(1.2.8) Show that for every pair of real numbers x and y such that $x < y$, there exists an irrational number s such that $x < s < y$.

Proof. The set of the rational numbers $\mathbb{Q}$ is known to be dense in the real numbers $\mathbb{R}$. This means that given $x, y \in \mathbb{R}$ and $x < y$, then there exists some rational number $r \in \mathbb{Q}$ such that $x < r < y$. Note that this is part of the Archimedean property.

Consider the two real numbers $\frac{x}{\sqrt{2}}$ and $\frac{y}{\sqrt{2}}$. Building off the earlier proofs in this course, it is known that $\sqrt{2}$ is irrational, and that the irrational numbers are a subset of $\mathbb{R}$. It is also known that $x < y$, so it follows that $\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$ as $\mathbb{R}$ is an ordered field. This is settled on Page 26 of *Basic Analysis I: Introduction to Real Analysis, Volume 1.* by Lebl, Jiri. By the density of the rationals in the reals, there exists some $r \in \mathbb{Q}$ such that $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$. Now, multiply the expression by $\sqrt{2}$, preserving the ordering though showing a different property. This yields the expression $x < \sqrt{2}r < y$. Both x and y are real numbers, and while $\sqrt{2}r$ is also a real number, it must be proved that it is irrational.

Again, it is known that r is a rational number and $\sqrt{2}$ is an irrational number. Assume to the contrary that $\sqrt{2}r$ is a rational number $q \in \mathbb{Q}$. In other words, $\sqrt{2}r = q$, where $r, q \in \mathbb{Q}$. Since r and q are both rational numbers, they can be expressed as $r = \frac{a}{b}$ and $q = \frac{c}{d}$, where $b, q \neq 0$ and $a, b, c, d \in \mathbb{Z}$. Using these ratios, solve for $\sqrt{2}$ from the expression $\sqrt{2}r = q$ as

$$\sqrt{2} = \frac{q}{r} = \frac{\frac{c}{d}}{\frac{a}{b}} = \frac{bc}{ad}$$

where r and q are substituted accordingly.

Note that this means $\sqrt{2} = \frac{bc}{ad}$, in which a contradiction is arrived at. Since $a, b, c, d \in \mathbb{Z}$ and by extension $cd, ab \in \mathbb{Z}$, then $\sqrt{2}$ must be rational as it can be expressed as a ratio of integers. This contradicts the assumption that $\sqrt{2}$ is irrational. Therefore $\sqrt{2}r$ is irrational.

Choosing $s = \sqrt{2}r$, it is shown that there exists some irrational number s between real numbers x and y given that $a < y$. $\square$

(1.2.9) Let A and B be two nonempty bounded sets of real numbers. Let $C \coloneqq \{a + b : a \in A, b \in B\}$. Show that C is a bounded set and that

$$\sup C = \sup A + \sup B \quad \text{and} \quad \inf C = \inf A + \inf B.$$

Proof. Since A and B are both bounded and nonempty subsets of the real numbers, then they must have finite bounds which exist. Hence, define the respective bounds as

$$\inf A = n_a, \sup A = N_A \quad \text{and} \quad \inf B = n_B, \sup B = N_B$$

In other words, for all $a \in A$ and $b \in B$, $n_a \leq a \leq N_A$ and $n_B \leq b \leq N_B$.

For any $c \in C$, there exists an $a \in A$ and $b \in B$ such that $c = a + b$ as given. Since $a \leq N_A$ and $b \leq N_B$, these inequalities can be added to yield $a + b \leq N_A + N_B$. Therefore $c \leq N_A + N_B$ and hence $N_A + N_B$ is an upper bound for c. Similarly, since $a \geq n_B$ and $b \geq n_B$, these inequalities can also be added to yield $a + b \geq n_A + n_B$. Therefore $c \geq n_A + n_B$ and hence $n_A + n_B$ is a lower bound for C. Since C has both an upper bound $N_A + N_B$ and a lower bound $n_A + n_B$, it follows that C is a bounded set.

For every $a \in A$ and $b \in B$, $c = a + b$. Since, by the definition of the supremum, $a \leq \sup A$ for all $a \in A$ and $b \leq \sup B$ for all $b \in B$, the inequalities can be combined to yield $a + b \leq \sup A + \sup B$. Hence, every element of C is less than or equal to $\sup A + \sup B$, and it follows that $\sup C \leq \sup A + \sup B$.

Now, choose an element $a' \in A$ who's arbitrarily close to $\sup A$ and similarly choose an element $b' \in B$ who's arbitrarily close to $\sup B$. Consider the pair a', b', which yields the sum $c' = a' + b'$. This sum c' will be arbitrarily close to $\sup A + \sup B$. Therefore, it follows that the least upper bound of $C, \sup C$, must be at least $\sup A + \sup B$. Thus, $\sup C \geq \sup A + \sup B$. Consequently $\sup C = \sup A + \sup B$, as both $\sup C \leq \sup A + \sup B$ and $\sup C \geq \sup A + \sup B$.

For any $a \in A$ and $b \in B$, $c = a + b$. Since $a \geq \inf A$ for all $a \in A$ and $b \geq \inf B$ for all $b \in B$ by definition, which can be combined to yield $a + b \geq \inf A + \inf B$. Hence, every element of C is greater than or equal to $\inf A + \inf B$, and it follows that $\inf C \geq \inf A + \inf B$.

Similarly to the argument above, choose an element $a' \in A$ who's arbitrarily close to $\inf A$ and similarly choose an element $b' \in B$ who's arbitrarily close to $\inf B$. Consider the pair a', b', which yields the sum $c' = a' + b'$. This sum c' will be arbitrarily close to $\inf A + \inf B$. Therefore, it follows that the greatest lower bound of $C, \inf C$, must be at most $\inf A + \inf B$. Thus, $\inf C \leq \inf A + \inf B$. Consequently $\inf C = \inf A + \inf B$, as both $\inf C \geq \inf A + \inf B$ and $\inf C \leq \inf A + \inf B$.

In summary, when given two nonempty bounded sets of real numbers, A and B with $C \coloneqq \{a + b : a \in A, b \in B\}$, it follows that C is bounded and that

$$\sup C = \sup A + \sup B \quad \text{and} \quad \inf C = \inf A + \inf B.$$

$\square$

(1.2.12) Prove Proposition 1.2.8. If $S \subset \mathbb{R}$ is nonempty and bounded above, then for every $\epsilon > 0$ there exists an $x \in S$ such that $(\sup S) - \epsilon < x \leq \sup S$.

Proof. Let $M := \sup S$ be the supremum of S. By definition of the supremum, M is the smallest number such that M is an upper bound of S where for all $x \in S$, it holds that $x \leq M$.

Assume to the contrary that no such $x \in S$ exists to satisfy $(\sup S) - \epsilon < x \leq \sup S$ for any $\epsilon > 0$. This means that for every $x \in S$, it holds that $x \leq M - \epsilon$. Consequently, $M - \epsilon$ would be an upper bound for S as $x \leq M - \epsilon$ for all $x \in S$. Thus, $M - \epsilon$ must be an upper bound. However, M is already assumed to be the supremum of S, meaning that M is the least upper bound. $M - \epsilon$ being an upper bound contradicts the fact that M is the least upper bound for the set S. Hence, the assumption that no such $x \in S$ exists to satisfy $(\sup S) - \epsilon < x \leq \sup S$ for any $\epsilon > 0$ is false.

$\square$

(1.3.5) Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be function with D being nonempty.

(a) Suppose $f(x) \leq g(y)$ for all $x \in D$ and $y \in D$. Show that

$$\sup_{x \in D} f(x) \leq \inf_{x \in D} g(x).$$

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. □

(b) Find a specific $D, f,$ and g, such that $f(x) \leq g(x)$ for all $x \in D$, but

$$\sup_{x \in D} f(x) > \inf_{x \in D} g(x).$$

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. $\square$

(1.3.7) Let D be a nonempty set. Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded functions.

(a) Show

$$\sup_{x \in D}(f(x) + g(x)) \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x) \quad \text{and} \quad \inf_{x \in D}(f(x) + g(x)) \geq \inf_{x \in D} f(x) + \inf_{x \in D} g(x)$$

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. □

(b) Find an example (or examples) where we obtain strict inequalities.