

MATH 321/421 HOMEWORK 1: DUE 09/04/2024

Directions: Show all of your work on each questions below. Proofs are to be written in full and grammatically correct sentences. The use of quantifiers in proofs such that $\forall, \exists, \implies$ in place of the written word will result in a point deduction from a problem for each use of such a quantifier. However, use of the notation $x \in A$ to denote an element x in a set A is perfectly acceptable. Each problem is worth 5 points.

- (1) Let F be an ordered field. Prove that, for any pair of elements $x, y \in F$ such that $0 < x < y$, it holds that $x^2 < y^2$.
- (2) Let S be an ordered set and $A \subset S$ be a finite subset.
 - (a) Prove that $\inf A$ and $\sup A$ exist.
 - (b) Prove that $\inf A, \sup A \in A$.

Hint: Use induction.

- (3) Let S be an ordered set, suppose that $B \subset S$ is bounded from above and from below, and let $A \subset B$ be non-empty. Suppose that all infs and sups exist. Prove that

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

- (4) Prove that $\sqrt{3}$ is irrational.
- (5) Prove the arithmetic-geometric means inequality: for any pair of non-negative real numbers x, y ,

$$\sqrt{xy} \leq \frac{x + y}{2},$$

and there is equality if and only if $x = y$.