In the following exercise, feel free to use what you know from calculus to find the limit, if it exists. But you must *prove* that you found the correct limit, or that the sequence is divergent.

(2.1.5) Is the sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ convergent? If so, what is the limit?

Limit Calculation with L'Hopital Rule

$$\begin{aligned}\lim_{n\to\infty}\frac{n}{n+1} &\to \frac{\infty}{\infty}\\ &\overset{LH}{=} \lim_{n\to\infty}\frac{\frac{d}{dn}(n)}{\frac{d}{dn}(n+1)}\\ &= \lim_{n\to\infty}\frac{1}{1}\\ &=1\end{aligned}$$

Scratch Work

$$\left|\frac{n}{n+1}-1\right| = \left|\frac{n}{n+1}-\frac{n+1}{n+1}\right| = \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right|$$

Proof. $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ is said to converge to some $x \in \mathbb{R}$ if for every $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ whenever $n \geq M$.

Let $\epsilon > 0$ be given and choose $x = 1$, so $\left|\frac{n}{n+1} - 1\right| < \epsilon$. If x_n is convergent, this is true for all $n \geq M$ where $M \in \mathbb{N}$. n is strictly positive as $n \in \mathbb{N}$, so by using the Scratch Work above it is said that

$$|x_n - x| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1}$$

By the definition of a convergent sequence, it must be shown that $\frac{1}{n+1} < \epsilon$ when $\epsilon > 0$ for some $n \geq M$, $M \in \mathbb{N}$. The desired inequality can be written as $\frac{1}{n+1} < \epsilon \equiv n+1 > \frac{1}{\epsilon} \equiv n > \frac{1}{\epsilon} - 1$. Choose $M = \lceil \frac{1}{\epsilon} - 1 \rceil$, then for all $n \geq M$, it is true that $\frac{1}{n+1} < \epsilon$ and that $\lim_{n\to\infty} \frac{n}{n+1} = 1$. Thus the sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ is convergent and converges to 1. $\square$

(2.1.9) Show that the sequence $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ is monotone and unbound. Then use Theorem 2.1.10, also known as the Monotone Convergence Theorem (MCT), to find the limit.

Show $x^{\frac{1}{3}}$ is an Increasing Function

Suppose $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$. Then, $f'(x) = \frac{d}{dx}\left(x^{1/3}\right) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

1. For $x > 0$, $\sqrt[3]{x^2}$ is positive, so $f'(x) = \frac{1}{3\sqrt[3]{x^2}} > 0$.
2. For $x < 0$, $\sqrt[3]{x^2}$ is also positive, and hence $f'(x) = \frac{1}{3\sqrt[3]{x^2}} > 0$.

In both cases, the derivative $f'(x)$ is positive, indicating that $f(x)$ is an increasing function.

Proof. $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ is given. Consider, for some arbitrary n, x_n and x_{n+1}. These two values of the sequence can be compared, and if $x_n > x_{n+1}$, then $\{x_n\}_{n=1}^{\infty}$ is decreasing.

$$\begin{aligned} x_n > x_{n+1} \equiv& \frac{1}{\sqrt[3]{n}} > \frac{1}{\sqrt[3]{n+1}} && \text{Substitute Given} \\ \equiv& \sqrt[3]{n+1} > \sqrt[3]{n} && \text{Cross Multiply} \end{aligned}$$

It is shown above that $x^{\frac{1}{3}}$ is an increasing function, so the statement $\sqrt[3]{n+1} > \sqrt[3]{n}$ is true. Consequently, this also proves that $x_n > x_{n+1}$. Thus, $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ is monotone decreasing as n is arbitrary. Since $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ is monotone decreasing, it is said to be monotone.

Take $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ as given, and examine the function which defines the sequence $\frac{1}{\sqrt[3]{n}} = n^{-\frac{1}{3}}$. If the sequence is unbounded, then $n^{-\frac{1}{3}} > M$ for some arbitrarily large M. Solve for n as

$$\begin{aligned} n^{-\frac{1}{3}} >& M && \text{Given} \\ n^{\frac{1}{3}} <& \frac{1}{M} && \text{Take the reciprocal of both sides} \\ n <& \left(\frac{1}{M}\right)^3 && \text{Cube both sides} \end{aligned}$$

For any $M > 0$, it is possible to find an $n \in \mathbb{R}$ such that $n < \left(\frac{1}{M}\right)^3$, and thus the original sequence inequality holds that $\frac{1}{\sqrt[3]{n}} > M$ for any n and arbitrarily large M. Hence the sequence $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ is unbounded.

The MCT states that if a sequence is monotone decreasing and bounded, then $\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$. As $n \to \infty$, $\frac{1}{\sqrt[3]{n}} \to 0$ as $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ is monotone decreasing and $n \in \mathbb{N}$. This means that 0 is the sequence's greatest lower bound (infimum) as it approaches but never reaches 0 as n gets arbitrarily large. Therefore $\left\{\frac{1}{\sqrt[3]{n}}\right\}_{n=1}^{\infty}$ is bounded below and $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = \inf\{x_n : n \in \mathbb{N}\} = 0$. □

(2.1.12) Prove Proposition 2.1.13:

Let $S \subset \mathbb{R}$ be a nonempty bounded set. Then there exist monotone sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that $x_n, y_n \in S$ and

$$\sup S = \lim_{n\to\infty} x_n \quad \text{and} \quad \inf S = \lim_{n\to\infty} .$$

Proof. It is given that S is nonempty and bounded, meaning that both $\sup S$ and $\inf S$ exist.

Suppose x_n is a sequence whose elements are a subset of S. □

(2.1.15) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined by

$$x_n := \begin{cases} n & \text{if } n \text{ is odd,} \\ \frac{1}{n} & \text{if } n \text{ is even.} \end{cases}$$

(a) Is the sequence bounded? (prove or disprove)

Proof. The sequence given is $\{x_n\}_{n=1}^{\infty}$ and it has two branches which leads to two different cases as follows:

1. Suppose n is odd. Corresponding terms in the given sequence $\{x_n\}_{n=1}^{\infty}$ are $1, 3, 7, \ldots, n$.
2. Suppose n is even. Corresponding terms in the given sequence $\{x_n\}_{n=1}^{\infty}$ are $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, \frac{1}{n}$.

In case 1, the corresponding terms are the set of all positive odd numbers. Assume to the contrary that there exists a greatest odd integer, say $M = 2m + 1$, for some $m \in \mathbb{Z}$. Take the integer M', who is represented by $M' = M + 2 = 2m + 3$. M' is clearly greater than M, so there is no greatest odd number. Since this subsequence of x_n is unbounded, then it implies that the entire sequence is unbounded. This proves that when n is odd, then the given sequence is unbounded and approaches infinity. □

(b) Is there a convergent subsequence? If so, find it.

Proof. It has been shown in part (a) that when n is odd, this subsequence is unbounded and diverges to infinity. Instead, take the subsequence when n is even, so $\{x_n\}_{n=2k}^{\infty}$ for $k \in \mathbb{N}$. As n increases it is observed that the terms get arbitrarily small and approach 0, which we will assume for now is the limit. Let $\epsilon > 0$ be given. By the archimedean property, there must exists some $M \in \mathbb{N}$ such that $0 < \frac{1}{M} < \epsilon$. Consequently, for every $n \geq M$, we have that $|x_n - 0| = \left|\frac{1}{n}\right| \leq \frac{1}{M} < \epsilon$ as required. Therefore the subsequence is convergent to 0 and the sequence given has a convergent subsequence as shown. □

(2.1.23) Suppose that $\{x_n\}_{n=1}^{\infty}$ is a monotone increasing sequence that has a convergent subsequence. Show that $\{x_n\}_{n=1}^{\infty}$ is convergent. *Note that Proposition 2.1.17 is an "if and only if" for monotone sequences.*

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. $\square$