(2.3.7) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences.

(a) Show that $\{x_n + y_n\}_{n=1}^{\infty}$ is bounded.

Proof. Suppose $M_1, M_2 \in \mathbb{R}$. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, the there exists some $M_1 > 0$ such that for all $n \in \mathbb{N}, |x_n| \leq M_1$. Similarly since $\{y_n\}_{n=1}^{\infty}$ is bounded, there exists some $M_2 > 0$ such that for all $n \in \mathbb{N}, |y_n| \leq M_2$.

Now consider the sequence $\{x_n + y_n\}_{n=1}^{\infty}$. By the triangle inequality, $|x_n + y_n| \leq |x_n| + |y_n|$ holds. We can then say that $|x_n + y_n| \leq |x_n| + |y_n| \leq M_1 + M_2$. Defining $M = M_1 + M_2$, this is equivalent to $|x_n + y_n| \leq M$. As M_1 and M_2 are arbitrary positive real numbers, M is also a positive real number. Thus, by definition the sequence $\{x_n + y_n\}_{n=1}^{\infty}$ is bounded, as required. □

(b) Show that

$$\left(\liminf_{n\to\infty} x_n\right) + \left(\liminf_{n\to\infty} y_n\right) \leq \liminf_{n\to\infty} (x_n + y_n)$$

Proof. Let $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, then $\{x_{n_k}\}_{k=1}^{\infty}$ must also be bounded. We can then choose a convergent subsequence $\left\{x_{n_{k_i}}\right\}_{i=1}^{\infty}$ of $\{x_{n_k}\}_{k=1}^{\infty}$, which is guaranteed to exist by the Bolzano-Weierstrass Theorem. Therefore $\lim_{i\to\infty}\left(x_{n_{k_i}}\right)$ exists and $\liminf_{n\to\infty} x_n \leq \lim_{i\to\infty}\left(x_{n_{k_i}}\right)$ by the definition of the limit inferior.

By similar reasoning we can let $\{y_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{y_n\}_{n=1}^{\infty}$ and can chose a subsequence $\left\{y_{n_{k_i}}\right\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty}\left(y_{n_{k_i}}\right)$ exists. Hence $\liminf_{n\to\infty} y_n \leq \lim_{i\to\infty}\left(y_{n_{k_i}}\right)$ by the definition of the limit inferior.

As you'd expect, we can also choose a subsequence $\left\{x_{n_{k_i}} + y_{n_{k_i}}\right\}_{i=1}^{\infty}$ of $\{x_{n_k} + y_{n_k}\}_{k=1}^{\infty}$ which is a subsequence of $\{x_n + y_n\}_{n=1}^{\infty}$. By the Bolzano-Weierstrass Theorem, we can choose this subsequence such that it converges and thus $\lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right) = \lim_{k\to\infty}(x_{n_k}) + \lim_{k\to\infty}(x_{n_k})$ and that $\liminf_{n\to\infty}(x_n + y_n) \geq \lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right)$ by definition of the limit inferior. Using the previously derived expressions, we have

$$\lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right) = \lim_{i\to\infty}\left(x_{n_{k_i}}\right) + \lim_{i\to\infty}\left(y_{n_{k_i}}\right)$$

$$\lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right) \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$$

$$\liminf_{n\to\infty}(x_n + y_n) \geq \lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right) \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$$

Therefore $(\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n) \leq \liminf_{n\to\infty}(x_n + y_n)$ as required. □

(c) Find an explicit $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that

$$\left(\liminf_{n\to\infty} x_n\right) + \left(\liminf_{n\to\infty} y_n\right) < \liminf_{n\to\infty} (x_n + y_n)$$

Proof. Let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$ for all $n \in \mathbb{N}$. Clearly both of these sequences are bounded above by 1 and below by -1.

$$x_n + y_n = (-1)^n + (-1)^{n+1} = (-1)^n + (-1)^n(-1) = (-1)^n(1-1) = 0$$

Thus $\liminf_{n\to\infty} (x_n + y_n) = 0$. The minimum value for x_n, y_n is -1 for all $n \in \mathbb{N}$, so $\liminf_{n\to\infty} x_n = -1$ and $\liminf_{n\to\infty} y_n = -1$. Therefore $\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n = -1 + -1 = -2$. Consequently,

$$\left(\liminf_{n\to\infty} x_n\right) + \left(\liminf_{n\to\infty} y_n\right) < \liminf_{n\to\infty} (x_n + y_n)$$
$$-2 < 0$$

showing that the strict inequality (no equality) holds for these choices of x_n, y_n. $\square$

(2.3.8) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences, and $\{x_n + y_n\}_{n=1}^{\infty}$ is bounded by the previous exercise.

(a) Show that

$$\left(\limsup_{n\to\infty} x_n\right) + \left(\limsup_{n\to\infty} y_n\right) \geq \limsup_{n\to\infty} (x_n + y_n)$$

Proof. Choose convergent subsequences $\left\{x_{n_{k_i}}\right\}_{i=1}^{\infty}, \left\{x_{n_{k_i}}\right\}_{i=1}^{\infty}$ of $\{x_{n_k}\}_{n=1}^{\infty}$ and $\{y_{n_k}\}_{n=1}^{\infty}$, respectively as done in the previous exercise. Therefore the limits and limit superiors can be related as $\limsup_{n\to\infty} x_n \geq \lim_{i\to\infty} x_{n_{k_i}}$ and $\limsup_{n\to\infty} y_n \geq \lim_{i\to\infty} y_{n_{k_i}}$ by the definition of the limit superior. Note that these limits are guaranteed to exist as the subsequences were chosen to converge and $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are given to be bounded.

Similarly, we can also choose a subsequence $\left\{x_{n_{k_i}} + y_{n_{k_i}}\right\}_{i=1}^{\infty}$ of $\{x_{n_k} + y_{n_k}\}_{k=1}^{\infty}$. By the Bolzano-Weierstrass Theorem, we can chose this subsequence to converge. In other words, $\lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right) = \lim_{k\to\infty}(x_{n_k}) + \lim_{k\to\infty}(x_{n_k})$. Also similarly to the above proof, $\limsup_{n\to\infty}(x_n + y_n) \leq \lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right)$ by definition of the limit superior. Consequently we have that,

$$\begin{aligned}
\lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right) &= \lim_{i\to\infty}\left(x_{n_{k_i}}\right) + \lim_{i\to\infty}\left(y_{n_{k_i}}\right) \\
\lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right) &\leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \\
\limsup_{n\to\infty}(x_n + y_n) \leq \lim_{i\to\infty}\left(x_{n_{k_i}} + y_{n_{k_i}}\right) &\leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n
\end{aligned}$$

Therefore, $(\limsup_{n\to\infty} x_n) + (\limsup_{n\to\infty} y_n) \geq \limsup_{n\to\infty}(x_n + y_n)$ as required. □

(b) Find an explicit $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that

$$\left(\limsup_{n\to\infty} x_n\right) + \left(\limsup_{n\to\infty} y_n\right) > \limsup_{n\to\infty} (x_n + y_n)$$

Proof. Let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$ for all $n \in \mathbb{N}$. Clearly both of these sequences are bounded above by 1 and below by -1.

$$x_n + y_n = (-1)^n + (-1)^{n+1} = (-1)^n + (-1)^n(-1) = (-1)^n(1-1) = 0$$

Thus $\limsup_{n\to\infty}(x_n + y_n) = 0$. The maximum value for x_n, y_n is 1 for all $n \in \mathbb{N}$, so $\limsup_{n\to\infty} x_n = 1$ and $\limsup_{n\to\infty} y_n = 1$. Therefore $\limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n = 1 + 1 = 2$. Consequently,

$$\begin{gathered}
\left(\limsup_{n\to\infty} x_n\right) + \left(\limsup_{n\to\infty} y_n\right) > \limsup_{n\to\infty} (x_n + y_n) \\
2 > 0
\end{gathered}$$

showing that the strict inequality (no equality) holds for these choices of x_n, y_n. □

(2.4.1) Prove that $\left\{\frac{n^2-1}{n^2}\right\}_{n=1}^{\infty}$ is Cauchy using directly the definition of Cauchy sequences.

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. □

(2.4.6) Suppose $|x_n - x_k| \leq \frac{n}{k^2}$ for all n and k. Show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy.

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. □

(2.4.8) True or false, prove or find a counterexample: If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then there exists an M such that for all $n \geq M$, we have $|x_{n+1} - x_n| \leq |x_n - x_{n-1}|$.

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. $\square$