Facts about Sequences

1. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be such that $a_n \leq x_n \leq b_n$ for all $n \in \mathbb{N}$. If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge to $x \in \mathbb{R}$, then $\{x_n\}_{n=1}^{\infty}$ must also converge to x.

2. Let $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ be convergent sequences. Suppose that $x_n \leq y_n$ for every $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} x_n \leq \lim\limits_{n\to\infty} y_n$.

3. If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence such that $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} x_n \geq 0$.

4. Let $a, b \in \mathbb{R}$ and $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence such that $a \leq x_n \leq b$ for all $n \in \mathbb{N}$. Then $a \leq \lim\limits_{n\to\infty} x_n \leq b$.

5. Addition, Subtraction, Multiplication, and Division can all be performed safely on limit expression provided they are both convergent and there is no division by 0.

6. Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence such that $x_n \geq 0$ for all $n \in \mathbb{N}$. then $\lim\limits_{n\to\infty} \sqrt{x_n} = \sqrt{\lim\limits_{n\to\infty} x_n}$.

7. If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence, then $\{|x_n|\}_{n=1}^{\infty}$ is a convergent sequence and $\lim\limits_{n\to\infty} |x_n| = \left|\lim\limits_{n\to\infty} x_n\right|$.

Convergence Tests

1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence, and suppose there is an $x \in \mathbb{R}$ and a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} a_n = 0$ and $|x_n - x| \leq a_n$ for all $n \in \mathbb{N}$. Then the sequence x_n converges to x.

2. Let $c > 0$. (1) If $c < 1$, then $\{c^n\}_{n=1}^{\infty}$ converges to zero. (2) If $c > 1$, then $\{c^n : n \in \mathbb{N}\}$ is an unbounded set of $\mathbb{R}$ and hence cannot converge.

3. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $x_n \neq 0$ for all $n \in \mathbb{N}$ and the limit $L := \lim\limits_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|$ exists. Then, (1) If $0 \leq L < 1$, then $\{x_n\}_{n=1}^{\infty}$ converges to 0. (2) If $L > 1$, then $\{x_n : n \in \mathbb{N}\}$ is unbounded, and so $\{x_n\}_{n=1}^{\infty}$ is divergent. (3) If $L = 1$, the test is inconclusive.

Limit Superior & Inferior

1. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence. For each $n \in \mathbb{N}$, define $a_n = \sup\{x_k : k \geq n\}$ and $b_n = \inf\{x_k : k \geq n\}$. Consider the sequences, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$. Define $\limsup\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} a_n$ and $\liminf\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} b_n$, provided both limits exist.

2. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence, and $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be as in the above definition. (1) The sequence $\{a_n\}_{n=1}^{\infty}$ is bounded and monotone decreasing, while $\{b_n\}_{n=1}^{\infty}$ is also bounded but is monotone increasing. In particular $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ both exist. (2)$\limsup_{n\to\infty} x_n = \inf\{a_n : n \in \mathbb{N}\} = \inf\{\sup\{a_k : k \geq n\} : n \in \mathbb{N}\}$ and $\liminf_{n\to\infty} x_n = \sup b_n : n \in \mathbb{N} = \sup\{\inf\{b_k : k \geq n\} : n \in \mathbb{N}\}$. (3) $\limsup_{n\to\infty} x_n \geq \liminf_{n\to\infty} x_n$.

3. Given a bounded sequence $\{x_n\}_{n=1}^{\infty}$, there exists subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ and $\{x_{m_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n$ and $\lim_{k\to\infty} x_{m_k} = \liminf_{n\to\infty} x_n$.

4. A bounded sequence $\{x_n\}_{n=1}^{\infty}$ converges if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$. In fact, $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$.

5. Suppose you have a bounded sequence $\{x_n\}_{n=1}^{\infty}$ and $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence. Then $\liminf_{n\to\infty} x_n \leq \liminf_{k\to\infty} x_{n_k} \leq \limsup_{k\to\infty} x_{n_k} \leq \limsup_{n\to\infty} x_n$.

6. A bounded sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$ if and only if every convergent subsequence converges to x.

Bolzano-Weierstrass Theorem

1. Every bounded sequence of real number must have a convergent subsequence.

2. We say that a sequence $\{x_n\}_{n=1}^{\infty}$ diverges to infinity if, for every $k \in \mathbb{R}$, there is some M such that $x_n > k$ whenever $n \geq M$. In this case, we write $\lim_{n\to\infty} x_n \coloneqq \infty$.

3. We say that a sequence $\{x_n\}_{n=1}^{\infty}$ diverges to negative infinity if, for every $k \in \mathbb{R}$, there is some M such that $x_n < k$ whenever $n \geq M$. In this case, we write $\lim_{n\to\infty} x_n \coloneqq -\infty$.

4. Let $\{x_n\}_{n=1}^{\infty}$ be an unbounded sequence of real numbers. Define the sequence of extended real number $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ by $a_n = \sup\{x_k : k \geq n\}$, $b_n = \inf\{x_k : k \geq n\}$. If each a_n and b_n is a real number, then $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n$ and $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n$.

Cauchy Sequences

1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\{x_n\}_{n=1}^{\infty}$ is said to be a cauchy sequence if, for every $\epsilon > 0$, there is some $M \in \mathbb{N}$ such that $|x_n - x_M| < M\epsilon$ whenever $n, m \geq M$.

2. Cauchy sequences are bounded.

3. A sequence of real numbers is convergent if and only if it is cauchy.

Series

1. Given a sequence $\{x_n\}_{n=1}^{\infty}$, we write the formal object $\sum_{n=0}^{\infty} x_n$ and call it a series. A series $\sum_{n=0}^{\infty} x_n$ converges if the sequence $\{S_k\}_{k=1}^{\infty}$ given by $S^k := \sum_{n=1}^{k} x_n = x_1 + x_2 + x_3 + \cdots + x_k$ converges. The numbers S_k are called partial sums. If $\sum_{n=0}^{\infty} x_n$ should converge, we write $\sum_{n=0}^{\infty} x_n = \lim_{k\to\infty} S^k = \lim_{k\to\infty} \sum_{n=0}^{k} x_n$. If the sequence $\left\{S^k\right\}_{k=1}^{\infty}$ diverges, then we say that $\sum_{n=0}^{\infty} x_n$ diverges.

2. Let $r \in \mathbb{R}$. The geometric series $\sum_{n=0}^{\infty} r^n$ converges if and only if $-1 < r < 1$. In particular $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ given that $-1 < r < 1$ and the series is convergent.

3. A series $\sum_{n=0}^{\infty} x_n$ converges if and only if its tails converge (i.e for $M \in \mathbb{N}$. the series $\sum_{n=M}^{\infty} x_n$ converges).

4. A series $\sum_{n=0}^{\infty} x_n$ is said to be cauchy if the sequence of partial sums $\left\{S^k\right\}_{k=1}^{\infty}$ is a cauchy sequence. For every $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that $\left|\sum_{k=1}^{m} x_n - \sum_{k=1}^{k} x_n\right| < \epsilon$ whenever $m, k \geq M$. Without loss of generality, we may suppose that $K > m$. Then we may write $\left|\sum_{n=m+1}^{k} x_n\right| < \epsilon$ for all $k, m \geq M$.

5. A series is cauchy if and only if for every $\epsilon > 0$ there is an $M \in \mathbb{N}$ such that $\left|\sum_{n=m+1}^{k} x_n\right| < \epsilon$ whenever $k, m \geq M$.

6. Let $\sum_{n=0}^{\infty} x_n$ be a convergent series. Then the sequence of terms $\{x_n\}_{n=1}^{\infty}$ converges and in fact $\lim_{n\to\infty} x_n = 0$. The converse is <u>not</u> true.

7. The summation is a linear operator.

8. If $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum_{n=0}^{\infty} x_n$ converges if and only if the sequence of partial sums in bounded above.

9. A series $\sum_{n=0}^{\infty} x_n$ converges absolutely if $\sum_{n=0}^{\infty} |x_n|$ converges. If a series converges, but doesn't converge absolutely, we say that it converges conditionally.

10. If $\sum_{n=0}^{\infty} x_n$ converges absolutely, then it also converges conditionally. The converse is <u>not</u> true.

Series Tests

1. *Comparison Test:* Let $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ be a pair of series such that $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$. (1) If $\sum_{n=0}^{\infty} y_n$ converges, then $\sum_{n=0}^{\infty} x_n$ converges. (2) If $\sum_{n=0}^{\infty} x_n$ diverges, then $\sum_{n=0}^{\infty} y_n$ diverges.

2. *P-series Test:* For $p \in \mathbb{R}$, the series given by $\sum_{n=0}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

3. Ratio Test: Let $\sum_{n=0}^{\infty} x_n$ be a series such that $x_n \neq 0$ for every $n \in \mathbb{N}$, and such that $L := \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$. If (1) If $L < 1$, then $\sum_{n=0}^{\infty} x_n$ converges absolutely. (2) If $L > 1$, then $\sum_{n=0}^{\infty} x_n$ diverges. (3) If $L = 1$, then the test is inconclusive.

Limits of Functions

1. Let $S \subset \mathbb{R}$. A number $x \in \mathbb{R}$ is called a cluster, or limit, point of S if, for every $\epsilon > 0$, the set $(x - \epsilon, x + \epsilon) \cap [S \setminus \{x\}]$ is nonempty. In other words, x is a cluster point of S if for every $\epsilon > 0$, there is some $y \in S$ with $y \neq x$, such that $|y - x| < \epsilon$, requiring that $y \in (x - \epsilon, x + \epsilon) \cap [S \setminus \{x\}]$.

2. Let S be a subset of $\mathbb{R}$ and $c \in \mathbb{R}$, then c is a cluster point of S if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in S \setminus \{c\}$ such that $\lim_{n\to\infty} x_n = c$.

3. Let $f : S \to \mathbb{R}$, with $S \subset \mathbb{R}$ be non empty and suppose that $c \in \mathbb{R}$ is a cluster point of S. Suppose $L \in \mathbb{R}$ is such that, for every $\epsilon > 0$, there exists some $\delta > 0$ for which $|f(x) - L| < \epsilon$ holds whenever $x \in S \setminus \{c\}$ satisfies $|x - c| < \delta$. We then say that $f(x)$

converges to L as $x \to c$, and we write $f(x) \to L$ as $x \to c$. We can this L a limit of $f(x)$ as x goes to c, and if L is unique we write $\lim_{x \to c} f(x) = L$. If no such L exists, we say that f diverges at c.

4. Let c be a cluster point of $S \in \mathbb{R}$ and let $f : S \to \mathbb{R}$ be a function such that $f(x)$ converges as x goes to c. Then the limit of $f(x)$ as $x \to c$ is unique.

5. Let $S \subset \mathbb{R}$, c be a cluster points of S, $f : S \to \mathbb{R}$ be a function, and $L \in \mathbb{R}$. Then $f(x) \to L$ as $x \to c$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S \setminus \{c\}$ for all n, and such that $\lim_{n \to \infty} x_n = c$, we have that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

6. $S \subset \mathbb{R}$, c be a cluster point of S, $f : S \to \mathbb{R}$ and $L \in \mathbb{R}$. Then $f(x) \to L$ as $x \to c$ if for every $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \in S \setminus \{c\}$ such that $|x - c| < \delta$.

7. Let $S \subset \mathbb{R}$ and c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ is a function such that the limit of $f(x)$ as $x \to c$ exists. Suppose there are two real numbers $a, b \in \mathbb{R}$ with $a \leq f(x) \leq b$ for all $x \in S \setminus \{c\}$. Then $a \leq \lim_{x \to c} f(x) \leq b$.

8. Let $S \subset \mathbb{R}$ and c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are functions such that the limits as $x \to c$ both exist. If $f(x) \leq g(x)$ holds for every $x \in S \setminus \{c\}$ then $\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$.

9. Limits are preserved across Addition, Subtraction, Multiplication, Division, and Absolute value provided there is no division by 0 and that both limits exist.

10. Let $f : S \to \mathbb{R}$ be a function and let $A \subset S \subset \mathbb{R}$. Define the function $f|_A = f(x)$ for $x \in A$. We call $f|_A$ the restriction of f to A.

11. Let $S \subset \mathbb{R}$, $c \in \mathbb{R}$, and $f : S \to \mathbb{R}$ be a function. Suppose $A \subset S$ is such that there is some $\alpha > 0$ satisfying $[A \setminus \{c\}] \cap (c - \alpha, c + \alpha) = [S \setminus \{c\}] \cap (c - \alpha, c + \alpha)$. Then (1) The point c is a cluster point of A if and only if c is a cluster point of S. (2) Supposing c is a cluster point of S, then $f(x) \to L$ as $x \to c$ if and only if $f|_A \to L$ as $x \to c$.

Continuous Functions

1. Suppose $S \subset \mathbb{R}$ and $c \in S$. We say $f : S \to \mathbb{R}$ is continuous at c is for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$. When f is continuous at all $c \in S$, then we say that f is a continuous function. If f is continuous for all $c \in A$, we say f is continuous on $A \subset S$. This implies that $f|_A$ is continuous, but the converse does not hold.

2. If for $f : S \to \mathbb{R}$ and $A \subset S$, f is continuous, then $f|_A$ is also continuous. The converse is false.

3. Consider a function $f : S \to \mathbb{R}$ defined on a set $S \subset \mathbb{R}$ and let $c \in S$. Then (1) If c is not a cluster point of S, then f is continuous at c. (2) If c is a cluster point of S, then f is continuous at c if and only if the limit of $f(x)$ as $x \to c$ exists and $\lim_{x \to c} f(x) = f(c)$. (3) The function f is continuous at c if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ where $x_n \in S$ and $\lim_{n\to\infty} x_n = c$, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(c)$.

4. The third statement above allows us to quickly apply what we know about limits of sequences to continuous functions and even prove that certain functions are continuous.

5. The Addition, Subtraction, Multiplication, and Division of functions continuous at some $c \in S$ results in a function continuous at c, given that there is no division by 0.

6. All polynomials are continuous.

7. Let $A, B \subset \mathbb{R}$ and $f : B \to \mathbb{R}$ and $g : A \to B$ be functions. If g is continuous at $c \in A$ and f is continuous at $g(c)$, then $f \circ g = f(g(x)) : A \to B$ is continuous at c.

8. Discontinuity at a point c is true when f is not continuous at c.

9. Let $f : S \to \mathbb{R}$ be a function and $c \in S$. Suppose that there exits a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in S$ for all n, and $\lim_{n\to\infty} x_n = c$ such that $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to $f(c)$. Then f is discontinuous at c.

10. $$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

    This function is discontinuous at all $c \in \mathbb{R}$.

11. $$f(x) := \begin{cases} \frac{1}{k} & \text{if } x \text{ is rational and in lowest terms} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

    This function is irrational c but discontinuous at all rational c.

12. A point is called a removable discontinuity if we could change the definition of its function by insisting that the point takes on a different value and obtain a continuous function.

Extreme Value Theorem

1. $f[a, b] \to \mathbb{R}$ is bounded is there exists a $B \in \mathbb{R}$ such that $|f(x)| \leq B$ for every $x \in [a, b]$.

2. A continuous function on a compact interval $f[a, b] \to \mathbb{R}$ is necessarily bounded.

3. (1) $f : S \to \mathbb{R}$ achieves an absolute minimum at $c \in S$ if $f(c) \leq f(x)$ for every $x \in S$. (2) $f : S \to \mathbb{R}$ achieves an absolute maximum at $c \in S$ if $f(c) \geq f(x)$ for every $x \in S$.

4. *Extreme Value Theorem:* A continuous function $f[a, b] \to \mathbb{R}$ achieves both an absolute minimum and an absolute maximum on $[a, b]$.
5. A compact interval $[a, b]$ is essential to the validity of the EVT. Continuity of f is also essential.