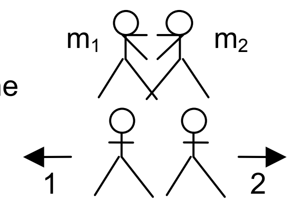
Chapter 11

Back to The Future III: Force and Center of Mass

- **We Begin to Deal with the Size of Bodies!**
- **Motivate CM Definition from Momentum Conservation**
- **Knowing Net Force tells us what the CM is Doing**
- **Potential Energy of Uniform Gravity is simply $M_{total}\, g\, y_{CM}$**

Center of Mass Motivation from Momentum Conservation



Predicting Positions as well as Velocities in an Old Example: Remember from Ch. 10 the two girls with masses m_1 and m_2 who were standing initially at rest next to each other in the middle of an ice-covered pond. One girl shoved the other and we were given what velocity girl 1 had. We were able to predict the other girl's velocity from momentum conservation:

$$m_1 v_1 + m_2 v_2 = 0$$

Now, if we are told how far one of the girls has moved, we can also predict how far the other girl has moved from momentum conservation. To do this, first rewrite the velocities as derivatives of positions:

$$m_1 \frac{dx_1}{dt} + m_2 \frac{dx_2}{dt} = 0$$

Multiplying the equation by dt to cancel it out, we get

$$m_1\, dx_1 + m_2\, dx_2 = 0$$

Finally, integrate this equation using $\int dx = \Delta x$ for the path taken in some time interval:

$$\boxed{m_1\, \Delta x_1 + m_2\, \Delta x_2 = 0}$$

If you have $\Delta x = 0$, then x = constant. Similarly, an equivalent mathematical statement for the box is

$$\boxed{m_1\, x_1 + m_2\, x_2 = \text{constant}}$$

For example, if we know the change Δx_1 in the position of girl 1 at some later time t, we can solve the first boxed equation for the change Δx_2 of girl 2 at the same time t: $m_2\, \Delta x_2 = - m_1\, \Delta x_1$ or

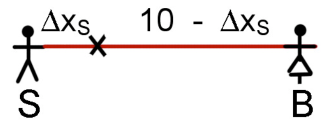
$$\Delta x_2 = - \frac{m_1}{m_2}\, \Delta x_1$$

- Comment: As expected, the two girls move in opposite directions (due to the above minus sign) and the heavier girl moves the smaller distance by the inverse ratio of their masses.

Predicting Positions in Another Example: Two skaters, Sam with mass 65 kg and Betty with mass 40kg (a kid), stand on an ice rink holding a pole with a length of 10 m and a mass that is negligible. Starting from the ends of the pole, the skaters pull themselves along the pole until they meet. Where do they meet?

Solution: We have the same situation as for the two girls (with no ice friction, **there's no external horizontal force, so total momentum is conserved and forever equal to zero since it was zero to start with**). Using subscripts S and B to refer to Sam and Betty, their changes in position must satisfy

$$m_S\,\Delta x_S + m_B\,\Delta x_B = 0$$



We could do all of this more quickly in terms of magnitudes but we want to motivate some formulas involving the x positions, so a choice of signs is in order. Let Sam move to the right in the positive x-axis direction ($\Delta x_S > 0$). The above equation demands Betty move oppositely ($\Delta x_B < 0$).

Also, for a pole length L, the distance they both move must add up to L: The magnitude of the distance Betty moves must therefore be

$$|\Delta x_B| = L - \Delta x_S$$

so, because we chose $\Delta x_B < 0$:

$$\Delta x_B = -(L - \Delta x_S)$$

Substitute this into the result from momentum conservation:

$$m_S\,\Delta x_S + m_B(-(L - \Delta x_S)) = 0$$

Isolating Δx_S :

$$\Delta x_S = \frac{m_B}{m_S + m_B} L = \frac{40}{65+40} 10 = 3.8 \text{ m}$$

Quicker solution: Use $m_S\,x_S + m_B\,x_B = \text{constant}$. This must be the same when Sam and Betty meet (when $x_S = x_B \equiv x_{meet}$) :

$$m_S\,x_S + m_B\,x_B = m_S\,x_{meet} + m_B\,x_{meet} = (m_S + m_B)\,x_{meet}$$

Solve for x_{meet}:

$$\boxed{x_{meet} = \frac{m_S\,x_S + m_B\,x_B}{m_S + m_B}}$$

Evaluating at the initial positions, $x_S = 0$, $x_B = L$, we get the same answer as above:

$$x_{meet} = \frac{m_S \cdot 0 + m_B\,L}{m_S + m_B} = \frac{m_B}{m_S + m_B} L$$

Thus we have a CM motivation from where "all particles meet:" If we had a system with a **zero net external force** in **3D**, and **zero total momentum**, then the previous results for two bodies in 1D generalize to N bodies in 3D:

$$m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4 + \ldots = \text{constant}$$

even if they move due to internal interactions (particles bouncing off of each other or people tied together with a rope on ice and pulling on the rope, etc.)

Then we define the Center of Mass

$$\boxed{\vec{r}_{CM} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4 + \ldots}{m_1 + m_2 + m_3 + m_4 + \ldots}}$$

or let's get more comfortable with summation notation. The above is equivalent to

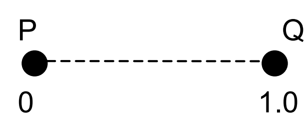
$$\boxed{\vec{r}_{CM} \equiv \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{M}} \quad \text{with} \quad \boxed{M \equiv \sum_{i=1}^{N} m_i \text{ , the total mass}}$$

Properties:

- Each particle position $\vec{r}_i$ enters the CM multiplied by (i.e., weighted by) the factor m_i / M
- **If the net external force on the system is zero, and if the total momentum of the particles is zero, the position $\vec{r}_{CM}$ will never change.** (See later for the more general remarks for nonzero momentum and nonzero external force.)
- If all the particles come together at some time due to their internal attraction, the common position is the $\vec{r}_{CM}$! Proof: if all $\vec{r}_i = \vec{r}_{common}$,

$$\vec{r}_{CM} \equiv \frac{\sum_{i=1}^{N} m_i \vec{r}_{common}}{M} = \frac{\vec{r}_{common}\sum_{i=1}^{N} m_i}{M} = \frac{\vec{r}_{common}\cancel{M}}{\cancel{M}} = \vec{r}_{common}$$

Let's go back to a simple two-body situation just to be sure of ourselves:



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Problem 11-1 *Minding your* P*'s and* Q*'s*

Two particles P *and* Q *are initially at rest 1.0 m apart.*
P *has a mass of 0.10 kg and* Q *a mass of 0.30 kg.* P *and* Q *attract each other with an* <u>*arbitrary*</u> *force that brings the two particles into contact. No external forces act. Even though we know nothing about the force, and it could even be ghastly complicated, the fact that there are no external forces means we can find easily where the particles collide!*

a) Where do they collide?

b) How fast was the CM moving when they collided?

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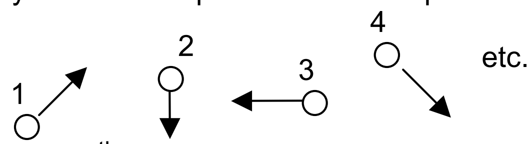
Total Momentum is just the Total Mass times the CM Velocity

Now that we have $\vec{r}_{CM}$, we can highlight a wonderful way to summarize what we have been doing. Consider a system made up of our bunch of particles:



In this system, the i^{th} particle has momentum $\vec{p}_i = m_i \vec{v}_i$. The **total** momentum can be rearranged as

$$\vec{p}_{total} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + m_4\vec{v}_4 + \ldots = \sum_{i=1}^{N} m_i \vec{v}_i$$

We can substitute $\vec{v}_i = \dfrac{d\vec{r}_i}{dt}$, with $\vec{r}_i$ the i^{th} particle's position, in the above to get

$$\vec{p}_{total} = \sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt}$$

Now take the time derivative outside the mass factors since they are constants ($\dfrac{dm_i}{dt} = 0$):

$$\vec{p}_{total} = \frac{d}{dt}\left(\sum_{i=1}^{N} m_i \vec{r}_i \right)$$

Aha! Rearrange the definition of $\vec{r}_{CM}$ to read $\sum_{i=1}^{N} m_i \vec{r}_i = M \vec{r}_{CM}$ and substitute:

$$\vec{p}_{total} = \frac{d}{dt}\left(M \vec{r}_{CM} \right)$$

But, as we said, all the masses are constant, so $\dfrac{dM}{dt} = 0$. So the derivative can come inside:

$$\vec{p}_{total} = M \frac{d\vec{r}_{CM}}{dt}$$

So, as we advertised in the caption,

$$\boxed{\vec{p}_{total} = M \vec{v}_{CM}} \quad \text{with} \quad \boxed{\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + m_4\vec{v}_4 + \ldots}{m_1 + m_2 + m_3 + m_4 + \ldots}}$$

Why is this wonderful? Well, here's why:

1) **This is completely general, independent of whether the net external force is zero. (See later.)**

2) **If the total momentum is conserved (i.e., constant), the CM velocity stays the same forever, too! (Need a zero net external force here.)**

3) **If the total momentum is conserved and, in particular, zero, the CM velocity is zero!**

Reduce Results to 1D for CM Position and CM Velocity

In going back to simpler 1D problems like the ones below, it is probably good to first put the various 1D results in one place:

The CM position in 1D is

$$\boxed{x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum_{i=1}^{N} m_i x_i}{M}}$$

The CM velocity in 1D involves the 1D total momentum

$$\boxed{\frac{dx_{CM}}{dt} = v_{CM} = \frac{p_{total}}{M} = \frac{m_1 v_1 + m_2 v_2 + m_3 v_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum_{i=1}^{N} m_i v_i}{M}}$$

When the 1D total momentum is conserved (zero net external 1D force) and is zero, we have

$x_{CM} =$ constant, or

$$m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots = \sum_{i=1}^{N} m_i x_i = \text{constant}$$

or, equivalently,

$$m_1 \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3 + \ldots = \sum_{i=1}^{N} m_i \Delta x_i = 0$$

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Problem 11-2

a) *For an ultra-quick problem in only 1D and involving only two particles, but still informative, please go back to Ch. 10, and look at the "fission" and "fusion" examples on pp. 10-3 and 10-4. Find the CM velocity* $v_{CM} = \dfrac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$ *before and after the collisions for each of the two cases and check that they are the same before and after in each case.*

Returning to Sam and Betty kinds of problems. Now it's Shark and Boat! Suppose a fisherman in a boat, which weighs 12,000 lb, catches a great white shark with a harpoon. The shark struggles but stops and dies while still at a distance of 1000 ft from the boat. The boat, initially at rest when the shark died, moves a total distance of 150 ft (relative to the shore) in the direction of the shark, while the fisherman pulls in the shark right up to the boat (by a rope attached to the harpoon).

b) *Calculate the shark's weight, neglecting water drag (it is not so bad to neglect the water drag force compared to the shark and boat weights).*

By the way, notice that we don't tell you about the sizes of the boat and the shark anywhere. Well, not only are they relatively small compared with the distances given in the problem, and so we pretend they are 'points' relative to the 1000-foot scale and even the 150-foot scale.

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Newton's Second Law for CM: $\vec{F}_{net}^{external} = M\vec{a}_{CM}$

We'll now prove a really great theorem: The CM point of any system obeys a Newton's second law of the following form:

$$\boxed{\vec{F}_{net}^{external} = M\vec{a}_{CM}}, \quad \text{where} \quad \vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{d^2\vec{r}_{CM}}{dt^2}$$

Theorem proof:

1) First show $\vec{F}_{net} = \vec{F}_{net}^{external}$ for the whole system: Go back to our bunch of small particles of p. 11-4. The net force for the system is the sum of the net forces on each particle:

$$\vec{F}_{net} = \vec{F}_{net\ on\ 1} + \vec{F}_{net\ on\ 2} + \vec{F}_{net\ on\ 3} + \ldots$$

Now the net force on each particle is itself a sum of the external forces and internal forces each particle experiences due to the other particles:

$$\begin{aligned}
\vec{F}_{net\ on\ 1} &= \vec{F}_{external\ on\ 1} + \vec{F}_{2\ on\ 1} + \vec{F}_{3\ on\ 1} + \ldots \\
\vec{F}_{net\ on\ 2} &= \vec{F}_{external\ on\ 2} + \vec{F}_{1\ on\ 2} + \vec{F}_{3\ on\ 2} + \ldots \\
\vec{F}_{net\ on\ 3} &= \vec{F}_{external\ on\ 3} + \vec{F}_{1\ on\ 3} + \vec{F}_{2\ on\ 3} + \ldots
\end{aligned}$$

etc.

But when we insert these individual force expressions into the net force $\vec{F}_{net}$ sum above, we have $\vec{F}_{2\ on\ 1} + \vec{F}_{1\ on\ 2} = 0$, $\vec{F}_{3\ on\ 1} + \vec{F}_{1\ on\ 3} = 0$, etc., all by Newton's third law. All the internal forces pair up and cancel. We are left with

$$\boxed{\vec{F}_{net} = \vec{F}_{external\ on\ 1} + \vec{F}_{external\ on\ 2} + \vec{F}_{external\ on\ 3} + \ldots = \vec{F}_{net}^{external}}$$

All the internal forces cancel; $\vec{F}_{net}$ depends only on the external forces.

2) Next show $\vec{F}_{net} = M\vec{a}_{CM}$: Since each particle i obeys its own second law, $\vec{F}_{net\ on\ i} = \frac{d\vec{p}_i}{dt}$, we can go back to the net force on the system and write it in terms of derivatives of momenta

$$\begin{aligned}
\vec{F}_{net} &= \vec{F}_{net\ on\ 1} + \vec{F}_{net\ on\ 2} + \vec{F}_{net\ on\ 3} + \ldots \\
&= \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + \frac{d\vec{p}_3}{dt} + \ldots \\
&= \frac{d}{dt}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \ldots) = \frac{d\vec{p}_{total}}{dt}
\end{aligned}$$

Now recall from p. 11-4, that $\vec{p}_{total} = M\vec{v}_{CM}$. So $d\vec{p}_{total}/dt = M\,d\vec{v}_{CM}/dt$ (since $dM/dt = 0$), but $d\vec{v}_{CM}/dt = \vec{a}_{CM}$. Thus

$$\boxed{\vec{F}_{net} = \frac{d\vec{p}_{total}}{dt} = M\vec{a}_{CM}}$$

Comparing the two boxed equations gives us the theorem: Q.E.D.

Comments about $\vec{F}_{net}^{external} = M\vec{a}_{CM}$:

- The net external force controls what the CM is doing!
- This says that the **CM point of a system moves like a single point particle whose mass is the total mass of the system, and whose acceleration is determined by Newton's second law with a net force equal to the external net force felt by the system!** It is as if the system were "squished" into the CM.
- Now no one in their right mind would say this all again. So obviously that explains why we'll say it all again: The CM point, given by $\vec{r}_{CM} \equiv \dfrac{\sum_{i=1}^{N} m_i \vec{r}_i}{M}$, is governed by $\vec{F}_{net}^{external} = M\vec{a}_{CM}$, with total mass $M \equiv \sum_{i=1}^{N} m_i$
- Finally, we verify the remark earlier in this chapter that, if the net external force is zero, the total momentum is constant, and the CM moves uniformly forever! In terms of the individual velocities and masses, this means that the CM velocity given by

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + m_4\vec{v}_4 + \ldots}{m_1 + m_2 + m_3 + m_4 + \ldots}$$

  would be constant if the net external force is zero.

- When we are talking about cars, balls, boats, etc., as "points," it is useful to visualize CM points, and it is sensible to draw free-body diagrams with forces all drawn as if they were applied at a single point, ignoring the size of the objects, if all we want is the overall motion!

The Potential Energy for an Arbitrary Body in Uniform Gravity

(Now that we have the CM variable, this is easy!)

Recall that the constant force mg leads to the potential energy U_g = mgy for a point mass m. Therefore the potential energy of a general body is the sum of potential energies for all the pieces:

$$U_{g,M} = \sum_{i=1}^{N} m_i g y_i = g \sum_{i=1}^{N} m_i y_i$$

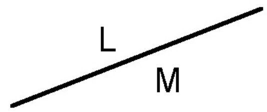
We recognize the sum from the CM definition: $y_{CM} = \dfrac{\sum_{i=1}^{N} m_i y_i}{M} \Rightarrow \sum_{i=1}^{N} m_i y_i = M y_{CM}$

Therefore, $U_{g,M} = g M y_{CM}$

Thus we have another "squished" result where, as far as the potential energy due to uniform gravity is concerned, **any body is just a point mass M completely concentrated at its CM point!**

$$\boxed{U(\text{uniform gravity, arbitrary body of mass M}) = M g y_{CM}}$$

Example of a uniform rod of mass M and length L:



Where is its CM? Well, your kid sister could tell you but we are compulsive and we will calculate it. Lay the rod along the x-axis, and realize that the sum in the definition of x_{CM} turns into an integral over the little differential pieces. Let $\lambda = M/L$ be the uniform mass per unit length. We start with

$$x_{CM} = \frac{\sum_{i=1}^{N} m_i x_i}{M}$$

If m_i comes from a piece Δx_i of the rod, then $m_i = \lambda \Delta x_i$. In the limit of tiny pieces, $\Delta x_i \rightarrow dx$ and the sum goes over to an integral:

$$x_{CM} \rightarrow \frac{\int_0^L \lambda\, dx\; x}{M} = \frac{\lambda \int_0^L x\; dx}{M}$$

But $\lambda = M/L$ and one of the first integrals you ever learn is

$$\int_0^L x\; dx = \left.\frac{x^2}{2}\right|_0^L = \frac{L^2}{2} - 0$$

Thus

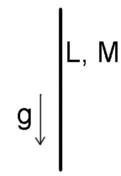
$$x_{CM} = M/L \;\frac{L^2/2}{M}$$

or

$$\boxed{x_{CM} = \frac{1}{2} L}$$

Never has so much been done with so many lines for such a simple result!

<u>What is its gravitational potential energy if the rod is vertical?</u>



Use the bottom of the rod as y = 0:

The answer is simply

$$\boxed{U_g \;=\; Mgy_{CM} \;=\; Mg\frac{L}{2}}$$

To be really compulsive again, let's check it. Follow the above procedure, with x replaced by y:

$$U_g \;=\; \sum m_i g y_i \;\;\rightarrow\;\; g\int_0^L \lambda\; dy\; y \;=\; g\lambda \int_0^L y\; dy \;=\; g\,\frac{M}{L}\left(\frac{L^2}{2}\right)$$

Therefore we confirm that

$$U_g \;=\; Mg\,\frac{L}{2}$$

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Problem 11-3 *Now for a crazily contrived problem – but it's fun and educational!*

An incredibly slender high-jumper is six feet tall. She is as thin as a very thin rod and uniformly thin. She is also so flexible that she can bend her body into a hairpin shape. For example, if she touched her toes, she would look like a bobby-pin.

a) *From a standing position this woman has the energy to elevate her center of mass two feet vertically upward from her standing position. If her weight is 100 lb, how much work does she do against gravity?*
b) *She is jumping in such a way that she is maneuvering her entire body over a thin horizontal bar. What is the highest bar she can get over? Hint: The answer is not 5 feet!*

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Problem 11-4 *Newton wants to entertain his two friends by showing them directly where the center of mass is between them using skateboards. He has the two folks hold a light rope (light so its mass doesn't muck things up by being too heavy itself) between them, then both pull on it until they are drawn right smack into each other. That's where their center of mass is! They can check it if they had made little crayon marks on the floor where they were originally standing. The important thing here is to aim their skates at each other so that there isn't too much friction.*

Newton now has three friends. Could you help Newt play a game to find out where the overall center of mass is of all three of his friends?

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