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Section 1.7: 1, 3, 5, 9, 13,19

1. $\frac{dy}{dt} = y^2 + a$

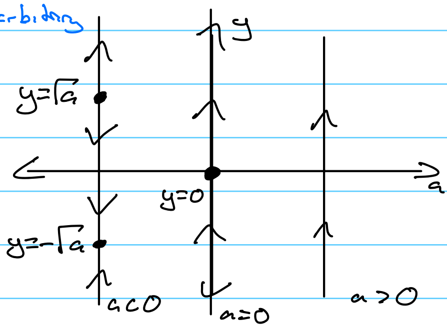
For these problems, the y-axis is arbitrary

$y^2 + a = 0 \qquad y = \frac{0 \pm \sqrt{-4a}}{2} = \frac{\pm\sqrt{-4a}}{2} = \frac{\pm(2)\sqrt{-a}}{2} = \pm\sqrt{-a}$

$\sqrt{-a}$: $a = 0 \rightarrow 1_{eq.} \qquad y^2 = 0 \rightarrow y = 0$

$a < 0 \rightarrow 2_{eq.} \qquad y^2 - 4 = (y+2)(y-2)$

$a > 0 \rightarrow$ no eq. $\qquad y^2 + 4 \rightarrow$ none



3. $\frac{dy}{dt} = y^2 - ay + 1$

$y^2 - ay + 1 = 0 \qquad y = \frac{a \pm \sqrt{a^2-4}}{2} = \frac{a \pm \sqrt{(a-2)(a+2)}}{2}$

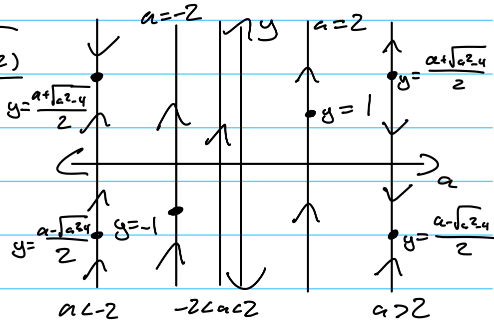
$\sqrt{(a-2)(a+2)}$

$+ \quad 0 \quad - \quad 0 \quad +$ on a number line with -2 and 2 marked

$a > 2$ or $a < -2 \rightarrow 2_{eq}$

$a = 2$ or $a = -2 \rightarrow 1_{eq}$

$-2 < a < 2 \rightarrow$ no eq.



5. $\frac{dy}{dt} = (y^2-\alpha)(y^2-4)$

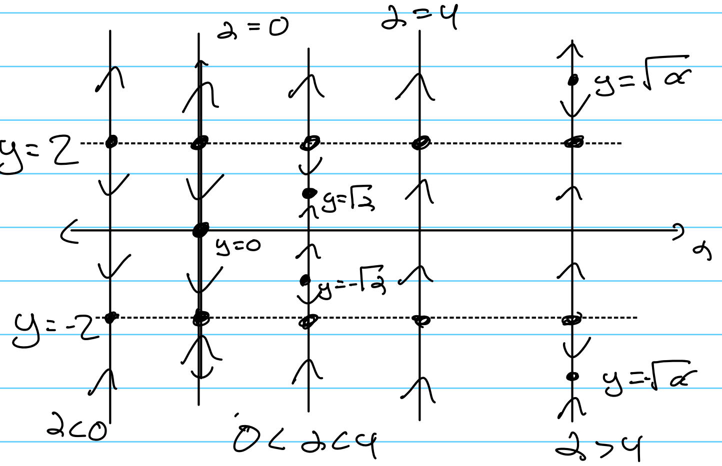
$\sqrt{\alpha}$: $\alpha = 4 \rightarrow 2_{eq.}$

$\alpha > 4 \rightarrow 4_{eq.}$

$0 < \alpha < 4 \rightarrow 4_{eq.}$

$\alpha = 0 \rightarrow 3_{eq.}$

$\alpha < 0 \rightarrow 2_{eq.}$



9. $\frac{dy}{dt} = \sin(y) + \alpha$ over $[-\pi, \pi]$

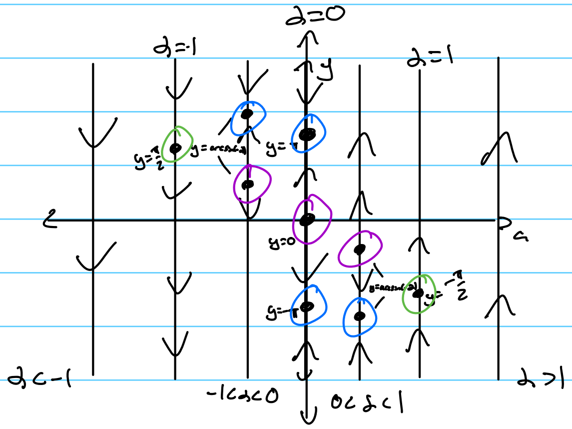
$0 = \sin(y) + \alpha$

α: $\alpha = 0 \rightarrow 3$ eq. points

$\alpha = \pm 1 \rightarrow 1$ eq. point

$\alpha > 1$ or $\alpha < -1 \rightarrow$ none

$0 < \alpha < 1$ or $-1 < \alpha < 0 \rightarrow 2$



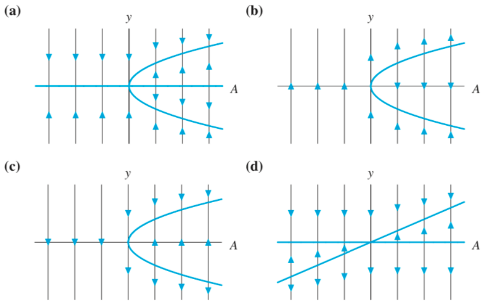
Bifurcation values $\alpha = [-1, 1]$

- Nodes in blue
- Sinks in green
- Sources in purple

13.

(i) $\frac{dy}{dt} = Ay - y^2$ (ii) $\frac{dy}{dt} = A + y^2$ (iii) $\frac{dy}{dt} = Ay - y^3$

(iv) $\frac{dy}{dt} = A - y^2$ (v) $\frac{dy}{dt} = y^2 - A$ (vi) $\frac{dy}{dt} = Ay + y^2$



i) cannot be A or B as its $A=0$ derivative is always negative.
when $A<0$, $-1y-y^2 \Rightarrow -y(y+1) \rightarrow y=0$ and $y=-1$
d must be the solution due to the above.
ii) must be b as $A=0$ is always positive
iii) must be a as $A=0$ has diffrent behavior around 0.
iv) must be c for the same reason as ii. It must follow similar behavior as well.
v) must be b for the same reason as ii and iv.
vi) $\frac{dy}{dt} = Ay+y^2 \Rightarrow \frac{dy}{dt} = y(y+A)$ must be b as derivative always positive for $A=0$

19. $\frac{dP}{dt} = 2P - \frac{P^2}{50} - 3\ell$

where ℓ is the # of licences issued

a) $P = \frac{-2 \pm \sqrt{4-4(-\frac{1}{50})(3\ell)}}{-\frac{2}{25}}$

$\sqrt{4 - \frac{2}{25}(3\ell)} > 0$

$-\frac{6\ell}{25} > -4$

$6\ell < 100 \quad \ell < 16.67$

Need equilibrium points for population.
17 licences too much. Max is 16.

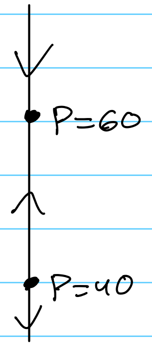
b) $\ell = 16$

$\left(\frac{dP}{dt} = 2P - \frac{P^2}{50} - 48\right) \times 50$

$\frac{dP}{dt} = -P^2 + 100P - 2400$

$= -(P^2 - 100P + 2400)$

$= -(P-60)(P-40)$



Initial > 60 → decrease to 60. Initial (40, 60) Increase to 60. Initial < 40 → decrease to 0.

c) It can be concluded that the fish population should be fine as long as the initial population is well above 40 fish. If it is too close, the amount of catches combined with the random environment changes would possibly lead to extinction. It may be wise to consider reducing licenses so our model has a lower minimum equilibrium point to counteract environmental factors.

Section 1.8: 3, 9, 11, 21, 25

3. $\frac{dy}{dt} = -3y + 4\cos 2t$ (NH) | $\frac{dy}{dt} = -3y$ (H)

$y_h(t)$: $y_h(t) = Ke^{\int a(t)\,dt}, K \in \mathbb{R} \Rightarrow y_h(t) = Ke^{-3t}$

$y_p(t)$: Guess $y_p(t) = A\cos(2t) + B\sin(2t)$

$y_p'(t) = -2A\sin(2t) + 2B\cos(2t)$

$\therefore -2A\sin(2t) + 2B\cos(2t) = -3A\cos(2t) - 3B\sin(2t) + 4\cos(2t)$

$-2A = -3B$ Solve system: $A = \frac{3B}{2}$

$2B = -3A + 4$ $2B = -3(\frac{3B}{2}) + 4$

$2B = \frac{-9B}{2} + 4$

$\frac{13B}{2} = 4 \Rightarrow B = \frac{8}{13}$ and $A = \frac{24}{26} = \frac{12}{13}$

$y_p(t) = \frac{8}{13}\cos(2t) + \frac{12}{13}\sin(2t)$

$\therefore$ general solution $= y_h(t) + y_p(t)$

$y(t) = Ke^{-3t} + \frac{12}{13}\cos(2t) + \frac{8}{13}\sin(2t)$

9. $\frac{dy}{dt} + y = \cos 2t, y(0) = 5$ (NH) $\frac{dy}{dt} = -y + \cos(2t)$ | (H) $\frac{dy}{dt} = -y$

$y_h(t) = Ke^{\int -dt} \Rightarrow y_h(t) = Ke^{-t}$

$y_p(t)$: Guess $y_p(t) = A\cos(2t) + B\sin(2t)$

$y_p'(t) = -2A\sin(2t) + 2B\cos(2t)$

$\therefore -2A\sin(2t) + 2B\cos(2t) = -A\cos(2t) - B\sin(2t) + \cos(2t)$

$-2A = -B$ Solve system: $A = \frac{1}{2}B$

$2B = -A + 1$ $2B = -\frac{1}{2}B + 1$

$\frac{5}{2}B = 1 \Rightarrow B = \frac{2}{5}, A = \frac{1}{5}$

$y_p(t) = \frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t)$

$\therefore$ general solution $= y_h(t) + y_p(t)$

$y(t) = Ke^{-t} + \frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t)$

IVP: $5 = Ke^{0} + \frac{1}{5}\cos 0 + \frac{2}{5}\sin 0$

$5 = K + \frac{1}{5}$ $K = \frac{24}{5}$

$\therefore y(t) = \frac{24}{5}e^{-t} + \frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t)$

11. $\frac{dy}{dt} - 2y = 7e^{2t}$, $y(0)=3$ (NH) $\frac{dy}{dt} = 2y + 7e^{2t}$ | (H) $\frac{dy}{dt} = 2y$

$y_h(t)$: $y_h(t) = Ke^{\int 2dt} = Ke^{2t}$

$y_p(t)$: $y_p(t) = Ce^{2t}$

$y'_p(t) = 2Ce^{2t}$

$2Ce^{2t} = 2Ce^{2t} + 7e^{2t}$ nope!

add t factor

$y_p(t) = t\,Ce^{2t}$

$y'_p(t) = 2tCe^{2t} + Ce^{2t}$

$2tCe^{2t} + Ce^{2t} = 2tCe^{2t} + 7e^{2t}$

$2 \cdot C = 2 \cdot C$

$C = 7$

$y_p(t) = 7te^{2t}$ $\Rightarrow y(t) = Ke^{2t} + 7te^{2t}$

IVP: $3 = Ke^{0} + 7(0)e^{0}$

$K=3$ $\therefore y(t) = 3e^{2t} + 7te^{2t}$

21. $\frac{dy}{dt} + 2y = t^2 + 2t + 1 + e^{4t}$, $y(0)=0$

(NH) $\frac{dy}{dt} = -2y + t^2 + 2t + 1 + e^{4t}$ | (H) $\frac{dy}{dt} = -2y$

$y_h(t) = Ke^{\int -2dt} = Ke^{-2t}$

$y_p(t)$: $y_p(t) = at^2 + bt + c + de^{4t}$

$y_p'(t) = 2at + b + 4de^{4t}$

$\underline{2at} + \underline{b} + \underline{4de^{4t}} = \underline{-2at^2} \underline{-2bt} \underline{-2c} \underline{-2de^{4t}} + \underline{t^2} + \underline{2t} + \underline{1} + \underline{e^{4t}}$

$$\left.\begin{aligned} 2a &= -2b + 2 \\ b &= -2c + 1 \\ 4d &= -2d + 1 \\ 0 &= -2a + 1 \end{aligned}\right\}$$

$6d = 1 \Rightarrow d = \frac{1}{6}$

$2a = 1 \Rightarrow a = \frac{1}{2}$

$b = \frac{2a-2}{-2} \Rightarrow b = \frac{1}{2}$

$c = \frac{b-1}{-2} = \frac{1}{4}$

$y_p(t) = \frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{4} + \frac{1}{6}e^{4t}$

general solution: $y = y_h + y_p$

$y(t) = ke^{-2t} + \frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{4} + \frac{1}{6}e^{4t}$

IVP: $0 = K + \frac{1}{4} + \frac{1}{6} \Rightarrow K = -\frac{5}{12}$ $\therefore y(t) = -\frac{5}{12}e^{-2t} + \frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{4} + \frac{1}{6}e^{4t}$

25. $\frac{dy}{dt} + 2y = b(t)$ where $(-1 < b(t) < 2)\ \forall t$

Solve (H) $\frac{dy}{dt} + 2y = b(t) \Rightarrow y' = -2y + b(t)$

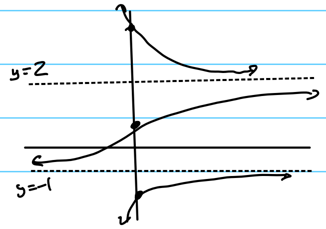
$y_h(t) = Ke^{\int -2dt} \Rightarrow y_h = Ke^{-2t}$

$y_p(t) + y_h(t) = y(t)$

$y(t) = Ke^{-2t} + y_p(t)$

If $y(0) = 4$ then $\frac{dy}{dt} = -8 + (-1, 2)$ as $b(t)$ is bounded by -1 and 2.

- At $y(0) > 2$ solution tends toward 2, the upper limit of b
- If $-1 < y(0) < 2$ then t will tend towards 2 as $t \to \infty$
- If $y(0) < 2$ then solution tends towards -1.



Section 1.9: 5, 9, 15, 19

5. $\frac{dy}{dt} - \frac{2t}{1+t^2} y = 3$

$\mu(t) = e^{\int \frac{-2t}{1+t^2} dt}$ $\quad u = 1+t^2$, $\frac{du}{dt} = 2t$

$= e^{-\ln(1+t^2)} = \frac{1}{1+t^2}$

$\mu(t) = \frac{1}{1+t^2}$

$\frac{dy}{dt}\left(\frac{1}{1+t^2}\right) - \frac{2t}{(1+t^2)^2} y = \frac{3}{1+t^2}$

$\int \left(y \cdot \frac{1}{1+t^2}\right)' = \int \frac{3}{1+t^2}$

$\frac{y}{1+t^2} = 3\tan^{-1}(t) + C$

$\therefore y = (1+t^2)(3\tan^{-1}(t) + C)$

9. $\frac{dy}{dt} + \frac{y}{t} = 2$, $y(1) = 3$

$\mu(t) = e^{\int \frac{1}{t} dt}$

$\mu(t) = e^{\ln t} = t$

$\frac{dy}{dt}(t) + \frac{yt}{t} = 2t$

$\int (y \cdot t)' = \int 2t$

$y \cdot t = t^2 + C$

$y = t + \frac{C}{t}$

$3 = 1 + \frac{C}{t} \Rightarrow C = 2$

$y = t + \frac{2}{t}$

15. $\frac{dy}{dt} = \frac{y}{t^2} + 4\cos t$

$\frac{dy}{dt} - \frac{y}{t^2} = 4\cos t$

$\mu(t) = e^{-\int t^{-2} dt} = e^{\frac{1}{t}}$

$\frac{dy}{dt} e^{\frac{1}{t}} - \frac{y}{t^2} e^{\frac{1}{t}} = 4e^{1/t}\cos t$

$\int \left(y \cdot e^{1/t}\right)' = \int 4 e^{1/t}\cos t$

$y \cdot e^{1/t} = \int 4 e^{1/t} \cos t \, dt$

$\therefore y = 4e^{-\frac{1}{t}} \int e^{1/t} \cos t \, dt$ $\quad +C$ baked into $\int$

19. $\frac{dy}{dt} = a\,ty + 4e^{-t^2}$

$\frac{dy}{dt} - aty = 4e^{-t^2}$

$\mu(t) = e^{-\int at\,dt} = e^{-\frac{at^2}{2}}$

$\frac{dy}{dt} e^{-\frac{at^2}{2}} - atye^{-\frac{at^2}{2}} = 4e^{-t^2} e^{-\frac{at^2}{2}}$

$\int \left(y \cdot e^{-\frac{at^2}{2}}\right)' = \int 4e^{-t^2 - \frac{at^2}{2}}$

$y \cdot e^{-\frac{at^2}{2}} = 4\int e^{-t^2 - \frac{at^2}{2}} dt$

No integrals required: $-t^2 - \frac{at^2}{2} = 0$

$\frac{a}{2} = -1$

$a = -2$

$a = -2$ eliminates integral solutions