2t} - (0.9975e^{-2t} + 0.0025e^{2t}))^2$

$= (0.005e^{-2t} - 0.005e^{2t})^2$

$= 0.000025e^{-4t} - 0.00005 + 0.000025e^{4t}$

$(y_2 - y_1)^2 = (-0.01e^{2t} - 0.01e^{2t})^2$

$= (-0.02e^{2t})^2$

$= 0.0004e^{4t}$

$(x_2 - x_1)^2 + (y_2 - y_1)^2 = 0.000025e^{-4t} - 0.00005 + 0.000025e^{4t} + 0.0004e^{4t}$

$= 0.000025e^{-4t} + 0.000425e^{4t} - 0.00005$

$1 = \sqrt{0.000025e^{-4t} + 0.000425e^{4t} - 0.00005}$

Using Mathematica: $t = -2.649$ or $t = 1.941$

Let $t \geq 0$, then 1 unit apart at $t = 1.941$

Section 3.4: 1, 3, 7, 11, 15, 22, 23

1. A has $\lambda = 1+3i$ and eigenvector $Y_0 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$

$$\vec{Y}(t) = K_1 e^{(1+3i)t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \begin{pmatrix} (2+i)K_1 e^{(1+3i)t} \\ K_1 e^{(1+3i)t} \end{pmatrix}$$

$$\vec{Y}(t) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))\vec{Y}_0 \longrightarrow \alpha = 1,\ \beta = 3,\ Y_0 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$\therefore \vec{Y}(t) = e^{t}(\cos 3t + i\sin 3t)\begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t(\cos 3t + i\sin 3t)(2+i) \\ e^t\cos 3t + ie^t\sin 3t \end{pmatrix} = \begin{pmatrix} e^t(2\cos 3t - \sin 3t + 2i\sin 3t + i\cos 3t) \\ e^t\cos 3t + ie^t\sin 3t \end{pmatrix}$$

Y_{re} and Y_{im} are both solutions of $\frac{dY}{dt}$

$$= \begin{pmatrix} e^t(2\cos 3t - \sin 3t) + ie^t(2\sin 3t + \cos 3t) \\ e^t\cos 3t + ie^t\sin 3t \end{pmatrix}$$

$$Y_{re} = e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix}$$

$$\therefore \vec{Y}(t) = e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} + ie^t\begin{pmatrix} 2\sin 3t + \cos 3t \\ \sin 3t \end{pmatrix}$$

$$Y_{im} = e^t\begin{pmatrix} 2\sin 3t + \cos 3t \\ \sin 3t \end{pmatrix}$$

General Sol: $\vec{Y}(t) = K_1 e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} + K_2 e^t\begin{pmatrix} 2\sin 3t + \cos 3t \\ \sin 3t \end{pmatrix}$

3. $\frac{dY}{dt} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} Y$, $Y_0 = (1,0)$

a) $p(\lambda) = \lambda^2 + 4 \quad \lambda = \pm 2i$

$\lambda_1 = 2i$ and $\lambda_2 = -2i$

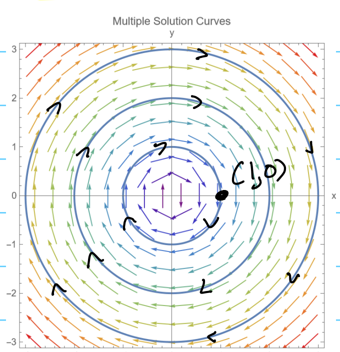
b) $\alpha = 0$ here so origin is a center

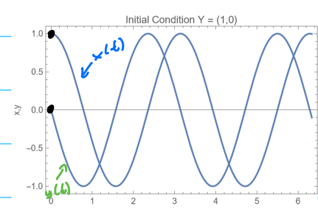
c) Natural Period $= 2\pi/\beta$ here $\beta = 2$

Natural Frequency $= \frac{\beta}{2\pi}$ here $\beta = 2$

Natural Period: π

Natural Frequency: $\frac{1}{\pi}$

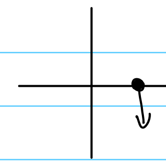




$t_0 = (1,0)$

d) Let $Y_0 = (1,0)$

$$\frac{dY}{dt}(1,0) = \begin{pmatrix} 2(0) \\ -2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$



This vector is roughly clockwise, so all others must be.

e) $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2i\begin{pmatrix} x \\ y \end{pmatrix}$

$2y = 2ix \qquad y = ix$

$-2x = 2iy \qquad y = -\frac{x}{i}$

$\begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\vec{Y}_1(t) = (\cos 2t + i\sin 2t)\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2t + i\sin 2t \\ i\cos 2t - \sin 2t \end{pmatrix} = \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + i\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$$

$\vec{Y}_{re} = \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix}$ and $\vec{Y}_{im} = \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$

$$\vec{Y}(t) = K_1\begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + K_2\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$$

$$= \begin{pmatrix} K_1\cos 2t + K_2\sin 2t \\ -K_1\sin 2t + K_2\cos 2t \end{pmatrix}$$

$Y_0 = (1,0)$: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} K_1 + 0 \\ 0 + K_2 \end{pmatrix} \qquad K_1 = 1,\ K_2 = 0$

$$\vec{Y}(t) = \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} \text{ for } Y_0 = (1,0)$$

7. $\frac{dY}{dt} = \begin{pmatrix} 2 & -6 \\ 2 & 1 \end{pmatrix} Y$, $Y_0 = (2, 1)$

a) $p(\lambda) = (2-\lambda)(1-\lambda) + 12$

$= 2 + \lambda^2 - \lambda - 2\lambda + 12$

$0 = \lambda^2 - 3\lambda + 14$

$\lambda = \frac{3 \pm \sqrt{9 - 4(14)}}{2} = \frac{3 \pm i\sqrt{47}}{2}$

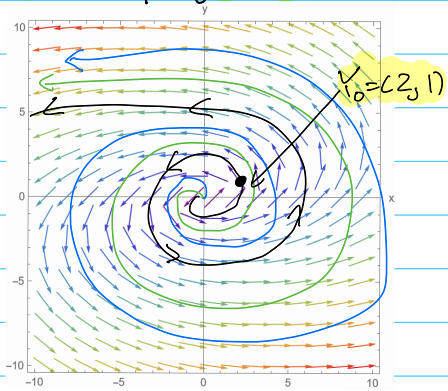
$\lambda_1 = \frac{3 + i\sqrt{47}}{2}$ and $\lambda_2 = \frac{3 - i\sqrt{47}}{2}$

b) $\alpha = \frac{3}{2} > 0$ so origin is a spiral source

c) $\beta = \frac{\sqrt{47}}{2}$

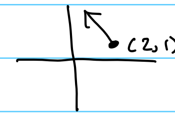
Natural Period: $4\pi / \sqrt{47}$

Natural Frequency: $\frac{\sqrt{47}}{4\pi}$



d) $\frac{dY}{dt} = \begin{pmatrix} 2x - 6y \\ 2x + y \end{pmatrix}$ @ $(2, 1)$

$= \begin{pmatrix} -2 \\ 5 \end{pmatrix}$



Vector points counterclockwise, all curves follow.

e) $\begin{pmatrix} 2 & -6 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{3}{2} + i\frac{\sqrt{47}}{2}\right) \begin{pmatrix} x \\ y \end{pmatrix}$

$2x - 6y = \frac{3}{2}x + xi\frac{\sqrt{47}}{2}$

$\frac{1}{2}x - 6y = xi\frac{\sqrt{47}}{2}$

$2x + y = \frac{3}{2}y + yi\frac{\sqrt{47}}{2}$ $\quad 2x = \frac{1}{2} + i\frac{\sqrt{47}}{2}$

$2x - \frac{1}{2}y = yi\frac{\sqrt{47}}{2}$ $\quad x = \frac{1}{4} + i\frac{\sqrt{47}}{4}$

$\begin{pmatrix} \frac{1}{4} + i\frac{\sqrt{47}}{4} \\ 1 \end{pmatrix}$

$\vec{Y}(t) = e^{\frac{3}{2}t}\left(\cos\left(t\frac{\sqrt{47}}{2}\right) + i\sin\left(t\frac{\sqrt{47}}{2}\right)\right)\begin{pmatrix} \frac{1}{4} + i\frac{\sqrt{47}}{4} \\ 1 \end{pmatrix}$

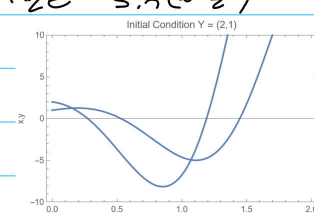
$= \begin{pmatrix} e^{3/2t}\left(\cos\left(t\frac{\sqrt{47}}{2}\right) + i\sin\left(t\frac{\sqrt{47}}{2}\right)\right)\left(\frac{1}{4} + i\frac{\sqrt{47}}{4}\right) \\ e^{3/2t}\left(\cos\left(t\frac{\sqrt{47}}{2}\right) + i\sin\left(t\frac{\sqrt{47}}{2}\right)\right) \end{pmatrix}$

$= e^{3/2t}\begin{pmatrix} \frac{1}{4}\cos\left(t\frac{\sqrt{47}}{2}\right) + \frac{1}{4}i\sin\left(t\frac{\sqrt{47}}{2}\right) + i\frac{\sqrt{47}}{4}\cos\left(t\frac{\sqrt{47}}{2}\right) - \frac{\sqrt{47}}{4}\sin\left(t\frac{\sqrt{47}}{2}\right) \\ \cos\left(t\frac{\sqrt{47}}{2}\right) + i\sin\left(t\frac{\sqrt{47}}{2}\right) \end{pmatrix}$

$\vec{Y}(t) = e^{3/2t}\begin{pmatrix} \frac{1}{4}\cos\left(t\frac{\sqrt{47}}{2}\right) - \frac{\sqrt{47}}{4}\sin\left(t\frac{\sqrt{47}}{2}\right) \\ \cos\left(t\frac{\sqrt{47}}{2}\right) \end{pmatrix} + ie^{3/2t}\begin{pmatrix} \frac{1}{4}\sin\left(t\frac{\sqrt{47}}{2}\right) + \frac{\sqrt{47}}{4}\cos\left(t\frac{\sqrt{47}}{2}\right) \\ \sin\left(t\frac{\sqrt{47}}{2}\right) \end{pmatrix}$

$\vec{Y}(t) = \begin{pmatrix} K_1 e^{3/2t}\left(\frac{1}{4}\left(\cos\left(t\frac{\sqrt{47}}{2}\right) - \sqrt{47}\sin\left(t\frac{\sqrt{47}}{2}\right)\right) + K_2\left(\sin\left(t\frac{\sqrt{47}}{2}\right) + \sqrt{47}\cos\left(t\frac{\sqrt{47}}{2}\right)\right)\right) \\ K_1 e^{3/2t}\cos\left(t\frac{\sqrt{47}}{2}\right) + K_2 e^{3/2t}\sin\left(t\frac{\sqrt{47}}{2}\right) \end{pmatrix}$

$\left.\begin{aligned} K_1 &= 1 \\ K_2 &= 0.255 \end{aligned}\right\}$ from Mathematica



11.a) $\frac{dY}{dt} = \begin{pmatrix} -3 & -5 \\ 3 & 1 \end{pmatrix} Y$, $Y_0 = (4, 0)$

$p(\lambda) = (-3-\lambda)(1-\lambda) + 15$

$= -3 + 3\lambda - \lambda + \lambda^2 + 15$

$0 = \lambda^2 + 2\lambda + 12$

$\lambda = \frac{-2 \pm \sqrt{4 - 4(12)}}{2} = \frac{-2 \pm 2i\sqrt{11}}{2} = -1 \pm i\sqrt{11}$

$\lambda_1 = -1 + i\sqrt{11}$: $\begin{pmatrix} -3 & -5 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (-1 + i\sqrt{11})\begin{pmatrix} x \\ y \end{pmatrix}$

$-3x - 5y = -x + xi\sqrt{11} \rightarrow y = -\frac{1}{5}(2x + xi\sqrt{11})$

$3x + y = -y + yi\sqrt{11} \rightarrow x = \frac{1}{3}(-2y + yi\sqrt{11})$

Let $y = 3$: $\begin{pmatrix} -2 + i\sqrt{11} \\ 3 \end{pmatrix}$

Continued on Next Page

$\vec{Y}(t) = e^{-t}\left(\cos(\sqrt{11}t) + i\sin(\sqrt{11}t)\right)\begin{pmatrix} -2 + i\sqrt{11} \\ 3 \end{pmatrix}$

$\vec{Y}(t) = e^{-t}\begin{pmatrix} \left(\cos(\sqrt{11}t) + i\sin(\sqrt{11}t)\right)(-2 + i\sqrt{11}) \\ 3\cos(\sqrt{11}t) + 3i\sin(\sqrt{11}t) \end{pmatrix}$

$= e^{-t}\begin{pmatrix} -2\cos\sqrt{11}t - 2i\sin(\sqrt{11}t) + i\sqrt{11}\cos(\sqrt{11}t) - \sqrt{11}\sin(\sqrt{11}t) \\ 3\cos\sqrt{11}t + 3i\sin\sqrt{11}t \end{pmatrix}$

$\vec{Y}(t) = K_1 e^{-t}\begin{pmatrix} -2\cos\sqrt{11}t - \sqrt{11}\sin\sqrt{11}t \\ 3\cos\sqrt{11}t \end{pmatrix}$

$+ K_2 i e^{-t}\begin{pmatrix} -2\sin\sqrt{11}t + \sqrt{11}\cos\sqrt{11}t \\ 3\sin\sqrt{11}t \end{pmatrix}$

$\vec{Y}(t) = K_1 e^{-t}\begin{pmatrix} -2\cos\sqrt{11}t - \sqrt{11}\sin\sqrt{11}t \\ 3\cos\sqrt{11}t \end{pmatrix} + K_2 e^{-t}\begin{pmatrix} -2\sin\sqrt{11}t + \sqrt{11}\cos\sqrt{11}t \\ 3\sin\sqrt{11}t \end{pmatrix}$

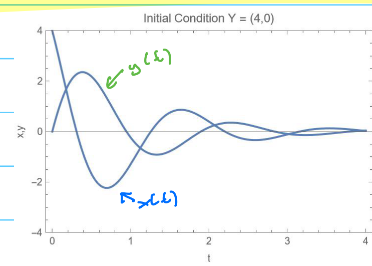
11 b) $\vec{Y}(t)=K_1e^{-t}\begin{pmatrix}-2\cos\sqrt{11}t-\sqrt{11}\sin\sqrt{11}t\\ 3\cos\sqrt{11}t\end{pmatrix}+K_2e^{-t}\begin{pmatrix}-2\sin\sqrt{11}t+\sqrt{11}\cos\sqrt{11}t\\ 3\sin\sqrt{11}t\end{pmatrix}$, $Y_0=(4,0)$

$$\begin{pmatrix}4\\0\end{pmatrix}=K_1e^{-t}\begin{pmatrix}-2\cos\sqrt{11}t-\sqrt{11}\sin\sqrt{11}t\\ 3\cos\sqrt{11}t\end{pmatrix}+K_2e^{-t}\begin{pmatrix}-2\sin\sqrt{11}t+\sqrt{11}\cos\sqrt{11}t\\ 3\sin\sqrt{11}t\end{pmatrix}$$

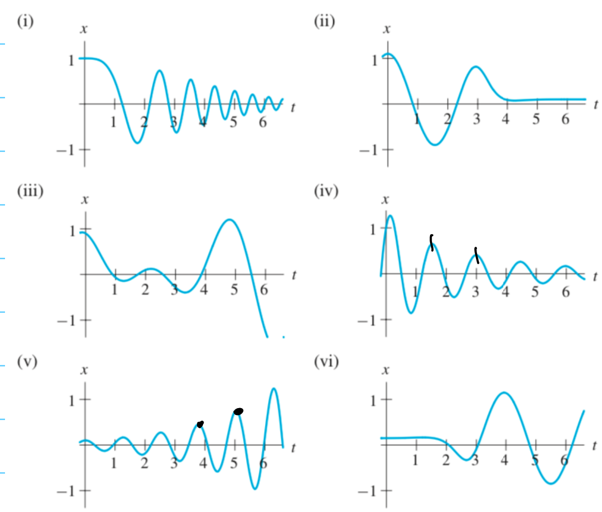
$$\left.\begin{array}{ll} 4=K_1(-2)+K_2(\sqrt{11}) & K_1=0 \\ 0=K_1(3)+K_2(0) & \therefore K_2=\frac{4}{\sqrt{11}}\end{array}\right\}\Rightarrow \vec{Y}(t)=\frac{4}{\sqrt{11}}e^{-t}\begin{pmatrix}-2\sin(\sqrt{11}t)+\sqrt{11}\cos(\sqrt{11}t)\\ 3\sin(\sqrt{11}t)\end{pmatrix}$$

c) $x(t)=\frac{4}{\sqrt{11}}e^{-t}\left(-2\sin\sqrt{11}t+\sqrt{11}\cos\sqrt{11}t\right)$

$y(t)=\frac{4}{\sqrt{11}}e^{-t}\left(3\sin\sqrt{11}t\right)$



15.



a) Must be Completely Periodic. (i) is not periodic @ t=0, and (ii), (iii), and (vi) are not periodic. Only non-real eigenvalues are (iv) and (v)

b) For graph (iv), Origin is a sink as $x\to 0$ as $t\to\infty$. Period ≈ 1.5

For graph (v), origin is a source as $x\to\infty$ as $t\to 0$. Period ≈ 1.3

c) (i) Oscilation Period is not constant. (ii) Oscillations stop eventually (iii) Amplitude is changing variable. (vi) Oscillation starts, without prior oscillations, at some time.

22. $x(t)=K_1\cos(\beta t)+K_2\sin(\beta t)$, show $x(t)=K\cos(\beta t-\phi)$ is true

Suppose $K=\sqrt{K_1^2+K_2^2}$ where $K_1=K\cos\phi$ and $K_2=K\sin\phi$

Substitute: $x(t)=K\cos\phi\cos\beta t+K\sin\phi\sin\beta t$

$x(t)=K(\cos\phi\cos\beta t+\sin\phi\sin\beta t)$

Cosine Identity: $x(t)=K\cos(\beta t-\phi)$. This is what we are looking to prove //

23. $\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$

a) $\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -pv - qy \end{cases}$ $\quad \frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}$

b) $p(\lambda) = (-\lambda)(-p-\lambda) + q$ $\quad \lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$ $\quad p^2 - 4q < 0$ for λ to be complex

$= \lambda p + \lambda^2 + q$

$0 = \lambda^2 + \lambda p + q$

$\therefore p^2 < 4q$ to create complex eigenvalues

c) $\frac{-p}{2} = \alpha$ Spiral Sink: $\alpha < 0 \therefore \frac{-p}{2} < 0 \to p > 0$

but $p^2 < 4q$ so $p < \pm 2\sqrt{q}$, assuming $q > 0$

$q > 0$ and $0 < p < 2\sqrt{q}$

Center: $\alpha = 0 \therefore \frac{-p}{2} = 0 \to p = 0$

but $p^2 < 4q$ so $0 < 4q \to q > 0$

$q > 0$ and $p = 0$

Spiral Center: $\alpha > 0 \therefore \frac{-p}{2} > 0 \to p < 0$

but $p^2 < 4q$ so $p < \pm 2\sqrt{q}$, assuming $q > 0$

$q > 0$ and $-2\sqrt{q} < p < 0$

d) Given some $(a, 0)$, its vector $\frac{d\vec{Y}}{dt}(a,0) = \begin{pmatrix} c \\ d \end{pmatrix}$ must have $c = 0$ and $d < 0$ in the first quadrant to be clockwise

$\begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ d \end{pmatrix}$

$0 = 0$ ✓

$-qa = d \to q = -\frac{d}{a}$

$q = -\frac{d}{a}$ for some $\begin{cases} a > 0 \\ d < 0 \end{cases}$

$\therefore$ q is positive, $q > 0$

Section 3.5: 3, 7, 10, 13, 15, 19, 21

3. $\frac{dY}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} Y$, $Y_0 = (1,0)$

a) $p(\lambda) = (-2-\lambda)(-4-\lambda) + 1$

$= 8 + 2\lambda + 4\lambda + \lambda^2 + 1$

$0 = \lambda^2 + 6\lambda + 9$

$0 = (\lambda+3)^2 \rightarrow \lambda = -3$

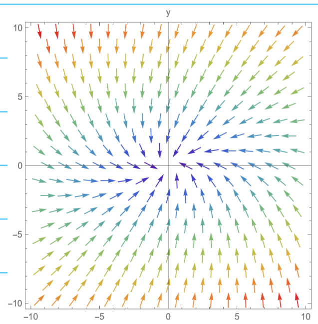
b) $\begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$

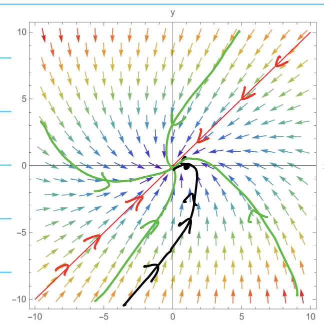
$-2x - y = -3x \Rightarrow y = x$

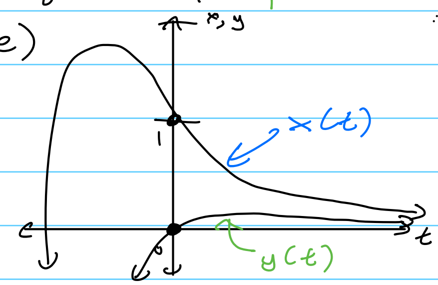
$x - 4y = -3y \Rightarrow y = x$

All eigenvectors in form $y = x$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$Y(t) = e^{\lambda t} V_0 + t e^{\lambda t} V_1$ $\quad V_1 = (A - \lambda I) V_0$

c) 

d) 

e) 

x 'swings', y moves consistently

7. $\frac{dY}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} Y$, $Y_0 = (1, 0)$

a) $\lambda = -3$: $\begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$

(from 3) $-2x - y = -3x \quad y = x \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$x - 4y = -3y \quad y = x$

$V_1 = (A - \lambda I) V_0$

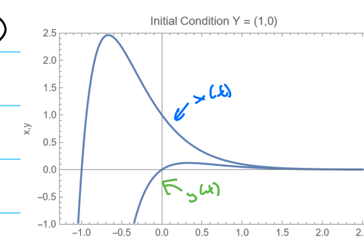
$V_1 = \begin{pmatrix} -2+3 & -1 \\ 1 & -4+3 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$\therefore V_1 = \begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 \end{pmatrix}$ $\quad \vec{Y}(t) = e^{-3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^{-3t}\begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 \end{pmatrix}$

b) $x_0 = 1$, $y_0 = 0$

$\begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 \end{pmatrix} = \begin{pmatrix} 1 - 0 \\ 1 - 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\therefore \vec{Y}(t) = e^{-3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

c) 

Very close to 3 sketch

10. a) $\lambda > 0$ $\quad \lim_{t \to \infty} t e^{\lambda t} = \infty$

b) $\lambda < 0$ $\quad \lim_{t \to \infty} t e^{\lambda t} = \lim_{t \to \infty} \frac{t}{e^{-\lambda t}} = 0$ as $e^{\lambda t} \to \infty$, so $\frac{a}{\infty} = 0$

13. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $AY = \lambda Y$ for all Y

Suppose $\vec{Y} = (1, 0)$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$a = \lambda$

$c = 0$

Suppose $\vec{Y} = (0, 1)$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$b = 0$

$d = \lambda$

For some $\vec{Y}_s$, $a = d = \lambda$ and $b = c = 0$ is true, so it is true for the system based on the given value $AY = \lambda Y$ is true for all Y.

15. $Y_1(t)=e^{\lambda t}V_0+te^{\lambda t}V_1$ and $Y_2(t)=e^{\lambda t}W_0+te^{\lambda t}W_1$

Let $t=0$: $Y_1(0)=V_0+0V_1$ and $Y_2(0)=W_0+0W_1$

$=V_0$ $\qquad$ $=W_0$

It is given that Y_1 and Y_2 are equal for all t

$\therefore$ If $Y_1(0)=Y_2(0)$ then $V_0=W_0$

Let $t=1$: $Y_1(1)=e^{\lambda}V_0+e^{\lambda}V_1$ and $Y_2(1)=e^{\lambda}W_0+e^{\lambda}W_1$

We have proved $V_0=W_0$, so substitute:

$Y_1(1)=e^{\lambda}W_0+e^{\lambda}V_1$ and $Y_2(1)=e^{\lambda}W_0+e^{\lambda}W_1$

$\therefore$ $Y_1(1)=e^{\lambda}(W_0+V_1)$ and $Y_2(1)=e^{\lambda}(W_0+W_1)$

Make use of the given, and set $Y_1=Y_2$

$e^{\lambda}(W_0+V_1)=e^{\lambda}(W_0+W_1)$, $e^{\lambda}\neq 0$ so you can divide

$W_0+V_1=W_0+W_1$

$\therefore$ $V_1=W_1$

As seen above, $t=1$ proves $V_1=W_1$ as the functions are equal $\forall t$

19. $\dfrac{dY}{dt}=\begin{pmatrix}4&2\\2&1\end{pmatrix}Y$

a) $p(\lambda)=(4-\lambda)(1-\lambda)-4$

$=4-4\lambda-\lambda+\lambda^2-4$

$=\lambda^2-5\lambda$

$=\lambda(\lambda-5)\rightarrow \lambda_1=0$ and $\lambda_2=5$

b) $\lambda_1=0$: $\begin{pmatrix}4&2\\2&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0\begin{pmatrix}x\\y\end{pmatrix}$

$4x+2y=0$ $\quad y=-2x$

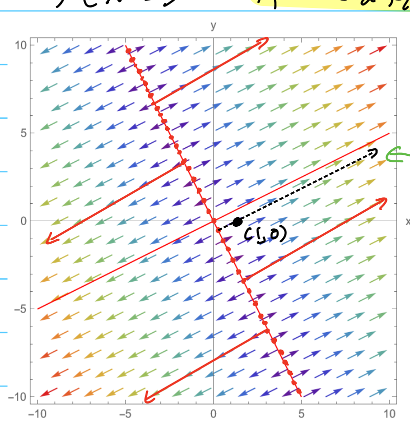
$2x+y=0$ $\quad y=-2x$ $\quad \begin{pmatrix}1\\-2\end{pmatrix}$

$\lambda_2=5$: $\begin{pmatrix}4&2\\2&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=5\begin{pmatrix}x\\y\end{pmatrix}$

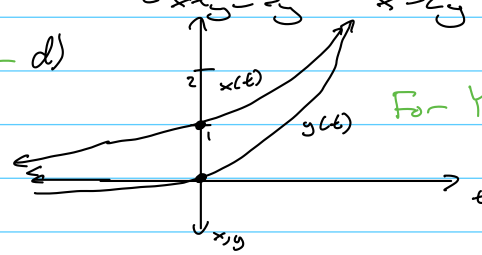
$4x+2y=5x$ $\quad x=2y$

$2x+y=5y$ $\quad x=2y$ $\quad \begin{pmatrix}2\\1\end{pmatrix}$

c)



d)



For $Y_0=(1,0)$

Should take the form of a typical linear solution

e) $Y(t)=K_1e^{0t}\begin{pmatrix}1\\-2\end{pmatrix}+K_2e^{5t}\begin{pmatrix}2\\1\end{pmatrix}$

$Y(t)=\begin{pmatrix}K_1+2K_2e^{5t}\\-2K_1+K_2e^{5t}\end{pmatrix}$

$Y_0=(1,0)$: $\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}K_1+2K_2\\-2K_1+K_2\end{pmatrix}$ $\quad K_2=2K_1$

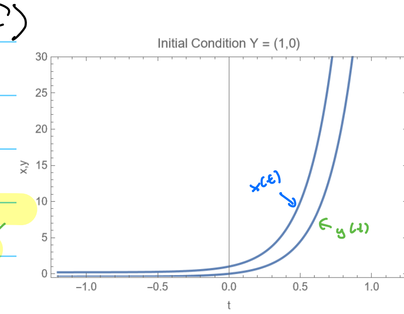
$1=K_1+2(2K_1)$

$1=5K_1$ $\quad K_1=\frac{1}{5}$

$K_1=\frac{1}{5}$, $K_2=\frac{2}{5}$

$\therefore$ $Y(t)=\begin{pmatrix}\frac{1}{5}+\frac{4}{5}e^{5t}\\-\frac{2}{5}+\frac{2}{5}e^{5t}\end{pmatrix}$

f)



f) For f, Graph is pretty close to sketch in d ✓

21. a) $\frac{dY}{dt} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} Y$

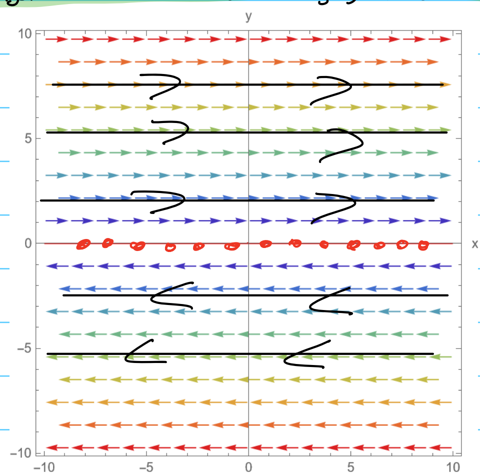
$p(\lambda) = (-\lambda)(-\lambda)$

$= \lambda^2 \to \lambda = 0$

$\lambda = 0: \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\begin{pmatrix} x \\ y \end{pmatrix}$

$2y = 0 \quad y = 0$

Eigenvectors all along with $y=0$, the x-axis



All horizontal Solutions

b) $\frac{dY}{dt} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} Y$

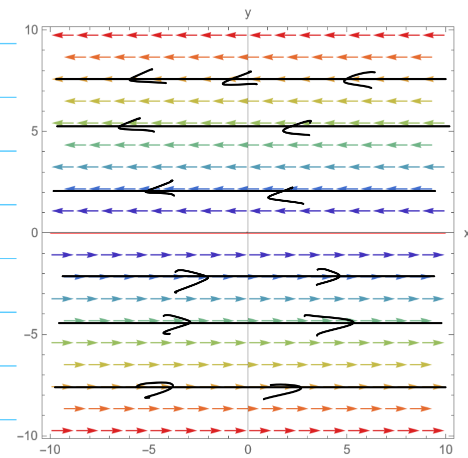
$p(\lambda) = (-\lambda)(-\lambda)$

$= \lambda^2 \to \lambda = 0$

$\lambda = 0: \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\begin{pmatrix} x \\ y \end{pmatrix}$

$-2y = 0 \quad y = 0$

Eigenvectors all along with $y=0$, the x-axis



Almost Identical to A, just reversed