Section 1.1 - Modeling Via Differential Equations

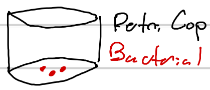
To build a Model:

- Clearly state all assumptions being made
- Completely describe all vars and parameters
  - dependent variable
  - independent variable (usually time)
  - Parameters: do not change with time but may be adjusted on a per model basis

Assumptions allow equations to be derived from vars and parameters

Example: Unlimited Population Growth

Assumptions needed to create more accurate models

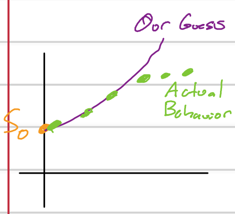


Predict the population of bacteria in the cup.

Variables: t = time (independent)

$S(t)$ = Population of bacteria

'rate of change' synonymous with derivative



Assumption: Number of new bacteria is proportional to the population itself

rate of change $= \frac{dS}{dt} = KS$

Two Equations: $\frac{dS}{dt} = KS$ ← differential Eq.

$S(0) = S_0$ ← Initial Condition

We need to find a function! $S(t)$

Guess: $S(t) = S_0 \cdot e^{Kt}$, so $\frac{dS}{dt} = K \cdot S_0 e^{Kt}$ ← Works for short time period but space/resources limiting!

Terminology

- A constant solution to a DE is called an equilibrium solution. A solution $P(t)$ st $\frac{dP}{dt} = 0$ for all t is an equilibrium solution
- The pair of equations $\frac{dP}{dt} = KP$ and $P(0) = P_0$ is called an initial value problem
- Ordinary DEs lack Partial Derivatives

Example: Limited Resources and the Logistic Population Model

Variables: t = time K = growth rate coef.

P = Population N = Carrying capacity

Model: $\frac{dP}{dt} = K\left(1 - \frac{P}{N}\right)P$

Called the Logistic Population Model, using defined vars

Assumptions: small population → rate of growth proportional to P

Population too large → rate of growth decreases

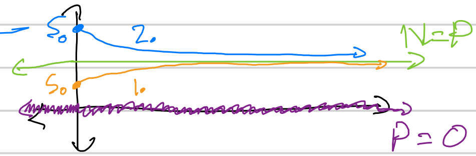
Our model is a first order DE, and it is nonlinear because of the P^2 term.

- A DE is nonlinear if the term involving the dependent variable is nonlinear
- The order of a DE is determined by the order of the derivative of the equation

What are the equilibrium solutions to our model?

We need the rate of change to be 0!

$\frac{dP}{dt} = KP\left(1 - \frac{P}{N}\right)$ $\frac{dP}{dt} = 0$ if $P = 0$ or $N = P$



1. If initial value lies btwn $P=0$ and $N=P$, Model approaches N like an asymptote
2. If initial value is larger than N, the derivate is negative, and approaches N

Example: Consider $\frac{dy}{dt} = y^4 + y^3 - 2y^2$

(a) Equilibrium Solutions, $\frac{dy}{dt} = 0$

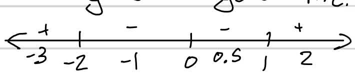
$y^4 + y^3 - 2y^2 = 0$

$y^2(y^2 + y - 2) = 0$

$y^2(y-1)(y+2) = 0$

$y = 0, y = 1, y = -2$

(b) values of y where $y(t)$ inc. and dec. $\frac{dy}{dt} = y^2(y-1)(y+2)$



Sign Diagrams helpful

Model Graph is derived from increasing and decreasing regions above

Predator - Prey Systems

Let: R = prey and F = predator (ex. rabbits and foxes)

Assumptions:
- If no foxes → rabbits reproduce proportional to population size
- If no rabbits → fox population decreases at rate prop. to pop. size
- Rate at which rabbits are being eaten by foxes is proportional to their interaction rate
- Fox births are proportional to the number of rabbits eaten

Parameters: (Assume all > 0)

α: growth rate coef. of rabbits (alpha)

Parameters do not change with time

γ: death rate coef. of foxes (gamma)

β: Constant that measures # of fox-rabbit interactions where the rabbit is eaten (beta)

δ: constant that measures the benefit to fox population of an eaten rabbit (delta)

Modeling:

$$\begin{cases} \frac{dR}{dt} = \alpha R - \beta RF \\ \frac{dF}{dt} = -\gamma F + \delta RF \end{cases}$$

This can be applied to other population types as well, not just rabbit-fox populations. Its just convenient for variable names

Definitions

1. Analytic methods: search for explicit formulas for solutions
2. Qualitative methods: Use of graphical models
3. Numerical methods: Use of technology

will appear in text.
Numerical is used a lot bc models are complicated

Section 1.2 - Separation of Variables

Standard form of first order DE: (*) $\frac{dy}{dt} = f(t, y)$

Special Cases

① $\frac{dy}{dt} = g(t)$, can be solved by integrating

② $\frac{dy}{dt} = h(y)$, called an autonomous DE

Example: Consider $\frac{dy}{dt} = y^2(2t+3)$

Is $y(t) = -\frac{1}{t^2+3t}$ a solution? What about $y(t) = 3t+2$

First eq: $y(t) = -\frac{1}{t^2+3t}$ in $\frac{dy}{dt} = y^2(2t+3)$

$\frac{dy}{dt} = \frac{2t+3}{(t^2+3t)^2}$ bc $y = -\frac{1}{t^2+3t}$ $\frac{dy}{dt} = y^2(2t+3)$, yes

Second eq: $y(t) = 3t+2$ in $\frac{dy}{dt} = y^2(2t+3)$

$\frac{dy}{dt} = 3 \neq (3t+2)^2(2t+3)$

Steps: ① Take derivative of equation
② Plug eq. into DE
③ Check Equivalence

Separable Equations work in the form $\frac{dy}{dt} = g(y) \cdot h(t)$

Example: Solve the IV problem $\frac{dy}{dt} = \frac{t^3}{y^2}$, $y(0) = 2$

① Separate: $y^2\,dy = t^3\,dt$

② Integrate: $\int y^2\,dy = \int t^3\,dt$

$\frac{1}{3}y^3 + C_1 = \frac{1}{4}t^4 + C_2$

> Don't forget constant
Constants can be combined to make a general constant

③ Simplify: $y^3 = \frac{3}{4}t^4 + C$

$y(t) = \left(\frac{3}{4}t^4 + C\right)^{1/3}$

④ Solve for C: $y(0) = C^{1/3} \rightarrow C^{1/3} = 2 \rightarrow C = 8$

⑤ Solution: $y(t) = \left(\frac{3}{4}t^4 + 8\right)^{1/3}$

You can check your solution by differentiating and plugging in, using the steps outlined in example 1.

Warning! there may be additional solutions corresponding to $h(y) = 0$

Example: Solve the IV problem $\frac{dy}{dt} = t^2 y^3$, $y(0) = 0$

① Separate: $\frac{dy}{y^3} = t^2\,dt$

Dividing by y^3 eliminates the $y=0$ possibility, so be careful!

② Integrate: $\int y^{-3}\,dy = \int t^2\,dt$

$-\frac{1}{2}y^{-2} + C_1 = \frac{1}{3}t^3 + C_2$

$y^{-2} = -\frac{2}{3}t^3 + C$

Always check for division by 0 when separating variables

③ Simplify: $y = \sqrt{\frac{1}{-\frac{2}{3}t^3 + C}}$

④ Solve for C: $y(0) = \sqrt{\frac{1}{C}}$ but $C = 0$ makes the function undefined!

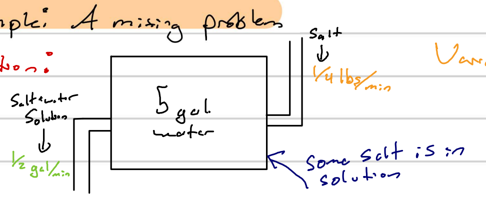
⑤ Solution: $y(t) = 0$ is the only solution

$\frac{d}{dt}(0) = 0$ and $\frac{dy}{dt}(0) = t^2(0)^3 = 0$

Changing the initial condition completely changes the solution

Example: A mixing problem

Situation:



Variables: t = time independent
$S(t)$ = amt. of salt dependent
$W = 5$ gal

DE Model: $\frac{dS}{dt} = \frac{1}{4} - \frac{1}{2}\left(\frac{S}{5}\right)$

rate of change of salt

$\begin{cases} \frac{dS}{dt} = \frac{1}{4} - \frac{S}{10} \\ S(0) = 0 \end{cases}$

Density of the mixture $= \frac{\text{amount of salt}}{\text{volume}}$

Units: $\frac{1\text{ gal}}{2\text{ min}}\left(\frac{S(t)\text{ salt}}{\text{gal}}\right) \rightarrow \frac{\text{salt}}{\text{min}}$ ✓

Solving: $\frac{dS}{dt} = \frac{1}{4} - \frac{S}{10} = f(S)$

Autonomous Equation

$\frac{dS}{\frac{1}{4} - \frac{S}{10}} = dt$

if you divide by vars, check for $x/0$ errors!

Consider $\frac{1}{4} - \frac{S}{10} = 0$ $S = \frac{10}{4} = \frac{5}{2}$ $S(t) = \frac{5}{2}$

$\int \frac{dS}{\frac{1}{4} - \frac{S}{10}} = \int dt$

$-10\ln\left|\frac{1}{4} - \frac{S}{10}\right| + C_1 = t + C_2$

$\ln\left|\frac{1}{4} - \frac{S}{10}\right| = -\frac{1}{10}t + C$

$K = \pm e^C$

$\frac{1}{4} - \frac{S}{10} = e^{-\frac{1}{10}t + C}$

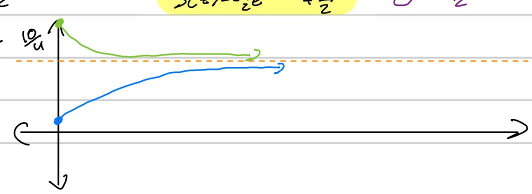
$S = Ke^{-\frac{1}{10}t} + \frac{5}{2}$

$S(t) = -\frac{5}{2}e^{-\frac{1}{10}t} + \frac{5}{2}$

$-\frac{S}{10} = e^{-\frac{1}{10}t + C} - \frac{1}{4}$ $S(0) = 0$

$0 = Ce^0 + \frac{5}{2}$

$C = -\frac{5}{2}$

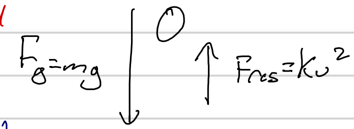


Example: Free falling object

Assumptions: Force of Air Resistance $\propto v^2$ ← Experimentally tested

Second Newton's Law: $\vec{F} = ma = m\frac{dv}{dt}$

$\frac{dv}{dt} = g - \frac{kv^2}{m}$ => $\vec{F} = mg - kv^2$



↘ $F = mg - kv^2$

Variables: t = time $v(t)$ = velocity

Model: $\frac{dv}{dt} = g - \frac{kv^2}{m} = g\left(1 - \frac{kv^2}{mg}\right)$ let $\alpha = \sqrt{\frac{k}{mg}}$

$\frac{dv}{dt} = g(1 - v^2\alpha^2)$

$\alpha = \sqrt{\frac{k}{mg}}$ to simplify integration → diff. of squares

Solving: $\int \frac{dv}{1-v^2\alpha^2} = \int g\,dt$ Partial Fraction Decomposition $\frac{1}{1-v^2\alpha^2} = \frac{\frac{1}{2}}{1+\alpha v} + \frac{\frac{1}{2}}{1-\alpha v}$

$\int \frac{dv}{1+\alpha v} + \int \frac{dv}{1-\alpha v} = 2gt + C$

$\frac{1}{\alpha}\ln|1+\alpha v| - \frac{1}{\alpha}\ln|1-\alpha v| = 2gt + C$

$\ln\left|\frac{1+\alpha v}{1-\alpha v}\right| = 2\alpha gt + C$

$\frac{1+\alpha v}{1-\alpha v} = Ke^{2\alpha gt}$ $K = \pm e^C$ bc $\ln|x|$ can be $-\infty < x < \infty$

$v(t) = \frac{1}{\alpha}\,\frac{Ke^{2\alpha gt} - 1}{Ke^{2\alpha gt} + 1}$ where $\alpha = \sqrt{\frac{k}{mg}}$

Terminal Velocity: $\lim_{t\to\infty} \frac{1}{\alpha}\,\frac{K - e^{-2\alpha gt}}{K + e^{-2\alpha gt}} = \frac{1}{\alpha} = \frac{1}{\sqrt{\frac{k}{mg}}} = \sqrt{\frac{mg}{k}}$

Derivation: Newton's law of cooling

Assumptions: Decrease in Temp $\propto$ the diff. btwn objects and environments temps

Variables: t = time, $T(t)$ = Temperature

Model: $T_0 = T(0) = 30F$ →

$$\begin{cases} \frac{dT}{dt} = -k(T - T_0) \\ T_0 = T_1 \end{cases}$$

Section 1.3 - Slope Fields

Suppose we have models:

① $\frac{dy}{dt} = \sin(t^2)$ $y(t) = \int \sin(t^2)\,dt$

② $\frac{dy}{dt} = \frac{1}{e^{y^2}}$ $t = \int e^{y^2}\,dy$

These are unsolvable by elementary methods so we need to use so-called numerical methods to solve

Slope Fields

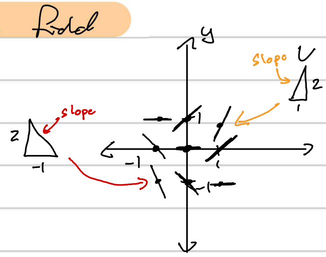
Let $\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(t_1) = y_1 \end{cases}$ At the point (t_1, y_1), $f(t_1, y_1)$ is the slope of the tangent line @ (t_1, y_1)

The values on the right side of the equation yield the slopes of the tangents at all points on the graph of a solution $y(t)$

Example: Constructing a slope field

Consider $y' = t + y$

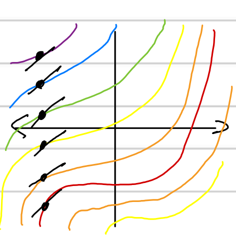
t	y	slope, y'
-1	0	-1
-1	-1	-2
-1	1	0
0	0	0
0	-1	-1
0	1	1
1	0	1
1	-1	0
1	1	2



Given an initial condition, we can sketch a potential solution curve. Graphing takes a while by hand, so technology is very useful here.

Special Case 1

① $\frac{dy}{dt} = f(t)$, where slopes only depend on t. Slopes are parallel along vertical lines.



Poorly Drawn, but slope doesn't depend on the y-coordinate!

Special Case 2 (autonomous)

$\frac{dy}{dt} = f(y)$. Slopes only depend on y, slopes parallel along horizontal lines. Slope doesn't depend on the t coordinate!

Section 1.4 - Euler's Method

Consider $\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$

Approximations made by starting at (t_0, y_0), drawing a tangent line with slope $f(t_0, y_0)$ ending at point (t_1, y_1) where $t_1 = t_0 + \Delta t$. Repeatable, leads to y_n approximations.

Note: Smaller Δt values result in more accurate tangent line approximations

Recursive Formulas: $t_{k+1} = t_k + \Delta t$

$y_{k+1} = y_k + f(t_k, y_k)\Delta t$ ← slope at (t_k, y_k) times Δt

} Simple, yet powerful

Example: Euler's Method by hand

$y' = 2y+1$, $y(0) = 3$. Compute $0 \le t \le 2$ with $\Delta t = 0.5$

① $t_0 = 0$, $t_1 = 0.5$

$y_1 = y_0 + \Delta t(2y_0+1)$

$= 3 + 0.5(7) = 6.5$

② $t_0 = 0$, $t_{-1} = -0.5$

$y_{-1} = 3 + -0.5(2y_0+1)$

$= 3 - 0.5(7) = -0.5$

③ $t_1 = 0.5$ $t_2 = 1$

$y_2 = 6.5 + 0.5(2 \cdot 6.5 + 1)$

$= 6.5 + 7 = 13.5$

④ $t_2 = 1$ $t_3 = 1.5$

$y_3 = 13.5 + 0.5(2 \cdot 13.5 + 1)$

$= 13.5 + 14 = 27.5$

⑤ $t_3 = 1.5$ $t_4 = 2$

$y_4 = 27.5 + 0.5(2 \cdot 27.5 + 1)$

$= 27.5 + 28 = 55.5$

For Comparison: $\frac{dy}{dt} = 2y+1$

$\int \frac{dy}{2y+1} = \int dt$ but $y = -\frac{1}{2}$ is lost in the space"

$\frac{1}{2}\ln|2y+1| = t + C$

$\ln|2y+1| = 2t + C$

$2y+1 = Ke^{2t}$

$y = Ke^{2t} - \frac{1}{2}$, $y(0) = 3$

$3 = K - \frac{1}{2} \rightarrow K = 3.5$

$y = 3.5e^{2t} - 0.5$

Generally, approximations are worse with rapidly growing functions or with unbounded models

Example: Eulers Method by hand

$y' = (3-y)(y+1)$, $y(0) = 4$, $\Delta t = 1$ with $0 \le t \le 5$

① $y(0) = 4$

② $t = 1$

$y = 4 + (-5) = -1$

③ $t = 2$

$y = -1 + (0) = -1$

$y_{k+1} = y_k + f(t_k, y_k)\Delta t$

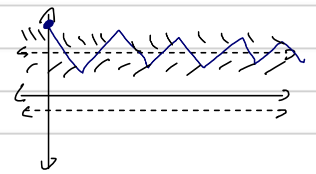
④ $t = 3$

$y = -1 + 0 = -1$

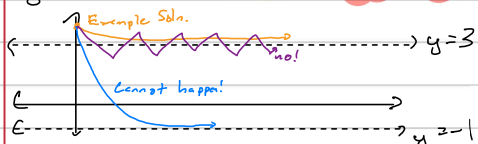
-1 is a general soln. but not one for this initial condition

The step size is so big that we get a false solution.

A step size of 0.5 doesn't work either, as it oscilates around $y = 3$



↑ Due to slopes around $y = 3$, shown above



Seeing how your graph should look, such as knowing $y(0) = 4$ will approach, but never = or cross 3, tells us which solutions to accept.

Section 1.5 - Existence and Uniqueness of Solutions

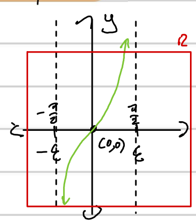
Suppose an equation $f(x)=x^5+5x-10$. Solve $f(x)=0$

$f(x)=x^5+5x-10$ Intermediate Value Thm.

$f(1)=-4$ and $f(2)=32$ $\therefore f(x)=0$ when $1<x<2$, now we can use numerical methods

Existence Thm: Suppose $f(t,y)$ is a cts. function on a open rectangle R in the ty-plane. If (t_0,y_0) is a point in R, then there exists an $\varepsilon>0$ and a function $y(t)$ defined for $t_0-\varepsilon<t<t_0+\varepsilon$ that solves the initial value problem

Example 1 $\frac{dy}{dt}=1+y^2$, $y(0)=0$, $t_0=0$ and $y_0=0$



Ash exists in some rectangle R

$$\int \frac{dy}{1+y^2}=\int dt$$

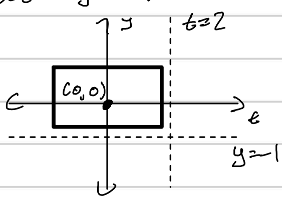
$\arctan y = t+C \Big|_{(0,0)} \therefore C=0$

$y=\tan t$

$\varepsilon=\frac{\pi}{2}$, tangent defined over $-\frac{\pi}{2}<t<\frac{\pi}{2}$

Example 2: Restricted Domain

$\frac{dy}{dt}=\frac{1}{(y+1)(t-2)}$, $y(0)=0$, not cts !!



Can be solved through integration, takes awhile so omitted

Uniqueness Thm: Suppose $f(t,y)$ and $\frac{\partial f}{\partial y}$ are cts. functions on an open rectangle R in the ty-plane that contains (t_0,y_0). Then the solutions cannot cross equilibrium sls.

If two solutions are ever in the same place at the same time, then they are the same function (for all t for which they are both defined.

Example 3: Uniqueness fails

Consider $y'=3y^{2/3}$ with $y(0)=0$)

Let $f(t,y)=3y^{2/3}$ cts.

$\frac{\partial f}{\partial y}=2y^{-1/3}$ not cts. @ $y=0$

$\therefore$ Uniqueness Thm. does not work here !!

Can be solved numerically or by integrating

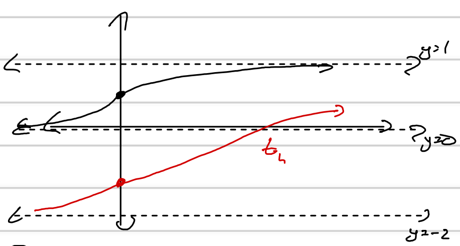
$$\int \frac{dy}{3y^{2/3}}=\int dt$$

Example 4: Role of Equilibrium Solutions

Consider $\frac{dy}{dt}=y^4+y^3-2y^2$ cts., $y(t_0)=y_0$

Uniqueness Thm. suggests

$\frac{\partial f}{\partial y}=4y^3+3y^2-4y$ cts. as well



Predicting initial conditions

if $0<y(t_0)<1$ then $0<y(t)<1$

entire curve squeezed b/twn solutions

$f(y)$ is cts for all points in the ty-plane

$y'=y^2(y^2+y-2)=y^2(y+2)(y-1) \Rightarrow y=0, y=-2, y=1$

A solution can never cross an equilibrium solution because of uniqueness theorem

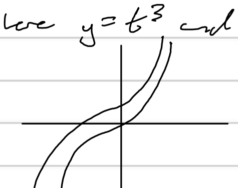
also guarantees solutions will not cross

↳ If a line $h(t)$ is drawn and crosses $y=0$, there would be two solutions at t_1

Example: Uniqueness Thm

Suppose $f(y,t)=y'$ where $y=t^3$ and $y=t^3+1$ are both solutions. What can be said about solutions

$$\begin{cases} y'=f(y,t) \\ f, \frac{\partial f}{\partial y} \text{ are cts in } ty \\ y(0)=a,\ 0<a<1 \end{cases}$$



$t^3<y(t)<t^3+1$ as it is impossible for solutions to cross the equilibrium solutions

Section 1.6 - Equilibria

We will consider autonomous DE in this section, in the form $\frac{dy}{dt} = f(y)$

Recall: Slopes are parallel along horizontal lines in the ty plane

↳ we can describe an entire field if we know the slopes along one vertical line only

This line is called the phase line, and contains compressed slope info for an autonomous DE

Recall: Equilibrium points are the y-values when $f(y) = 0$.

Note: There can be holes in a phase line

Theorem

so Existence and Uniqueness apply

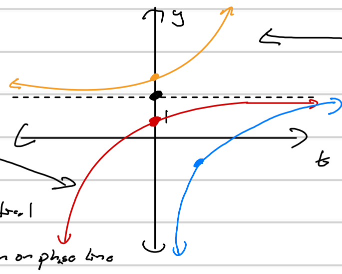
Consider a function $\frac{dy}{dt} = f(y)$ where $f(y)$ are cts and differentiable. Suppose $y(t)$ is a solution

1. If $f(y(0)) = 0$, then $y(0)$ is an equilibrium point and $y(t) = y(0)$ for all t
2. If $f(y(0)) > 0$, then $y(t)$ is increasing for all t and either $y(t) \to \infty$, or $y(t) \to$ larger eq. point
3. If $f(y(0)) < 0$, then $y(t)$ is decreasing for all t and either $y(t) \to -\infty$, or $y(t) \to$ smaller eq. point

Example: Phase Lines

Draw the phase line for $\begin{cases} y' = (y-1)^2 \\ y(0) = \frac{1}{2} \end{cases}$

$\lim_{t \to \infty} y(t) = 1$



← This doesn't always approach inf, as there could be a vertical asymptote

Actual sln of $y = 1 + \frac{1}{C - t}$, with the vertical asymptote @ $t = C$, which cannot be seen on phase line

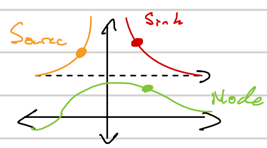
Example: Stinky ass

Draw the phase line for $\begin{cases} y' = \frac{1}{y+1} \\ y(0) = x \end{cases}$

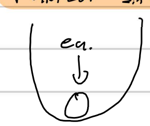
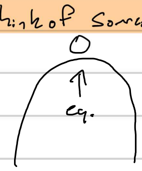
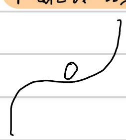
undefined @ $y = -1$!!

No eq. points

Classification of Eq. points let y_0 be an initial condition



1. A sink is where y_0 asymptotically approaches y_0 as $t \to \infty$
2. A source is where y_0 asymptotically approaches y_0 as $t \to -\infty$
3. A node is neither a source nor a sink

Think of sinks as	Think of sources as	Think of nodes
		
it is very stable	unstable, loses eq. easily	half stable

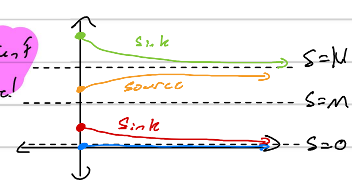
Modified Logistic Population Model

$$\boxed{\frac{dS}{dt} = kS\left(1 - \frac{S}{N}\right)\left(\frac{S}{M} - 1\right)},$$

N is carrying capacity, M is sparsity

- $S(t)$ is population and k is growth rate coef

Population can't be negative!



Section 1.7 - Bifurcations

In this section we are interested in $\frac{dy}{dt} = f_\mu(y)$ where μ is the parameter of interest.

A parameter where there is a significant change in behaviours is a bifurcation

Example: Find Equilibrium points for $y' = y^2 + 3y + \mu$

$f_\mu(y) = 0$

Equilibrium points: $y = \frac{-3 \pm \sqrt{9 - 4\mu}}{2}$

- if $\mu = \frac{9}{4}$ => one solution
- if $\mu > \frac{9}{4}$ => No real solution
- if $\mu < \frac{9}{4}$ => 2 solutions

$\frac{9}{4}$ is a location of large change.

$\mu = \frac{9}{4}$ is a bifurcations point

Bifurcation Plot - A graph in the μy plane

① Plot parameter μ along horizontal axis

② Draw phase line will μ eq. points

Example: bifurcation plot

Draw bifurcation diagram for $y' = y^3 - \alpha y$

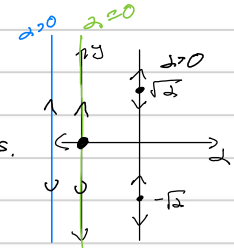
$y' = f_\alpha(y) = y^3 - \alpha y$

- if $\alpha > 0$: 3 eq sols.
- if $\alpha < 0$: 1 eq sol
- if $\alpha = 0$: 1 eq. sols

$0 = y^3 - \alpha y$

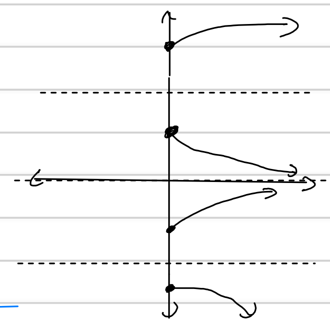
$0 = y(y^2 - \alpha)$

$0 = y(y - \sqrt{\alpha})(y + \sqrt{\alpha})$

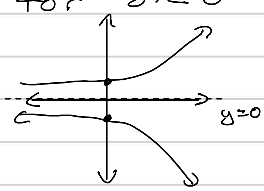


The behaviour is the same regardless of α choice if in range

for $\alpha > 0$



for $\alpha < 0$



Notice the Dramatic Difference in graph.
This is why its a bifurcation

$f_C(P) = kP(1 - \frac{P}{N}) - C$ where C is a parameter of fish being caught.

autonomous, f does not depend on t

Example: Application to Sustainability. Use above eqn

$$\frac{dP}{dt} = kP(1 - \frac{P}{N}) - C$$

Solve: $f_C(P) = kP(1 - \frac{P}{N})$

C is just a constant

$0 = kP - \frac{K}{N}P^2 - C \quad * (-\frac{N}{K})$

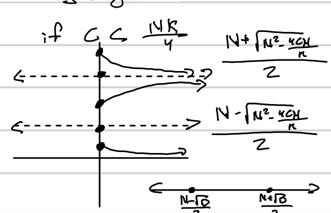
$0 = P^2 - NP + \frac{CN}{k}$

$P = \frac{N \pm \sqrt{N^2 - 4(\frac{CN}{k})}}{2}$

Solve the discriminant

$N^2 = \frac{4CN}{K} \Rightarrow C = \frac{NK}{4}$

if $C < \frac{NK}{4}$



- if $C < \frac{NK}{4}$ 2 solutions
- if $c > \frac{NK}{4}$ no eq.
- if $c = \frac{NK}{4}$ one eq.

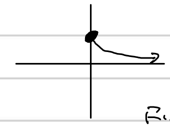
$F_C(P) = P^2 + kP - C = 0$

$P_1 = \frac{N - \sqrt{P^2 - \frac{4CN}{k}}}{2}$

$P_2 = \frac{N + \sqrt{P^2 - \frac{4CN}{k}}}{2}$

Plug into eqn and factor to get behavior

if $C > \frac{NK}{4}$



$f(t) < 0$ for all t

Fish die!!

Section 1.8 - Linear Equations

Linear DEs

$\frac{dy}{dt} = a(t)y + b(t)$, where a and b are arbitrary functions of t

Examples: $y' = e^{2t}y - \sin t$ ✓ $y' = ty^2$ ✓

$y' = 3y + 4$ ✓ $\frac{dP}{dt} = kP(1-\frac{P}{N})$ ✗

$y' = t^2 y$ ✓

notice how the function of t doesn't matter. Only care about dependent variable

- If $b(t) = 0$ for all t, the equation is called homogeneous (or unforced)
  - ↳ otherwise called nonhomogeneous (or forced)
- If $a(t)$ is constant, the linear equation is called a constant-coefficient equation.

The Linearity Principle (Homogeneous Case)

If $y_h(t)$ is a solution of the homogeneous linear equation $\frac{dy}{dt} = a(t)y$, then any constant multiple of $y_h(t)$ is a solution.

Given: $y'(t) = a(t)y$

$y_h(t)$ - is a solution

Prove: $k \cdot y_h(t)$ - is also a solution

$(k \cdot y_h(t))' = k \cdot y_h'(t) = a(t) \cdot (k \cdot y_h(t))$

General Solution to Homogeneous Equation: $y_h(t) = ke^{\int a(t)dt}$

Suppose: $\frac{dy}{dt} = a(t) \cdot y$

$\int \frac{dy}{y} = \int a(t)dt$

$\ln|y| = \int a(t)dt$

$|y| = e^{\int a(t)dt} \cdot e^C$

$y(t) = k \cdot e^{\int a(t)dt}$ where $k = \pm e^C$

but $y(t) = 0$ is a solution

$\therefore y(t) = k \cdot e^{\int a(t)dt}$, $k \in \mathbb{R}$

The extended linearity principle

(NH) $\frac{dy}{dt} = a(t)y + b(t)$

(H) $\frac{dy}{dt} = a(t)y$

The general solution to (NH) is $y = y_h + y_p$, where y_h is the general solution to (H) and y_p is a particular solution to (NH)

Given: $y_p(t)$ is particular solution to (NH)

$y_h(t)$ is general solution to (H)

Prove: $y(t) = y_p(t) + y_h(t)$ is also a solution to (NH)

$\frac{dy}{dt} = (y_p(t) + y_h(t))' = y_p'(t) + y_h'(t) = a(t)y_p(t) + b(t) + a(t)y_h(t)$

$= a(t)(y_p(t) + y_h(t)) + b(t)$

$\therefore (y_p(t) + y_h(t))$ is a solution to NH

Two solutions of NH, their difference is a solution to the corresponding H.

Algorithm for solving linear equations

1. Find the general solution $y_h(t)$ of H
2. Find one particular solution to NH by guessing for now
3. Find the general solution by adding general solution of H to particular solution of NH

$y = y_h(t) + y_p(t)$

Example 1: Solve the IVP $y' = 3y - e^{4t}$, $y(0) = 5$

(NH): $y' = 3y - e^{4t}$

(H): $y' = 3y$

$a(t) = 3$

$b(t) = -e^{4t}$

$\therefore y_h(t) = k \cdot e^{\int 3dt} = k \cdot e^{3t}$

Guess for NH: $y(t) = -e^{4t}$, $y' = -4e^{4t}$

$-4e^{4t} = -3e^{4t} - e^{4t}$ ✓

$y(t) = -e^{4t} + ke^{3t}$

$y(0) = -1 + k(1) = 5$

$k = 6$

$\therefore \boxed{y(t) = -e^{4t} + 6e^{3t}}$

Example 2: Find solution to $y' = 3y + \sin 2t$ no initial condition, just general!

(NH): $y' = 3y + \sin 2t$

(H): $y' = 3y$ $\qquad y_h = ke^{3t}$

$a(t) = 3$

$b(t) = \sin 2t$

Guess: $y_p(t)$ of $y' = 3y + \sin(2t)$

Try $\Rightarrow$ $y_p(t) = A\cos 2t + B\sin 2t$

$\therefore y'_p(t) = -2A\sin 2t + 2B\cos 2t \overset{\text{eq NH}}{=} 3(A\cos 2t + B\sin 2t) + \sin 2t$

$-2A\sin 2t + 2B\cos 2t = 3A\cos 2t + 3B\sin 2t + \sin 2t$

$\sin 2t$: $-2A = 3B + 1$

$\cos 2t$: $2B = 3A$ $\quad \therefore B = \frac{3A}{2}$

$-2A = \frac{9A}{2} + 1$

$-\frac{13A}{2} = 1$ $\qquad \therefore A = -\frac{2}{13}, B = -\frac{3}{13}$

$y_p(t) = -\frac{2}{13}\cos 2t - \frac{3}{13}\sin 2t$

general solution:

$y(t) = ke^{3t} - \frac{2}{13}\cos 2t - \frac{3}{13}\sin 2t$

Example 3: Sometimes guessing not be successful

(NH): $y' = 4y - e^{4t}$

(H): $y' = 4y$ $\qquad y = ke^{4t}$

$a(t) = 4$

$b(t) = -e^{4t}$

Guess: $y_p(t) = C \cdot e^{4t}$

$4Ce^{4t} = 4Ce^{4t} - e^{4t}$

$\cancel{-e^{4t} = 0}$ $\Leftarrow$ can never happen

If this happens, add t factor

$\therefore y_p(t) = C \cdot t \cdot e^{4t}$

$y'_p(t) = Ce^{4t} + 4Cte^{4t}$

$\Rightarrow C \cdot e^{4t} + C \cdot 4te^{4t} = 4Cte^{4t} - e^{4t}$

$C = -1$

$\therefore y_p(t) = -te^{4t}$

Section 1.9 - Integrating Factors

Using Integrating Factors to solve 1st order NHDE

① Rewrite NH DE in the form:

$$\text{(NH)}\quad \frac{dy}{dt} + g(t)y = b(t) \quad \leftarrow \text{Standard Form}$$

② Multiply Both sides by some function $\mu(t)$

$$\mu(t)\frac{dy}{dt} + \mu(t)g(t)y = \mu(t)b(t)$$

$\mu(t)$ is called an integrating factor

③ Simplify using product rule $\frac{d\mu}{dt} = \mu g$

$$\frac{d}{dt}\left(\mu(t)y(t)\right) = \frac{d\mu}{dt}y + \frac{dy}{dt}\mu$$

④ Simplify

$$\frac{d\mu}{dt}y = \mu(t)g(t)y(t)$$

⑤ Solve for μ

$$\int \frac{d\mu}{\mu} = \int g \Rightarrow \ln|\mu| = \int g(t)\,dt \Rightarrow \mu = ke^{\int g(t)dt}$$

Condensed Steps

① Make sure its in the standard form
② Compute the integrating factor $\mu(t) = e^{\int g(t)dt}$
③ Multiply both sides by $\mu(t)$
④ Integrate both sides with respect to t
⑤ Solve for $y(t)$ by dividing by $\mu(t)$

TL;DR:

Dont Forget abt constant when evaluating

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)b(t)\,dt, \text{ where } \mu(t) = e^{\int g(t)dt}$$

Example 1: $\frac{dy}{dt} = \frac{3}{t}y + t^5$ for $t > 0$

① $\frac{dy}{dt} \underbrace{- \frac{3}{t}}_{g(t)} y = t^5,\ t > 0$ Standard form

② $\mu(t) = e^{\int -\frac{3}{t}dt}$ ← Dont need a constant here

$\mu(t) = e^{-3\ln(t)}$ $t > 0$, so no $|t|$

$\mu(t) = t^{-3}$

③ $\underbrace{y' t^{-3} - 3t^{-4}y}_{} = t^2$

Conversful product rule $(y \cdot t^{-3})' = t^2$

④ $\int (y \cdot t^{-3})' = \int t^2$

$y \cdot t^{-3} = \frac{t^3}{3} + C$

⑤ $y = t^3 C + \frac{t^6}{3}$

Example 2: $\frac{dy}{dt} = (\sin t)y + 4$

① $\frac{dy}{dt} - (\sin t)y = 4$

② $\mu(t) = e^{\int(-\sin t)dt} = e^{\cos t}$

③ $\frac{dy}{dt}e^{\cos t} - e^{\cos t}\sin t\, y = 4$

$(y \cdot e^{\cos t})' = 4e^{\cos t}$

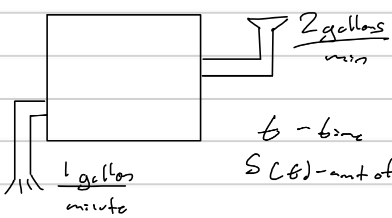
④ $\int (y \cdot e^{\cos t})'dt = \int 4e^{\cos t}dt$

$y \cdot e^{\cos t} = 4\int e^{\cos t}dt$

⑤ $y(t) = 4e^{-\cos t} \cdot \int e^{\cos t}dt$

If the integral becomes too challenging you can only leave it in the integral form.

Example 3: Salt in a tank... again



Volume = 15 + t

density = $\frac{St}{\text{Volume}} = \frac{S(t)}{15+t}$

t - time

$S(t)$ - amt of salt

$$\frac{dS}{dt} = 1 - \frac{S(t)}{15+t},\ S(0) = 6 \qquad S(15) = ?$$

① $\frac{dS}{dt} + \frac{S}{15+t} = 1$

② $\mu(t) = e^{\int \frac{1}{15+t}dt}$

$\mu(t) = e^{\ln(15+t)} = 15+t$

③ $\frac{dS}{dt}(15+t) + S = 15+t$

$(S \cdot (15+t))' = 15+t$

④ $\int (S \cdot (15+t))'dt = \int (15+t)dt$

$S(t) \cdot (15+t) = 15t + \frac{1}{2}t^2 + C$

⑤ $S(t) = \frac{15t + \frac{1}{2}t^2 + C}{15+t}$

IVP: $S(t) = \frac{15t + \frac{1}{2}t^2 + C}{15+t},\ S(0) = 6$

$6 = \frac{C}{15} \therefore C = 90 \Rightarrow S(t) = \frac{15t + \frac{1}{2}t^2 + 90}{15+t}$

Practice on Integrating Factors

1. $\frac{dy}{dt} = -\frac{y}{t} + 2$

$\frac{dy}{dt} + \frac{y}{t} = 2$

$\mu(t) = e^{\int \frac{1}{t}dt} = t$

$\frac{dy}{dt}t + y = 2t$

$\int (t \cdot y)' = \int 2t\,dt$

$t \cdot y = t^2 + C$

$y = t + C/t$

2. $\frac{dy}{dt} = \frac{3}{t}y + t^5$

$\frac{dy}{dt} - \frac{3}{t}y = t^5$

$\mu(t) = e^{\int -\frac{3}{t}dt} = t^{-3}$

$\frac{dy}{dt}t^{-3} - \frac{3}{t^4}y = t^2$

$\int (y \cdot t^{-3})' = \int t^2\,dt$

$y \cdot t^{-3} = \frac{1}{3}t^3 + C$

$y(t) = \frac{1}{3}t^6 + Ct^3$

3. $\frac{dy}{dt} = -\frac{y}{1+t} + t^2$

$\frac{dy}{dt} + \frac{y}{1+t} = t^2$

$\mu(t) = e^{\int \frac{1}{1+t}dt}$

$\mu(t) = 1+t$

$\frac{dy}{dt}(1+t) + y = t^2 + t^3$

$\int (y \cdot (1+t))' = \int t^2 + t^3\,dt$

$y \cdot (1+t) = \frac{1}{3}t^3 + \frac{1}{4}t^4 + C$

$y = \frac{t^3}{3(1+t)} + \frac{t^4}{4(1+t)} + \frac{C}{1+t}$

4. $\frac{dy}{dt} = -2ty + 4e^{-t^2}$

$\frac{dy}{dt} + 2ty = 4e^{-t^2}$

$\mu(t) = e^{\int 2t\,dt}$

$\mu(t) = e^{t^2}$

$\frac{dy}{dt}e^{t^2} + 2te^{t^2}y = 4$

$\int (y \cdot e^{t^2})' = \int 4\,dt$

$y \cdot e^{t^2} = 4t + C$

$y = 4te^{-t^2} + Ce^{-t^2}$

5. $\frac{dy}{dt} - \frac{2t}{1+t^2}y = 3$

$\mu(t) = e^{-\int \frac{2t}{1+t^2}dt}$ $u = 1+t^2$, $\frac{du}{dt} = 2t$

$\mu(t) = e^{-\int \frac{1}{u}du} = \frac{1}{1+t^2}$

$\frac{dy}{dt}\left(\frac{1}{1+t^2}\right) - \frac{2t}{(1+t^2)^2}y = \frac{3}{1+t^2}$

$\int \left(y \cdot \frac{1}{1+t^2}\right)' = \int \frac{3}{1+t^2}$

$y \cdot \frac{1}{1+t^2} = 3\arctan t + C$

$y = (1+t^2)(3\arctan t + C)$

6. $\frac{dy}{dt} - \frac{2}{t}y = t^3e^t$

$\mu(t) = e^{-\int \frac{2}{t}dt}$

$\mu(t) = t^{-2}$

$\frac{dy}{dt}(t^{-2}) - \frac{2}{t^3}y = te^t$

$\int \left(y \cdot \frac{1}{t^2}\right)' = \int te^t$

$y \cdot \frac{1}{t^2} = te^t - e^t + C$

$y = t^3e^t - t^2e^t + Ct^2$

7. $\frac{dy}{dt} = \frac{-y}{1+t} + 2$, $y(0) = 3$

$\frac{dy}{dt} + \frac{y}{1+t} = 2$

$\mu(t) = e^{\int \frac{1}{1+t}dt} = 1+t$

$\frac{dy}{dt}(1+t) + y = 2(1+t)$

$(y \cdot (1+t))' = 2(1+t)$

$\int (y \cdot (1+t))' = \int 2 + 2t\,dt$

$y(1+t) = 2t + t^2 + C$

$y = \frac{2t}{1+t} + \frac{t^2}{1+t} + \frac{C}{1+t}$

$3 = \frac{C}{1}$, $C = 3$

$y(t) = \frac{2t}{1+t} + \frac{t^2}{1+t} + \frac{3}{1+t}$

8. $\frac{dy}{dt} = \frac{1}{t+1}y + 4t^2 + 4t$, $y(1) = 10$

$\frac{dy}{dt} - \frac{y}{t+1} = 4t^2 + 4t$

$\mu(t) = e^{\int -\frac{1}{t+1}dt} = \frac{1}{t+1}$

$\frac{dy}{dt}\left(\frac{1}{t+1}\right) - \frac{y}{(t+1)^2} = \frac{4t(t+1)}{t+1}$

$\int \left(y \cdot \left(\frac{1}{t+1}\right)\right)' = \int 4t$

$y \cdot \frac{1}{t+1} = 2t^2 + C$

$y = (t+1)(2t^2 + C)$

$10 = 2(2 + C)$

$10 = 2C + 4$, $C = 3$

$y(t) = (t+1)(2t^2 + 3)$

13. $\frac{dy}{dt} = (\sin t)y + 4$

$\frac{dy}{dt} - (\sin t)y = 4$

$\mu(t) = e^{\int -\sin t\,dt} = e^{\cos t}$

$\frac{dy}{dt}e^{\cos t} - \sin t\, e^{\cos t}y = 4e^{\cos t}$

$\int (y \cdot e^{\cos t})' = \int 4e^{\cos t}$

$y \cdot e^{\cos t} = 4\int e^{\cos t}$

$y(t) = 4e^{-\cos t}\int e^{\cos t}dt$

16. $\frac{dy}{dt} = y + 4\cos t^2$

$\frac{dy}{dt} - y = 4\cos t^2$

$\mu(t) = e^{\int -1\,dt} = e^{-t}$

$\frac{dy}{dt}e^{-t} - ye^{-t} = 4e^{-t}\cos t^2$

$\int (y \cdot e^{-t})' = \int 4e^{-t}\cos t^2$

$y \cdot e^{-t} = 4\int e^{-t}\cos t^2$

$y = 4e^{t}\int e^{-t}\cos t^2\,dt$

17. $\frac{dy}{dt} = -\frac{y}{e^{t^2}} + \cos t$

$\frac{dy}{dt} + \frac{y}{e^{t^2}} = \cos t$

$\mu(t) = e^{\int e^{t^2}dt}$!!

$\frac{dy}{dt}e^{\int e^{t^2}dt} + \frac{y}{e^{t^2}}e^{\int e^{t^2}dt} = e^{\int e^{t^2}dt}\cos t$

$\int \left(y \cdot e^{\int e^{t^2}dt}\right)' = \int e^{\int e^{t^2}dt}\cos t$

$y \cdot e^{\int e^{t^2}dt} = \int e^{\int e^{t^2}dt}\cos t\,dt$

$y(t) = e^{-\int e^{t^2}dt}\int e^{\int e^{t^2}dt}\cos t\,dt$

19. $\frac{dy}{dt} = aty + 4e^{-t^2}$

$\frac{dy}{dt} - aty = 4e^{-t^2}$

$\mu(t) = e^{\int -at\,dt} = e^{-\frac{1}{2}at^2}$

$\frac{dy}{dt}e^{-\frac{a}{2}t^2} - ate^{-\frac{a}{2}t^2}y = 4e^{-t^2}e^{-\frac{a}{2}t^2}$

$e^{-t^2 - \frac{a}{2}t^2} \overset{?}{=} 1 \rightarrow -t^2 - \frac{a}{2}t^2 = 0$ $a = -2$

20. $\frac{dy}{dt} = t^r y + 4$

$\frac{dy}{dt} - t^r y = 4$

$\mu(t) = e^{-\int t^r dt} = e^{-\frac{t^{r+1}}{r+1}}$

$\frac{dy}{dt}e^{-\frac{t^{r+1}}{r+1}} - t^r y e^{-\frac{t^{r+1}}{r+1}} = 4e^{-\frac{t^{r+1}}{r+1}}$

$\left(y \cdot e^{-\frac{t^{r+1}}{r+1}}\right)' = 4\int e^{-\frac{t^{r+1}}{r+1}}$

$-\frac{t^{r+1}}{r+1} = 0$ $t^{r+1} = 0$

$r = -1$!! $r = 0$ works

but integral

$r = -1$: $\mu = e^{-\int \frac{1}{t}dt} = \frac{1}{t}$

$\frac{dy}{dt}\left(\frac{1}{t}\right) - \frac{y}{t^2} = 4\left(\frac{1}{t}\right)$

Still need to integrate !!

Section 2.1 - Modeling Via Systems

Recall: Predator - Prey Model ← common model

Parameters: α = growth rate of rabbits
γ = death rate of foxes
β = Number of rabbit deaths, roughly
δ = benefit to fox population from rabbit

$$\begin{cases} \frac{dR}{dt} = \alpha R - \beta RF \\ \frac{dF}{dt} = -\gamma F + \delta RF \end{cases}$$

This is a first order system of DEs
A solution to this is a pair of functions that satisfy both equations!
Also given a pair of IVs as well

Equilibrium solutions (R,F) are solutions if they make $\frac{dR}{dt}$ and $\frac{dF}{dt} = 0$

Example 1: parameters given for predator-prey model

Given: $\alpha = 2$, $\beta = 1.2$, $\gamma = 1$, $\delta = 0.9$

$$\frac{dR}{dt} = 2R - 1.2RF \qquad \frac{dF}{dt} = -F + 0.9RF$$

Equilibrium Solutions: have to get one of two

$$\begin{cases} 2R - 1.2RF = 0 \\ -F + 0.9RF = 0 \end{cases} \Rightarrow \begin{cases} R(2 - 1.2F) = 0 \\ F(0.9R - 1) = 0 \end{cases} \Rightarrow R=0,\ F=0,\quad R = \frac{1}{0.9},\ F = \frac{2}{1.2}$$

Plotting rabbits and foxes shows a graph independent of time, called a phase portrait

Two equilibrium solutions found: $(R,F) = (0,0)$ $(1.11, 1.67)$

The RF-plane is called the phase plane, analogous to a phase line

Phase Portraits: A graph in the RF-plane of solutions to multiple initial value problems for a system

The models above aren't fully accurate, don't account for carrying capacity. We look to logistic!

Logistic Modification

$$\frac{dR}{dt} = \alpha R\left(1 - \frac{R}{N}\right) - \beta RF \quad \text{and} \quad \frac{dF}{dt} = -\gamma F + \delta RF$$

Example 2: Examining Logistic Model

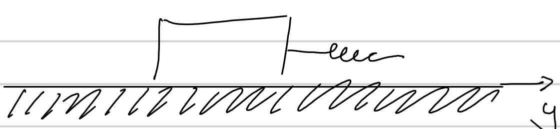
$(0,0)$ is called a trivial solution for these models

$$\begin{cases} 2R\left(1 - \frac{R}{2}\right) - 1.2FR = 0 \\ -F - 0.9RF = 0 \end{cases}$$

← Could solve by factoring or numerical methods, but it is omitted

A phase portrait of these curves shows they all approach equilibrium

Undamped Mass-Spring System



N2L
$$F_{total} = ma = m\frac{d^2y}{dt^2}$$

Hooke's Law
$$F_{spring} = -ky$$

$$m\frac{d^2y}{dt^2} = -ky$$
$$\Downarrow$$
$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

Second Order DE

IV of $y(0)=0$ for this problem

Logically this should be a sinusoid solution

$\frac{d^2y}{dt^2} - \frac{k}{m}y = 0$ is called the simple undamped harmonic oscillator

We can turn second order / higher order DEs to lower ones with other variables

$v = \frac{dy}{dt}$ and $a = \frac{dv}{dt}$, giving us $\frac{dv}{dt} = -\frac{k}{m}y$ where $\frac{dy}{dt} = v$

add this because we used the relationship

This gives us a first order system which is better for analysis

Example: Guessing

Consider $\begin{cases} \frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \\ y(t) = \cos(\beta t) \end{cases}$. what β is $y(t) = \cos(\beta t)$ a solution

plug in $-\beta^2\cos(\beta t) + \frac{k}{m}\cos(\beta t) = 0 \quad \therefore \beta = \pm\sqrt{\frac{k}{m}}$

Cosine is even so ± doesn't matter

$$y(t) = \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

Section 2.2 - Geometry of Systems

Consider Predator-Prey Model Example:

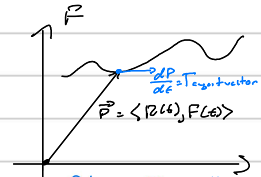
$$\frac{dR}{dt} = 2R - 1.2RF \qquad \frac{dF}{dt} = -F + 0.9RF$$

Solution vector $\vec{P}(t) = \begin{pmatrix} R(t) \\ F(t) \end{pmatrix}$

vector valued function

Then, Calc 3 vector derivative rules

$$\frac{d\vec{P}}{dt} = \begin{pmatrix} \frac{dR}{dt} \\ \frac{dF}{dt} \end{pmatrix} = \begin{pmatrix} 2R - 1.2RF \\ -F + 0.9RF \end{pmatrix}$$



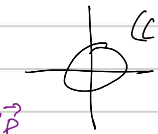
Slope Fields in R

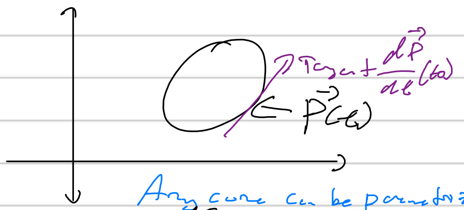
Remember that any curve can be parametrized

Instead of a slope field we got a vector field

Recall Circle Parametrization

Circle $(\cos t, \sin t)$, $0 \le t \le 2\pi$





Any curve can be parametrized by vector functions

When all vectors on the vector plot of a system are the same length, this is called a direction field

Example: $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x + x^3 = 0$

This technique can be applied to any order!

a) convert to a system

$\frac{dx}{dt} = y$, $\frac{d^2x}{dt^2} = \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = -2y + 3x - x^3$

System, 2 functions $\rightarrow$ $\begin{cases} \frac{dx}{dt} = y(t) \\ \frac{dy}{dt} = -2y + 3x - x^3 \end{cases}$

b) $\vec{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $\frac{d\vec{y}}{dt} = \begin{pmatrix} y(t) \\ -2y + 3x - x^3 \end{pmatrix} \longrightarrow$ vector field

Equilibrium when $\frac{d\vec{y}}{dt} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{cases} y = 0 \\ -2y + 3x - x^3 = 0 \end{cases} \Rightarrow y = 0$

$3x - x^3 = 0$

$x(3 - x^2) = 0$

$x = 0$, $x = +\sqrt{3}$ and $x = -\sqrt{3}$

$(0, 0) \leftarrow$ Trivial

$(\sqrt{3}, 0)$

$(-\sqrt{3}, 0)$

Section 2.3 - Damped Harmonic Oscillators

In this model, assume the strength of the damping force is proportional to velocity

The damping force is: $-b\frac{dy}{dt}$ where $b>0$ is the damping constant

$y(t)$ - position $\quad \frac{dy}{dt}$ - velocity $\quad \frac{d^2y}{dt^2}$ - acceleration

N2L yields: $m\frac{d^2y}{dt^2} = \underbrace{-b\frac{dy}{dt}}_{\text{Damping force}}\underbrace{-Ky}_{\text{Spring}} \Leftrightarrow m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + Ky = 0 \Leftrightarrow \boxed{my'' + by' + Ky = 0}$

$F = ma$

Since mass $\neq 0$: $\frac{d^2y}{dt^2} + \frac{b}{m}\left(\frac{dy}{dt}\right) + \frac{K}{m}y = 0$

for simplicity we say $\frac{b}{m} = p$ and $\frac{K}{m} = q$ $\Rightarrow y'' + py' + qy = 0$

Transform $y'' + py' + qy = 0$ to a System...

$\frac{dy}{dt} = v \qquad \frac{d^2y}{dt^2} = \frac{dv}{dt}$

$$\begin{cases} \frac{dv}{dt} = -pv - qy \\ \frac{dy}{dt} = v \end{cases}$$

We can observe Behaviors of solutions for this by making use of "VectorPlot" in Mathematica

$\therefore \vec{y} = \begin{pmatrix} v \\ y \end{pmatrix} = \begin{pmatrix} -pv - qy \\ v \end{pmatrix}$ (idea from Section 2.2)

Example: Guess-and-test method

Let $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$

Guess some exp function ← It should be either a trig function or exponential

Let $y(t) = \alpha e^{st}$

$y' = \alpha s e^{st}$ and $y'' = \alpha s^2 e^{st}$

$\therefore \cancel{\alpha} s^2 e^{st} + 5\cancel{\alpha} s e^{st} + 6\cancel{\alpha} e^{st} = 0$

$e^{st}(s^2 + 5s + 6) = 0$ $\quad e^{st} \neq 0$ so division works

This is called a characteristic equation $\Rightarrow s^2 + 5s + 6 = 0$

$s = \frac{-5 \pm \sqrt{25-24}}{2}$, $s = -3$, $s = -2$

$\rightarrow y_1(t) = e^{-2t}$, $y_2(t) = e^{-3t}$

$y_1 + y_2$ is also a solution

$(y_1 + y_2)'' + 5(y_1 + y_2)' + 6(y_1 + y_2) = 0$

$\therefore$ General Solution: $\boxed{y(t) = \alpha e^{-2t} + \beta e^{-3t}}$

for $\alpha, \beta \in \mathbb{R}$

Consider $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \alpha e^{-2t} + \beta e^{-3t} = 0$ ← Motion, or oscillation, will stop over time

Section 2.4 - Decoupled Systems

A system of DEs is decoupled if the rate of change of at least one of the dependent variables depends on its own value.

Example: A completely decoupled system

$$\begin{cases} \frac{dx}{dt} = 3x \\ \frac{dy}{dt} = -2y \end{cases}$$

Separable!

1. $\frac{dx}{x} = 3dt$, $x(t) = K_1 e^{3t}$

2. $\frac{dy}{y} = 2dt$, $y(t) = K_2 e^{-2t}$

Vector Notation

If $\vec{y} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ then, $\frac{d\vec{y}}{dt} = \begin{pmatrix} 3x \\ -2y \end{pmatrix}$

Plotting this reveals $(0,0)$ is a source

Example: Partially Decoupled System

$$\begin{cases} \frac{dx}{dt} = -5x \\ \frac{dy}{dt} = 2x - 3y \end{cases}$$

1. $x(t) = K_1 e^{-5t}$

$\therefore \frac{dy}{dt} = 2K_1 e^{-5t} - 3y$ linear eq!

$\frac{dy}{dt} + 3y = 2K_1 e^{-5t}$

Can either solve by doing NH/H or by integrating factors

Solve linear: $y' + 3y = 2K_1 e^{-5t}$ (NH)

$y' + 3y = 0$ (H)

$y_h(t) = K_2 e^{-3t}$

Guess: $y_p(t) = Ce^{-5t}$, $y_p'(t) = -5Ce^{-5t}$

$\therefore -5Ce^{-5t} + 3Ce^{-5t} = 2K_1 e^{-5t}$

$-2C = 2K_1 \Rightarrow C = -K_1$

general solution $y(t) = y_h(t) + y_p(t)$

$y(t) = K_2 e^{-3t} - K_1 e^{-5t}$

$$\begin{cases} x(t) = K_1 e^{-5t} \\ y(t) = K_2 e^{-3t} - K_1 e^{-5t} \end{cases}$$

Section 2.5 - Euler's Method for Systems

Consider the system

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases}$$

A solution curve $\vec{Y}(t) = (x(t), y(t))$ whose tangent vector agrees w/ the vector field:

$\vec{F}(\vec{Y}) = (f(x,y), g(x,y))$

A vector of derivatives, "tangent vector"

with Initial Condition (x_0, y_0)

Algorithm for Euler's Method

1. Select a step size Δt, starting at (x_0, y_0). Use $\Delta t \vec{F}(x_0, y_0)$ to make the first step

   $(x_1, y_1) = (x_0, y_0) + \Delta t \vec{F}(x_0, y_0)$

2. Compute $\vec{F}(x_1, y_1)$ and do the following

   $(x_2, y_2) = (x_1, y_1) + \Delta t \vec{F}(x_1, y_1)$

3. Repeat until you don't need to anymore

Very similar to Euler's method from Chapter 1, just with coordinates instead.

$\vec{F}(x_0, y_0)$ is equivalent to:

$\left(\frac{dx}{dt}(x_0, y_0), \frac{dy}{dt}(x_0, y_0)\right)$

↑ Derivatives evaluated at point

Section 2.6 - Existence and Uniqueness for Systems

Non autonomous Systems

$$\frac{d\vec{P}}{dt} = \vec{F}(t, \vec{P}) = \begin{pmatrix} f(t, x, y) \\ g(t, x, y) \end{pmatrix}$$

A vector field $\vec{F}(t, \vec{P})$ is **continuously differentiable** if

$$\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \text{ are cts.}$$

Connection to singular DE

If a vector field $\vec{F}$ is cts and differentiable, then a solution exists around an initial value and it is unique on the interval.

Geometrically: Two solutions cannot start at the same place at the same time

Same can be said for autonomous systems, but we don't need to check $\frac{\delta f}{\delta t}$ or $\frac{\delta g}{\delta t}$

It is wise to check these things before applying numerical methods

Example 1: Show that $x(t) \to \infty$ when $t \to \infty$

$$\begin{cases} \frac{dx}{dt} = x^2 + y = f \\ \frac{dy}{dt} = xy = g \end{cases} \qquad \vec{F} = \begin{pmatrix} x^2+y \\ xy \end{pmatrix} \text{ is cts. differentiable}$$

$(x_0, y_0) = (0, 1)$ $\quad \frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 1 \,\Big|\, \frac{\partial g}{\partial x} = y, \frac{\partial g}{\partial y} = x$

$\frac{dx}{dt} \geq 0$ $\quad \forall t > 0$ $\quad$ x and y are increasing functions

$\frac{dy}{dt} \geq 0$

You can deduce this from the initial condition $(0,1)$

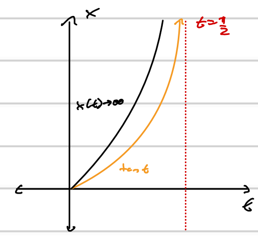
Consider $\frac{dx}{dt} = x^2 + 1$, $x(0) = 0$

$$\int \frac{dx}{x^2+1} = \int dt$$

$\arctan(x) = t$

$x(t) = \tan(t)$

$\lim_{t \to \frac{\pi}{2}^-} \tan(t) = \infty$



Example 2

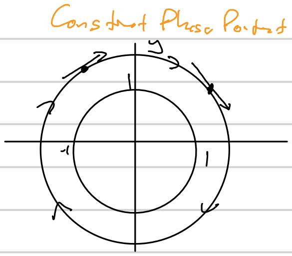
$$\begin{cases} \frac{dx}{dt} = y(x^2+y^2-1) \\ \frac{dy}{dt} = -x(x^2+y^2-1) \end{cases}$$

Equilibriums: $(0,0)$

$$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = -\frac{x}{y}$$

Solve $\begin{cases} y\,dy = -x\,dx \\ x^2 + y^2 = C \end{cases}$

Construct Phase Portrait



Curves are all circles with radius C centered @ $(0,0)$

Vector @ $(1,1)$ $\langle 1, -1 \rangle$

Vector @ $(-1,1)$ $\langle 1, 1 \rangle$

Direction determined by vectors at points!

Test 1 - Friday 3/1

- Covers chapter 1 and 2
- Practice exams on Canvas
  - ↳ Solutions provided down the line
- Extra office hours
-