(2.2.3) Prove that if $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence, $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n\right)^k.$$

Proof. Prove by induction. Allow the base case to be $k = 1$. Therefore, we have that $\lim_{n\to\infty} x_n^1 = (\lim_{n\to\infty} x_n)^1$, which simplifies to $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_n$, which is true trivially as $x_n^1 = x_n$. Now suppose that $k = 2$. Defining $x \coloneqq (\lim_{n\to\infty} x_n)$, and $(\lim_{n\to\infty} x_n)^2 = x^2$, we can then write, making use of the product of limit property, $\lim_{n\to\infty} x_n^2 = \lim_{n\to\infty}(x_n \cdot x_n) = (\lim_{n\to\infty} x_n) \cdot (\lim_{n\to\infty} x_n) = x \cdot x = x^2$. This is known to hold as $\{x_n\}_{n=1}^{\infty}$ is given to converge. Therefore both base cases of $k = 1, 2$ hold true.

Assume that the statement holds true for all $k \leq m$, where $m \in \mathbb{N}$. We now have that $\lim_{n\to\infty} x_n^m = x^m$ is true by assumption and want to show that $\lim_{n\to\infty} x_n^{m+1} = x^{m+1}$. By the properties of exponents we have that $x_n^{m+1} = x_n^m \cdot x_n$, so we can use the above limit property to say $\lim_{n\to\infty} x_n^{m+1} = \lim_{n\to\infty} (x_n^m \cdot x_n) = (\lim_{n\to\infty} x_n^m) \cdot (\lim_{n\to\infty} x_n)$. Using the inductive hypothesis, we have that $\lim_{n\to\infty} x_n^m = x^m$ and can show that

$$\left(\lim_{n\to\infty} x_n^m\right) \cdot \left(\lim_{n\to\infty} x_n\right) = x^m \cdot x = x^{m+1}.$$

Note that we defined $x \coloneqq \lim_{n\to\infty} x_n$.

Therefore, through the use of limit and exponent properties along with principles of induction, we have shown that if $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence, $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n\right)^k.$$

□

(2.2.5) Let $x_n := \frac{n-\cos(n)}{n}$. Use the squeeze lemma to show that $\{x_n\}_{n=1}^{\infty}$ converges and find the limit.

Proof. We begin by rewriting x_n using basic algebraic properties as

$$x_n = \frac{n}{n} - \frac{\cos n}{n} = 1 - \frac{\cos n}{n}$$

It is known that the cosine function is bounded between -1 and 1. in other words, $|\cos n| \leq 1$. Using this, we can then create upper and lower bound functions for x_n in order to make use of the squeeze lemma. These functions can be derived as

$$\begin{aligned}
|-\cos n| &\leq 1 && \text{Add negative sign to } \cos \\
-1 \leq -\cos n &\leq 1 && \text{Definition of Absolute Value} \\
-\frac{1}{n} \leq -\frac{\cos n}{n} &\leq \frac{1}{n} && \text{Divide all terms by } n \\
1 - \frac{1}{n} \leq 1 - \frac{\cos n}{n} &\leq 1 + \frac{1}{n} && \text{Add 1 to all terms} \\
1 - \frac{1}{n} \leq x_n &\leq 1 + \frac{1}{n} && \text{Substitute } x_n
\end{aligned}$$

Therefore x_n is bounded below by $a_n = 1 - \frac{1}{n}$ and above by $b_n = 1 + \frac{1}{n}$. Consequently we can say $a_n \leq x_n \leq b_n$. It must now be shown that these bounds converge to the same limit. In other words,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Let $\epsilon > 0$ be given. By the archimedean property, there must exist some $M \in \mathbb{N}$ such that $0 < \frac{1}{M} < \epsilon$. Consequently, for every $n > M$, we have that $|x_n - 0| = \left|\frac{1}{n}\right| \leq \frac{1}{m} \leq \epsilon$, showing that $\lim_{n\to\infty} \frac{1}{n} = 0$ as required. Note that the sequence $\{1\}_{n=1}^{\infty}$ is a constant sequence and converges to 1.

We can rewrite, with the addition property of limits, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n}$. These limits can be evaluated as $\lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n} = 1 - 0 = 1$.

We can similarly rewrite $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}$. These limits can be evaluated as $\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n} = 1 + 0 = 1$.

Since both $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge to 1, then $\{x_n\}_{n=1}^{\infty} = \left\{\frac{n-\cos(n)}{n}\right\}_{n=1}^{\infty}$ must also converge to 1 as $a_n \leq x_n \leq b_n$, as required.

□

(2.2.9) Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence, $x \in \mathbb{R}$, and $x_n \neq x$ for all $n \in \mathbb{N}$. Suppose the limit

$$L \coloneqq \lim_{n\to\infty} \frac{|x_{n+1} - x|}{|x_n - x|}$$

exists and $L < 1$. Show that $\{x_n\}_{n=1}^{\infty}$ converges to x.

Proof. Since $L \coloneqq \lim_{n\to\infty} \frac{|x_{n+1}-x|}{|x_n-x|}$ and $L < 1$, if we let $\epsilon > 0$ be given, there exists some $M \in \mathbb{N}$ such that for all $n \geq M$, $\left|\frac{|x_{n+1}-x|}{|x_n-x|} - L\right| < \epsilon$. This implies the following by the definition of the absolute value

$$-\epsilon < \frac{|x_{n+1} - x|}{|x_n - x|} - L < \epsilon$$
$$L - \epsilon < \frac{|x_{n+1} - x|}{|x_n - x|} - L < L + \epsilon$$

Since $L < 1$ is given, we can choose an $\epsilon > 0$ such that $L + \epsilon < 1$. $L > 0$ must be true as $\frac{|x_{n+1}-x|}{|x_n-x|} > 0$ for all $n \in \mathbb{N}$ by definition of the absolute value. In other words, $0 < L + \epsilon < 1$. Define $\beta \coloneqq L + \epsilon$, yielding that $0 < \beta < 1$. We can now say that, for sufficiently large $n \in \mathbb{N}$ such that $n \geq M$, $\frac{|x_{n+1}-x|}{|x_n-x|} < \beta$ or equivalently that $|x_{n+1} - x| < \beta\,|x_n - x|$.

We want to show by induction that $|x_{n+k} - x| < \beta^k\,|x_n - x|$ for any $k \in \mathbb{N}$. We start with the base case $k = 1$, which is true as $|x_{n+1} - x| < \beta\,|x_n - x|$ holds by definition of the limit. Assume this holds true for $k \in \mathbb{N}$, and we want to show it also holds for $k + 1$. In other words, assume $|x_{n+k} - x| < \beta^k\,|x_n - x|$ to be true, and use this hypothesis to show $|x_{n+k+1} - x| < \beta^{k+1}\,|x_n - x|$ is also true. We know that for successive terms $n + k + 1$ and $n + k$, $|x_{n+k+1} - x| < \beta\,|x_{n+k} - x|$ is true. Because $\beta \in \mathbb{R}$, we can substitute the inductive hypothesis to say that $|x_{n+k+1} - x| < \beta\,|x_{n+k} - x| < \beta\left(\beta^k\,|x_n - x|\right)$. We can then conclude that, after combining the exponent, $|x_{n+k+1} - x| < \beta^{k+1}\,|x_n - x|$. This shows that for all $k \in \mathbb{N}$, $|x_{n+k} - x| < \beta^k\,|x_n - x|$.

Since $0 < \beta < 1$, we can now observe that as $k \to \infty$, $\beta \to 0$. In other words, $\lim_{n\to\infty} |x_{n+k} - x| = 0$, which implies that there exists some $\alpha > 0$ for which $|x_{n+k} - x| < \alpha$ for all $n \geq N$, $N \in \mathbb{N}$. Since this sequence is the k-tail of $\{x_n\}_{n=1}^{\infty}$ and is convergent to x by definition, the sequence $\{x_n\}_{n=1}^{\infty}$ must also be convergent to x.

□

(2.2.12) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences.
a) Suppose $\{a_n\}_{n=1}^{\infty}$ is bounded and $\{b_n\}_{n=1}^{\infty}$ converges to 0. Show that $\{a_nb_n\}_{n=1}^{\infty}$ converges to 0.

Proof. $\{a_n\}_{n=1}^{\infty}$ is given to be bounded, so we can say that there exists an $M > 0, M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. $\{b_n\}_{n=1}^{\infty}$ is given to converge to 0, so we can say that for any arbitrary $\epsilon > 0$, there exists an $N \in \mathbb{N}$ for which $|b_n| < \epsilon$ for all $n \geq N$. We need to show that $\{a_nb_n\}_{n=1}^{\infty}$ converges to 0, meaning that for any arbitrary $\epsilon > 0$, there exists a $k \in \mathbb{N}$ such that for all $n \geq K$, $|a_nb_n - 0| < \epsilon$.

Since $\{a_n\}_{n=1}^{\infty}$ is bounded and that absolute value is friendly with multiplication, we have $|a_nb_n| = |a_n|\,|b_n| \leq M\,|b_n|$. Since $\{b_n\}_{n=1}^{\infty}$ converges to zero, we can make $|b_n|$ arbitrarily small, and can choose $|b_n| < \frac{\epsilon}{M}$ as this holds for all $n \geq N$ and $\epsilon > 0$ is arbitrary and $M > 0$ is given. Thus we have that $|a_nb_n| = |a_n|\,|b_n| \leq M\,|b_n| < M \cdot \frac{\epsilon}{M} = \epsilon$. Thus for all $n \geq K$, we have that $|a_nb_n| < \epsilon$ which is equivalent to $|a_nb_n - 0| < \epsilon$, as required. □

b) Find an example where $\{a_n\}_{n=1}^{\infty}$ is unbounded, $\{b_n\}_{n=1}^{\infty}$ converges to 0, and $\{a_nb_n\}_{n=1}^{\infty}$ is not convergent.

Proof. Choose $a_n = n$ and $b_n = \frac{\cos n}{n}$. In this case $\{a_n\}_{n=1}^{\infty}$ is unbounded as there is no greatest natural number and $n \in \mathbb{N}$. For $\{b_n\}_{n=1}^{\infty}$, we perform similar manipulations as done in problem 2. Cosine is bounded by 1, meaning that $|\cos n| \leq 1$. We can use the squeeze lemma to show that $\frac{\cos n}{n}$ converges as

$$\begin{aligned}
|\cos n| &\leq 1 && \text{Defining Cosine}\\
-1 \leq \cos n &\leq 1 && \text{Definition of absolute value}\\
-\frac{1}{n} \leq \frac{\cos n}{n} &\leq \frac{1}{n} && \text{Divide by } n \text{ as it preserves ordering}
\end{aligned}$$

We can see that $\frac{\cos n}{n}$ is bounded below by $-\frac{1}{n}$ and above by $\frac{1}{n}$. It has been shown that $\frac{1}{n}$ converges to 0, so $-\frac{1}{n} = -\left(\frac{1}{n}\right)$ also converges to 0. Since both bounds of $\frac{\cos n}{n}$ converge to the same limit, 0, we can say that $\frac{\cos n}{n}$ also converges to 0 by the squeeze lemma. Consequently, $\{b_n\}_{n=1}^{\infty}$ is convergent and converges to 0.

We can now define $\{a_nb_n\}_{n=1}^{\infty} = \left\{n \cdot \frac{\cos n}{n}\right\}_{n=1}^{\infty} = \{\cos n\}_{n=1}^{\infty}$. The cosine function alone oscillates between -1 and 1, so it is not convergent just as $\{(-1)^n\}_{n=1}^{\infty}$ does not converge. Therefore $\{a_nb_n\}_{n=1}^{\infty}$ is not convergent as required.

□

c) Find an example where $\{a_n\}_{n=1}^{\infty}$ is bounded, $\{b_n\}_{n=1}^{\infty}$ converges to some $x \neq 0$, and $\{a_nb_n\}_{n=1}^{\infty}$ is not convergent.

Proof. Suppose $\{a_n\}_{n=1}^{\infty} = \cos n$ and $\{b_n\}_{n=1}^{\infty} = 1$. $\{a_n\}_{n=1}^{\infty}$ is bounded as $|\cos n| \leq 1$, which is the definition of boundedness as we can choose $M = 1$ for $|\cos n| \leq M$. $\{b_n\}_{n=1}^{\infty}$ is the constant sequence 1, and consequently only takes on values of 1 for all $n \in \mathbb{N}$. Therefore b_n converges to 1 and $1 \neq 0$.

We now have that $\{a_nb_n\}_{n=1}^{\infty} = \{\cos n \cdot 1\}_{n=1}^{\infty} = \{\cos n\}_{n=1}^{\infty}$. This is the same sequence as $\{a_n\}_{n=1}^{\infty}$, which is not convergent. Therefore $\{a_nb_n\}_{n=1}^{\infty}$ is not convergent as required. $\square$

(2.2.15) Prove $\lim_{n\to\infty}(n^2+1)^{\frac{1}{n}} = 1$

Proof. We can approach this similar to Example 2.2.14 from the textbook. We begin by rewriting the expression as $(n^2+1)^{\frac{1}{n}} = n^{\frac{2}{n}}\left(1+\frac{1}{n^2}\right)^{\frac{1}{n}}$. We can first analyze the limit of $n^{\frac{2}{n}}$. We can say that $n^{\frac{2}{n}} = e^{\frac{2\ln n}{n}}$. As $n \to \infty$, $\frac{2\ln n}{n} \to 0$. This can be proved using the squeeze theorem as demonstrated in 2.2.5 & 2.2.15b. Therefore, $\frac{2\ln n}{n} \to 0$ implies that $n^{\frac{2}{n}} \to e^0 = 1$.

Next we analyze $\left(1+\frac{1}{n^2}\right)^{\frac{1}{n}}$. We can take the natural log of this expression to yield $\ln\left(1+\frac{1}{n^2}\right)^{\frac{1}{n}}$. We can then leverage the fact that $\ln(1+x) \approx x$ as x gets sufficiently small. As $\frac{1}{n^2}$ approaches 0 as $n \to \infty$, we can say that $\ln\left(1+\frac{1}{n^2}\right)^{\frac{1}{n}} \approx \frac{1}{n^2}$ at the limit. Therefore, we have that

$$\ln\left(1+\frac{1}{n^2}\right)^{\frac{1}{n}} = \frac{1}{n}\ln\left(1+\frac{1}{n^2}\right) \approx \frac{1}{n}\cdot\frac{1}{n^2} = \frac{1}{n^3}$$

Note that this approximation is only true as $n \to \infty$, the limit.

Let $\epsilon > 0$ be given. By the archimedean property, there exists some $M \in \mathbb{N}$ such that $0 < M < \epsilon$. Consequently, for every $n \geq M$, we have that $\left|\frac{1}{n^3} - 0\right| = \left|\frac{1}{n^3}\right| \leq \frac{1}{M} < \epsilon$ as required. Therefore $\frac{1}{n^3}$ converges to 0, and hence as $n \to \infty$, $\left(1+\frac{1}{n^2}\right)^{\frac{1}{n}} \to 0$ implies $e^0 = 1$.

Now we combine the results into a limit using the product property of limits as

$$\lim_{n\to\infty} n^{\frac{2}{n}}\left(1+\frac{1}{n^2}\right)^{\frac{1}{n}} = \lim_{n\to\infty} n^{\frac{2}{n}} \cdot \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{\frac{1}{n}} = 1\cdot 1 = 1$$

Therefore, we can conclude that $\lim_{n\to\infty}(n^2+1)^{\frac{1}{n}} = 1$ as required.

$\square$