Chapter 15

We now consider oscillations of physical systems. They arise because all stable systems have restoring forces that fight any disturbance of their equilibrium positions. And a great model for these restoring forces is the simple spring!

We start with the force of a spring and put it into Newton's second law and solve that f = ma equation for the general motion. This simple model is really powerful and gives not only insight, but good quantitative prediction, for many general situations.

Oscillations

Much of our world oscillates; much is periodic.

Examples:

- -heart
- -coordinate of anything going in a circle
- -AC electricity at any point on the wire
- -various car engine parts
- -atoms in a solid at room temperature
- -point on a violin string
- -pendulum
- -day/night cycle
- -etc.

The period and frequency variables, T, f=1/T, $\omega = 2\pi f$, discussed in Chapter 12 apply here, too. A system comes back to its initial state in the period T, i.e., repeats in time T seconds

$$T \equiv \text{period} = \frac{\text{number of seconds}}{\text{one cycle}}$$

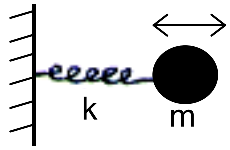
The frequency f is the number of repeats/second = cycles/second = cps $\equiv$ Hz (Hertz)

$$\text{(cycle) frequency} \equiv f = \frac{1}{T} = \frac{\text{number of cycles}}{\text{second}} = \text{number of Hz}$$

In most of these examples, there is a restoring force that can be approximated by an effective spring model. Equivalently, their motion can be at least approximately modeled by "simple harmonic motion" (SHM), that is, by the same motion as that of a spring.

We want to find x(t) ***for a spring force*** as a fundamental tool to be added to our arsenal ***analogous*** to the $x(t) = x_0 + v_0 t + \frac{1}{2}at^2$ ***for a constant force*** (note we will again need two initial constants just as we needed x_0 and v_0 in the constant force case).

GOAL: Find x(t) for a mass m tied to a spring (no gravity here so m just moves back and forth horizontally)



Springs – Hooke's "Law"

If F(x) is the force by the spring on the mass m, for the "spring constant" k, and if x = 0 is the spring equilibrium position, then F is proportional to the displacement x with proportionality constant – k:

$\boxed{F = -k\,x}$ for mass m at position x
(the positive direction of x is to the right)

Check that it "restores:"
$$\begin{cases} \text{put m at } x = +d \;\Rightarrow\; F = -kd < 0 \\ \qquad\qquad \Rightarrow \text{ spring pulls inward} \qquad (\leftarrow F\text{, checks}) \\ \text{put m at } x = -d \;\Rightarrow\; F = +kd > 0 \\ \qquad\qquad \Rightarrow \text{ spring pushes outward} \qquad (\rightarrow F\text{, checks}) \end{cases}$$

Now Newton says

$$\underbrace{F}_{-kx} = m\,\underbrace{a}_{\frac{d^2x}{dt^2}}$$

The most famous 2nd-order differential equation in the world follows:

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

Forgetting the k/m factor for the moment, what two famous functions are equal to the negative of their second derivatives? Guess before going to the next page. And you can also guess how the k/m factor might be accounted for.

A SOLUTION:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \text{is satisfied by} \quad x = \cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$\text{check}: \frac{dx}{dt} = -\sin\left(\sqrt{\frac{k}{m}}\,t\right)\frac{d}{dt}\left(\sqrt{\frac{k}{m}}\,t\right) = -\sin\left(\sqrt{\frac{k}{m}}\,t\right)\sqrt{\frac{k}{m}}$$

$$\frac{d^2x}{dt^2} = -\cos\left(\sqrt{\frac{k}{m}}t\right)\sqrt{\frac{k}{m}}\sqrt{\frac{k}{m}} = -\frac{k}{m}\cos\left(\sqrt{\frac{k}{m}}t\right) = -\frac{k}{m}x \quad \checkmark$$

ANOTHER SOLUTION (you could check it if you want):

$$x = \sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

A MOST GENERAL SOLUTION:

$$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$$

$$\text{with } \boxed{\omega = \sqrt{\frac{k}{m}}} \text{ and constants } C_1, C_2$$

(the justification for using an “angular frequency” notation will become apparent later)

Comment:

This most general solution for a linear force of the form of F = - kx can be compared with the most general solution for a constant force of the form $F = F_0$:

$$x = x_0 + v_0 t + \frac{1}{2}\frac{F_0}{m}t^2$$

which is, of course, well known to us and also involves two constants as the second law (involving a second-order derivative) always will.

or equivalently (a more popular choice),

$$\boxed{x(t) = A\cos(\omega t + \phi)}$$

$$\text{That is, } x(t) = A\big(\cos(\omega t)\cos\phi - \sin(\omega t)\sin\phi\big)$$
$$= \underbrace{A\cos\phi}_{C_1}\cos(\omega t) - \underbrace{A\sin\phi}_{C_2}\sin(\omega t)$$

(so one can use either the A, ϕ representation or the C_1 , C_2 representation)

REMARKS:

1) The constants A and ϕ can be found given, for example, with two initial pieces of information

$$x(0),\ \frac{dx(0)}{dt} \equiv v(0),\ \text{“initial data”}$$

This is just like the constant acceleration situation where the initial conditions were used to find the two constants there. Given ω, the two boxed equations shown below can be used to find the two unknowns A and ϕ :

$$\boxed{x(0) = A\cos\phi} \quad \Rightarrow \quad \frac{dx}{dt}(t) = -\omega A\sin(\omega t + \phi) \equiv v(t)$$
$$\therefore \frac{dx}{dt}(0) \equiv \boxed{v(0) = -\omega A\sin\phi}$$

2) Clearly, the position x repeats itself every T sec PROVIDED $\omega T = 2\pi$. That is,

$$t \to t + T \Rightarrow x(t) \to x(t+T) = A\cos(\omega t + \omega T + \phi)$$
$$\therefore \quad x(t+T) = x(t) \ \text{ if } \ \omega T = 2\pi \ \ \text{(smallest T)}$$

$$\therefore \quad \boxed{T = \frac{2\pi}{\omega} = \frac{1}{f} \quad \Leftrightarrow \quad \omega = 2\pi f}$$

with units : ω = radians / sec = angular frequency

Interpretation of A

A = amplitude = maximum magnitude of x (A > 0)

AMPLITUDE EXAMPLE:

The motion of the piston in an automobile engine is approximately simple harmonic. Suppose that the piston travels back and forth over a total distance of 8.50 cm. What are its maximum acceleration and maximum speed if the engine is turning over at its highest safe rate of 6000 revolutions per minute (rpm)?

SOLUTION:

The piston travels vertically, say, with the vertical height, $h(t) = A\cos(\omega t + \phi)$. Then we know the most positive h can be is when the cosine is +1, and the most negative it can be is when the cosine is - 1:

$$-1 \le \cos(\omega t + \phi) \le 1 \quad \Rightarrow \quad -A \le h \le A$$

$$\Rightarrow \quad 2A = 8.50 \text{ cm}$$

$$f = 6000 \text{ rpm} = \frac{6000}{60} \text{rps} = 100 \text{ cps}$$

$$\Rightarrow \ \omega = 2\pi(100) \text{ rad/s}$$

$$\therefore \ A = 4.25 \text{ cm}, \quad \omega = 628 \text{ s}^{-1}$$

$$\rightarrow \frac{dh}{dt} \equiv \dot{h} = -\omega A \sin(\omega t + \phi) \Rightarrow -\omega A \le \dot{h} \le \omega A$$

$$\therefore \quad \text{max speed} = \omega A = 628 \ (0.0425 \text{ m}) = 26.7 \ {}^{m}\!/_{s}$$

$$\sim 60 \text{ mph!}$$

$$\rightarrow \frac{d^2h}{dt^2} \equiv \ddot{h} = -\omega^2 A \cos(\omega t + \phi) \ \Rightarrow \ -\omega^2 A \le \ddot{h} \le \omega^2 A$$

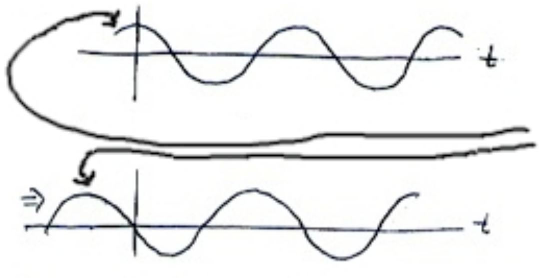
$$\therefore \quad \text{max acceleration} = \omega^2 A = 628 \ (26.7) = 1.68 \times 10^4 \text{ m/s}^2$$

Interpretation of ϕ

The angle ϕ ≡ phase and it tells us how much the pattern is shifted. That is, if we change the phase, we shift the peaks left or right in the sinusoid

e.g. $\phi = 0 \;\Rightarrow\; x = A\cos(\omega t)$, which is plotted in the first figure below

$\phi = \pi/2 \;\Rightarrow\; x = -A\sin(\omega t)$, which is plotted in the second figure



Find peak at $t = -\dfrac{\phi}{\omega}$

in general

$$\phi > 0 \;\Rightarrow\; \text{shifts the peak to the left}$$

$$\phi < 0 \;\Rightarrow\; \text{shifts the peak to the right}$$

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Problem 15-1. *Experience shows that roughly one-fourth of the passengers in an airliner can be expected to suffer motion sickness if the airliner bounces up and down with a peak acceleration of 0.3 gee (gee = 9.8 m/s^2) and a frequency of about 0.3 Hz. Assume that this up-and-down motion is simple harmonic motion. What is the amplitude, in m, of the motion?*

Problem 15-2. *A mass oscillates in SHM according to the equation, x = 4cos(6t + π/4) in SI units.*

Find in terms of SI units: (a) the amplitude, (b) the phase in degrees, (c) the frequency f, (d) the period T, (e) the velocity as a function of time, (f) the acceleration as a function of time, and (g) the displacement of the mass at the time t = 0.3 s.

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Example of Phase and Amplitude Determination Suppose that we have the following initial data and angular frequency:

$$x(0) = -5.0\,\text{m}, \quad \dot{x}(0) = -20\,\text{m/s} \quad \text{with} \quad \omega = 3.0\text{s}^{-1}$$

What is A? What is ϕ?

SOLUTION:

$$\text{Recall } x(t) = A\cos(\omega t + \phi) \quad \Rightarrow x(0) = A\cos\phi$$

$$\dot{x}(t) = -\omega A\sin(\omega t + \phi) \;\Rightarrow \dot{x}(0) = -\omega A\sin\phi$$

Above data $\Rightarrow \boxed{A\cos\phi = -5}$ and $-\omega A\sin\phi = -20 \Rightarrow \boxed{A\sin\phi = +\dfrac{20}{3}}$

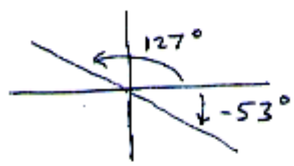
To get A:

$$\underbrace{(A\cos\phi)^2}_{25} + \underbrace{(A\sin\phi)^2}_{\left(\frac{20}{3}\right)^2} = A^2(\cos^2\phi + \sin^2\phi) = A^2$$

$$\Rightarrow \quad A = \sqrt{69.4} = 8.3$$

To get ϕ:

$$\left.\begin{array}{l} \dfrac{\cancel{A}\sin\phi}{\cancel{A}\cos\phi} = \tan\phi \\ \underbrace{\qquad\qquad}_{\frac{+20/3}{-5}} = -1.33 \end{array}\right\} \Rightarrow \phi = -53^\circ \text{ or } 127^\circ$$



WHICH?

To see which, go back to individual equations

S	A
T	C

$\cos\phi < 0$
$\sin\phi > 0$ $\Rightarrow$

$\Rightarrow \phi = 127^\circ$ (2nd quadrant)

$\therefore$ A = 8.3 m and ϕ = 127° (not -53° !)

In case you're wondering, the quadrant map with A, C, T, and S in the quadrants shows which of the sine, cosine, and tangent functions are positive in which quadrant: A = all, C = cosine, T =tangent, and S = sine. "All Case Teachers Succeed" (: is a way to remember it!

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Problem 15-3

Suppose we have the following initial data and angular frequency: $x(0) = -10\text{m}$, $\dot{x}(0) = +40\text{m/s}$, *and* $\omega = 5.0\text{s}^{-1}$. *What are the amplitude* A (in m) *and the phase* ϕ *(in degrees)*?

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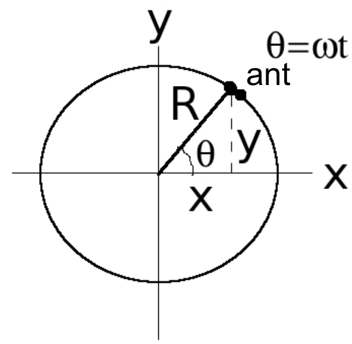
UNIFORM CIRCULAR MOTION VERSUS SIMPLE HARMONIC MOTION

Why does the uniform circular motion of Chapter 12 seem to be related to the simple harmonic motion in this Chapter? (The same sorts of angular frequency ω relationships seem to arise, for example.) Well, consider the Cartesian coordinates as a function of time, x(t) and y(t), for an ant in circular motion with constant ω. Let $x(0) = R$, $y(0) = 0$, to start:

$$\left.\begin{array}{l} x(t) = R\cos(\omega t) \\ y(t) = R\sin(\omega t) \end{array}\right\} \text{ for } x(0) = R,\ y(0) = 0$$



More generally,

$$x = R\cos(\omega t + \phi_0)$$

$$y = R\sin(\omega t + \phi_0)$$

THIS SHOWS THAT EACH COORDINATE DISPLACEMENT LOOKS LIKE SIMPLE HARMONIC MOTION!

$$\left.\begin{array}{l} \text{For a spring, } \omega = \sqrt{\dfrac{k}{m}} \\ \text{For a circle, } \omega = \text{angular velocity such that } v = \omega R \end{array}\right\} \text{Both are radians/sec}$$

In fact, look at circular motion on edge (have a toy car go in circles on a table and then you crouch down and look at it from the side) :

$$\text{e.g. look on edge } \parallel\ x \text{ (and } \perp\ y)$$

$$\Rightarrow\ R\cos(\omega t) \text{ motion } \Rightarrow \text{ SHM!}$$

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Problem 15-4 *Consider the ant's circular motion given by* $x(t) = R\cos(\omega t)$, $y(t) = R\sin(\omega t)$.

a) *As a warm-up, what is* $x^2 + y^2$ *and how is it related to the radius* R*? (sorry for the trivial question, but we compulsively want to tell a complete story here)*

b) *If* ω *is a constant, what is* $\dot{x}^2 + \dot{y}^2$ *and how is it related to the speed* $v = \omega R$ *of the ant?*

c) *And, continuing to consider* ω *as constant, what is* $\ddot{x}^2 + \ddot{y}^2$ *and how is it related to the centripetal acceleration* $a_c = v^2/R = \omega^2 R$ *of the ant?*

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