

LCR

Damped and Forced Oscillators

revised March, 2023

Learning Objectives:

During this lab, you will

1. communicate scientific results in writing.
2. estimate the uncertainty in a quantity that is calculated from quantities that are uncertain.
3. fit an exponentially decaying sinusoidal model to experimental data.
4. fit a Lorentzian model to experimental data.
5. compare quantities measured by different experiments.
6. study the behavior of oscillating circuits.

A. Introduction

An inductor is usually a coil or toroid of wire that can store energy in its magnetic field. The behavior of an inductor in a circuit is governed by Faraday's Law of Induction, which can be used to demonstrate that EMFs across an inductor are associated with changes in the current passing through it.

In this lab you will examine the behavior of an inductor in an electronic circuit and how, by combining an inductor with a capacitor and resistor, one can produce and control oscillations. Understanding the behavior of an LCR circuit should also help you appreciate the many mechanical analogs of this system, basically anything that vibrates or shakes.

An electronic oscillator, like its mechanical equivalent, has a natural or resonant frequency $\omega_R = 1/\sqrt{L C}$ at which it tends

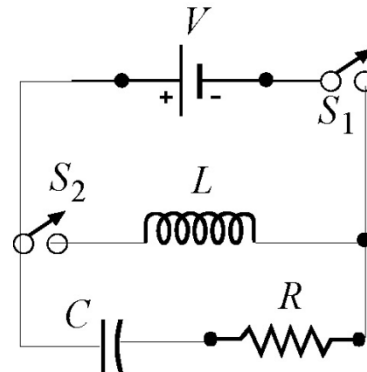


Figure 1: LCR (damped oscillator) circuit diagram.

to oscillate. A periodic driving force delivered at just this frequency will cause large oscillations. If the driving force is delivered at any other frequency $\omega \neq \omega_R$, the system will oscillate with reduced amplitude at the frequency ω of the driving force, not at its natural frequency. You will investigate in this lab some important properties of the forced, damped oscillator.

You must write a paper for this lab worth 60 points.

B. Theory

B.1. Damped Oscillator

Figure 1 shows a schematic diagram of a circuit with a resistor, capacitor, and inductor which can be connected in series by closing switch S_2 to form a *damped LC oscillator*. When switch S_2 is open and switch S_1 is first closed, the capacitor is connected to the voltage source, V , and charges through resistor R to this voltage. When S_1 is subsequently opened and S_2 is closed, the voltage across the capacitor decays back to zero but may oscillate while doing so, depending on the values of the components. The behavior of the voltage as a function of time is illustrated in Figure 2 (see next page).

A damped LC oscillator is equivalent to a damped mechanical oscillator formed from a mass on a spring with a shock absorber. Both systems obey the same differential

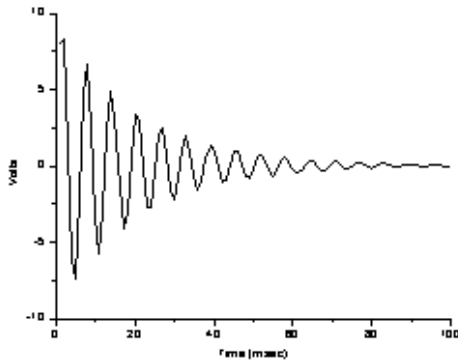


Figure 2: Voltage decay of an LCR circuit. equation, in the form of Eq. 1 for the charge on the capacitor in the damped LC oscillator and Eq. 2 for its mechanical equivalent; only the symbols are different.

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = 0 \quad (1)$$

$$M \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0 \quad (2)$$

One can see from these equations that the inductance L takes the place of mass, providing inertia to the system. The capacitance (actually $1/C$), takes the place of the spring constant k , supplying a restoring force. The resistance R substitutes for the damping force b to resist motion, converting the kinetic energy of the conduction electrons into heat.

A solution to Eq. 1 is an exponentially damped sine wave (Fig. 2)

$$Q = Q_0 e^{-(t/\tau)} \sin(\omega t + \phi). \quad (3)$$

with

$$\omega' = \sqrt{\omega^2 - \frac{1}{\tau^2}} = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \quad (4)$$

where the parameters are

Q_0 = charge at $t = 0$ if $\phi = \pi/2$;

$\tau = 2L/R$ is the decay time or time for the *envelope* or exponential term to decrease by $1/e$;

$\omega^2 = 1/LC$ where ω is the resonant angular frequency of the undamped ($R = 0$) LC circuit. This shifts to ω' with damping;

ϕ = the phase of the signal, which describes where along the sine function the wave starts at $t = 0$.

Note that you will be measuring the voltage, V_C , across the capacitor and not the charge, Q , on it. However, $Q = CV_C$ so this difference is unimportant and Eq. 3 also applies to the voltage, with Q replaced by V_C . (The Theory section of your paper should show the expected variation of voltage with time.)

We can classify damped harmonic systems in one of three ways based on if ω'^2 is positive (underdamped) zero (critically damped) or negative (over damped). Alternatively many use the damping ratio ζ defined as

$$\zeta = \frac{R}{2} \sqrt{\frac{C}{L}} \quad (5)$$

to represent the class of the damped harmonic system. If $\zeta < 1$, it is underdamped, if $\zeta = 1$, it is critically damped and if $\zeta > 1$, it is overdamped.

B.2. Forced Oscillator

The forced, damped oscillator circuit is shown schematically in Figure 3. It is essentially the same as the circuit for the damped

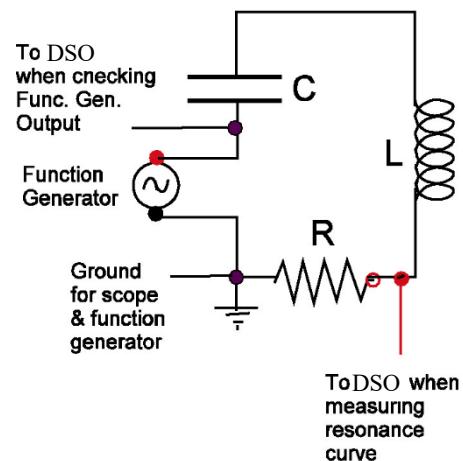


Figure 3: Forced, damped oscillator.

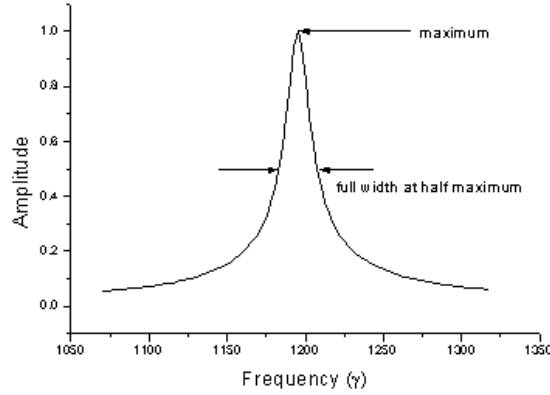


Figure 4: Resonance curve.

oscillator with a function generator replacing the battery. There is no switch since we shall only be concerned with the *steady-state* solution, *i.e.*, we are not concerned with what happens when the function generator is first connected to the *LCR* circuit, but only in the operation of the circuit after it has been running for a time which is long compared to the period of the *LC* oscillations.

The amount of charge on the capacitor in this circuit is described by the differential equation

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E_m \cos \omega t \quad (6)$$

where $\omega = 2\pi f$ is the angular frequency of the driving oscillator. (*This equation is similar to the equation for the forced mechanical oscillator.*) The solution of Eq. 6 for the current ($I = dQ/dt$) in the circuit is

$$I = I_m \cos(\omega t + \varphi) \quad (7)$$

where I_m is the amplitude of the current and φ is the phase of the current relative to the phase of the driving oscillator. Note that the current oscillates at the **angular** frequency ω of the driver and not at the natural frequency of the circuit. The amplitude of the current is given by

$$I_m = \frac{V_m}{D} \quad (8)$$

with

$$D = \sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2}$$

I_m is a maximum when D is a minimum, which occurs when the term in parenthesis above is zero and the driving frequency ω obeys the relation

$$\omega L = \frac{1}{\omega C} \quad i.e., \quad \omega = \sqrt{\frac{1}{LC}} = \omega_R \quad (9)$$

These relationships show that the current is greatest when the angular frequency ω of the driving force equals the resonant frequency of the circuit ω_R .

From Eq. (8) we can compute the gain voltage across the resistor (the ratio V_m/V_R) from Ohm's law since $V_R = I R$. This yields

$$\frac{V_R}{V_m} = \frac{R_R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \quad (10)$$

A graph of the gain vs. ω is referred to as a *resonance curve*, as illustrated in Figure 4. The signal is described as *at resonance* when its frequency gives the maximum amplitude shown in Fig. 4. The width of the curve can be characterized by its *full width at half maximum* $\Delta\omega$, often referred to as its *half width*. (*This terminology can be confusing—it does not mean half the width of the curve, but the full width at half the maximum amplitude.*)

The *Quality Factor* or “*Q*” of a resonant circuit is related to the relative energy loss, $\Delta U/U$, during each oscillation cycle by

$$\Delta U/U = 2\pi/Q \quad (11)$$

Q can be expressed in terms of the components of the circuit as

$$Q = \frac{\omega_R L}{R} = \frac{1}{R} \sqrt{\frac{L}{C}} \quad (12)$$

The Q of the circuit can also be related to the full width at half maximum $\Delta\omega$ by

$$\frac{\Delta\omega}{\omega_R} = \frac{c}{Q} \quad (13)$$

where c is a constant (*roughly* $\sqrt{3}$), so Q is

$$Q = \frac{c\omega_R}{\Delta\omega}. \quad (14)$$

Combining equations 12 and 14, give

$$\Delta\omega = \frac{cR}{L}. \quad (15)$$

One can see from Eq. 14 that a large Q corresponds to a small $\Delta\omega$ or a very sharply peaked resonance curve. This also explains why the Q is often called the quality factor. The higher the Q of a resonant circuit, the better it is for selecting a single frequency. For example, an FM radio tuner needs a Q of about 1000 in order to distinguish stations spaced at, for example, 101.5 and 101.7 MHz.

We wish to investigate the gain, as a function of the driving frequency ω for values of ω which are close to the resonant frequency ω_R .

C. Damped Oscillator

C.1. Apparatus

You will construct the LCR circuit on a Pasco board using a 80-100 mH inductor, recognizable as many layers of wire wound on a cylindrical core. Since the inductor is made from a long length of thin wire, it has a non-negligible resistance (*about* 200 Ω) which should be measured and added to the

resistance R in your calculations. You will use an LC meter to measure the inductance of the inductor. The manufacturer of the LC meters claims they have an accuracy of $\pm 2\% \pm 1$ digit.

You will also use resistors (two 1 k Ω in parallel and a 100 Ω), capacitors (0.47 μ F and 0.022 μ F) and a DMM with capacitance measuring capability. The value of the capacitors may not be clear from their labeling but you can check this with your LC Meter. You should also use your DMM to more accurately determine the values of your resistors. Data will be acquired and analyzed using *LoggerPro*.

C.2. Procedure

Construct the circuit of Figure 1 as per the circuit layout in Figure 5. Your first circuit should use a 0.022 μ F capacitor and an inductor (Although we do not have a resistor, remember that you still have the resistance of the inductor in this circuit.) Use only one of the batteries on the Pasco board.

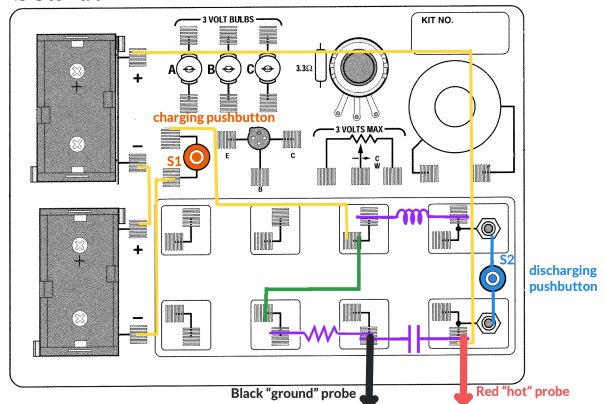


Figure 5: Circuit layout for the damped oscillator.

The pushbuttons on the Pasco board are closed only while being pressed. S_1 charges the capacitor through the resistor and S_2 discharges the capacitor through the inductor and resistor. (To check that you have constructed the circuit correctly, you *may* temporarily connect a DMM across the capacitor. This will let you verify that the

capacitor charges to the correct voltage when you press S_1 . It will not let you follow the discharge cycle in detail, since this occurs too quickly for the DMM circuitry. You should **disconnect** the DMM before continuing with the experiment since the capacitor can discharge through the DMM.)

C.3. *LoggerPro* Directions

Start the *LoggerPro* program and load the file ..._E and M Labs \\LCR. Acquiring a clean plot takes some patience. The general idea is to **reset *LoggerPro* using the mouse, press S_1 to charge the capacitor, then press S_2 , and then release S_1 to run a discharge cycle.** The *LoggerPro* trigger circuit is not very sophisticated and does not function reliably but it **will** work eventually. While *LoggerPro* is trying to trigger on your signal, the message “Waiting for Trigger” will flash on the screen - this is normal. You may have to charge and discharge a several times before a proper plot appears. You can keep alternating between pressing the charge and discharge button until a plot appears. Since the time constant is measured in milliseconds, you can do this very quickly, pressing each button several times a second until *LoggerPro* triggers properly. If you encounter triggering problems in run 3 below, you should change the triggering to decreasing across 1.45 V. If you are still having problems, connect both batteries (3 V) in the circuit.

If a poor plot appears, you will have to delete it clicking the “Collect” button. A poor plot might have the first few oscillations cut off at the top or glitches somewhere in the curve. You might find that your data ranges up to 10 V even though your battery only supplies 1.5 V; this is not necessarily a problem and you may use a “clean” curve even if this occurs. The signals can be larger than the battery voltage because of the transient effects that occur when you include a

pushbutton in the circuit and because the voltage across the inductor depends on the time rate of change of the current and not just on the original battery voltage. *LoggerPro* cannot properly handle signals larger than 10 V. They will instead chop off the data at 10 V or possibly even display it as 0 V. If the first cycles of your data look odd, like squared-off sine waves or with abrupt drops to zero or spikes, discard this data and acquire another plot.

When you get a successful oscillation from setup 1, analyze it as described in Section D. After finishing that analysis, acquire and analyze the three additional setups (2-4) described below, so that you have four sets of data altogether.

1. $C = 0.022 \mu\text{F}$, $R = 0 \Omega$.
2. $C = 0.47 \mu\text{F}$, $R = 0 \Omega$
3. $C = 0.47 \mu\text{F}$, $R = 100 \Omega$
4. $C = 0.47 \mu\text{F}$, $R = 500 \Omega$

Finally take a look at the response from the following two setups.

5. $C = 0.47 \mu\text{F}$, $R = 1 \text{ k}\Omega$
6. $C = 0.47 \mu\text{F}$, $R = 2 \text{ k}\Omega$

D. Damped Oscillator Analysis

D.1. Curve Fitting

You need to **perform a fit to Eq. 3 for each of your four sets of data. Make rough estimates, within a factor of two or so, of the maximum amplitude, angular frequency ω , time constant τ and phase ϕ of your measured signals.** These values will serve as starting values for the least-squares fitting procedure. Such approximate starting values are required when fitting any non-trivial curve, since least squares fitting will normally fail if started with arbitrary values. You should be able to make suitable estimates “by eye,” without the use of a calculator, just by looking at your plot and understanding the significance of each term in Eq. 3. *(If you are confused about how to estimate these*

parameters, the Appendix I at the end of this Lab provides some guidance.)

D.1.1. LoggerPro Curve Fitting

Use the *LoggerPro* software to do the fitting. The first step is to **enter the proper equation.** (The exponentially damped sine wave isn't built into *LoggerPro*; however, your Lab Director has put it into the file for this experiment). **Go to ANALYSIS/CURVE FIT and scroll to the bottom of the list and select the equation:**

$$y = A \cdot \exp(-t/L) \cdot \sin(W \cdot t - P)$$

(If someone has maliciously deleted it, you can click on "define function" and type the above in.) Note the correspondence of the terms in this equation with those in Eq. 3. *LoggerPro* is not case sensitive, so you cannot use T for τ ; we have chosen to use L (for capacitance) for that variable.

Make sure the Fit Type is "Automatic" and click "Try Fit." If the fit in the mini-window looks OK, click "OK." Sometimes *LoggerPro* has a hard time finding a good fit from its default parameters. If your fit doesn't look good, try adjusting the starting parameters to your rough estimates of the values for A (i.e., V_0), W (i.e., ω), L (i.e., τ) and P (i.e., ϕ), in this order and trying the fit again. When you are satisfied that the fit is as good as it will get (*the parameters displayed by LoggerPro should stop changing*), press OK. This action should paste the fitting parameters into the plot (including uncertainties), which you can then **save to the L: drive. Include these plots in your paper.**

D.2. Data Analysis

For the data from trials 1-4, compare your fitted values for ω' and τ to those you calculate from the values of your electronic components.

How much of a difference should there be between ω' and ω based on the components' values?

What is the effect of increasing R ? In Eq. 4, it should be possible to make ω' equal to 0 by making $(R/2L)^2 = (1/LC)$. This is called *critical damping* and describes a system that no longer oscillates. For all 6 trials compute ζ (Eq. 5). Classify each system as underdamped, critically damped, or overdamped. Thinking of a mechanical analog of the damped harmonic oscillator, this is how you'd like your car's suspension to behave when you go over a bump; it bounces once and then stops. If you make R even bigger (*heavy duty shocks*), to increase the damping, the circuit will take longer to decay towards zero as the RC time constant begins to dominate the $2L/R$ term. This is called *overdamping* and would be like pushing down on a car's suspension and having it take several seconds to slowly return to equilibrium.

What is the effect of decreasing C ? Should this have an effect on the damping time constant τ ? Did it?

E. Resonant Circuit

E.1. Apparatus

You will construct the oscillator diagrammed in Fig. 3 on a Pasco board using a 80-100 mH inductor, a 0.0047 μ F capacitor and a 1 k Ω resistor. Drive the circuit with a function generator and observe the signals with an oscilloscope.

E.2. Measuring Procedure

Use your DMM to check the values of the resistor and capacitor. Measure the resistance of the inductor if you haven't already done so. **Wire the inductor, capacitor and resistor in series with the function generator** as illustrated in Fig. 3. Note the common ground point for the function generator and scope. This is

essential; connecting grounds to two separate points in a circuit shorts out everything in between and can damage your electronics, your lab grade, and your relationship with the Lab Director.

Connect the Ch2 output of the function generator in the circuit, because channel 2 gives a higher amplitude than channel 1. Make sure to select Ch2 on the function generator by pressing the CH1/2 button when you set frequencies and amplitudes. “CH2 Waveform” should display in the middle left of the screen. Set the function generator frequency to 10 kHz and adjust the function generator’s amplitude control to 16 V_{pp}. With the scope connected across the function generator, make sure you see the signal on the scope. Press the blue AUTO button if you cannot find the signal by turning the Vertical and Horizontal controls. Record the reading for frequency and amplitude from both the function generator and the scope in your notebook.

Now move the “hot” wire of the scope between the resistor and the inductor. The signal on the scope will be very small, so press AUTO again to find the signal. **Find the resonant frequency of the circuit by varying the frequency of the signal generator to locate the peak amplitude.** At resonance, the amplitude decreases on either side of the maximum. When you use the dial on the function generator to vary the frequency, you will notice that only the most significant digit of the frequency value changes. Turn the dial up and down until you notice the amplitude on the scope becoming maximum. You will have to periodically find the signal on the scope again, either by pressing the AUTO button, or by using the Vertical and Horizontal controls on the scope. When you find the maximum with the “coarse” dial control, start using the keypad

on the function generator to find a more accurate value for the resonant frequency.

Acquire the data necessary to make your own plot similar to Fig. 4, a plot of amplitude (voltage) versus frequency. Start by decreasing and then increasing the frequency until the amplitude of the oscillation is about 10% of its maximum value. After you’ve established the 10% endpoints of your graph, take about 20 coordinate pairs (frequency and amplitude) across the resonant curve as the amplitude increases from ~10% of maximum on the low frequency side to maximum and then decreases to ~20% of maximum on the high frequency side of the resonance. Be certain to take points when the amplitude is at its maximum and at about half of its maximum since you will need these for your analysis.

E.3. Analysis

Use Origin to plot a graph of the Gain versus frequency as in Fig. 4. (Be certain to enter your data in *Origin* in numerical order from lowest frequency to highest; otherwise *Origin*’s fitting may fail. The gain is the voltage across the resistor divided by the input voltage from the function generator. To do this, simply divide all of the data points by the value of the input voltage. You can do this with the ADD NEW COLUMN and SET COLUMN VALUES features of Origin.) **Fit the peak function (Appendix II) to your data** (Select ANALYSIS → FITTING → NON-LINEAR CURVE FIT → OPEN DIALOG and create a new function (See the Appendix II below for an explanation of the peak function.)

Compare the “theoretical” and predicted values of ω_R and Q using your measured values for R, L and C. Be certain to include the resistance of the inductor in your calculation (it adds in series to the resistor). Also don’t forget that $\omega = 2\pi f$. Don’t be surprised

if the experimental value is a little off the theoretical value. In this case, one of the problems is that the function generator adds significant resistance to the circuit. **Use A to determine the resistance of the function generator. Add this to your total resistance; recalculate the predicted Q of the circuit.**

Appendix I. Estimating Parameters

Your LCR data should be similar to the curve plotted in Fig. 2 and described by Eq. 3. Three of the parameters, the amplitude, angular frequency and phase are determined largely by the sine term while the time constant is contained in the exponential term. The exponential term goes to unity as $t \rightarrow 0$, so examining the data near the beginning of the curve lets you isolate the first three terms mentioned above.

The amplitude A (i.e., V_0) is approximately the amplitude (peak to zero) of the first cycle of the sine wave.

The angular frequency ω (i.e., ω) can be found from the time it takes to complete one cycle, the period T . The angular frequency $\omega = 2\pi f$ where f is the frequency. $f = 1/T$, so $\omega = 2\pi/T$. Estimate the period from your plot, invert it and multiply by 6 to get a rough estimate of ω .

The time constant τ or L (i.e., τ) is deduced from the exponentially decreasing aspect of the *envelope* of the curve. At $t = \tau = T$, the amplitude of the curve, at its local peak, should have fallen to $1/e$ of its final value. It will be sufficient to estimate the time at which the amplitude is $1/3$ of its initial height.

The phase P (i.e., ϕ) is probably the hardest of these parameters to estimate accurately. A phase of 0° corresponds to a sine wave which starts at $V = 0$ and initially increases. (Note that you will probably need this value in radians, not degrees.) A phase of 90° describes a cosine. A phase of 180°

describes an inverted sine wave, one that starts going negative rather than positive. Increasing the phase angle (up to a maximum of 360° or 2π radians) causes the starting point of the plot to move further along an imaginary sine curve, effectively shifting the plotted curve to its left. If you can estimate the phase to within $\pi/2$ radians, *LoggerPro* will have no problem completing the fit.

Appendix II. Bandpass Peak Function

It turns out that this form of the peak function in Eq. (10) is numerically difficult to fit; however, we can rewrite this in terms of Q , ω_R and $A = R_L/R$ as

$$\frac{V_R}{V_{in}} = \frac{A}{\sqrt{1 + Q^2 \frac{(\omega^2 - \omega_R^2)^2}{\omega^2 \omega_R^2}}} \quad (16)$$

Note that in the expression above we are using **angular frequencies ($\omega = 2\pi f$)**.

This function is not pre-programmed as a fitting function in *Origin*. So you will have to create the function yourself. For your convenience the function is in *Origin's* language below with independent variable f , dependent variable y , and parameters A , Q , and f_R (Select ANALYSIS \rightarrow FITTING \rightarrow NON-LINEAR CURVE FIT \rightarrow OPEN DIALOG and create a new function. Refer to Appendix IV (ORIGIN) F.4 for details on how to add your own fitting function. Remember that as with any non-linear curve fit you will need good initial guesses for the parameters so that the function will fit.

$$y = A / (\text{sqrt}(1 + Q^2 * (f^2 - f_R^2)^2 / (f^2 * f_R^2))) + C$$