PROBLEM 1 [24 points]
State whether the following statements are true or false. Provide a short explanation.

(a) The initial value problem

$$\begin{cases} \frac{dx}{dt} = ax = f(x), \quad \frac{\partial f}{\partial x} = a \text{ - continuous} \\ x(0) = b \end{cases}$$

has a unique solution for any a and b.

Answer $\boxed{\text{Yes/True}}$

(b)The system

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = \sin(xy) \end{cases}$$

$y=0$, $\sin(0)=0$

$(0,0)$

has only one equilibrium point.

Answer $\boxed{\text{True}}$

(c) If $x_0(t)$ is a solution to an autonomous equation $\frac{dx}{dt} = f(x)$, then any function of the form $x_c(t) = x_0(t-c)$ is also a solution.

autonomous equation

Answer $\boxed{\text{True}}$

(d) Any linear homogeneous equation is separable.

Answer $\boxed{\text{False}}$

(e) If a linear homogeneous system with constant coefficients $\frac{d\vec{Y}}{dt} = A\vec{Y}$ has a sink at the origin, then the system $\frac{d\vec{Y}}{dt} = -A\vec{Y}$ must have a sink there as well.

$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ $\lambda_1 = -1$, $\lambda_2 = -2$

$-A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\lambda_1 = 1$, $\lambda_2 = 2$

Answer False

(f) The system

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \end{cases}$$

is a Hamiltonian system.

Skip

Answer

(g) If a linear homogeneous system with constant coefficients $\frac{d\vec{Y}}{dt} = A\vec{Y}$ has a zero among its eigenvalues, it has infinitely many equilibria, possibly all points of a line through the origin.

Answer True

(h) For any continuous bounded functions f and g it is true that if $\mathcal{L}[f](s) = \mathcal{L}[g](s)$ then $f(t) = g(t) + C$, where C is an arbitrary constant.

$L^{-1}[f]$ is unique

Answer False

PROBLEM 2 [25 points]
Solve the initial value problem

$$\begin{cases} \frac{dy}{dt} = \frac{y^2}{1+t} \\ y(0) = 0 \end{cases}$$

(a) Find the general solution to the ODE

(b) Find the particular solution satisfying the given initial condition.

(c) What does the existence/uniqueness theorem guarantee for this initial value problem? Are your answers consistent with that?

a) $\int \frac{dy}{y^2} = \int \frac{dt}{1+t}$ Don't forget about $y=0$!

$-\frac{1}{y} = \ln|1+t| + C$

$y = -\frac{1}{\ln|1+t| + C}$ and $y=0$

b) $y(0)=0$ only for $y(t)=0$

c) $f(y,t) = \frac{y^2}{1+t}$ $\frac{\partial f}{\partial y} = \frac{2y}{1+t^2}$

f and $\frac{\partial f}{\partial y}$ are continuous around $(0,0)$

so, from Existence and Uniqueness Theorem solution exists and is unique, which is $y(t)=0$ only.

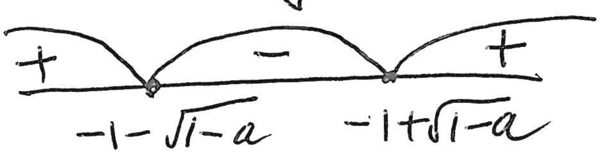
PROBLEM 3 [25 points]
Consider the equation $\frac{dy}{dt} = y^2 + 2y + a$, where $a \in \mathbb{R}$ is a parameter.

(a) Determine the equilibrium solutions - their values and number may depend on a; draw the phase line for each of the equilibrium configurations, determine their type.

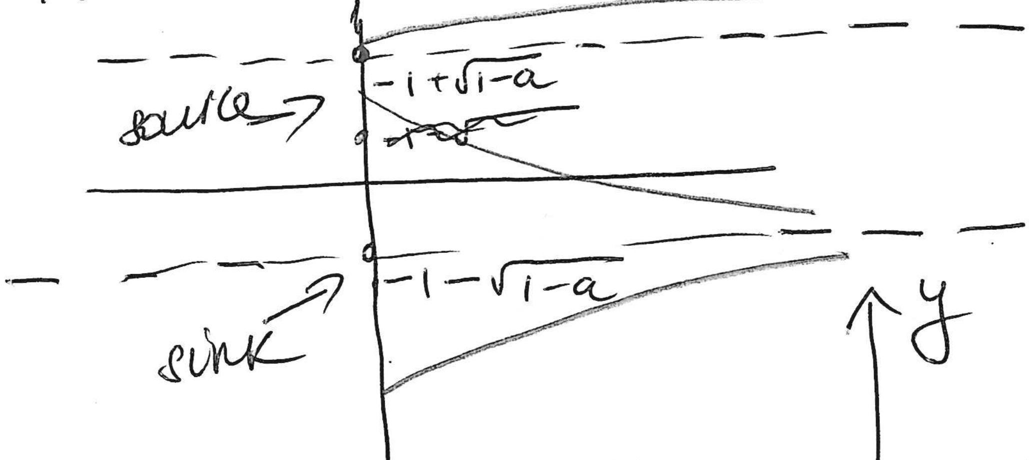
(b) Draw the bifurcation diagram.

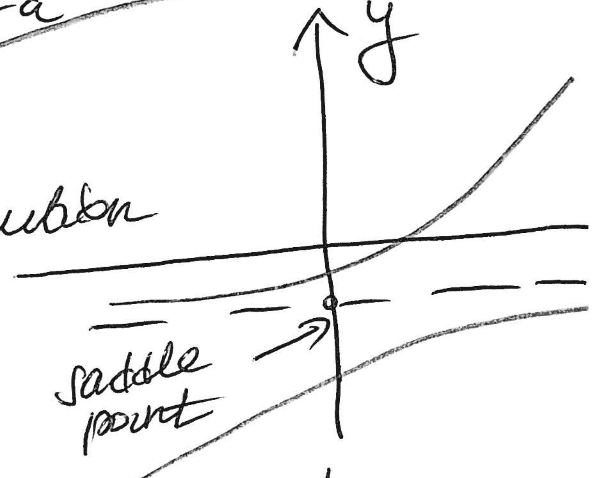
a) Explore $y^2+2y+a=0$. $y=\frac{-2\pm\sqrt{4-4a}}{2}$

$=-1\pm\sqrt{1-a}$.



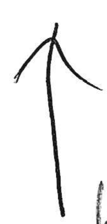
If $4-4a>0 \iff a<1$: 2 solutions :





if $a=1$: $y=-\frac{1}{2}$ 1 solution

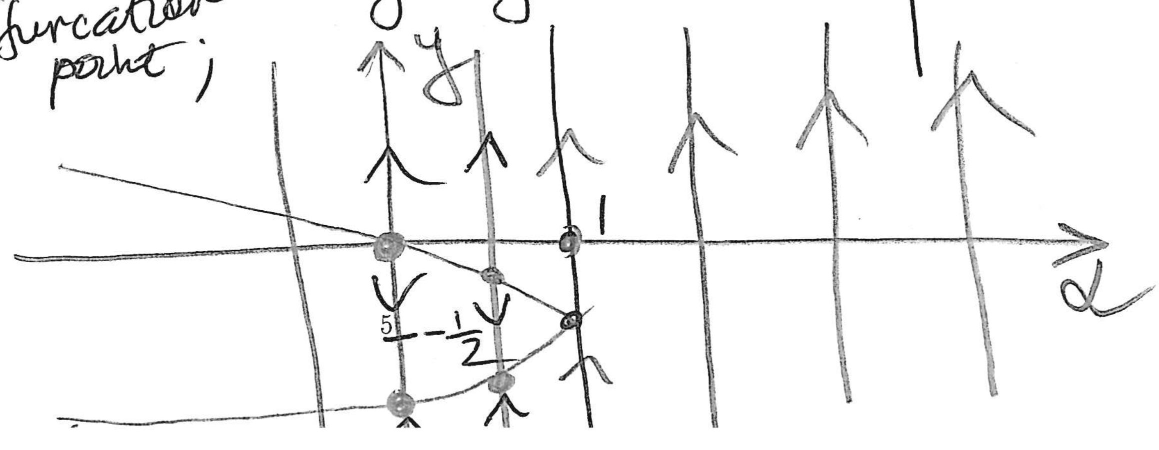
if $a>1$: NO solutions :



$y^2+2y+a>0$

$a=1$ is bifurcation point;

b)



PROBLEM 4 [25 points]
Find the general solution to each equation and determine if the resonance effect is present.

$$(a)\frac{d^2y}{dt^2} + 9y = -2\cos(3t)$$

$$(b)\frac{d^2y}{dt^2} + 9y = 3e^{3t}$$

a) Homogeneous: $y'' + 9y = 0$
$s^2 + 9 = 0 \quad s = \pm i3$
$y_h = k_1 \cos 3t + k_2 \sin 3t$

External force $-2\cos 3t$ has the same frequency, so we have the presence of resonance effect.
$y_p = t(\alpha \cos 3t + \beta \sin 3t)$;
$y''_p = -6\alpha \sin 3t + 6\beta \cos 3t - 9t\alpha \cos 3t - 9t\beta \sin 3t$
$y''_p + 9y_p = -6\alpha \sin 3t + 6\beta \cos 3t = -2\cos 3t$
so, $\alpha = 0$ and $\beta = -\frac{1}{3}$;
$y_p = -\frac{1}{3} t \sin 3t$
$y(t) = k_1 \cos 3t + k_2 \sin 3t - \frac{1}{3} t \sin 3t$

b) $y_p = \alpha e^{3t}$, $y''_p = 9\alpha e^{3t}$
$18\alpha = 3$, $\alpha = 1/6$

$y = k_1 \cos 3t + k_2 \sin 3t + \frac{1}{6} e^{3t}$
External force $3e^{3t}$ is not periodic. No resonan

PROBLEM 5 [20 points]

$$\begin{cases} \frac{dx}{dt} = x - 2y \\ \frac{dy}{dt} = 3y \end{cases}$$

(a) Find the general solution to the system

(b) Determine the type of equilibrium at $(0,0)$

(c) Draw the phase portrait of the system. Can the solution trajectories on this portrait intersect? Why?

a) $A = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$, $\det\begin{pmatrix} 1-\lambda & -2 \\ 0 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) = 0$

$\lambda_1 = 1$, $\lambda_2 = 3$

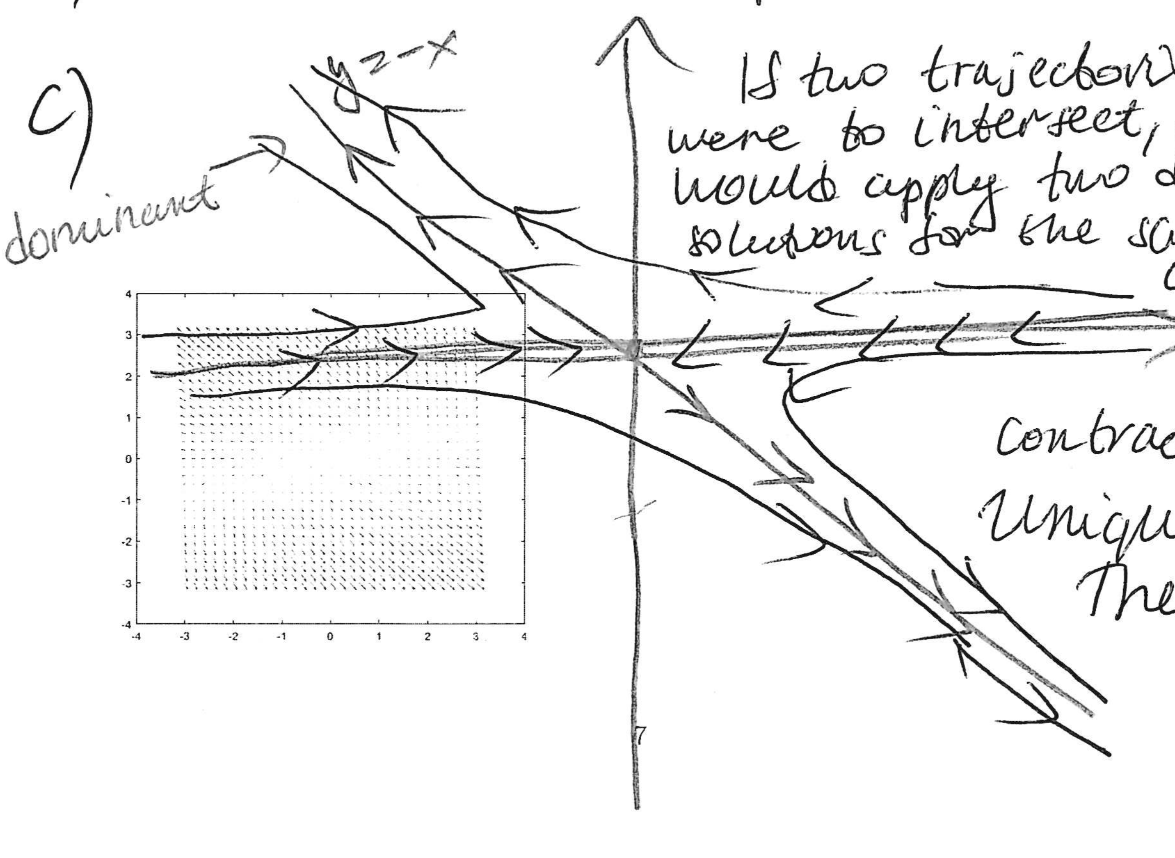
For $\lambda_1 = 1$ $\quad$ $\begin{matrix} 0v_1 - 2v_2 = 0 \\ 0v_1 + 2v_2 = 0 \end{matrix}$ $\quad$ $\begin{matrix} v_1 = 1 \\ v_2 = 0 \end{matrix}$ $\quad$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = 3$: $\begin{cases} -2v_1 - 2v_2 = 0 \\ 0v_1 + 0v_2 = 0 \end{cases}$ $\quad$ $\begin{matrix} v_1 = -1 \\ v_2 = +1 \end{matrix}$ $\quad$ $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

General solution $\vec{y}(t) = k_1 e^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{3t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

b) both eigenvalues are positive : source.

c)



If two trajectories were to intersect, that would apply two different solutions for the same initial condition. That contradicts to Uniqueness Theorem.

Problem 6 [20 points]

$$\begin{cases} \frac{dx}{dt} = -y \\ \frac{dy}{dt} = 9x \end{cases}$$

Find the general solution to the system. Determine the type of equilibrium at $(0,0)$.

$A = \begin{pmatrix} 0 & -1 \\ 9 & 0 \end{pmatrix}$ $\det \begin{pmatrix} 0-\lambda & -1 \\ 9 & 0-\lambda \end{pmatrix} = \lambda^2 + 9 = 0$

$\lambda = \pm 3i$, purely imaginary. The general solution is periodic.

Eigenvector for $\lambda = 3i$: $\begin{pmatrix} -3i & -1 \\ 9 & -3i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-3ix - y = 0 \Rightarrow y = -3ix$ $\vec{v} = \begin{pmatrix} 1 \\ -3i \end{pmatrix}$

$\vec{y}(t) = e^{3it}\begin{pmatrix} 1 \\ -3i \end{pmatrix} = (\cos 3t + i\sin 3t)\begin{pmatrix} 1 \\ -3i \end{pmatrix} =$

$= \begin{pmatrix} \cos 3t \\ 3\sin 3t \end{pmatrix} + i\begin{pmatrix} \sin 3t \\ -3\cos 3t \end{pmatrix}.$

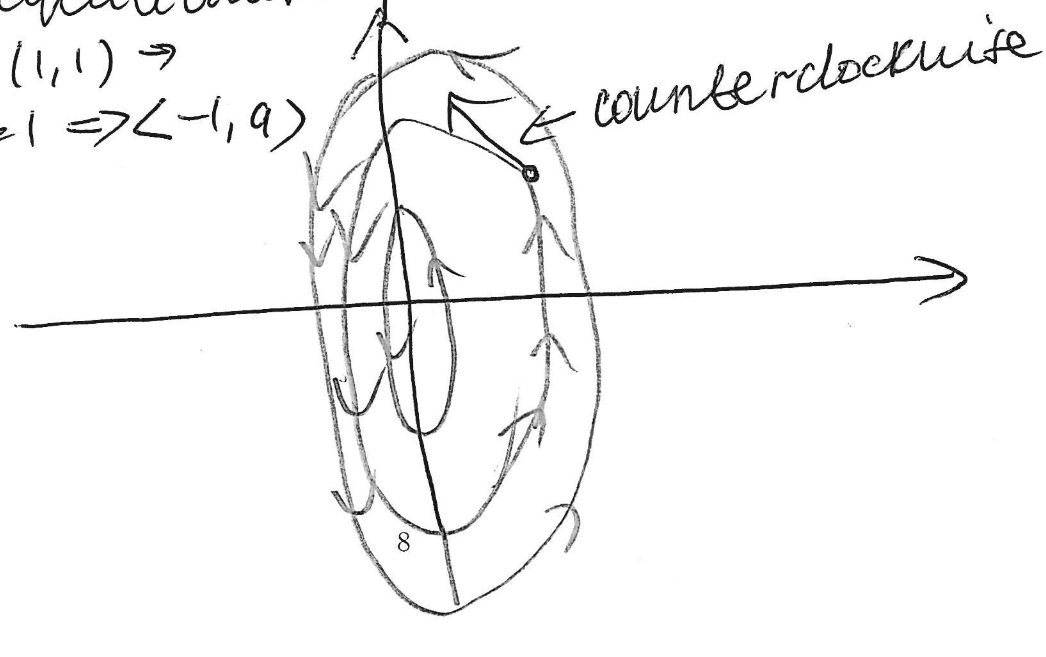
So, general real-valued solution:

$\vec{y}(t) = K_1\begin{pmatrix} \cos 3t \\ 3\sin 3t \end{pmatrix} + K_2\begin{pmatrix} \sin 3t \\ -3\cos 3t \end{pmatrix}$

The equilibrium point $(0,0)$ is a center.

Take $(1,1) \rightarrow$

$x = 1, y = 1 \Rightarrow \langle -1, 9 \rangle$



PROBLEM 7 [20 points]

Describe how the behavior of the system (the phase portrait and stability near (0,0)) evolves as the parameter a is changing from $-\infty$ to ∞.

$$\begin{cases} \frac{dx}{dt} = x + y \\ \frac{dy}{dt} = ay \end{cases}$$

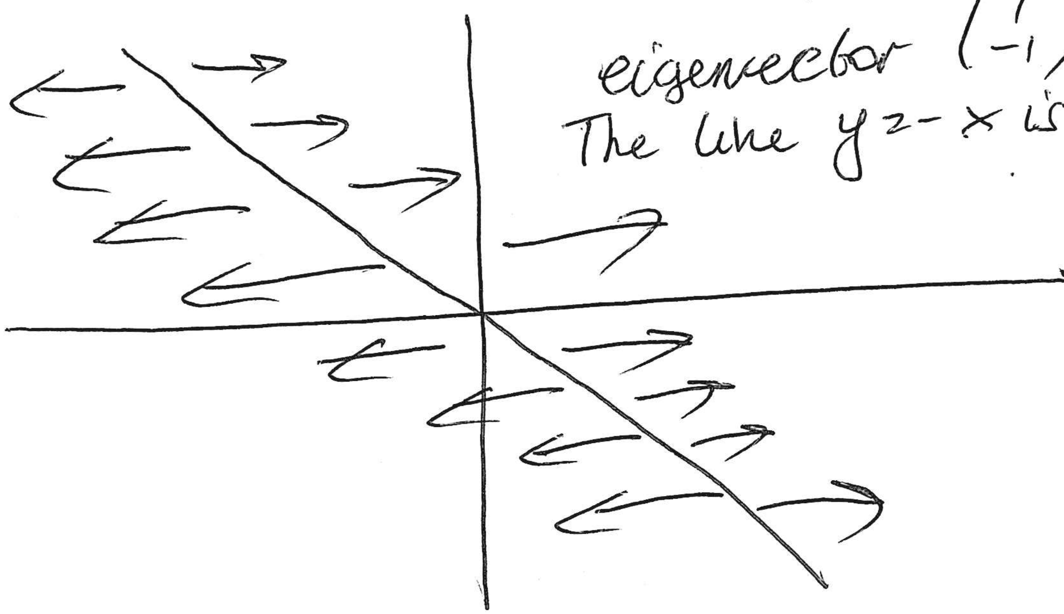
where $a \in \mathbb{R}$ is a parameter.

$A = \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix}$ Eigenvalues are $\lambda_1 = 1$, $\lambda_2 = a$

If $a > 0$: Both eigenvalues are positive.
$(0,0)$ is a source

if $a = 0$: $\lambda_2 = 0$, $\lambda_1 = 1$ $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
The line $y = -x$ is equilibrium line



if $a < 0$, $\lambda_1 < 0 < \lambda_2$ saddle point.

if $a = 1$: $\lambda_1 = \lambda_2 = 1$: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

eigenvector: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $y = 0$

PROBLEM 8[25 points]

(a) Determine the equilibrium solutions of the system;

(b) linearize the system about those equilibrium solutions and determine their stability (if possible)

$$\begin{cases} \frac{dx}{dt} = x^2 + y \\ \frac{dy}{dt} = y - xy \end{cases}$$

a) $\begin{cases} x^2+y=0 \\ y-xy=0 \end{cases}$ $y=-x^2$

$-x^2+x^3 = x^2(x-1)$

2 equilibria: $(0,0)$, $(1,-1)$

b) Jacobian: $\begin{pmatrix} 2x & 1 \\ -y & 1-x \end{pmatrix}$

$J(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ $\lambda_1 = 0$ (impossible to determine the stability)

$J(1,-1) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ $(2-\lambda)(-\lambda)-1 =$

$\lambda = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$ $= \lambda^2 - 2\lambda - 1 = 0$

$\lambda_1 < 0 < \lambda_2$ Saddle point.

PROBLEM 9[16 points]

Solve the following initial value problem using the Laplace tranform

$$\frac{dy}{dt} = -y + 2u_3(t), y(0) = 4$$

$$L[y'] = sL[y] - y(0) = -L[y] + 2\frac{e^{-3s}}{s}$$

$$L[y](s+1) = 4 + 2\frac{e^{-3s}}{s}$$

$$L[y] = \frac{4}{s+1} + 2\frac{e^{-3s}}{s(s+1)} \quad ; \quad \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$y(t) = L^{-1}\left[\frac{4}{s+1}\right] + 2L^{-1}\left[\frac{e^{-3s}}{s}\right] - 2L^{-1}\left[\frac{e^{-3s}}{s+1}\right]$$

$$= \cancel{4e^{t} + 2u_3(t) - 2u_3(t)e^{-(t+3)}}$$

$$= 4e^{-t} + 2u_3(t) - 2u_3(t)e^{-(t-3)}$$