# Chapter 1

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Read Chapter 0 to refresh Basics of Set Theory for Chapter 3

Homeworks 250pts
- To be submitted weekly on Canvas (x10)
- Submit in LaTeX :)

3 Exams, No Final!

Brush up on LaTeX

# Real Numbers

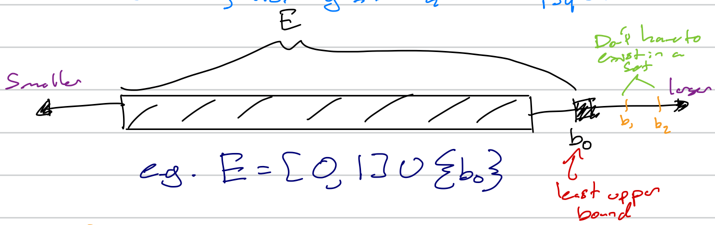
**Notations:** $\mathbb{N}$ denotes the set of natural numbers w/o the element $0$
$\mathbb{Z}$ denotes the set of integers
$\mathbb{R}$ denotes the set of real numbers $\qquad \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$
$\mathbb{Q}$ denotes the set of rational numbers

## Section 1 Basic Properties

**Def:** An ordered set is a set $S$ with a relation $<$ s.t.
(i) (trichotomy) For every pair $x, y \in S$ exactly one of the following hold:
$x<y$, $x=y$, or $y<x$
(ii) (transitivity) If $x, y, z$ belong to $S$, and satisfy $x<y$ and $y<z$, then $x<z$
- We write $x \leq y$ to mean that $x<y$ or $x=y$
- we define greater than ($>$) or $\geq$ in a similar fashion

**Examples**
- $\mathbb{Z}$ is an ordered set
- $\mathbb{Q}$ is an ordered set

we say that $y > x$, $x, y \in \mathbb{Q}$ iff $y-x$ is a positive rational number, i.e. $y-x = \frac{p}{q}$ where $p, q \in \mathbb{N}$




e.g. $E = [0,1] \cup \{b_0\}$

Greatest lower bound is same idea but on the other side :)

**Def:** Let $E \subset S$, where $S$ is an ordered set
(i) If $\exists\, b \in S$ s.t. $x \leq b \;\forall x \in E$, we say $E$ is bounded above and we call $b$ an upper bound for $E$
(ii) If $\exists\, b \in S$ s.t. $b \leq x \;\forall x \in E$, we say $E$ is bounded from below and we call $b$ a lower bound for $E$
(iii) If $\exists\, b_0 \in S$, an upper bound for $E$, s.t. $b_0 \leq b \;\forall$ upper bounds $b \in S$ of $E$, we say $b_0$ is the least upper bound, or supremum of $E$. This element $b_0 := \sup E$. $:=$ is 'denoted by'
(iv) If $\exists\, b_0 \in S$, a lower bound for $E$, s.t. $b_0 \geq b \;\forall$ lower bound $b$ of $E$, we say that $b_0$ is the greatest lower bound, or infimum, of $E$. Denote it by $b_0 := \inf E$

**More Examples**
(1) $S = \mathbb{Q}$, $E = \{x \in \mathbb{Q} : x < 1\}$
$1 \in \mathbb{Q}$ is an upper bound for $E$, & is in fact the least upper bound for $E$. However $1 \in \mathbb{Q} / E$, so $E$ does not contain it
$0$ is the additive identity on its own

(2) $S = \mathbb{Q}$, $E = \{x \in \mathbb{Q} : x \leq 1\}$
This set contains its supremum
(3) $S = \mathbb{Q}$, $E = \{x \in \mathbb{Q} : x \geq 0\}$
This set has no upper bound, so it cannot have a least upper bound. It does however have a greatest lower bound being $0 \in \mathbb{Q}$.

Allows us to take limits

**Def:** An ordered set $S$ has the least upper bound property if every non-empty subset $E \subset S$ that is bound above has a least upper bound. This is also called completeness.

**Example:** $\mathbb{Q}$ does not satisfy the least upper bound property

Claim: $\sqrt{2}$ is an irrational number

Pf/ Consider the set $E = \{x \in \mathbb{Q} : x^2 < 2\}$ pf by contradiction

(Fun Fact:) $\sqrt{2}$ exists & is the supremum for $E$.

Assume to the contrary, that there is an $x \in \mathbb{Q}$ s.t. $x^2 = 2$, and let $x = \frac{n}{m}$, $n, m \in \mathbb{Z} \setminus \{0\}$, in lowest terms
Then $m^2 = 2n^2$, where by $m^2$ is seen to be divisible by 2 $\therefore$ $m$ is also divisible by 2. Now $m^2 = 2k$, $k \in \mathbb{N}$.
$\therefore (2k)^2 = 2n^2 \rightarrow 4k^2 = 2n^2$. Dividing both sides by 2, we see $n^2$ is also divisible by 2 $\rightarrow$ contradiction

# Fields

**Def:** A set $F$ is said to be a **field** if it has two operations (closed) $+$ & $\cdot$, & if it satisfies the axioms:

Addition

(A1) If $x, y \in F$, then $x+y \in F$
(A2) $x+y = y+x \ \forall x,y \in F$
(A3) $(x+y)+z = x+(y+z) \ \forall x,y,z \in F$
(A4) $\exists$ an element $0 \in F$ s.t. $x+0 = 0+x = x \ \forall x \in F$
(A5) $\forall x \in F$, $\exists -x \in F$ s.t. $x+(-x) = -x+x = 0$

Multiplication

(M1) $x,y \in F \Rightarrow x \cdot y \in F$
(M2) $x \cdot y = y \cdot x \ \forall x,y \in F$
(M3) $(x\cdot y)\cdot z = x\cdot(y\cdot z) \ \forall x,y,z \in F$
(M4) $\exists$ an element $1 \in F$ s.t. $x \cdot 1 = 1 \cdot x = x \ \forall x \in F$
(M5) $\forall x \in F, x \neq 0 \ \exists \frac{1}{x} \in F$ s.t. $\frac{1}{x}\cdot x = x \cdot \frac{1}{x} = 1$

Distribution

(D) $x(y+z) = xy + xz \ \forall x,y,z \in F$

**Examples:**
- $\mathbb{Q}$ form of a field
- $\mathbb{Z}$ are not a field (cannot multiply)
- $\mathbb{R}$ are a field

**Def:** A field $F$ is said to be an **ordered field** if $F$ is an ordered set s.t.
1. For $x,y,z \in F$, $x<y \rightarrow x+z < y+z$
2. For $x,y \in F$, $x>0$ & $y>0$ both imply $xy>0$

- If $x>0$, then $x$ is said to be positive, and if $x<0$ then $x$ is said to be negative
  - we also say $x$ is nonnegative if $x \geq 0$, & nonpositive if $x \leq 0$

Can be defined analogously to the other operations $\{>, <, \geq, \leq\}$

**Proposition:** Let $F$ be an ordered field, and $x,y,z,w \in F$, then:
1. If $x>0$, then $-x<0$ (and vice-versa)
2. If $x>0$, & $y<z$, then $x\cdot y < x \cdot z$ ← preserves ordering
3. If $x<0$ and $y<z$, then $x\cdot y > xz$ ← flips ordering
4. If $x \neq 0$, then $x^2 > 0$ Note: ④ $\Rightarrow 1>0$
5. If $0<x<y$, then $0<\frac{1}{y}<\frac{1}{x}$
6. If $0<x<y$, then $x^2<y^2$
7. If $x \leq y$ & $z \leq w$, then $x+z \leq y+w$

Do not submit proofs written in symbols!

**Pf/:** We settle items ① and ⑤

① The inequality $x>0$, & item ① in definition of an order field imply that $0+(-x) < x+(-x)$
Now, as $x+(-x)=0$ because $F$ is a field, we conclude that $-x<0$ whenever $x>0$, as required

Every line should be carefully explained

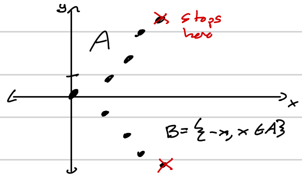
⑤ First note that $\frac{1}{x}$ is nonzero as a multiplicative inverse of a nonzero element in a field.
Suppose $\frac{1}{x}<0$, then $-\frac{1}{x}>0$ by part ① of the proposition. By part ② of def. of an ordered field, $x\cdot(-\frac{1}{x})>0$, which means $-1>0$, a contradiction. (using item ① of proposition) Therefore $1/x > 0$. A similar argument shows $1/y > 0$
Thus, $1/x \cdot 1/y > 0$ (by ② in def of ordered field)
Finally, we can conclude $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)x < \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)(y)$ So, $0<\frac{1}{y}<\frac{1}{x}$
commutative and associative

**Homework Correction**

#1 Given $x,y \in F$, $F$ an ordered field
$0<x<y$, prove $x^2<y^2$

→ Sup exists

**Proposition:** Let $F$ be an ordered field with the least upper bound property
↳ Let $A \subset F$ be a non-empty set bounded below. Then $\inf A$ exists in $F$.




Pf/ Consider $B = \{-x \,;\, x \in A\}$

Let $b \in F$ be a generic lower bound for $A$; this means given any $x \in A$, we know that $x \geq b$. In other words, $\forall\, x \in A$, $-x \leq -b$. Consequently $-b$ is an upper bound for $B$. Since $B$ is a non-empty & bounded from above set in an ordered field $F$ that satisfies the completeness property, $\sup B$ exists as an element in $F$. $C := \sup B$

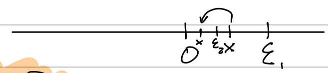
$A = \{x \in \mathbb{Q} : x^2 < 2\}$

Note that $C \leq -b$. $\forall\, y : y \in B$, we know that $y \leq C$. By the definition of $B$, we know $-x \leq C$ $\forall\, x \in A$. Putting everything together: $-x \leq C \leq -b$ $\forall\, x \in A$. Multiply by $-1$, we know $x \geq -c \geq b$ $\forall\, x \in A$, so $-c = \inf A$. //

## Section 2 The set of Real Numbers

**Theorem:** There exists a unique ordered field $\mathbb{R}$ with the least upper bound property that contains $\mathbb{Q}$. This field $\mathbb{R}$ is called the set of real numbers.

**Proposition:** If $x \in \mathbb{R}$ such that $x \leq \varepsilon$ holds $\forall\, \varepsilon \in \mathbb{R}$, when $\varepsilon > 0$, then $x \leq 0$




Forces x negative

Pf/ If $x > 0$, then $0 < \frac{x}{2} < x$. Choosing $\varepsilon = \frac{x}{2}$ results in a contradiction //

**Example:** The set $A = \{x \in \mathbb{Q} : x^2 < 2\}$ has a least upper bound $\sup A$ that does not belong to the rational numbers.

Claim: There is a unique number $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $r^2 = 2$. We denote this number by $r = \sqrt{2}$.

Pf/ We begin by showing $A$ is bounded from above and non-empty. Note that $A \neq \emptyset$, since $1 \in A$. The equation $x \geq 2$ implies that $x^2 \geq 4$, so if $x^2 < 2$, then $x < 2$. Therefore $A$ is bounded from above. Thus, as $\mathbb{R}$ satisfies the least upper bound property, $r := \sup A$ exists as an element.

Goal: Show $r^2 = 2$

We'll do this by showing that $r^2 \geq 2$ and $r^2 \leq 2$

Step 1: $r^2 \geq 2$.

Choose an $s > 0$ such that $s^2 < 2$. We will search for an $h > 0$ such that $(s+h)^2 < 2$. Since $2 - s^2 > 0$, we see that $\frac{2-s^2}{2s+1} > 0$. Choose $h$ such that $0 < h < \frac{2-s^2}{2s+1}$. We may also choose $h < 1$. Then we can estimate

$$\begin{aligned} (s+h)^2 - s^2 &= h(2s+h) \\ &< h(2s+1) && \text{as } 0<h<1 \\ &< 2 - s^2 && \text{as } h < \tfrac{2-s^2}{2s+1} \end{aligned}$$

Consequently, $(s+h)^2 < 2$. Thus, $s + h \in A$. Hence $s < r := \sup A$. As $s > 0$ is arbitrary with $s^2 < 2$, it follows that $r^2 \geq 2$.

Step 2: $r^2 \leq 2$. Apply similar logic

By Steps 1 and 2, we know that $r^2 = 2$.

Uniqueness follows as usual. ▨

Written in textbook pretty clearly

The set $\mathbb{R}/\mathbb{Q}$ which is non-empty is called the set of irrational numbers.

# Archimedean Property

Theorem:

(i) (Archimedean Property): If $x, y \in \mathbb{R}$ and $x > 0$, there must exist a natural number $n$ s.t. $nx > y$

↳ For every natural number, you can find a real number smaller than it

(ii) ($\mathbb{Q}$ is dense in $\mathbb{R}$): If $x, y \in \mathbb{R}$ and $x < y$, $\exists\, r \in \mathbb{Q}$ s.t. $x < r < y$

↳ Between any real numbers you can always find a rational number between

Proof: We begin with the proof of item (i)

Dividing $nx > y$ by $x$, item (i) asserts that $\forall t \in \mathbb{R}$, $t := y/x$, we can find $n \in \mathbb{N}$ such that $n > t$. In other words (i) asserts that $\mathbb{N} \subset \mathbb{R}$ is not bounded from above

Assume to the contrary that $\mathbb{N}$ is indeed bounded from above as a subset of $\mathbb{R}$. By the completeness of $\mathbb{R}$, there is a finite number $b := \sup \mathbb{N}$. As $b$ is the least upper bound of the natural numbers, $(b-1)$ can no longer be an upper bound for $\mathbb{N}$. Therefore, there must be some $m \in \mathbb{N}$ such that $m > b-1$. Adding 1 to both sides, & noting that $m+1 \in \mathbb{N}$, $m+1 > b$, a contradiction $\boxtimes$

We proceed to the proof of item (ii).

First, we suppose that $x \geq 0$. Then $y - x > 0$. By part (i), there is a natural number $n$ such that $n(y-x) > 1$, or $(y-x) > 1/n$.

Again using part (i), the set $A = \{k \in \mathbb{N} : k \geq nx\}$ is non-empty. By the well-ordering property of the natural numbers $A$ has a least element $m$. Then $m \geq nx$. If $m > 1$, then $m-1 \geq 0$ and $m-1 \leq nx$ as $x \geq 0$. In other words,

$$m - 1 \leq nx \quad \text{or} \quad m \leq nx + 1$$

On the other hand, $n(y-x) > 1$ so we obtain $ny > 1 + nx$. Consequently, $ny \geq 1 + nx > m$; hence $y > \frac{m}{n}$. Putting everything together, we know that $x < \frac{m}{n} < y$. Choose $r := m/n$.

Suppose now that $x < 0$.

- If $y > 0$, choose $r = 0$
- If $y \leq 0$, then $0 \leq -y < -x$, and from the first case we can choose $q \in \mathbb{Q}$ such that $-y < q < -x$. Take $r = -q$ in this case $\boxtimes$

Corollary

$\inf\{1/n : n \in \mathbb{N}\} = 0$

Proof: Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. The set $A$ is non-empty & and $1/n > 0$ for every $n \in \mathbb{N}$.

Therefore, 0 is a lower bound for $A$, and so $b := \inf A$ exists. We also know that $b \geq 0$. By the archimedean property, there must exist an $n \in \mathbb{N}$ such that $na > 1$, with $a > 0$ being arbitrary. In other words, for any positive number $a$, there is some $n \in \mathbb{N}$ for which $a > 1/n$. Thus, $a$ cannot be a lower bound for $A$, whenever $a > 0$. Consequently, $b = \inf A = 0$. $\boxtimes$

# Using sup & inf

- For $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, define the translation of $A$ by $x$ via

$$x + A = \{x + a : a \in A\} \quad \text{and then} \quad xA = \{xa : a \in A\}$$

**Proposition** Let $A \subset \mathbb{R}$ be non-empty

(i) If $x \in \mathbb{R}$ and $A$ is bounded above, then $\sup(x+A) = x + \sup A$

(ii) If $x \in \mathbb{R}$ and $A$ is bounded below, then $\inf(x+A) = x + \inf A$

(iii) If $x \geq 0$ and $A$ is bounded above, then $\sup(xA) = x \cdot (\sup A)$

(iv) If $x \geq 0$ and $A$ is bounded below, then $\inf(xA) = x \cdot (\inf A)$

(v) If $x < 0$, and $A$ is bounded below, then $\sup(xA) = x \cdot (\inf A)$

(vi) If $x < 0$, and $A$ is bounded above, then $\inf(xA) = x \cdot (\sup A)$

**Proof** We prove (i)

Let $b$ be an upper bound for $A$, that is, $a \leq b \ \forall a \in A$. Thus, $x + a \leq x + b$ $\forall x \in A$. Thus, $x + b$ is an upper bound for $x + A$. Hence, $\sup(x+A) \leq x + b$.

- Choosing $b := \sup A$, we conclude that $\sup(x+A) \leq x + \sup A$

Next, let $c$ be an upper bound for $x + A$. This means $z \leq c \ \forall z \in x + A$. Note that $z = x + w$ for some $w \in A$. So $w \leq c - x \ \forall w \ w \in A$. Then $c - x$ is an upper bound for $A$. In particular, $\sup A \leq c - x \ \forall$ upper bound $c$ of $x + A$.

- choosing $c := \sup(x+A)$, we conclude that $\sup(A) + x \leq \sup(x+A)$ $\boxtimes$

**Proposition:** Let $a, b$ be any pair of non-empty subsets of $\mathbb{R}$ s.t. $x \leq y$ whenever $x \in A$ and $y \in B$. Then, $A$ is bounded above, $B$ is bounded below, and $\sup(A) \leq \inf(B)$

**Proof:** Any element of $A$ is a lower bound for $B$.

Moreover, since $B$ is non-empty and bounded below, the completeness property of $\mathbb{R}$ guarantees $\inf(B)$ exists. Therefore, $x \leq \inf B \ \forall x \in A$. So $\inf B$ is an upper bound for $A$, and we conclude that $\sup A \leq \inf B$ $\boxtimes$

**Question:** Given two sets $A, B$ in $\mathbb{R}$ such that $x < y \ \forall x \in A, y \in B$. Does it hold that $\sup A < \inf B$? NO!!!

Counterexample: Choose $A = \{0\}$, $B = \{\frac{1}{n}, n \in \mathbb{N}\}$. Then $\inf B = \sup A = 0$

**Proposition:** If $S \subset \mathbb{R}$ is non-empty and bound from above, then $\forall \varepsilon > 0, \exists x \in S$ s.t. $\sup S - \varepsilon < x \leq \sup S$

# Extended Real Numbers

Def: Let $A \subseteq \mathbb{R}$

(i) If $A = \emptyset$, define $\sup A := -\infty$

(ii) If $A$ is not bounded above & non-empty, then $\sup A := +\infty$

(iii) If $A$ is empty, define $\inf A := +\infty$

(iv) If $A$ is not bounded below & non-empty, then $\inf A := -\infty$

The set $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$ is called the extended Real Numbers & it can be made an ordered set via: $-\infty < \infty$, $-\infty < x < \infty$ $\forall x \in \mathbb{R}$

Def: When $A \subset \mathbb{R}$ non-empty and bounded above, and $\sup A \in A$, then $\sup A$ is called the maximum of $A$, and it is denoted by $\max A$.

Not always the case

If $A \subset \mathbb{R}$ non-empty and bounded below, and $\inf A \in A$, then $\inf A$ is called the minimum of $A$ and is denoted $\min A$.

Fact: Any non-empty finite subset of $\mathbb{R}$ has a minimum and a minimum and maximum

↳ Proved by induction on HW1

# Absolute Value & Functions

For any $x \in \mathbb{R}$, define $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Proposition:

(i) $|x| \geq 0$, with equality iff $x = 0$.

(ii) $|-x| = |x|$ for all $x \in \mathbb{R}$ symmetry, not negativity

(iii) $|xy| = |x| \cdot |y|$ for all $x, y \in \mathbb{R}$

(iv) $|x^2| = x^2$ $\forall x \in \mathbb{R}$

(v) $|x| \leq y$ iff $-y \leq x \leq y$

(vi) $-|x| \leq x \leq |x|$ $\forall x \in \mathbb{R}$

Proposition (Triangle Inequality): For any pair $x, y \in \mathbb{R}$,

$$|x+y| \leq |x| + |y|$$

Proof: By (vi) of the previous proposition, we know that $-|x| \leq x \leq |x|$ & $-|y| \leq x \leq |y|$.
Addition of these two equations yields $-(|x|+|y|) \leq x+y \leq (|x|+|y|)$
Apply item (v) $\blacksquare$

Corollary: For any pair $x, y \in \mathbb{R}$, the following hold:

(i) (Reverse triangle ineq): $\big||x| - |y|\big| \leq |x-y|$

(ii) $|x-y| \leq |x| + |y|$

Proof: We settle item (i)

Set $x = a-b$, $y = b$ for some arbitrary pair $a, b \in \mathbb{R}$. Applying the triangle ineq. we get

$$|a| = |\underbrace{a-b}_{x} + \underbrace{b}_{y}| \leq |a-b| + |b|,$$

or equivalently, that

$$|a| - |b| \leq |a-b|$$

Switching the roles of a and b, we also have

$$|b| - |a| \leq |a-b|$$

Apply item (v) of the previous proposition to get the desired result $\blacksquare$

**Corollary:** Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$ Then

Then $|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots |x_n|$ Induction Proof

**Example:** Find a number $M$ such that $|x^2 - 9x + 1| \leq M$ for $-1 \leq x \leq 5$.

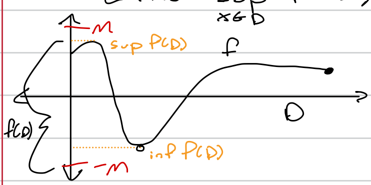
Solution: For any $x \in \mathbb{R}$, the triangle ineq gives

$|x^2 - 9x + 1| \leq |x^2| + 9|x| + 1$

For those $-1 \leq x \leq 5$, the maximum of $|x^2| + 9|x| + 1$ occurs when $x = 5$.

So choose $M = 5^2 + 9(5) + 1 = 71$. ← Not the best $M$ but it holds

**Def:** Suppose $f: D \to \mathbb{R}$ is a function. We say that $f$ is bounded if there is some $M > 0$ such that $|f(x)| \leq M$ for all $x \in D$. For function $f: D \to \mathbb{R}$, we write $\sup\limits_{x \in D} f(x) := \sup f(D)$ & $\inf\limits_{x \in D} f(x) := \inf f(D)$




sup and inf still don't need to exist as part of the image set

**Example:** Let $D = \{x : -1 \leq x \leq 5\}$ & $f(x) = x^2 - 9x + 1$

Using calculus I: $\sup\limits_{x \in D} f(x) = \sup\limits_{-1 \leq x \leq 5} [x^2 - 9x + 1] = 11$ Just take derivative to find min and max.

$\inf\limits_{x \in D} f(x) = \inf\limits_{-1 \leq x \leq 5} [x^2 - 9x + 1] = -\frac{77}{4}$

**Proposition:** Given a pair of bounded functions $f, g: D \to \mathbb{R}$, with $D$ being non-empty, such that $f(x) \leq g(x)$ for all $x \in D$, it holds that

$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x) \quad \& \quad \inf_{x \in D} f(x) \leq \inf_{x \in D} g(x)$$

Caution: The $x$ on LHS of these inequalities is different than the $x$ on the RHS

For example, the first should be thought of as: $\sup\limits_{x \in D} f(x) \leq \sup\limits_{y \in D} g(y)$

**Proof:** Suppose $b$ is an upper bound for $g(D)$. Then, for every $x \in D$ we have $f(x) \leq g(x) \leq b$ based on the propositions assumption, so $b$ is an upper bound for $f(D)$.

In other words, $f(x) \leq b$ for every $x \in D$. Thus for all $x \in D$, $f(x) \leq \sup\limits_{y \in D} g(y)$.

Consequently, $\sup\limits_{x \in D} f(x) \leq \sup\limits_{y \in D} g(y)$ ∎

**Remark:** Under the Hypothesis of this proposition, the ineq $\sup\limits_{x \in D} f(x) \leq \inf\limits_{y \in D} g(y)$ is false

Look up counter example in Homework

- Look at $x$ and $x+1$ from $0 \to 10$
- Or look at $1/n$

# Intervals

## Intervals in $\mathbb{R}$

Given $a, b \in \mathbb{R}$ $a \le b$ set

- $[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$ closed interval
- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ open interval
- $(a, b] = \{x \in \mathbb{R} : a < x \le b\}$ half open interval
- $[a, b) = \{x \in \mathbb{R} : a \le x < b\}$ "

All such intervals are called bounded

## Unbounded Intervals

Given $a, b \in \mathbb{R}$ $a \le b$ set

- $[a, \infty) = \{x \in \mathbb{R} : a \le x < \infty\}$ closed interval
- $(a, \infty) = \{x \in \mathbb{R} : a < x < \infty\}$ open interval
- $(-\infty, b] = \{x \in \mathbb{R} : -\infty < x \le b\}$ closed interval
- $(-\infty, b) = \{x \in \mathbb{R} : -\infty < x < b\}$ open interval
- $(-\infty, \infty) = \mathbb{R}$ open interval

**Proposition:** A set $I \subset \mathbb{R}$ is an interval iff $I$ contains at least two points, and for all $a, b \in I$ and $c \in \mathbb{R}$ such that $a < c < b$, we have that $c \in I$.

**Theorem:** $\mathbb{R}$ is an uncountable set

# Chapter 2

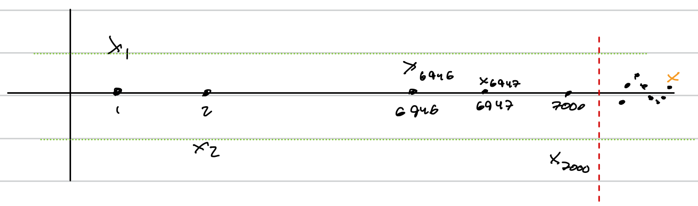
# Sequences and Series

**Def:** A sequence of real numbers is any function $x: \mathbb{N} \to \mathbb{R}$. Instead of using $x(n)$, we use the notation $x_n$ to denote the $n^{th}$ element of the sequence.
→ To denote the sequence, we will use $\{x_n\}_{n=1}^{\infty}$, $\{x_n\}_n$, $\{x_n\}$, $\{x_n\}_1^{\infty}$ interchangeably
↳ Use interchangeably based on needs of proof. Royden uses $\{x_n\}_1^{\infty}$ too

A sequence is bounded if there exists $\{M>0: B\in\mathbb{R}\}$ such that $|x_n| \le M$ for every $n \in \mathbb{N}$
In other words the set $\{x_n : n \in \mathbb{N}\} \subset \mathbb{R}$ is a bounded set
↑ subset of the real line

**Examples:**
✓ (i) $\{\frac{1}{n}\}_{n=1}^{\infty}$ is bounded.
Choose $M=1$
✓ (ii) Let $c \in \mathbb{R}$. Define the constant sequence $\{c_n\}_{n=1}^{\infty} = \{c, c, c, \dots, c, \dots\}$
Choose $M = |c|$ will do the trick
✗ (iii) $\{n\}_{n=1}^{\infty}$ is not bounded
✓ (iv) $\{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1 \dots\}$
Choose $M \geq 1$

**Def:** A sequence $\{x_n\}_1^{\infty}$ is said to converge to some $x \in \mathbb{R}$ if for every $\varepsilon > 0$, there exist $M \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n \geq M$.

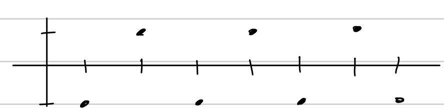



The number $x$ is called a limit of $\{x_n\}_1^{\infty}$ & we write
$$x := \lim_{n\to\infty} x_n$$
A sequence that converges is said to be convergent. If a sequence does not converge, it is said to be divergent or it diverges

**Example:** The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges & $\lim_{n\to\infty} \frac{1}{n} = 0$

**Proof:** Let $\varepsilon > 0$ be given. By the archimedean property, there must exist some $M \in \mathbb{N}$ such that $0 < \frac{1}{M} < \varepsilon$. Consequently, for every $n \geq M$, we have that $|x_n - 0| = |\frac{1}{n}| \leq \frac{1}{M} < \varepsilon$, as required. ∎

**Example:** The sequence $\{(-1)^n\}_{n=1}^{\infty}$ diverges ← the alternating sequence



**Proof:** Assume to the contrary that $\{(-1)^n\}_{n=1}^{\infty}$ converges to some $x \in \mathbb{R}$ & let $\varepsilon = \frac{1}{2}$.
There must exist an $M \in \mathbb{N}$ such that $|(-1)^n - x| < \frac{1}{2}$ whenever $n \geq M$.

- For each even $n \geq M$, we get $\frac{1}{2} > |\overset{x_n}{1} - x|$ & $|x_{n+1} - x| = |\overset{1ex}{-1} - x| < \frac{1}{2}$

$$\frac{1}{2} + \frac{1}{2} > |1-x| + |-1-x| \geq |1 - x + 1 + x| = 2$$
↑ Use triangle inequality

$1 > 2$ Contradiction ∎

Tricks used here in this proof useful for entire semester

**Proposition:** A convergent sequence has a unique limit.

**Proof:** Suppose that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence, & that $x, y \in \mathbb{R}$ are limits of $\{x_n\}_1^{\infty}$. Let $\varepsilon > 0$ be arbitrary. Since $\{x_n\}_1^{\infty}$ converges to $x$, there must exist an $M_1 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon/2$. Similarly, as $\{x_n\}_1^{\infty}$ converges to $y$, there must be some $M_2 \in \mathbb{N}$ such that $|x_n - y| < \varepsilon/2$ whenever $n \geq M_2$.

Apply Def Twice

Let $M := \max\{M_1, M_2\}$. Then by the triangle inequality

$$|x - y| \leq |x_n - x| + |x_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

whenever $n \geq M$. Since $\varepsilon > 0$ can be made arbitrarily small, we conclude that $x = y$.

**Proposition:** A convergent sequence is bounded

**Proof:** Suppose that $\{x_n\}$ converges to $x$. For $\varepsilon = 1$ there must exist some $M \in \mathbb{N}$ such that $|x_n - x| < 1$ whenever $n \geq M$. Thus, for $n \geq M$, $|x_n| \leq |x_n - x| + |x| < 1 + |x|$.

The set $\{|x_n| : n = 1, 2, \ldots, M\}$ is finite & nonempty so it has a maximum say $\max\{|x_n| : n = 1, 2, 3, \ldots M\} = V$

Therefore, for every $n \in \mathbb{N}$, $|x_n| \leq \max\{V, 1 + |x|\}$. Consequently, $\{x_n\}$ is a bounded sequence.

Caution: Bounded sequences are not guaranteed to converge.

**Example:** Show that $\left\{\frac{n^2+1}{n^2+n}\right\}_1^{\infty}$ converges & that $\lim_{n \to \infty} \frac{n^2+1}{n^2+n} = 1$

Scratch work:

$$\left|\frac{n^2+1}{n^2+n} - 1\right|, \quad \left|\frac{n^2+1-(n^2+n)}{n^2+n}\right| = \left|\frac{-1+n}{n^2+n}\right| \leq \frac{1+|n|}{n(n+1)} = \frac{1+n}{n(n+1)} = \frac{1}{n}$$

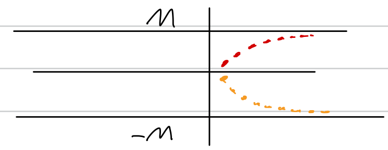
$\Rightarrow$ We know that $\left\{\frac{1}{n}\right\}_1^{\infty}$ converges to zero ✓

Expect one question like this on the exam.

**Proof:** Let $\varepsilon > 0$ be given. Since $\left\{\frac{1}{n}\right\}_1^{\infty}$ converges to 0, there must exist an $M \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ whenever $n \geq M$. Therefore, by the triangle inequality, $\left|\frac{n^2+1}{n^2+n} - 1\right| \leq \frac{1+n}{n(n+1)} = \frac{1}{n} < \varepsilon$ whenever $n \geq \mathbb{N}$. Since $\varepsilon > 0$ is arbitrary, we know that $\left\{\frac{n^2+1}{n^2+n}\right\}_{n=1}^{\infty}$ converges to 1. ▨

**Def:** A sequence $\{x_n\}_{n\geq 1}^{\infty}$ is said to be monotone increasing if for all $n \in \mathbb{N}$, $x_n \leq x_{n+1}$, and is said to be monotone decreasing if $x_n \geq x_{n+1}$ for every $n \in \mathbb{N}$. If a sequence $\{x_n\}_1^{\infty}$ is either one of these types, then $x_n$ is said to be monotone.

**Examples:** $\{\frac{1}{n}\}_1^{\infty}$ is monotone decreasing
$\{n\}_1^{\infty}$ is monotone increasing




You can write MCT

**Monotone Converge Theorem:** A monotone sequence $\{x_n\}_1^{\infty}$ is convergent if and only if it is bounded. Furthermore, if $\{x_n\}_1^{\infty}$ is monotone increasing and bounded, then $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$.

On the other hand, if $\{x_n\}_1^{\infty}$ is monotone decreasing and bounded, then $\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$

**Proof:** We will prove the theorem in the instance when $\{x_n\}_1^{\infty}$ is monotone increasing

Suppose that the sequence $\{x_n\}_{n\geq 1}^{\infty}$ is bounded; that is the set $\{x_n : n \in \mathbb{N}\}$.
Let $x := \sup\{x_n : n \in \mathbb{N}\}$.

Let $\varepsilon > 0$ be given. As $x$ is the sup of $\{x_n : n \in \mathbb{N}\}$, there must be at least one element $x_M \in \{x_n : n \in \mathbb{N}\}$ such that $x_M > x - \varepsilon$. As $\{x_n\}_1$ is monotone increasing, we know $x_n \geq x_M$ whenever $n \geq M$. Consequently, for any $n \geq M$,

$$|x_n - x| = x - x_n \leq x - x_M < \varepsilon.$$

Therefore $\{x_n\}_1^{\infty}$ converges to $x$. We have already proven the other direction: every convergent sequence is bounded.

**Example:** Consider the sequence $\{\frac{1}{\sqrt{n}}\}_{n\geq 1}^{\infty}$.

This sequence is bounded from below by $0$, as $\frac{1}{\sqrt{n}} \geq 0$ for every $n \in \mathbb{N}$. It is also monotone decreasing, as $\sqrt{n} \leq \sqrt{n+1}$ for every $n \in \mathbb{N}$ (you would need to explain why). Then $\frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}}$ for every $n \in \mathbb{N}$.

It follows from the MCT, that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = \inf\{\frac{1}{\sqrt{n}} : n \in \mathbb{N}\}$ Homework: Show $\inf\{\frac{1}{\sqrt{n}} : n \in \mathbb{N}\} = 0$

**Proposition:** Let $S \subset \mathbb{R}$ be a non-empty bounded set. Then there exist sequences $\{x_n\}$, $\{y_n\}$ $x_n, y_n \in S$ for every $n \in \mathbb{N}$, such that

$$\sup S = \lim_{n\to\infty} x_n \quad \& \quad \inf S = \lim_{n\to\infty} y_n$$

# Exam 1: Sheet $\mathcal{G}$, S×ll w/ Definitions & thm. handwritten

**Def:** For a sequence $\{x_n\}_{n\geq 1}^{\infty}$, the $K$-tail, $K\in\mathbb{N}$, or just tail of the sequence is the sequence starting at $K+1$ usually written as $\{x_{n+K}\}_{n\geq 1}^{\infty}$ or $\{x_n\}_{n=K+1}^{\infty}$

**Proposition:** Let $\{x_n\}$ be a sequence. Then the following are equivalent:

(i) The sequence $\{x_n\}_{n\geq 1}^{\infty}$ converges

(ii) The $K$-tail $\{x_{n+K}\}_{n\geq 1}^{\infty}$ converges for every $K\in\mathbb{N}$

(iii) The $K$-tail $\{x_{n+K}\}_{n\geq 1}^{\infty}$ converges for some $K\in\mathbb{N}$

Furthermore, if any (and hence all) of the limits exist, then for every $K\in\mathbb{N}$,

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+K}$.

**Proof:** The implication (ii) => (iii) is immediate. We will show that (i) implies (ii) and that (iii) implies (i). We begin with (i) → (ii)

Suppose that $\{x_n\}_{n\geq 1}$ converges to some $x\in\mathbb{R}$. Let $K\in\mathbb{N}$ be arbitrary, and define $y_n = x_{n+K}$ for each $n\in\mathbb{N}$. Goal: Show $\{y_n\}_{n\geq 1}^{\infty}$ converges to $x$. Given any $\varepsilon>0$, there is an $M\in\mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n\geq M$. Note that $n\geq M$ implies $n+K\geq M$. Therefore $|y_n - x| = |x_{n+K} - x| < \varepsilon$ for every $n\geq M$. Hence the sequence $\{y_n\}_{n\geq 1}^{\infty}$ converges. This completes that (i) → (ii). We next prove that (iii) → (i)

Let $\bar{K}\in\mathbb{N}$ be the necessary $K$ for which (iii) holds. Define $y_n = x_{n+\bar{K}}$, & assume that $\{y_n\}_{n\geq 1}^{\infty}$ converges to $y\in\mathbb{R}$. We need to show that $\{x_n\}$ converges to $y$. Given $\varepsilon>0$, there is some $M\in\mathbb{N}$ such that $|y_n - y| < \varepsilon$ for all $n\geq M$. Set $M' = M+\bar{K}$. Then $n\geq M'$ implies that $n-\bar{K}\geq M$. Thus, $|x_n - y| = |y_{n-\bar{K}} - y| < \varepsilon$ whenever $n\geq M'$ as required.

**Definition:** Let $\{x_n\}_{n\geq 1}^{\infty}$ be a sequence. Let $\{n_i\}_{i\geq 1}^{\infty}$ be a strictly increasing seq. of natural numbers, i.e. $n_i < n_{i+1}$ for all $i\in\mathbb{N}$. The sequence $\{x_{n_i}\}_{i\geq 1}^{\infty}$ is called a subsequence of $\{x_n\}_{n\geq 1}^{\infty}$

**Example:** $\{(-1)^{2i}, i\in\mathbb{N}\} = \{1, 1, 1\ldots\}$ is a subsequence of $\{(-1)^n, n\in\mathbb{N}\}$

**Proposition:** If $\{x_n\}_{n\geq 1}^{\infty}$ is a convergent sequence, then every subsequence of $\{x_n\}_{n\geq 1}^{\infty}$ must converge to the <u>same</u> limit

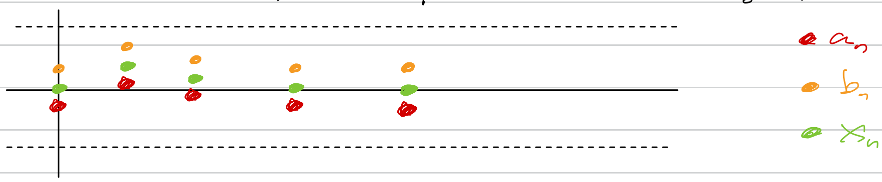
This word is important

**Proof:** Suppose $\lim_{n\to\infty} x_n = x$ and let $\varepsilon>0$ be arbitrary. There must exist some $M\in\mathbb{N}$ for which $|x_n - x| < \varepsilon$ whenever $n\geq M$. By induction, it follows that $n_i\geq i$ for any $i\in\mathbb{N}$. Consequently $i\geq M$ implies that $n_i\geq M$. Thus for all $i\geq M$, $|x_{n_i} - x| < \varepsilon$ as needed. $\blacksquare$

# This is Exam 1 Cutoff!!!

# Facts about Sequences

**Squeeze Lemma:** Let $\{a_n\}_{n\geq1}^{\infty}, \{b_n\}_{n\geq1}^{\infty}, \{x_n\}_{n\geq1}^{\infty}$ be such that $a_n \leq x_n \leq b_n$ for all $n \in \mathbb{N}$. If $\{a_n\}_1^{\infty}$ and $\{b_n\}_1^{\infty}$ both converge to $x \in \mathbb{R}$, then $\{x_n\}_1^{\infty}$ must also converge to $x$.




**Proof:** Let $\varepsilon > 0$ be given. As $\{a_n\}_{n\geq1}^{\infty}$ converges to $x$, there must be an $M_1 \in \mathbb{N}$ such that $|a_n - x| < \varepsilon$ holds for all $n \geq M_1$. As $\{b_n\}_{n\geq1}^{\infty}$ converges to $x$, there is an $M_2 \in \mathbb{N}$ such that $|b_n - x| < \varepsilon$ for those $n \geq M_2$. Set $M$ to be the maximum among $M_1$ and $M_2$, $M := \max(M_1, M_2)$, and suppose that $n \geq M$. For those such $n$, it holds that $x - \varepsilon < a_n$ & $b_n < x + \varepsilon$. Thus,

$$x - \varepsilon < a_n \leq x_n \leq b_n < x + \varepsilon$$

In other words $-\varepsilon < x_n - x < \varepsilon$ whenever $n \geq M$. This is equivalent to $|x_n - x| < \varepsilon$ whenever $n \geq M$, as required. ∎

**Lemma:** Let $\{x_n\}_1^{\infty}, \{y_n\}_1^{\infty}$ be convergent sequences. Suppose that $x_n \leq y_n$ for every $n \in \mathbb{N}$. Then the $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.

Absolute value equiv $|x_n - x|$

**Proof:** Set $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$. Let $\varepsilon > 0$ be given. Choose $M_1 \in \mathbb{N}$ such that $|x_n + x| < \frac{\varepsilon}{2}$ whenever $n \geq M_1$ and $M_2 \in \mathbb{N}$ such that $|y_n - y| < \frac{\varepsilon}{2}$ whenever $n \geq M_2$. Set $M = \max\{M_1, M_2\}$. For those $n \geq M$, it holds that $x_n - x < \varepsilon/2$ & $y_n - x < \frac{\varepsilon}{2}$. Adding these inequalities we get $(y_n - x_n)(x - y) < \varepsilon$ or that $y_n - x_n < y - x + \varepsilon$

whenever $n \geq M$. Because $x_n \leq y_n$, this becomes $0 \leq y_n - x_n < y - x + \varepsilon$ for those $n \geq M$. Therefore $0 < y - x + \varepsilon$. In other words, $x - y < \varepsilon$. As $\varepsilon > 0$ is arbitrary, we conclude that $x - y \leq 0$. ∎

**Corollary**

(i) If $\{x_n\}$ is a convergent sequence such that $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n \geq 0$

(ii) Let $a, b \in \mathbb{R}$ and $\{x_n\}$ be a convergent sequence such that $a \leq x_n \leq b$ for all $n \in \mathbb{N}$. Then $a \leq \lim_{n\to\infty} x_n \leq b$

**Algebra of Limits**: Let $\{x_n\}$, $\{y_n\}$ be convergent sequences. Then the following hold

(i) $\lim_{n\to\infty} [x_n + y_n] = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$ Addition

(ii) $\lim_{n\to\infty} [x_n - y_n] = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n$ Subtraction

(iii) $\lim_{n\to\infty} [x_n y_n] = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$ Multiplication

(iv) If $y_n \neq 0$ & $\lim_{n\to\infty} y_n \neq 0$ for every $n \in \mathbb{N}$, then Quotient
$$\lim_{n\to\infty}\left[\frac{x_n}{y_n}\right] = \left(\frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}\right)$$

**Proof** We settle (i)

(i): Suppose $x = \lim_{n\to\infty} x_n$ & $y = \lim_{n\to\infty} y_n$. Let $\varepsilon > 0$ be given. Choose $M_1 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n \geq M_1$, and choose $M_2 \in \mathbb{N}$ such that $|y_n - y| < \varepsilon$ whenever $n \geq M_2$. Define $M = \max\{M_1, M_2\}$ Then, for every $n \geq M$, the triangle inequality implies that
$$|(x_n + y_n) - (x+y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| < \varepsilon + \varepsilon = 2\varepsilon$$
Since 2 is independent of $\varepsilon$, we have that the sequence given by $\{x_n + y_n\}_{n=1}^{\infty}$ converges to $x+y$ as required. ∎

(iii) Let $\varepsilon > 0$ be given. Set $K = \max\{|x|, |y|, \varepsilon/3, 1\}$. Using the fact that $\{x_n\}$ converges to $x$ and $\{y_n\}$ converges to $y$, we may select a pair $M_1, M_2 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n \geq M_1$ and $|y_n - y| < \varepsilon$ whenever $n \geq M_2$. Set $M = \max\{M_1, M_2\}$. Then, for any such $n \geq M$,
$$|x_n y_n - xy| = |(x_n - x + x)(y_n - y + y) - xy| = |(x_n - x)y + (y_n - y)x + (x_n - x)(y_n - y)|$$
$$\leq |(x_n - x)y| + |(y_n - y)x| + |(x_n - x)(y_n - y)|$$
$$= |y| \cdot |x_n - x| + |x| \cdot |y_n - y| + |x_n - x| \cdot |y_n - y|$$
$$< |y| \cdot \varepsilon + |x| \cdot \varepsilon + \varepsilon^2$$
∎

(iv) We may use part (iii) to get the result after proving the next claim

Claim: If $\{y_n\}$, $y_n \neq 0$ for all $n \in \mathbb{N}$, & $\{y_n\}$ converges to $y \neq 0$, then $\{1/y_n\}$ converges to $1/y$

**Proof Claim**: Let $\varepsilon > 0$ be given. As $|y| \neq 0$, $K_\varepsilon = \min\{|y|^2 \cdot \varepsilon/2, |y|/2\} > 0$. As $\{y_n\}$ converges to $y$, we may select $M \in \mathbb{N}$ such that $|y_n - y| < K_\varepsilon$ whenever $n \geq M$. Consequently, for every $n \geq M$, we have $|y| = |y + y_n - y_n| \leq |y_n - y| + |y_n| < \frac{|y|}{2} + |y_n|$. Thus, for those $n \geq M$, $\frac{|y|}{2} < |y_n|$ which is equivalent to $\frac{1}{|y_n|} < \frac{2}{|y|}$. To complete the proof, for every $n \geq M$,
$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y - y_n}{y_n y}\right| = \frac{|y - y_n|}{|y_n||y|} \leq \frac{2}{|y|} \cdot \frac{|y_n - y|}{|y|} = |y_n - y| \frac{2}{|y|^2}$$
$$< \frac{|y|^2 \cdot \frac{\varepsilon}{2}}{|y|} \cdot \frac{2}{|y|} = \varepsilon$$
∎

**Proposition:** Let $\{x_n\}$ be a convergent sequence such that $x_n \geq 0$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n}$.

**Proof:** Let $\{x_n\}$ converge to $x$. As $x_n \geq 0$ for all $n$, we know that $x \geq 0$. Let $\varepsilon > 0$ be given.

<u>Case (1)</u> $x = 0$

We need to show that $\{\sqrt{x_n}\}$ converges to $0$. As $\{x_n\}$ converges to $0$, there is some $M \in \mathbb{N}$ such that $|x_n - 0| < \varepsilon^2$ for every $n \geq M$. Then for every $n \geq M$,

nonneg. $\rightarrow$ $|\sqrt{x_n}| = \sqrt{x_n} < \sqrt{\varepsilon^2} = \varepsilon$.

<u>Case (2)</u> $x > 0$

Because $x > 0$, we know that $\sqrt{x} > 0$. Choose $M_2 \in \mathbb{N}$ such that $|x_n - x| < \sqrt{x}\,\varepsilon$ whenever $n \geq M_2$. Then, for every $n \geq M_2$, $|\sqrt{x_n} - \sqrt{x}| = \left|\left(\frac{(\sqrt{x_n} - \sqrt{x})}{1}\right)\left(\frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}}\right)\right|$

$$= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} < \frac{|x_n - x|}{\sqrt{x}} < \frac{\varepsilon}{|x|} \quad \blacksquare$$

**Proposition:** If $\{x_n\}$ is a convergent sequence, then $\{|x_n|\}_1^\infty$ is a convergent sequence, and

$$\lim_{n\to\infty} |x_n| = \left|\lim_{n\to\infty} x_n\right|$$

**Proof:** Suppose $\{x_n\}$ converges to $x$. Given any $\varepsilon > 0$, choose $M \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n \geq M$. By the Reverse Triangle Inequality, for every $n \geq M$, we have $||x_n| - |x|| \leq |x_n - x| < \varepsilon$, as required. $\blacksquare$

**Example:** Show that $\left\{\left|\sqrt{1 + \frac{1}{n}} - \frac{100}{n^2}\right|\right\}_{n \geq 1}^{\infty}$ converges to $1$

Sketch of the idea: $\lim_{n\to\infty} \left|\sqrt{1 + \frac{1}{n}} - \frac{100}{n^2}\right|$

(i) $\lim_{n\to\infty} \frac{1}{n} = 0 \implies \lim_{n\to\infty} [1 + \frac{1}{n}] = 1$ & as $1 + \frac{1}{n} \geq 0$ $\forall n \in \mathbb{N}$, then $\left\{\sqrt{1 + \frac{1}{n}}\right\}_1^\infty$ converges to $\sqrt{1} = 1$.

Squeeze Lemma $\rightarrow$ (ii) Since $\frac{100}{n^2} < \frac{1}{n}$ $\forall n \in \mathbb{N}$, then $0 \leq \lim_{n\to\infty} \left(\frac{100}{n^2}\right) \leq 100 \lim_{n\to\infty} \frac{1}{n} = 0$

Therefore by items (i) and (ii), $\lim_{n\to\infty} \left|\sqrt{1 + \frac{1}{n}} - \frac{100}{n^2}\right|$

$$= \lim_{n\to\infty} \left|\sqrt{\lim_{n\to\infty}(1 + \tfrac{1}{n})} - \lim_{n\to\infty} \frac{100}{n^2}\right|$$

$$= |\sqrt{1} - 0| = 1$$

# Convergence Tests

Dominated Convergence

**Proposition:** Let $\{x_n\}_1^\infty$ be a sequence, and suppose there is an $x \in \mathbb{R}$ and a sequence $\{a_n\}_1^\infty$ such that $\lim_{n\to\infty} a_n = 0$ and $|x_n - x| \le a_n$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges to $x$.

**Proof:** Let $\varepsilon > 0$ be given. As each $a_n \ge 0$ and $\{a_n\}$ converges to 0, there is some $M > 0$ for which $a_n < \varepsilon$ whenever $n \ge M$. By assumption, $|x_n - x| \le a_n$ for each $n \in \mathbb{N}$. Therefore $|x_n - x| < \varepsilon$ whenever $n \ge M$. ☒

**Proposition:** Let $c > 0$
(i) If $c < 1$, then $\{c^n\}_{n\ge1}^\infty$ converges to zero
(ii) If $c > 1$, then $\{c^n : n \in \mathbb{N}\}$ is an unbounded set of $\mathbb{R}$, and so $\{c^n\}_{n\ge1}^\infty$ cannot converge

**Proof:** First assume $0 < c < 1$. As $c > 0$, $c^n > 0$ for each $n \in \mathbb{N}$. As $c < 1$, we know that $c^{n+1} < c^n$ for every $n \in \mathbb{N}$. Therefore $\{c^n\}_1^\infty$ is a monotone decreasing sequence that is bounded from below by 0. Hence by MCT, it must converge to $x := \inf\{c^n : n \in \mathbb{N}\}$. Since $\lim_{n\to\infty} c^n = x$, necessarily, the 1-tail $\{c^{1+n}\}_{n\ge1}^\infty$ must converge, & in fact $\lim_{n\to\infty} c^n = \lim_{n\to\infty} c^{1+n} = x$. Since, for each $n \in \mathbb{N}$, $c^{n+1} = c \cdot c^n$, by taking limits, we get $x = cx$, or equivalently $(1-c)x = 0$. Either $1 - c = 0$, a contradiction, or $x = 0$

$= c \lim_{n\to\infty} c^n = c \cdot x$

Suppose now that $c > 1$. Let $B > 0$ be arbitrary. As $c > 1$, $\frac{1}{c} < 1$ and so by item (i) the sequence $\{1/c^n\}_{n\ge1}^\infty$ must converge to 0. Therefore there must be an $M \in \mathbb{N}$ such that $1/c^n < \frac{1}{B}$ whenever $n \ge M$, which is equivalent to saying $c^n > B$ whenever $n \ge M$. As $B$ is arbitrary, the set $\{c^n : n \in \mathbb{N}\}$ must be unbounded.

**Ratio Test for Sequences**

Let $\{x_n\}_1^\infty$ be a sequence such that $x_n \ne 0$ for all $n \in \mathbb{N}$ & the limit

$$L := \lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right|$$

exists. Then
(i) If $0 \le L < 1$, then $\{x_n\}_1^\infty$ converges to 0.
(ii) If $L > 1$, then $\{x_n : n \in \mathbb{N}\}$ is unbounded, & so $\{x_n\}$ is divergent
(iii) If $L = 1$, the test is inconclusive

item (i)

**Proof:** Suppose that $L < 1$. Choose a number $r \in \mathbb{Q}$ such that $L < r < 1$ (density of $\mathbb{Q}$). We want to compare $\{x_n\}$ with $\{r^n\}_{n\ge1}^\infty$. Now $\underbrace{r - L}_{\text{treat as } \varepsilon} > 0$, so there exists an $M \in \mathbb{N}$ such that

$$\frac{|x_{n+1}|}{|x_n|} - L \le \left| \frac{|x_{n+1}|}{|x_n|} - L \right| < r - L$$

whenever $n \ge M$

Therefore, $\frac{|x_{n+1}|}{|x_n|} < r$ whenever $n \ge M$. For each $n \ge M+1$

$$|x_n| = |x_M| \frac{|x_{M+1}|}{|x_M|} \cdots \frac{|x_{n-1}|}{|x_{n-2}|} \frac{|x_n|}{|x_{n-1}|} < |x_M| \cdot \underbrace{r \cdot r \cdots r}_{n-M \text{ times}} = |x_M| \cdot r^{n-M} = [|x_M| r^{-M}] \cdot r^n$$

The sequence $\{r^n\}_{n\ge1}^\infty$ must converge to 0 by the previous proposition & so $\{|x_M| \cdot r^{-M} \cdot r^n\}_{n\ge1}^\infty$ must also converge to zero. Therefore, the M-tail $\{x_{M+n}\}_{n\ge1}^\infty$ must converge to zero and hence $\{x_n\}_1^\infty$ must converge to the same limit, 0.

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Item (iii) Suppose $L > 1$, & choose $r$ such that $1 < r < L$. Then $\overbrace{L - r > 0}^{\text{use as } \varepsilon}$. Pick $M \in \mathbb{N}$ such that

$$-\left(\frac{|x_{n+1}|}{|x_n|} - L\right) \leq \left|\frac{|x_{n+1}|}{|x_n|} - L\right| < L - r \quad \text{whenever } n \geq M.$$

Therefore, for each $n \geq M$, $\frac{|x_{n+1}|}{|x_n|} > r$. Again, for each $n \geq M$, $|x_n| > |x_M| \cdot r^{n-M}$

same logic as before

This is equivalent to $[|x_M| \cdot r^{-M}] \cdot r^N$. By the previous proposition, the set $\{r^n : n \in \mathbb{N}\}$ is unbounded because $r > 1$.

Therefore, $\{x_n : n \in \mathbb{N}\}$ is unbounded, for if it were there would exist some $B > 0$ such that $|x_n| \leq B$ for every $n \in \mathbb{N}$, which would imply that $r^N < \frac{B}{|x_M|} r^M$, $n \geq M$ a contradiction. ☒