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Section 3.6: 1, 5, 7, 11, 13, 19, 25, 39

1. $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} - 7y = 0$

Guess: $y(t) = e^{st}$

$s^2e^{st} - 6se^{st} - 7e^{st} = 0$

$e^{st}(s^2 - 6s - 7) = 0$

$(s-7)(s+1) = 0$; $s_1 = 7, s_2 = -1$

$y(t) = K_1e^{7t} + K_2e^{-t}$

5. $\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 25y = 0$

Guess: $y(t) = e^{st}$

$s^2e^{st} + 8se^{st} + 25e^{st} = 0$

$e^{st}(s^2 + 8s + 25) = 0$

$s = \frac{-8 \pm \sqrt{64 - 4(25)}}{2}$; $s = -4 \pm 3i$

$y(t) = e^{-4t}(\cos(3t) + i\sin(3t))$

$y(t) = K_1e^{-4t}\cos(3t) + K_2e^{-4t}\sin(3t)$

7. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = 0$ $\begin{cases} y(0) = 6 \\ y'(0) = -2 \end{cases}$

Guess: $y(t) = e^{st}$

$e^{st}(s^2 + 2s - 3) = 0$

$(s+3)(s-1) = 0$; $s_1 = -3, s_2 = 1$

$y(t) = K_1e^{-3t} + K_2e^{t}\big|_{t=0} = K_1 + K_2$

$y'(t) = -3K_1e^{-3t} + K_2e^{t}\big|_{t=0} = -3K_1 + K_2$

$\begin{pmatrix} 6 \\ -2 \end{pmatrix} = \begin{pmatrix} K_1 + K_2 \\ -3K_1 + K_2 \end{pmatrix}$ $\quad K_2 = 3K_1 - 2$, $\quad 6 = K_1 + 3K_1 - 2$

$K_2 = 4 \Leftarrow K_1 = 2$

$\therefore y(t) = 2e^{-3t} + 4e^{t}$

11. $\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 16y = 0$ $\begin{cases} y(0) = 3 \\ y'(0) = 11 \end{cases}$

Guess: $y(t) = e^{st}$

$e^{st}(s^2 - 8s + 16) = 0$

$(s-4)^2 = 0$; $s = 4$

$y(t) = K_1e^{4t} + K_2te^{4t}$

$y'(t) = 4K_1e^{4t} + K_2e^{4t} + 4K_2te^{4t}$

$\begin{pmatrix} 3 \\ 11 \end{pmatrix} = \begin{pmatrix} K_1 + 0 \\ 4K_1 + K_2 + 0 \end{pmatrix}$ $\quad K_1 = 3$, $\quad 11 = 12 + K_2$, $\quad K_2 = -1$

$\therefore y(t) = 3e^{4t} - te^{4t}$

$y'' + py' + qy = 0 \quad p = \frac{b}{m} \quad q = \frac{k}{m}$

13. a) $\frac{d^2y}{dt^2} + 8y' + 7y = 0$ $\begin{cases} y(0) = -1 \\ v(0) = 5 \end{cases}$

or as a system $\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -7y - 8v \end{cases}$

b) $A = \begin{pmatrix} 0 & 1 \\ -7 & -8 \end{pmatrix}$ $\quad p(\lambda) = (-\lambda)(-8-\lambda) + 7 = \lambda^2 + 8\lambda + 7$

$0 = (\lambda + 7)(\lambda + 1)$; $\lambda_1 = -7, \lambda_2 = -1$

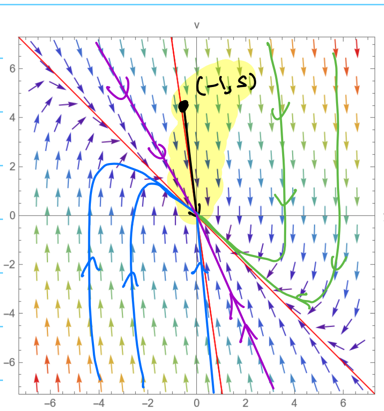
$\lambda_1 = -7$: $\begin{pmatrix} 0 & 1 \\ -7 & -8 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix} = -7\begin{pmatrix} y \\ v \end{pmatrix}$ $\quad$ $\lambda_2 = -1$: $\begin{pmatrix} 0 & 1 \\ -7 & -8 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix} = -\begin{pmatrix} y \\ v \end{pmatrix}$

$v = -7y$, $-7y - 8v = -7v$ $\quad \begin{pmatrix} 1 \\ -7 \end{pmatrix}$ $\qquad$ $v = -y$, $-7y - 8v = -v$ $\quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c) $b > 0$ as $b = 8$ and $b^2 - 4mk > 0$ as $b^2 - 4mk = 36$

$\therefore$ The Oscillator is Overdamped

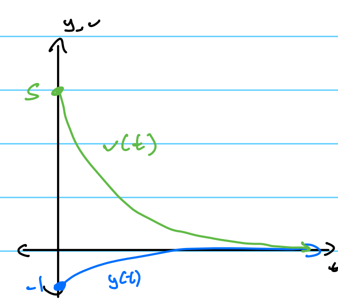
d)



e) Both y and v approach 0

$v \to 0$ from positive

$y \to 0$ from negative



19. $2y''+0y'+3y=0$ $\begin{cases} y(0)=2 \\ v(0)=-3 \end{cases}$

$2y''+3y=0$

a) $\begin{cases} \dfrac{dy}{dt}=v \\ \dfrac{dv}{dt}=-\dfrac{3}{2}y \end{cases}$

b) $A=\begin{pmatrix} 0 & 1 \\ -\frac{3}{2} & 0 \end{pmatrix}$ $p(\lambda)=\lambda^2+\frac{3}{2}$

$\lambda=\pm i\sqrt{3/2}$

$\lambda=\sqrt{3/2}$ $\therefore \begin{pmatrix} 0 & 1 \\ -\frac{3}{2} & 0 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}=i\sqrt{3/2}\begin{pmatrix} y \\ v \end{pmatrix}$

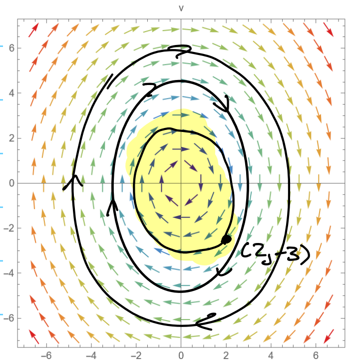
$v=iy\sqrt{3/2}$

$-\frac{3}{2}y=iv\sqrt{3/2}$

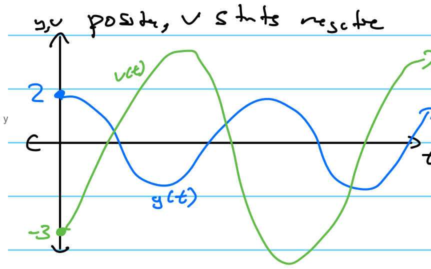
$\begin{pmatrix} 1 \\ i\sqrt{3/2} \end{pmatrix}$

c) $b=0$, undamped. Period: $\frac{2\pi}{\beta}$ $\beta=\sqrt{3/2}$ $\therefore 2\pi\sqrt{\frac{2}{3}}$

d)



e) Oscillations. y starts positive, v starts negative



25. $2y''+3y'+y=0$ $\begin{cases} y(0)=0 \\ v(0)=3 \end{cases}$

Guess: $y(t)=e^{st}$

a) $e^{st}(2s^2+3s+1)=0$

$s=\dfrac{-3\pm\sqrt{9-4(2)}}{4}$ $s_1=-1,\ s_2=-\frac{1}{2}$

$y(t)=k_1e^{-t}+k_2e^{-\frac{1}{2}t}$

b) $v=-k_1e^{-t}-\frac{1}{2}k_2e^{-\frac{1}{2}t}$

$\begin{pmatrix} k_1+k_2 \\ -k_1-\frac{1}{2}k_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ $k_1=-k_2$, $k_2-\frac{1}{2}k_2=3$ $\therefore k_2=6$, $k_1=-6$

$y(t)=-6e^{-t}+6e^{-\frac{1}{2}t}$ $v(t)=6e^{-t}-3e^{-\frac{1}{2}t}$

c)

v will increase slightly then approach 0 as $t\to\infty$. y will decrease to 0 over time.

39. $my''+0y'+2y=0$

$\begin{cases} \dfrac{dy}{dt}=v \\ \dfrac{dv}{dt}=-\dfrac{2}{m}y \end{cases}$

$A=\begin{pmatrix} 0 & 1 \\ -\frac{2}{m} & 0 \end{pmatrix}$ $p(\lambda)=\lambda^2+\frac{2}{m}$

$\lambda=\pm i\sqrt{\frac{2}{m}}$

$\beta=\sqrt{\frac{2}{m}}$ and $P=\frac{2\pi}{\beta}$

Mass when $P=1$?

$1=\dfrac{2\pi}{\sqrt{\frac{2}{m}}}$

$\sqrt{\frac{2}{m}}=2\pi$

$\frac{2}{m}=4\pi^2 \Rightarrow m=\dfrac{1}{2\pi^2}$

Section 3.7: 3, 7, 9, 13

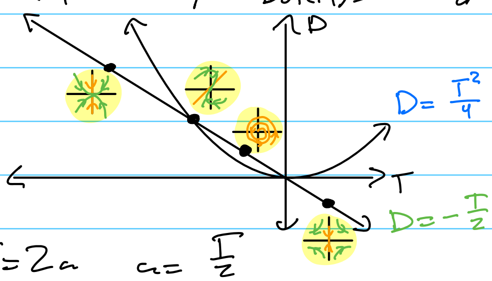
3. $\frac{d\vec{Y}}{dt} = \begin{pmatrix} a & a^2+a \\ 1 & a \end{pmatrix} \vec{Y}$

$A = \begin{pmatrix} a & a^2+a \\ 1 & a \end{pmatrix}$; $Tr(A) = 2a$

$Det(A) = a^2 - a^2 - a = -a$

$\sqrt{T^2-4D}$

a)



$T = 2a \qquad a = \frac{T}{2}$

$D = -a \qquad D = -\frac{T}{2}$

c) $a = 0$ and $a = -1$ are bifurcation values as calculated in (b)

b) $p(\lambda) = (a-\lambda)(a-\lambda) - a^2 - a$

$= a^2 + \lambda^2 - 2a\lambda - a^2 - a$

$= \lambda^2 - 2a\lambda - a$

$\lambda = \frac{2a \pm \sqrt{4a^2+4a}}{2} = a \pm \sqrt{a(a+1)}$

$+ \quad -1 \quad - \quad 0 \quad +$

$a < -1$: Real Sink

$a = -1$: Repeated $\lambda < 0$

$-1 < a < 0$: Complex, spiral sink $\lambda < 0$

$a = 0$: Repeated $\lambda = 0$

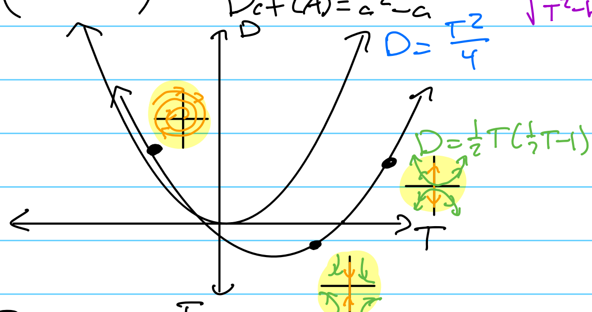
$a > 0$: Saddle

7. $\frac{d\vec{Y}}{dt} = \begin{pmatrix} a & 1 \\ a & a \end{pmatrix} \vec{Y}$

$A = \begin{pmatrix} a & 1 \\ a & a \end{pmatrix}$; $Tr(A) = 2a$

$Det(A) = a^2 - a$

$\sqrt{T^2-4D}$



$T = 2a \qquad a = \frac{T}{2}$

$D = \frac{1}{4}T^2 - \frac{1}{2}T = \frac{1}{2}T(\frac{1}{2}T - 1)$

b) $p(\lambda) = (a-\lambda)^2 - a$

$= \lambda^2 - 2a\lambda + a^2 - a$

$\lambda = \frac{2a \pm \sqrt{4a^2 - 4a^2 + 4a}}{2} = a \pm \sqrt{a}$

$a < 0$: spiral sink

$a = 0$: Repeated $\lambda = 0$

$0 < a < 1$: Saddle

$a = 1$: Two Eigenvalue with $\lambda_1 = 0$

$a > 1$: Source

c) Bifurcations at $a = 0$ and $a = 1$ as per (b)

9. $\frac{dY}{dt} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} Y$ $\qquad p(\lambda) = \lambda^2 - T\lambda + D$

$Tr(A) = 2a$, $Det(A) = a^2 - b^2$

$p(\lambda) = \lambda^2 - 2a\lambda + a^2 - b^2$

$\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2-b^2)}}{2} = a \pm \sqrt{b^2} = a \pm |b|$

$\lambda = a \pm |b|$

$|b|$, $\sqrt{b^2}$ cannot be imaginary!

$a = \pm b$ then $\lambda = 0$, $b = 0$ and $a \neq 0$ then repeated λ, Repeated $\lambda = 0$ when $a = b = 0$. Source when $a > |b|$ Sink when $a < -|b|$. Saddle for all other values

13. $m\frac{d^2y}{dt^2}+\frac{dy}{dt}+2y=0$

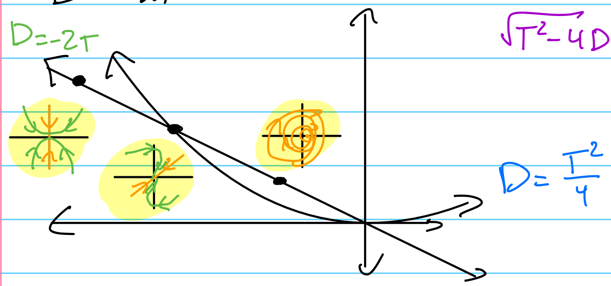
where $b=1$, $k=2$, and $0<m<\infty$

a) $\begin{cases} \frac{dy}{dt}=v \\ \frac{dv}{dt}=-\frac{2}{m}y-\frac{1}{m}v \end{cases}$

b) $A=\begin{pmatrix} 0 & 1 \\ -\frac{2}{m} & -\frac{1}{m} \end{pmatrix}$ $\quad Tr(A)=-\frac{1}{m}$ $\quad Det(A)=\frac{2}{m}$

$T=-\frac{1}{m}$ so $m=-\frac{1}{T}$

$D=\frac{2}{m}$ so $D=-2T$



c) $b^2-4km=0$

$b=1$, $k=2$

$1-8m=0$

$m=\frac{1}{8}$

$0<m<\frac{1}{8}$: $b^2-4km>0$ → overdamped

$m=\frac{1}{8}$: $b^2-4km=0$ → critically damped

$m>\frac{1}{8}$: $b^2-4km<0$ → underdamped