Spring 2024 Math 224

Instructions: show all steps to get full credits.

Problem 1. Given the linear system

$$\frac{dx}{dt} = 2y,$$
$$\frac{dy}{dt} = -2x,$$

1. Find the eigenvalues.
   Solution.
   - The coefficient matrix $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.
   - $A - \lambda I = \begin{bmatrix} -\lambda & 2 \\ -2 & -\lambda \end{bmatrix}$.
   - $\det(A - \lambda I) = \lambda^2 + 4$.
   - Solve for the eigenvalues:
     $$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm\sqrt{-4} = \pm\sqrt{4} \cdot \sqrt{-1} = \pm 2i$$
   - Denote $\lambda_1 = 2i$, $\lambda_2 = -2i$.

2. Determine if the origin is a spiral sink, a spiral source, or a center.
   Answer. The real part $\alpha = 0$, so the origin is a center.

3. Determine the natural period and natural frequency of the oscillations.
   Answer.
   - The complex part $\beta = 2$. The period is $\frac{2\pi}{\beta} = \frac{2\pi}{2} = \pi$.
   - The frequency is $\frac{1}{\text{the period}} = \frac{1}{\pi}$.

4. Determine the direction of the oscillations in the phase plane, that is, determine whether the solutions go clockwise or counter-clockwise.
   Answer.
   - Test the slope at any point other than $(0, 0)$: For example, we choose $(x, y) = (1, 0)$.

- Use the equations in the system:

$$\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle 2y, -2x\rangle = \langle 2\cdot 0, -2\cdot 1\rangle = \langle 0, -2\rangle$$

- $\langle 0, -2\rangle$ indicates the clockwise direction of the oscillation.

5. Find the general solution.
Solution.

- We choose $\lambda_1 = 2i$.
- A complex eigenvector can be computed by

$$\begin{aligned}(A - \lambda_1 I)\vec{v}_1 = \vec{0} \ &\Rightarrow \begin{bmatrix} -2i & 2 \\ -2 & 2i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \quad \begin{aligned} -2i\alpha + 2\beta &= 0 \\ -2\alpha + 2i\beta &= 0 \end{aligned} \\ &\Rightarrow \quad \alpha = i\beta \end{aligned}$$

- Choose $\beta = 1$, so $\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

- $\mathbf{Y}_1(t) = e^{\lambda_1 t}\vec{v}_1 = e^{2it}\begin{bmatrix} i \\ 1 \end{bmatrix}$.

- Use Euler's formula and decompose:

$$\begin{aligned} e^{2it}\begin{bmatrix} i \\ 1 \end{bmatrix} &= (\cos(2t) + i\sin(2t))\begin{bmatrix} i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix} + i\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} \end{aligned}$$

- The general solution:

$$\mathbf{Y}(t) = k_1\begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix} + k_2\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}$$

Hence,

$$\begin{aligned} x(t) &= -k_1\sin(2t) + k_2\cos(2t) \\ y(t) &= k_1\cos(2t) + k_2\sin(2t) \end{aligned}$$

6. Find the particular solution with the initial value $\mathbf{Y}(0) = (1, 0)$.
Solution. For $(x(0), y(0)) = (1, 0)$, $k_1 = 0$ and $k_2 = 1$. Therefore,

$$(x(t), y(t)) = (\cos(2t), \sin(2t))$$

Problem 2. Given the linear system

$$\frac{dx}{dt} = 6x + 2y,$$
$$\frac{dy}{dt} = -x + 8y,$$

1. Find the eigenvalues.
   Solution.

   - The coefficient matrix $A = \begin{bmatrix} 6 & 2 \\ -1 & 8 \end{bmatrix}$.
   - $A - \lambda I = \begin{bmatrix} 6-\lambda & 2 \\ -1 & 8-\lambda \end{bmatrix}$.
   - $\det(A - \lambda I) = \lambda^2 - 14\lambda + 50$.
   - Solve for the eigenvalues: $\lambda_1 = 7 + i$ and $\lambda_2 = 7 - i$.

2. Determine if the origin is a spiral sink, a spiral source, or a center.
   Answer. The real part $\alpha = 7 > 0$, so the origin is a spiral source.

3. Determine the natural period and natural frequency of the oscillations.
   Answer.

   - The complex part $\beta = 1$. The period is $\frac{2\pi}{\beta} = \frac{2\pi}{1} = 2\pi$.
   - The frequency is $\frac{1}{\text{the period}} = \frac{1}{2\pi}$.

4. Determine the direction of the oscillations in the phase plane, that is, determine whether the solutions go clockwise or counter-clockwise.
   Answer.

   - Test the slope at any point other than $(0,0)$: For example, we choose $(x, y) = (1, 0)$.
   - Use the equations in the system:

   $$\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle 6x + 2y, -x + 8y \rangle = \langle 6 \cdot 1 + 2 \cdot 0, -1 + 8 \cdot 0 \rangle = \langle 6, -1 \rangle$$

   - $\langle 6, -1 \rangle$ indicates the clockwise direction of the oscillation.

5. Find the general solution.
   Solution.

- We choose $\lambda_1 = 7 + i$.
- A complex eigenvector can be computed by

$$(A - \lambda_1 I)\vec{v}_1 = \vec{0} \Rightarrow \begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \begin{aligned} (-1-i)\alpha + 2\beta &= 0 \\ -\alpha + (1-i)\beta &= 0 \end{aligned}$$
$$\Rightarrow \alpha = (1-i)\beta$$

- Choose $\beta = 1$, so $\vec{v}_1 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$.
- $\mathbf{Y}_1(t) = e^{\lambda_1 t}\vec{v}_1 = e^{(7+i)t}\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$.
- Use Euler's formula and decompose:

$$\begin{aligned} e^{(7+i)t}\begin{bmatrix} 1-i \\ 1 \end{bmatrix} &= (e^{7t}\cos(t) + ie^{7t}\sin(t))\begin{bmatrix} 1-i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{7t}\cos(t) + e^{7t}\sin(t) \\ e^{7t}\cos(t) \end{bmatrix} + i\begin{bmatrix} -e^{7t}\cos(t) + e^{7t}\sin(t) \\ e^{7t}\sin(t) \end{bmatrix} \end{aligned}$$

- The general solution:

$$\mathbf{Y}(t) = k_1\begin{bmatrix} e^{7t}\cos(t) + e^{7t}\sin(t) \\ e^{7t}\cos(t) \end{bmatrix} + k_2\begin{bmatrix} -e^{7t}\cos(t) + e^{7t}\sin(t) \\ e^{7t}\sin(t) \end{bmatrix}$$

Hence,

$$\begin{aligned} x(t) &= (k_1 - k_2)e^{7t}\cos(t) + (k_1 + k_2)e^{7t}\sin(t) \\ y(t) &= k_1 e^{7t}\cos(t) + k_2 e^{7t}\sin(t) \end{aligned}$$

6. Find the particular solution with the initial value $\mathbf{Y}(0) = (1, -1)$.
Solution. $(x(0), y(0)) = (1, -1)$.

$$\begin{aligned} k_1 - k_2 &= 1 \\ k_1 &= -1 \end{aligned}$$

Hence, $k_1 = -1$, $k_2 = 1$.

$$\begin{aligned} x(t) &= -2e^{7t}\cos(t) \\ y(t) &= -e^{7t}\cos(t) + e^{7t}\sin(t) \end{aligned}$$

Problem 3. Given the linear system

$$\frac{dx}{dt} = -3x + 3y,$$
$$\frac{dy}{dt} = y - 5x,$$

1. Find the eigenvalues.
   Solution.
   - The coefficient matrix $A = \begin{bmatrix} -3 & 3 \\ -5 & 1 \end{bmatrix}$.
   - $A - \lambda I = \begin{bmatrix} -3-\lambda & 3 \\ -5 & 1-\lambda \end{bmatrix}$.
   - $\det(A - \lambda I) = \lambda^2 + 2\lambda + 12$.
   - Solve for the eigenvalues: $\lambda_1 = -1 + \sqrt{11}i$ and $\lambda_2 = -1 - \sqrt{11}i$.

2. Determine if the origin is a spiral sink, a spiral source, or a center.
   Answer. The real part $\alpha = -1 < 0$, so the origin is a spiral sink.

3. Determine the natural period and natural frequency of the oscillations.
   Answer.
   - The complex part $\beta = \sqrt{11}$. The period is $\dfrac{2\pi}{\sqrt{11}}$.
   - The frequency is $\dfrac{1}{\text{the period}} = \dfrac{\sqrt{11}}{2\pi}$.

4. Determine the direction of the oscillations in the phase plane, that is, determine whether the solutions go clockwise or counter-clockwise.
   Answer.
   - Test the slope at any point other than $(0, 0)$: For example, we choose $(x, y) = (1, 0)$.
   - Use the equations in the system:
   $$\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle -3x + 3y, y - 5x \rangle = \langle -3, -5 \rangle$$
   - $\langle -3, -5 \rangle$ indicates the clockwise direction of the oscillation.

5. Find the general solution.
   Solution.

- We choose $\lambda_1 = -1 + \sqrt{11}i$.
- A complex eigenvector can be computed by

$$
\begin{aligned}
(A - \lambda_1 I)\vec{v}_1 = \vec{0} \ &\Rightarrow \ \begin{bmatrix} -2-\sqrt{11}i & 3 \\ -5 & 2-\sqrt{11}i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&\Rightarrow \quad \begin{array}{c} (-2-\sqrt{11}i)\alpha + 3\beta = 0 \\ -5\alpha + (2-\sqrt{11}i)\beta = 0 \end{array} \\
&\Rightarrow \quad \alpha = \tfrac{2-\sqrt{11}i}{5}\beta
\end{aligned}
$$

- Choose $\beta = 5$, so $\vec{v}_1 = \begin{bmatrix} 2-\sqrt{11}i \\ 5 \end{bmatrix}$.
- $\mathbf{Y}_1(t) = e^{\lambda_1 t}\vec{v}_1 = e^{(-1+\sqrt{11}i)t}\begin{bmatrix} 2-\sqrt{11}i \\ 5 \end{bmatrix}$.
- Use Euler's formula and decompose:

$$
\begin{aligned}
e^{(-1+\sqrt{11}i)t}\begin{bmatrix} 2-\sqrt{11}i \\ 5 \end{bmatrix} &= (e^{-t}\cos(\sqrt{11}t) + ie^{-t}\sin(\sqrt{11}t))\begin{bmatrix} 2-\sqrt{11}i \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} 2e^{-t}\cos(\sqrt{11}t) + \sqrt{11}e^{-t}\sin(\sqrt{11}t) \\ e^{-t}\cos(\sqrt{11}t) \end{bmatrix} \\
&\quad + i\begin{bmatrix} -\sqrt{11}e^{-t}\cos(\sqrt{11}t) + 2e^{-t}\sin(\sqrt{11}t) \\ e^{-t}\sin(\sqrt{11}t) \end{bmatrix}
\end{aligned}
$$

- The general solution:

$$
\begin{aligned}
\mathbf{Y}(t) = k_1 &\begin{bmatrix} 2e^{-t}\cos(\sqrt{11}t) + \sqrt{11}e^{-t}\sin(\sqrt{11}t) \\ e^{-t}\cos(\sqrt{11}t) \end{bmatrix} \\
+ k_2 &\begin{bmatrix} -\sqrt{11}e^{-t}\cos(\sqrt{11}t) + 2e^{-t}\sin(\sqrt{11}t) \\ 5e^{-t}\sin(\sqrt{11}t) \end{bmatrix}
\end{aligned}
$$

  Hence,

$$
\begin{aligned}
x(t) &= (2k_1 - \sqrt{11}k_2)e^{-t}\cos(\sqrt{11}t) + (\sqrt{11}k_1 + 2k_2)e^{-t}\sin(\sqrt{11}t) \\
y(t) &= 5k_1e^{-t}\cos(\sqrt{11}t) + 5k_2e^{-t}\sin(\sqrt{11}t)
\end{aligned}
$$

6. Find the particular solution with the initial value $\mathbf{Y}(0) = (4, 0)$.
Solution. $(x(0), y(0)) = (4, 0)$.

$$
\begin{aligned}
2k_1 - \sqrt{11}k_2 &= 4 \\
5k_1 &= 0
\end{aligned}
$$

Hence, $k_1 = 0$, $k_2 = -4/\sqrt{11}$.

$$x(t) = 4e^{-t}\cos(\sqrt{11}t) - \frac{8}{\sqrt{11}}e^{-t}\sin(\sqrt{11}t)$$
$$y(t) = -\frac{20}{\sqrt{11}}e^{-t}\sin(\sqrt{11}t)$$

Problem 4. Given the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\mathbf{Y}$$

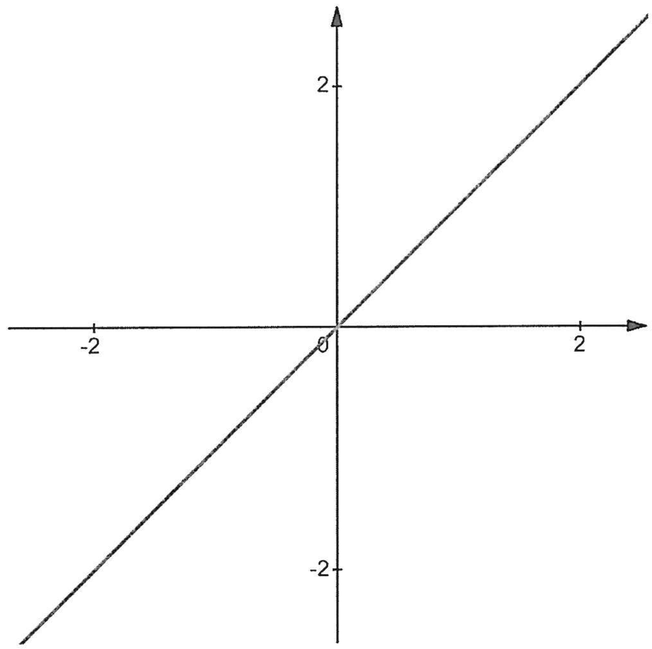
1. Find the eigenvalue.
   Solution.
   - The coefficient matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$.
   - $A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix}$.
   - $\det(A - \lambda I) = \lambda^2 - 6\lambda + 9$.
   - Solve for the eigenvalues: $\lambda = 3$ is the repeated eigenvalue.

2. Find an eigenvector.
   Solution. $\vec{v}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.
   - Compute $\vec{v}_1$:

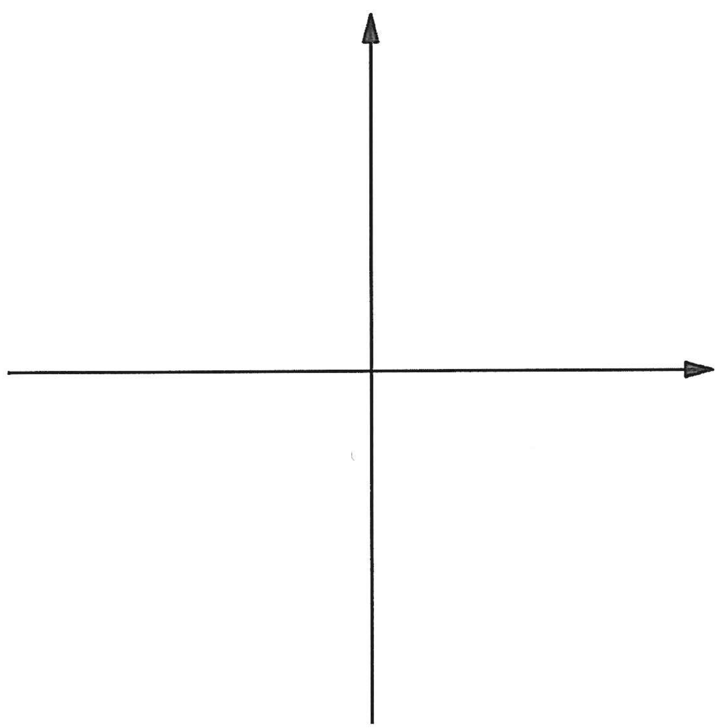
$$\begin{aligned} \vec{v}_1 &= (A - 3I)\vec{v}_0 \\ &= \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \begin{bmatrix} -x_0 + y_0 \\ -x_0 + y_0 \end{bmatrix} \end{aligned}$$

   - We can choose $x_0 = 0$ and $y_0 = 1$. An eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3. Sketch the phase portrait including the solution curve with the initial value $\mathbf{Y}(0) = (1, 0)$.
Solution.



4. Sketch the $x(t)$ graph in the tx-plane and $y(t)$ graph in the ty plane of the solution curve with the initial value $\mathbf{Y}(0) = (1, 0)$.
Solution.



5. Find the general solution.
Solution. The general solution:

$$\begin{aligned}\mathbf{Y}(t) &= e^{\lambda t}\vec{v}_0 + te^{\lambda t}\vec{v}_1 \\ &= e^{3t}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + te^{3t}\begin{bmatrix} -x_0 + y_0 \\ -x_0 + y_0 \end{bmatrix}\end{aligned}$$

Hence,

$$\begin{aligned}x(t) &= x_0 e^{3t} + (-x_0 + y_0)te^{3t} \\ y(t) &= y_0 e^{3t} + (-x_0 + y_0)te^{3t}\end{aligned}$$

6. Find the particular solution with the initial value $\mathbf{Y}(0) = (1, 0)$.
Solution. $\mathbf{Y}(0) = (x_0, y_0) = (1, 0)$, that is, $x_0 = 1$ and $y_0 = 0$. Hence,

$$\begin{aligned}x(t) &= e^{3t} - te^{3t} \\ y(t) &= -te^{3t}\end{aligned}$$

Problem 5. Given the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{Y}$$

1. Find the eigenvalue.
   Solution.

   - The coefficient matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$.
   - $A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix}$.
   - $\det(A - \lambda I) = \lambda^2 + 2\lambda + 1$.
   - Solve for the eigenvalues: $\lambda = -1$ is the repeated eigenvalue.

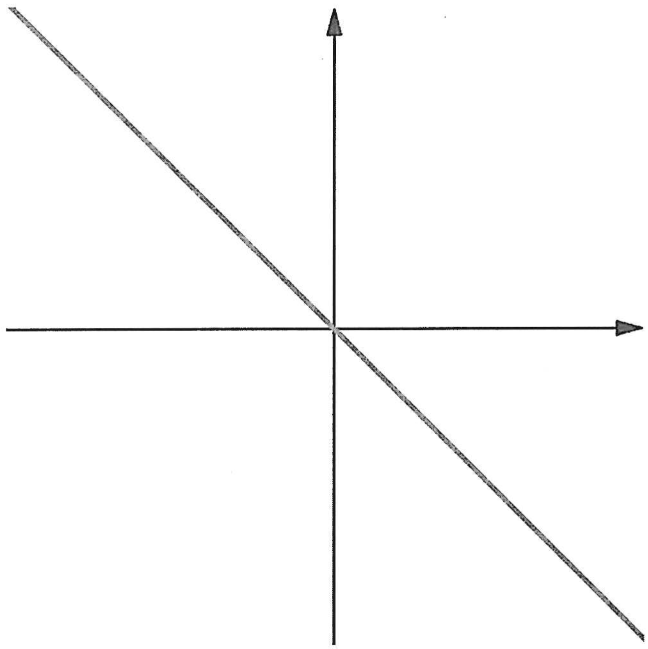
2. Find an eigenvector.
   Solution. $\vec{v}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

   - Compute $\vec{v}_1$:

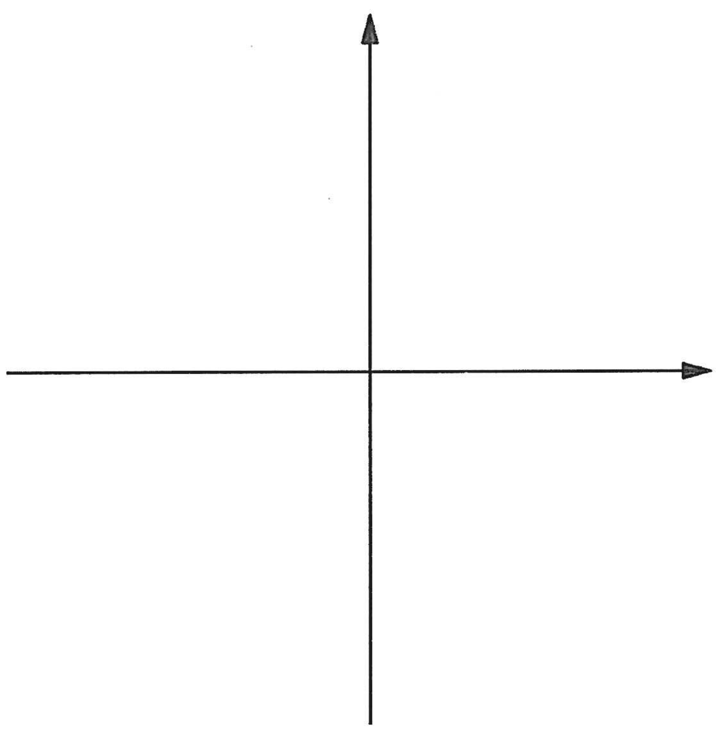
$$\begin{aligned} \vec{v}_1 &= (A - (-1)I)\vec{v}_0 \\ &= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \begin{bmatrix} x_0 + y_0 \\ -x_0 - y_0 \end{bmatrix} \end{aligned}$$

   - We can choose $x_0 = 1$ and $y_0 = 0$. An eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

3. Sketch the phase portrait including the solution curve with the initial value $\mathbf{Y}(0) = (1, 0)$.
Solution.



4. Sketch the $x(t)$ graph in the tx-plane and $y(t)$ graph in the ty plane of the solution curve with the initial value $\mathbf{Y}(0) = (1, 0)$.
Solution.



5. Find the general solution.

Solution. The general solution:

$$\begin{aligned}\mathbf{Y}(t) &= e^{\lambda t}\vec{v}_0 + te^{\lambda t}\vec{v}_1 \\ &= e^{-t}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + te^{-t}\begin{bmatrix} x_0 + y_0 \\ -x_0 - y_0 \end{bmatrix}\end{aligned}$$

Hence,

$$\begin{aligned}x(t) &= x_0 e^{-t} + (x_0 + y_0)te^{-t} \\ y(t) &= y_0 e^{-t} + (-x_0 - y_0)te^{-t}\end{aligned}$$

6. Find the particular solution with the initial value $\mathbf{Y}(0) = (1, 0)$.

Solution. $\mathbf{Y}(0) = (x_0, y_0) = (1, 0)$, that is, $x_0 = 1$ and $y_0 = 0$. Hence,

$$\begin{aligned}x(t) &= e^{-t} + te^{-t} \\ y(t) &= -te^{-t}\end{aligned}$$