(1) Let F be an ordered field. Prove that, for any pair of elements $x, y \in F$ such that $0 < x < y$, it holds that $x^2 < y^2$.

Proof. Consider the expression $x^2 < y^2$,

$$
\begin{aligned}
0 <& y^2 - x^2 && \text{Subtract } x^2 \text{ from both sides} \\
y^2 - x^2 >& 0 && \text{Rearrange for clarity} \\
(y - x)(y + x) >& 0 && \text{Factor the expression}
\end{aligned}
$$

Note that it suffices to show that $y^2 - x^2 > 0$, as this is derived from a basic algebraic operation, which is assumed to hold true. It is given that $y > x > 0$, which can be leveraged to prove the inequality. Both factors must be individually contended with and multiplied to yield strictly positive results. Firstly, $y - x > 0$ is always true for all x, y, as it is given that $y > x$. Secondly, $y + x > 0$ is guaranteed to be true for all x, y, as the sum of two positive elements of an ordered field, given to be F here, is positive. Therefore both factors of the expression are positive. By the definition of an ordered field F, if two elements $a, b \in F$ are both positive, meaning $a > 0$ and $b > 0$, then it is implied that $ab > 0$. Defining $a = y - x$ and $b = y + x$, the two positive factors, it is clear that $ab > 0$, and by extension, $(y - x)(y + x) > 0$. Consequently, $y^2 - x^2 > 0$, and hence $y^2 > x^2$ by fundamental properties of algebra which are assumed to hold true. Therefore, for any $x, y \in F$, where F is an ordered field, and $0 < x < y$, it holds that $x^2 < y^2$. □

(2) Let S be an ordered set and $A \subset S$ be a finite subset.
(a) Prove that $\inf A$ and $\sup A$ exist.

Proof. Suppose a set A has n elements, written as $\{a_0, a_1, \ldots, a_n\}$.

Base Case:
Consider the set with one element, $A = \{a\}$. Note this is the $(n = 1)$ case. Trivially, it is noted that this set A has $\inf A$ and $\sup A$ are both equal to a, as a is both the largest and smallest element in A. Hence, $\inf A$ and $\sup A$ exist.

Inductive Hypothesis:
Assume that for any set B, who's a finite subset of S, $B \subset S$, with $n \geq 1$ elements, both $\inf B$ and $\sup B$ exist.

Inductive Step:
Consider a set $A = \{a_1, a_2, a_3, \ldots, a_{n+1}\}$. Note this is set of $(n + 1)$ elements. By the inductive hypothesis, the subset $B = \{a_1, a_2, a_3, \ldots, a_n\}$ has both a supremum and an infimum.

To find $\inf A$, compare a_{n+1} with $\inf B$. There are two cases:

- *Case 1:* $a_{n+1} \geq \inf B$. In this case, the $n+1$ element is greater than $\inf B$, so it cannot be the greatest lower bound, hence $\inf B$ is the infimum of set A.

- *Case 2:* $a_{n+1} < \sup B$. In this case, the $n+1$ element is less than $\inf B$, so the infimum exists and is equal to a_{n+1}.

To find $\sup A$, compare a_{n+1} with $\sup B$. There are two similar possible cases:

- *Case 1:* $a_{n+1} \leq \sup B$. In this case, the $n + 1$ element does not exceed $\sup B$, so the supremum of A remains equal to $\sup B$, which exists by the inductive hypothesis.

- *Case 2:* $a_{n+1} > \sup B$. In this case, the $n+1$ element exceeds, or is greater than, every element in B, making this value the new supremum for set A.

Conclusion:
Therefore by induction, the statement is true for any finite subset $A \subset S$ for an ordered set S. Both $\inf A$ and $\sup A$ exist. $\square$

(b) Prove that $\inf A, \sup A \in A$.

Proof. Using the proof in part (a), $\inf A$ and $\sup A$ both exist for any finite subset A of the ordered set S.

The infimum of A is an element of set A:
Since A is defined as a finite subset, then it must have a smallest element because it is a subset of an ordered set which, by definition, requires all elements to be able to be compared. Define m as this element, whereby $m \leq a$ for all $a \in A$. By definition, $\inf A$ is the greatest lower bound of A, meaning that $\inf A \leq a$ for all $a \in A$. Since m is defined as the smallest element of A, $m \leq \inf A$ as a result. However, $\inf A \leq m$ as m is an element of A and $\inf A$ is a lower bound. Note that both values are less than or equal to each other. Therefore, $\inf A = m$, and consequently $\inf A$ is an element of A.

The supremum of A is an element of set A:
Similarly, A also has a greatest as it is finite. Define M as this element, where $M \geq a$ for all $a \in A$. By definition $\sup A$ is the least upper bound of A, meaning $\sup A \geq a$ for all $a \in A$. Since M is defined as the greatest element of A, $M \geq \sup A$ as a result. However, $\sup A \geq M$ as M is an element of A and $\sup A$ is an upper bound. Following similar logic to the above reasoning, $\sup A = M$, and consequently $\sup A$ is an element of A.

Conclusion:
Therefore, both $\inf A$ and $\sup A$ are elements of A, where A is a finite subset of the ordered set S. $\square$

(3) Let S be an ordered set, suppose that $B \subset S$ is bounded from above and from below, and let $A \subset B$ be non-empty. Suppose that all infs and sups exist. Prove that

$$\inf B \leq \inf A \leq \sup A \leq \sup B$$

Proof. Assuming that A is a non-empty set, and that all infs and sups exist, it is known that every element in A is also an element in B. Therefore, any lower bound of B is also a lower bound of A, and any upper bound of B is also an upper bound of A.

Step 1: Prove $\inf B \leq \inf A$

As stated above, any lower bound of B is also a lower bound of A. Since $\inf B$ is the greatest lower bound of B, and A is made up of elements from set B, then $\inf B$ must be less than or equal to any lower bound of A, including $\inf A$.

Step 2: Prove $\inf A \leq \sup A$

By definition, $\inf A$ is the greatest lower bound of A, and $\sup A$ is the least upper bound of A. This allows two basic cases:

- *Case 1:*
  If A has only one element, then that one element, say m, is both the lower and upper bound, or $\inf A$ and $\sup A$ at the same time. Therefore the two are equal.

- *Case 2:*
  If A has n number of elements, then it has an upper bound and lower bound, as it is a subset of B, which is bounded from above and below. By the same definition listed above, $\inf A \leq \sup A$, as $\inf A$ is a lower bound and $\sup A$ is an upper bound.

Step 3: Prove that $\sup A \leq \sup B$

As stated above, any upper bound of B is also an upper bound for A. Since $\sup B$ is the least upper bound of B, and A is a subset of B, then $\sup B$ must be greater than or equal to any upper bound of A, including $\sup A$.

It should be noted that the case-by-case logic used in step two can be applied to the other steps, though the logic is redundant and not as necessary when comparing $\inf$*s and* $\sup$*s of two different sets.*

Conclusion:

As S is an ordered set, and is the parent set to A and B, apply transitivity to the three inequalities proved in the above steps. Hence given a subset B of an ordered set S which is bounded from above and below and a subset A of B which is non-empty, it follows that:

$$\inf B \leq \inf A \leq \sup A \leq \sup B$$

Assuming that all infs and sups exist. ☐

(4) Prove that $\sqrt{3}$ is irrational.

Proof. Assume to the contrary, that there is an $x \in \mathbb{Q}$ such that $x^2 = 3$, and that this chosen x is rational. Let $x = \frac{m}{n}$, where $m, n \in \mathbb{Z} \setminus \{0\}$, and assume x is in lowest terms. So $(\frac{m}{n})^2 = 3$, by substitution and we note that $m^2 = 3n^2$, where m^2 is seen to be divisible by 3 and consequently m is also divisible by 3. Write $m = 3k$ for some integer $k \neq 0$ and so, by substitution, $(3k)^2 = 3n^2$. Divide the expression by 3 and note that $3k^2 = n^2$, and hence n is also divisible by 3. Both m *and* n are divisible by 3, so $\frac{m}{n}$ cannot be in lowest terms. Thus we have arrived at a contradiction , showing that $\sqrt{3}$ is irrational. □

(5) Prove the arithmetic-geometric means inequality: for any pair of non-negative real numbers x, y,

$$\sqrt{xy} \leq \frac{x+y}{2}$$

and there is equality if and only if $x = y$.

Proof. Note that $x, y \in \mathbb{R}$, and $x, y \geq 0$ as they are non-negative. Apply basic algebraic manipulation:

$$\begin{aligned}
\sqrt{xy} &\leq \frac{x+y}{2} && \text{Given} \\
2\sqrt{xy} &\leq (x+y) && \text{Multiply both sides by 2} \\
(2\sqrt{xy})^2 &\leq (x+y)^2 && \text{Square both sides} \\
4xy &\leq x^2 + 2xy + y^2 && \text{Expand both expressions} \\
x^2 + 2xy + y^2 &\geq 4xy && \text{Rearrange inequality} \\
x^2 - 2xy + y^2 &\geq 0 && \text{Combine like terms} \\
\implies (x-y)^2 &\geq 0 && \text{Factor binomial expression}
\end{aligned}$$

It can be noted that the above steps could be repeated in reverse order to further emphasize the equality of the first and last lines of the algebra above.

There are now two clear cases derived from the above expression as follows:

- *Case 1:*
  Consider the case where $x \neq y$. In this case, any choice of x and y which abide by the aforementioned restrictions, yield a positive value by Proposition 4. This states that any non-zero value in an ordered field, in this case the real numbers, yields a positive value when squared. $(x - y)$ only yields a value of 0 if $x = y$, which is neglected in this case. Consequently, The inequality holds true for this case.

- *Case 2:*
  Consider the case where $x = y$. In this case, any choice of x and y which are both equal and abide by the aforementioned restrictions results in 0 when they are subtracted. The square of zero is also zero, and hence the inequality holds true in this case as $0 \leq 0$. Evaluating the inequality using an $'='$ as:

  $$(x-y)^2 = 0 \implies x - y = 0 \implies x = y$$

  more elegantly shows that equality is only reached when $x = y$.

Both of the above cases are true, therefore the arithmetic-geometric means inequality is true for any pair of non-negative real numbers x, y. $\sqrt{xy} \leq \frac{x+y}{2}$ is true for any such choice, and equal if and only if $x = y$. $\square$