



Chapter 1



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Read Chapter 0 to refresh Basics of Set Theory for Chapter 3

Homeworks 250pts

- To be submitted weekly on Canvas (x10)
- Submit in LaTeX

3 Exams, No Final!

Brush up on Latex

Real Numbers

Notations: \mathbb{N} denotes the set of natural numbers w/o the element 0

\mathbb{Z} denotes the set of integers

\mathbb{R} denotes the set of real numbers

\mathbb{Q} denotes the set of rational numbers

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

Section 1 Basic Properties

Def: An ordered set is a set S with a relation \leq s.t.

(i) (Trichotomy) For every pair $x, y \in S$ exactly one of the following hold:

$$x < y, x = y \text{ or } y < x$$

(ii) (Transitivity) If x, y, z belong to S , and satisfy $x \leq y$ and $y \leq z$, then $x \leq z$

- we write $x \leq y$ to mean that $x \leq y$ or $x = y$

- we define greater than ($>$) or \geq in a similar fashion

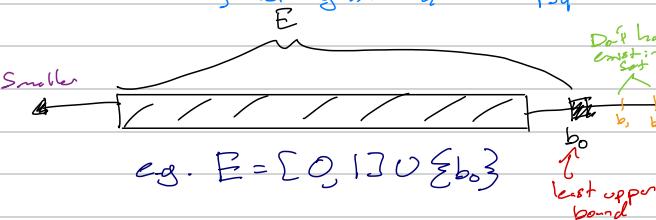
Examples

• \mathbb{Z} is an ordered set

• \mathbb{Q} is an ordered set

we say that $y > x, x, y \in \mathbb{Q}$

iff $y - x$ is a positive rational number, i.e. $y - x = \frac{p}{q}$ where $p, q \in \mathbb{N}$



Greatest lower bound is same idea
but on the other side

Def: Let $E \subseteq S$, where S is an ordered set

(i) If $\exists b \in S$ s.t. $x \leq b \forall x \in E$,

we say E is bounded above and we call b an upper bound for E

(ii) If $\exists b \in S$ s.t. $b \leq x \forall x \in E$,

we say E is bounded from below and we call b a lower bound for E

(iii) If $\exists b_0 \in S$, an upper bound for E , s.t. $b_0 \leq b \forall$ upper bounds $b \in S$ of E , we say b_0 is the least upper bound, or supremum of E . This element $b_0 := \sup E$.

(iv) If $\exists b_0 \in S$, a lower bound for E , s.t. $b_0 \geq b \forall$ lower bound b of E , we say that b_0 is the greatest lower bound, or infimum of E . Denote it by $b_0 := \inf E$

More Examples

(1) $S = \mathbb{Q}, E = \{x \in \mathbb{Q} : x \leq 1\}$

1 ∈ \mathbb{Q} is an upper bound for E , & is in fact the least upper bound for E . However $1 \notin E$, so E does not contain it

0 is the additive identity on its own

(2) $S = \mathbb{Q}, E = \{x \in \mathbb{Q} : x \leq 1\}$

This set contains its supremum

(3) $S = \mathbb{Q}, E = \{x \in \mathbb{Q} : x \geq 0\}$

This set has no upper bound, so it cannot have a least upper bound. It does however have a greatest lower bound being 0 ∈ \mathbb{Q} .

Def: An ordered set S has the least upper bound property if every non-empty subset $E \subseteq S$ that is bounded above has a least upper bound. This is also called completeness.

Example: \mathbb{Q} does not satisfy the least upper bound property

Pf: Consider the set $E = \{x \in \mathbb{Q} : x^2 \geq 2\}$ Pf by contradiction

Assume to the contrary, that there is an $x \in \mathbb{Q}$ s.t. $x^2 \geq 2$, and let $x = \frac{m}{n}$, $n, m \in \mathbb{Z} \setminus \{0\}$, in lowest terms. Then $m^2 \geq 2n^2$, where by m^2 is seen to be divisible by 2 ∵ m is also divisible by 2. Now $m^2 = 2k, k \in \mathbb{N}$. ∴ $(2k)^2 = 2n^2 \Rightarrow 4k^2 = 2n^2$. Dividing both sides by 2, we see n^2 is also divisible by 2 → contradiction

Claim: $\sqrt{2}$ is an irrational number

Fact:

$\sqrt{2}$ exists &
is the supremum
for E .

Fields

Defn: A set F is said to be a **field** if it has two operations (closed)

+ & ; & it satisfies the axioms:

Addition (A1) If $x, y \in F$, then $x+y \in F$

(A2) $x+y = y+x \forall x, y \in F$

(A3) $(x+y)+z = x+(y+z) \forall x, y, z \in F$

(A4) \exists an element $0 \in F$ s.t.

$$x+0 = 0+x = x \forall x \in F$$

(A5) $\forall x \in F, \exists -x \in F$ s.t.

$$x+(-x) = -x+x = 0$$

Multiplication

(M1) $x, y \in F \Rightarrow xy \in F$

(M2) $x \cdot y = y \cdot x \forall x, y \in F$

(M3) $(x \cdot y) \cdot z = x \cdot (y \cdot z) \forall x, y, z \in F$

(M4) \exists an element $1 \in F$ s.t.

$$x \cdot 1 = 1 \cdot x = x \forall x \in F$$

(M5) $\forall x \in F, x \neq 0 \exists \frac{1}{x} \in F$

s.t. $\frac{1}{x} \cdot x = x \cdot \frac{1}{x} = 1$

Distributivity

(D) $x(y+z) = xy + xz \forall x, y, z \in F$

Examples:

- \mathbb{Q} form of a field

- \mathbb{Z} are not a field

(cannot multiply)

- \mathbb{R} are a field

Defn: A field F is said to be an ordered field if F is an ordered set s.t.

① For $x, y \in F$, $x < y \rightarrow x+z < y+z$

② For $x, y \in F$, $x > 0$ & $y > 0$ both imply $xy > 0$

• If $x > 0$, then x is said to be positive, and if $x < 0$ then x is said to be negative

↳ we also say x is non-negative if $x \geq 0$, & nonpositive if $x \leq 0$

Can be defined analogously to the other operations $\{+, -, \cdot, \leq\}$

Proposition: Let F be an ordered field, and $x, y, z, w \in F$, then:

① If $x > 0$, then $-x < 0$ (and vice-versa)

② If $x > 0$ & $y < 0$, then $xy < 0$ \Leftarrow preserves ordering

③ If $x < 0$ and $y < 0$, then $xy > 0$ \Leftarrow flips ordering

④ If $x \neq 0$, then $x^2 > 0$

Note: $④ \Rightarrow 1 > 0$

⑤ If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$

⑥ If $0 < x < y$, then $x^2 < y^2$

⑦ If $x \leq y$ & $z \leq w$, then $x+z \leq y+w$

Do not submit proofs written in symbols!

PF: We settle items ① and ②

① The inequality $x > 0$ & item ① in definition of an ordered field imply that $0+(-x) < x+(-x)$

Now, as $x+(-x)=0$ because F is a field, we conclude that $-x < 0$ whenever $x > 0$, as required

Every loc should be carefully explained

⑤ First note that $\frac{1}{x}$ is nonzero as a multiplicative inverse of a nonzero element in a field. Suppose $\frac{1}{x} < 0$, then $-\frac{1}{x} > 0$ by part ① of the proposition. By part ② of defn of an ordered field, $x+(-\frac{1}{x}) > 0$, which means $-1 > 0$, a contradiction. (using item ① of proposition) Therefore $\frac{1}{x} > 0$. A similar argument shows $\frac{1}{y} > 0$

Thus, $\frac{1}{x} \cdot \frac{1}{y} > 0$ (by ② in def of ordered field)

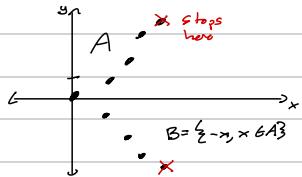
Finally, we can conclude $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)x < \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)(y)$ so, $0 < \frac{1}{y} < \frac{1}{x}$

Homework **Completion**

#1 Given $x, y \in F$, F an ordered field

$0 < x < y$, prove $x^2 < y^2$

Proposition: Let F be an ordered field with the least upper bound property.
 ↳ Let $A \subset F$ be a non-empty set bounded below. Then $\inf A$ exists in F .



Pf Consider $B = \{z - x : z \in A\}$

Let $b \in F$ be a generic lower bound for A , this means given any $x \in A$, we know that $x \geq b$. In other words, $\forall x \in A$, $-x \leq -b$. Consequently $-b$ is an upper bound for B . Since B is a non-empty & bounded from above set in an ordered field F that satisfies the completeness property, $\sup B$ exists as an element in F . $C := \sup B$
 Note that $C \leq -b$. Why? $y \in B$, we know that $y \leq C$. By the definition of B , we know $-y \leq -C \quad \forall x \in A$. Putting everything together: $-x \leq -C \leq -b \quad \forall x \in A$. Multiplying by -1 , we know $x \geq C \geq b \quad \forall x \in A$, so $C = \inf A$. //

Section 2 The set of Real Numbers

Theorem: There exists a unique ordered field \mathbb{R} with the least upper bound property that contains \mathbb{Q} . This field \mathbb{R} is called the set of real numbers.

Proposition: If $x \in \mathbb{R}$ such that $x \leq \varepsilon$ holds $\forall \varepsilon > 0$, then $x \leq 0$

Pf If $x > 0$, then $0 < \frac{x}{2} < x$. Choosing $\varepsilon = \frac{x}{2}$ results in a contradiction. //

Example: The set $A = \{x \in \mathbb{R} : x^2 \geq 2\}$ has a least upper bound $\sup A$ that does not belong to the rational numbers.

Claim: There is a unique number $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $r^2 = 2$. We denote this number by $r = \sqrt{2}$.

Pf We begin by showing A is bounded from above and non-empty. Note that $A \neq \emptyset$, since $1 \in A$. The equation $x \geq 2$ implies that $x^2 \geq 4$, so if $x^2 \geq 2$, then $x < 2$. Therefore A is bounded from above. Thus, as \mathbb{R} satisfies the least upper bound property, $r := \sup A$ exists as an element.

Goal: Show $r^2 = 2$

We'll do this by showing that $r^2 \geq 2$ and $r^2 \leq 2$

Step 1: $r^2 \geq 2$.

Choose an $\varepsilon > 0$ such that $\varepsilon^2 < 2$. we will search for an $h > 0$ such that $(s+h)^2 < 2$. Since $2 - \varepsilon^2 > 0$, we see that $\frac{2-\varepsilon^2}{2+\varepsilon} > 0$. Choose h such that $0 < h < \frac{2-\varepsilon^2}{2s+h}$.

We may also choose $h < 1$. Then we can estimate $(s+h)^2 - s^2 = h(2s+h)$

$$< h(2s+1) \text{ as } 0 < h \\ < 2 - \varepsilon^2 \text{ as } h < \frac{2-\varepsilon^2}{2s+1}$$

Consequently, $(s+h)^2 < 2$. Thus, $s+h \in A$. Hence $s < r := \sup A$. As $s > 0$ is arbitrary with $s^2 \geq 2$, it follows that $r^2 \geq 2$.

Step 2: $r^2 \leq 2$. Apply similar logic

By Steps 1 and 2, we know that $r^2 = 2$.

Uniqueness follows as usual.

Written in textbook
pretty clearly

The set \mathbb{R}/\mathbb{Q} which is nonempty
is called the set of irrational numbers.

Archimedean Property

Theorem:

(i) (Archimedean Property): If $x, y \in \mathbb{R}$ and $x > 0$ then there must exist a natural number n s.t. $nx > y$.

↳ For any natural number $g > 0$ we can find a real number smaller than $+g$.

(ii) (\mathbb{Q} is dense in \mathbb{R}): If $x, y \in \mathbb{R}$ and $x < y$, $\exists r \in \mathbb{Q}$ s.t. $x < r < y$.

↳ Between any real numbers you can always find a rational in between.

Proof. We begin with the proof of item (i).

Dividing $nx > y$ by x , item (i) asserts that $\forall t \in \mathbb{R}, t := \frac{y}{x}$, we can find $n \in \mathbb{N}$ such that $n > t$. In other words (i) asserts that $\mathbb{N} \subset \mathbb{R}$ is not bounded from above.

Assume to the contrary that \mathbb{N} is indeed bounded from above as a subset of \mathbb{R} . By the completeness of \mathbb{R} , there is a least upper bound $b := \sup \mathbb{N}$. As b is the least upper bound of the natural numbers, $(b-1)$ cannot longer be an upper bound for \mathbb{N} . Therefore, there must be some $m \in \mathbb{N}$ such that $m > b-1$. Adding 1 to both sides, & noting that $m+1 \in \mathbb{N}, m+1 > b$, a contradiction. \square

We proceed to the proof of item (ii).

First, we suppose that $x \geq 0$. Then $y - x \geq 0$. By part (i), there is a natural number n such that $n(y-x) > 1$, or $(y-x) > \frac{1}{n}$.

Again using part (i), the set $A = \{k \in \mathbb{N} : k \geq nx\}$ is non-empty. By the well-ordering property of the natural numbers A has a least element m . Then $m \geq nx$. If $m=0$, then $m-1=0$ and $m-1 < nx$ as $x > 0$. In other words,

$$m-1 \leq nx \text{ or } m \leq nx+1$$

On the other hand, $n(y-x) > 1$ so we obtain $ny > 1+nx$. Consequently, $ny \geq 1+nx > m$; hence $y > \frac{m}{n}$. Putting everything together, we know that $x < \frac{m}{n} \leq y$. Choose $r := \frac{m}{n}$.

Suppose now that $x < 0$.

• If $y \geq 0$, choose $r=0$

• If $y \leq 0$, then $0 \leq -y \leq -x$, and from the first case we can choose $q \in \mathbb{Q}$ such that $-y < q \leq -x$. Take $r=-q$ in this case. \square

Corollary

Proof. Let $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. The set A is non-empty, $\inf \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = 0$ and $\frac{1}{n} > 0$ for every $n \in \mathbb{N}$.

Therefore, 0 is a lower bound for A , and so $b = \inf A$ exists. We also know that $b \geq 0$. By the Archimedean property, there must exist an $n \in \mathbb{N}$ such that $n > 1/a$, with $a > 0$ being arbitrary. In other words, for any positive number a , there is some $n \in \mathbb{N}$ for which $a > \frac{1}{n}$. Thus, a cannot be a lower bound for A , whenever $a > 0$. Consequently, $b = \inf A = 0$. \square

Using sup & inf

- For $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, define the translation of A by x via
 $x+A = \{x+a : a \in A\}$ and thus $xA = \{xa : a \in A\}$

Proposition Let $A \subseteq \mathbb{R}$ be nonempty

- (i) If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x+A) = x + \sup A$
- (ii) If $x \in \mathbb{R}$ and A is bounded below, then $\inf(x+A) = x + \inf A$
- (iii) If $x > 0$ and A is bounded above, then $\sup(xA) = x \cdot (\sup A)$
- (iv) If $x > 0$ and A is bounded below, then $\inf(xA) = x \cdot (\inf A)$
- (v) If $x < 0$, and A is bounded below, then $\sup(xA) = x \cdot (\inf A)$
- (vi) If $x < 0$, and A is bounded above, then $\inf(xA) = x \cdot (\sup A)$

Proof We prove (i)

Let b be an upper bound for A , that is, $a \leq b \forall a \in A$. Thus, $x+a \leq x+b \forall x \in A$. Thus, $x+b$ is an upper bound for $x+A$. Hence, $\sup(x+A) \leq x+b$.

Choosing $b := \sup A$, we conclude that $\sup(x+A) \leq x+\sup A$

Next, let c be an upper bound for $x+A$. This means $z \leq c \forall z \in x+A$.

Note that $z = x+w$ for some $w \in A$. So $w \leq c-x \forall w \in A$. Thus $c-x$ is an upper bound for A . In particular, $\sup A \leq c-x$ is an upper bound of $x+A$.

Choosing $c := \sup(x+A)$, we conclude that $\sup(x+A) \geq x \leq \sup(x+A)$ \square

Proposition: Let A, B be any pair of nonempty subsets of \mathbb{R} s.t. $x \leq y \forall x \in A$ and $y \in B$. Then, A is bounded above, B is bounded below, and $\sup(A) \leq \inf(B)$

Proof: Any element of A is a lower bound for B .

Moreover, since B is nonempty and bounded below, the completeness property of \mathbb{R} guarantees $\inf(B)$ exists. Therefore, $x \leq \inf B \forall x \in A$. So $\inf B$ is an upper bound for A , and we conclude that $\sup A \leq \inf B$ \square

Question: Given two sets $A, B \subseteq \mathbb{R}$ such that $x \leq y \forall x \in A, y \in B$. Does it hold that $\sup A \leq \inf B$? No!!!

Counterexample: Choose $A = \{0\}$, $B = \{\frac{1}{n}, n \in \mathbb{N}\}$. Then $\inf B = \sup A = 0$

Proposition: If $S \subseteq \mathbb{R}$ is nonempty and bounded from above, then $\forall \epsilon > 0$, $\exists x \in S$ s.t.
 $\sup S - \epsilon < x \leq \sup S$

Extended Real Numbers

Def: Let $A \subseteq \mathbb{R}$

(i) If $A = \emptyset$, define $\sup A := -\infty$

(ii) If A is not bounded above & non-empty, then $\sup A := +\infty$

(iii) If A is empty, define $\inf A := +\infty$

(iv) If A is not bounded below & non-empty, then $\inf A := -\infty$

The set $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$ is called the **extended Real Numbers**. It can be made an ordered set via $-\infty < 0, -\infty < x < 0 \forall x \in \mathbb{R}$

Notation: the case

Def: When $A \subset \mathbb{R}$ non-empty and bounded above, and $x \in A$, then $\sup A$ is called the **maximum of A** and is denoted by $\max A$.

If $A \subset \mathbb{R}$ non-empty and bounded below, and $x \in A$, then $\inf A$ is called the **minimum of A** and is denoted $\min A$.

Fact: Any non-empty finite subset of \mathbb{R} has a maximum and a minimum and a unique one.

↳ Proved by induction on H_w

Absolute Value & Functions

For any $x \in \mathbb{R}$, define $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Proposition:

(i) $|x| \geq 0$, with equality if $x=0$.

(ii) $|-x| = |x|$ for all $x \in \mathbb{R}$ *symmetry, not negative*

(iii) $|xy| = |x| \cdot |y|$ for all $x, y \in \mathbb{R}$

(iv) $|x^2| = x^2 \forall x \in \mathbb{R}$

(v) $|x| \leq y$ iff $-y \leq x \leq y$

(vi) $-|x| \leq x \leq |x| \forall x \in \mathbb{R}$

Proposition (Triangle Inequality): For any pair $x, y \in \mathbb{R}$,

$$|x+y| \leq |x| + |y|$$

Proof: By (vi) of the previous proposition, we know that $-|x| \leq x \leq |x|$ & $-|y| \leq y \leq |y|$.

Addition of these two equations yields $-(|x| + |y|) \leq x+y \leq (|x| + |y|)$

Apply item (v)

□

Corollary: For any pair $x, y \in \mathbb{R}$, the following hold:

(i) (Reverse triangle ineq): $||x|-|y|| \leq |x-y|$

(ii) $|x-y| \leq |x| + |y|$

Proof: We settle item (i)

Set $x=a-b, y=b$ for some arbitrary pair $a, b \in \mathbb{R}$. Applying the triangle ineq. we get

$$|a| = |a-b+b| \leq |a-b| + |b|,$$

or equivalently, that

$$|a| - |b| \leq |a-b|$$

Switching the roles of a and b , we also have

$$|b| - |a| \leq |a-b|$$

Apply item (v) of the previous proposition to get the desired result



Corollary: Let $x_1, x_2, \dots, x_n \in \mathbb{R}$ then

$$\text{Then } |x_1 + x_2 + \dots + x_n| \leq |x_1| + |x_2| + \dots + |x_n|$$

Induction Proof

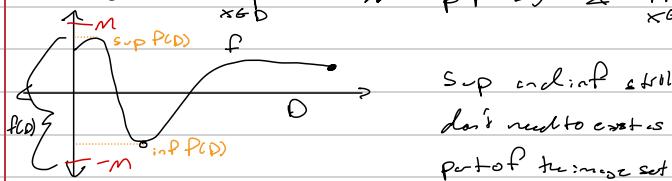
Example: Find a number M such that $|x^2 - 9x + 1| \leq M$ for $-1 \leq x \leq 5$.

Solution: For any $x \in \mathbb{R}$, the triangle inequality gives

$$|x^2 - 9x + 1| \leq |x^2| + 9|x| + 1$$

For those $-1 \leq x \leq 5$, the maximum of $|x^2| + 9|x| + 1$ occurs when $x=5$.
So choose $M = 8^2 + 9(5) + 1 = 71$. ← Not the best M
but it holds

Defn: Suppose $f: D \rightarrow \mathbb{R}$ is a function. We say that f is bounded if there is some $M > 0$ such that $|f(x)| \leq M$ for all $x \in D$. For functions $f: D \rightarrow \mathbb{R}$, we write $\sup_{x \in D} f(x) := \sup f(D)$ & $\inf_{x \in D} f(x) := \inf f(D)$



Sup and inf still
don't need to exist as
part of the image set

Example: Let $D = \{x : -1 \leq x \leq 5\} \subset \mathbb{R}$ & $f(x) = x^2 - 9x + 1$

$$\text{Using calculus i.e. } \sup_{x \in D} f(x) = \sup_{-1 \leq x \leq 5} [x^2 - 9x + 1] = 1$$

Just take domain to find
min and max.

$$\inf_{x \in D} f(x) = \inf_{-1 \leq x \leq 5} [x^2 - 9x + 1] = -\frac{89}{4}$$

Proposition: Given a pair of bounded functions $f, g: D \rightarrow \mathbb{R}$, with D being non-empty, such that $f(x) \leq g(x)$ for all $x \in D$, it holds that

$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x) \quad \& \quad \inf_{x \in D} f(x) \leq \inf_{x \in D} g(x)$$

Caution: The x on LHS of these inequalities is different than the x on RHS

For example, the first should be thought of as: $\sup_{x \in D} f(x) \leq \sup_{y \in D} g(y)$

Proof: Suppose b is an upper bound for $g(D)$. Then, for every $x \in D$ we have $f(x) \leq g(x) \leq b$ based on the proposition's assumption, so b is an upper bound for $f(D)$.

In other words, $f(x) \leq b$ for every $x \in D$. Thus for all $x \in D$, $f(x) \leq \sup_{y \in D} g(y)$.

$$\text{Consequently, } \sup_{x \in D} f(x) \leq \sup_{y \in D} g(y)$$



Remark: Under the hypothesis of the proposition, the inequality $\sup_{x \in D} f(x) \leq \inf_{y \in D} g(y)$ is false

Cook up counter example in Homework

- Look at x and y from $0 \rightarrow 10$
- Or look at y

Intervals

Intervals in \mathbb{R}

Given $a, b \in \mathbb{R}$ $a \leq b$ set

- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ closed interval
- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ open interval
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ half open interval
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$..

All such intervals are called bounded

Unbounded intervals

Given $a, b \in \mathbb{R}$ $a \leq b$ set

- $[a, \infty) = \{x \in \mathbb{R} : a \leq x < \infty\}$ closed interval
- $(a, \infty) = \{x \in \mathbb{R} : a < x < \infty\}$ open interval
- $(-\infty, b] = \{x \in \mathbb{R} : -\infty < x \leq b\}$ closed interval
- $(-\infty, b) = \{x \in \mathbb{R} : -\infty < x < b\}$ open interval
- $(-\infty, \infty) = \mathbb{R}$ open interval

Proposition: A set $I \subset \mathbb{R}$ is an interval if \mathbb{R} I contains at least two points, and for all $a, b \in I$ and $c \in \mathbb{R}$ such that $a \leq c \leq b$, we have that $c \in I$.

Theorem: \mathbb{R} is an uncountable set

Chapter 2



Sequences and Series

D.R: A sequence of real numbers is any function $x: \mathbb{N} \rightarrow \mathbb{R}$. Instead of using $x(n)$, we use the notation x_n to denote the n^{th} element of the sequence.
 → To denote the sequence, we will use $\{x_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{200}$, $\{x_n\}_1^{\infty}$, $\{x_n\}_1^{200}$, $\{x_n\}_{n=1}^{200}$, interchangeably.
 ↳ Use interchangeably based on needs of proof Based on uses $\{x_n\}_{n=1}^{\infty}$, too

A sequence is bounded if there exists $\exists M > 0$ such that $|x_n| \leq M$ for every $n \in \mathbb{N}$. In other words, the set $\{x_n : n \in \mathbb{N}\} \subset \mathbb{R}$ is a bounded set.

Experiments

Simples

✓ (i) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is bounded.

Choose $M=1$

✓ (ii) Let $c \in \mathbb{R}$. Define the constant sequence $\{c_n\}_{n=1}^{\infty} = \{c, c, c, \dots, c, \dots\}$

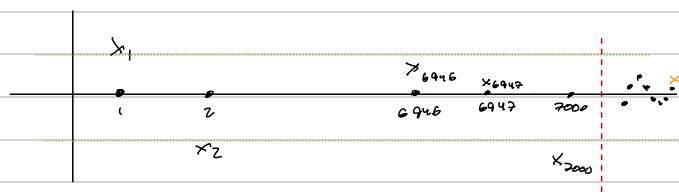
Choose $M = 10$ will do the trick

* (iii) $\{n^{\frac{1}{n}}\}_{n=1}^{\infty}$ is not bounded

$$\checkmark (-1) \left\{ (-1)^n \right\}_{n=1}^{\infty} = \left\{ -1, 1, -1, 1, \dots \right\}$$

Choose is M>1

Def: A sequence $\{x_n\}_{n=1}^{\infty}$ is said to converge to some $x \in \mathbb{R}$ if for any $\epsilon > 0$, there exist $M \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ whenever $n \geq M$.



The number x is called as limit of $\{x_n\}_{n=1}^{\infty}$ if we write

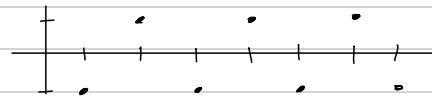
$$x_i = \lim_{n \rightarrow \infty} x_n$$

A sequence that converges is said to be convergent. If a sequence does not converge, it is said to be divergent or it diverges.

Example: The sequence $\left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$ converges to $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

Proof: Let $\varepsilon > 0$ be given. By the Archimedean property, there must exist some $M \in \mathbb{N}$ such that $0 < \frac{1}{m} < \varepsilon$. Consequently, for every $n \geq M$, we have that $|x_n - 0| = \left|\frac{1}{n}\right| \leq \frac{1}{m} < \varepsilon$, as required. The other direction

Example The sequence $\{(-1)^n\}_{n=1}^{\infty}$ diverges



Proof: Assume to the contrary that $\{\varepsilon_i\}_{i=1}^{\infty}$ converges to some $\varepsilon \in \mathbb{R}$ & let $\varepsilon > \frac{1}{2}$.

There must exist a $M \in \mathbb{N}$ such that $|(-1)^n - x| < \frac{1}{2}$ whenever $n \geq M$.

- For each $n \geq M$, we get $\frac{1}{2} > |1-x| \quad \text{and} \quad |x_{n+1}-x| = |-1-x| < \frac{1}{2}$

$$\frac{1}{2} + \frac{1}{2} > |1-x| + |-1-x| \geq |1-x+(-1-x)| = 2$$

$|1-x| > 2$ Contradiction

Use triangle inequality

Tools used here in the proof useful for future semester

Proposition: A convergent sequence has a unique limit.

Proof: Suppose that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence, so that $x, y \in \mathbb{R}$ are limits of $\{x_n\}_{n=1}^{\infty}$. Let $\epsilon > 0$ be arbitrary. Since $\{x_n\}_{n=1}^{\infty}$ converges to x , there must exist an $M, G N$ such that $|x_n - x| < \frac{\epsilon}{2}$. Similarly, as $\{x_n\}_{n=1}^{\infty}$ converges to y , there must be some $M_2 \geq N$ such that $|x_n - y| < \frac{\epsilon}{2}$ whenever $n \geq M_2$.
Let $M := \max\{M, M_2\}$. Then by the triangle inequality,
$$|x - y| \leq |x_n - x| + |x_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

whenever $n \geq M$. Since $\epsilon > 0$ can be made arbitrarily small, we conclude that $x = y$.

Proposition: A convergent sequence is bounded

Proof: Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to x . For $\epsilon = 1$ there must exist some $M \in \mathbb{N}$ such that $|x_n - x| < 1$ whenever $n \geq M$. Thus, for $n \geq M$, $|x_n| \leq |x_n - x| + |x| < 1 + |x|$.

The set $\{|x_n| : n = 1, 2, \dots, M\}$ is finite & nonempty so it has a maximum $\sup_{n \leq M} \{|x_n| : n = 1, 2, \dots, M\} = V$

Therefore, for every $n \in \mathbb{N}$, $|x_n| \leq \max\{V, |x|\}$. Consequently, $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence.

Caution: Bounded sequences are not guaranteed to converge.

Example: Show that $\left\{\frac{n^2+1}{n^2+n}\right\}_{n=1}^{\infty}$ converges & that $\lim_{n \rightarrow \infty} \frac{n^2+1}{n^2+n} = 1$

Scratch work:

$$\left| \frac{n^2+1}{n^2+n} - 1 \right| = \left| \frac{n^2+1 - (n^2+n)}{n^2+n} \right| = \left| \frac{-1+n}{n^2+n} \right| \leq \frac{|-1+n|}{n(n+1)} = \frac{|1-n|}{n(n+1)} = \frac{1-n}{n(n+1)} = \frac{1}{n}$$

→ We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ converges to zero ✓
But we question like this on the exam

Proof: Let $\epsilon > 0$ be given. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ converges to 0, there must exist an $M \in \mathbb{N}$ such that $\left| \frac{1}{n} - 0 \right| \leq \frac{1}{n} = \frac{1}{n} < \epsilon$ whenever $n \geq M$. Therefore, by the triangle inequality, $\left| \frac{n^2+1}{n^2+n} - 1 \right| \leq \frac{1}{n} = \frac{1}{n} < \epsilon$ whenever $n \geq M$. Since $\epsilon > 0$ is arbitrary, we know that $\sum_{n=1}^{\infty} \frac{n^2+1}{n^2+n}$ converges to 1. □

Dari A sequence $\{x_n\}_{n=1}^{\infty}$ is said to be **monotone increasing**: if for all $n \in \mathbb{N}$, $x_n \leq x_{n+1}$, and is said to be **monotone decreasing** if $x_n \geq x_{n+1}$ for any $n \in \mathbb{N}$. If a sequence $\{x_n\}_{n=1}^{\infty}$ is either one of these types, then x_n is said to be **monotone**.

Examples: $\{\frac{1}{n}\}_{n=1}^{\infty}$ is monotone decreasing
 $\{\frac{1}{n}\}_{n=1}^{\infty}$ is monotone increasing

You can write MCT

Monotone convergence theorem: A monotone sequence $\{x_n\}_{n=1}^{\infty}$ is convergent if and only if it is bounded. Furthermore, if $\{x_n\}_{n=1}^{\infty}$ is monotone increasing and bounded, then $\lim_{n \rightarrow \infty} x_n = \sup_{n \in \mathbb{N}} \{x_n\}_{n=1}^{\infty}$.

On the other hand, if $\{x_n\}_{n=1}^{\infty}$ is monotone decreasing and bounded, then $\lim_{n \rightarrow \infty} x_n = \inf_{n \in \mathbb{N}} \{x_n\}_{n=1}^{\infty}$

Proof: We will prove the theorem in the instance when $\{x_n\}_{n=1}^{\infty}$ is monotone increasing.

Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded; that is the set $\{x_n : n \in \mathbb{N}\}$. Let $x := \sup_{n \in \mathbb{N}} \{x_n\}_{n=1}^{\infty}$.

Let $\epsilon > 0$ be given. As x is the sup of $\{x_n\}_{n \in \mathbb{N}}$, there must be at least one element $x_m \in \{x_n\}_{n \in \mathbb{N}}$ such that $x_m > x - \epsilon$. As $\{x_n\}_{n \in \mathbb{N}}$ is monotone increasing, we know $x_n \geq x_m$ whenever $n \geq m$. Consequently, for any $n \geq m$,

$$|x_n - x| = x - x_n \leq x - x_m \leq \epsilon.$$

Therefore $\{x_n\}_{n=1}^{\infty}$ converges to x . We have already proven the other direction: every convergent sequence is bounded.

Example: Consider the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$.

This sequence is bounded from below by 0, as $\frac{1}{n} \geq 0$ for every $n \in \mathbb{N}$. It is also monotone decreasing, as $\frac{1}{n} \leq \frac{1}{n+1}$ for every $n \in \mathbb{N}$ (you could need to explain why). Then $\frac{1}{n} \geq \frac{1}{n+1}$ for every $n \in \mathbb{N}$.

It follows from the MCT that $\lim_{n \rightarrow \infty} \frac{1}{n} = \inf_{n \in \mathbb{N}} \{\frac{1}{n} : n \in \mathbb{N}\}$ **Homework:** Show $\inf_{n \in \mathbb{N}} \{\frac{1}{n} : n \in \mathbb{N}\} = 0$

Proposition: Let $S \subset \mathbb{R}$ be a non-empty bounded set. Then there exist sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $x_n, y_n \in S$ for every $n \in \mathbb{N}$, such that

$$\sup S = \lim_{n \rightarrow \infty} x_n \quad \inf S = \lim_{n \rightarrow \infty} y_n$$

Exam 1 Sheet Q5x6 w/ Definitions & them handwritten

Def: For a sequence $\{x_n\}_{n=1}^{\infty}$, the **K-tail**, **KGN**, or just **tail** of the sequence is the sequence starting at x_{k+1} usually written as $\{x_{n+k}\}_{n=1}^{\infty}$ or $\{x_n\}_{n=k+1}^{\infty}$

Proposition: Let $\{x_n\}$ be a sequence. Then the following are equivalent:

(i) The sequence $\{x_n\}_{n=1}^{\infty}$ converges.

(ii) The K-tail $\{x_{n+k}\}_{n=1}^{\infty}$ converges for every KGN.

(iii) The K-tail $\{x_{n+k}\}_{n=1}^{\infty}$ converges for some KGN.

Furthermore, if any (and hence all) of the limits exist, then for every KGN,

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{n+k},$$

Proof: The implication (ii) \Rightarrow (iii) is immediate. We will show that (i) implies (ii) and that (iii) implies (i). We begin with (i) \Rightarrow (ii).

Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to some $s \in \mathbb{R}$. Let $K \in \mathbb{N}$ be arbitrary, and define $y_n = x_{n+K}$ for each $n \in \mathbb{N}$. Goal: Show $\{y_n\}_{n=1}^{\infty}$ converges to s .

Given any $\epsilon > 0$, there is an $M \in \mathbb{N}$ such that $|x_n - s| < \epsilon$ whenever $n \geq M$.

Note that $n \geq M$ implies $n+K \geq M$. Therefore $|y_n - s| = |x_{n+K} - s| < \epsilon$ for every $n \geq M$.

Hence the sequence $\{y_n\}_{n=1}^{\infty}$ converges. This completes that (i) \Rightarrow (ii). We next prove that (iii) \Rightarrow (i).

Let $K \in \mathbb{N}$ be the necessary K for which (iii) holds. Define $y_n = x_{n+K}$, assume that $\{y_n\}_{n=1}^{\infty}$ converges to $y \in \mathbb{R}$. We need to show that $\{x_n\}_{n=1}^{\infty}$ converges to y . Given $\epsilon > 0$, there is some $M \in \mathbb{N}$ such that $|y_n - y| < \epsilon$ for all $n \geq M$. Set $M' = M + K$. Then $n \geq M'$ implies that $n - K \geq M$. Thus, $|x_n - y| = |y_{n-K} - y| < \epsilon$ whenever $n \geq M'$ as required.

Definition: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Let $\{n_i\}_{i=1}^{\infty}$ be a strictly increasing seq. of natural numbers, i.e. $n_i \leq n_{i+1}$ for all $i \in \mathbb{N}$. The sequence $\{x_{n_i}\}_{i=1}^{\infty}$ is called a **subsequence** of $\{x_n\}_{n=1}^{\infty}$.

Example: $\{-1\}^{2^i}, i \in \mathbb{N} \equiv \{1, -1, 1, -1, \dots\}$ is a subsequence of $\{(-1)^n, n \in \mathbb{N}\}$

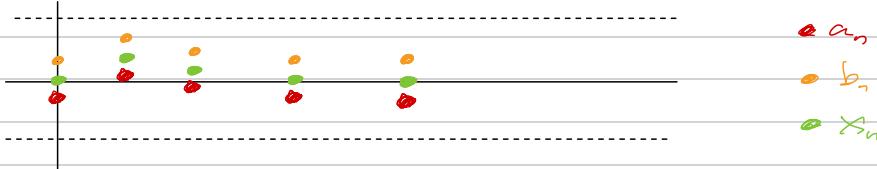
Proposition: If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence, then every subsequence of $\{x_n\}_{n=1}^{\infty}$ must converge to the same limit.

Proof: Suppose $\lim_{n \rightarrow \infty} x_n = s$ and let $\epsilon > 0$ be arbitrary. There must exist some $M \in \mathbb{N}$ for which $|x_n - s| < \epsilon$ whenever $n \geq M$. By induction, it follows that $n_i \geq i$ for any $i \in \mathbb{N}$. Consequently $i \geq M$ implies that $n_i \geq M$. Thus, for all $i \geq M$, $|x_{n_i} - s| < \epsilon$ as needed. □

This is Exam 1 Cut off PP!!!

Facts about Sequences

Squeeze Lemma: Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$ be such that $a_n \leq x_n \leq b_n$ for all $n \in \mathbb{N}$. If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge to $x \in \mathbb{R}$, then $\{x_n\}_{n=1}^{\infty}$ must also converge to x .



Proof: Let $\epsilon > 0$ be given. As $\{a_n\}_{n=1}^{\infty}$ converges to x , there must be an $M_1 \in \mathbb{N}$ such that $|a_n - x| < \epsilon$ holds for all $n \geq M_1$. As $\{b_n\}_{n=1}^{\infty}$ converges to x , there is an $M_2 \in \mathbb{N}$ such that $|b_n - x| < \epsilon$ for those $n \geq M_2$. Set M to be the maximum among M_1 and M_2 , $M := \max(M_1, M_2)$, and suppose that $n \geq M$. For these such n , it holds that $x - \epsilon < a_n \leq x_n \leq b_n < x + \epsilon$. Thus,

$$x - \epsilon < a_n \leq x_n \leq b_n < x + \epsilon$$

In other words, $-\epsilon < x_n - x < \epsilon$ whenever $n \geq M$. This is equivalent to $|x_n - x| < \epsilon$ whenever $n \geq M$, as required. \blacksquare

Lemma: Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ be convergent sequences. Suppose that $x_n \leq y_n$ for every $n \in \mathbb{N}$. Then the $\lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n$.

Proof: Set $x = \lim_{n \rightarrow \infty} x_n$ and $y = \lim_{n \rightarrow \infty} y_n$. Let $\epsilon > 0$ be given. Choose $M_1, M_2 \in \mathbb{N}$ such that $|x_n - x| < \frac{\epsilon}{2}$ whenever $n \geq M_1$ and $M_2 \in \mathbb{N}$ such that $|y_n - y| < \frac{\epsilon}{2}$ whenever $n \geq M_2$. Set $M = \max\{M_1, M_2\}$. For those $n \geq M$, it holds that $x_n - x < \frac{\epsilon}{2}$ and $y_n - y < \frac{\epsilon}{2}$. Adding these inequalities we get $(y_n - x_n)(x - y) < \epsilon$ or that $y_n - x_n < y - x + \epsilon$

whenever $n \geq M$. Because $x_n \leq y_n$, this becomes $0 \leq y_n - x_n < y - x + \epsilon$ for these $n \geq M$. Therefore $0 \leq y - x + \epsilon$. In other words, $y - x \leq \epsilon$. As $\epsilon > 0$ is arbitrary, we conclude that $y - x \leq 0$. \blacksquare

Corollary

(i) If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence such that $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\lim_{n \rightarrow \infty} x_n \geq 0$

(ii) Let $a, b \in \mathbb{R}$ and $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence such that $a \leq x_n \leq b$ for all $n \in \mathbb{N}$. Then $a \leq \lim_{n \rightarrow \infty} x_n \leq b$

Algebra of Limits: Let $\{x_n\}$, $\{y_n\}$ be convergent sequences. Then the following hold

$$(\text{i}) \lim_{n \rightarrow \infty} [x_n + y_n] = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n$$

Addition

$$(\text{ii}) \lim_{n \rightarrow \infty} [x_n - y_n] = \lim_{n \rightarrow \infty} x_n - \lim_{n \rightarrow \infty} y_n$$

Subtraction

$$(\text{iii}) \lim_{n \rightarrow \infty} [x_n y_n] = (\lim_{n \rightarrow \infty} x_n) (\lim_{n \rightarrow \infty} y_n)$$

Multiplication

(iv) If $y \neq 0$ & $\lim_{n \rightarrow \infty} y_n \neq 0$ for any $n \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} \left[\frac{x_n}{y_n} \right] = \left(\frac{\lim_{n \rightarrow \infty} x_n}{\lim_{n \rightarrow \infty} y_n} \right)$$

Quotient

Proof we will prove (i)

(i): Suppose $x = \lim_{n \rightarrow \infty} x_n$ & $y = \lim_{n \rightarrow \infty} y_n$. Let $\epsilon > 0$ be given. Choose $M_1, M_2 \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ whenever $n \geq M_1$, and choose $M_2 \in \mathbb{N}$ such that $|y_n - y| < \epsilon$ whenever $n \geq M_2$. Define $M = \max\{M_1, M_2\}$. Then, for every $n \geq M$, the triangle inequality implies that

$$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| < \epsilon + \epsilon = 2\epsilon$$

Since ϵ is independent of n , we have that the sequence given by $\{x_n + y_n\}_{n=1}^{\infty}$ converges to $x + y$ as required. □

(ii) Let $\epsilon > 0$ be given. Set $K = \max\{|x|, |y|, \frac{\epsilon}{3}, 1\}$. Using the fact that $\{x_n\}$ converges to x and $\{y_n\}$ converges to y , we may select a pair $M_1, M_2 \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ whenever $n \geq M_1$ and $|y_n - y| < \epsilon$ whenever $n \geq M_2$. Set $M = \max\{M_1, M_2\}$. Then, for any such $n \geq M$,

$$\begin{aligned} |x_n y_n - xy| &= |(x_n - x)(y_n - y) + (x_n - x)y + (y_n - y)x| \\ &\leq |(x_n - x)y| + |(y_n - y)x| + |(x_n - x)(y_n - y)| \\ &= |y| \cdot |x_n - x| + |x| \cdot |y_n - y| + |x_n - x| \cdot |y_n - y| \\ &< |y| \cdot \epsilon + |x| \cdot \epsilon + \epsilon^2 \end{aligned}$$
□

(iv) We may use part (ii) to get the result after proving the next claim.

Claim: If $\{y_n\}, y \neq 0$ for all $n \in \mathbb{N}$, & $\{y_n\}$ converges to $y \neq 0$, then $\{\frac{1}{y_n}\}$ converges to $\frac{1}{y}$.

Proof: Let $\epsilon > 0$ be given. As $|y| \neq 0$, $K = \min\{\frac{1}{|y|^2}, \frac{1}{|y|}, \frac{1}{\epsilon^2}\} > 0$. As $\{y_n\}$ converges to y , we may select $M \in \mathbb{N}$ such that $|y_n - y| \leq K\epsilon$ whenever $n \geq M$. Consequently, for any $n \geq M$, we have

$$|y| = |y + y_n - y_n| \leq |y_n - y| + |y_n| < \frac{\epsilon}{2} + |y_n|. \text{ Thus, for these } n \geq M, \frac{|y|}{2} < |y_n|$$

which is equivalent to $\frac{1}{|y_n|} \leq \frac{2}{|y|}$. To complete the proof, for any $n \geq M$, $\left| \frac{1}{y_n} - \frac{1}{y} \right| = \left| \frac{y - y_n}{y y_n} \right| \geq \frac{|y - y_n|}{|y_n||y|} \leq \frac{|y - y_n|}{|y_n|} \cdot \frac{1}{|y|} = \frac{|y - y_n|}{|y|} \cdot \frac{|y|}{|y_n|^2} \leq \frac{|y - y_n|}{|y|} \cdot \frac{\epsilon}{|y|^2} \leq \frac{\epsilon}{|y|} \cdot \frac{2}{|y|^2} = \frac{2\epsilon}{|y|^3} = \epsilon$



Proposition: Let $\{x_n\}$ be a convergent sequence such that $x_n \geq 0$ for all n and
 Then $\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{\lim_{n \rightarrow \infty} x_n}$.

Proof: Let $\{x_n\}$ converge to x . As $x_n \geq 0$ for all n , we know that $x \geq 0$. Let $\varepsilon > 0$ be

given.

Case (1) $x = 0$

We need to show that $\{\sqrt{x_n}\}$ converges to 0. As $\{x_n\}$ converges to 0, there is some $M \in \mathbb{N}$ such that $|x_n - 0| < \varepsilon^2$ for every $n \geq M$. Then for every $n \geq M$,

$$\Rightarrow |\sqrt{x_n}| = \sqrt{x_n} < \sqrt{\varepsilon^2} = \varepsilon.$$

Case (2) $x > 0$

Because $x > 0$, we know that $\sqrt{x} \geq 0$. Choose $M_2 \in \mathbb{N}$ such that $|x_n - x| < \sqrt{\varepsilon}$ whenever $n \geq M_2$. Then, for any $n \geq M_2$, $|\sqrt{x_n} - \sqrt{x}| = \left| \frac{(\sqrt{x_n} - \sqrt{x})}{1} \right| \left| \frac{(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}} \right| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} < \frac{|x_n - x|}{\sqrt{x}} < \frac{\varepsilon}{\sqrt{x}}$. \square

Proposition: If $\{x_n\}$ is a constant sequence, then $\{\sqrt{|x_n|}\}$ is a constant sequence, and $\lim_{n \rightarrow \infty} |\sqrt{x_n}| = |\lim_{n \rightarrow \infty} x_n|$

Proof: Suppose $\{x_n\}$ converges to x . Given any $\varepsilon > 0$, choose $M \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n \geq M$. By the Reverse Triangle Inequality, for every $n \geq M$, we have $||x_n| - |x|| \leq |x_n - x| < \varepsilon$, as required. \square

Example: Show that $\{\sqrt{1 + \frac{1}{n}} - \frac{100}{n^2}\}_{n=1}^{\infty}$ converges to 1

Sketch of the idea: $\lim_{n \rightarrow \infty} \left| \sqrt{1 + \frac{1}{n}} - \frac{100}{n^2} \right|$

(i) $\lim_{n \rightarrow \infty} \frac{1}{n} = 0 \Rightarrow \lim_{n \rightarrow \infty} [1 + \frac{1}{n}] = 1$ as $1 + \frac{1}{n} \geq 0 \forall n$.

Squeeze Lemma then $\{\sqrt{1 + \frac{1}{n}}\}_{n=1}^{\infty}$ converges to $\sqrt{1} = 1$.

(ii) Since $\frac{100}{n^2} < \frac{1}{n} \forall n \in \mathbb{N}$, then $0 \leq \lim_{n \rightarrow \infty} \left(\frac{100}{n^2} \right) \leq \lim_{n \rightarrow \infty} \frac{1}{n} = 0$

Therefore by items (i) and (ii), $\lim_{n \rightarrow \infty} \left| \sqrt{1 + \frac{1}{n}} - \frac{100}{n^2} \right|$

$$= \lim_{n \rightarrow \infty} \left| \sqrt{\lim_{n \rightarrow \infty} (1 + \frac{1}{n})} - \lim_{n \rightarrow \infty} \frac{100}{n^2} \right|$$

$$= |\sqrt{1} - 0| = 1$$

Convergence Tests

Dominated Convergence: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence, and suppose there is an $M > 0$ and a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\lim_{n \rightarrow \infty} a_n = 0$ and $|x_n - x| \leq a_n$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x .

Proof: Let $\epsilon > 0$ be given. As each $a_n \geq 0$ and $\{a_n\}_{n=1}^{\infty}$ converges to 0, there is some $M > 0$ for which $a_n < \epsilon$ whenever $n \geq M$. By assumption, $|x_n - x| \leq a_n$ for each $n \in \mathbb{N}$. Therefore $|x_n - x| \leq \epsilon$ whenever $n \geq M$. \square

Proposition: Let $c > 0$

(i) If $c < 1$, then $\{c^n\}_{n=1}^{\infty}$ converges to zero

(ii) If $c > 1$, then $\{c^n\}_{n \in \mathbb{N}}$ is an unbounded set of \mathbb{R} , and so $\{c^n\}_{n=1}^{\infty}$ cannot converge.

Proof: First assume $0 < c < 1$. As $c > 0$, $c^n > 0$ for each $n \in \mathbb{N}$. As $c < 1$, we know that $c^{n+1} < c^n$ for every $n \in \mathbb{N}$. Therefore $\{c^n\}_{n=1}^{\infty}$ is a monotone decreasing sequence that is bounded from below by 0. Hence by MCT, it must converge to $\liminf_{n \rightarrow \infty} c^n = 0$. Since $\lim_{n \rightarrow \infty} c^n = 0$, necessarily, to $1 - c < 1$, $\{c^{n+1}\}_{n=1}^{\infty}$ must converge, & in fact $\lim_{n \rightarrow \infty} c^{n+1} = \lim_{n \rightarrow \infty} c^{1+n} = 0$. Since, for each $n \in \mathbb{N}$, $c^{n+1} = c \cdot c^n$, by taking limits, we get $x = cx$, or equivalently $(1-c)x = 0$. Either $1-c=0$, or $c=1$, or $x=0$.

Suppose now that $c > 1$. Let $B > 0$ be arbitrary. As $c > 1$, $\frac{1}{c} < 1$ and so by item

(i) the sequence $\{\frac{1}{c^n}\}_{n=1}^{\infty}$ must converge to 0. Therefore there must be $M \in \mathbb{N}$ such that $\frac{1}{c^n} < \frac{1}{B}$ whenever $n \geq M$, which is equivalent to saying $c^n > B$ whenever $n \geq M$. As B is arbitrary, the set $\{c^n\}_{n \in \mathbb{N}}$ must be unbounded.

Ratio Test for Sequences

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $x_n \neq 0$ for all $n \in \mathbb{N}$ & turbulent

$$L := \lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right|$$

exists. Then

(i) If $0 < L < 1$, then $\{x_n\}_{n=1}^{\infty}$ converges to 0.

(ii) If $L > 1$, then $\{x_n\}_{n \in \mathbb{N}}$ is unbounded, & so $\{x_n\}_{n=1}^{\infty}$ is divergent

(iii) If $L=1$, the test is inconclusive.

item (i)

Proof: Suppose that $L < 1$. Choose a number $r \in \mathbb{Q}$ such that $L < r < 1$ (density of \mathbb{Q}).

We want to compare $\{x_n\}_{n=1}^{\infty}$ with $\{r^n\}_{n=1}^{\infty}$. Now $r-L > 0$, so there exists $M \in \mathbb{N}$ such that

$$\frac{|x_{n+1}|}{|x_n|} - L \leq \left| \frac{|x_{n+1}|}{|x_n|} - L \right| < r-L$$

whenever $n \geq M$

Therefore, $\frac{|x_{n+1}|}{|x_n|} < r$ whenever $n \geq M$. For each $n \geq M$,

$$|x_n| = |x_M| \frac{|x_{M+1}|}{|x_M|} \cdots \frac{|x_{n-1}|}{|x_{n-1}|} \frac{|x_n|}{|x_{n-1}|} < |x_M| \underbrace{r \cdots r}_{n-M \text{ times}} = |x_M| \cdot r^{n-M} = [|x_M| \cdot r^{-M}] \cdot r^n$$

The sequence $\{r^n\}_{n=1}^{\infty}$ must converge to 0 by the previous proposition & so $\{|x_M| \cdot r^{-M} \cdot r^n\}_{n=1}^{\infty}$ must also converge to zero. Therefore, the M -tail $\{x_{n+1}\}_{n=M}^{\infty}$ must converge to zero and hence $\{x_n\}_{n=1}^{\infty}$ must converge to the same limit, 0.

Item (ii) Suppose $L > 1$, & choose r such that $1 < r < L$. Then $L - r > 0$. Pick $M \in \mathbb{N}$ such that $\frac{|x_{n+1}| - r}{|x_n|} \leq \frac{|x_{n+1}|}{|x_n|} - 1 < L - r$ whenever $n \geq M$.

Therefore, for each $n \geq M$, $\frac{|x_{n+1}|}{|x_n|} \geq r$. Again, for each $n \geq M$, $|x_n| = |x_M| \cdot r^{n-M}$

This is equivalent to $[|x_M| \cdot r^{n-M}] \cdot r^n$. By the previous proposition, the set $\{r^n : n \in \mathbb{N}\}$ is unbounded because $r > 1$.

Therefore, $\{x_n : n \in \mathbb{N}\}$ is unbounded, for if it were true there would exist some $B > 0$ such that $|x_n| \leq B$ for any $n \in \mathbb{N}$, which would imply that $r^n < \frac{B}{|x_M|} r^M$, $n \geq M$ a contradiction. \square

Example: $\lim_{n \rightarrow \infty} \sqrt{n} = \infty$

Proof: Let $\epsilon > 0$ be given & consider the sequence $\{\frac{n}{(1+\epsilon)^n}\}_{n=1}^{\infty}$. Compute for each

$$\forall n \in \mathbb{N}, \frac{\frac{n+1}{(1+\epsilon)^{n+1}}}{\frac{n}{(1+\epsilon)^n}} = \frac{n+1}{n} \cdot \frac{1}{1+\epsilon} \xrightarrow[n \rightarrow \infty]{\text{L'Hopital}} \left(\frac{\frac{n+1}{(1+\epsilon)^{n+1}}}{\frac{n}{(1+\epsilon)^n}} \right) \xrightarrow[n \rightarrow \infty]{\text{L'Hopital}} \left(\frac{n+1}{n} \cdot \frac{1}{1+\epsilon} \right)$$

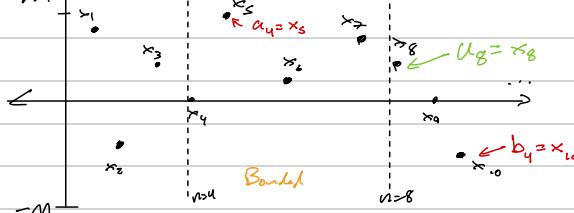
Note that $\frac{n+1}{n} = 1 + \frac{1}{n}$, so $\{\frac{n+1}{n}\}_{n=1}^{\infty}$ converges to 1. Therefore,

$$\lim_{n \rightarrow \infty} (\star) = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right) \cdot \lim_{n \rightarrow \infty} \left(\frac{1}{1+\epsilon} \right) = \frac{1}{1+\epsilon} < 1$$

By the ratio test, $\{\frac{n}{(1+\epsilon)^n}\}_{n=1}^{\infty}$ converges to zero. Thus we may choose an $M \in \mathbb{N}$ such that $\frac{n}{(1+\epsilon)^n} < 1$ whenever $n \geq M$. This is equivalent to saying that $n < (1+\epsilon)^n$ for all $n \geq M$. Therefore, for any $n \geq M$, $\sqrt{n} < 1 + \epsilon$. As $n \geq M$, for any n , we know $1 \leq \sqrt{n} \leq 1 + \epsilon$, we get the desired result sending $\epsilon \rightarrow 0$.

Limit Superior, Limit Inferior, & Bolzano Weierstrass Thm

If $\{x_n\}_{n=1}^{\infty}$ is bounded, then $\{x_{n+1}\}_{n=1}^{\infty}$ is a bounded set in \mathbb{R}



Moreover, for each $n \in \mathbb{N}$, the set $\{x_k : k \geq n\}$ is also what

Definition: Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence. For each $n \in \mathbb{N}$, define

$$a_n = \sup \{x_k : k \geq n\} \quad \text{and} \quad b_n = \inf \{x_k : k \geq n\}. \quad \leftarrow \begin{array}{l} \text{Cutting sequence} \\ \text{Find } b_n \text{ and } a_n \text{ first} \end{array}$$

Consider the sequences, $\{a_n\}_{n=1}^{\infty}$ & $\{b_n\}_{n=1}^{\infty}$. Define $\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} a_n$ (Limsup) & $\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} b_n$ (Liminf) provided these limits both exist.

Proposition: Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence, and $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be as in the above definition.

(i) The sequence $\{a_n\}_{n=1}^{\infty}$ is bounded & monotone decreasing, while $\{b_n\}_{n=1}^{\infty}$ is also bounded but is monotone increasing. In particular, $\limsup_{n \rightarrow \infty} x_n$ and $\liminf_{n \rightarrow \infty} x_n$ both exist.

Come from MCT (ii) $\limsup_{n \rightarrow \infty} x_n = \inf \{a_n : n \in \mathbb{N}\} = \inf \{ \sup \{x_k : k \geq n\} : n \in \mathbb{N} \}$

$$\liminf_{n \rightarrow \infty} x_n = \sup \{b_n : n \in \mathbb{N}\} = \sup \{ \inf \{x_k : k \geq n\} : n \in \mathbb{N} \}$$

$$(iii) \limsup_{n \rightarrow \infty} x_n \geq \liminf_{n \rightarrow \infty} x_n$$

Proof: We show that $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence.

For each $n \in \mathbb{N}$, $a_n = \sup \{x_k : k \geq n\}$ & $a_{n+1} = \sup \{x_k : k \geq n+1\}$. As $\{x_k : k \geq n\} \supseteq \{x_k : k \geq n+1\}$, we know that $a_{n+1} \leq a_n$ for each n . Therefore, $\{a_n\}_{n=1}^{\infty}$ is a bounded & decreasing sequence. By the MCT, $\{a_n\}_{n=1}^{\infty}$ must converge to $\inf \{a_n : n \in \mathbb{N}\}$.

The argument that $\{b_n\}_{n=1}^{\infty}$ is increasing is similar. Thus prove (i) & item (ii). For item (iii), for each $n \in \mathbb{N}$, $b_n = \inf \{x_k : k \geq n\} \leq \sup \{x_k : k \geq n\} = a_n$. Since limits preserve inequalities, item (i) implies that

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} b_n \leq \lim_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} x_n$$



Example: Let $\{x_n\}_{n=1}^{\infty}$ be given by
 $x_n = \begin{cases} \frac{n+1}{n}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$, Compute the limit inferior and superior

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\inf \{x_k : k \geq n\} \right) = \lim_{n \rightarrow \infty} 0 = 0$$

$$\{0, 0, \frac{4}{3}, 0, \frac{6}{5}, 0, \frac{8}{7}, 0, \dots\}$$

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\sup \{x_k : k \geq n\} \right)$$

$$\sup \{x_k : k \geq n\} = \begin{cases} \frac{n+1}{n}, & n \text{ odd} \\ \frac{n+2}{n}, & n \text{ even} \end{cases}$$

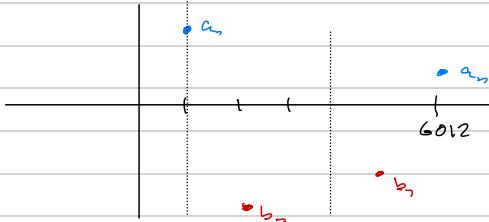
$$\text{we know } \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1 \neq \lim_{n \rightarrow \infty} \frac{n+2}{n} = 1.$$

Remark: The sequence $\{x_n\}_{n=1}^{\infty}$ does not converge!

Recall Definition of \limsup & \liminf

Defn: Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence. For each $n \in \mathbb{N}$ define

$$a_n = \sup \{x_k : k \geq n\} \quad b_n = \inf \{x_k : k \geq n\}$$



Consider $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$. Define

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} a_n$$

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} b_n$$

Theorem: Given a bounded sequence $\{x_n\}_{n=1}^{\infty}$, there exist subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ & $\{x_{m_k}\}_{k=1}^{\infty}$ of $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} x_{n_k} = \limsup_{n \rightarrow \infty} x_n \quad \& \quad \lim_{n \rightarrow \infty} x_{m_k} = \liminf_{n \rightarrow \infty} x_n$$

Proofs almost the same. Refer

Proof: Define $a_n = \sup \{x_k : k \geq n\}$ for each $n \in \mathbb{N}$, & set $x := \limsup_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} a_n$.

We define a subsequence inductively. Pick $n_1 = 1$.

Suppose that n_1, n_2, \dots, n_k have been chosen for some $k \geq 2$. Pick $m > n_k + 1$ such that $a_{m+1} - x_m < \frac{1}{k}$. Such an m exists because $a_{m+1} = \sup \{x_k : k \geq m+1\}$. Set $n_{k+1} = m$. The subsequence $\{x_{n_{k+1}}\}_{k=1}^{\infty}$ has thus been defined. We need to show $\lim_{n \rightarrow \infty} x_{n_k} = x$.

For each $k \geq 2$, we know that $a_{n_k} \geq x_{n_k} \& a_{n_k} \leq x_{n_{k+1}}$. Therefore, for every $k \geq 2$,

$$|a_{n_k} - x_{n_k}| = |a_{n_k} - x_{n_{k+1}}| \leq a_{n_{k+1}} - x_{n_{k+1}} \leq \frac{1}{k}.$$

Let $\epsilon > 0$ be given. As $\{a_n\}_{n=1}^{\infty}$ converges to x , the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ must also converge to x . There exists an $M \in \mathbb{N}$ such that $|a_{n_k} - x| < \frac{\epsilon}{2}$ whenever $k \geq M$. Fix

$M_2 \in \mathbb{N}$ such that $\frac{1}{M_2} \leq \frac{\epsilon}{2}$. Take $M = \max\{M_1, M_2, 2\}$. Then for all $k \geq M$,

$$|x - x_{n_k}| = |a_{n_k} - x_{n_k} + a_{n_k} - x| \leq |a_{n_k} - x_{n_{k+1}}| + |a_{n_{k+1}} - x| \leq \frac{1}{k} + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ as required.}$$

Proposition: A bounded sequence $\{x_n\}_{n=1}^{\infty}$ converges if and only if

$$\limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n$$

$$\text{In fact, } \limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} f(x_n)$$

Proof: For each $n \in \mathbb{N}$, define $a_n = \sup_{k \geq n} \{x_k\}_{k=1}^{\infty}$ & $b_n = \inf_{k \geq n} \{x_k\}_{k=1}^{\infty}$. In particular, for each $n \in \mathbb{N}$, we know that $b_n \leq x_n \leq a_n$. Assume that $\limsup_{n \rightarrow \infty} x_n$ & $\liminf_{n \rightarrow \infty} x_n$ coincide. Then, since $\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} a_n$ & $\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} b_n$, we may apply the squeeze lemma to get that

$$\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} a_n$$

Conversely, assume that $\{x_n\}_{n=1}^{\infty}$ converges to some $x \in \mathbb{R}$. According to the previous theorem, there must be some subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ that converges to $\limsup_{n \rightarrow \infty} x_n$. However, as $\lim_{n \rightarrow \infty} x_n$ necessarily $\lim_{n \rightarrow \infty} x_{n_k} = x$. Therefore $x = \limsup_{n \rightarrow \infty} x_n$ by the uniqueness of the limit.

Similarly for the \liminf you can extract a subsequence to show $x = \liminf_{n \rightarrow \infty} x_n$ \square

Proposition: Suppose you have a bounded sequence $\{x_n\}_{n=1}^{\infty}$ & $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence.

Then $\liminf_{n \rightarrow \infty} x_n \leq \liminf_{k \rightarrow \infty} x_{n_k} \leq \limsup_{k \rightarrow \infty} x_{n_k} \leq \limsup_{n \rightarrow \infty} x_n$

Proposition: A bounded sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$ iff every convergent subsequence converges to x .

Bolzano-Weierstrass Theorem:

Purely Geometric Theorem / Idea

Every bounded sequence of real numbers must have a convergent subsequence.

Proof: We have shown that any bounded sequence of real numbers must exhibit a subsequence that converges to its limit superior \square

Def: We say that a sequence $\{x_n\}_{n=1}^{\infty}$ diverges to infinity if, for every $K \in \mathbb{R}$, there is some M such that $x_n > K$ whenever $n \geq M$. In this case we write $\lim_{n \rightarrow \infty} x_n = \infty$.

Similarly, we say that a sequence $\{x_n\}_{n=1}^{\infty}$ diverges to negative infinity if, for every $K \in \mathbb{R}$, there is some M such that $x_n < K$ whenever $n \geq M$. In this case, we write $\lim_{n \rightarrow \infty} x_n = -\infty$.

Proposition: Suppose $\{x_n\}_{n=1}^{\infty}$ is a monotone and unbounded sequence. Then

$$\lim_{n \rightarrow \infty} x_n = \begin{cases} \infty & \text{if } \{x_n\}_{n=1}^{\infty} \text{ is increasing} \\ -\infty & \text{if } \{x_n\}_{n=1}^{\infty} \text{ is decreasing} \end{cases}$$

Example: (i) $\{n^{2/3}\}_{n=1}^{\infty}$ diverges to ∞

(ii) $\{-n^2\}_{n=1}^{\infty}$ diverges to $-\infty$

(iii) $\{n + \frac{1}{n}\}_{n=1}^{\infty}$

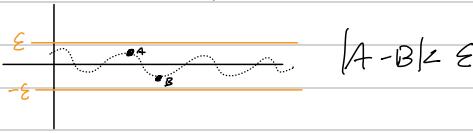
Def Let $\{x_n\}$ be unbounded set of real numbers. Define the sequence of extended real numbers $\{a_n\}$ & $\{b_n\}$ by

$$a_n = \sup \{x_k : k \geq n\}, \quad b_n = \inf \{x_k : k \geq n\}$$

If each a_n & b_n is a real number, then $\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} a_n$ & $\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} b_n$.

Cauchy Sequences

Def: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\{x_n\}_{n=1}^{\infty}$ is said to be a **Cauchy sequence** if, for any $\epsilon > 0$, there exists some $M \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ whenever $n, m \geq M$.



- Examples:**
- (i) Given any $C \in \mathbb{R}$, the sequence $\{c, c, \dots\}_{n=1}^{\infty}$ is a Cauchy sequence.
 - (ii) $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a Cauchy sequence.
 - (iii) $\{(-1)^n\}_{n=1}^{\infty}$ is not a Cauchy sequence.

Proposition Cauchy sequences are bounded

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. For $\epsilon = 1$, we may select an $M \in \mathbb{N}$ such that $|x_n - x_M| < 1$ whenever $n, m \geq M$. For these sorts $n \geq M$, the reverse triangle inequality implies

$$|x_n| - |x_M| \leq |x_n - x_M| \leq 1.$$

Therefore, for any $n \geq M$, $|x_n| \leq 1 + |x_M|$. Define $B := \max\{|x_1|, |x_2|, \dots, |x_M|, 1 + |x_M|\}$. Then, $|x_n| \leq B$ for every $n \in \mathbb{N}$, as required. □

Theorem (Completeness) A sequence of real numbers is convergent if it is Cauchy.

Proof: Suppose $\{x_n\}_{n=1}^{\infty}$ converges to an $x \in \mathbb{R}$. For each $\epsilon > 0$, choose $M \in \mathbb{N}$ for which $|x_n - x| < \frac{\epsilon}{2}$ whenever $n \geq M$. Then for all pairs $n, m \geq M$,

$$|x_n - x_m| = |(x_n - x) - (x_m - x)| \leq |x_n - x| + |x_m - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Conversely, assume that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. We need to show that $\{x_n\}_{n=1}^{\infty}$ converges. As we have just shown, $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. Therefore, $a := \limsup_{n \rightarrow \infty} x_n$ & $b := \liminf_{n \rightarrow \infty} x_n$ both exist. We have proven a theorem that guarantees the existence of subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ and $\{x_{m_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that

$$\lim_{k \rightarrow \infty} x_{n_k} = a \quad \& \quad \lim_{k \rightarrow \infty} x_{m_k} = b, \quad \text{End Goal: } a = b = 0$$

when $k \geq M$, so is x_n

Let $\epsilon > 0$ be given. Choose $M_1 \in \mathbb{N}$ such that $|x_n - a| < \frac{\epsilon}{3}$ whenever $n \geq M_1$. Choose $M_2 \in \mathbb{N}$ for which $|x_{m_k} - b| < \frac{\epsilon}{3}$ whenever $k \geq M_2$. Using the fact that $\{x_n\}_{n=1}^{\infty}$ is Cauchy, we have the sequence of an $M_3 \in \mathbb{N}$ such that $|x_n - x_m| < \frac{\epsilon}{3}$ whenever $n, m \geq M_3$. Set $M := \max\{M_1, M_2, M_3\}$. If $n \geq M$, then $n, m \geq M$. Therefore for any $k \geq M$,

$$|a - b| = |a - x_{n_k} + x_{n_k} - x_{m_k} + x_{m_k} - b| \leq |a - x_{n_k}| + |b - x_{m_k}| + |x_{n_k} - x_{m_k}| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, necessarily $a = b$ and thus

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n, \quad \text{as required.}$$

□

Series

Defn: Given a sequence $\{x_n\}_{n=1}^{\infty}$ we write the formal object

$$\sum_{n=1}^{\infty} x_n \text{, or also } \sum_{n=1}^{\infty} x_n \text{ and call it a series}$$

A series $\sum_{n=1}^{\infty} x_n$ converges (or is convergent) if the sequence $\{S_k\}_{k=1}^{\infty}$ given by $S_k = \sum_{n=1}^k x_n = x_1 + x_2 + \dots + x_k$ converges. The numbers S_k are called partial sums. If $\sum_{n=1}^{\infty} x_n$ should converge, we write

$$\sum_{n=1}^{\infty} x_n = \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} \sum_{n=1}^k x_n$$

If the sequence $\{S_k\}_{k=1}^{\infty}$ diverges, then we say that $\sum_{n=1}^{\infty} x_n$ diverges (or is divergent).

Example: The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to 1, that is $\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{2^k} = 1$

Proof: For each $K \in \mathbb{N}$ the identity $\sum_{n=1}^K \frac{1}{2^n} + \frac{1}{2^{K+1}} = 1$ holds. Set for each $K \in \mathbb{N}$, $S_K = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^K}$. Then we can write $|1 - S_K| = |1 - \sum_{n=1}^K \frac{1}{2^n}| = \left| \frac{1}{2^{K+1}} + \sum_{n=1}^{K+1} \frac{1}{2^n} - \sum_{n=1}^K \frac{1}{2^n} \right| = \frac{1}{2^{K+1}}$ for each $K \in \mathbb{N}$. The sequence $\{\frac{1}{2^n}\}_{n=1}^{\infty}$ converges to zero. Therefore $\{|1 - S_K|\}_{K=1}^{\infty}$ also converges to zero. Consequently $\{S_K\}_{K=1}^{\infty}$ must converge to 1, as required. \square

Proposition: Let $r \in \mathbb{R}$. The geometric series $\sum_{n=0}^{\infty} r^n$ converges if $|r| < 1$.

Proof: For each $K \in \mathbb{N}$ we have that $\sum_{n=0}^{\infty} r^n = \frac{1-r^K}{1-r}$. Can use induction to show that the series converges. This is a homework assignment question. \square

Proposition: A series $\sum_{n=1}^{\infty} x_n$ converges if & only if its tail converges, i.e. for $M \in \mathbb{N}$, the series $\sum_{n=M}^{\infty} x_n$ converges.

Proof: For $K \geq M$, write $\sum_{n=1}^K = \sum_{n=1}^M + \sum_{n=M}^K x_n$. The expression $\sum_{n=1}^M$ is a fixed number, so we can finish the proof using properties & results about sequences. \square

Def: A series $\sum_{n=1}^{\infty} x_n$ is said to be Cauchy if the sequence of partial sums $\{S_k\}_{k=1}^{\infty}$ is a Cauchy sequence.

for every $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $|\sum_{n=m+1}^{\infty} x_n| < \epsilon$ whenever $n, k \geq m$.

Without loss of generality, we may suppose that $k \geq m$. Then we may write $|\sum_{n=m+1}^k x_n| < \epsilon$ for all $n, k \geq m$.

Proposition: A series is Cauchy if and only if for every $\epsilon > 0$ there is a $M \in \mathbb{N}$ such that $|\sum_{n=M+1}^{\infty} x_n| < \epsilon$ whenever $n, k \geq M$.

Example: If $r \geq 1$ or $r \leq -1$, the geometric series $\sum_{n=0}^{\infty} r^n$ diverges.

Proof: For each $n \in \mathbb{N}$, we know that $|r|^n \geq |\pm 1|^n = 1$. Necessarily, $\sum_{n=0}^{\infty} r^n$ is not Cauchy. So it cannot converge. \square

Proposition Let $\sum_{n=1}^{\infty} x_n$ be a convergent series. Then the sequence of sums $\sum_{n=1}^N x_n$ converges and in fact $\lim_{N \rightarrow \infty} \sum_{n=1}^N x_n = 0$. Carefully, Cauchy's is not true.

Proof: As $\sum_{n=1}^{\infty} x_n$ is convergent, it must also be Cauchy. Given $\epsilon > 0$, there exists some $M > 0$ such that $|x_{n+1}| = |\sum_{i=n+1}^{n+M} x_i| < \epsilon$ whenever $n \geq M$.

$$x_{n+1} = \sum_{i=1}^{n+1} x_i - \sum_{i=1}^n x_i \Rightarrow x_1 - x_2 + x_3 - x_4 + \dots + x_n - x_{n+1} + x_{n+2}$$

Example: The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges despite the fact that the sum of its terms converges to zero.

Proof It is sufficient to show that the partial sums are unbounded. We consider the partial sums $n = 2^k$:

$$\begin{aligned} S_1 &= 1 & S_2 &= (1 + \frac{1}{2}) & S_4 &= (1 + \frac{1}{2}) + (\frac{1}{3} + \frac{1}{4}) \\ S_2 &= 1 + \frac{1}{2} & S_8 &= (1 + \frac{1}{2}) + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) & \dots & S_{2^k} = 1 + \sum_{i=1}^{2^k} \left[\sum_{m=2^{i-1}+1}^{2^i} \frac{1}{m} \right] \end{aligned}$$

Note that $\frac{1}{3} + \frac{1}{4} \geq \frac{1}{4}$, $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq \frac{1}{8}$, \dots

$$\sum_{m=2^{i-1}+1}^{2^i} \frac{1}{m} \geq \sum_{m=2^{i-1}+1}^{2^i} \frac{1}{2^i} = 2^{i-1} \cdot \frac{1}{2^i} = \frac{1}{2}$$

Therefore, $S_{2^k} = 1 + \sum_{i=1}^{2^k} \left[\sum_{m=2^{i-1}+1}^{2^i} \frac{1}{m} \right] \geq 1 + \sum_{i=1}^{2^k} \frac{1}{2} = 1 + \frac{2^k}{2} = 1 + 2^{k-1}$.

The sequence $\{\sum_{n=1}^{2^k} x_n\}_{k=1}^{\infty}$ is unbounded by the archimedean property. \square

Property (Linearity of Series) Let $\lambda \in \mathbb{R}$, and suppose $\sum_{n=1}^{\infty} x_n$ & $\sum_{n=1}^{\infty} y_n$ are convergent.

- (i) $\sum_{n=1}^{\infty} (\lambda x_n)$ is convergent and $\sum_{n=1}^{\infty} (\lambda x_n) = \lambda \sum_{n=1}^{\infty} x_n$
- (ii) $\sum_{n=1}^{\infty} (x_n + y_n)$ is convergent and $\sum_{n=1}^{\infty} (x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$

Proposition If $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} x_n$ converges iff the sequence of partial sums is bounded above.

Def: A series $\sum_{n=1}^{\infty} x_n$ converges absolutely if $\sum_{n=1}^{\infty} |x_n|$ converges. If a series converges, but doesn't converge absolutely, we say that it converges conditionally.

Proposition: If $\sum_{n=1}^{\infty} x_n$ converges absolutely, then it also converges (conditionally).

Proof: A series converges iff it is Cauchy. Therefore, as $\sum_{n=1}^{\infty} |x_n|$ is convergent, it is also Cauchy. Given $\epsilon > 0$, there is an $M \in \mathbb{N}$ such that $\sum_{i=k+1}^n |x_i| = |\sum_{i=k+1}^n x_i| < \epsilon$ whenever $k \geq M$ and $n \geq k$.

Applying the triangle inequality for finite sums,

$$\left| \sum_{i=1}^n x_i \right| \leq \sum_{i=1}^n |x_i| \geq \epsilon \text{ whenever } k \geq M.$$

Here, $\sum_{n=1}^{\infty} x_n$ is Cauchy as required. \square

Remark: The converse statement is not true.

