(2.5.2) Prove Proposition 2.5.5, that is, for $-1 < r < 1$, prove

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

given that the geometric series $\sum_{n=0}^{\infty} r^n$ converges.

Proof. Let S_k be the partial sum of the series up to the k-th term of the series, that is $S_k = \sum n = 0\infty = 1 + r + r^2 + \cdots + r^k$. We must first show that $S_k = \frac{1-r^{k+1}}{1-r}$ for a finite k and $r \neq 1$. For the base case, take $k = 0$ so $S_0 = 1$ and the formula gives $S_0 = \frac{1-r^1}{1-r} = 1$, so this holds for $k = 0$. Assume this holds for some k, so $S_k = 1 + r + r^2 + \cdots + r^k = \frac{1-r^{k+1}}{1-r}$. We must show that this holds for $k + 1$, or that $S_{k+1} = 1 + r + r^2 + \cdots + r^{k+1} = \frac{1-r^{k+2}}{1-r}$. Notice that $S_{k+1} = S_k + r^{k+1}$, so we have $S_{k+1} = \frac{1-r^{k+1}}{1-r} + r^{k+1}$. We can then simply this expression as

$$\begin{aligned}
S_{k+1} =& \frac{1-r^{k+1}}{1-r} + \frac{(1-r)r^{k+1}}{1-r} && \text{Common denominator} \\
=& \frac{1-r^{k+1}+r^{k+1}-r^{k+2}}{1-r} && \text{Expand the second term} \\
=& \frac{1-r^{k+2}}{1-r} && \text{as required}
\end{aligned}$$

This proves that the partial sums $S_k = \frac{1-r^{k+1}}{1-r}$. If we take the limit of the partial sums as $k \to \infty$, we can use the fact that since $-1 < r < 1$, $\lim_{k\to\infty} r^{k+1} = 0$. Therefore we have that

$$\begin{aligned}
\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1-r^{k+1}}{1-r} & \\
=& \frac{1}{1-r} \cdot \lim_{k\to\infty} \left(1 - r^{k+1}\right) && \text{Factor out scalar} \\
=& \frac{1}{1-r} \left(1 - \lim_{k\to\infty} r^{k+1}\right) && \text{Limit algebra} \\
=& \frac{1}{1-r} \cdot (1-0) && \text{Use } \lim_{k\to\infty} r^{k+1} = 0
\end{aligned}$$

Thus showing that as the number of terms in the partial sums tends to ∞, we have that $\lim_{k\to\infty} S_k = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. Hence given $-1 < r < 1$, $\sum_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$, as required. □

(2.5.3) Decide the convergence or divergence of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{9n+1}$

Proof. First, notice that $\frac{3}{9n+1} > 0$ for all $n \in \mathbb{N}$. We will compare this to $\frac{1}{n}$, which is also positive for all n. We can use the limit comparison test as follows

$$\lim_{n\to\infty} \frac{\frac{3}{9n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n}{9n+1} = \lim_{n\to\infty} \frac{3}{9+\frac{1}{n}} = \frac{3}{9}$$

Note that the last equality holds as $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$. Since we have used the limit comparison test to compare $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{3}{9n+1}$, we have shown that $\displaystyle\sum_{n=1}^{\infty} \frac{3}{9n+1}$ diverges as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and both series must either both converge or diverge if $0 < \displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} < \infty$. □

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Proof. Notice again that $\frac{1}{2n-1} > 0$ for all $n \in \mathbb{N}$. We will compare this to $\frac{1}{n}$, which is also positive for all n. We can use the limit comparison test as follows

$$\lim_{n\to\infty} \frac{\frac{1}{2n-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}$$

Since we have used the limit comparison test to compare $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$, we have shown that the series diverges as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as $\frac{1}{2}$ is a positive constant. □

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Proof. We will prove this using the alternating series test, which is proved on J. Lebl pp.101. The test states that if $\{x_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence of positive real number such that $\displaystyle\lim_{n\to\infty} x_n = 0$, then $\displaystyle\sum_{n=1}^{\infty}(-1)^n x_n$ converges. We must show, therefore, that given $x_n = \frac{1}{n^2}$, $\{x_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence where $x_n \geq 0$ for all $n \in \mathbb{N}$ and that $\displaystyle\lim_{n\to\infty} \frac{1}{n^2} = 0$.

Firstly, note that $\frac{1}{n^2} \geq 0$ for all $n \in \mathbb{N}$ as $n^2 \geq 0$. Next, we define $x_{n+1} = \frac{1}{(n+1)^2}$, and we know that $n^2 \leq (n+1)^2$ for all n and hence $\frac{1}{n^2} \geq \frac{1}{(n+1)^2}$. Therefore $x_n \geq x_{n+1}$ so we have shown that x_n is monotone decreasing and positive for all n. We can formally write that x_n is monotone decreasing and bounded below by zero, so $\inf x_n = 0$, and hence the MCT tells us that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{1}{n^2} = 0$. Therefore as $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$ and $\frac{1}{n^2}$ is monotone decreasing, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges. □

(d) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Proof. First note that $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$. This can be supported by

$$\frac{1}{n} - \frac{1}{n+1} = \frac{(n+1)}{n(n+1)} - \frac{n}{n(n+1)} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)}$$

Hence we can express the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ as the telescoping sum

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1} \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots && \text{Write out the first few terms} \\
&= 1 && \text{All terms after 1 cancel out}
\end{aligned}$$

Therefore the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges as it is a telescoping series and more specifically converges to 1. □

(e) $\sum_{n=1}^{\infty} ne^{-n^2}$

Proof. We will choose to compare the series $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$ to $\sum_{n=1}^{\infty} \frac{n}{n^3}$. First note that both ne^{-n^2} and $\frac{n}{n^3}$ are positive for all $n \in \mathbb{N}$. To justify this choice and , compare e^{n^2} to n^3 (continuous functions) as

$$\lim_{n\to\infty} \frac{e^{n^2}}{n^3} \overset{LH}{=} \lim_{n\to\infty} \frac{\frac{d}{dn} e^{n^2}}{\frac{d}{dn} n^3} = \lim_{n\to\infty} \frac{2ne^{n^2}}{3n} \to \infty \qquad \left(\frac{\text{Exponential}}{\text{Linear}} \to \infty\right).$$

The limit of the terms tends to infinity showing that $e^{n^2} \geq n^3$ and hence $\frac{1}{e^{n^2}} \leq \frac{1}{n^3}$. Consequently we have that $\sum_{n=1}^{\infty} ne^{-n^2} \leq \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test with $p = 2$. In summary we have shown that $0 \leq \sum_{n=1}^{\infty} ne^{-n^2} \leq \frac{1}{n^2}$. Therefore by the comparison test, $\sum_{n=1}^{\infty} ne^{-n^2}$ converges. □

(2.5.14) Suppose $\sum_{n=1}^{\infty} x_n$ converges and $x_n \geq 0$ for all n. Prove that $\sum_{n=1}^{\infty} x_n^2$ converges.

Proof. To determine the convergence of $\sum_{n=1}^{\infty} x_n^2$, we can compare it to $\sum_{n=1}^{\infty} x_n$. Because $\sum_{n=1}^{\infty} x_n$ is convergent, it implies the as n gets sufficiently large, $x_n \to 0$. In other words, the convergence of $\sum_{n=1}^{\infty} x_n$ implies that $\lim_{n\to\infty} x_n = 0$. Therefore we have that $0 < x < 1$ for sufficiently large n. Notice that for $\sum_{n=1}^{\infty} x_n^2$, we have $x_n^2 \leq x_n$ for sufficiently large n. Note that we have to prove this expression, and also that we only care about long term behavior as that is what ultimately determines convergence. We therefore have

$$
\begin{aligned}
x^2 &\leq x && \text{Want to show for } 0 < x < 1 \\
x - x^2 &\geq 0 && \text{Subtract } x \text{ from both sides} \\
x(1-x) &\geq 0 && \text{Factor the expression}
\end{aligned}
$$

The following statement is true as $0 < x < 1$ is given and hence x and $1-x$, the factors, are both always positive. Therefore $x^2 \leq x$ holds for all $0 < x < 1$.

Let $M \in \mathbb{N}$ be given such that $n \geq M$, thus we are comparing the tails of the series when $0 < x < 1$ holds. We therefore have that $0 \leq x_n^2 \leq x_n$ for all $n \geq M$. Note that $\sum_{n=1}^{\infty} x_n$ converges for $n \geq M$ as a series converges if and only if its tails converge. This also means that if a series tails converge, then the series also converges. Since $\sum_{n=1}^{\infty} x_n$ converges and the terms in the long term bound the terms of $\sum x_n^2$, $\sum_{n=1}^{\infty} x_n^2$ must also converge by the comparison test. $\square$

(3.1.1) Find the limit (and prove it of course) or prove that the limit does not exist.
(a) $\lim_{x \to c} \sqrt{x}$, for $c \geq 0$

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. $\square$

(b) $\lim_{x \to c} x^2 + x + 1$, for $c \in \mathbb{R}$

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. $\square$

(c) $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. $\square$

(d) $\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \cos\left(\frac{1}{x}\right)$

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. $\square$

(e) $\lim_{x \to 0} \sin(0) \cos\left(\frac{1}{x}\right)$

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. $\square$

(3.1.2) Prove Corollary 3.1.10, that is, let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ is a function such that the limit of $f(x)$ as x goes to c exists. Suppose there are two real numbers a and b such that

$$a \leq f(x) \leq b \qquad \text{for all } x \in S \setminus \{c\}$$

Then

$$a \leq \lim_{x \to c} f(x) \leq b.$$

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. □