(3.1.5) Let $A \subset S$. Show that if c is a cluster point of A, then c is a cluster point of S. *Note the difference from Proposition 3.1.15*

Proof. A point c is a cluster point of a set A if every neighborhood or c contains a point of A that is distinct from c. Similarly, c is a cluster point is a cluster point of S if every neighborhood of c contains a point of S distinct from c. Assume that c is a cluster point of A. This means that for every $\epsilon > 0$, there is some $x \in A$ with $x \neq c$ such that $|x - c| < \epsilon$. More specifically it means that $x \in (c - \epsilon, c + \epsilon) \cap [S \setminus \{c\}]$. Since $A \subset S$, every point $a \in A$ must also be a point in S by definition. Therefore, for every $x \in (c - \epsilon, c + \epsilon) \cap [S \setminus \{c\}]$, we know that $x \in A \subset S$. This implies that every ϵ yields an x abiding by the previous description such that $x \in S \setminus \{c\}$, meaning that c must be a cluster point of S. $\square$

(3.1.12) Prove Proposition 3.1.17, that is, Let $S \subset \mathbb{R}$ be such that c is a cluster point of both $S \cap (-\infty, c)$ and $S \cap (c, \infty)$, let $f : S \to \mathbb{R}$ be a function, and let $L \in \mathbb{R}$. Then c is a cluster point of S and

$$\lim_{x \to c} f(c) = L \qquad \text{if and only if} \qquad \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

Proof. We begin with the forward direction. Assume that $\lim_{x \to c} f(x) = L$. By the definition of the limit, for every $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$. We can split this into two parts:

1. For the left-hand limit, $x \to c^-$: If $x < c$ and $0 < c - x < \delta$, then $|f(x) - L| < \epsilon$. Thus, $\lim_{x \to c^-} f(x) = L$.

2. For the right-hand limit, $x \to c^+$: If $x > c$ and $0 < x - c < \delta$, then $|f(x) - L| < \epsilon$. Thus, $\lim_{x \to c^+} f(x) = L$.

Since both limits equal L, we conclude that $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$.

Now we must prove the reverse direction. Suppose that $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$. By the definition of left- and right-hand limits, we have:

1. For the left-hand limit, given $\epsilon > 0$, there exists some $\delta_1 > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < c - x < \delta_1$.

2. For the right-hand limit, given the same arbitrarily chosen $\epsilon > 0$, there exists some $\delta_2 > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta_1$.

We must connect δ_1 and δ_2. Let $\delta = \min(\delta_1, \delta_2)$. This ensures that for all x sufficiently close to c, the function values $f(x)$ are close to L as this x satisfies $0 < |x - c| < \delta$ (as it was said to be sufficiently close). Here, x will either fall into the interval $(c - \delta, c)$ (for $x < c$) or $(c, c + \delta)$ (for $x > c$). Therefore for such x we can say $|f(x) - L| < \epsilon$. By definition this shows that $\lim_{x \to c} f(c) = L$, as required. $\square$

(3.2.2) Using the definition of continuity directly prove that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) \coloneqq \frac{1}{x}$ is continuous.

Proof. To directly prove that the function $f : (0, \infty) \to \mathbb{R}$ given by $f(x) \coloneqq \frac{1}{x}$ is continuous, we must show that at any point $c \in (0, \infty)$, the epsilon-delta definition of continuity holds. That is, we need to show that for every $\epsilon > 0$, there exists some $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

We begin by computing $|f(x) - f(c)| = \left|\frac{1}{x} - \frac{1}{c}\right|$. We can simplify this by finding a common denominator as $\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{c-x}{xc}\right| = \frac{|c-x|}{|xc|}$. Now we must show that this expression is lees than ϵ, or that $\frac{|c-x|}{|xc|} < \epsilon$. We can rewrite the previous inequality as $|c - x| < \epsilon\,|xc|$.

We must now find δ, which requires controlling both $|x - c|$ and $|xc|$. As $c > 0$, set $|x - c| < \delta$. We can expand this to be $-\delta < x - c < \delta \implies c - \delta < x < c + \delta$. If we take $\delta \leq \frac{c}{2}$, then this ensures that x remains close to c. Thus we have $c - \frac{c}{2} < x < c + \frac{c}{2} \implies \frac{c}{2} < c < \frac{3c}{2}$. As this x must be greater than $\frac{c}{2}$, we can bound $|xc|$ as $|xc| > \left(\frac{c}{2}c\right) = \frac{c^2}{2}$. We can substitute these back into $|c - x| < \epsilon\,|xc|$ to yield $|c - x| < \epsilon\frac{c^2}{2}$. This allows us to choose $\delta = \min\left(\frac{c}{2}, \frac{\epsilon c^2}{2}\right)$.

We can now proceed by verifying this choice of δ. We know from above that if $|x - c| < \delta$, then $|c - x| < \frac{\epsilon c^2}{2}$. We also have that $|xc| > \frac{c^2}{2}$, so

$$|f(x) - f(c)| = \frac{|c - x|}{|xc|} < \frac{\frac{\epsilon c^2}{2}}{\frac{c^2}{2}} = \epsilon.$$

and hence $f(x) \coloneqq \frac{1}{x}$ is continuous at every $c \in (0, \infty)$, as required. $\square$

(3.2.13) Let $f : S \to \mathbb{R}$ be a function and $c \in S$, such that for every sequence $\{x_n\}_{n=1}^{\infty}$ in S with $\lim_{n\to\infty} x_n = c$, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges. Show that f is continuous at c.

Proof. We know that f satisfies the property that for every sequence $\{x_n\}_{n=1}^{\infty}$ in S with $\lim_{n\to\infty} x_n = c$, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges. We must show continuity at c, meaning that $\lim_{x\to c} f(x) = f(c)$.

Assume for contradiction that f is not continuous at c. By definition of continuity, it must consequently be the case that $\lim_{x\to c} f(x) \neq f(c)$. This implies that there exists some sequence $\{x_n\}_{n=1}^{\infty}$ in S such that $\lim_{n\to\infty} x_n = c$, but that the corresponding sequence $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to $f(c)$. However, we have assumed that for every sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = c$, the corresponding sequence $\{f(x_n)\}_{n=1}^{\infty}$ does, in fact, converge. Thus we arrive at a contradiction indicating that our assumption of f not being continuous is false. Therefore f is continuous at c, as required. $\square$

(3.2.15) Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) = 0$, and suppose $f : \mathbb{R} \to \mathbb{R}$ is such that $|f(x) - f(y)| \leq g(x - y)$ for all x and y. Show that f is continuous.

Proof. We are given that $|f(x) - f(y)| \leq g(x-y)$, and letting $y = c$, we have $|f(x) - f(c)| \leq g(x - c)$. We can do this as this inequality is given to hold true for all $x, y \in \mathbb{R}$, and we know $c \in \mathbb{R}$. We proceed by analyzing the behavior of g as $x \to c$. In other words, we must see the behavior of $\lim\limits_{x\to c} g(x - c)$. Since $x \to c$ implies that $x - c \to 0$, we have that $\lim\limits_{x\to c} g(x - c) = g(0) = 0$.

We will proceed with the use of the squeeze theorem. Since $|f(x) - f(c)| \leq g(x - c)$, we have that $-g(x - c) \leq f(x) - f(c) \leq g(x - c)$. We take the limit of this expression as:

$$\lim_{x\to c} (-g(x - c) \leq f(x) - f(c) \leq g(x - c)) = -\lim_{x\to c} g(x - c) \leq \lim_{x\to c} f(x) - f(c) \leq \lim_{x\to c} g(x - c).$$

Therefore we have that $\lim\limits_{x\to c} f(x)-f(c) = 0$ by the squeeze theorem as $\lim\limits_{x\to c} g(x-c) = g(0) = 0$. We thus conclude that $\lim\limits_{x\to c} |f(x) - f(c)| = 0$. As $f(c)$ is not dependent on x, we can rewrite this expression as $\lim\limits_{x\to c} |f(x)| = |f(c)|$. This shows that f is continuous at c, and since c was chosen arbitrarily, we therefore have that f is continuous everywhere on $\mathbb{R}$. $\square$