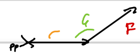
Cycle 3

More on Rotations

Summary of Important Collisions

Torque: $\tau_p = rF\sin\theta$



N2L Rot: $\tau_{Net} = I\alpha$

R.C: $v = \omega r$ and $a = \alpha r$

Values for Rotational Inertia (I)

Textbook / Labs may call it moment of Inertia

$I = C m R^2$, for some const C

Common Values for I:

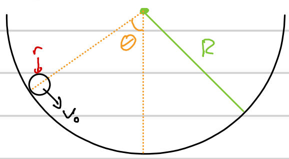
$I_{Hoop} = mR^2$ $I_{Hollow\ Sphere} = \frac{2}{3} mR^2$

$I_{Disk} = \frac{1}{2} mR^2$ $I_{Rod\ CM} = \frac{1}{12} m\ell^2$

$I_{Solid\ Sphere} = \frac{2}{5} mR^2$ $I_{Rod\ End} = \frac{1}{3} m\ell^2$

Good to have on cheatsheet to reference easily

Example: Hollow ball rolls in bowl



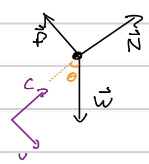
Given

- m, r, v_0, θ of ball, no slip/slide
- R of Bowl
- $I = \frac{2}{3} m r^2$

Find Normal Force, Friction force, a

Assume $R \gg r$

FBD Ball



N2L c-coord centripetal

$F_c = m a_c$

$N - W\cos\theta = m\frac{v_0^2}{R}$ ← radius of bowl as motion dictated by the bowl

$N = mg\cos\theta + \frac{m v_0^2}{R}$

$$\boxed{N = m\left(g\cos\theta + \frac{v_0^2}{R}\right)}$$

N2L v-coord tangental

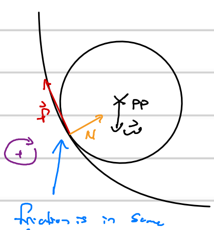
$F_v = m a_v$

$W\sin\theta - f = m a_v$

Stop!

Now Contend with Rotations!

xFBD Ball



friction is in same direction as coordinate sys so torque is positive

N2L Rot

$\tau_{Net} = I\alpha$

$\tau_w + \tau_N + \tau_p = I\alpha$

$0 + 0 + rf\sin(90°) = I\alpha$

$\cancel{r} f = \frac{2}{3} m r^{\cancel{2}} \alpha$ $\alpha = \frac{a_v}{r}$ R.C

$f = \frac{2}{3} m r\left(\frac{a_v}{r}\right)$

Stop! $f = \frac{2}{3} m a_v$

① τ_w: $0 w\sin\theta = 0$

② τ_N: $rN\sin(180°) = 0$

Checking G in torque

force is positive for torque if its aligned with coordinate system.

Two Equations, Two Unknowns

① $mg\sin\theta - f = m a_v$

② $f = \frac{2}{3} m a_v$

$\cancel{m} g\sin\theta - \frac{2}{3}\cancel{m} a_v = \cancel{m} a_v$

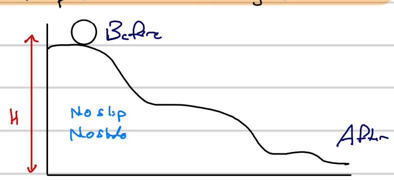
$g\sin\theta = \frac{5}{3} a_v$

$$\boxed{a_v = \frac{3}{5} g\sin\theta}$$

$f = \frac{2}{3} m\left(\frac{3}{5} g\sin\theta\right)$

$$\boxed{f = \frac{2}{5} mg\sin\theta}$$

Example: Ball rolling on slide



Given: Release @ rest

$m, R, I = \frac{2}{5} mR^2$ of ball

H of ball initially

Translational Kinetic: $K_T = \frac{1}{2} m v^2$

Rotational Kinetic: $K_R = \frac{1}{2} I\omega^2$

$\therefore K_{Tot} = K_T + K_R = \frac{1}{2} m v^2 + \frac{1}{2} I\omega^2$

Friction does "no work" @ point of contact

$E_{Tot} = E'_{Tot}$

$U + K_T + K_R = U' + K_T' + K_R'$

$mgy + \frac{1}{2} m v^2 + \frac{1}{2} I\omega^2 = mgy' + \frac{1}{2} v'^2 + \frac{1}{2} I\omega'^2$

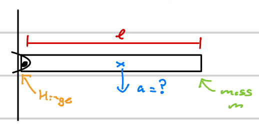
$y = H,\ y' = 0,\ v = 0,\ \omega = 0,\ \omega = \frac{v}{r}$ R.C

$mgH + 0 + 0 = 0 + \frac{1}{2} m v_p^2 + \frac{1}{2}\left(\frac{2}{5} m\cancel{R^2}\right)\left(\frac{v_T}{\cancel{r}}\right)^2$

$\cancel{m} gH = \frac{1}{2}\cancel{m} v_p^2 + \frac{1}{5}\cancel{m} v_T^2$ $v_T = v_p$

$gH = \frac{7}{10} v_p^2 \rightarrow \boxed{v_p = \sqrt{\frac{10}{7} gH}}$

Rod on Hinge

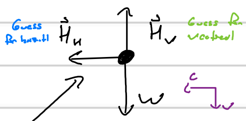


① Find Acceleration of CoM @ release. Rod starts @ rest!

Not worth spending a lot of time determining direction of H components

CoM is moving on circular path so use v/c-coordinates.

FBD Rod



Hinge Force is completely unknown! Just assign two arbitrary forces to represent it. One is vertical one is horizontal.

If you mis-guess direction, answer is negated. Just note that and move on to get positive back.

N2L Horizontal c-coord

$F_c = m a_c$

$H_H = m \frac{v^2}{``R"}$ @ rest so $v=0$

$\therefore H_H = 0$ woah!

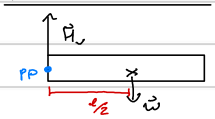
N2L Vertical v-coord

$F_v = m a_v$

$w - H_v = m a_v$

$mg - H_v = m a_v$ Stop!

x FBD Rod



(2)

Neglect H_H bc it is 0 from N2L

N2L Rot

$\tau_{Net} = I\alpha$ Known for Rod

$\tau_v + \tau_w = \frac{1}{3}m\ell^2\alpha$

$0 + \frac{\ell}{2}mg(\sin 90°) = \frac{1}{3}m\ell^2\alpha$

$\frac{1}{2}\cancel{\ell}\cancel{m}g = \frac{1}{3}\cancel{m}\cancel{\ell^2}\left(\frac{a_v}{\cancel{\ell}/2}\right)$

$\frac{1}{2}g = \frac{1}{3}2a_v$

$\boxed{a_v = \frac{3}{4}g}$

R.C.

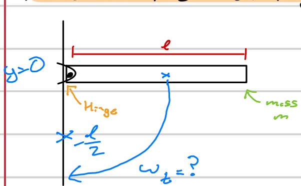
$a = \alpha r$

$\alpha = \frac{a}{r}$

r is radial distance from pp to CoM here!

$r = \frac{\ell}{2}$

Another Rod on Hinge



Hinge Force does no work bc its @ point of contact. No Force applied over a distance

CofME

Conditions: Weights conservative and Hinge does no work @ point of contact!

Choose End of Rod as Reference, no translational K!

Before = After

$E_{Tot} = E'_{Tot}$

$U + K_T + K_R = U' + K'_T + K'_R$

$U + 0 + K_{Rod} = U' + 0 + K'_{Rod}$

$mgy + 0 + \frac{1}{2}I\omega^2 = mgy' + 0 + \frac{1}{2}I\omega'^2$

$0 + 0 + 0 = mg\left(-\frac{\ell}{2}\right) + \frac{1}{2}\left(\frac{1}{3}m\ell^2\right)\omega_f^2$

Algebra!

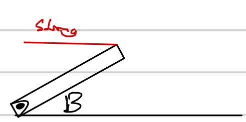
$\frac{1}{2}\cancel{m}g\cancel{\ell} = \frac{1}{6}\cancel{m}\ell^{\cancel{2}}\omega_f^2$

$\boxed{\omega_f = \sqrt{\frac{3g}{\ell}}}$

If you choose a different Reference Point, Its different!

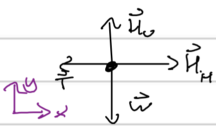
You will still get the right answer if you use a different reference point!

Hinge with a String



$a=0,\ \alpha=0$

FBD Rod



At rest! so H_x and the must hold rod in place

N2L y-dir

$F_y = m\cancel{a_y}$ 0

$H_v - w = 0$

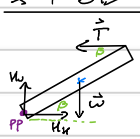
$\boxed{H_v = mg}$

N2L x-dir

$F_x = m\cancel{a_x}$ 0

$H_H - T = 0$ Stop!

x FBD Rod



$\tau_H = 0$

$\tau_T = \ell T\sin\beta$

$\tau_w = \left(\frac{\ell}{2}\right)mg\sin(90-\beta)$

$\tau_w = \frac{\ell}{2}mg\cos\beta$ Trig!

Statics

Static Friction

$f_s \le \mu_s N$

Constrained

Kinetic Friction

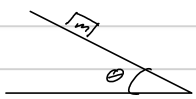
$f_k = \mu_k N$

Equality

For Statics, $a = 0$ and $\alpha = 0$

Use N2L to find N → N2L to find f_s → Apply constraint

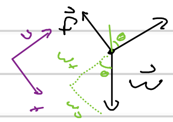
Example



Given: m, θ, μ_s

Find θ_{max}

FBD Block



N2L y-dir

$F_y = m\cancel{a_y}^{0}$

$N - W\cos\theta = 0$

$N = mg\cos\theta$

N2L x-dir

$F_x = m\cancel{a_x}^{0}$

$W\sin\theta - f_s = 0$

$f_s = mg\sin\theta$

Example shows $f_s \le \mu_s N$ is not useless! Good for determining maximums!

No an Equal Sign!

Apply Constraint: $f_s \le \mu_s N$

$mg\sin\theta \le \mu_s mg\cos\theta$

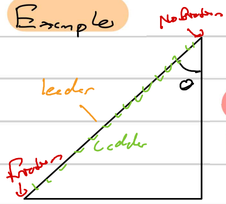
$\frac{\sin\theta}{\cos\theta} \le \mu_s$

$\tan\theta \le \mu_s$

$\theta \le \arctan\mu_s$

$\therefore \boxed{\theta_{max} = \arctan(\mu_s)}$

Example

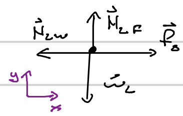


Given

m, θ, L

Find all forces

FBD Ladder



N2L y-dir

$F_y = m\cancel{a_y}^{0}$

$N_{LF} - W = 0$

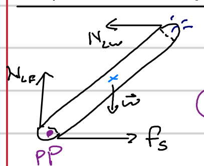
$\boxed{\begin{aligned} N_{LF} &= mg \quad (1) \\ W &= mg \quad (2) \end{aligned}}$

N2L x-dir

$F_x = m\cancel{a_x}^{0}$

$f_s - N_{LW} = 0$ Stop!

FBD Ladder



Place PP where it kills most forces

N2L Rot

$\tau_{Net} = I\cancel{\alpha}^{0}$

$\tau_{N_{LF}} + \tau_{f_s} + \tau_W + \tau_{N_{LW}} = 0$

$0 + 0 - \frac{L}{2}W\sin\theta + LN_{LW}\sin(90-\theta) = 0$

$-\frac{1}{2}\cancel{L}mg\sin\theta + \cancel{L}N_{LW}\cos\theta = 0$

$N_{LW}\cos\theta = \frac{1}{2}mg\sin\theta$

$\boxed{N_{LW} = \frac{1}{2}mg\tan\theta}$ (3)

Recall: $\sin(90-\theta) = \cos\theta$

$\cos(90-\theta) = \sin\theta$

$f_s - N_{LW} = 0$

$f_s = N_{LW}$

$\boxed{f_s = \frac{1}{2}mg\tan\theta}$ (4)

Center of Mass and Matter Distribution

CoM:

$\vec{r} \equiv \frac{m_1 r_1 + m_2 r_2 + \dots m_n r_n}{M}$

$x_{cm} \equiv \frac{m_1 x_1 + m_2 x_2 + \dots m_n x_n}{M}$

$a_{cm} \equiv \frac{d}{dt}(\vec{v}_{cm}) = \frac{m_1 a_1 + m_2 a_2 + \dots m_n a_n}{M}$

$M \equiv m_1 + m_2 + \dots m_n$

$v_{cm} \equiv \frac{d}{dt}(\vec{x}_{cm}) = \frac{m_1 v_1 + m_2 v_2 \dots m_n v_n}{M}$

Good definitions to have in your back pocket

Continuous Distribution of Matter

$$\boxed{\begin{aligned} \vec{r}_{cm} &\equiv \frac{1}{M}\sum_{i=1}^{n} m_i r_i \\ \vec{r}_{cm} &\equiv \frac{1}{M}\int_{volume} r\,dm \end{aligned}}$$

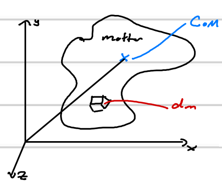
where $M = \sum_{i=1}^{n} m_i$

Uniform Density

$\rho = \frac{m}{V}$

Not Uniform Density

$\rho = \frac{dm}{dV} \rightarrow dm = \rho\,dV$



$x_{cm} = \frac{1}{M}\int_{volume} x\,dm$

$= \frac{1}{M}\int x\rho\,dV$

$= \frac{1}{M}\iiint x\rho\,dx\,dy\,dz$

Triple Integral ON

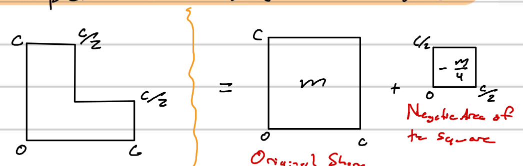
Calculus 3 CoM

$\bar{x}_{cm} \equiv \frac{1}{M}\iiint x\rho\,dx\,dy\,dz$

$\bar{y}_{cm} \equiv \frac{1}{M}\iiint y\rho\,dx\,dy\,dz$

$\bar{z}_{cm} \equiv \frac{1}{M}\iiint z\rho\,dx\,dy\,dz$

Example: A cut-out Square from a Square



Find x_{cm}?

Original Shape

Negative Area of the Square

$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m(\frac{c}{2}) + (-\frac{m}{4})(\frac{c}{2})}{m + (-\frac{m}{4})}$

$= \frac{\frac{mc}{2} - \frac{mc}{16}}{\frac{3m}{4}} = \frac{2\cancel{m}c - \frac{\cancel{m}c}{4}}{3\cancel{m}} = \frac{\frac{7}{4}c}{3} = \boxed{\frac{7c}{12}}$

Recall: $\vec{F}_{\substack{Net\\Sys\\Ext}} = m\vec{a}_{cm}$

Special Cases

① If $\vec{F}_{\substack{Net\\Sys\\Ext}} = 0$ means $\vec{a}_{cm} = 0$, $\vec{v}_{cm} = \text{constant}$

Individual Forces don't need to be 0, just need to cancel

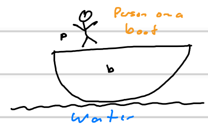
then $\vec{P}_{Tot} = m\vec{v}_{cm} = \text{constant}$ C of LM

This means system is isolated

② If $\vec{v}_{cm} = 0$ means $P_{Tot} = 0$

$V_{cm} = 0 \rightarrow$ CoM is not changing!

Thus $\vec{r}_{cm} = \text{constant}$ in CoMRF



$$x_{cm} = \frac{m_p x_p + m_b x_b}{m_p + m_b}$$

x_{cm} is constant so if you move one way, the boat will move one way to keep x_{cm} constant

Recall: Elastic $\rightarrow K = K'$

Moving to CoMRF — center of mass reference frame

$$v' = v - v_{cm}$$

velocity in CoMRF; CoMRF in "Lab" Frame; Com in "Lab" Frame

Used in Ubox Method!

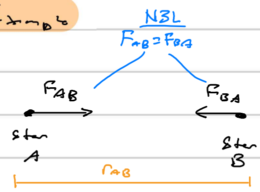
Universal Gravity Revisited

$$F_{UG} = \frac{GMm}{r^2}$$

"attractive"

distance of separation

Example

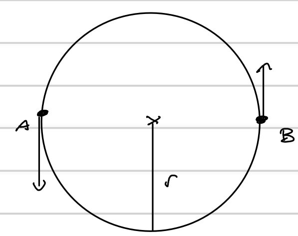


$$F_{AB} = F_{BA} = F_{UG} = \frac{GM_AM_B}{r_{AB}^2}$$

$a_A = a_B = a$ is true!

$$a = \frac{v^2}{r}$$

Find v_A, v_B



N2L

Let $m_A = m_B = m$

$$F_{Net} = ma$$

$$\frac{GMm}{r^2} = m\frac{v^2}{r}$$

$r^2 \leftarrow$ distance of separation; $r \leftarrow$ radius

$$\frac{GM^2}{(2r)^2} = m\frac{v^2}{r}$$

$$\frac{GM^2}{4r^2} = \frac{mv^2}{r}$$

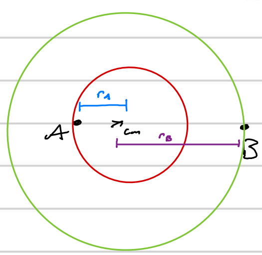
$$\frac{GM}{4r} = v^2$$

$$\boxed{v = \sqrt{\frac{GM}{4r}}}$$

Both stars move w/ the same velocity

$v_A = v_B = v$

Example: $m_A > m_B$



Find $v_A, v_B, \omega_A, \omega_B$

$$x_{cm} = \frac{m_Ax_A + m_Bx_B}{m_A + m_B} = 0$$

$$= \frac{m_A(r_A) + m_B(r_B)}{m_A + m_B} = 0$$

Distance between

$$m_Ar_A + m_Br_B = 0$$

$$\frac{r_A}{r_B} = \frac{m_B}{m_A}$$

N2L $F_{Net} = ma$

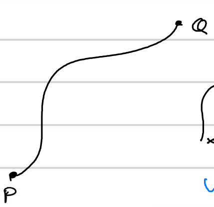
$$\frac{GM_AM_B}{(r_A + r_B)^2} = \underbrace{m_A\frac{v_A^2}{r_A}}_{\text{Replace A with B for stars}}$$

radius

$$\frac{GM_Br_A}{(r_A + r_B)^2} = v_A^2 \rightarrow \boxed{v_A = \sqrt{\frac{GM_Br_A}{(r_A + r_B)^2}}}$$

R.C $v = \omega r \rightarrow \omega = \frac{v}{r}$

Path Integrals and Conservative Forces



Work Definition

$$W_P = \int \vec{F} \cdot d\vec{r}$$

$$\int_{x_P}^{x_Q} F_x\,dx + \int_{y_P}^{y_Q} F_y\,dy + \int_{z_P}^{z_Q} F_z\,dz$$

Work is Conservative; Force depends on path

Conservative Forces

① Force is conservative if work done by that force from P to Q is independent of path taken

② The work done by that force around a closed loop is zero

Example

Given $F = Axy\,\hat{i} + Bz\,\hat{j}$, is conservative?

$$\int_a^b Axy\,dx + \int_a^b Bz\,dy$$

$$\tfrac{1}{2}Ax^2y\Big|_a^b + Bzy\Big|_a^b$$

$W \neq 0$ so not conservative

Example

$$F = \int Ax\,\hat{i} + \int By^3\,\hat{j} + \int Ce^{-bz}\,\hat{k}$$

$$\frac{x^2}{2}\Big| \quad \frac{y^4}{4}\Big| \quad -be^{-bz}\Big|$$

Bridge Concepts

Newton's Laws: 1st, N2L, N3L

Conservative Laws: CofME, CofLM, CofAM

Classical Work Energy Theorem

$$W_T = W_{F_1} + W_{F_2} + W_3 \dots = \Delta K$$

$$W_T = \int F_{Net}\,dr = \Delta K$$

Momentum Impulse Relation

$$F_{Net} = \frac{dP}{dt}$$

$$\int F_{Net}\,dt = \Delta p$$

Spring Force

$$F_{sp} = Kx \qquad \int Kx\,dx$$

$$\rightarrow \tfrac{1}{2}Kx^2$$