(2.5.2) Prove Proposition 2.5.5, that is, for $-1 < r < 1$, prove

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

given that the geometric series $\sum_{n=0}^{\infty} r^n$ converges.

Proof. Let S_k be the partial sum of the series up to the k-th term of the series, that is $S_k = \sum n = 0\infty = 1 + r + r^2 + \cdots + r^k$. We must first show that $S_k = \frac{1-r^{k+1}}{1-r}$ for a finite k and $r \neq 1$. For the base case, take $k = 0$ so $S_0 = 1$ and the formula gives $S_0 = \frac{1-r^1}{1-r} = 1$, so this holds for $k = 0$. Assume this holds for some k, so $S_k = 1 + r + r^2 + \cdots + r^k = \frac{1-r^{k+1}}{1-r}$. We must show that this holds for $k+1$, or that $S_{k+1} = 1 + r + r^2 + \cdots + r^{k+1} = \frac{1-r^{k+2}}{1-r}$. Notice that $S_{k+1} = S_k + r^{k+1}$, so we have $S_{k+1} = \frac{1-r^{k+1}}{1-r} + r^{k+1}$. We can then simply this expression as

$$\begin{aligned}
S_{k+1} =& \frac{1-r^{k+1}}{1-r} + \frac{(1-r)r^{k+1}}{1-r} && \text{Common denominator} \\
=& \frac{1-r^{k+1} + r^{k+1} - r^{k+2}}{1-r} && \text{Expand the second term} \\
=& \frac{1-r^{k+2}}{1-r} && \text{as required}
\end{aligned}$$

This proves that the partial sums $S_k = \frac{1-r^{k+1}}{1-r}$. If we take the limit of the partial sums as $k \to \infty$, we can use the fact that since $-1 < r < 1$, $\lim_{k\to\infty} r^{k+1} = 0$. Therefore we have that

$$\begin{aligned}
\lim_{k\to\infty} S_k = \lim_{k\to\infty} & \frac{1-r^{k+1}}{1-r} \\
=& \frac{1}{1-r} \cdot \lim_{k\to\infty} \left(1 - r^{k+1}\right) && \text{Factor out scalar} \\
=& \frac{1}{1-r} \left(1 - \lim_{k\to\infty} r^{k+1}\right) && \text{Limit algebra} \\
=& \frac{1}{1-r} \cdot (1-0) && \text{Use } \lim_{k\to\infty} r^{k+1} = 0
\end{aligned}$$

Thus showing that as the number of terms in the partial sums tends to ∞, we have that $\lim_{k\to\infty} S_k = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. Hence given $-1 < r < 1$, $\sum_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$, as required. $\square$

(2.5.3) Decide the convergence or divergence of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{9n+1}$

Proof. First, notice that $\frac{3}{9n+1} > 0$ for all $n \in \mathbb{N}$. We will compare this to $\frac{1}{n}$, which is also positive for all n. We can use the limit comparison test as follows

$$\lim_{n\to\infty} \frac{\frac{3}{9n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n}{9n+1} = \lim_{n\to\infty} \frac{3}{9+\frac{1}{n}} = \frac{3}{9}$$

Note that the last equality holds as $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$. Since we have used the limit comparison test to compare $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{3}{9n+1}$, we have shown that $\displaystyle\sum_{n=1}^{\infty} \frac{3}{9n+1}$ diverges as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and both series must either both converge or diverge if $0 < \displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} < \infty$. □

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Proof. Notice again that $\frac{1}{2n-1} > 0$ for all $n \in \mathbb{N}$. We will compare this to $\frac{1}{n}$, which is also positive for all n. We can use the limit comparison test as follows

$$\lim_{n\to\infty} \frac{\frac{1}{2n-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}$$

Since we have used the limit comparison test to compare $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$, we have shown that the series diverges as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as $\frac{1}{2}$ is a positive constant. □

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Proof. We will prove this using the alternating series test, which is proved on J. Lebl pp.101. The test states that if $\{x_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence of positive real number such that $\displaystyle\lim_{n\to\infty} x_n = 0$, then $\displaystyle\sum_{n=1}^{\infty}(-1)^n x_n$ converges. We must show, therefore, that given $x_n = \frac{1}{n^2}$, $\{x_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence where $x_n \geq 0$ for all $n \in \mathbb{N}$ and that $\displaystyle\lim_{n\to\infty} \frac{1}{n^2} = 0$.

Firstly, note that $\frac{1}{n^2} \geq 0$ for all $n \in \mathbb{N}$ as $n^2 \geq 0$. Next, we define $x_{n+1} = \frac{1}{(n+1)^2}$, and we know that $n^2 \leq (n+1)^2$ for all n and hence $\frac{1}{n^2} \geq \frac{1}{(n+1)^2}$. Therefore $x_n \geq x_{n+1}$ so we have shown that x_n is monotone decreasing and positive for all n. We can formally write that x_n is monotone decreasing and bounded below by zero, so $\inf x_n = 0$, and hence the MCT tells us that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \dfrac{1}{n^2} = 0$. Therefore as $\lim_{n\to\infty} \dfrac{1}{n^2} = 0$ and $\frac{1}{n^2}$ is monotone decreasing, the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2}$ converges. □

(d) $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)}$

Proof. First note that $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$. This can be supported by

$$\frac{1}{n} - \frac{1}{n+1} = \frac{(n+1)}{n(n+1)} - \frac{n}{n(n+1)} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)}$$

Hence we can express the series $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)}$ as the telescoping sum

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1} \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots && \text{Write out the first few terms} \\
&= 1 && \text{All terms after 1 cancel out}
\end{aligned}$$

Therefore the series $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)}$ converges as it is a telescoping series and more specifically converges to 1.

□

(e) $\sum_{n=1}^{\infty} ne^{-n^2}$

Proof. We will choose to compare the series $\sum_{n=1}^{\infty} \dfrac{n}{e^{n^2}}$ to $\sum_{n=1}^{\infty} \dfrac{n}{n^3}$. First note that both ne^{-n^2} and $\frac{n}{n^3}$ are positive for all $n \in \mathbb{N}$. To justify this choice and , compare e^{n^2} to n^3 (continuous functions) as

$$\lim_{n\to\infty} \frac{e^{n^2}}{n^3} \overset{LH}{=} \lim_{n\to\infty} \frac{\frac{d}{dn} e^{n^2}}{\frac{d}{dn} n^3} = \lim_{n\to\infty} \frac{2ne^{n^2}}{3n} \to \infty \qquad \left(\frac{\text{Exponential}}{\text{Linear}} \to \infty\right).$$

The limit of the terms tends to infinity showing that $e^{n^2} \geq n^3$ and hence $\frac{1}{e^{n^2}} \leq \frac{1}{n^3}$. Consequently we have that $\sum_{n=1}^{\infty} ne^{-n^2} \leq \sum_{n=1}^{\infty} \dfrac{n}{n^3} = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$, and $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges by the p-series test with $p = 2$. In summary we have shown that $0 \leq \sum_{n=1}^{\infty} ne^{-n^2} \leq \dfrac{1}{n^2}$. Therefore by the comparison test, $\sum_{n=1}^{\infty} ne^{-n^2}$ converges. □

(2.5.14) Suppose $\sum_{n=1}^{\infty} x_n$ converges and $x_n \geq 0$ for all n. Prove that $\sum_{n=1}^{\infty} x_n^2$ converges.

Proof. To determine the convergence of $\sum_{n=1}^{\infty} x_n^2$, we can compare it to $\sum_{n=1}^{\infty} x_n$. Because $\sum_{n=1}^{\infty} x_n$ is convergent, it implies the as n gets sufficiently large, $x_n \to 0$. In other words, the convergence of $\sum_{n=1}^{\infty} x_n$ implies that $\lim_{n\to\infty} x_n = 0$. Therefore we have that $0 < x < 1$ for sufficiently large n. Notice that for $\sum_{n=1}^{\infty} x_n^2$, we have $x_n^2 \leq x_n$ for sufficiently large n. Note that we have to prove this expression, and also that we only care about long term behavior as that is what ultimately determines convergence. We therefore have

$$
\begin{aligned}
x^2 &\leq x && \text{Want to show for } 0 < x < 1 \\
x - x^2 &\geq 0 && \text{Subtract } x \text{ from both sides} \\
x(1-x) &\geq 0 && \text{Factor the expression}
\end{aligned}
$$

The following statement is true as $0 < x < 1$ is given and hence x and $1 - x$, the factors, are both always positive. Therefore $x^2 \leq x$ holds for all $0 < x < 1$.

Let $M \in \mathbb{N}$ be given such that $n \geq M$, thus we are comparing the tails of the series when $0 < x < 1$ holds. We therefore have that $0 \leq x_n^2 \leq x_n$ for all $n \geq M$. Note that $\sum_{n=1}^{\infty} x_n$ converges for $n \geq M$ as a series converges if and only if its tails converge. This also means that if a series tails converge, then the series also converges. Since $\sum_{n=1}^{\infty} x_n$ converges and the terms in the long term bound the terms of $\sum x_n^2$, $\sum_{n=1}^{\infty} x_n^2$ must also converge by the comparison test. □

(3.1.1) Find the limit (and prove it of course) or prove that the limit does not exist.
(a) $\lim\limits_{x \to c} \sqrt{x}$, for $c \geq 0$

Proof. Assume $f(x) = \sqrt{x}$ and $f : S \to \mathbb{R}$ where $S = [0, \infty\}$. We will suggest that as the square root is a continuous function, $L = \sqrt{c}$. Therefore we must show that for every $\epsilon > 0$, there exists some $\delta > 0$ for which $|f(x) - L| = |\sqrt{x} - \sqrt{c}| < \epsilon$ holds whenever $x \in S \setminus \{c\}$ satisfies $|x - c| < \delta$.

We can multiply $|\sqrt{x} - \sqrt{c}|$ by its conjugate to get $|\sqrt{x} - \sqrt{c}| = \frac{|x-c|}{|\sqrt{x}+\sqrt{c}|}$. Note that since $\sqrt{x}$ and $\sqrt{c}$ are both always positive, we can ignore the absolute value of its sum and can further say that $\sqrt{x} + \sqrt{c} \geq \sqrt{c}$. Consequently $\frac{1}{\sqrt{x}+\sqrt{c}} \leq \frac{1}{\sqrt{c}}$, leading us to $|\sqrt{x} - \sqrt{c}| \leq \frac{|x-c|}{\sqrt{c}}$. To ensure the inequality $|\sqrt{x} - \sqrt{c}| < \epsilon$ is satisfied, we need $\frac{|x-c|}{\sqrt{c}} < \epsilon$ to hold as it bounds $|\sqrt{x} - \sqrt{c}|$. This gives us $|x - c| < \epsilon\sqrt{c}$ and thus we choose $\delta = \epsilon\sqrt{c}$. Note that for $c = 0$, we have $|\sqrt{x} - 0| = \sqrt{x}$ and to ensure $\sqrt{x} < \epsilon$, we need $x < \epsilon^2$. Hence we can choose $\delta = \epsilon^2$ to guarantee that for $x < \delta$, we have $\sqrt{x} < \epsilon$. Therefore $\lim\limits_{x \to c} \sqrt{x} = c$, as required. ☐

(b) $\lim\limits_{x \to c} x^2 + x + 1$, for $c \in \mathbb{R}$

Proof. Let $f(x) = x^2 + x + 1$, and we want to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$, it holds that

$$\begin{aligned} |f(x) - f(c)| = \left|(x^2 + x + 1) - (c^2 + c + 1)\right| &= \left|x^2 - c^2 + x - c\right| \\ &= |(x - c)(x + c) + (x - c)| \\ &= |x - c| \cdot |x + c + 1| < \epsilon \end{aligned}$$

Note that we can rewrite $|x + c + 1| = |(x - c) + (2c + 1)|$, and by the triangle inequality we have $|(x - c) + (2c + 1)| \leq |x - c| + |2c + 1|$. As $|x - c| < \delta$, we have $|x + c + 1| \leq |2c + 1| + \delta$. Consequently we can write $|x - c| \cdot |x + c + 1| \leq |x - c| \cdot (|2c + 1| + \delta) < \epsilon$, and hence we need to fulfill $|x - c| \cdot (|2c + 1| + \delta) < \epsilon$. Choose $\delta = \min\left(1, \frac{\epsilon}{|2c+1|+1}\right)$. This choice ensures that $|x - c|$ is kept sufficiently small and that the product $|x - c| \cdot (|2c + 1| + \delta)$ is less than ϵ. Thus $\lim\limits_{x \to c} x^2 + x + 1 = c^2 + c + 1$ as required. ☐

(c) $\lim\limits_{x \to 0} x^2 \cos\left(\dfrac{1}{x}\right)$

Proof. Let $f(x) = x^2 \cos\left(\frac{1}{x}\right)$. We will use the squeeze theorem to prove its limit. First note that the cosine function is bounded by 1, meaning that $|\cos n| \leq 1$, where $n = \frac{1}{x}$ in this case (holds for all $x \neq 0$). We can say that $-1 \leq \cos\frac{1}{x} \leq 1$. Multiplying this expression by x^2 maintains the validity as $x^2 \geq 0$ for all x, and hence $-x^2 \leq f(x) \leq x^2$. Therefore $f(x)$ is bounded below by $-x^2$ and above by x^2 and $f(x)$ will converge to the limits of its bounds if they are equal. Taking the limits of the bounding functions gives us $\lim\limits_{x \to 0} -x^2 = 0$ and $\lim\limits_{x \to 0} x^2 = 0$. Since $f(x)$ is squeezed between $-x^2$ and x^2 as $x \to 0$, $\lim\limits_{x \to 0} -x^2 \leq \lim\limits_{x \to 0} f(x) \leq \lim\limits_{x \to 0} x^2$. Hence by the squeeze theorem we have $\lim\limits_{x \to 0} f(x) = 0$ and thus $\lim\limits_{x \to 0} x^2 \cos\left(\dfrac{1}{x}\right) = 0$, as required. ☐

(d) $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)$

Proof. Note that individual components of the function $f(x) = \sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)$, $\sin\left(\frac{1}{x}\right)$ and $\cos\left(\frac{1}{x}\right)$ are both bounded by the interval $[-1, 1]$. Note that the $f(x)$ and its components are discussed in this context with $x \neq 0$. Therefore their product cannot exceed these bounds and hence we have that $-1 \leq \sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right) \leq 1$, equivalently $-1 \leq f(c) \leq 1$. Since $f(x)$ oscillates between -1 and 1, as those are its bounds due to the behavior of sine and cosine, $f(x)$ cannot converge to a limit. Hence $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)$ does not exist, as required. □

(e) $\lim_{x \to 0} \sin(x)\cos\left(\frac{1}{x}\right)$

Proof. Let $f(x) = \sin(x)\cos\left(\frac{1}{x}\right)$. As per our argument in part (c), we know that $|\cos n| \leq 1$ and hence $-1 \leq \cos\frac{1}{x} \leq 1$ for all $x \neq 0$. This is acceptable as the function does not need to contain its limit, but it must approach it. To show this behavior, we multiply the previous expression by $\sin x$ to get an expression contending with $f(x)$. Thus we have $-\sin x \leq \sin(x)\cos\left(\frac{1}{x}\right) \leq \sin x$. This means that $f(x)$ is bounded below by $-\sin x$ and above by $\sin x$, and we can use the squeeze theorem as done in part (c) to prove convergence. Take the limits of the bounds to get $\lim_{x \to 0} -\sin x = 0$ and $\lim_{x \to 0} \sin x$. This is known to be true as $\sin x$ is continuous and approaches 0 as x approaches 0. Since $\sin(x)\cos\left(\frac{1}{x}\right)$ is squeezed by $-\sin x$ and $\sin x$, it must converge to their limits, which agree to be 0. Therefore, $\lim_{x \to 0} \sin(x)\cos\left(\frac{1}{x}\right) = 0$, as required. □

(3.1.2) Prove Corollary 3.1.10, that is, let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ is a function such that the limit of $f(x)$ as x goes to c exists. Suppose there are two real numbers a and b such that

$$a \leq f(x) \leq b \qquad \text{for all } x \in S \setminus \{c\}$$

Then

$$a \leq \lim_{x \to c} f(x) \leq b.$$

Proof. Let $L := \lim_{x \to c} f(x)$ as it is given to exist. This means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $|x - c| < \delta$ for $x \in S \setminus \{c\}$. We want to show that given two real numbers a, b, with $a \leq f(x) \leq b$ for $x \in S \setminus \{c\}$, that $a \leq L \leq b$. We can express $|f(x) - L| < \epsilon$ as $L - \epsilon < f(x) < L + \epsilon$ by the definition of the absolute value. We will break up this equality as $L - \epsilon < f(x)$ and $f(x) < L + \epsilon$ in order to better reach our end goal. Since $f(x) \leq b$ is given, we have $L - \epsilon < f(x) \leq b$ which is equivalent to $L - \epsilon \leq b \implies L \leq b + \epsilon$. Similarly for a, we are given that $a \leq f(x)$ and hence $a \leq f(x) < L + \epsilon$ which is equivalent to $a \leq L + \epsilon \implies a - \epsilon \leq L$. Note that in the previous derivations of the inequalities, we can choose $\leq$ as a relation as limits allow for, but do not require, equality. We can recombine the inequalities $a - \epsilon \leq L$ and $L \leq b + \epsilon$ to yield $a - \epsilon \leq L \leq b + \epsilon$. Since $\epsilon > 0$ is arbitrary, we can say generally that $a \leq L \leq b$, and hence $a \leq \lim_{x \to c} f(x) \leq b$, as required. $\square$