(3.1.5) Let $A \subset S$. Show that if c is a cluster point of A, then c is a cluster point of S. *Note the difference from Proposition 3.1.15*

Proof. A point c is a cluster point of a set A if every neighborhood or c contains a point of A that is distinct from c. Similarly, c is a cluster point is a cluster point of S if every neighborhood of c contains a point of S distinct from c. Assume that c is a cluster point of A. This means that for every $\epsilon > 0$, there is some $x \in A$ with $x \neq c$ such that $|x - c| < \epsilon$. More specifically it means that $x \in (c - \epsilon, c + \epsilon) \cap [S \setminus \{c\}]$. Since $A \subset S$, every point $a \in A$ must also be a point in S by definition. Therefore, for every $x \in (c - \epsilon, c + \epsilon) \cap [S \setminus \{c\}]$, we know that $x \in A \subset S$. This implies that every ϵ yields an x abiding by the previous description such that $x \in S \setminus \{c\}$, meaning that c must be a cluster point of S. $\square$

(3.1.12) Prove Proposition 3.1.17, that is, Let $S \subset \mathbb{R}$ be such that c is a cluster point of both $S \cap (-\infty, c)$ and $S \cap (c, \infty)$, let $f : S \to \mathbb{R}$ be a function, and let $L \in \mathbb{R}$. Then c is a cluster point of S and

$$\lim_{x \to c} f(c) = L \qquad \text{if and only if} \qquad \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. □

(3.2.2) Using the definition of continuity directly prove that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) \coloneqq \frac{1}{x}$ is continuous.

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. $\square$

(3.2.13) Let $f : S \to \mathbb{R}$ be a function and $c \in S$, such that for every sequence $\{x_n\}_{n=1}^{\infty}$ in S with $\lim\limits_{n\to\infty} x_n = c$, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges. Show that f is continuous at c.

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. □

(3.2.15) Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) = 0$, and suppose $f : \mathbb{R} \to \mathbb{R}$ is such that $|f(x) - f(y)| \leq g(x - y)$ for all x and y. Show that f is continuous.

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. □