Sections 2.1–2.3, 2.5, 2.6 are omitted here, since they are either exactly the same as Single Equation Systems or are covered in-depth in chapters 3 and 4.

Final Exam Topic Summary

Chapter 1

Section 1.1: Modeling with Differential Equations

- Start by defining dependent/independent variables and parameters
- Use dimensional analysis (units) to describe the situation accurately
- Reference Common Models to the right if needed
- Examine equilibrium points ($\frac{dy}{dt}=0$) and graph requested solutions
- Use proper techniques to solve the specific equation.
- Standard Form of a DE: $\frac{dy}{dt}-g(t)y=b(t)$

Common DE Models

- Logistic Population Model: $\frac{dP}{dt}=KP(1-\frac{P}{N})$
- Modified Logistic Model: $\frac{dS}{dt}=KS(1-\frac{S}{N})(\frac{S}{M}-1)$

N is carrying capacity, M is sparsity, K is growth coef.

- Exponential Population Model: $\frac{dS}{dt}=K\cdot S_0e^{Kt}$ or $\frac{dS}{dt}=KS$
  - ↳ Half Life: $\frac{1}{2}r_0=r_0e^{kt}$ — Does not account for carrying capacity
- Standard Predator Prey System

$$\begin{cases}\frac{dR}{dt}=\alpha R-\beta RF & \text{prey, eg. rabbits}\\ \frac{dF}{dt}=-\gamma F+\delta RF & \text{predator, eg. foxes}\end{cases}$$

- Logistic Predator Prey System

$$\begin{cases}\frac{dR}{dt}=\alpha R(1-\frac{R}{N})-\beta RF\\ \frac{dF}{dt}=-\gamma F+\delta RF\end{cases}$$

Constants

- α: Prey growth rate
- β: # of interactions where prey eaten
- δ: benefit to predators per prey eaten
- γ: predator death rate

Section 1.2–1.4: Basics of Solving DEs

- Special Cases: ① $\frac{dy}{dt}=g(t)$ – can be solved by separating
  ② $\frac{dy}{dt}=h(y)$ — called autonomous
- Important Reminder: If $\ln|y|=t+C$, then $y=Ke^t$ for $K=\pm e^C$!
- Slope Fields: ① $\frac{dy}{dt}=g(t)$ – Slopes depend only on t ⇒ Parallel along vertical lines
  ② $\frac{dy}{dt}=h(y)$ – Slopes depend only on y ⇒ Parallel along horizontal lines
- Euler's Method: $t_{k+1}=t_k+\Delta t$
  $y_{k+1}=y_k+\frac{dy}{dt}(t_k,y_k)\Delta t$
  Watch out for solutions that cross eq lines!
  ↳ Poor step sizes lead to poor approximations!!

Section 1.5–1.6: Basics of Analyzing Equilibria

Applies for systems as well:

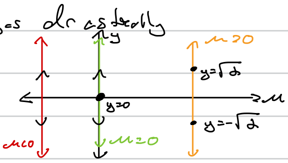
- Existence Thm: Given $\frac{dy}{dt}=f(t,y)$, solutions exist if, and only if, f is cts around point of interest.
- Uniqueness Thm: Given $\frac{dy}{dt}=f(t,y)$, solutions are unique, meaning that no solution ever intersects $\forall t$, if, and only if, all partial derivatives of f exist and are cts at/around point of interest.
- Classifying Equilibria: (when $\frac{dP}{dt}=0$)
  ① $f(y(0))=0$, then $y(0)$ is an eq solution and $y(t)=y(0)\ \forall t$
  ② $f(y(0))>0$, then $y(t)$ is increasing $\forall t$ and either $y(t)\to\infty$ or larger eq.
  ③ $f(y(0))<0$, then $y(t)$ is decreasing $\forall t$ and either $y(t)\to-\infty$ or smaller eq.

Phase Lines: ← sink, ← node, ← source

- Stability of Equilibria: sink > node > source ← picture a ball on hill / in a bowl

Section 1.7: Bifurcation Diagrams

- Bifurcation Points: points or parameter values where DE solution behavior changes drastically
- Plots done usually in μ-y plane.
  ① Determine parameter (μ) values which cause system to change drastically
  ② Draw vertical lines at determined μ values and indicate behavior



Section 1.8–1.9: Linear Equations

Assume given $\frac{dy}{dt}=a(t)y+b(t)$

- Algorithm (guessing):
  ① Determine solution $y_h(t)$ to (H) where (H): $\frac{dy}{dt}=a(t)y$
  ② Guess solution $y_p(t)$ to (NH) where (NH): $\frac{dy}{dt}=a(t)y+b(t)$
  ③ Determine general solution $y(t)=y_h(t)+y_p(t)$ ← Extended Linearity Principle
- Algorithm (integrating factors):
  ① Write DE in standard form → $\frac{dy}{dt}-a(t)y=b(t)$
  ② Compute Integrating Factor given by $\mu(t)=e^{\int a(t)dt}$
  ③ Multiply both sides of DE by $\mu(t)$
  ④ Apply Reverse Product Rule $(y\cdot\mu)'$ and Integrate both sides w/ respect to t
  ⑤ Solve for $y(t)$ by dividing both sides by $\mu(t)$

Basics of Guessing/Solving

- $y_h(t)=Ke^{\int a(t)dt}$
- $y_p(t)=Ce^{st}$ if $b(t)=e^{st}$ ← Add t factor if guess fails
- $y_p(t)=A\sin(\omega t)+B\cos(\omega t)$ if $b(t)=\frac{\cos}{\sin}(\omega t)$

Chapter 2

Section 2.4: Decoupled Systems

← Defined by at least one rate depends on its own value in a system

- Completely Decoupled: $\begin{cases}\frac{dx}{dt}=f(x)\\ \frac{dy}{dt}=g(y)\end{cases}$ Separate to solve!
- Partially Decoupled: $\begin{cases}\frac{dx}{dt}=f(x)\\ \frac{dy}{dt}=g(x)+h(y)\end{cases}$ Solve separable equation, then substitute solution and use proper techniques to solve for other equation

Chapter 3

Section 3.1-3.2: Basics of Linear Systems

- If $\det(A) \neq 0$, only eq. point for system is $(0,0)$ ← trivial solution
- $A\vec{Y} = \lambda \vec{Y} \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ for an eigenvector λ ← $\lambda\begin{pmatrix} x \\ y \end{pmatrix}$ where $\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector
  - ↳ solve for eigenvector $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ in this way if you have λ
- $\vec{Y}(t) = K_1 e^{\lambda_1 t}\vec{V}_1 + K_2 e^{\lambda_2 t}\vec{V}_2$ for eigenvalues λ_1, λ_2 and eigenvectors $\vec{V}_1, \vec{V}_2$
- Derive λ from characteristic polynomial: $p(\lambda) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - cb$

Section 3.3: Phase Portraits for Real Eigenvalues

- Classifications of the origin eq. point:
  1. Sink: if $\lambda_1 < \lambda_2 < 0$ both neg.
  2. Source: if $0 < \lambda_1 < \lambda_2$ both pos.
  3. Saddle: if $\lambda_1 < 0$ and $\lambda_2 > 0$ one each

  Solution curves tend towards the dominant eigenvector

Section 3.4: Complex Eigenvalues

- Euler's Formula: $e^{a+ib} = e^a e^{ib} = e^a(\cos b + i \sin b)$
- Analogously: $e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos(\beta t) + i \sin(\beta t))$ ← use this to solve DEs
- Given $\frac{d\vec{Y}}{dt} = A\vec{Y}$ with $\lambda_{1,2} = \alpha \pm i\beta$, solutions in complex form are:

  $\vec{Y}(t) = e^{\alpha t}(\cos \beta t + i \sin \beta t)\vec{V}$ ← $Y(t) = K_1 e^{\alpha t} \vec{Y}_{re} + K_2 e^{\alpha t}\vec{Y}_{im}$

  ↳ Natural period $= \frac{2\pi}{\beta}$ – oscillations of cos/sin functions

  Natural Frequency $= \frac{\beta}{2\pi}$ – cycles per unit time

  Angular Frequency $= \beta$
- Behavior of solutions based on α
  1. $\alpha < 0$ – origin is a spiral sink — solutions spiral to origin
  2. $\alpha = 0$ – origin is a center — solutions are periodic
  3. $\alpha > 0$ – origin is a spiral source — solutions spiral from origin

Section 3.5: Repeated Eigenvalues

Plug in λ to $\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$

- Solution is $\vec{Y}(t) = e^{\lambda t}V_0 + te^{\lambda t}V_1$ where $V_1 = (A - \lambda I)V_0$ Assume $V_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

  ↳ V_0 and V_1 typically take place of K_1 and K_2

Section 3.6: Summary

Second Order DEs

$ay'' + by' + cy = 0$

1. Characteristic Equation

$as^2 + bs + c = 0$

2. Solve based on value of s
   1. $s_1 \neq s_2 \in \mathbb{R}$ — $y(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$
   2. $s_{1,2} = \alpha \pm \beta i$ — $y(t) = K_1 e^{\alpha t}\cos \beta t + K_2 e^{\alpha t} \sin \beta t$
   3. $s_1 = s_2 \in \mathbb{R}$ — $y(t) = K_1 e^{st} + K_2 t e^{st}$

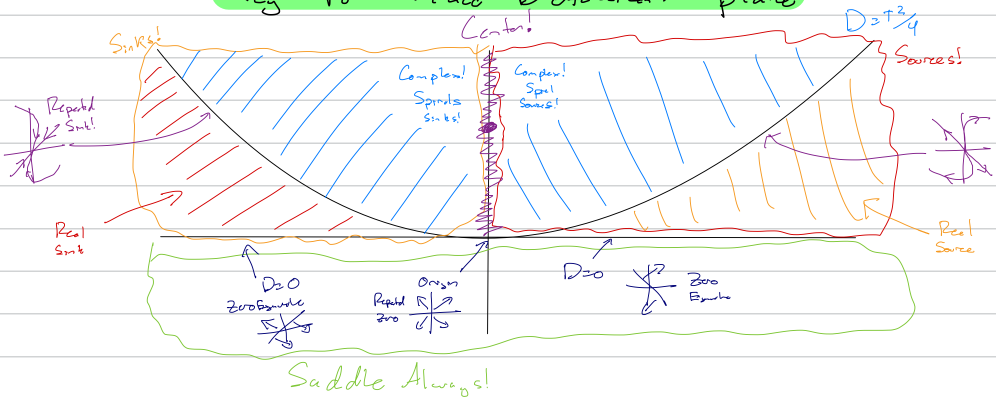
$my'' + by' + ky = 0$

↳ **Un Forced Harmonic Oscillators**

1. $b^2 - 4km < 0$ — $s = \alpha \pm i\beta$ — underdamped
2. $b^2 - 4km > 0$ — two negatives — overdamped
3. $b^2 - 4km = 0$ — one negative s — critically damped

$my'' + ky = 0$ ↳ 4. $b = 0$ — $s = \pm i\sqrt{\frac{k}{m}}$

Key for Trace Determinant plane



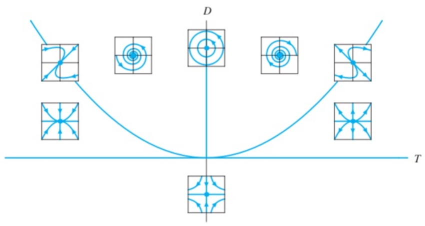
More on Next Pg →

Section 3.7: Trace Determinant Plane

- Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: ① Trace $Tr(A) = a + d$ so $p(\lambda) = \lambda^2 - T\lambda + D$ ← Useful for determining eigenvalues quickly!
  ② Determinant $Det(A) = ab - cd$
- Roots of $p(\lambda)$ are $\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} \rightarrow \lambda = \frac{T}{2} \pm \sqrt{\frac{T^2}{4} - D}$

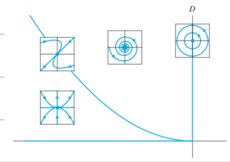
Qualitative Analysis			
$T^2 - 4D > 0$	two real	$D < T^2/4$	← below parabola
$T^2 - 4D = 0$	one repeated	$D = T^2/4$	← on parabola
$T^2 - 4D < 0$	two complex	$D > T^2/4$	← above parabola

- Case A: $D > T^2/4$
  - $T < 0$ spiral sink
  - $T = 0$ center
  - $T > 0$ spiral source
- Case B: $D = T^2/4$
  - $T < 0$ repeated root sink
  - $T > 0$ repeated root source
- Case C: $D < T^2/4$
  - (i) Suppose $T > 0$
    ① $D > 0$; $\lambda_1 > 0 \rightarrow$ real source
    ② $D < 0$; $\lambda_1 < 0 \rightarrow$ sink
  - (ii) Suppose $T < 0$
    ① $D > 0$; two negative eigenvalues
    ② $D < 0$; one λ of each sign
  - (iii) Suppose $T = 0$
    $\lambda = \pm \frac{1}{2}\sqrt{-4D}$; $D < 0$ to be real



Harmonic Oscillators in TD plane

$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \rightarrow \begin{pmatrix} 0 & 1 \\ -k/m & -b/m \end{pmatrix}$



Chapter 4

Section 4.1: Forced Harmonic Oscillators

- $y'' + py' + qy = g(t)$ has homogeneous (H): $y'' + qy' + py = 0$ with solution $y_h(t)$
- Guess solution to (NH) to get $y_p(t)$
- Solution to DE is $y(t) = y_h(t) + y_p(t)$

Section 4.2: Sinusoidal Forcing

- Follow (H) solution method from 4.1 to get $y_h(t)$
- Either guess $y_p(t) = \alpha\cos\omega t + B\sin\omega t$ or $y_p(t) = \alpha e^{i\omega t}$ after converting forcing function to imaginary plane

Getting Real Solution: ① If $g(t)$ was $a\sin\omega t$, then $y_p(t) = Y_{im}(t)$
② If $g(t)$ was $a\cos\omega t$, then $y_p(t) = Y_{re}(t)$

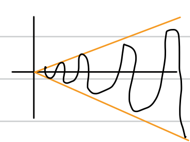
Section 4.3: Undamped Forcing and Resonance

- (NH): $y'' + qy' = g(t)$
- Guess (NH) sol with $y_p(t) = \alpha e^{i\omega t}$ ← add t factor(s) if guess has same terms (H) solution
- Suppose $g(t) = a\cos\omega t$ or $a\sin\omega t$; solution is near resonant if frequency of solution is close to frequency of $g(t)$. Called Resonant if frequencies are equal
- Possible Simplification

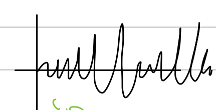
$y(t) = K_1\cos Bt + K_2\sin Bt \rightarrow y(t) = K\cos(Bt - \varphi)$

This is true iff $K_1 = K\cos\varphi$ and $K_2 = K\sin\varphi$

Resonant Case



Near Resonant



'Beating'

Chapter 5

Section 5.1 - 5.2: Equilibrium Point Analysis

- Given a system $\begin{cases} \frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy \\ \frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy \end{cases}$

  Cannot solve with any analytical techniques

Note: Increase in either x or y has a negative impact on other species

- Algorithm for Point Analysis
  ① Compute Jacobian for system $J_s = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$
  ② Evaluate $J_s(x, y)$ for eq. point
  ③ Solve for eigenvalues of J_s and determine solution behavior
  ④ Repeat for all other equilibrium points

Nullclines

- x-nullcline: set of all points (x, y) where $f(x, y) = 0$
- y-nullcline: set of all points (x, y) where $g(x, y) = 0$

For a system $\begin{cases} \frac{dx}{dt} = f(x, y) \\ \frac{dy}{dt} = g(x, y) \end{cases}$ Equilibrium points occur when nullclines intersect

- On vector fields, vectors are horizontal on y-nullcline and vertical on x-nullcline

Chapter 6

Section 6.1 – 6.4: Laplace Transforms

- Sufficient Conditions: ① $y(t)$ is piecewise cts. for $t \geq 0$
  ② $y(t)$ is of exponential order, meaning $|y(t)| \leq Me^{ct}\ \forall t$ bounded by this!
- Laplace Transforms are Linear operators
- Table of Images and Inverses given, be comfortable with partial fractions and t-axis or s-axis shifts

Section 6.1–6.2 Images

- $\mathcal{L}[e^{at}] = \frac{1}{s-a},\ s > a$
- $\mathcal{L}[y'] = s\mathcal{L}[y] - y(0)$
- $\mathcal{L}[t] = \frac{1}{s^2}$
- $\mathcal{L}[a] = \frac{a}{s}$
- $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$
- $\mathcal{L}[u_a(t)] = \frac{e^{-as}}{s}$
- $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}\mathcal{L}[f(t)]$

Section 6.3–6.4 Images

- $\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0)$
- $\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$
- $\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$
- $\mathcal{L}[e^{at}\sin(\omega t)] = \frac{\omega}{(s-a)^2+\omega^2}$
- $\mathcal{L}[e^{at}\cos(\omega t)] = \frac{s-a}{(s-a)^2+\omega^2}$
- $\mathcal{L}[t^k f(t)] = (-1)^k * \frac{d^k F(s)}{ds^k}$
  ↳ where $F(s) = \mathcal{L}[f(t)]$
- $\mathcal{L}[\delta_a(t)] = e^{-as}$ ← no subscript implies $a=0$

Fun Identities

- n^{th} order derivative Image
  ① $\mathcal{L}[y^{n}] = s\mathcal{L}[y^{(n-1)}] - y^{(n-1)}$
  ② $\mathcal{L}[y^{n}] = s^n\mathcal{L}[y] - \sum_{k=0}^{n-1} s^{(n-1-k)} * y^{k}(0)$
- Laplace Image of Periodic Functions

Period T means $f(t+T) = f(t)\ \forall t$

$$\therefore \mathcal{L}[f(t)] = \frac{1}{1-e^{-Ts}}\int_0^T f(t)e^{-st}\,dt$$

Relationship of δ-Dirac and Heaviside Functions: $\delta_a(t) = \frac{d}{dt}(u_a(t))$

Good way to think about it ☺

Given Laplace Tables on Formula Sheet

Table 6.1
Frequently Encountered Laplace Transforms.

$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$	$y(t) = \mathcal{L}^{-1}[Y]$	$Y(s) = \mathcal{L}[y]$
$y(t) = e^{at}$	$Y(s) = \frac{1}{s-a}\quad (s > a)$	$y(t) = t^n$	$Y(s) = \frac{n!}{s^{n+1}}\quad (s > 0)$
$y(t) = \sin\omega t$	$Y(s) = \frac{\omega}{s^2+\omega^2}$	$y(t) = \cos\omega t$	$Y(s) = \frac{s}{s^2+\omega^2}$
$y(t) = e^{at}\sin\omega t$	$Y(s) = \frac{\omega}{(s-a)^2+\omega^2}$	$y(t) = e^{at}\cos\omega t$	$Y(s) = \frac{s-a}{(s-a)^2+\omega^2}$
$y(t) = t\sin\omega t$	$Y(s) = \frac{2\omega s}{(s^2+\omega^2)^2}$	$y(t) = t\cos\omega t$	$Y(s) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$y(t) = u_a(t)$	$Y(s) = \frac{e^{-as}}{s}\quad (s > 0)$	$y(t) = \delta_a(t)$	$Y(s) = e^{-as}$

Table 6.2
Rules for Laplace Transforms:
Given functions $y(t)$ and $w(t)$ with $\mathcal{L}[y] = Y(s)$ and $\mathcal{L}[w] = W(s)$ and constants α and a.

Rule for Laplace Transform	Rule for Inverse Laplace Transform
$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0) = sY(s) - y(0)$	
$\mathcal{L}[y+w] = \mathcal{L}[y] + \mathcal{L}[w] = Y(s) + W(s)$	$\mathcal{L}^{-1}[Y+W] = \mathcal{L}^{-1}[Y] + \mathcal{L}^{-1}[W] = y(t) + w(t)$
$\mathcal{L}[\alpha y] = \alpha\mathcal{L}[y] = \alpha Y(s)$	$\mathcal{L}^{-1}[\alpha Y] = \alpha\mathcal{L}^{-1}[Y] = \alpha y(t)$
$\mathcal{L}[u_a(t)y(t-a)] = e^{-as}\mathcal{L}[y] = e^{-as}Y(s)$	$\mathcal{L}^{-1}[e^{-as}Y] = u_a(t)y(t-a)$
$\mathcal{L}[e^{at}y(t)] = Y(s-a)$	$\mathcal{L}^{-1}[Y(s-a)] = e^{at}\mathcal{L}^{-1}[Y] = e^{at}y(t)$

Note

$\mathcal{L}[y'']$ is also given!