Ordered Set: A set S with relation $<$ s.t.

(i) (Trichotomy) $\forall x,y \in S$, exactly one of $x<y$, $x=y$, $y<x$ holds ← other inequalities exist too

(ii) (Transitivity) If $x,y,z \in S$ and $x<y$ & $y<z$, then $x<z$

⇒ An ordered set has the least upper bound property if every non-empty subset $E \subset S$ that is bounded above has a least upper bound (supremum). This is also called completeness.

Field: A set F is a field if it has $+$ & $\cdot$ operations and:

(A1) If $x,y \in F$, then $x+y \in F$

(A2) $x+y=y+x \; \forall x,y \in F$

(A3) $(x+y)+z=x+(y+z) \; \forall x,y,z \in F$

(A4) $\exists$ element $0 \in F$ s.t. $x+0=0+x=x \; \forall x \in F$

(A5) $\forall x \in F$, $\exists -x \in F$ s.t. $x+(-x)=-x+x=0$

(M1) If $x,y \in F$, then $x \cdot y \in F$

(M2) $x \cdot y = y \cdot x \; \forall x \in F$

(M3) $(y \cdot x) \cdot z = y \cdot (x \cdot z) \; \forall x,y,z \in F$

(M4) $\exists$ element $1 \in F$ s.t. $x \cdot 1 = 1 \cdot x = x \; \forall x \in F$

(M5) $\forall x \in F, x \neq 0 \; \exists \frac{1}{x} \in F$ s.t. $\frac{1}{x} \cdot x = x \cdot \frac{1}{x} = 1$

distribution (D) $x(y+z) = xy + xz \; \forall x,y,z \in F$

Ordered Field: A field F which is an ordered set where

(i) $\forall x,y,z \in F$, $x<y \to x+z<y+z$

(ii) $\forall x,y \in F$, $x>0$ & $y>0$ both imply $xy>0$

Archimedean Property

(i) If $x,y \in \mathbb{R}$ and $x>0$, $\exists$ an $n \in \mathbb{N}$ s.t. $nx>y$ — You can find a natural number greater than any real number

(ii) If $x,y \in \mathbb{R}$ and $x<y$, $\exists\, r \in \mathbb{Q}$ s.t. $x<r<y$ — $\mathbb{Q}$ is dense in $\mathbb{R}$

Def: Let $A \subseteq \mathbb{R}$ — $\mathbb{R}$ is an uncountable set!

(i) if $A = \emptyset = \{\}$, $\sup A := -\infty$

(ii) If A is not bounded above and non-empty, $\sup A := +\infty$

(iii) if $A = \emptyset$, $\inf A := +\infty$

(iv) if A is not bounded below and non-empty, $\inf A = -\infty$

Corollary: For every pair $x,y \in \mathbb{R}$ it holds that

(i) (Triangle Inequality) $|x-y| \le |x| + |y|$

(ii) (Reverse Triangle Inequality) $\big||x|-|y|\big| \le |x-y|$

Prop: (i) $|xy| = |x| \cdot |y| \; \forall x,y \in \mathbb{R}$

(ii) $|x|^2 = x^2 \; \forall x \in \mathbb{R}$

(iii) $|x| \ge 0$ and equal if $x=0$

(iv) $-|x| \le x \le |x| \; \forall x \in \mathbb{R}$

(v) $|x| \le y$ iff $-y \le x \le y$

Bounded Function: $f: D \to \mathbb{R}$ is bounded if for $M>0$, $|f(x)| \le M$ for all $x \in D$. It holds that $\sup_{x \in D} f(x) = \sup f(D)$ and $\inf_{x \in D} f(x) = \inf f(D)$

Prop: A set is an interval iff it contains at least two points and $\forall a,b \in I$ ($I \subset \mathbb{R}$) and $c \in \mathbb{R}$ such that $a<c<b$, we have that $c \in I$.

Convergence: A sequence $\{x_n\}_1^\infty$ is said to converge to some $x \in \mathbb{R}$ if for every $\varepsilon > 0$, $\exists$ an $M \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n \ge M$. The number x is called a limit of $\{x_n\}_1^\infty$ and is denoted $x := \lim_{n\to\infty} x_n$.

Monotonicity: A sequence $\{x_n\}_1^\infty$ is said to be monotone increasing if $\forall n \in \mathbb{N}$, $x_n \le x_{n+1}$. A sequence is said to be monotone decreasing if $\forall n \in \mathbb{N}$, $x_n \ge x_{n+1}$. If either one is true, the sequence is said to be monotone.

Monotone Convergence Thm (MCT): A monotone sequence $\{x_n\}_1^\infty$ is convergent iff it is bounded.

(i) If $\{x_n\}$ is monotone incr. and bounded: $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$

(ii) if $\{x_n\}$ is monotone decr. and bounded: $\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$

K-tail: For a sequence $\{x_n\}_{n=1}^\infty$, the K-tail, $K \in \mathbb{N}$, or just tail of the sequence is starting at $K+1$ usually written as $\{x_{n+K}\}_{n=1}^\infty$ or $\{x_n\}_{n=K+1}^\infty$

Subsequence: Let $\{x_n\}_{n=1}^\infty$ be a sequence and let $\{n_i\}_{i=1}^\infty$ be a strictly increasing seq. of $\mathbb{N}$ i.e. $n_i \le n_{i+1} \; \forall i \in \mathbb{N}$. The sequence $\{x_{n_i}\}_{i=1}^\infty$ is called a subsequence of $\{x_n\}_{n=1}^\infty$

Prop: If $\{x_n\}_{n=1}^\infty$ is a convergent sequence, all of its subsequences must converge too. Must be to the same limit!

Let $E \subset S$, S is an ordered set

(i) If $\exists\, b \in S$ s.t. $x \le b \; \forall x \in E$, E is bounded above and b is an upper bound for E

(ii) If $\exists\, b \in S$ s.t. $b \le x \; \forall x \in E$, E is bounded below and b is a lower bound for E

(iii) if $\exists\, b_0 \in S$, an upper bound s.t. $b_0 \le b \; \forall$ upper bounds, b_0 is the least upper bound; $b_0 := \sup E$

(iv) If $\exists\, b_0 \in S$, a lower bound s.t. $b_0 \ge b \; \forall$ lower bounds, b_0 is the greatest lower bound; $b_0 := \inf E$

(v) A set is bounded if it is bounded both above and below.

Prop: Let F be an ordered field

(i) Ordering is preserved w/ positives

(ii) negation flips ordering

(iii) If $x \neq 0$, $x^2 > 0$

(iv) If $0<x<y$, then $0<\frac{1}{y}<\frac{1}{x}$

(v) if $0<x<y$, then $x^2<y^2$

(vi) if $x \le y$ & $z \le w$, then $x+z \le y+w$

Prop: Algebra may be done on sup/infs

Irrational Numbers: The set $\mathbb{R}/\mathbb{Q}$

Prop: $\sup S - \varepsilon < x \le \sup S$ if S bounded from above, $\forall \varepsilon > 0 \; \exists x \in S$

Prop: $x \in A$, $y \in A$ and $x \le y$ and bounded, then $\sup A \le \inf B$ for $A, B \subseteq \mathbb{R}$

Extended Reals: $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$ is an ordered set; $-\infty < \infty$, $-\infty < x < \infty$

min/max: $\max A = \sup A$ iff $\sup A \in A$ and $\min A = \inf A$ iff $\inf A \in A$. Any nonempty finite subset of $\mathbb{R}$ has a min and max.

Prop: Given $f,g: D \to \mathbb{R}$ with $f(x) \le g(x) \; \forall x \in D$, it holds that $\sup_{x\in D} f(x) \le \sup_{x \in D} g(x)$ & $\inf_{x\in D} f(x) \le \inf_{x \in D} g(x)$

x on LHS $\neq$ x on RHS sometimes!

Prop: A convergent sequence has a unique limit and is bounded. Bounded sequences are not guaranteed to converge!

Prop: Let $S \subset \mathbb{R}$ be non-empty & bounded. Then $\exists \{x_n\}, \{y_n\}$, $x_n, y_n \in S$ $\forall n \in \mathbb{N}$ such that $\sup S = \lim_{n\to\infty} x_n$ & $\inf S = \lim_{n\to\infty} y_n$

The following are equivalent

Prop: Let $\{x_n\}_1^\infty$ be a sequence.

(i) $\{x_n\}_1^\infty$ converges

(ii) $\{x_{n+K}\}_{n=1}^\infty$ converges for every $K \in \mathbb{N}$

(iii) $\{x_{n+K}\}_{n=1}^\infty$ converges for some $K \in \mathbb{N}$

If any (and hence all) of the limits exist, then $\forall K \in \mathbb{N}$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+K}$