Chapter 1

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Read Chapter 0 to Refresh Basics of Set Theory for Chapter 3

Homeworks 250pts
- To be submitted weekly on Canvas (x10)
- Submit in LaTeX :)

3 Exams, No Final!

Brush up on LaTeX

Real Numbers

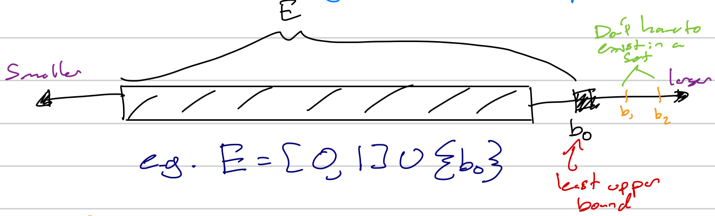
Notations: $\mathbb{N}$ denotes the set of natural numbers w/o the element 0
$\mathbb{Z}$ denotes the set of integers
$\mathbb{R}$ denotes the set of real numbers $\qquad \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$
$\mathbb{Q}$ denotes the set of rational numbers

Section 1 Basic Properties

Def: An ordered set is a set S with a relation $<$ s.t.
(i) (trichotomy) For every pair $x, y \in S$ exactly one of the following hold:
$x < y$, $x = y$, or $y < x$
(ii) (transitivity) If x, y, z belong to S, and satisfy $x<y$ and $y<z$, then $x<z$
- We write $x \leq y$ to mean that $x<y$ or $x=y$
- we define greater than ($>$) or $\geq$ in a similar fashion

Examples
- $\mathbb{Z}$ is an ordered set
- $\mathbb{Q}$ is an ordered set

we say that $y > x$, $x, y \in \mathbb{Q}$ iff $y - x$ is a positive rational number, i.e. $y - x = \frac{p}{q}$ where $p, q \in \mathbb{N}$



e.g. $E = [0,1] \cup \{b_0\}$

Greatest lower bound is same idea but on the other side :)

Def: Let $E \subset S$, where S is an ordered set
(i) If $\exists\ b \in S$ s.t. $x \leq b\ \forall x \in E$, we say E is bounded above and we call b an upper bound for E
(ii) If $\exists\ b \in S$ s.t. $b \leq x\ \forall x \in E$, we say E is bounded from below and we call b a lower bound for E
(iii) If $\exists\ b_0 \in S$, an upper bound for E, s.t. $b_0 \leq b\ \forall$ upper bounds $b \in S$ of E, we say b_0 is the least upper bound, or supremum of E. This element $b_0 := \sup E$. ($:=$ is 'denoted by')
(iv) If $\exists\ b_0 \in S$, a lower bound for E, s.t. $b_0 \geq b\ \forall$ lower bound b of E, we say that b_0 is the greatest lower bound, or infimum, of E. Denote it by $b_0 := \inf E$

More Examples

(1) $S = \mathbb{Q}$, $E = \{x \in \mathbb{Q} : x < 1\}$
$1 \in \mathbb{Q}$ is an upper bound for E, & is in fact the least upper bound for E. However $1 \in \mathbb{Q} / E$, so E does not contain it
0 is the additive identity on its own

(2) $S = \mathbb{Q}$, $E = \{x \in \mathbb{Q} : x \leq 1\}$
This set contains its supremum

(3) $S = \mathbb{Q}$, $E = \{x \in \mathbb{Q} : x \geq 0\}$
This set has no upper bound, so it cannot have a least upper bound. It does however have a greatest lower bound being $0 \in \mathbb{Q}$.

Allows us to take limits

Def: An ordered set S has the least upper bound property if every non-empty subset $E \subset S$ that is bound above has a least upper bound. This is also called completeness.

Example: $\mathbb{Q}$ does not satisfy the least upper bound property

Claim: $\sqrt{2}$ is an irrational number

(for now) Fact: $\sqrt{2}$ exists & is the supremum for E.

Pf/ Consider the set $E = \{x \in \mathbb{Q} : x^2 < 2\}$

pf by contradiction

Assume to the contrary, that there is an $x \in \mathbb{Q}$ s.t. $x^2 = 2$, and let $x = \frac{n}{m}$, $n, m \in \mathbb{Z} \setminus \{0\}$, in lowest terms
Then $m^2 = 2n^2$, where by m^2 is seen to be divisible by 2 $\therefore$ m is also divisible by 2. Now $m = 2K$, $K \in \mathbb{N}$.
$\therefore (2K)^2 = 2n^2 \rightarrow 4K^2 = 2n^2$. Dividing both sides by 2, we see n^2 is also divisible by 2 $\rightarrow$ contradiction

Fields

Def: A set F is said to be a **field** if it has two operations (closed) $+$ & $\cdot$, & if it satisfies the axioms:

Addition

(A1) If $x, y \in F$, then $x+y \in F$

(A2) $x+y = y+x \;\; \forall x,y \in F$

(A3) $(x+y)+z = x+(y+z) \;\; \forall x,y,z \in F$

(A4) $\exists$ an element $0 \in F$ s.t. $x+0 = 0+x = x \;\; \forall x \in F$

(A5) $\forall x \in F$, $\exists -x \in F$ s.t. $x+(-x) = -x+x = 0$

Multiplication

(M1) $x, y \in F \Rightarrow x \cdot y \in F$

(M2) $x \cdot y = y \cdot x \;\; \forall x,y \in F$

(M3) $(x \cdot y) \cdot z = x \cdot (y \cdot z) \;\; \forall x,y,z \in F$

(M4) $\exists$ an element $1 \in F$ s.t. $x \cdot 1 = 1 \cdot x = x \;\; \forall x \in F$

(M5) $\forall x \in F, x \neq 0 \;\exists \frac{1}{x} \in F$ s.t. $\frac{1}{x} \cdot x = x \cdot \frac{1}{x} = 1$

Distribution

(D) $x(y+z) = xy + xz \;\; \forall x,y,z \in F$

Examples:

- $\mathbb{Q}$ form of a field
- $\mathbb{Z}$ are not a field (cannot multiply)
- $\mathbb{R}$ are a field

Def: A field F is said to be an **ordered field** if F is an ordered set s.t.

① For $x,y,z \in F$, $x < y \rightarrow x+z < y+z$

② For $x, y \in F$, $x > 0$ & $y > 0$ both imply $xy > 0$

- If $x > 0$, then x is said to be positive, and if $x < 0$ then x is said to be negative
  - ↳ we also say x is nonnegative if $x \geq 0$, & nonpositive if $x \leq 0$

Can be defined analogously to the other operations $\{>, <, \geq, \leq\}$

Proposition: Let F be an ordered field, and $x,y,z,w \in F$, then:

① If $x > 0$, then $-x < 0$ (and vice-versa)

② If $x > 0$, & $y < z$, then $x \cdot y < x \cdot z$ ← preserves ordering

③ If $x < 0$ and $y < z$, then $x \cdot y > x z$ ← flips ordering

④ If $x \neq 0$, then $x^2 > 0$

Note: ④ $\Rightarrow$ $1 > 0$

⑤ If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$

⑥ If $0 < x < y$, then $x^2 < y^2$

⑦ If $x \leq y$ & $z \leq w$, then $x+z \leq y+w$

Do not submit proofs written in symbols!

Pf/: We settle items ① and ⑤

① The inequality $x > 0$ & item ① in definition of an order field imply that $0 + (-x) < x + (-x)$. Now, as $x + (-x) = 0$ because F is a field, we conclude that $-x < 0$ whenever $x > 0$, as required.

Every line should be carefully explained

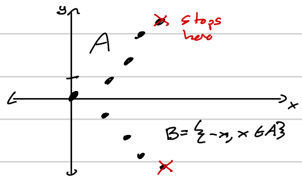
⑤ First note that $\frac{1}{x}$ is nonzero as a multiplicative inverse of a nonzero element in a field. Suppose $\frac{1}{x} < 0$, then $-\frac{1}{x} > 0$ by part ① of the proposition. By part ② of def. of an ordered field, $x \cdot (-\frac{1}{x}) > 0$, which means $-1 > 0$, a contradiction. (using item ① of proposition) Therefore $1/x > 0$. A similar argument shows $1/y > 0$.
Thus, $1/x \cdot 1/y > 0$ (by ② in def of ordered field)
Finally, we can conclude $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)x < \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)(y)$ So, $0 < \frac{1}{y} < \frac{1}{x}$
commutative and associative

Homework Correction

#1 Given $x, y \in F$, F an ordered field $0 < x < y$, prove $x^2 < y^2$

→ sup exists

Proposition: Let F be an ordered field with the least upper bound property.
↳ Let $A \subset F$ be a non-empty set bounded below. Then $\inf A$ exists in F.



Pf/ Consider $B=\{-x; x\in A\}$

Let $b \in F$ be a generic lower bound for A; this means given any $x\in A$, we know that $x \geq b$. In other words, $\forall\, x \in A$, $-x \leq -b$. Consequently $-b$ is an upper bound for B. Since B is a non-empty & bounded from above set in an ordered field F that satisfies the completeness property, $\sup B$ exists as an element in F. $C := \sup B$

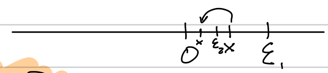
$A = \{x \in \mathbb{Q} : x^2 < 2\}$

Note that $C \leq -b$. $\forall y : y \in B$, we know that $y \leq C$. By the definition of B, we know $-x \leq C$ $\forall\, x\in A$. Putting everything together: $-x \leq C \leq -b$ $\forall\, x \in A$. Multiply by -1, we know $x \geq -c \geq b$ $\forall\, x \in A$, so $-c = \inf A$. //

Section 2 The set of Real Numbers

Theorem: There exists a unique ordered field $\mathbb{R}$ with the least upper bound property that contains $\mathbb{Q}$. This field $\mathbb{R}$ is called the set of real numbers.

Proposition: If $x \in \mathbb{R}$ such that $x \leq \varepsilon$ holds $\forall\, \varepsilon \in \mathbb{R}$, when $\varepsilon > 0$, then $x \leq 0$



Forces x negative

Pf/ If $x > 0$, then $0 < \frac{x}{2} < x$. Choosing $\varepsilon = \frac{x}{2}$ results in a contradiction //

Example: The set $A = \{x \in \mathbb{Q} : x^2 < 2\}$ has a least upper bound $\sup A$ that does not belong to the rational numbers.

Claim: There is a unique number $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $r^2 = 2$. We denote this number by $r = \sqrt{2}$.

Pf/ We begin by showing A is bounded from above and non-empty. Note that $A \neq \emptyset$, since $1 \in A$. The equation $x \geq 2$ implies that $x^2 \geq 4$, so if $x^2 < 2$, then $x < 2$. Therefore A is bounded from above. Thus, as $\mathbb{R}$ satisfies the least upper bound property, $r := \sup A$ exists as an element r.

Goal: Show $r^2 = 2$

We'll do this by showing that $r^2 \geq 2$ and $r^2 \leq 2$

Step 1: $r^2 \geq 2$.

Choose an $s > 0$ such that $s^2 < 2$. We will search for an $h > 0$ such that $(s+h)^2 < 2$. Since $2 - s^2 > 0$, we see that $\frac{2-s^2}{2s+1} > 0$. Choose h such that $0 < h < \frac{2-s^2}{2s+1}$. We may also choose $h < 1$. Then we can estimate

$$\begin{aligned}(s+h)^2 - s^2 &= h(2s+h)\\ &< h(2s+1) \quad \text{as } 0<h<1\\ &< 2-s^2 \quad \text{as } h < \tfrac{2-s^2}{2s+1}\end{aligned}$$

Consequently, $(s+h)^2 < 2$. Thus, $s+h \in A$. Hence $s < r := \sup A$. As $s > 0$ is arbitrary with $s^2 < 2$, it follows that $r^2 \geq 2$.

Step 2: $r^2 \leq 2$. Apply similar logic

By Steps 1 and 2, we know that $r^2 = 2$.

Uniqueness follows as usual. $\boxed{}$

written in textbook pretty clearly

The set $\mathbb{R} / \mathbb{Q}$ which is non-empty is called the set of irrational numbers.

Archimedean Property

Theorem:

(i) (Archimedean Property): If $x, y \in \mathbb{R}$ and $x > 0$, there must exist a natural number n s.t. $nx > y$

↳ For every natural number, you can find a real number smaller than it

(ii) ($\mathbb{Q}$ is dense in $\mathbb{R}$): If $x, y \in \mathbb{R}$ and $x < y$, $\exists\, r \in \mathbb{Q}$ s.t. $x < r < y$

↳ Between any real numbers you can always find a rational # between

Proof: We begin with the proof of item (i)

Dividing $nx > y$ by x, item (i) asserts that $\forall t \in \mathbb{R}$, $t := y/x$, we can find $n \in \mathbb{N}$ such that $n > t$. In Other words (i) asserts that $\mathbb{N} \subset \mathbb{R}$ is not bounded from above

Assume to the contrary that $\mathbb{N}$ is indeed bounded from above as a subset of $\mathbb{R}$. By the completeness of $\mathbb{R}$, there is a finite number $b := \sup \mathbb{N}$. As b is the least upper bound of the natural numbers, $(b-1)$ can no longer be an upper bound for $\mathbb{N}$. Therefore, there must be some $m \in \mathbb{N}$ such that $m > b-1$. Adding 1 to both sides, & noting that $m+1 \in \mathbb{N}$, $m+1 > b$, a contradiction ☒

We proceed to the proof of item (ii).

First, we suppose that $x \geq 0$. Then $y - x > 0$. By part (i), there is a natural number n such that $n(y-x) > 1$, or $(y-x) > 1/n$.

Again using part (i), the set $A = \{k \in \mathbb{N} : k \geq nx\}$ is non-empty. By the well-ordering property of the natural numbers A has a least element m. Then $m \geq nx$. If $m > 1$, then $m - 1 \geq 0$ and $m - 1 \leq nx$ as $x \geq 0$. In other words,

$$m - 1 \leq nx \quad \text{or} \quad m \leq nx + 1$$

On the other hand, $n(y-x) > 1$ so we obtain $ny > 1 + nx$. Consequently, $ny \geq 1 + nx > m$; hence $y > \frac{m}{n}$. Putting everything together, we know that $x < \frac{m}{n} < y$. Choose $r := m/n$.

Suppose now that $x < 0$.

- If $y > 0$, choose $r = 0$
- If $y \leq 0$, then $0 \leq -y < -x$, and from the first case we can choose $q \in \mathbb{Q}$ such that $-y < q < -x$. Take $r = -q$ in this case ☒

Corollary

$\inf\{1/n : n \in \mathbb{N}\} = 0$

Proof Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. The set A is non-empty & and $1/n > 0$ for every $n \in \mathbb{N}$.

Therefore, 0 is a lower bound for A, and so $b := \inf A$ exists. We also know that $b \geq 0$. By the archimedean property, there must exist an $n \in \mathbb{N}$ such that $na > 1$, with $a > 0$ being arbitrary. In other words, for any positive number a, there is some $n \in \mathbb{N}$ for which $a > 1/n$. Thus, a cannot be a lower bound for A, whenever $a > 0$. Consequently, $b = \inf A = 0$. ☒

Using sup & inf

- For $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, define the translation of A by x via

$x + A = \{x + a : a \in A\}$ and then $xA = \{xa : a \in A\}$

Proposition Let $A \subset \mathbb{R}$ be non-empty

(i) If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x+A) = x + \sup A$

(ii) If $x \in \mathbb{R}$ and A is bounded below, then $\inf(x+A) = x + \inf A$

(iii) If $x \geq 0$ and A is bounded above, then $\sup(xA) = x \cdot (\sup A)$

(iv) If $x \geq 0$ and A is bounded below, then $\inf(xA) = x \cdot (\inf A)$

(v) If $x < 0$, and A is bounded below, then $\sup(xA) = x \cdot (\inf A)$

(vi) If $x < 0$, and A is bounded above, then $\inf(xA) = x \cdot (\sup A)$

Proof We prove (i)

Let b be an upper bound for A, that is, $a \leq b$ $\forall a \in A$. Thus, $x + a \leq x + b$ $\forall x \in A$. Thus, $x + b$ is an upper bound for $x + A$. Hence, $\sup(x+A) \leq x + b$.

- Choosing $b := \sup A$, we conclude that $\sup(x+A) \leq x + \sup A$

Next, let c be an upper bound for $x + A$. This means $z \leq c$ $\forall z \in x + A$. Note that $z = x + w$ for some $w \in A$. So $w \leq c - x$ $\forall w$ $w \in A$. Then $c - x$ is an upper bound for A. In particular, $\sup A \leq c - x$ $\forall$ upper bound c of $x + A$.

- choosing $c := \sup(x+A)$, we conclude that $\sup(A) + x \leq \sup(x+A)$ $\boxtimes$

Proposition: Let a, b be any pair of non-empty subsets of $\mathbb{R}$ s.t. $x \leq y$ whenever $x \in A$ and $y \in B$. Then, A is bounded above, B is bounded below, and $\sup(A) \leq \inf(B)$

Proof: Any element of A is a lower bound for B.

Moreover, since B is non-empty and bounded below, the completeness property of $\mathbb{R}$ guarantees $\inf(B)$ exists. Therefore, $x \leq \inf B$ $\forall x \in A$. So $\inf B$ is an upper bound for A, and we conclude that $\sup A \leq \inf B$ $\boxtimes$

Question: Given two sets A, B in $\mathbb{R}$ such that $x < y$ $\forall x \in A, y \in B$. Does it hold that $\sup A < \inf B$? NO!!!

Counterexample: Choose $A = \{0\}$, $B = \{\frac{1}{n}, n \in \mathbb{N}\}$. Then $\inf B = \sup A = 0$

Proposition: If $S \subset \mathbb{R}$ is non-empty and bound from above, then $\forall \varepsilon > 0$, $\exists x \in S$ s.t. $\sup S - \varepsilon < x \leq \sup S$

Extended Real Numbers

Def: Let $A \subseteq \mathbb{R}$

(i) If $A = \emptyset$, define $\sup A := -\infty$

(ii) If A is not bounded above & non-empty, then $\sup A := +\infty$

(iii) If A is empty, define $\inf A := +\infty$

(iv) If A is not bounded below & non-empty, then $\inf A := -\infty$

The set $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$ is called the extended Real Numbers & it can be made an ordered set via: $-\infty < \infty$, $-\infty < x < \infty$ $\forall x \in \mathbb{R}$

Not always the case

Def: When $A \subset \mathbb{R}$ non-empty and bounded above, and $\sup A \in A$, then $\sup A$ is called the maximum of A, and it is denoted by $\max A$.

If $A \subset \mathbb{R}$ non-empty and bounded below, and $\inf A \in A$, then $\inf A$ is called the minimum of A and is denoted $\min A$.

Fact: Any non-empty finite subset of $\mathbb{R}$ has a minimum and a minimum and maximum

↳ Proved by induction on HW1

Absolute Value & Functions

For any $x \in \mathbb{R}$, define $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Proposition:

(i) $|x| \geq 0$, with equality iff $x = 0$.

(ii) $|-x| = |x|$ for all $x \in \mathbb{R}$ symmetry, not negativity

(iii) $|xy| = |x| \cdot |y|$ for all $x, y \in \mathbb{R}$

(iv) $|x^2| = x^2$ $\forall x \in \mathbb{R}$

(v) $|x| \leq y$ iff $-y \leq x \leq y$

(vi) $-|x| \leq x \leq |x|$ $\forall x \in \mathbb{R}$

Proposition (Triangle Inequality): For any pair $x, y \in \mathbb{R}$,

$$|x+y| \leq |x| + |y|$$

Proof: By (vi) of the previous proposition, we know that $-|x| \leq x \leq |x|$ & $-|y| \leq x \leq |y|$.
Addition of these two equations yields $-(|x|+|y|) \leq x+y \leq (|x|+|y|)$
Apply item (v) $\boxtimes$

Corollary: For any pair $x, y \in \mathbb{R}$, the following hold:

(i) (Reverse triangle ineq): $\big||x|-|y|\big| \leq |x-y|$

(ii) $|x-y| \leq |x| + |y|$

Proof: We settle item (i)

Set $x = a-b$, $y = b$ for some arbitrary pair $a, b \in \mathbb{R}$. Applying the triangle ineq. we get

$$|a| = |\underbrace{a-b}_{x}+\underbrace{b}_{y}| \leq |a-b| + |b|,$$

or equivalently, that

$$|a| - |b| \leq |a-b|$$

Switching the roles of a and b, we also have

$$|b| - |a| \leq |a-b|$$

Apply item (v) of the previous proposition to get the desired result $\boxtimes$

Corollary: Let $x_1, x_2, \dots, x_n \in \mathbb{R}$. Then

Then $|x_1 + x_2 + \dots + x_n| \le |x_1| + |x_2| + \dots |x_n|$ Induction Proof

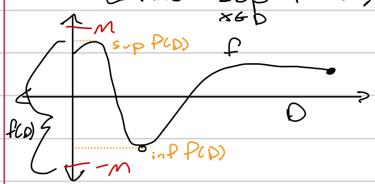
Example: Find a number M such that $|x^2 - 9x + 1| \le M$ for $-1 \le x \le 5$.

Solution: For any $x \in \mathbb{R}$, the triangle ineq gives
$|x^2 - 9x + 1| \le |x^2| + 9|x| + 1$

For those $-1 \le x \le 5$, the maximum of $|x^2| + 9|x| + 1$ occurs when $x = 5$.

So choose $M = 5^2 + 9(5) + 1 = 71$. ← Not the best M but it holds

Def: Suppose $f: D \to \mathbb{R}$ is a function. We say that f is bounded if there is some $M > 0$ such that $|f(x)| \le M$ for all $x \in D$. For function $f: D \to \mathbb{R}$, we write $\sup_{x \in D} f(x) := \sup f(D)$ & $\inf_{x \in D} f(x) := \inf f(D)$



sup and inf still don't need to exist as part of the image set

Example: Let $D = \{x : -1 \le x \le 5\}$ & $f(x) = x^2 - 9x + 1$

Using calculus: $\sup_{x \in D} f(x) = \sup_{-1 \le x \le 5} [x^2 - 9x + 1] = 11$ Just take derivative to find min and max.

$\inf_{x \in D} f(x) = \inf_{-1 \le x \le 5} [x^2 - 9x + 1] = -\frac{77}{4}$

Proposition: Given a pair of bounded functions $f, g: D \to \mathbb{R}$, with D being non-empty, such that $f(x) \le g(x)$ for all $x \in D$, it holds that

$$\sup_{x \in D} f(x) \le \sup_{x \in D} g(x) \quad \& \quad \inf_{x \in D} f(x) \le \inf_{x \in D} g(x)$$

Caution: The x on LHS of these inequalities is different than the x on the RHS

For example, the first should be thought of as: $\sup_{x \in D} f(x) \le \sup_{y \in D} g(y)$

Proof: Suppose b is an upper bound for $g(D)$. Then, for every $x \in D$ we have $f(x) \le g(x) \le b$ based on the propositions assumption, so b is an upper bound for $f(D)$.

In other words, $f(x) \le b$ for every $x \in D$. Thus for all $x \in D$, $f(x) \le \sup_{y \in D} g(y)$.

Consequently, $\sup_{x \in D} f(x) \le \sup_{y \in D} g(y)$ ∎

Remark: Under the Hypothesis of this proposition, the ineq $\sup_{x \in D} f(x) \le \inf_{y \in D} g(y)$ is false

Look up Counter example in Homework
- Look at x and $x+1$ from $0 \to 10$
- Or look at $1/n$

Intervals

Intervals in $\mathbb{R}$

Given $a, b \in \mathbb{R}$ $a \leq b$ set

- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ closed interval
- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ open interval
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ half open interval
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ "

All such intervals are called bounded

Unbounded Intervals

Given $a, b \in \mathbb{R}$ $a \leq b$ set

- $[a, \infty) = \{x \in \mathbb{R} : a \leq x < \infty\}$ closed interval
- $(a, \infty) = \{x \in \mathbb{R} : a < x < \infty\}$ open interval
- $(-\infty, b] = \{x \in \mathbb{R} : -\infty < x \leq b\}$ closed interval
- $(-\infty, b) = \{x \in \mathbb{R} : -\infty < x < b\}$ open interval
- $(-\infty, \infty) = \mathbb{R}$ open interval

Proposition: A set $I \subset \mathbb{R}$ is an interval iff I contains at least two points, and for all $a, b \in I$ and $c \in \mathbb{R}$ such that $a < c < b$, we have that $c \in I$.

Theorem: $\mathbb{R}$ is an uncountable set

Chapter 2

Sequences and Series

Def: A sequence of real numbers is any function $x : \mathbb{N} \to \mathbb{R}$. Instead of using $x(n)$, we use the notation x_n to denote the n^{th} element of the sequence.

→ To denote the sequence, we will use $\{x_n\}_{n=1}^{\infty}$, $\{x_n\}_n$, $\{x_n\}$, $\{x_n\}_1^{\infty}$ interchangeably

↳ Use interchangeably based on needs of proof. Royden uses $\{x_n\}_1^{\infty}$ too

A sequence is bounded if there exists $\{M>0 : B\in\mathbb{R}\}$ such that $|x_n| \le M$ for every $n \in \mathbb{N}$.
In other words the set $\{x_n : n \in \mathbb{N}\} \subset \mathbb{R}$ is a bounded set
↑ subset of the real line

Examples:

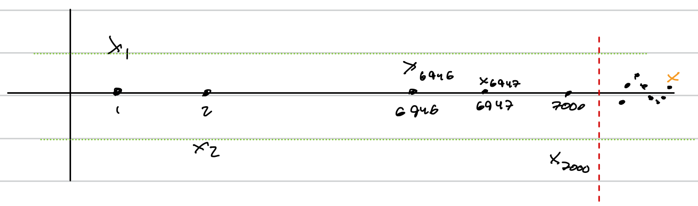
✓ (i) $\{\frac{1}{n}\}_{n=1}^{\infty}$ is bounded.
Choose $M=1$

✓ (ii) Let $c \in \mathbb{R}$. Define the constant sequence $\{c_n\}_{n=1}^{\infty} = \{c, c, c, \dots, c, \dots\}$
Choose $M = |c|$ will do the trick

✗ (iii) $\{n\}_{n=1}^{\infty}$ is <u>not</u> bounded

✓ (iv) $\{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1 \dots\}$
Choose $M \ge 1$

Def: A sequence $\{x_n\}_1^{\infty}$ is said to converge to <u>some</u> $x \in \mathbb{R}$ if for every $\varepsilon > 0$, there exist $M \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n \ge M$.



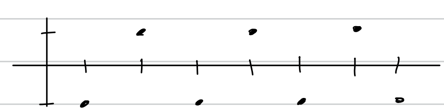
The number x is called <u>a limit</u> of $\{x_n\}_1^{\infty}$ & we write
$$x := \lim_{n\to\infty} x_n$$

A sequence that converges is said to be <u>convergent</u>. If a sequence does not converge, it is said to be <u>divergent</u> or <u>it diverges</u>

Example: The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges & $\lim_{n\to\infty} \frac{1}{n} = 0$

Proof: Let $\varepsilon > 0$ be given. By the archimedean property, there must exist some $M \in \mathbb{N}$ such that $0 < \frac{1}{M} < \varepsilon$. Consequently, for every $n \ge M$, we have that $|x_n - 0| = |\frac{1}{n}| \le \frac{1}{M} < \varepsilon$, as required. ∎

Example: The sequence $\{(-1)^n\}_{n=1}^{\infty}$ diverges
(the alternating sequence)



Proof: Assume to the contrary that $\{(-1)^n\}_{n=1}^{\infty}$ converges to some $x \in \mathbb{R}$ & let $\varepsilon = \frac{1}{2}$.
Then must exist an $M \in \mathbb{N}$ such that $|(-1)^n - x| < \frac{1}{2}$ whenever $n \ge M$.

- For each <u>even</u> $n \ge M$, we get $\frac{1}{2} > |\overset{x_n}{1} - x|$ & $|x_{n+1} - x| = |\overset{1ex}{-1} - x| < \frac{1}{2}$

$$\frac{1}{2} + \frac{1}{2} > |1-x| + |-1-x| \ge |1 - x + 1 + x| = 2$$
↑ Use triangle inequality

$$1 > 2 \quad \text{Contradiction}$$
∎

Tricks used here in this proof useful for entire semester

Proposition: A convergent sequence has a unique limit.

Proof: Suppose that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence, & that $x, y \in \mathbb{R}$ are limits of $\{x_n\}_1^{\infty}$. Let $\varepsilon > 0$ be arbitrary. Since $\{x_n\}_1^{\infty}$ converges to x, there must exist an $M_1 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon/2$. Similarly, as $\{x_n\}_1^{\infty}$ converges to y, there must be some $M_2 \in \mathbb{N}$ such that $|x_n - y| < \varepsilon/2$ whenever $n \geq M_2$.

Apply Def Twice

Let $M := \max\{M_1, M_2\}$. Then by the triangle inequality

$$|x-y| \leq |x_n - x| + |x_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

whenever $n \geq M$. Since $\varepsilon > 0$ can be made arbitrarily small, we conclude that $x = y$.

Proposition: A convergent sequence is bounded

Proof: Suppose that $\{x_n\}$ converges to x. For $\varepsilon = 1$ there must exist some $M \in \mathbb{N}$ such that $|x_n - x| < 1$ whenever $n \geq M$. Thus, for $n \geq M$, $|x_n| \leq |x_n - x| + |x| < 1 + |x|$.

The set $\{|x_n| : n = 1, 2, \dots, M\}$ is finite & nonempty so it has a maximum say $\max\{|x_n| : n = 1, 2, 3, \dots M\} = V$

Therefore, for every $n \in \mathbb{N}$, $|x_n| \leq \max\{V, 1 + |x|\}$. Consequently, $\{x_n\}$ is a bounded sequence.

Caution: Bounded sequences are not guaranteed to converge.

Example: Show that $\left\{\frac{n^2+1}{n^2+n}\right\}_1^{\infty}$ converges & that $\lim_{n\to\infty} \frac{n^2+1}{n^2+n} = 1$

Scratch work:

$$\left|\frac{n^2+1}{n^2+n} - 1\right|, \quad \left|\frac{n^2+1-(n^2+n)}{n^2+n}\right| = \left|\frac{-1+n}{n^2+n}\right| \leq \frac{1+|n|}{n(n+1)} = \frac{1+n}{n(n+1)} = \frac{1}{n}$$

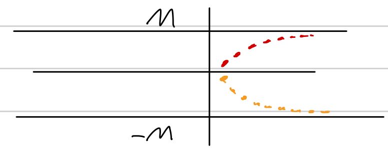
$\Rightarrow$ We know that $\left\{\frac{1}{n}\right\}_1^{\infty}$ converges to zero

Expect one question like this on the exam.

Proof: Let $\varepsilon > 0$ be given. Since $\left\{\frac{1}{n}\right\}_1^{\infty}$ converges to 0 there must exist an $M \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ whenever $n \geq M$. Therefore, by the triangle inequality, $\left|\frac{n^2+1}{n^2+n} - 1\right| \leq \frac{1+n}{n(n+1)} = \frac{1}{n} < \varepsilon$ whenever $n \geq \mathbb{N}$. Since $\varepsilon > 0$ is arbitrary, we know that $\left\{\frac{n+1}{n^2+n}\right\}_{n=1}^{\infty}$ converges to 1. ▨

Def: A sequence $\{x_n\}_{n=1}^{\infty}$ is said to be monotone increasing if for all $n \in \mathbb{N}$, $x_n \leq x_{n+1}$, and is said to be monotone decreasing if $x_n \geq x_{n+1}$ for every $n \in \mathbb{N}$. If a sequence $\{x_n\}_1^{\infty}$ is either one of these types, then x_n is said to be monotone.

Examples: $\{\frac{1}{n}\}_1^{\infty}$ is monotone decreasing
$\{n\}_1^{\infty}$ is monotone increasing



You can write MCT

Monotone Converge Theorem: A monotone sequence $\{x_n\}_1^{\infty}$ is convergent if and only if it is bounded. Furthermore, if $\{x_n\}_1^{\infty}$ is monotone increasing and bounded, then $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$.

On the other hand, if $\{x_n\}_1^{\infty}$ is monotone decreasing and bounded, then $\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$

Proof: We will prove the theorem in the instance when $\{x_n\}_1^{\infty}$ is monotone increasing

Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded; that is the set $\{x_n : n \in \mathbb{N}\}$. Let $x := \sup\{x_n : n \in \mathbb{N}\}$.

Let $\varepsilon > 0$ be given. As x is the sup of $\{x_n : n \in \mathbb{N}\}$, there must be at least one element $x_M \in \{x_n : n \in \mathbb{N}\}$ such that $x_M > x - \varepsilon$. As $\{x_n\}$ is monotone increasing, we know $x_n \geq x_M$ whenever $n \geq M$. Consequently, for any $n \geq M$,

$$|x_n - x| = x - x_n \leq x - x_M < \varepsilon.$$

Therefore $\{x_n\}_1^{\infty}$ converges to x. We have already proven the other direction: every convergent sequence is bounded.

Example: Consider the sequence $\{\frac{1}{\sqrt{n}}\}_{n\geq 1}^{\infty}$.

This sequence is bounded from below by 0, as $\frac{1}{\sqrt{n}} \geq 0$ for every $n \in \mathbb{N}$. It is also monotone decreasing, as $\sqrt{n} \leq \sqrt{n+1}$ for every $n \in \mathbb{N}$ (you would need to explain why). Then $\frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}}$ for every $n \in \mathbb{N}$.

It follows from the MCT that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = \inf\{\frac{1}{\sqrt{n}} : n \in \mathbb{N}\}$ Homework: Show $\inf\{\frac{1}{\sqrt{n}} : n \in \mathbb{N}\} = 0$

Proposition: Let $S \subset \mathbb{R}$ be a non-empty bounded set. Then there exist sequences $\{x_n\}, \{y_n\}$ $x_n, y_n \in S$ for every $n \in \mathbb{N}$, such that

$$\sup S = \lim_{n\to\infty} x_n \quad \& \quad \inf S = \lim_{n\to\infty} y_n$$

Exam 1: | Sheet 8.5x11 w/ Definitions & thm. handwritten

Def: For a sequence $\{x_n\}_{n\geq 1}^{\infty}$, the K-tail, $K\in\mathbb{N}$, or just tail of the sequence is the sequence starting at $K+1$ usually written as $\{x_{n+K}\}_{n\geq 1}^{\infty}$ or $\{x_n\}_{n=K+1}^{\infty}$

Proposition: Let $\{x_n\}$ be a sequence. Then the following are equivalent:

(i) The sequence $\{x_n\}_{n\geq 1}^{\infty}$ converges

(ii) The K-tail $\{x_{n+K}\}_{n\geq 1}^{\infty}$ converges for every $K\in\mathbb{N}$

(iii) The K-tail $\{x_{n+K}\}_{n\geq 1}^{\infty}$ converges for some $K\in\mathbb{N}$

Furthermore, if any (and hence all) of the limits exist, then for every $K\in\mathbb{N}$,

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+K}$.

Proof: The implication (ii) => (iii) is immediate. We will show that (i) implies (ii) and that (iii) implies (i). We begin with (i) -> (ii)

Suppose that $\{x_n\}_{n\geq 1}^{\infty}$ converges to some $x\in\mathbb{R}$. Let $K\in\mathbb{N}$ be arbitrary, and define $y_n = x_{n+K}$ for each $n\in\mathbb{N}$. Goal: Show $\{y_n\}_{n\geq 1}^{\infty}$ converges to x. Given any $\varepsilon>0$, there is an $M\in\mathbb{N}$ such that $|x_n-x|<\varepsilon$ whenever $n\geq M$. Note that $n\geq M$ implies $n+K\geq M$. Therefore $|y_n-x|=|x_{n+K}-x|<\varepsilon$ for every $n\geq M$. Hence the sequence $\{y_n\}_{n\geq 1}^{\infty}$ converges. This completes that (i) -> (ii). We next prove that (iii) -> (i)

Let $\bar{K}\in\mathbb{N}$ be the necessary K for which (iii) holds. Define $y_n = x_{n+\bar{K}}$, & assume that $\{y_n\}_{n\geq 1}^{\infty}$ converges to $y\in\mathbb{R}$. We need to show that $\{x_n\}$ converges to y. Given $\varepsilon>0$, there is some $M\in\mathbb{N}$ such that $|y_n-y|<\varepsilon$ for all $n\geq M$. Set $M'=M+\bar{K}$. Then $n\geq M'$ implies that $n-\bar{K}\geq M$. Thus, $|x_n-y|=|y_{n-\bar{K}}-y|<\varepsilon$ whenever $n\geq M'$ as required.

Definition: Let $\{x_n\}_{n\geq 1}^{\infty}$ be a sequence. Let $\{n_i\}_{i\geq 1}^{\infty}$ be a strictly increasing seq. of natural numbers, i.e. $n_i < n_{i+1}$ for all $i\in\mathbb{N}$. The sequence $\{x_{n_i}\}_{i\geq 1}^{\infty}$ is called a subsequence of $\{x_n\}_{n\geq 1}^{\infty}$

Example: $\{(-1)^{2i}, i\in\mathbb{N}\} = \{1,1,1\ldots\}$ is a subsequence of $\{(-1)^n, n\in\mathbb{N}\}$

Proposition: If $\{x_n\}_{n\geq 1}^{\infty}$ is a convergent sequence, then every subsequence of $\{x_n\}_{n\geq 1}^{\infty}$ must converge to the same limit

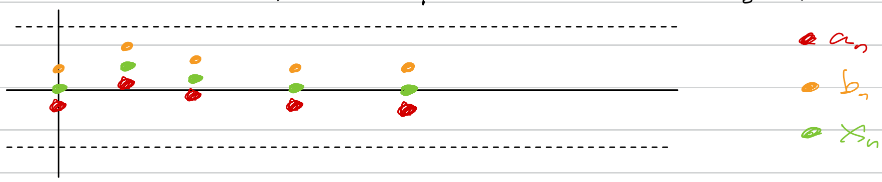
This word is important

Proof: Suppose $\lim_{n\to\infty} x_n = x$ and let $\varepsilon>0$ be arbitrary. There must exist some $M\in\mathbb{N}$ for which $|x_n-x|<\varepsilon$ whenever $n\geq M$. By induction, it follows that $n_i\geq i$ for any $i\in\mathbb{N}$. Consequently $i\geq M$ implies that $n_i\geq M$. Thus for all $i\geq M$, $|x_{n_i}-x|<\varepsilon$ as needed. $\square$

This is Exam 1 Cutoff!!!

Facts about Sequences

Squeeze Lemma: Let $\{a_n\}_{n\geq 1}^{\infty}, \{b_n\}_{n\geq 1}^{\infty}, \{x_n\}_{n\geq 1}^{\infty}$ be such that $a_n \leq x_n \leq b_n$ for all $n \in \mathbb{N}$. If $\{a_n\}_1^{\infty}$ and $\{b_n\}_1^{\infty}$ both converge to $x \in \mathbb{R}$, then $\{x_n\}_1^{\infty}$ must also converge to x.



Proof: Let $\varepsilon > 0$ be given. As $\{a_n\}_{n\geq 1}^{\infty}$ converges to x, there must be an $M_1 \in \mathbb{N}$ such that $|a_n - x| < \varepsilon$ holds for all $n \geq M_1$. As $\{b_n\}_{n\geq 1}^{\infty}$ converges to x there is an $M_2 \in \mathbb{N}$ such that $|b_n - x| < \varepsilon$ for those $n \geq M_2$. Set M to be the maximum among M_1 and M_2, $M := \max(M_1, M_2)$, and suppose that $n \geq M$. For these such n, it holds that $x - \varepsilon < a_n$ & $b_n < x + \varepsilon$. Thus,

$$x - \varepsilon < a_n \leq x_n \leq b_n < x + \varepsilon$$

In other words $-\varepsilon < x_n - x < \varepsilon$ whenever $n \geq M$. This is equivalent to $|x_n - x| < \varepsilon$ whenever $n \geq M$, as required. ∎

Lemma: Let $\{x_n\}_1^{\infty}, \{y_n\}_1^{\infty}$ be convergent sequences. Suppose that $x_n \leq y_n$ for every $n \in \mathbb{N}$. Then the $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.

Absolute value equiv $|x_n - x|$

Proof: Set $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$. Let $\varepsilon > 0$ be given. Choose $M_1 \in \mathbb{N}$ such that $\boxed{|x_n + x|} < \frac{\varepsilon}{2}$ whenever $n \geq M_1$ and $M_2 \in \mathbb{N}$ such that $|y_n - y| < \frac{\varepsilon}{2}$ whenever $n \geq M_2$. Set $M = \max\{M_1, M_2\}$. For those $n \geq M$, it holds that $x_n - x < \varepsilon/2$ & $y_n - x < \frac{\varepsilon}{2}$. Adding these inequalities we get $(y_n - x_n)(x - y) < \varepsilon$ or that $y_n - x_n < y - x + \varepsilon$ whenever $n \geq M$. Because $x_n \leq y_n$, this becomes $0 \leq y_n - x_n < y - x + \varepsilon$ for those $n \geq M$. Therefore $0 < y - x + \varepsilon$. In other words, $x - y < \varepsilon$. As $\varepsilon > 0$ is arbitrary, we conclude that $x - y \leq 0$. ∎

Corollary

(i) If $\{x_n\}$ is a convergent sequence such that $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n \geq 0$

(ii) Let $a, b \in \mathbb{R}$ and $\{x_n\}$ be a convergent sequence such that $a \leq x_n \leq b$ for all $n \in \mathbb{N}$. Then $a \leq \lim_{n\to\infty} x_n \leq b$

Algebra of Limits: Let $\{x_n\}$, $\{y_n\}$ be convergent sequences. Then the following hold

(i) $\lim_{n\to\infty}[x_n+y_n] = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$ — Addition

(ii) $\lim_{n\to\infty}[x_n-y_n] = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n$ — Subtraction

(iii) $\lim_{n\to\infty}[x_n y_n] = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$ — Multiplication

(iv) If $y_n \neq 0$ & $\lim_{n\to\infty} y_n \neq 0$ for every $n\in\mathbb{N}$, then — Quotient

$$\lim_{n\to\infty}\left[\frac{x_n}{y_n}\right] = \left(\frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}\right)$$

Proof We settle (i)

(i): Suppose $x = \lim_{n\to\infty} x_n$ & $y = \lim_{n\to\infty} y_n$. Let $\varepsilon > 0$ be given. Choose $M_1 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n \geq M_1$, and choose $M_2 \in \mathbb{N}$ such that $|y_n - y| < \varepsilon$ whenever $n \geq M_2$. Define $M = \max\{M_1, M_2\}$. Then, for every $n \geq M$, the triangle inequality implies that

$$|(x_n+y_n)-(x+y)| = |(x_n-x)+(y_n-y)| \leq |x_n-x| + |y_n-y| < \varepsilon+\varepsilon = 2\varepsilon$$

Since 2 is independent of ε, we have that the sequence given by $\{x_n+y_n\}_{n=1}^{\infty}$ converges to $x+y$ as required. ∎

(iii) Let $\varepsilon > 0$ be given. Set $K = \max\{|x|, |y|, \varepsilon/3, 1\}$. Using the fact that $\{x_n\}$ converges to x and $\{y_n\}$ converges to y, we may select a pair $M_1, M_2 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n \geq M_1$ and $|y_n - y| < \varepsilon$ whenever $n \geq M_2$. Set $M = \max\{M_1, M_2\}$. Then, for any such $n \geq M$,

$$\begin{aligned}|x_n y_n - xy| &= |(x_n - x + x)(y_n - y + y) - xy| = |(x_n-x)y + (y_n-y)x + (x_n-x)(y_n-y)| \\ &\leq |(x_n-x)y| + |(y_n-y)x| + |(x_n-x)(y_n-y)| \\ &= |y|\cdot|x_n-x| + |x|\cdot|y_n-y| + |x_n-x|\cdot|y_n-y| \\ &< |y|\cdot\varepsilon + |x|\cdot\varepsilon + \varepsilon^2\end{aligned}$$

∎

(iv) We may use part (iii) to get the result after proving the next claim

Claim: If $\{y_n\}$, $y_n \neq 0$ for all $n\in\mathbb{N}$, & $\{y_n\}$ converges to $y \neq 0$, then $\{1/y_n\}$ converges to $1/y$

Proof Claim: Let $\varepsilon > 0$ be given. As $|y| \neq 0$, $K_\varepsilon = \min\{|y|^2\cdot \varepsilon/2, |y|/2\} > 0$. As $\{y_n\}$ converges to y, we may select $M \in \mathbb{N}$ such that $|y_n - y| < K_\varepsilon$ whenever $n \geq M$. Consequently, for every $n \geq M$, we have $|y| = |y + y_n - y_n| \leq |y_n - y| + |y_n| < \frac{|y|}{2} + |y_n|$. Thus, for those $n \geq M$, $\frac{|y|}{2} < |y_n|$ which is equivalent to $\frac{1}{|y_n|} < \frac{2}{|y|}$. To complete the proof, for every $n \geq M$,

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y-y_n}{y_n y}\right| = \frac{|y-y_n|}{|y_n||y|} \leq \frac{2}{|y|}\cdot\frac{|y_n-y|}{|y|} = |y_n-y|\frac{2}{|y|^2} < \frac{|y|^2\cdot\frac{\varepsilon}{2}}{|y|}\cdot\frac{2}{|y|} = \varepsilon$$

∎

Proposition: Let $\{x_n\}$ be a convergent sequence such that $x_n \geq 0$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n}$.

Proof: Let $\{x_n\}$ converge to x. As $x_n \geq 0$ for all n, we know that $x \geq 0$. Let $\varepsilon > 0$ be given.

Case (1) $x = 0$

We need to show that $\{\sqrt{x_n}\}$ converges to 0. As $\{x_n\}$ converges to 0, there is some $M \in \mathbb{N}$ such that $|x_n - 0| < \varepsilon^2$ for every $n \geq M$. Then for every $n \geq M$,

nonneg to $\rightarrow$ $|\sqrt{x_n}| = \sqrt{x_n} < \sqrt{\varepsilon^2} = \varepsilon.$

Case (2) $x > 0$

Because $x > 0$, we know that $\sqrt{x} > 0$. Choose $M_2 \in \mathbb{N}$ such that $|x_n - x| < \sqrt{x}\,\varepsilon$ whenever $n \geq M_2$. Then, for every $n \geq M_2$, $|\sqrt{x_n} - \sqrt{x}| = \left|\left(\frac{(\sqrt{x_n} - \sqrt{x})}{1}\right)\left(\frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}}\right)\right|$

$$= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} < \frac{|x_n - x|}{\sqrt{x}} < \frac{\varepsilon}{|x|} \quad \blacksquare$$

Proposition: If $\{x_n\}$ is a convergent sequence, then $\{|x_n|\}_1^\infty$ is a convergent sequence, and

$$\lim_{n\to\infty} |x_n| = \left|\lim_{n\to\infty} x_n\right|$$

Proof: Suppose $\{x_n\}$ converges to x. Given any $\varepsilon > 0$, choose $M \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n \geq M$. By the Reverse Triangle Inequality, for every $n \geq M$, we have $||x_n| - |x|| \leq |x_n - x| < \varepsilon$, as required. $\blacksquare$

Example: Show that $\left\{\left|\sqrt{1 + \frac{1}{n}} - \frac{100}{n^2}\right|\right\}_{n=1}^\infty$ converges to 1

Sketch of the idea: $\lim_{n\to\infty} \left|\sqrt{1 + \frac{1}{n}} - \frac{100}{n^2}\right|$

(i) $\lim_{n\to\infty} \frac{1}{n} = 0 \implies \lim_{n\to\infty} [1 + \frac{1}{n}] = 1$ & as $1 + \frac{1}{n} \geq 0 \ \forall n \in \mathbb{N}$, then $\{\sqrt{1 + \frac{1}{n}}\}_1^\infty$ converges to $\sqrt{1} = 1$.

Squeeze Lemma $\rightarrow$ (ii) Since $\frac{100}{n^2} < \frac{1}{n} \ \forall n \in \mathbb{N}$, then $0 \leq \lim_{n\to\infty} \left(\frac{100}{n^2}\right) \leq 100 \lim_{n\to\infty} \frac{1}{n} = 0$

Therefore by items (i) and (ii), $\lim_{n\to\infty} \left|\sqrt{1 + \frac{1}{n}} - \frac{100}{n^2}\right|$

$$= \lim_{n\to\infty} \left|\sqrt{\lim_{n\to\infty}(1 + \frac{1}{n})} - \lim_{n\to\infty} \frac{100}{n^2}\right|$$

$$= |\sqrt{1} - 0| = 1$$

Convergence Tests

Dominated Convergence

Proposition: Let $\{x_n\}_1^\infty$ be a sequence, and suppose there is an $x\in\mathbb{R}$ and a sequence $\{a_n\}_1^\infty$ such that $\lim_{n\to\infty} a_n = 0$ and $|x_n - x| \le a_n$ for all $n\in\mathbb{N}$. Then the sequence $\{x_n\}$ converges to x.

Proof: Let $\varepsilon>0$ be given. As each $a_n\ge 0$ and $\{a_n\}$ converges to 0, there is some $M>0$ for which $a_n<\varepsilon$ whenever $n\ge M$. By assumption, $|x_n-x|\le a_n$ for each $n\in\mathbb{N}$. Therefore $|x_n-x|<\varepsilon$ whenever $n\ge M$. ∎

Proposition: Let $c>0$

(i) If $c<1$, then $\{c^n\}_{n\ge1}^\infty$ converges to zero

(ii) If $c>1$, then $\{c^n : n\in\mathbb{N}\}$ is an unbounded set of $\mathbb{R}$, and so $\{c^n\}_{n\ge1}^\infty$ cannot converge.

Proof: First assume $0<c<1$. As $c>0$, $c^n>0$ for each $n\in\mathbb{N}$. As $c<1$, we know that $c^{n+1}<c^n$ for every $n\in\mathbb{N}$. Therefore $\{c^n\}_1^\infty$ is a monotone decreasing sequence that is bounded from below by 0. Hence by MCT, it must converge to $x := \inf\{c^n : n\in\mathbb{N}\}$. Since $\lim_{n\to\infty} c^n = x$, necessarily, the 1-tail $\{c^{1+n}\}_{n\ge1}^\infty$ must converge, & in fact $\lim_{n\to\infty} c^n = \lim_{n\to\infty} c^{1+n} = x$. Since, for each $n\in\mathbb{N}$, $c^{n+1} = c\cdot c^n$, by taking limits, we get $x = cx$, or equivalently $(1-c)x=0$. Either $1-c=0$, a contradiction, or $x=0$.

$= c\lim_{n\to\infty} c^n = c\cdot x$

Suppose now that $c>1$. Let $B>0$ be arbitrary. As $c>1$, $\frac{1}{c}<1$ and so by item (i) the sequence $\{1/c^n\}_{n\ge1}^\infty$ must converge to 0. Therefore there must be an $M\in\mathbb{N}$ such that $1/c^n<\frac{1}{B}$ whenever $n\ge M$, which is equivalent to saying $c^n>B$ whenever $n\ge M$. As B is arbitrary, the set $\{c^n : n\in\mathbb{N}\}$ must be unbounded.

Ratio Test for Sequences

Let $\{x_n\}_1^\infty$ be a sequence such that $x_n\ne 0$ for all $n\in\mathbb{N}$ & the limit

$$L := \lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right|$$

exists. Then

(i) If $0\le L<1$, then $\{x_n\}_1^\infty$ converges to 0.

(ii) If $L>1$, then $\{x_n : n\in\mathbb{N}\}$ is unbounded, & so $\{x_n\}$ is divergent

(iii) If $L=1$, the test is inconclusive

item (i)

Proof: Suppose that $L<1$. Choose a number $r\in\mathbb{Q}$ such that $L<r<1$ (density of $\mathbb{Q}$). We want to compare $\{x_n\}$ with $\{r^n\}_{n\ge1}^\infty$. Now $\underbrace{r-L}_{\text{treat as }\varepsilon}>0$, so there exists an $M\in\mathbb{N}$ such that

$$\frac{|x_{n+1}|}{|x_n|}-L \le \left|\frac{|x_{n+1}|}{|x_n|}-L\right| < r-L$$

whenever $n\ge M$

Therefore, $\frac{|x_{n+1}|}{|x_n|}<r$ whenever $n\ge M$. For each $n\ge M+1$

$$|x_n| = |x_M|\frac{|x_{M+1}|}{|x_M|}\cdots\frac{|x_{n-1}|}{|x_{n-2}|}\frac{|x_n|}{|x_{n-1}|} < |x_M|\cdot\underbrace{r\cdot r\cdots r}_{n-M\text{ times}} = |x_M|\cdot r^{n-M} = \left[|x_M| r^{-M}\right]\cdot r^n$$

The sequence $\{r^n\}_{n\ge1}^\infty$ must converge to 0 by the previous proposition & so $\{|x_M|\cdot r^{-M}\cdot r^n\}_{n\ge1}^\infty$ must also converge to zero. Therefore, the M-tail $\{x_{M+n}\}_{n\ge1}^\infty$ must converge to zero and hence $\{x_n\}_1^\infty$ must converge to the same limit, 0. ∎

Item (iii) Suppose $L>1$, & choose r such that $1<r<L$. Then $\overbrace{L-r}^{\text{uses as } \varepsilon}>0$. Pick $M\in\mathbb{N}$ such that

$$-\left(\frac{|x_{n+1}|}{|x_n|}-L\right)\le\left|\frac{|x_{n+1}|}{|x_n|}-L\right|<L-r \quad \text{whenever } n\ge M.$$

Therefore, for each $n\ge M$, $\frac{|x_{n+1}|}{|x_n|}>r$. Again, for each $n\ge M$, $|x_n|>|x_M|\cdot r^{n-M}$
Same logic as before

This is equivalent to $[|x_M|\cdot r^{-M}]\cdot r^N$. By the previous proposition, the set $\{r^n : n\in\mathbb{N}\}$ is unbounded because $r>1$.
Therefore, $\{x_n : n\in\mathbb{N}\}$ is unbounded, for if it were, there would exist some $B>0$ such that $|x_n|\le B$ for every $n\in\mathbb{N}$, which would imply that $r^N<\frac{B}{|x_M|}r^M$, $n\ge M$ a contradiction. ∎

Example: $\lim_{n\to\infty}\sqrt[n]{n}=1$

Proof: Let $\varepsilon>0$ be given & consider the sequence $\left\{\frac{n}{(1+\varepsilon)^n}\right\}_{n=1}^{\infty}$. Compute for each $n\in\mathbb{N}$,

$$\frac{\frac{n+1}{(1+\varepsilon)^{n+1}}}{\frac{n}{(1+\varepsilon)^n}}=\frac{n+1}{n}\cdot\frac{1}{1+\varepsilon}\to\lim_{n\to\infty}\underbrace{\left(\frac{\frac{n+1}{(1+\varepsilon)^{n+1}}}{\frac{n}{(1+\varepsilon)^n}}\right)}_{\star}=\lim_{n\to\infty}\left(\frac{n+1}{n}\cdot\frac{1}{1+\varepsilon}\right)$$

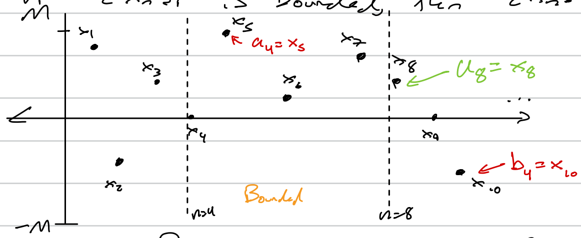
Note that $\frac{n+1}{n}=1+\frac{1}{n}$, so $\left\{\frac{n+1}{n}\right\}_{n=1}^{\infty}$ converges to 1. Therefore,

$$\lim_{n\to\infty}(\star)=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)\cdot\lim_{n\to\infty}\left(\frac{1}{1+\varepsilon}\right)=\frac{1}{1+\varepsilon}<1$$

By the ratio test, $\left\{\frac{n}{(1+\varepsilon)^n}\right\}_{n=1}^{\infty}$ converges to zero. Thus we may choose an $M\in\mathbb{N}$ such that $\frac{n}{(1+\varepsilon)^n}<1$ whenever $n\ge M$. This is equivalent to saying that $n<(1+\varepsilon)^n$ for all $n\ge M$. Therefore, for every $n\ge M$, $\sqrt[n]{n}<1+\varepsilon$. As $n\ge M$, for every n, we know $1\le\sqrt[n]{n}<1+\varepsilon$, we get the desired result sending ε to 0.

Limit Superior, Limit Inferior, & Bolzano Weierstrass Thm

If $\{x_n\}_1^\infty$ is bounded, then $\{x_n : n \in \mathbb{N}\}$ is a bounded set in $\mathbb{R}$



Moreover, for each $n \in \mathbb{N}$, the set $\{x_k : k \geq n\}$ is also bounded

Definition: Let $\{x_n\}_1^\infty$ be a bounded sequence. For each $n \in \mathbb{N}$, define

$$a_n = \sup\{x_k : k \geq n\} \quad \& \quad b_n = \inf\{x_k : k \geq n\}.$$ ← cutting sequence and finding biggest & smallest

Consider the sequences, $\{a_n\}_1^\infty$ & $\{b_n\}_1^\infty$. Define $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n$ ← lim superior
& $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n$ ← lim inferior, provided these limits both exist

Proposition: Let $\{x_n\}_1^\infty$ be a bounded sequence, and $\{a_n\}_{n\geq1}^\infty$ & $\{b_n\}_{n\geq1}^\infty$ be as in the above definitions

Come from MCT

(i) The sequence $\{a_n\}$ is bounded & monotone decreasing, while $\{b_n\}_{n\geq1}^\infty$ is also bounded but is monotone increasing. In particular, $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ both exist

(ii) $\limsup_{n\to\infty} x_n = \inf\{a_n : n \in \mathbb{N}\} = \inf\{\sup\{a_k : k \geq n\} : n \in \mathbb{N}\}$
&
$\liminf_{n\to\infty} x_n = \sup\{b_n : n \in \mathbb{N}\} = \sup\{\inf\{b_k : k \geq n\} : n \in \mathbb{N}\}$

(iii) $\limsup_{n\to\infty} x_n \geq \liminf_{n\to\infty} x_n$

Proof: We show that $\{a_n\}_{n\geq1}^\infty$ is a decreasing sequence.

For each $n \in \mathbb{N}$, $a_n = \sup\{a_k : k \geq n\}$ & $a_{n+1} = \sup\{a_k : k \geq n+1\}$. As $\{a_k : k \geq n\} \supseteq \{a_k : k \geq n+1\}$, we know that $a_n \geq a_{n+1}$ for each n. Therefore, $\{a_n\}_{n\geq1}^\infty$ is a bounded & decreasing sequence. By the MCT, $\{a_n\}_{n\geq1}^\infty$ must converge to the $\inf\{a_n : n \in \mathbb{N}\}$.

The argument that $\{b_n\}_{n\geq1}^\infty$ is increasing is similar. This proves (i) & item (ii)

For item (iii), for each $n \in \mathbb{N}$, $b_n = \inf\{x_k : k \geq n\} \leq \sup\{x_k : k \geq n\} = a_n$. Since limits preserve inequalities, item (i) implies that

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n = \limsup_{n\to\infty} x_n$$

∎

Example: Let $\{x_n\}_{n=1}^{\infty}$ be given by

$x_n = \begin{cases} \frac{n+1}{n}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$, Compute the limit inferior and superior

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf\{x_k : k\ge n\}\right) = \lim_{n\to\infty} 0 = 0$$

$$\{2, 0, \tfrac{4}{3}, 0, \tfrac{6}{5}, 0, \tfrac{8}{7}, 0 \dots\}$$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup\{x_k : k\ge n\}\right)$$

For each $n\in\mathbb{N}$

$\sup\{x_k : k\ge n\} = \begin{cases} \frac{n+1}{n} & n \text{ odd} \\ \frac{n+2}{n+1} & n \text{ even} \end{cases}$

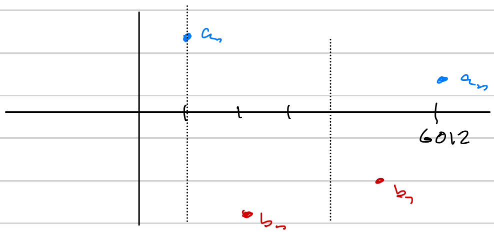
we know $\lim_{n\to\infty} \frac{n+1}{n} = 1$ & $\lim_{n\to\infty} \frac{n+2}{n+1} = 1$. Therefore $\limsup_{n\to\infty} x_n = 1$.

Remark: The sequence $\{x_n\}_{n=1}^{\infty}$ does not converge!

Recall Definition of limsup & liminf

Def: Let $\{x_n\}_1^{\infty}$ be a bounded sequence. For each $n\in\mathbb{N}$ define

$a_n = \sup\{x_k : k\ge n\}$ $b_n := \inf\{x_k : k\ge n\}$



Consider $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$. Define

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} a_n$$

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} b_n$$

Theorem: Given a bounded sequence $\{x_n\}_{n=1}^{\infty}$, there exist subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ & $\{x_{m_k}\}_{k=1}^{\infty}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n \quad \& \quad \lim_{k\to\infty} x_{m_k} = \liminf_{n\to\infty} x_n$$

Proofs almost the same as before

Proof: Define $a_n = \sup\{x_k : k\ge n\}$ for each $n\in\mathbb{N}$, & set $x := \limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n$.
We define a subsequence inductively. Pick $n_1 = 1$.
Suppose that $n_1, n_2, \dots, n_{k-1}$ have been chosen for some $k\ge 2$. Pick $m\ge n_{k-1}+1$ such that $a_{(n_{k-1}+1)} - x_m < 1/k$. Such an m exists because $a_{n_{k-1}+1} = \sup\{x_\ell : \ell\ge n_{k-1}+1\}$. Set $n_k := m$. The subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ has thus been defined.
We need to show $\lim_{k\to\infty} x_{n_k} = x$.

For each $k\ge 2$, we know that $a_{n_{k-1}} \ge a_{n_k}$ & $a_{n_k}\ge x_{n_k}$. Therefore, for every $k\ge 2$, $|a_{n_k} - x_{n_k}| = a_{n_k} - x_{n_k} \le a_{n_{k-1}+1} - x_{n_k} \le 1/k$.

Let $\varepsilon > 0$ be given. As $\{a_n\}_{n=1}^{\infty}$ converges to x, the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ must also converge to x. There exists an $M_1\in\mathbb{N}$ such that $|a_{n_k} - x| < \frac{\varepsilon}{2}$ whenever $k\ge M_1$. Find $M_2\in\mathbb{N}$ such that $1/M_2 \le \varepsilon/2$. Take $M = \max\{M_1, M_2, 2\}$. Then, for all $k\ge M$, $|x - x_{n_k}| = |a_{n_k} - x_{n_k} + a_{n_k} - x| \le |a_{n_k} - x_{n_k}| + |a_{n_k} - x| \le 1/k + \varepsilon/2 \le \varepsilon/2 + \varepsilon/2 = \varepsilon$ as required. $\boxtimes$

Proposition: A bounded sequence $\{x_n\}_{n=1}^{\infty}$ converges if and only if
$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$$
In fact, $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$

Proof: For each $n \in \mathbb{N}$, define $a_n = \sup\{x_k : k \geq n\}$ & $b_n = \inf\{x_k : k \geq n\}$. In particular, for each $n \in \mathbb{N}$, we know that $b_n \leq x_n \leq a_n$. Assume that $\limsup_{n\to\infty} x_n$ & $\liminf x_n$ coincide. Then, since $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n$ & $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n$, we may apply the squeeze lemma to get that
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$$
Conversely, assume that $\{x_n\}_{n=1}^{\infty}$ converges to some $x \in \mathbb{R}$. According to the previous theorem, there must be some subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ that converges to $\limsup_{n\to\infty} x_n$. However, as $\lim_{n\to\infty} x_n$ necessarily, $\lim_{k\to\infty} x_{n_k} = x$. Therefore $x = \limsup_{n\to\infty} x_n$ by the uniqueness of the limit.

Similarly, for the liminf you can extract a subsequence to show $x = \liminf_{n\to\infty} x_n$ $\blacksquare$

Proposition: Suppose you have a bounded sequence $\{x_n\}_{n=1}^{\infty}$ & $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence. Then
$$\liminf_{n\to\infty} x_n \leq \liminf_{k\to\infty} x_{n_k} \leq \limsup_{k\to\infty} x_{n_k} \leq \limsup_{n\to\infty} x_n$$
HWS

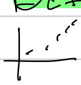
Proposition: A bounded sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$ iff every convergent subsequence converges to x.

Bolzano-Weierstrass Theorem: Purely Geometric Theorem/Idea

Every bounded sequence of real numbers must have a convergent subsequence.

Proof: We have shown that any bounded sequence of real numbers must exhibit a subsequence that converges to its limit superior $\blacksquare$

Def: We say that a sequence $\{x_n\}$ diverges to infinity if, for every $K \in \mathbb{R}$, there is some M such that $x_n > K$ whenever $n \geq M$. In this case we write $\lim_{n\to\infty} x_n := \infty$.



Similarly, we say that a sequence $\{x_n\}$ diverges to negative infinity if, for every $K \in \mathbb{R}$, there is some M such that $x_n < K$ whenever $n \geq M$. In this case, we write $\lim_{n\to\infty} x_n = -\infty$.

Proposition: Suppose $\{x_n\}_{n=1}^{\infty}$ is a monotone and unbounded sequence. Then
$$\lim_{n\to\infty} x_n = \begin{cases} \infty & \text{if } \{x_n\} \text{ is increasing} \\ -\infty & \text{if } \{x_n\} \text{ is decreasing} \end{cases}$$

Examples: (i) $\{n^2\}$ diverges to ∞

(ii) $\{-n\}$ diverges to $-\infty$

(iii) $\{n + \frac{1}{n}\}$

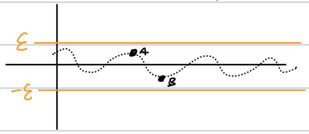
Def Let $\{x_n\}$ be unbounded of real numbers. Define the sequence of extended real numbers $\{a_n\}$ & $\{b_n\}$ by

$$a_n = \sup\{x_k : k \geq n\}, \quad b_n = \inf\{x_k : k \geq n\}$$

If each a_n & b_n is a real number, then $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n$ & $\liminf x_n = \lim_{n\to\infty} b_n$.

Cauchy Sequences

Def: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\{x_n\}$ is said to be a **Cauchy sequence** if, for any $\varepsilon > 0$, there is some $M \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ whenever $n, m \geq M$.



$|A - B| < \varepsilon$

Examples: (i) Given any $c \in \mathbb{R}$, the sequence $\{c, c, c, \ldots\}$ is a Cauchy sequence

(ii) $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a Cauchy sequence

(iii) $\{(-1)^n\}_{n=1}^{\infty}$ is <u>not</u> a Cauchy sequence

Proposition Cauchy sequences are bounded

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. For $\varepsilon = 1$, we may select an $M \in \mathbb{N}$ such that $|x_m - x_n| < 1$ whenever $n, m \geq M$. For those such $n \geq M$, the reverse triangle inequality implies

$$|x_n| - |x_M| \leq |x_n - x_M| < 1.$$

Therefore, for any $n \geq M$, $|x_n| < 1 + |x_M|$. Define $B := \max\{|x_1|, |x_2|, \ldots |x_M|, 1 + |x_M|\}$. Then, $|x_n| \leq B$ for every $n \in \mathbb{N}$, as required. ∎

Theorem (completeness) A sequence of real numbers is convergent iff it is Cauchy.

Proof: Suppose $\{x_n\}_{n=1}^{\infty}$ converges to an $x \in \mathbb{R}$. For each $\varepsilon > 0$, choose $M \in \mathbb{N}$ for which $|x_n - x| < \varepsilon/2$ whenever $n \geq M$. Then for all pairs $n, m \geq M$,

$$|x_m - x_n| = |(x_m - x) - (x_n - x)| \leq |x_m - x| + |x_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence.

Conversely, assume that $\{x_n\}$ is a Cauchy sequence. We need to show that $\{x_n\}$ converges. As we have just shown, $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. Therefore, $a := \limsup_{n\to\infty} x_n$ & $b := \liminf_{n\to\infty} x_n$ both exist. We have proven a theorem that guarantees the existence of subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ and $\{x_{m_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that

$\lim_{k\to\infty} x_{n_k} = a$ & $\lim_{k\to\infty} x_{m_k} = b$, End Goal: $a - b = 0$

when $k \geq M_1$, so is x_{n_k}

Let $\varepsilon > 0$ be given. Choose $M_1 \in \mathbb{N}$ such that $|x_{n_k} - a| < \varepsilon/3$ whenever $k \geq M_1$. Choose $M_2 \in \mathbb{N}$ for which $|x_{m_k} - b| < \varepsilon/3$ whenever $k \geq M_2$. Using the fact that $\{x_n\}_{n=1}^{\infty}$ is Cauchy, we have the sequence of an $M_3 \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon/3$ whenever $n, m \geq M_3$. Set $M := \max\{M_1, M_2, M_3\}$. If $k \geq M$, then $m_k, n_k \geq M$. Therefore, for any $k \geq M$,

$$|a - b| = |a - x_{n_k} + x_{n_k} + x_{m_k} - x_{m_k} - b| \leq |a - x_{n_k}| + |b - x_{m_k}| + |x_{n_k} - x_{m_k}| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, necessarily $a = b$ and thus

$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, as required ∎

Series

Def: Given a sequence $\{x_n\}_{n\geq 1}^{\infty}$ we write the formal object
$\sum_{n=0}^{\infty} x_n$, or lazily $\sum_1^{\infty} x_n$ and call it a series.

A series $\sum_1^{\infty} x_n$ converges (or is convergent) if the sequence $\{S_k\}_{k=1}^{\infty}$ given by $S_k := \sum_{n=1}^{k} x_n = x_1 + x_2 + \dots + x_k$ converges. The numbers S_k are called partial sums. If $\sum_1^{\infty} x_n$ should converge, we write

$$\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{n=1}^{k} x_n$$

If the sequence $\{S_k\}_{k=1}^{\infty}$ diverges, then we say that $\sum_1^{\infty} x_n$ diverges (or is divergent)

Example: The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to 1, that is: $\sum_1^{\infty} \frac{1}{2^n} = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{2^n} = 1$

Proof For each $k \in \mathbb{N}$ the identity $\sum_{n=1}^{k} \frac{1}{2^n} + \frac{1}{2^k} = 1$ holds. Set for each $k \in \mathbb{N}$, $S_k = 1/2 + 1/4 + \dots + 1/2^k$. Then we can write $|1 - S_k| = \left|1 - \sum_{n=1}^{k} \frac{1}{2^n}\right| = \left|\frac{1}{2^k} + \sum_{n=1}^{k} \frac{1}{2^n} - \sum_{n=1}^{k} \frac{1}{2^n}\right| = |1/2^k|$ for each $k \in \mathbb{N}$. The sequence $\{\frac{1}{2^k}\}_{k=1}^{\infty}$ converges to zero. Therefore $\{|1 - S_k|\}_{k=1}^{\infty}$ also converges to zero. Consequently $\{S_k\}_{k=1}^{\infty}$ must converge to 1, as required. ▨

Proposition: Let $r \in \mathbb{R}$. The geometric series $\sum_{n=0}^{\infty} r^n$ converges iff $-1 < r < 1$.

Proof For each $k \in \mathbb{N}$ verify that $\sum_{n=0}^{\infty} r^n = \frac{1 - r^k}{1 - r}$. Can use induction then show that the series converges. This is a homework assignment question. ▨

Proposition A series $\sum_1^{\infty} x_n$ converges iff its tails converge. i.e. for $M \in \mathbb{N}$, the series $\sum_{n=M}^{\infty} x_n$ converges

Proof For $k \geq M$, write $\sum_{n=1}^{k} = \sum_{n=1}^{M-1} x_n + \sum_{n=M}^{k} x_n$. The expression $\sum_1^{M-1} x_n$ is a fixed number, so we can finish the proof using properties & results about sequences. ▨

Def: A series $\sum_1^{\infty} x_n$ is said to be Cauchy if the sequence of partial sums $\{S_k\}_{k=1}^{\infty}$ is a Cauchy sequence:
for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that $\left| \underbrace{\sum_{k=1}^{m} x_n}_{S_m} - \underbrace{\sum_{k=1}^{k} x_n}_{S_k} \right| < \varepsilon$ whenever $m, k \geq M$.

Without loss of generality, we may suppose that $k > m$. Then we may write $\left|\sum_{n=m+1}^{k} x_n\right| < \varepsilon$ for all $k, m \geq M$.

Proposition: A series is Cauchy if and only if for every $\varepsilon > 0$ there is an $M \in \mathbb{N}$ such that $\left|\sum_{n=m+1}^{k} x_n\right| < \varepsilon$ whenever $k, m \geq M$.

Example: If $r \geq 1$ or $r \leq -1$, the geometric series $\sum_0^{\infty} r^n$ diverges.

Proof For each $n \in \mathbb{N}$, we know that $|r|^n \geq |\pm 1|^n = 1$. Necessarily, $\sum_0^{\infty} r^n$ is not Cauchy, so it cannot converge. ▨

Proposition Let $\sum_1^\infty x_n$ be a convergent series. Then the sequence of terms $\{x_n\}_1^\infty$ converges and in fact $\lim_{n\to\infty} x_n = 0$. Careful, Converse is not true

Proof: As $\sum_1^\infty x_n$ is convergent, it must also be cauchy. Given $\varepsilon > 0$, there exists some $M \geq 0$ such that $|x_{n+1}| = |\sum_{i=n+1}^{n+1} x_i| < \varepsilon$ whenever $n \geq M$.

$x_{n+1} = \sum_{i=1}^{n+1} x_i - \sum_{i=1}^{n} x_i = $ ↗ ↖ $x_1 - x_1 + x_2 - x_2 + \dots + x_n - x_n + x_{n+1}$

Example: The harmonic series $\sum_1^\infty \frac{1}{n}$ diverges despite the fact that the series of its terms converge to zero

Proof It is sufficient to show that the partial sums are unbounded. We consider the partial sums $n = 2^k$:

$S_1 = 1$ $\quad S_4 = (1+\frac{1}{2}) + (\frac{1}{3}+\frac{1}{4})$

$S_2 = 1 + 1/2$ $\quad S_8 = (1+\frac{1}{2}) + (\frac{1}{3}+\frac{1}{4}) + (\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})$ $\dots$ $S_{2^k} = 1 + \sum_{i=1}^{k}\left[\sum_{m=2^{i-1}+1}^{2^i} \frac{1}{m}\right]$

Notice that $1/3 + 1/4 \geq 1/4$, $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} \geq \frac{1}{8}$ $\dots$ $\sum_{m=2^{k-1}+1}^{2^k} \frac{1}{m} \geq \sum_{m=2^{k-1}+1}^{2^k} \frac{1}{2^k} = 2^{k-1} \cdot \frac{1}{2^k} = \frac{1}{2}$.

Therefore, $S_{2^k} = 1 + \sum_{i=1}^{k}\left[\sum_{m=2^{i-1}+1}^{2^i} \frac{1}{m}\right] \geq 1 + \sum_{i=1}^{k} \frac{1}{2} = 1 + \frac{k}{2}$

The sequence $\{S_{2^k}\}_{n=1}^\infty = \{\frac{k}{2}\}_{k=1}^\infty$ is unbounded by the archimedean property. ∎

Property (Linearity of Series) Let $\alpha \in \mathbb{R}$, and suppose $\sum_1^\infty x_n$ & $\sum_1^\infty y_n$ are convergent.

(i) $\sum_1^\infty (\alpha x_n)$ is convergent and $\sum_1^\infty (\alpha x_n) = \alpha \sum_1^\infty x_n$

(ii) $\sum_1^\infty (x_n + y_n)$ is convergent and $\sum_1^\infty (x_n + y_n) = \sum_1^\infty x_n + \sum_1^\infty y_n$

Proposition If $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum_1^\infty x_n$ converges iff the sequence of partial sums is bounded above

Def: A series $\sum_1^\infty x_n$ converges absolutely if $\sum_1^\infty |x_n|$ converges. If a series converges, but doesn't converge absolutely, we say that it converges conditionally.

Proposition If $\sum_1^\infty x_n$ converges absolutely, then it also converges (conditionally).

Proof: A series converges iff it is Cauchy. Therefore, as $\sum_1^\infty |x_n|$ is convergent it is also Cauchy. Given $\varepsilon > 0$, there is an $M \in \mathbb{N}$ such that $\sum_{i=K+1}^{n} |x_i| = \left|\sum |x_i|\right| < \varepsilon$ whenever $K \geq M$ and $n > K$.

Applying the triangle inequality for finite sums,

$$\left|\sum_{i=K+1}^{n} x_n\right| \leq \sum_{i=K+1}^{n} |x_i| < \varepsilon \quad \text{whenever } K \geq M.$$

∎

Hence, $\sum_1^\infty x_n$ is Cauchy as required

Remark: The converse statement is not true

Comparison Test: Let $\sum_1^\infty x_n$ & $\sum_1^\infty y_n$ be a pair of series such that $0 \le x_n \le y_n$ for all $n \in \mathbb{N}$

(i) If $\sum_1^\infty y_n$ converges, then $\sum_1^\infty x_n$ converges

(ii) If $\sum_1^\infty x_n$ diverges, then $\sum_1^\infty y_n$ diverges

Proof: Because $x_n, y_n \ge 0$ for every n, the partial sums are monotonically increasing. Because $x_n \le y_n$ for every n, the partial sums also satisfy $\sum_1^k x_n \le \sum_1^k y_n$ (✱) for each $k \in \mathbb{N}$. Now suppose that $\sum_1^\infty y_n$ converges. This is equivalent to the sequence of the partial sums $\{\sum_1^k y_n\}_{k=1}^\infty$ converging. Therefore there must be some $B \ge 0$ such that $\sum_1^k y_k \le B$ for every $k \in \mathbb{N}$. By (✱), we ✓ that $\sum_1^k x_n \le B$ for every $k \in \mathbb{N}$. Thus, $\{\sum_1^k x_n\}_{k=1}^\infty$ is a monotonically increasing sequence that is bounded from above. By MCT, it must thus converge, & so $\sum_1^\infty x_n$ thus converges.

On the other hand, if $\sum_1^\infty x_n$ diverges, the sequence of partial sums is unbounded, $\{\sum_1^k x_n\}_{k=1}^\infty$ is unbounded. By (✱) it follows that $\{\sum_1^k y_n\}_{k=1}^\infty$ is also unbounded.

P-series test: For $p \in \mathbb{R}$, the series given by $\sum \frac{1}{n^p}$ converges iff $p > 1$.

Example: The series $\sum_1^\infty \frac{1}{n^2+1}$ converges

Proof: For each $n \in \mathbb{N}$, $\frac{1}{n^2+1} \le \frac{1}{n^2}$, According to the p-series test, $\sum_1^\infty \frac{1}{n^2}$ converges. By the comparison test, $\sum_1^\infty \frac{1}{n^2+1}$ also converges. ∎

Ratio Test Let $\sum_1^\infty x_n$ be a series such that $x_n \ne 0$ for every $n \in \mathbb{N}$, & such that

$$L := \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} \text{ exists}$$

(i) If $L < 1$, then $\sum_1^\infty x_n$ converges absolutely

(ii) If $L > 1$, then $\sum_1^\infty x_n$ diverges

(iii) If $L = 1$, then the test is void/inconclusive

Proof If $L > 1$, the ratio test for sequences tells us that $\{x_n\}_{n=1}^\infty$ diverges, whereby $\sum_1^\infty x_n$ must diverge.

Suppose $L < 1$. Automatically, we know that $L \ge 0$. Choose $r \in (L, 1)$. Because $r - L > 0$, we may use the fact $L = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$ to find an $M \in \mathbb{N}$ sufficiently large such that

$$\frac{|x_{n+1}|}{|x_n|} - L \le \left| \frac{|x_{n+1}|}{|x_n|} - L \right| < r - L \quad \text{whenever } n \ge M$$

Therefore, for each $n \ge M$, $\frac{|x_{n+1}|}{|x_n|} < r$. For those $n \ge M+1$, write $|x_n| < [|x_M| \cdot r^{-M}] \cdot r^n$. For each $k > M$ the partial sums

$$\sum_1^k |x_n| = \sum_1^M |x_n| + \sum_{M+1}^k |x_n| = \sum_{n=1}^M |x_n| + \sum_{n=M+1}^k [|x_M| \cdot r^{-M}] \cdot r^n$$

$$= \underbrace{\sum_{n=1}^{M} |x_n|}_{\text{a number}} + |x_M| \cdot r^{-M} \sum_{n=M+1}^k r^n$$

$$\le \sum_{n=1}^M |x_n| + |x_M| \cdot r^{-M} \sum_1^\infty r^n$$

Finally, $\sum_1^M |x_n|$ & $|x_M| \cdot r^{-M}$ are independent of k and $\sum_1^\infty r^n$ is a geometric series so $\sum_1^\infty |x_n|$ must converge absolutely. ∎

Chapter 3

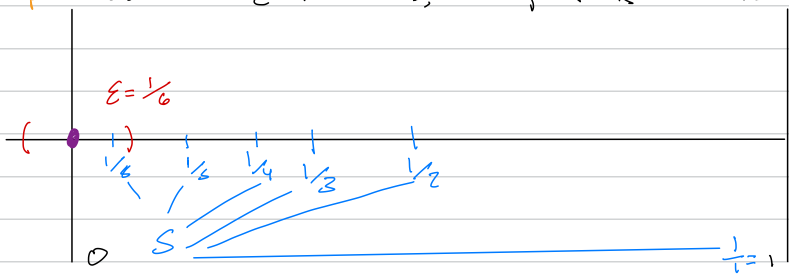
Limits of Functions

Also referred to as a limit point

Def: Let $S \subset \mathbb{R}$ non-empty. A number $x \in \mathbb{R}$ is called a cluster point of S if, for every $\varepsilon > 0$, the set $(x-\varepsilon, x+\varepsilon) \cap [S \setminus \{x\}]$ is nonempty. In other words, x is a cluster point of S, if for every $\varepsilon > 0$, there is some $y \in S$, with $y \neq x$, such that $|y-x| < \varepsilon$, meaning that $y \in (x-\varepsilon, x+\varepsilon) \cap [S \setminus \{x\}]$

Example: Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. The point $x=0$ is a cluster point of S.

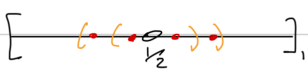
This follows from the archimedean property



Example: Every pt of $[0,1]$ is a cluster pt $(0,1)$

Example: The cluster points of $\mathbb{Q}$ are all the points in $\mathbb{R}$. This follows from density

Example:



If we include -1 into the sets -1 is not a cluster point as it stands alone !!

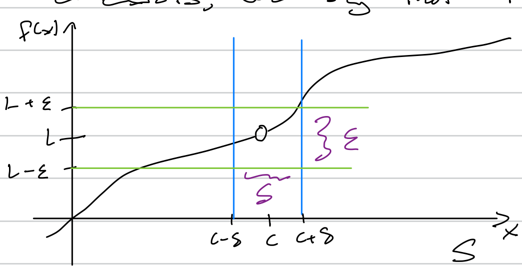
$1/2$ is a cluster pt $S = [0, 1/2) \cup (1/2, 1]$

Proposition: Let S be a subset of $\mathbb{R}$ and $c \in \mathbb{R}$, then c is a cluster point of S iff there is a sequence $\{x_n\}_{n\geq 1}^{\infty}$ with $x_n \in S \setminus \{c\}$ such that $\lim_{n\to\infty} x_n = c$.

Proof: Suppose that c is a cluster point. Then choosing $\varepsilon = \frac{1}{n}$ for each n, there must be a point $x_n \in S$, $x_n \neq c$, such that $|x_n - c| < \frac{1}{n}$. Because $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to zero, necessarily, the sequence $\{x_n\}_{n\geq 1}^{\infty}$ must converge to c.

Suppose that $\{x_n\}$, $x_n \in S \setminus \{c\}$ is a sequence converging to c. Given any $\varepsilon > 0$, there is an $M \in \mathbb{N}$ such that $|x_n - c| < \varepsilon$ whenever $n \geq M$. In particular, $x_M \in (c-\varepsilon, c+\varepsilon) \cap [S \setminus \{c\}]$ and so c is a cluster point of S. ▨

Def: Let $f: S \to \mathbb{R}$, with $S \subset \mathbb{R}$ be non-empty, and suppose that $c \in \mathbb{R}$ is a cluster point of S. Suppose $L \in \mathbb{R}$ is such that, for every $\varepsilon > 0$, there exists some $\delta > 0$ for which $|f(x) - L| < \varepsilon$ holds whenever $x \in S \setminus \{c\}$ satisfies $|x-c| < \delta$. We then say that $f(x)$ converges to L as $x \to c$, and we write $f(x) \to L$ as $x \to c$. We call this L a limit of $f(x)$ as x goes to c, and if L is unique, we write $\lim_{x\to c} f(x) = L$. If no such L exists, we say that f diverges at c.



Showing this box should get closer and closer to a specific value, but the function doesn't need to be defined there.

Box gets smaller and smaller, squeezing down and down, no where to run

Proposition: Let c be a cluster point of $S \subset \mathbb{R}$ and let $f: S \to \mathbb{R}$ be a function such that $f(x)$ converges as x goes to c. Then the limit of $f(x)$ as $x \to c$ is unique.

Usual trick is to assume two points and show they must be equal.

Proof: Suppose that L_1 & L_2 are two such limits. Let $\varepsilon > 0$ be given. As $f(x) \to L_1$ as $x \to c$, there must be a $\delta_1 > 0$ such that $|f(x) - L_1| < \varepsilon/2$ whenever $x \in S \setminus \{c\}$ such that $|x - c| < \delta_1$. Choose $\delta_2 > 0$ such that $|f(x) - L_2| < \varepsilon/2$ whenever $x \in S \setminus \{c\}$ satisfies $|x-c| < \delta_2$. Set $\delta = \min\{\delta_1, \delta_2\}$.

Suppose that $x \in S \setminus \{c\}$ is such that $|x - c| < \delta$; such an x exists because $\delta > 0$ and c is a cluster point of S. Then

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2|$$
$$\leq |f(x) - L_1| + |f(x) - L_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we know that $L_1 = L_2$, as required. ∎

Example: Consider the function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. For any $c \in \mathbb{R}$, $\lim_{x \to c} f(x) = c^2$.

Proof: Let $\varepsilon > 0$ be given and fix $c \in \mathbb{R}$. Set $\delta = \min\{1, \frac{\varepsilon}{2|c|+1}\} > 0$. Take any $x \neq c$ such that $|x - c| < \delta$. In particular, $|x - c| < 1$. By the reverse triangle inequality, we know $|x| - |c| \leq |x - c| < 1$. Add $2|c|$ to both sides, we get $|x| + |c| < 2|c| + 1$ for any $x \in S \setminus \{c\}$ satisfying $|x - c| < \delta$. Estimate

$$|f(x) - c^2| = |x^2 - c^2| = |(x+c)(x-c)| = |x+c| \cdot |x-c| \leq [|x| + |c|] \cdot |x - c| \leq [2|c|+1] \cdot |x-c| < [2|c|+1] \frac{\varepsilon}{2|c|+1} = \varepsilon$$ ∎

Sequential Convergence

Proposition: Let $S \subset \mathbb{R}$, c be a cluster point of S, $f: S \to \mathbb{R}$ be a function, and $L \in \mathbb{R}$. Then $f(x) \to L$ as $x \to c$ iff for every sequence $\{x_n\}$ such that $x_n \in S \setminus \{c\}$ for all n, and such that $\lim_{n\to\infty} x_n = c$ we have that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

Further, $S \subset \mathbb{R}$, c a cluster point of S, $f: S \to \mathbb{R}$ & $L \in \mathbb{R}$. Then $f(x) \to L$ as $x \to c$

(*) if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in S \setminus \{c\}$ such that $|x - c| < \delta$.

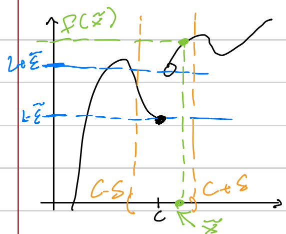
Guaranteed to exist as per prior proof

Proof: Suppose $f(x) \to L$ as $x \to c$, and let $\{x_n\}$ be a sequence as given above. We must show that $\{f(x_n)\}_{n=1}^{\infty}$ converges to L. Let $\varepsilon > 0$ be given. Choose a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ holds whenever $x \in S \setminus \{c\}$ satisfies $|x - c| < \delta$. As $\{x_n\}_{n=1}^{\infty}$ converges to c, there is a $M \in \mathbb{N}$ for which $|x_n - c| < \delta$ whenever $n \geq M$. Thus, for any $n \geq M$ $|f(x_n) - L| < \varepsilon$, as required.

We must show that $\{f(x_n)\}_{n=1}^{\infty}$ converges to L. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in S \setminus \{c\}$ satisfies $|x - c| < \delta$. Using the fact that $\lim_{n\to\infty} x_n = c$, we can find an $M \in \mathbb{N}$ such that $|x_n - c| < \delta$ whenever $n \geq M$. Therefore, for each $n \geq M$, we know that $|f(x_n) - L| < \varepsilon$ as required.

Proof of Reverse Direction of Prop

Proof by contrapositive



• The reverse direction will be proved by contrapositive. Suppose that $f(x) \not\to L$ as $x \to c$. The negation of the definition of convergence of a function (*) is
• that there exists an $\tilde{\varepsilon} > 0$ such that for every $\delta > 0$, there must be an $\tilde{x} \in S \setminus \{c\}$ with $|\tilde{x} - c| < \delta$ such that $|f(\tilde{x}) - L| \geq \tilde{\varepsilon}$. We will use $\delta = \frac{1}{n}$, & $n \in \mathbb{N}$ to construct the sequence. Using the fact that c is a cluster point of S, there must be some $x_n \in (c - \frac{1}{n}, c + \frac{1}{n}) \cap [S \setminus \{c\}]$ for every $n \in \mathbb{N}$. In other words, this sequence $\{x_n\}$ satisfies that $|x_n - c| < 1/n$ & $x_n \in S \setminus \{c\}$ for every $n \in \mathbb{N}$. We may also require that $|f(x_n) - L| \geq \varepsilon$ for each $n \in \mathbb{N}$. Therefore the sequence $\{f(x_n)\}_{n=1}^{\infty}$ cannot converge to L. $\blacksquare$

Examples: (i) $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ DNE

(ii) $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$

Proofs

(i): Define $\{x_n\}_{n=1}^{\infty}$ by $x_n = \frac{1}{n\pi + \frac{\pi}{2}}$. Now $\lim_{n \to \infty} \frac{1}{n\pi + \pi/2} = 0$. However for each $n \in \mathbb{N}$, $\sin\left(\frac{1}{x_n}\right) = \sin\left(n\pi + \pi/2\right) = (-1)^n$. Therefore $\{\sin(x_n)\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$ does not converge. By the previous proposition, $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist $\blacksquare$

(ii) The domain $f(x) = x\sin\left(\frac{1}{x}\right)$ is $S = \mathbb{R} \setminus \{0\}$ and $c = 0$ is a cluster point of this sets. To verify the limit it is enough to consider any sequence $\{x_n\}$, $x_n \in S \setminus \{0\}$, converging to 0. We need to show $\{f(x_n)\}_{n=1}^{\infty}$ converges to 0. For each $x \in S$, we know that $\left|x_n \sin\left(\frac{1}{x_n}\right)\right| = |x| \cdot \left|\sin\left(\frac{1}{x_n}\right)\right| \leq |x|$. As $\{x_n\}_{n=1}^{\infty}$ converges to 0, necessarily from this estimate we know $\{f(x_n)\}_{n=1}^{\infty}$ converges to 0 as required. $\blacksquare$

Corollary: Let $S \subset \mathbb{R}$ and c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ is a function such that the limit of $f(x)$ as $x \to c$ exists. Suppose there are two real numbers $a, b \in \mathbb{R}$ with $a \leq f(x) \leq b$ for all $x \in S \setminus \{c\}$. Then $a \leq \lim_{x \to c} f(x) \leq b$.

Corollary: Let $S \subset \mathbb{R}$ & let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ & $g : S \to \mathbb{R}$ are functions such that the limits as $x \to c$ both exist. If

$$f(x) \leq g(x) \text{ holds for every } x \in S \setminus \{c\} \text{ then } \lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$$

Proof: Choose an arbitrary sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S \setminus \{c\}$ and $\lim_{n \to \infty} x_n = c$. Define $L_1 = \lim_{x \to c} f(x)$ and $L_2 = \lim_{x \to c} g(x)$. Sequential convergence guarantees that $\{f(x_n)\}_{n=1}^{\infty}$ converges to L_1 and $\{g(x_n)\}_{n=1}^{\infty}$ converges to L_2. By assumption, we know that $f(x_n) \leq g(x_n)$ for every $n \in \mathbb{N}$. Therefore,

$$L_1 = \lim_{n \to \infty} f(x_n) \leq \lim_{n \to \infty} g(x_n) = L_2$$

$\blacksquare$

Corollary: Let $S \subset \mathbb{R}$ & c be a cluster point of S. Suppose $f: S \to \mathbb{R}$ and $g: S \to \mathbb{R}$ are functions such that their limits as $x \to c$ both exist. Then

(i) $\lim\limits_{x\to c}[f(x) \pm g(x)] = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x)$

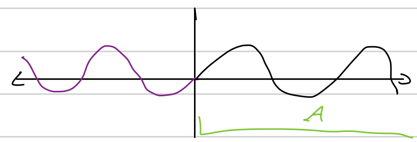
(ii) $\lim\limits_{x\to c}[f(x) g(x)] = [\lim\limits_{x\to c} f(x)] \cdot [\lim\limits_{x\to c} g(x)]$

(iii) $\lim\limits_{x\to c}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$ provided $g(x) \neq 0$ $\forall x$ and $\lim\limits_{x\to c} g(x) \neq 0$

(iv) $\lim\limits_{x\to c} |f(x)| = |\lim\limits_{x\to c} f(x)|$

Def: Let $f: S \to \mathbb{R}$ be a function and let $A \subset S \subset \mathbb{R}$.
Define the function $f|_A : A \to \mathbb{R}$ by $f|_A(x) = f(x)$ for $x \in A$.
We call $f|_A$ the restriction of f to A.

Example: $f(x) = \sin(x)$ then $f|_A = \sin(x), x \geq 0$
$S = \mathbb{R}$, $A = [0, \infty)$

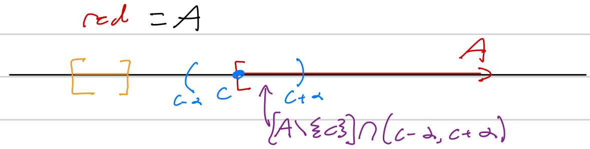


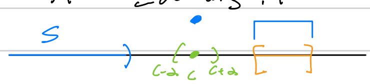
Proposition Let $S \subset \mathbb{R}$, $c \in \mathbb{R}$, & $f: S \to \mathbb{R}$ be a function. Suppose $A \subset S$ is such that there is some $\alpha > 0$ satisfying $[A \setminus \{c\}] \cap (c-\alpha, c+\alpha) = [S \setminus \{c\}] \cap (c-\alpha, c+\alpha)$. Then

(i) The point c is a cluster point of A if and only if c is a cluster point of S.

(ii) Supposing c is a cluster point of S, then $f(x) \to L$ as $x \to c$ if and only if $f|_A \to L$ as $x \to c$.

orange + red = S
red = A





Proof: (i) Suppose that c is a cluster point of A. This means that for every $\varepsilon > 0$, the set $(c-\varepsilon, c+\varepsilon) \cap [A \setminus \{c\}]$ is nonempty. As $A \subset S$, necessarily the sets $(c-\varepsilon, c+\varepsilon) \cap [S \setminus \{c\}]$ are nonempty for every $\varepsilon > 0$. So c is a cluster point of S.

Suppose now that c is a cluster point of S. Then for any $\varepsilon \in (0, \alpha]$, we know that $[A \setminus \{c\}] \cap (c-\varepsilon, c+\varepsilon) = [S \setminus \{c\}] \cap (c-\varepsilon, c+\varepsilon)$ is a non-empty set. Therefore the set $[A \setminus \{c\}] \cap (c-\varepsilon, c+\varepsilon)$ is nonempty for every $\varepsilon > 0$. Thus c is a cluster point of A. $\blacksquare$

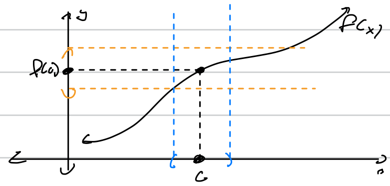
(ii) Assume c is a cluster point of S (and by (i) also a cluster point of A). Suppose that $f(x) \to L$ as $x \to c$. As $A \subset S$, if $x \in A \setminus \{c\}$, then $x \in S \setminus \{c\}$, and so $f|_A(x) := f(x) \to L$, $x \in A$ as $x \to c$. This result is immediate.

Next Assume $f|_A \to L$ as $x \to c$. Let $\varepsilon > 0$ be given. There must be a $\delta' > 0$ such that $|f|_A(x) - L| < \varepsilon$ whenever $x \in A \setminus \{c\}$ satisfies $|x-c| < \delta'$. Take $\delta = \min\{\delta', \alpha\}$. Now suppose $x \in A \setminus \{c\}$ is such that $|x-c| < \delta$. Then $|f(x) - L| = |f|_A(x) - L| < \varepsilon$, as required. $\blacksquare$

Continuous Functions

Def: Suppose $S \subset \mathbb{R}$ and $c \in S$. We say $f: S \to \mathbb{R}$ is continuous at c if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $x \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \varepsilon$. When $f: S \to \mathbb{R}$ is continuous at all $c \in S$, then we say that f is a continuous function.

If f is continuous for all $c \in A$, we say f is continuous on $A \subset S$. This implies that $f|_A$ is continuous, but the converse does not hold.



Remark: If $f: S \to \mathbb{R}$, $A \subset S$, is continuous, then $f|_A$ is also continuous. The converse is false.

Proposition: Consider a function $f: S \to \mathbb{R}$ defined on a set $S \subset \mathbb{R}$ and let $c \in S$. Then

(i) If c is not a cluster point of S, then f is continuous at c.

(ii) If c is a cluster point of S, then f is cts. at c iff the limit of $f(x)$ as $x \to c$ exists and
$$\lim_{x \to c} f(x) = f(c).$$

(iii) The function f is cts. at c iff for every sequence $\{x_n\}_{n=1}^{\infty}$ where $x_n \in S$ and $\lim_{n\to\infty} x_n = c$, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(c)$.

Proof (i): Suppose c is not a cluster point of S. Then there exists a $\delta > 0$ such that $S \cap (c - \delta, c + \delta) = \{c\}$. For any $\varepsilon > 0$, simply pick this given δ. The only $x \in S$ such that $|x - c| < \delta$ is $x = c$. Then $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon$. $(\cdot)$ δ

(ii): Suppose c is a cluster point of S. Let us first suppose that $\lim_{x \to c} f(x) = f(c)$. Then for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $x \in S \setminus \{c\}$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. Also $|f(c) - f(c)| = 0 < \varepsilon$, so the definition of continuity at c is satisfied. On the other hand, suppose f is cts. at c. For every $\varepsilon > 0$, there exists a $\delta > 0$ such that for $x \in S$ where $|x - c| < \delta$, we have $|f(x) - f(c)| < \varepsilon$. Then the statement is, of course, still true if $x \in S \setminus \{c\} \subset S$. Therefore $\lim_{x \to c} f(x) = f(c)$.

(iii): First suppose f is cts. at c. Let $\{x_n\}$ be a sequence such that $x_n \in S$ and $\lim_{n\to\infty} x_n = c$. Let $\varepsilon > 0$ be given. Find a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ for all $x \in S$ where $|x - c| < \delta$. Find an $M \in \mathbb{N}$ such that for $n \geq M$, we have $|x_n - c| < \delta$. Then for $n \geq M$, we have that $|f(x_n) - f(c)| < \varepsilon$, so $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(c)$.

We prove the other direction by contrapositive. Suppose f is not cts. at c. Then there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there exists an $x \in S$ such that $|x - c| < \delta$ and $|f(x) - f(c)| \geq \varepsilon$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ as follows. Let $x_n \in S$ be such that $|x_n - c| < \frac{1}{n}$ and $|f(x_n) - f(c)| \geq \varepsilon$. Now $\{x_n\}$ is a sequence in S such that $\lim_{n\to\infty} x_n = c$ and such that $|f(x_n) - f(c)| \geq \varepsilon$ for all $n \in \mathbb{N}$. Thus $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to $f(c)$. It may generally converge or not converge, but it definitely doesn't converge to $f(c)$. $\boxtimes$

Remark: Item (iii) allows us to quickly apply what we know about limits of sequences to cts. functions and even prove that certain functions are cts.

Example: The function $f: (0, \infty) \to \mathbb{R}$ defined by $f(x) := \frac{1}{x}$ is cts.

c is a cluster point because there's always something despite shrinking

Proof Fix $c \in (0, \infty)$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $(0, \infty)$ such that $\lim_{n\to\infty} x_n = c$. Then
$$f(c) = \frac{1}{c} = \frac{1}{\lim_{n\to\infty} x_n} = \lim_{n\to\infty} \frac{1}{x_n} = \lim_{n\to\infty} f(x_n).$$
Thus f is cts. at c. As f is continuous at all $c \in (0, \infty)$, f is cts. $\boxtimes$

Proposition Let $f: \mathbb{R} \to \mathbb{R}$ be a polynomial. That is,

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0$$

for some constants $a_0, a_1, \dots a_d$. Then f is continuous.

Proof Fix $c \in \mathbb{R}$. Let $\{x_n\}$ be a sequence such that $\lim_{n\to\infty} x_n = c$. Then

$$\begin{aligned} f(c) &= a_d c^d + a_{d-1} c^{d-1} + \dots + a_1 c + a_0 \\ &= a_d \left(\lim_{n\to\infty} x_n\right)^d + a_{d-1}\left(\lim_{n\to\infty} x_n\right)^{d-1} + \dots + a_1\left(\lim_{n\to\infty} x_n\right) + a_0 \\ &= \lim_{n\to\infty}\left(a_d x_n^d + a_{d-1} x_n^{d-1} + \dots + a_1 x_n + a_0\right) = \lim_{n\to\infty} f(x_n) \end{aligned}$$

Thus f is cts. at c. As f is cts. at all $c \in \mathbb{R}$, f is cts.

Proposition Let $f: S \to \mathbb{R}$ and $g: S \to \mathbb{R}$ be functions cts. at $c \in S$.

(i) The function $h: S \to \mathbb{R}$ defined $h(x) := f(x) \pm g(x)$ is cts. at c

(ii) The function $h: S \to \mathbb{R}$ defined $h(x) := f(x)g(x)$ is cts. at c

(iii) If $g(x) \neq 0$ for all $x \in S$, the function $h: S \to \mathbb{R}$ given by $h(x) := \frac{f(x)}{g(x)}$ is is cts at c.

Example: $\sin(x)$ and $\cos(x)$ are cts. at $\mathbb{R}$

Proof: Let $c \in \mathbb{R}$ be arbitrary. For each $x \in \mathbb{R}$, $|\sin(x)| \le |x|$, $|\cos(x)| \le 1$, & $|\sin(x)| \le 1$

Therefore for every $x \in \mathbb{R}$,

$$\begin{aligned} |\sin(x) - \sin(c)| &= \left|2\sin\left(\tfrac{x-c}{2}\right)\cos\left(\tfrac{x+c}{2}\right)\right| \\ &\le 2\left|\sin\left(\tfrac{x-c}{2}\right)\right| \\ &\le 2\left|\tfrac{x-c}{2}\right| = |x-c| \end{aligned}$$

sum/diff formulas

Cosine for HW

Given $\varepsilon > 0$, choose $\delta = \varepsilon$ to conclude that $|\sin(x) - \sin(c)| \le |x-c| < \delta = \varepsilon$ whenever $|x-c| < \delta$. So $\sin(x)$ is cts at c. Since c is arbitrary, $\sin(x)$ is a continuous function.

Recall: $[f \circ g](x) = f(g(x))$ ⟸ function composition

Proposition Let $A, B \subset \mathbb{R}$ and $f: B \to \mathbb{R}$ and $g: A \to B$ be functions. If g is cts at $c \in A$ and f is cts at $g(c)$, then $f \circ g: A \to \mathbb{R}$ is cts. at c

Proof Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in A such that $\lim_{n\to\infty} x_n = c$. As g is cts at c, we have $\{g(x_n)\}_{n=1}^{\infty}$ converges to $g(c)$. As f is cts. at $g(c)$, we have $\{f(g(x_n))\}_{n=1}^{\infty}$ converges to $f(g(c))$. Thus $f \circ g$ is cts. at c. ∎

Example: Claim: $(\sin(1/x))^2$ is a cts. function on $(0, \infty)$.

on $\mathbb{R}$ too, only interested in this range for this example

Proof The function $1/x$ is cts. on $(0,\infty)$ and $\sin(x)$ is cts. on $(0,\infty)$. Hence the composition $\sin(1/x)$ is cts. Also x^2 is cts. on $[-1, 1]$ (the range of sin). Thus the composition $(\sin(1/x))^2$ is continuous on $(0,\infty)$. ∎

Def: When f is not cts at c, we say f is **discontinuous at c**, or that it has a discontinuity at c.

Proposition Let $f: S \to \mathbb{R}$ be a function and $c \in S$. Suppose there exists a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in S$ for all n, and $\lim_{n\to\infty} x_n = c$ such that $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to $f(c)$. Then f is discts at c.

Example: The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} -1 & \text{if } x<0, \\ 1 & \text{if } x \geq 0 \end{cases}$ is not continuous at 0.

Consider $\{\frac{-1}{n}\}_{n=1}^{\infty}$ which converges to 0. For each $n \in \mathbb{N}$, $f(-\frac{1}{n}) = -1$, however $f(0)=1$. So f is not continuous at 0.

Proof Consider $\{\frac{-1}{n}\}_{n=1}^{\infty}$ which converges to 0. Then $f(\frac{-1}{n}) = -1$ for every n, and so $\lim_{n\to\infty} f(\frac{1}{n}) = 1$ for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} f(\frac{1}{n}) = f(0) = 1$. So $\{f(x_n)\}_{n=1}^{\infty}$ may converge to $f(0)$ for some specific sequence $\{x_n\}_{n=1}^{\infty}$ going to 0, despite the function being discontinuous at 0.

another solution

Finally, consider $f(\frac{(-1)^n}{n}) = (-1)^n$. This sequence diverges. Jumps!

Dirichlet Function

Example: Take $f(x) := \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$. The function is discts at all $c \in \mathbb{R}$.

Proof If c is rational, take a sequence $\{x_n\}_{n=1}^{\infty}$ of irrational numbers such that $\lim_{n\to\infty} x_n = c$. Then $f(x_n) = 0$ and so $\lim_{n\to\infty} f(x_n) = 0$, but $f(c) = 1$. If c is irrational take a sequence of rational numbers $\{x_n\}_{n=1}^{\infty}$ that converges to c. Then $\lim_{n\to\infty} f(x_n) = 1$, but $f(c) = 0$. ∎

Thomae Function / Popcorn Function

Example: Take $f: (0,1) \to \mathbb{R}$ as $f(x) := \begin{cases} 1/k & \text{if } x = m/k, \text{ where } m, k \in \mathbb{N} \text{ and have no common divisors} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Claim: f is cts at all irrational c and discts at all rational c.

Proof Let $c = m/k$ be rational in lowest terms. Take a sequence of irrational numbers $\{x_n\}$ such that $\lim_{n\to\infty} x_n = c$. Then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$, but $f(c) = 1/k \neq 0$. So f is discts at c.

rationals are dense, so this is allowed

Now let c be irrational, so $f(c) = 0$. Take a sequence $\{x_n\}_{n=1}^{\infty}$ in $(0,1)$ such that $\lim_{n\to\infty} x_n = c$. Given $\varepsilon > 0$, find $K \in \mathbb{N}$ such that $1/K < \varepsilon$ by the archimedean property. If $m/k \in (0,1)$ and $m,k \in \mathbb{N}$, then $0 < m < k$. So there are only finitely many rational numbers in $(0,1)$ whose denominator k in lowest terms is less than K. As $\lim_{n\to\infty} x_n = c$, every number not equal to c can appear at most finitely many times in $\{x_n\}_{n=1}^{\infty}$. Hence there is an M such that for $n \geq M$, all numbers x_n that are rational have a denominator larger than or equal to K. Thus for $n \geq M$,
$$|f(x_n) - \underset{f(c)}{0}| = f(x_n) \leq 1/K < \varepsilon.$$
Therefore, f is cts at irrational c.

Example: Define $g: \mathbb{R} \to \mathbb{R}$ by $g(x) := 0$ if $x \neq 0$ and $g(0) := 1$. Then g is not cts. at 0, but cts everywhere else. The point $x=0$ is called a **removable discontinuity**. That is be if we would change the definition of g, by insisting that $g(0)=0$, we would obtain a cts. function.

On the other hand, let f be the function $f(x) = \begin{cases} -1 & \text{if } x<0, \\ 1 & \text{if } x \geq 0 \end{cases}$. Then f does not have a removable discontinuity at 0. No matter how we would define $f(0)$ the function would still not be cts. The difference is that $\lim_{x\to 0} g(x)$ exists while $\lim_{x\to 0} f(x)$ does not.

Further, let $A := \{0\}$, then $g|_A$ is cts, while g is not cts on A. Similarly, if $B := \mathbb{R} \setminus \{0\}$, then $g|_B$ is also cts, and g is in fact cts. on B.

Extreme Value Theorem

Def: $f:[a,b]\to\mathbb{R}$ is bounded if there exists $B\in\mathbb{R}$ such that $|f(x)|\le B$ for every $x\in[a,b]$.

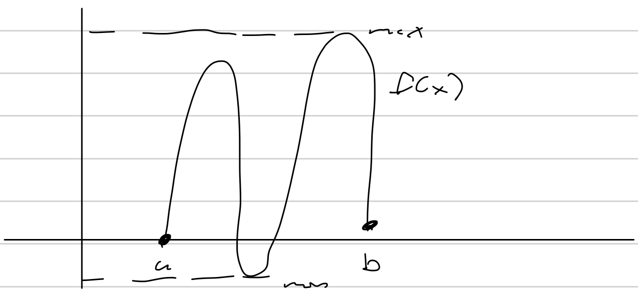
Lemma: A continuous function on a compact interval $f:[a,b]\to\mathbb{R}$ is necessarily bounded

Proof: We proceed by contrapositive. Suppose that f is not bounded. Then, for each $n\in\mathbb{N}$, there is an $x_n\in[a,b]$ for which $|f(x_n)|\ge n$. The sequence $\{x_n\}$ is bounded because $a\le x_n\le b$ for every $n\in\mathbb{N}$. By the Bolzano-Weierstrass Theorem, there must be a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_n\}$. Set $x=\lim_{i\to\infty}x_{n_i}$. Because $a\le x_{n_i}\le b$ for every $i\in\mathbb{N}$, we know that $x\in[a,b]$. The sequence $\{f(x_{n_i})\}_{i=1}^{\infty}$ is unbounded because $|f(x_{n_i})|\ge n_i\ge i$ for every i. Thus, f cannot be cts. at $x\in[a,b]$. $\boxtimes$

Def: (i) $f:S\to\mathbb{R}$ achieves an absolute maximum at $c\in S$ if $f(c)\ge f(x)$ for every $x\in S$.
(ii) $f:S\to\mathbb{R}$ achieves an absolute minimum at $c\in S$ if $f(c)\le f(x)$ for every $x\in S$.

Extreme Value Theorem (EVT)

A cts function $f:[a,b]\to\mathbb{R}$ achieves both an absolute maximum and an absolute minimum on $[a,b]$.



Proof: We proved in the preceding lemma that such a function f is necessarily bounded. Therefore, the set $f([a,b])=\{f(x):x\in[a,b]\}$ is a bounded subset of $\mathbb{R}$, & thus $\inf f([a,b])$ and $\sup f([a,b])$ both exist. That is, there exists sequences $\{f(x_n)\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty}f(x_n)=\inf f([a,b])$ & $\lim_{n\to\infty}f(y_n)=\sup f([a,b])$, where $x_n,y_n\in[a,b]$ for every n. Because $a\le x_n\le b$ and $a\le y_n\le b$, the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are bounded. Apply the Bolzano-Weierstrass theorem (compactness) to extract subsequences $\{x_{n_k}\}_{k=1}^{\infty}$, and $\{y_{n_k}\}_{k=1}^{\infty}$ from $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, respectively, that converge, and set $x=\lim_{k\to\infty}x_{n_k}$ & $y=\lim_{k\to\infty}y_{n_k}$. For each k, we know that $a\le x_{n_k}\le b$ and $a\le y_{n_k}\le b$. Therefore by taking limits as $k\to\infty$, we have that $x,y\in[a,b]$. Since the limit of a subsequence is the same as the limit of the original sequence, continuity guarantees that, $f(x)=\lim_{k\to\infty}f(x_{n_k})=\lim_{n\to\infty}f(x_n)=\inf f([a,b])$, and so $f(x)$ is the minimum $f(x)=\min_{z\in[a,b]}f(z)$. Similar logic holds for the maximum. $\boxtimes$

Remark:

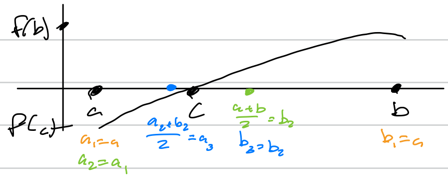
(i) A compact interval $[a,b]$ is essential to the validity of EVT
- $f:(0,\infty)\to\mathbb{R}$, $f(x)=x$
- $f:(0,1)\to\mathbb{R}$, $f(x)=1/x$

(ii) Continuity of f is also essential
- $f(x)=\begin{cases}1/x, & 0<x\le 1\\ 0, & x=0\end{cases}$

Intermediate Value Theorem

Intermediate Value Theorem (IVT)

Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Suppose $y\in\mathbb{R}$ such that $f(a)<y<f(b)$ or $f(a)>y>f(b)$. Then there is some point $c\in(a,b)$ such that $f(c)=y$.



Lemma: Let $f:[a,b]\to\mathbb{R}$ be a cts. function such that $f(a)<0$ and $f(b)\geq 0$. Then there is a $c\in(a,b)$ such that $f(c)=0$

Proof: We define two sequences inductively $\{a_n\}$ & $\{b_n\}$: via bisection method

(i) $a_1=a$ and $b_1=b$

(ii) if $f(\frac{a_n+b_n}{2})\geq 0$, set $a_{n+1}=a_n$ and $b_{n+1}=\frac{a_n+b_n}{2}$

(iii) if $f(\frac{a_n+b_n}{2})<0$, set $a_{n+1}=\frac{a_n+b_n}{2}$ and $b_{n+1}=b_n$

If $a_n<b_n$, then $a_n<\frac{a_n+b_n}{2}<b_n$, so $a_{n+1}<b_{n+1}$. Induction gives that $a_n<b_n$ for all n. Furthermore, $a_n\leq a_{n+1}$ and $b_n\geq b_{n+1}$ for every $n\in\mathbb{N}$. Therefore $\{a_n\}$ & $\{b_n\}$ are monotone sequences, and actually $a\leq a_n<b$ and $a<b_n\leq b$ for every n, so they are also bounded. By the MCT they each must converge to $c=\lim_{n\to\infty}a_n$ and $d=\lim_{n\to\infty}b_n$, and $a<c\leq d<b$. We need to show that $c=d$. For each n, notice that $b_{n+1}-a_{n+1}=\frac{b_n-a_n}{2}$. Proceeding inductively (or iteratively) we get $b_n-a_n=\frac{b_1-a_1}{2^{n-1}}=2^{1-n}(b_1-a_1)=2^{1-n}(b-a)$. As the sequence $\{2^{1-n}(b-a)\}_{n=1}^{\infty}$ converges to 0, we have $d-c=\lim_{n\to\infty}[b_n-a_n]=(b-a)\lim_{n\to\infty}\frac{1}{2^{n-1}}=0$, so $d=c$. By construction, $f(a_n)<0$ and $f(b_n)\geq 0$ for every n. As $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=c$, continuity of f gives: $f(c)=\lim_{n\to\infty}f(a_n)\leq 0$ and $f(c)=\lim_{n\to\infty}f(b_n)\geq 0$. Therefore $f(c)=0$ with $a<c<b$, as required. $\boxtimes$

Proof of IVT: If $f(a)<y<f(b)$, then define $g(x)=f(x)-y$ for $x\in[a,b]$. Then $g:[a,b]\to\mathbb{R}$ is continuous and $g(a)<0<g(b)$. By the previous lemma, there is some $c\in(a,b)$ such that $g(c)=0$, or equivalently $f(c)=y$. The other case is similar. $\boxtimes$

Remark: If $f:S\to\mathbb{R}$ is cts then $f|_A$ for any $A\subset S$ is also cts. In particular, $[a,b]\subset S$, then $f|_{[a,b]}$ is cts, & we can use the IVT.

Corollary: If $f:[a,b]\to\mathbb{R}$ is continuous, then the direct image $f([a,b])$ is a closed and bounded (compact) interval, or is a single point