In the following exercise, feel free to use what you know from calculus to find the limit, if it exists. But you must *prove* that you found the correct limit, or that the sequence is divergent.

(2.1.5) Is the sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ convergent? If so, what is the limit?

Limit Calculation with L'Hopital Rule

$$\begin{aligned}\lim_{n\to\infty}\frac{n}{n+1} &\to \frac{\infty}{\infty}\\ &\overset{LH}{=} \lim_{n\to\infty}\frac{\frac{d}{dn}(n)}{\frac{d}{dn}(n+1)}\\ &= \lim_{n\to\infty}\frac{1}{1}\\ &=1\end{aligned}$$

Scratch Work

$$\left|\frac{n}{n+1}-1\right| = \left|\frac{n}{n+1}-\frac{n+1}{n+1}\right| = \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right|$$

Proof. $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ is said to converge to some $x \in \mathbb{R}$ if for every $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ whenever $n \geq M$.

Let $\epsilon > 0$ be given and let $x = 1$ as it is the proposed limit, so $\left|\frac{n}{n+1}-1\right| < \epsilon$. If x_n is convergent, this is true for all $n \geq M$ where $M \in \mathbb{N}$. n is strictly positive as $n \in \mathbb{N}$, so by using the Scratch Work above it is said that

$$|x_n - x| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1}$$

By the definition of a convergent sequence, it must be shown that $\frac{1}{n+1} < \epsilon$ when $\epsilon > 0$ for some $n \geq M$, $M \in \mathbb{N}$. The desired inequality can be written as $\frac{1}{n+1} < \epsilon \equiv n+1 > \frac{1}{\epsilon} \equiv n > \frac{1}{\epsilon} - 1$. Choose $M = \lceil \frac{1}{\epsilon} - 1 \rceil$, then for all $n \geq M$, it is true that $\frac{1}{n+1} < \epsilon$ and that $\lim_{n\to\infty}\frac{n}{n+1} = 1$. Thus the sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ is convergent and converges to 1. $\square$

(2.1.9) Show that the sequence $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ is monotone and unbound. Then use Theorem 2.1.10, also known as the Monotone Convergence Theorem (MCT), to find the limit.

Show $x^{\frac{1}{3}}$ is an Increasing Function

Suppose $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$. Then, $f'(x) = \frac{d}{dx}\left(x^{1/3}\right) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

1. For $x > 0$, $\sqrt[3]{x^2}$ is positive, so $f'(x) = \frac{1}{3\sqrt[3]{x^2}} > 0$.
2. For $x < 0$, $\sqrt[3]{x^2}$ is also positive, and hence $f'(x) = \frac{1}{3\sqrt[3]{x^2}} > 0$.

In both cases, the derivative $f'(x)$ is positive, indicating that $f(x)$ is an increasing function.

Proof. $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ is given. Consider, for some arbitrary n, x_n and x_{n+1}. These two values of the sequence can be compared, and if $x_n \geq x_{n+1}$, then $\{x_n\}_{n=1}^{\infty}$ is decreasing.

$$\begin{aligned} x_n \geq x_{n+1} \equiv & \frac{1}{\sqrt[3]{n}} \geq \frac{1}{\sqrt[3]{n+1}} && \text{Substitute Given} \\ \equiv & \sqrt[3]{n+1} \geq \sqrt[3]{n} && \text{Cross Multiply} \end{aligned}$$

It is shown above that $x^{\frac{1}{3}}$ is an increasing function, so the statement $\sqrt[3]{n+1} > \sqrt[3]{n}$ is true. Consequently, this also proves that $x_n \geq x_{n+1}$. Thus, $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ is monotone decreasing as n is arbitrary. Since $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ is monotone decreasing, it is said to be monotone.

Take $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ as given, and examine the function which defines the sequence $\frac{1}{\sqrt[3]{n}} = n^{-\frac{1}{3}}$. If the sequence is unbounded, then $n^{-\frac{1}{3}} > M$ for some arbitrarily large M. Solve for n as

$$\begin{aligned} n^{-\frac{1}{3}} > & M && \text{Given} \\ n^{\frac{1}{3}} < & \frac{1}{M} && \text{Take the reciprocal of both sides} \\ n < & \left(\frac{1}{M}\right)^3 && \text{Cube both sides} \end{aligned}$$

For any $M > 0$, it is possible to find an $n \in \mathbb{R}$ such that $n < \left(\frac{1}{M}\right)^3$, and thus the original sequence inequality holds that $\frac{1}{\sqrt[3]{n}} > M$ for any arbitrarily large M where $n < \left(\frac{1}{M}\right)^3$. Hence the sequence $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ is unbounded.

The MCT states that if a sequence is monotone decreasing and bounded, then $\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$. As $n \to \infty$, $\frac{1}{\sqrt[3]{n}} \to 0$ as $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ is monotone decreasing and $n \in \mathbb{N}$. This means that 0 is the sequence's greatest lower bound (infimum) as it approaches but never reaches 0 as n gets arbitrarily large. Therefore $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ is bounded below and $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = \inf\{x_n : n \in \mathbb{N}\} = 0$. □

(2.1.12) Prove Proposition 2.1.13:

Let $S \subset \mathbb{R}$ be a nonempty bounded set. Then there exist monotone sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that $x_n, y_n \in S$ and

$$\sup S = \lim_{n\to\infty} x_n \quad \text{and} \quad \inf S = \lim_{n\to\infty} y_n.$$

Proof. It is given that $x_n \in S$. Define $M \coloneqq \sup S$, meaning that M is the least upper bound of S. Note that $\sup S$ is guaranteed to exist as S is bounded. This means that $x \leq M$ for all $x \in S$ and that for every $\epsilon > 0$, there exists $x \in S$ such that $x > M - \epsilon$. Consider subsets of S, say S_n, whose elements are all less than or equal to $M - \frac{1}{n}$, given as $S_n = \{x \in S\}$, where $x \leq M - \frac{1}{n}$. Choose x_n, the nth element of $\{x_n\}_{n=1}^{\infty}$, to be the supremum of each of these subsets. This is given as $x_n \coloneqq \sup\{x \in S\}$, where $x \leq M - \frac{1}{n}$. It is known that for all $n \in \mathbb{N}$, $\frac{1}{n}$ decreases.. Consequently, $M - \frac{1}{n}$ increases as n increases, and hence S_n gets larger as the upper bound of the subset is moving closer to M. Thus, as the upper bound of the subsets grows, so does x_n, showing that $x_n \leq x_{n+1}$ and $\{x_n\}_{n=1}^{\infty}$ is guaranteed to be monotone increasing for these choices of x_n. Let $\epsilon > 0$ be given. There exists $x \in S$ such that $x > M - \epsilon$ and hence $\sup S = \lim_{n\to\infty} x_n$.

Similarly, it is given that $y_n \in S$, and that $\inf S$ is guaranteed to exist as S is bounded. Define $m \coloneqq \inf S$, meaning that m is the greatest lower bound of S and that for some $x \in S$, $x \geq m$ for all $x \in S$. It also holds that $x < m + \epsilon$ for every $\epsilon > 0$. Similar to the supremum, define $y_n \coloneqq \inf\{x \in S\}$, where $x \geq m + \frac{1}{n}$. Note that this means y_n is the infimum of all the the elements in the subset of S whose elements are greater than or equal to $m + \frac{1}{n}$. Since $m + \frac{1}{n}$ decreases as n increases, as explained above, the minimum value in the subset S_n also decreases. Thus, $y_n \geq y_{n+1}$ making $\{y_n\}$ a decreasing sequence. Let $\epsilon > 0$ be given. There exists $x \in S$ such that $x < m + \epsilon$ and hence $\inf S = \lim_{n\to\infty} y_n$. ☐

(2.1.15) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined by

$$x_n := \begin{cases} n & \text{if } n \text{ is odd,} \\ \frac{1}{n} & \text{if } n \text{ is even.} \end{cases}$$

(a) Is the sequence bounded? (prove or disprove)

Proof. The sequence given is $\{x_n\}_{n=1}^{\infty}$ and it has two branches which leads to two different cases as follows:

1. Suppose n is odd. Corresponding terms in the given sequence $\{x_n\}_{n=1}^{\infty}$ are $1, 3, 7, \ldots, n$.
2. Suppose n is even. Corresponding terms in the given sequence $\{x_n\}_{n=1}^{\infty}$ are $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, \frac{1}{n}$.

In case 1, the corresponding terms are the set of all positive odd numbers. Assume to the contrary that there exists a greatest odd integer, say $M = 2m + 1$, for some $m \in \mathbb{Z}$. Take the integer M', who is represented by $M' = M + 2 = 2m + 3$. M' is clearly greater than M, so there is no greatest odd number. Since this subsequence of x_n is unbounded, then it implies that the entire sequence is unbounded. This proves that when n is odd, then the given sequence is unbounded and approaches infinity. $\square$

(b) Is there a convergent subsequence? If so, find it.

Proof. It has been shown in part (a) that when n is odd, this subsequence is unbounded and diverges to infinity. Instead, take the subsequence when n is even, so $\{x_n\}_{n=2k}^{\infty}$ for $k \in \mathbb{N}$. As n increases it is observed that the terms get arbitrarily small and approach 0, which we will assume for now is the limit. Let $\epsilon > 0$ be given. By the archimedean property, there must exists some $M \in \mathbb{N}$ such that $0 < \frac{1}{M} < \epsilon$. Consequently, for every $n \geq M$, we have that $|x_n - 0| = \left|\frac{1}{n}\right| \leq \frac{1}{M} < \epsilon$ as required. Therefore the subsequence is convergent to 0 and the sequence given has a convergent subsequence as shown. $\square$

(2.1.23) Suppose that $\{x_n\}_{n=1}^{\infty}$ is a monotone increasing sequence that has a convergent subsequence. Show that $\{x_n\}_{n=1}^{\infty}$ is convergent. *Note that Proposition 2.1.17 is an "if and only if" for monotone sequences.*

Proof. $\{x_n\}_{n=1}^{\infty}$ is given to be monotone increasing, meaning that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. It is also given that there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, where it can be said that $\lim_{n\to\infty} x_{n_k} = L$ for some $L \in \mathbb{R}$. Let $\epsilon > 0$ be arbitrarily given. Since $\{x_{n_k}\}_{k=1}^{\infty}$ converges to L, there must exist an $M \in \mathbb{N}$ such that $|x_{n_k} - L| < \epsilon$ for all $k \geq M$. By the definition of the absolute value, it is said that $-\epsilon < x_{n_k} - L < \epsilon$, and hence implied that $L - \epsilon < x_{n_k} < L + \epsilon$ for all $k \geq M$. For any $n \leq n_k$, it can be said that $x_n \leq x_{n_k}$ as $\{x_n\}_{n=1}^{\infty}$ is monotone increasing. This inequality can be combined with the one above to show that $x_n \leq x_{n_k} < L + \epsilon$ and thus x_n is bounded by $L + \epsilon$. This implies that $\{x_n\}_{n=1}^{\infty}$ is bounded by L as ϵ is arbitrary. By the MCT, if $\{x_n\}_{n=1}^{\infty}$ is monotone increasing and bounded, which has been shown, then $\lim_{n\to\infty} x_n = \sup\{x_n\}$ for $n \in \mathbb{N}$. Since the sequence converges to L, it is the least upper bound and it can be said that $\lim_{n\to\infty}\{x_n\} = L$. Hence, $\{x_n\}$ is convergent. $\square$