(2.3.7) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences.

(a) Show that $\{x_n + y_n\}_{n=1}^{\infty}$ is bounded.

Proof. Suppose $M_1, M_2 \in \mathbb{R}$. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, the there exists some $M_1 > 0$ such that for all $n \in \mathbb{N}, |x_n| \leq M_1$. Similarly since $\{y_n\}_{n=1}^{\infty}$ is bounded, there exists some $M_2 > 0$ such that for all $n \in \mathbb{N}, |y_n| \leq M_2$.

Now consider the sequence $\{x_n + y_n\}_{n=1}^{\infty}$. By the triangle inequality, $|x_n + y_n| \leq |x_n| + |y_n|$ holds. We can then say that $|x_n + y_n| \leq |x_n| + |y_n| \leq M_1 + M_2$. Defining $M = M_1 + M_2$, this is equivalent to $|x_n + y_n| \leq M$. As M_1 and M_2 are arbitrary positive real numbers, M is also a positive real number. Thus, by definition the sequence $\{x_n + y_n\}_{n=1}^{\infty}$ is bounded, as required. $\square$

(b) Show that

$$\left(\liminf_{n\to\infty} x_n\right) + \left(\liminf_{n\to\infty} y_n\right) \leq \liminf_{n\to\infty} (x_n + y_n)$$

Proof. Let $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, then $\{x_{n_k}\}_{k=1}^{\infty}$ must also be bounded. We can then choose a convergent subsequence $\left\{x_{n_{k_i}}\right\}_{i=1}^{\infty}$ of $\{x_{n_k}\}_{k=1}^{\infty}$, which is guaranteed to exist by the Bolzano-Weierstrass Theorem. Therefore $\lim_{i\to\infty}\left(x_{n_{k_i}}\right)$ exists and $\liminf_{n\to\infty} x_n \leq \lim_{i\to\infty}\left(x_{n_{k_i}}\right)$ by the definition of the limit inferior.

By similar reasoning we can let $\{y_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{y_n\}_{n=1}^{\infty}$ and can chose a subsequence $\left\{y_{n_{k_i}}\right\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty}\left(y_{n_{k_i}}\right)$ exists. Hence $\liminf_{n\to\infty} y_n \leq \lim_{i\to\infty}\left(y_{n_{k_i}}\right)$ by the definition of the limit inferior.

As you'd expect, we can also choose a subsequence $\left\{x_{n_{k_i}} + y_{n_{k_i}}\right\}_{i=1}^{\infty}$ of $\{x_{n_k} + y_{n_k}\}_{k=1}^{\infty}$ which is a subsequence of $\{x_n + y_n\}_{n=1}^{\infty}$. By the Bolzano-Weierstrass Theorem, we can choose this subsequence such that it converges and thus $\lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right) = \lim_{k\to\infty}(x_{n_k}) + \lim_{k\to\infty}(x_{n_k})$ and that $\liminf_{n\to\infty}(x_n + y_n) \geq \lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right)$ by definition of the limit inferior. Using the previously derived expressions, we have

$$\begin{aligned}
\lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right) &= \lim_{i\to\infty}\left(x_{n_{k_i}}\right) + \lim_{i\to\infty}\left(y_{n_{k_i}}\right) \\
\lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right) &\geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \\
\liminf_{n\to\infty}(x_n + y_n) \geq \lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right) &\geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n
\end{aligned}$$

Therefore $(\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n) \leq \liminf_{n\to\infty}(x_n + y_n)$ as required. $\square$

(c) Find an explicit $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that

$$\left(\liminf_{n\to\infty} x_n\right) + \left(\liminf_{n\to\infty} y_n\right) < \liminf_{n\to\infty}(x_n + y_n)$$

Proof. Let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$ for all $n \in \mathbb{N}$. Clearly both of these sequences are bounded above by 1 and below by -1.

$$x_n + y_n = (-1)^n + (-1)^{n+1} = (-1)^n + (-1)^n(-1) = (-1)^n(1-1) = 0$$

Thus $\liminf_{n\to\infty}(x_n + y_n) = 0$. The minimum value for x_n, y_n is -1 for all $n \in \mathbb{N}$, so $\liminf_{n\to\infty} x_n = -1$ and $\liminf_{n\to\infty} y_n = -1$. Therefore $\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n = -1 + -1 = -2$. Consequently,

$$\left(\liminf_{n\to\infty} x_n\right) + \left(\liminf_{n\to\infty} y_n\right) < \liminf_{n\to\infty}(x_n + y_n)$$
$$-2 < 0$$

showing that the strict inequality (no equality) holds for these choices of x_n, y_n. □

(2.3.8) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences, and $\{x_n + y_n\}_{n=1}^{\infty}$ is bounded by the previous exercise.

(a) Show that

$$\left(\limsup_{n\to\infty} x_n\right) + \left(\limsup_{n\to\infty} y_n\right) \geq \limsup_{n\to\infty} (x_n + y_n)$$

Proof. Choose convergent subsequences $\left\{x_{n_{k_i}}\right\}_{i=1}^{\infty}, \left\{x_{n_{k_i}}\right\}_{i=1}^{\infty}$ of $\{x_{n_k}\}_{n=1}^{\infty}$ and $\{y_{n_k}\}_{n=1}^{\infty}$, respectively as done in the previous exercise. Therefore the limits and limit superiors can be related as $\limsup_{n\to\infty} x_n \geq \lim_{i\to\infty} x_{n_{k_i}}$ and $\limsup_{n\to\infty} y_n \geq \lim_{i\to\infty} y_{n_{k_i}}$ by the definition of the limit superior. Note that these limits are guaranteed to exist as the subsequences were chosen to converge and $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are given to be bounded.

Similarly, we can also choose a subsequence $\left\{x_{n_{k_i}} + y_{n_{k_i}}\right\}_{i=1}^{\infty}$ of $\{x_{n_k} + y_{n_k}\}_{k=1}^{\infty}$. By the Bolzano-Weierstrass Theorem, we can chose this subsequence to converge. In other words, $\lim_{i\to\infty} \left(x_{n_{k_i}} + y_{n_{k_i}}\right) = \lim_{k\to\infty} (x_{n_k}) + \lim_{k\to\infty} (x_{n_k})$. Also similarly to the above proof, $\limsup_{n\to\infty} (x_n + y_n) \leq \lim_{i\to\infty} \left(x_{n_{k_i}} + y_{n_{k_i}}\right)$ by definition of the limit superior. Consequently we have that,

$$\begin{aligned}
\lim_{i\to\infty} \left(x_{n_{k_i}} + y_{n_{k_i}}\right) &= \lim_{i\to\infty} \left(x_{n_{k_i}}\right) + \lim_{i\to\infty} \left(y_{n_{k_i}}\right) \\
\lim_{i\to\infty} \left(x_{n_{k_i}} + y_{n_{k_i}}\right) &\leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \\
\limsup_{n\to\infty} (x_n + y_n) \leq \lim_{i\to\infty} \left(x_{n_{k_i}} + y_{n_{k_i}}\right) &\leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n
\end{aligned}$$

Therefore, $(\limsup_{n\to\infty} x_n) + (\limsup_{n\to\infty} y_n) \geq \limsup_{n\to\infty} (x_n + y_n)$ as required. □

(b) Find an explicit $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that

$$\left(\limsup_{n\to\infty} x_n\right) + \left(\limsup_{n\to\infty} y_n\right) > \limsup_{n\to\infty} (x_n + y_n)$$

Proof. Let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$ for all $n \in \mathbb{N}$. Clearly both of these sequences are bounded above by 1 and below by -1.

$$x_n + y_n = (-1)^n + (-1)^{n+1} = (-1)^n + (-1)^n(-1) = (-1)^n(1-1) = 0$$

Thus $\limsup_{n\to\infty} (x_n + y_n) = 0$. The maximum value for x_n, y_n is 1 for all $n \in \mathbb{N}$, so $\limsup_{n\to\infty} x_n = 1$ and $\limsup_{n\to\infty} y_n = 1$. Therefore $\limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n = 1 + 1 = 2$. Consequently,

$$\begin{gathered}
\left(\limsup_{n\to\infty} x_n\right) + \left(\limsup_{n\to\infty} y_n\right) > \limsup_{n\to\infty} (x_n + y_n) \\
2 > 0
\end{gathered}$$

showing that the strict inequality (no equality) holds for these choices of x_n, y_n. □

(2.4.1) Prove that $\left\{\frac{n^2-1}{n^2}\right\}_{n=1}^{\infty}$ is Cauchy using directly the definition of Cauchy sequences.

Proof. We need to show that there exists some $M \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ whenever $n, m \geq M$ for any arbitrary $\epsilon > 0$. We have $x_n = \frac{n^2-1}{n^2} = 1 - \frac{1}{n^2}$. Then we also have,

$$\begin{aligned}
|x_n - x_m| &= \left|\frac{n^2-1}{n^2} - \frac{m^2-1}{m^2}\right| \\
&= \left|\left(1 - \frac{1}{n^2}\right) - \left(1 - \frac{1}{m^2}\right)\right| \\
&= \left|-\frac{1}{n^2} + \frac{1}{m^2}\right| \\
&= \left|\frac{1}{m^2} - \frac{1}{n^2}\right| \\
&= \left|\frac{n^2 - m^2}{m^2n^2}\right| && \text{Common Denominator} \\
&= \left|\frac{(n-m)(n+m)}{m^2n^2}\right| && \text{Difference of Squares} \\
&= \frac{|n-m|\,(n+m)}{m^2n^2}
\end{aligned}$$

Note that the last statement is true as $n, m > 0$, and hence their sum and product of their squares are both positive by properties of ordered fields.

Firstly, we chose n, m such that they are close together. Now note that $n, m \geq M$, so then n, m are both at least M, and we can say that $n + m \leq M + M = 2M$. Now with the previous restriction of n, m, we can bound $|n - m|$ by some $K \in \mathbb{R}$ such that $|n - m| \leq K$ for all choices of n, m where n, m are sufficiently close. We can combine these ideas to write

$$|x_n - x_m| = \frac{|n-m|\,(n+m)}{m^2n^2} \leq \frac{K \cdot 2M}{m^2n^2}$$

Now note that for any $n, m \geq M$, we have that $m^2 \geq M^2$ and $n^2 \geq N^2$ by properties previously discussed in class. Therefore $m^2n^2 \geq m^2 \cdot n^2 \geq M^4$, and hence

$$|x_n - x_m| \leq \frac{K \cdot 2M}{m^2n^2} \leq \frac{K \cdot 2M}{M^4} = \frac{2K}{M^3}$$

as $\frac{1}{m^2n^2} \leq \frac{1}{M^4}$ follows from $m^2n^2 \geq M^2$.

Therefore we need to find an M such that $\frac{2K}{M^3} < \epsilon$ is true for any $\epsilon > 0$ to fulfill the criteria for a Cauchy sequence. We can perform some algebra to find that $M > \left(\frac{2K}{\epsilon}\right)^{\frac{1}{3}}$. Thus for any arbitrary $\epsilon > 0$ and $n, m \geq M$, we can choose $M >> \left(\frac{2K}{\epsilon}\right)^{\frac{1}{3}}$ such that $|x_n - x_m| < \epsilon$. Hence $\left\{\frac{n^2-1}{n^2}\right\}_{n=1}^{\infty}$ is Cauchy, as required. $\square$

(2.4.6) Suppose $|x_n - x_k| \leq \frac{n}{k^2}$ for all n and k. Show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy.

Proof. To show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy, we must show that for all n, k, there exists an M such that $n, k \geq M$ where $|x_n - x_k| \leq \frac{n}{k^2} < \epsilon$ for an arbitrary $\epsilon > 0$.

Given the condition that $|x_n - x_k| \leq \frac{n}{k^2}$ for all n and k, we can assume without loss of generality that $n \geq k$. As both $n, k \geq M$, we know that $k \geq M$ and hence $k^2 \geq M^2$. Therefore $\frac{1}{k^2} \leq \frac{1}{M^2}$ and thus $|x_n - x_k| \leq \frac{n}{M^2}$.

We then want to choose M such that $\frac{n}{M^2} < \epsilon$. Since we have $n \geq M$, we can see that

$$\frac{n}{M^2} \leq \frac{M}{M^2} < \epsilon \implies \frac{1}{M} < \epsilon \implies M > \frac{1}{\epsilon}$$

Thus we can choose an $M > \frac{1}{\epsilon}$ and therefore for all $n, k \geq M$, we can conclude that

$$|x_n - x_k| \leq \frac{n}{k^2} \leq \frac{M}{M^2} < \epsilon$$

showing that $\{x_n\}_{n=1}^{\infty}$ is Cauchy, as required. □

(2.4.8) True or false, prove or find a counterexample: If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then there exists an M such that for all $n \geq M$, we have $|x_{n+1} - x_n| \leq |x_n - x_{n-1}|$.

Proof. Let $\epsilon_0 > 0$ be given. Since $\{x_n\}_{n=1}^{\infty}$ is given as a Cauchy sequence, by definition, there must exists some $N_0 \in \mathbb{N}$ such that for all $n \geq N_0 + 1$, $|x_n - x_{n-1}| < \epsilon_0$. Now choose $\epsilon_1 = |x_n - x_{n-1}|$. For this ϵ_1, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have that $|x_{n+1} - x_n| \leq \epsilon_1 = |x_n - x_{n-1}|$. There fore we can choose an $M = N_1$ such that for all $n \geq M$, we have that $|x_{n+1} - x_n| \leq |x_n - x_{n-1}|$, as required. $\square$