PHYS 121: Final Exam Announcement and Ten Final Exam Practice Problems

Please note: The Final Exam is Tuesday, May 7 2024 from 3:30 PM EST to 6:30 PM EDT in the Allen Ford Auditorium You are allowed *four pages* (front and back) for your own hand-written notes. Also a calculator is allowed but you will not need it. Students will be assigned seats. Please bring your student ID card – you may be asked to verify your identity to a proctor.

Please see next page for important rules and guidelines for the Final Exam

Some Important Rules and Guidelines Regarding the PHYS 121 Final Exam (please read):

- The Final Exam is **required** for ***all*** students. All PHYS 121 students must sit for the Final Exam. You cannot earn a passing letter grade for PHYS 121 unless you submit a Final Exam.
- The Final Exam is scheduled for 3:30 PM EST to 6:30 PM EDT on Tuesday, May 7, 2024.
- Students should make sure to arrive early so that they can find their seats.
- ***Extremely important. Get enough sleep the night before. Your performance will definitely take a hit if you are too sleepy to work. If you oversleep the final exam you may not pass the class. Make sure you get enough sleep. Make sure you are awake and ready to take the exam at the assigned time. This is absolutely critically important.***
- The same rules apply as before. No books, handouts, or other materials are permitted. You are allowed *four pages* (front and back) for your own hand-written notes. Your notes need to be generated by hand only. No computer-generated or pad-generated output allowed. Also a calculator is allowed but you will not need it.
- Students may be asked to present an identification card at any time during the exam. Students should plan to have their CWRU student ID card (or some other photo ID) handy during the final exam
- Students should expect to be assigned a seating location in the auditorium. Details on seating assignments will go out to students by May 5.
- The exam will contain at least six problems but not more than eight problems for a total of 150 points. At most one of the exam problems will contend directly with materials that were presented during the last week of class regarding fictitious forces, Einstein's gravity and cosmology The style and format of the exam will be the same as for the previous Hour Exams.
- Please re-read Course Document #10: *Mr. Covault's Comments on Taking Physics Exams*. Many good reminders there.

Important: Do not look at solutions! Read First:

Practice problems can be useful for students. However, there are pitfalls to using practice problems and so these general guidelines might be helpful:

- First and foremost, all students are encouraged to attempt to work all problems "under exam conditions". In other words, ***work each problem without notes or books, and without looking at the solutions.*** Ideally, give yourself about 15 to 20 minutes and try to complete each problem. Doing the practice problems under exam conditions will tell you if you are "well-prepared" to do well on the exam as it will actually be given.

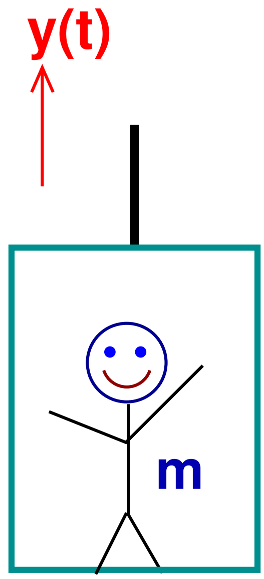
- Again, *resist any temptation to look at the solutions before you have made a decent effort to solve the problem on your own.*. This is really important. Student who quickly look at the solutions before they actually work the problems risk fooling themselves into thinking they know how to solve the problem. **Physics is learned by doing, not observing.** Simply looking at solutions before you try the problems as preparation for an exam is like trying to learn how to ride a bicycle by watching someone else riding. You tell yourself "Oh I understand that. I can do that." But you are fooling yourself. The only way to know if you have mastered a problem is to demonstrate that you have actually done the problem on your own.

- So give yourself a fixed period of time to work one or more problems. Use the whole time to work the problem even if you are stuck. At the end of the time, look at the solution. First, make sure that you are using the correct ***Main Physics Concept*** for any given part of a problem. This is most important. If you pick the wrong Main Physics Concept you will get nowhere. Study the solution. Make sure you understand it. Make sure you understand where you went wrong. Then *put the problem away for at least 12 to 24 hours.* Now, say, the next day, try to solve the problem – again without looking at the solution. Maybe you will get further. Repeat until you have mastered the problem and you can solve it quickly without looking at the solution at all.

- If you are struggling with a Main Concept, better to make sure you get some help with that Concept instead of doing many practice problems. Every student has different learning preferences. Some students do best by studying the materials. Others do well with practice problems. You want to develop a study strategy that is best for you.

- Remember the goal of the course is to have students *demonstrate that they understand the Main Physics Concept by successfully solving problems.* We are not looking for "the right answer". We are looking for you to show us that you understand the main concepts and can apply these correctly to solve a wide variety of problems. The problems on the exam will be *different* from the practice problems, guaranteed. Do not "memorize solutions" to problems. Instead, make sure you understand the problems so that you can tackle new problems using the same Physics Concepts – problems that may seem both familiar and unfamiliar. Mastery means being ready to apply the Main Physics Concepts to new problems that you have not yet encountered, working your way methodically towards the correct solution quickly and with confidence.

Problem 1: One-Dimensional Motion (40 points)



You are in an elevator. Your mass is given as m. The motion of the elevator is given by the following equation for the vertical position y as a function of time during the interval $0 < t < T$ (where T is a given positive parameter) as follows:

$$y(t) = A + Bt - Ct^2 + Dt^3$$

where A, B, C and D are given positive parameters each in appropriate units.

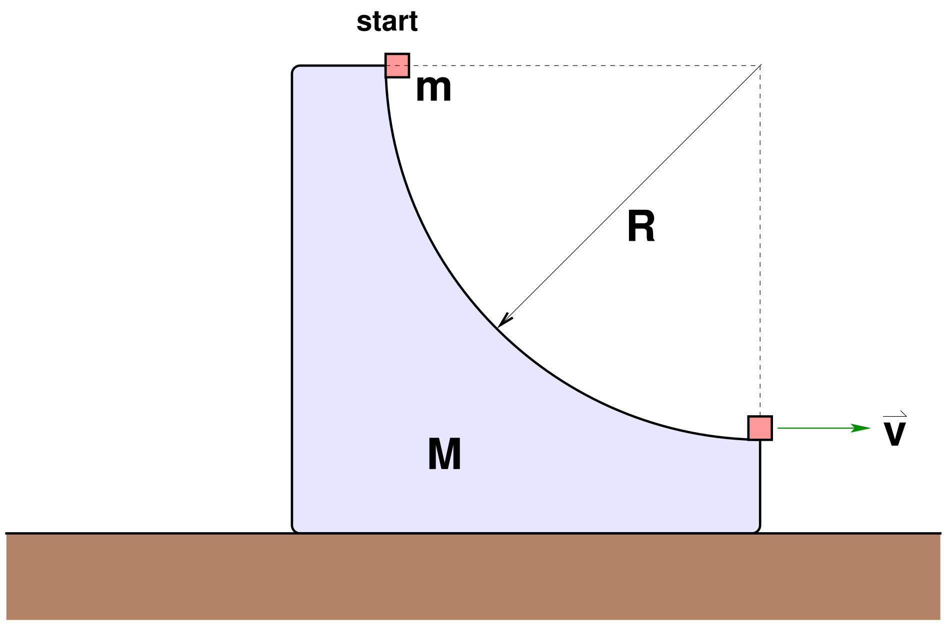
Part (a) – (10 points): What your **velocity** at the instant of time when $t = 0$? Present your answer in terms of the given parameters. Explain your work.

Part (b) – (10 points): What your **acceleration** at the instant of time when $t = T$? Present your answer in terms of the given parameters. Explain your work.

Part (c) – (10 points): Suppose there exists an instant in time $t = t_n$ during the interval $0 < t_n < T$ when the **net force on you is precisely zero**. Write down an expression for t_n in terms of the given parameters. Explain your work.

Part (d) – (10 points): Write down an expression for your **Effective Weight** as a function of time in terms of the given parameters. Based on this result, describe in one or two sentences your ***experience in the elevator*** during the interval from $0 < t < T$. Explain your work.

Problem 2: An Inverse Collision (40 points)



A small cube of given mass m is released from rest at the "start" position as shown and then slides down a *frictionless* circular path of given radius R cut into a large block of given mass M which is initially at rest on a *frictionless* horizontal surface.

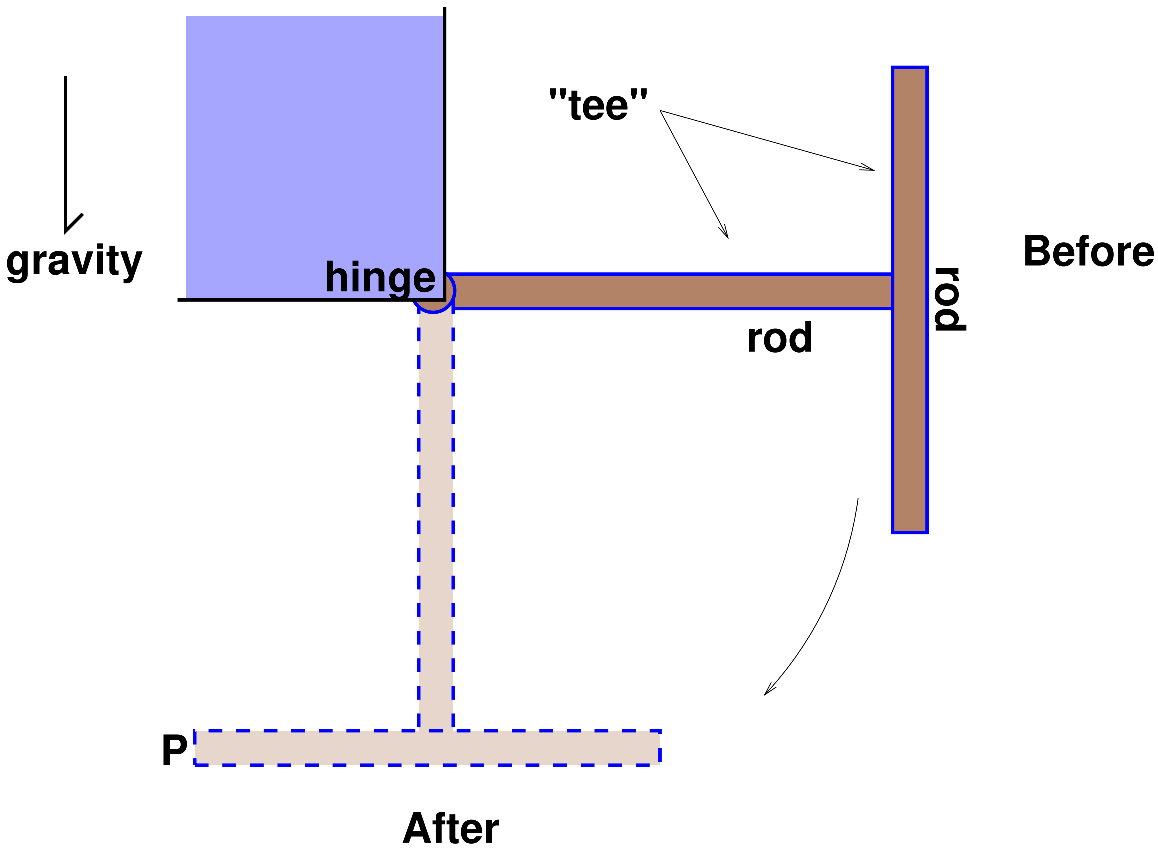
Find the magnitude of the **horizontal velocity** v **of the small cube** as it leaves the block. Very important: Explain your work. You must present a clear, coherent and logical method, correctly identifying and justifying the correct physics concepts to receive full credit. Give your answer in terms of the given parameters. Note that in this problem the small cube m and the large block M are both free to move.

Important: Completely and clearly explain your work.

Note: you should assume that the physical size of the small cube is much smaller than the radius R, but you may *not* assume that the mass of the small cube is small compared to the mass of the large block. In other words, treat the small cube as a "point mass".

Hint: There is no hope whatsoever of trying to solve this problem using Newton's Force Laws. Instead ask yourself this: What is conserved here?

Problem 3: Getting Torqued and Teed



A **rigid body** is comprised of two thin rods glued together so that the end of one rod abuts the center of the other rod at a right angle to make a "T"-shaped structure as shown. Each rod has a given mass m and a given length L. The "tee" is connected to an ideal hinge and shown and released at rest.

Part (a) – (10 points): Determine x_{cm} which is the position of the center-of-mass of the entire rigid body "tee" relative to the hinge. Give your answer in term of the given parameters. Explain your work.

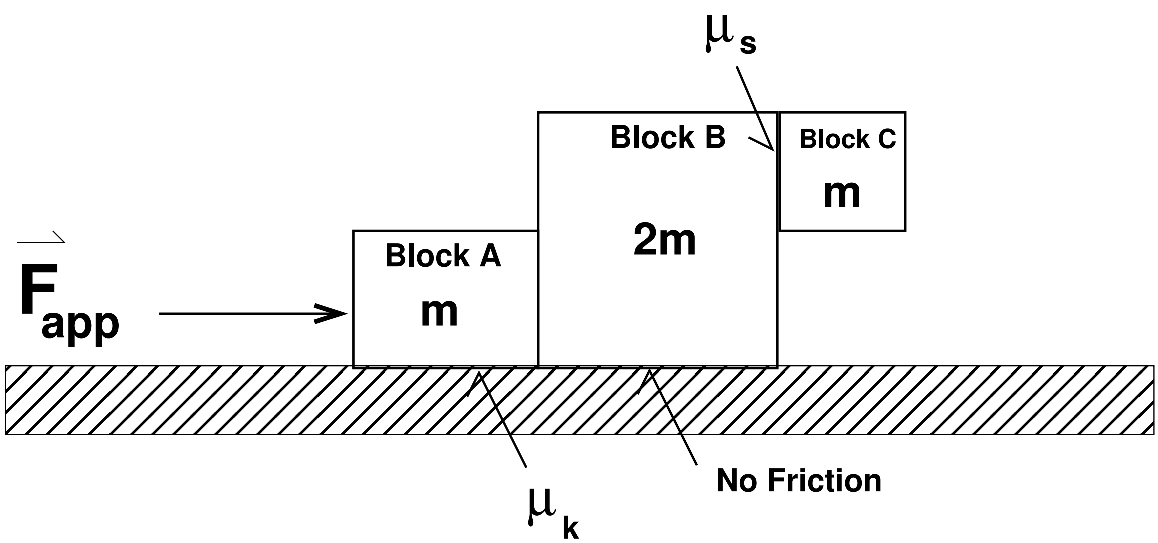
Part (b) – (10 points): Determine the **rotational inertia** I_T for the entire rigid body "tee" as calculated about a pivot point defined at the hinge. Give your answer in term of the given parameters. Explain your work. Hint: the **Parallel Axis Theorem** might be useful here.

Part (c) – (10 points): The "tee" is ***released from rest*** in the position as shown. Determine the **angular acceleration** α of the rigid body "tee" immediately after release. Give your answer in term of the given parameters. See Hint below. Explain your work.

Part (d) – (10 points): Harder: A small dab of paint is applied to the rigid body at the position P as shown. Determine the **maximum speed** of the dab of paint as the rigid body moves. Give your answer in term of the given parameters. See Hint below. Explain your work.

Possibly Useful Hint for Parts (c) and (d): To simplify your algebra, do your calculations in terms of x_{cm} and I_T treating these as "givens" and then subsequently plug in your answers from Parts (a) and (b) at the very end.

Problem 4: Three Accelerating Blocks with Friction (40 points)



A given force F_{app} is applied horizontally to Block A with given mass m which is immediately adjacent to Block B (with given mass $2m$) and Block C (with given mass m). The force is applied so that Block C is held in place vertically by friction. The coefficient of static friction between Block B and Block C is given as μ_s. The coefficient of kinetic friction between Block A and the horizontal surface is given as μ_k. There is *zero friction* between Block B and the horizontal surface.

Part (a) – (10 points): Draw a complete and appropriately labeled Free Body Diagram for each of the three blocks. ***Also*** draw a complete and appropriately labeled Free Body Diagram for the ***whole system*** of blocks.

Part (b) – (5 points): Calculate the magnitude and direction of the **force of Friction on Block A**. Present your answer in terms of the given parameters. Explain your work.

Part (c) – (5 points): Calculate the magnitude and direction of the **acceleration of Block B**. Present your answer in terms of the given parameters. Explain your work.

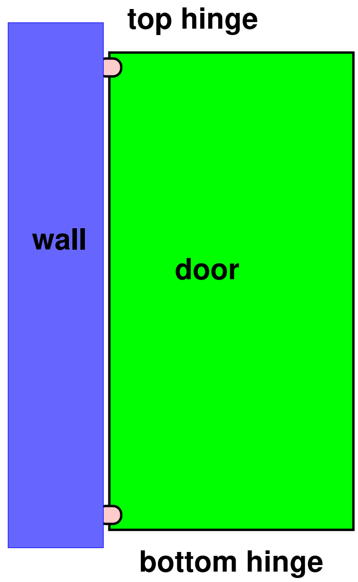
Part (d) – (5 points): Calculate the magnitude and direction of the **Normal force exerted on Block B due to Block A**. Present your answer in terms of the given parameters. Explain your work.

Part (e) – (5 points): Calculate the magnitude and direction of the **Normal force exerted on Block B due to Block C**. Present your answer in terms of the given parameters. Explain your work.

Part (f) – (5 points): Calculate the magnitude and direction of the **force of Friction on Block C**. Present your answer in terms of the given parameters. Explain your work.

Part (g) – (5 points): What is the **constraint** on the magnitude of F_{app} that ensures that Block C will ***not slide downward?*** Express your answer as inequality. Present your answer in terms of the given parameters. Explain your work.

Problem 5: Door on two hinges (40 points):



A thin uniform rectangular door of given mass m and given height h and given width w is connected to a wall by two hinges each positioned very close to the upper and lower edges as shown.

The bottom hinge applies a force to the door to keep the door attached to the wall. The bottom hinge is designed to apply both a horizontal force, $\vec{H}_h$, and a vertical force, $\vec{H}_v$, as needed to ensure that the bottom of the door stays connected to the wall. In contrast the top hinge can only apply a horizontal force $\vec{H}_t$ to the door. The top hinge is designed to always apply zero vertical force on the door.

Finally, assume further that the door is swinging about the hinge with a constant given rotational speed of Ω.

In terms of the given parameters m, h, w, and Ω, calculate the direction and magnitude of all three hinge forces $\vec{H}_v$, $\vec{H}_h$, and $\vec{H}_t$. Explain your work.

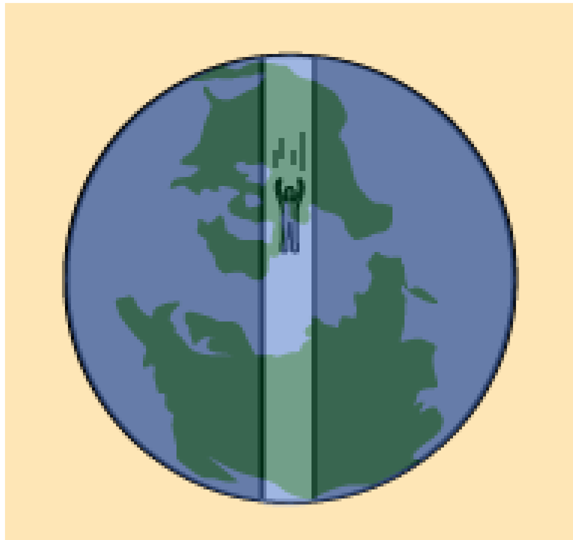
Useful Fact: The moment of inertia (rotational inertia) of the door relative to the axis of rotation at the hinges is given by $I = \frac{1}{3}mw^2$.

Helpful Hint: There are equivalent ways to represent the magnitude of the torque:

$$\tau_F \equiv rF\sin\phi = r_\perp F = Fr_\perp$$

where the subscript $_\perp$ means "the perpendicular component" (perpendicular relative to the other vector). So for example $Fr_\perp$ means "the magnitude of the Force vector times the magnitude of the component of the radial vector that is perpendicular to the Force vector." This fact allows you to calculate the torque without contending with the angle ϕ explicitly.

Problem 6 Falling Through The Earth – 40 points



Suppose you could make arrangements to have a tunnel dug straight down through the center of the earth as shown above. The walls of the tunnel would be supported by some solid material with remarkable properties so as to protect you from the crushing pressures and heat at the center of the planet. Also we will ignore air resistance. Consider what would happen if you jumped into this hole:

Newton showed that the gravitational force on small body at the surface of a large spherically symmetrical mass is equivalent the force that would result if all of that mass were concentrated at the central point.

Newton *also* showed that the gravitational force on any small body that is located within any uniform spherically symmetric hollow shell of matter is zero.

These two facts together mean that gravitational force on a body at radius $r < R$ that is *embedded* within a spherically symmetric solid object of radius R is equivalent to the gravitational force that would result assuming that the *mass enclosed* with the sphere of radius r were concentrated at the central point of the sphere. (This result is sometimes called "Gauss' Law").

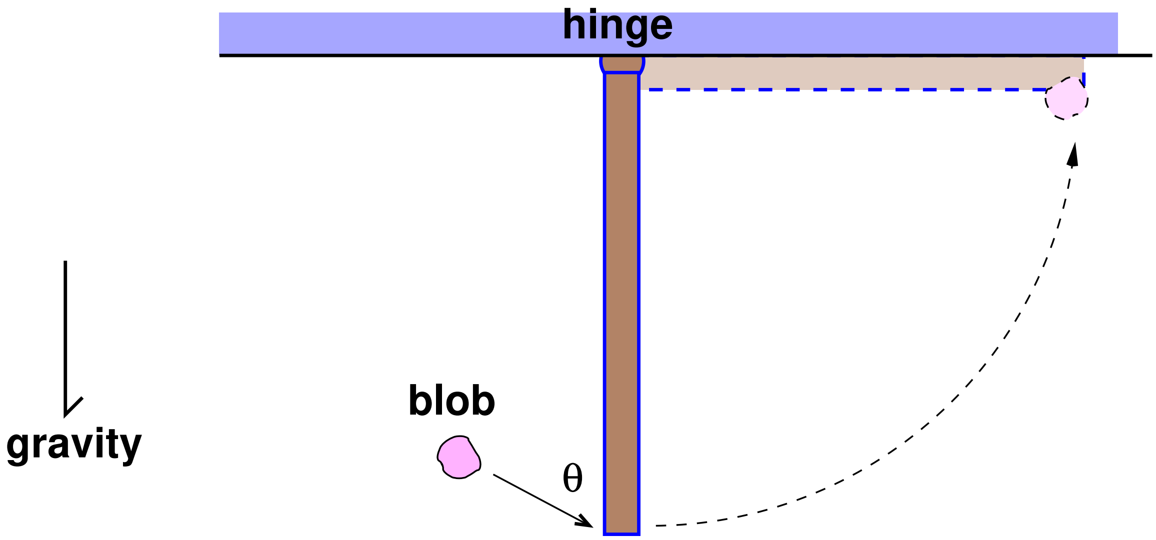
Part (a) – 10 pts: Assume that M and R are the given mass and radius of the Earth. Suppose your body mass is given as m. Assuming that the Earth can be approximately represented as a solid sphere of *uniform mass density*, calculate the force of Universal Gravity on you due to the Earth as a function of r where r is the distance from you to the center of the Earth as you fall down the hole. Explain your work.

Part (b) – 10 pts: Write down the **Equation of Motion** for your body once you have jumped into the hole. Neglect any air resistance or friction. Explain how you worked this out.

Part (c) – 10 pts: What is the **solution** $r(t)$ to the Equation of Motion? Explain how you know this.

Part (d) – 10 pts: Explain your "sensations" as you fall through the exact center-point of the Earth. Is there an experiment that you can do that will clearly indicate when you have moved past the center point? Here assume that the walls of the tunnel are very smooth so that you cannot actually tell how fast they are moving past you at any given time.

Problem 7: Clay Bob Collides with a Rod on a Hinge (40 points)

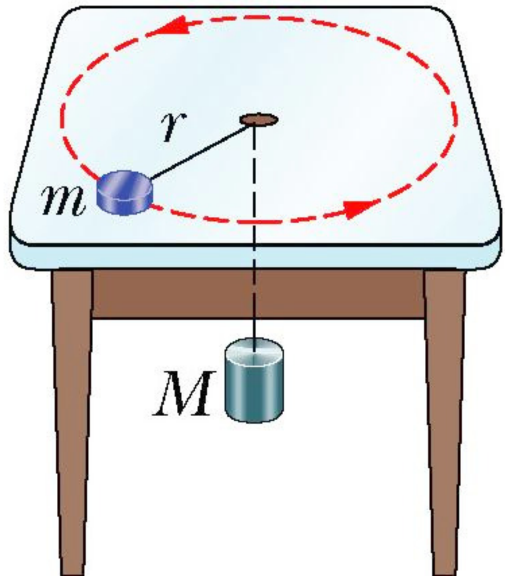


A thin rod of given mass m and given length L hangs vertically from the ceiling attached to an ideal hinge. A small blob of clay also of the same mass m strikes the very end of the rod at a given angle θ relative to the vertical. The blob *sticks* to the rod.

If we define v_i as the *speed* of the blob just before it strikes the rod, then what is the minimum value of v_i required to ensure that the rod will swing up and make contact with the ceiling? Give your answer in terms of the given parameters. Explain your work.

Note, this is a technically rich problem. You will have to break it into steps. Be as clear as possible at each step about what you are doing.

Problem 8: (40 points): Hole in the Table



A small puck with given mass m is placed on a frictionless table and is attached to a massless string which drops down through a hole in the center of table. A large block of unknown mass M is hung from the end of the string. The small puck is made to move in a circle with constant given radius r and constant given speed v.

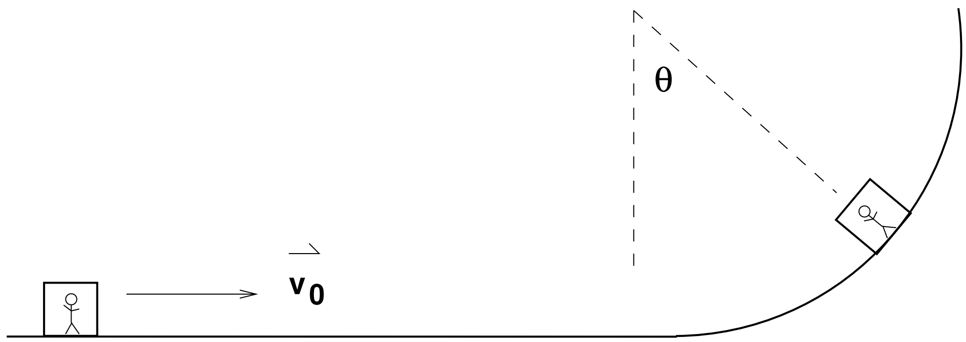
Part a) – (15 points): What is the **mass** M of the large block? Give your answer in terms of the given parameters. Explain your work.

Part b) – (15 points): The large block is now pulled slightly so the radius of the orbit circle decreases gradually from the given value of r to a new smaller given value of r' . Calculate the **new speed** v' of the puck. Give your answer in terms of the given parameters. Explain your work.

Part c) – (10 points): How much **work** was done by pulling on the large block to change the radius from r to r'? Give your answer in terms of the given parameters. Explain your work.

Important Hint: the force of tension on the string is *not constant* as the large block is pulled. This means you ***cannot*** simply calculate the work by multiplying a force times a distance.

Problem 9: Bill in a Box – 40 points



Suppose Bill Nye is in a box that is moving with constant speed v_0 on a perfectly frictionless horizontal surface as shown. The box has one-way windows so that you can see into the box but Bill cannot see outside and does not know what is going on.

The box approaches a frictionless circular ramp as show. The radius of the ramp is R which is much larger than the dimensions of the box. The box with Bill goes up the ramp, stops for an instant at some maximum angle θ_{max}, and then slides back down the ramp going the other way.

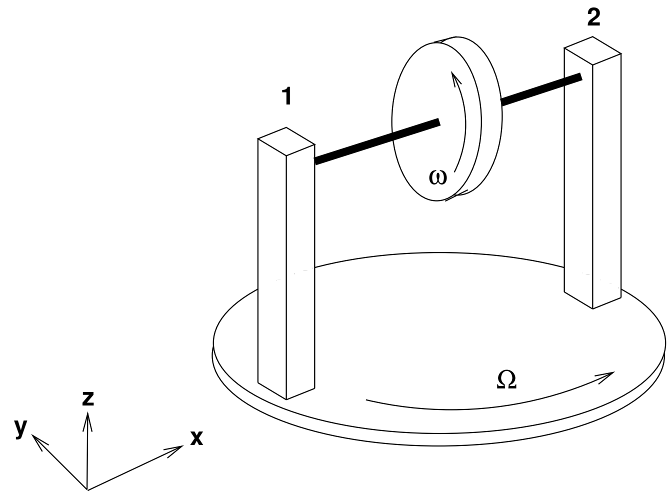
Part (a) – (10 points) : Determine θ_{max} as a function of v_0. Possibly useful hint: Consider defining the center of the circle as $y = 0$. This will make it relatively easy to specify y as a function of θ. **Important: Explain your work.**

Part (b) – (10 points) : Determine the total acceleration on the box in terms of given parameters v_0 and R at the precise instant when the box is located at θ_{max}. Hint: here the speed is zero and the problem is very similar to a block on a frictionless ramp. **Important: Explain your work.** Give your answer in terms of the given parameters.

Part (c) – (10 points) : A little Harder: Determine the total acceleration on the box as a function of θ, v_0 and R for any arbitrary value of θ that the box will have on the way up the ramp ($0 < \theta < \theta_{max}$). Possibly useful hint: consider calculating both centripetal and tangential components of acceleration. **Important: Explain your work.** Give your answer in terms of the given parameters.

Part (d) – (10 points) : Even Harder Still: What does Bill experience during this trip? Does Bill feel the room tip as the box goes up the ramp? And if so, in which direction? If Bill takes a penny out of his pocket and releases it at some instant while he is on the ramp at angle θ. which way – as seen in *his* frame (the box) – will the penny fall? Can you make a qualitative plot that shows the magnitude and direction of the "effective gravity" in the box for Bill as a function of box position during the trip? **Important: Explain your work.** You must present a reasoned explanation in terms of physics concept and your calculations to receive full credit.

Problem 10: Gyroscope on a Turntable (40 points)



A **gyroscope** is constructed from a rapidly spinning uniform disk with given spin angular speed ω given mass M and given radius R. The gyroscope is mounted between two pylons which are themselves firmly attached to a large sturdy horizontal turntable. The length of the axle between the pylons is 2ℓ with the disk placed exactly midway between the ends and directly over the center of the underlying turntable. The turntable is motor-driven so as to maintain a fixed uniform rotation with given angular speed Ω as shown. Assume that the bearings of the axle are ideal so that the angular spin speed of the disk remains constant. Also assume that $\omega >> \Omega$ so that we can assume that the total angular momentum is completely dominated by the spin rotation of the disk.

Important: On this particular problem, the angular frequency Ω is not the "free precession frequency" corresponding to a gyroscope hanging on a string or whatever. Here, the gyroscope is being *driven* around at a *given* angular precession rate of Ω by the turntable.

Part (a) (10 points): What is the **angular momentum** vector $\vec{L}$ of the gyroscope at the instant shown in the figure above? Give your answer as a **vector expression**, indicating both magnitude and direction in terms of the given coordinate system as shown. Explain how you know this.

Part (b) (10 points): What is the **torque** vector $\vec{\tau}$ of the gyroscope at the instant shown in the figure above? Give your answer as a **vector expression**, indicating both magnitude and direction in terms of the given coordinate system. Explain how you know this. Hint: You do *not* (yet) know the forces exerted on the gyroscope. Therefore, to calculate the torque you ***cannot*** use the "Definition of Torque" and instead you ***must*** use the fact that this system is (by definition) a *gyroscope* with a given known precession rate Ω.

Part (c) (10 points): If Ω were zero, then there would be equal contact forces downward on each vertical pylon due to the "dead weight" of the gyroscope. However, since Ω is non-zero, the gyroscope will apply additional forces to the pylons. **Will the total contact force applied by the gyroscope onto pylon "1" be greater than, equal to, or less than the total contact force applied by the gyroscope onto pylon "2"?** Explain your reasoning. Also, **determine the magnitude and direction of the contact force on each of the two pylons due to the gyroscope.** Show your work. Hint: the best place to put a pivot point here is at the center-of-mass of the gyroscope. Another Hint: the contact forces exerted on the two pylons are *purely vertical* and not horizontal. This is what is needed to ensure that the torque has the correct direction. There are ***no*** horizontal forces applied to the pylons at all.

Part (d) (10 points): Based on first principles, what is the **total *power*** (e.g., Work per unit time) that must be applied to the drive axle in order to maintain a steady rotational rate Ω for the turntable? Explain your reasoning.

Solution to Problem 1:

Part (a): (10 points)

This is a problem in **One-Dimensional Kinematics.** We are given the position as a function of time. To get the velocity, **by definition** we take the ***derivative of the position with respect to time***:

$$y = A + Bt - Ct^2 + Dt^3$$

$$v(t) \equiv \frac{dy}{dt} = B - 2Ct + 3Dt^2$$

Plugging in for $t = 0$ we get:

$$\boxed{v(t = 0) = B}$$

Part (b): (10 points)

We continue to apply **One-Dimensional Kinematics. By definition** we can calculate the acceleration by taking the ***derivative of the velocity with respect to time***:

$$a(t) \equiv \frac{dv}{dt} = -2C + 6Dt$$

Plugging in for $t = T$ we get:

$$\boxed{a(t = T) = -2C + 6DT}$$

Part (c): (10 points)

If the ***net force is zero*** then we know by **Newton's Second Law** that the acceleration must also be zero:

$$F_{net} = ma = 0$$
$$a(t = t_n) = 0$$

We use our result from **Part (b)**: to express the acceleration at time $t = t_n$:

$$a(t = t_n) = -2C + 6Dt_n = 0$$

Solving for t_n:

$$-2C + 6Dt_n = 0$$
$$6Dt_n = 2C$$
$$t_n = \frac{2C}{6D}$$
$$\boxed{t_n = \frac{C}{3D}}$$

Solution to Problem 1 Continued...

Part (d): (10 points)

We consider the **Effective Weight as calculated in the non-inertial reference frame of the elevator.** If your mass is m then the Effective Weight is given by the (vector) sum of the Real Weight plus the Fictitious Weight:

$$\vec{W}_{eff} = \vec{W}_{real} + \vec{W}_{fict}$$
$$\vec{W}_{eff} = -mg\hat{\boldsymbol{\jmath}} - m\vec{a}$$

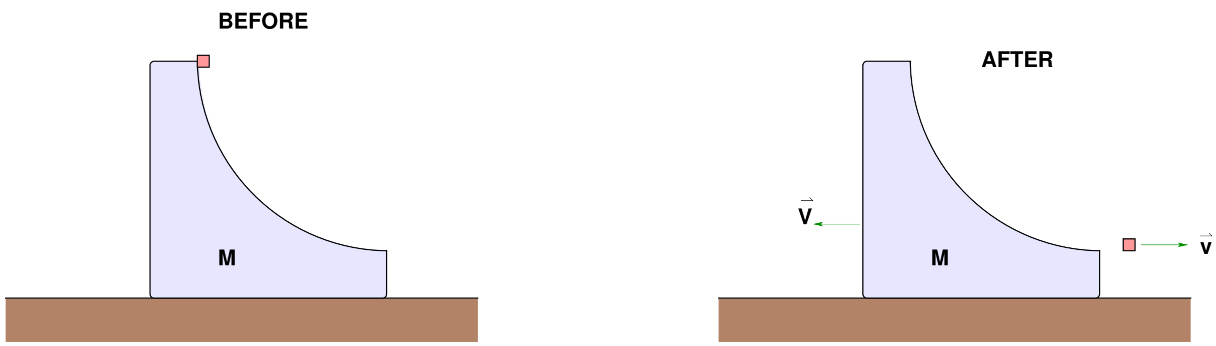
Using our result from Part (b):

$$\vec{W}_{eff} = -mg\hat{\boldsymbol{\jmath}} - m(-2C + 6Dt)\hat{\boldsymbol{\jmath}}$$

$$\vec{W}_{eff} = -m(g - 2C + 6Dt)\hat{\boldsymbol{\jmath}}$$
$$W_{eff} = m(g - 2C + 6Dt)$$

At time $t- = 0$ the magnitude of the Effective Weight is $m(g - 2C)$ (less than mg) but increasing ***linearly with time*** to a value $m(g - 2C + 6DT)$ (greater than mg). In other words, during the interval, you start out feeling $\boxed{\textit{\textbf{lighter than normal}}}$ but as time progresses, you steadily feel $\boxed{\text{increasingly heavy}}$.

Solution to Problem 2:



The key is to see this problem as an **inverse collision** where both (horizontal) momentum and energy are conserved.

In the **horizontal direction** we can say that momentum is conserved. There are no net horizontal forces on the system.

$$P_{tot} = P'_{tot}$$
$$p_M + p_m = p'_M + p'_m$$
$$0 + 0 = MV + mv$$
$$V = -v\frac{m}{M}$$

Solution continues next page....

Next we consider **Conservation of Energy**: which works fine since the only forces here are Weight (conservative) and Normal (which here does no work).

$$E_{tot} = E'_{tot}$$
$$U + K = U' + K'$$
$$mgy + 0 = mgy' + \frac{1}{2}mv^2 + \frac{1}{2}MV^2$$
$$mgR + 0 = 0 + \frac{1}{2}mv^2 + \frac{1}{2}MV^2$$
$$mgR = \frac{1}{2}mv^2 + \frac{1}{2}MV^2$$
$$mgR = \frac{1}{2}mv^2 + \frac{1}{2}M\left(-v\frac{m}{M}\right)^2$$
$$v^2 + \frac{M}{m}\left(-v\frac{m}{M}\right)^2 = 2gR$$
$$v^2 + v^2\left(\frac{m}{M}\right) = 2gR$$
$$v^2\left[1 + \left(\frac{m}{M}\right)\right] = 2gR$$
$$v^2 = \frac{2gR}{1 + \left(\frac{m}{M}\right)}$$
$$\boxed{v = \sqrt{\frac{2gR}{1 + \left(\frac{m}{M}\right)}}}$$

Note to Grader: Students who fail completely to contend with the motion of the large block have made a **major conceptual error** which will have a large impact on their ability to solve the problem using the physics concepts we want them to use.

Some students argue that they can use Conservation of Energy to calculate the velocity of the small block **relative** to the large block. This, while clever, is conceptually incorrect. The large block does not represent an "inertial reference frame" – indeed while the small block is sliding down, the large block is *accelerating* to the left.

The only correct way to use Conservation of Energy is to consider the kinetic energy of both the large block and the small block at the same time.

Solution to Problem 3:

Part (a): (5 points)

The "tee" is a ***system*** that is made from two different thin rods. By **symmetry** we know that the left (horizontal) rod has a center-of-mass point located at a distance $x_L = L/2$ corresponding to half-way along the rod. Likewise, the center-of-mass point for the right (vertical) rod is located at the right end of the left rod $x_R = L$. We use the **Definition of the Center-of-Mass** which tells is that x_{cm} is calculated as the *weighted average position* of each component (treating each rod as a point-mass):

$$x_{cm} = \frac{m_L x_L + m_R x_R}{m_R + m_L}$$
$$x_{cm} = \frac{M(L/2) + ML}{2M}$$
$$x_{cm} = \frac{3L/2}{2}$$
$$x_{cm} = \left(\frac{3}{4}\right) L$$

Therefore, since $x_{cm} = KL$ we see immediately that

$$\boxed{K = \frac{3}{4}}$$

Guidelines for grader:

- **5 pts:** Correct answer includes **Definition of Center-of-mass**.
- **3 pts:** Correct answer but missing definition.
- **3 pts:** Correct set up but incorrect algebra or other manipulation.
- **2 pts:** Failure to contend with both rods.
- **at most 1 pts:** Some other incorrect physics ideas.

Solution continues next page...

Part (b): (5 points)

We are asked to determine a *numerical* constant C where $I = CMR^2$ for the tee.[1] The total rotational inertial of the system of two rods must be equal to the **sum of that for each rod**,[2] as calculated about the hinge pivot point:

$$I_{PP} = I_L + I_R$$

Here I_L is just that for a thin rod about one end: $I_L = \frac{1}{3}ML^2$. For the right rod, however, we need to use the **Parallel Axis Theorem** since the center of mass is displaced from the pivot-point:

$$I_R = I_{cm} + MD^2 = \frac{1}{12}ML^2 + ML^2$$

Altogether then:

$$I_{PP} = \frac{1}{3}ML^2 + \left(\frac{1}{12}ML^2 + ML^2\right)$$

$$I_{PP} = \left(\frac{4}{12}\right) ML^2 + \left(\frac{13}{12}\right) ML^2)$$

$$I_{PP} = \left(\frac{17}{12}\right) ML^2$$

Therefore, since $I_{pp} = CML^2$ we see that

$$\boxed{C = \frac{17}{12}}$$

Guidelines for grader:

- **5 pts:** Correct answer includes both **Sum** and **Parallel Axis Theorem** concepts.
- **3 pts:** Correct answer but missing explanations.
- **3 pts:** Correct set up but incorrect algebra or other manipulation.
- **2 pts:** Failure to contend with both rods properly.
- **at most 1 pts:** Some other incorrect physics ideas.

Solution continues next page...

[1] Several students took a disastrous path of applying Newton's Second Law to Part (b) here. When we ask for a "numerical constant" we are literally looking for a number. Applying Newton's Second Law does not yield the value of rotational inertia here, but even if it did, the "answer" comes out in terms of symbolic parameters such as the mass, length and so forth. Also a "five point" part tells you that the work for this must be short and quick.

[2] To be clear, we have a fundamental definition of the rotational inertia as a integral, which of course is just a sum. In this problem, we have two bodies move together as one system. So the moment of inertia, which is an integral of the sum of the two bodies, is also the sum of the integrals. In other words, we just **add** the rotational inertia as calculate for each separate body about the given pivot point.

Part (c): (12 points)

Here we use **Newton's Second Law in Rotational Form**.

$$\tau_{net} = I\alpha$$

The only forces on the "tee" are the (unknown) Hinge force and the force of Weight. So we need to find these two torques:

$$\tau_{\vec{H}} + \tau_{\vec{W}} = I\alpha$$

To calculate the two torques we use the **Definition of Torque:**

$$\tau_{\vec{F}} \equiv rF\sin\phi$$

where r is the distance from the pivot point to the location where the force is applied.

For the Hinge force, since this is applied at zero distance from the pivot point, *the torque due to the Hinge force is zero*.

The force of Weight is applied to the "tee" at the position of the center-of-mass point in the downward (perpendicular) direction. We the distance to the center-of-mass is $r = KL$ So the torque due to the Weight force is just the weight force $W_{tee} = M_{tee}g = 2Mg$ and so:

$$0 + (KL)(2Mg)\sin(90°) = I\alpha$$
$$2KLMg = I\alpha$$

Solving for α and plugging in for the rotational inertia:

$$\alpha = \frac{2KLMg}{I} = \frac{2KLMg}{CML^2}$$

$$\boxed{\alpha = \left(\frac{K}{C}\right)\frac{2g}{L}}$$

Plugging in our results from parts (a) and (b):

$$\alpha = \left(\frac{3/4}{17/12}\right)\frac{2g}{L} = \left(\frac{3}{4}\right)\left(\frac{12}{17}\right)\frac{2g}{L} = \left(\frac{3}{1}\right)\left(\frac{3}{17}\right)\frac{2g}{L}$$

$$\boxed{\alpha = \left(\frac{18}{17}\right)\frac{g}{L}}$$

Guidelines for grader:

- **12 pts:** Correct answer includes explanation and correct calculation of the torque due to the Weight.
- **11 pts:** Correct answer except assigning Weight = Mg (incorrect) instead of $2Mg$ (correct).
- **10 pts:** Correct answer except force applied at $(L/2)$ instead of Center-of-Mass point.
- **6 to 9 pts:** Basic concept and setup correct but incorrect method.
- **at most 3 pts:** Some other incorrect physics ideas.

Solution continues next page...

Part (d): (12 points)

Since we want the *angular speed* a good concept to consider is **Conservation of Mechanical Energy**. First however, we must consider the ***Condition for Conservation of Energy*** which is that all of the forces on the system are *either* Conservative *or* do no work. The two forces on the "tee" are Weight (which is Conservative) and the Hinge Force (which is *not* Conservative *but* does no work in this problem since the force is not applied over any finite distance). So we have ***met the Condition*** and we apply the Conservation Law:

$$“BEFORE'' = “AFTER''$$

$$E_{tot} = E'_{tot}$$

$$U + K = U' + K'$$

$$mgy + \frac{1}{2}I\omega^2 = mgy' + \frac{1}{2}I\omega'^2$$

Here y represents the position of ***the center-of-mass of the system***. We can chose any zero-point that we wish. For simplicity we will chose $y = 0$ as the hinge point. In the "Before" case we start at $y = 0$ and we end at $y' = (-KL)$. Note that here $m = M_{sys} = 2M$.

For kinetic energy, since the entire system rotates about the hinge, and since we have already chosen the hinge as the *reference point* for calculating the rotational inertia, the proper definition of kinetic energy as being ***purely rotational***:[3]
$K_{tot} = K_{rot} = \frac{1}{2}I\omega^2$. in the "Before" case, release at rest, $\omega = 0$ and in the "After" case $\omega' = \omega_{max}$ which is what we want to find. Putting this all in:

$$(2M)g(0) + \frac{1}{2}I(0)^2 = (2M)g(-KL) + \frac{1}{2}I\omega_{max}^2$$

$$0 + 0 = -2mgKL + \frac{1}{2}I\omega_{max}^2$$

Solving for ω_{max} and plugging in the rotational inertia, the rest is algebra:

$$\frac{1}{2}I\omega_{max}^2 = 2MgKL$$

$$I\omega_{max}^2 = 4MgKL$$

[3]In general, the total kinetic energy is a sum of translational and rotational kinetic energy, and in this case, one could argue that the "tee" is both translating and rotating. This would be the correct argument if we had calculated the rotational inertia I_{tot} at a reference point that corresponds to a position on the "tee" that is moving translationally, such as the center-of-mass point. But in this case – as in the case of any rotational motion about a fixed hinge or axis – if we define the reference point at the hinge, then – by definition – this point of the body is not moving translationally, and so the body has zero translational kinetic energy and the motion is correctly described as "purely rotational" in terms of kinetic energy. In other words $K_{tot} = K_{rot} = \frac{1}{2}I_{pp}\omega^2$ only and the translational kinetic energy $\frac{1}{2}mv^2$ is set to zero. Note that in principle, we are *free* to choose a different reference point – such as the center-of-mass point of the "tee" – and in this case we would have both non-zero rotational and translational kinetic energy. But in this case we would need to use the value for the rotational inertia that corresponds to the reference point, and we would need to calculate this, since $I_{cm} \neq I_{pp}$. Choosing to include translational kinetic energy without also specifying the reference point and the associated rotational inertia as calculated about that reference point is incorrect.

$$\omega_{max}^2 = \frac{4MgKL}{I}$$

$$\omega_{max}^2 = \frac{4MgKL}{CML^2}$$

$$\omega_{max}^2 = \left(\frac{K}{C}\right)\frac{4g}{L}$$

$$\boxed{\omega_{max} = \sqrt{\left(\frac{K}{C}\right)\frac{4g}{L}}}$$

Plugging in K and C as before:

$$\omega_{max} = \sqrt{\left(\frac{9}{17}\right)\frac{4g}{L}}$$

$$\omega_{max} = \sqrt{\left(\frac{36}{17}\right)\frac{g}{L}}$$

$$\boxed{\omega_{max} = 6\sqrt{\frac{g}{17L}}}$$

Guidelines for grader:

- **12 pts:** Correct answer includes explanations.
- **11 pts:** Correct answer but incorrectly used M instead of $2M$ for the system mass.
- **10 pts:** Correct answer but forgot to explicitly check whether we have **met the Conditions** for application of Conservation of Energy.
- **10 pts:** Correct answer but incorrectly setting $y = y_{cm}$ equal to either length of rod or half-length of rod instead of position of center-of-mass of rod.
- **at most 9 pts:** Incorrect calculation of total kinetic energy (e.g., non-zero translational kinetic energy, etc.)
- **at most 6 pts:** Correct concept and setup up but incorrect method.
- **at most 2 pts:** Some other incorrect physics idea.

Solution continues next page...

Part (e): (6 points)

To get the linear speed at Point P we apply the **Rolling Constraint:**

$$v = \omega r$$

Here ω is just ω_{max} that we calculated in Part (d). The value of "r" is the distance ***from* the pivot point *to* Point P**. This distance, call it r_P is given by applying the Pythagorean Theorem to the "tee":

$$r_P = \sqrt{L^2 + (L/2)^2} = \sqrt{L^2 + \frac{L^2}{4}} = \sqrt{\frac{5L^2}{4}} = \left(\frac{\sqrt{5}}{2}\right) L$$

And so:

$$v_p = (\omega_{max})(r_P)$$

$$v_p = \left[\sqrt{\left(\frac{K}{C}\right)\frac{4g}{L}}\right]\left[\left(\frac{\sqrt{5}}{2}\right) L\right]$$

$$v_p = \frac{1}{2}\sqrt{\frac{(K)(4g)(5)(L^2)}{(C)(L)}}$$

$$v_p = \boxed{\sqrt{\frac{5KgL}{C}}}$$

$$v_p = \sqrt{5\left(\frac{3}{4}\right)\left(\frac{12}{17}\right) gL}$$

$$v_p = \sqrt{\left(\frac{36}{4}\right)\left(\frac{5}{17}\right) gL}$$

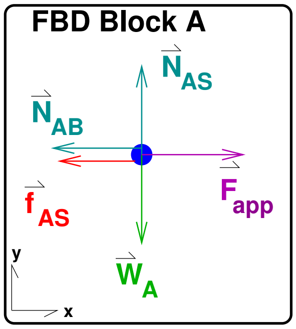
$$\boxed{v_p = 3\sqrt{\frac{5gL}{17}}}$$

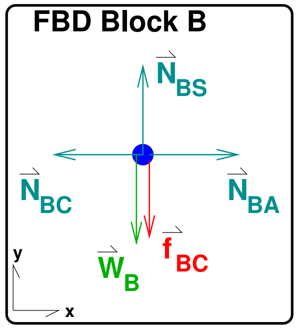
Guidelines for grader:

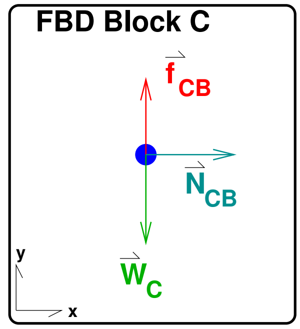
- **6 pts:** Correct answer includes both **Rolling Constraint** and **correct radial distance** concepts explained.
- **3 pts:** Correct concept of Rolling constraint but incorrectly used either the length of the rod or the half-length of the rod as the radial distance instead of applying the Pythagorean theorem.
- **at most 1 pts:** Some other incorrect physics ideas.

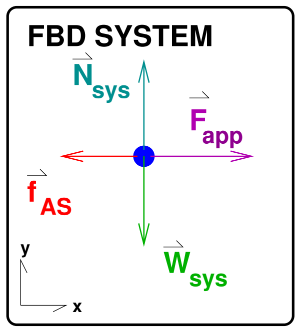
Solution to Problem 4:

Part (a): (10 points)









For the three blocks:

- There are five forces on Block A:
  - $\vec{W}_A$ The *Weight* force on Block A (downward),
  - $\vec{F}_{app}$ The *Applied* Force (rightward),
  - $\vec{N}_{AS}$ The *Normal* Force on Block A due to the horizontal surface (upward)
  - $\vec{f}_{AS}$ The *Friction* Force on Block A due to the horizontal surface (leftward), and
  - $\vec{N}_{AB}$ The *Normal* Force on Block A due to Block B (leftward).
- There are five forces on Block B:
  - $\vec{W}_B$ The *Weight* force on Block B (downward)
  - $\vec{N}_{BS}$ The *Normal* Force on Block B due to the horizontal surface (upward),
  - $\vec{N}_{BA}$ The *Normal* Force on Block B due to Block A (rightward),
  - $\vec{N}_{BC}$ The *Normal* Force on Block B due to Block C (leftward), and
  - $\vec{f}_{BC}$ The *Friction* Force on Block B due to Block C (downward).
- There are three forces on Block C:
  - $\vec{W}_C$ The *Weight* force on Block C (downward),

- $\vec{N}_{CB}$ The *Normal* Force on Block C due to Block B (rightward), and
- $\vec{f}_{CB}$ The *Friction* Force on Block C due to Block A (upward).

For the system as a whole:

- There are four (external) forces on the system as a whole:
  - $\vec{W}_{sys}$ The *Weight* force on the system where $W_{sys} = W_A + W_B + W_C$,
  - $\vec{N}_{sys}$ The *Normal* force on the system where $N_{sys} = N_AS + N_BS$,
  - $\vec{F}_{app}$ The *Applied* Force, and
  - $\vec{f}_{AS}$ The *Friction* Force on Block A due to the horizontal surface

Note to grader: The vertical forces on Block B and the vertical forces on the System as a whole are of no consequences to this problem and can be missing and/or incorrect with no consequence to the grading.

Part (b): (5 points)

We apply **Newton's Second Law** to Block A, first in the vertical direction:

$$F_{Ax} = m_A a_y$$
$$N_{AS} - W_A = 0$$
$$N_{AS} = W_A$$
$$N_{AS} = mg$$

And now we just apply the special rule for **Kinetic Friction**: $f_k = \mu_k N$

$$f_{AS} = \mu_k N_{AS}$$
$$\boxed{f_{AS} = \mu_k mg \qquad \text{leftward}}$$

Part (c): (5 points)

To get the acceleration we apply **Newton's Second Law** to the **entire system** in the vertical direction, using the FBD as show:

$$F_{sys,x} = M_{sys} a_x$$

$$F_{app} - f_{AS} = (m_A + m_B + m_C)a_x$$
$$F_{app} - \mu_k mg = (m + 2m + m)a_x$$
$$F_{app} - \mu_k mg = 4ma_x$$
$$a_x = \frac{F_{app} - \mu_k mg}{4m}$$

Finally, we note that since the three blocks move together, the acceleration any block, including Block B, is the same as the acceleration of the entire system:

$$\boxed{a_B = a_x = \frac{F_{app} - \mu_k mg}{4m}}$$

Part (d): (5 points)

We want N_{BA} There are at least two ways to do this.

Option 1: Most Elegant: We apply **Newton's Second Law** to the **(sub)-system** that is ***Block B and Block C together*** in the horizontal direction: In this case there is only one external force on this (sub)-system, and that's the one we want:

$$F_{sysBC,x} = M_{sysBC}\, a_x$$
$$N_{BA} = (3m)\left(\frac{F_{app} - \mu_k mg}{4m}\right)$$
$$\boxed{N_{BA} = \frac{3(F_{app} - \mu_k mg)}{4} \qquad \text{rightward}}$$

Option 2: We apply **Newton's Second Law** to Block A in the horizontal direction:

$$F_{Ax} = m_A a_X$$
$$F_{app} - f_{AB} - N_{AB} = (m)\left(\frac{F_{app} - \mu_k mg}{4m}\right)$$
$$F_{app} - \mu_k mg - N_{AB} = \frac{F_{app} - \mu_k mg}{4}$$
$$N_{AB} = (F_{app} - \mu_k mg) - \frac{F_{app} - \mu_k mg}{4}$$
$$N_{AB} = \frac{3(F_{app} - \mu_k mg)}{4}$$

Then we apply Newton's Third Law which tells us $N_{BA} = N_{AB}$:

$$\boxed{N_{BA} = \frac{3(F_{app} - \mu_k mg)}{4} \qquad \text{rightward}}$$

Part (e): (5 points)

We want N_{BC}. The easiest way to get this is to consider the forces on Block C. We apply **Newton's Second Law** to Block C in the horizontal direction:

$$F_{Cx} = m_C a_x$$
$$N_{CB} = (m)\left(\frac{F_{app} - \mu_k mg}{4m}\right)$$
$$N_{CB} = \frac{F_{app} - \mu_k mg}{4}$$

And then **Newton's Third Law** tells us:

$$N_{CB} = N_{BC}$$

So

$$\boxed{N_{BC} = \frac{F_{app} - \mu_k mg}{4} \qquad \text{leftward}}$$

Part (f): (5 points)

We apply **Newton's Second Law** to Block C in the vertical direction:

$$F_{Cy} = m_C a_y$$
$$F_{Cy} = 0$$
$$f_{CB} - W_C = 0$$
$$f_{CB} - mg = 0$$
$$\boxed{f_{CB} = mg \qquad \text{upward}}$$

Part (g): (5 points)

We apply the **Constraint for Static Friction:** $f_k \leq \mu_s N$. Here we apply this to the forces between Block B and Block C:

$$f_{BC} \leq \mu_s N_{CB}$$

Plugging in results from Parts (e) and (f):

$$mg \leq \mu_s \left(\frac{F_{app} - \mu_k mg}{4} \right)$$

Re-arranging:

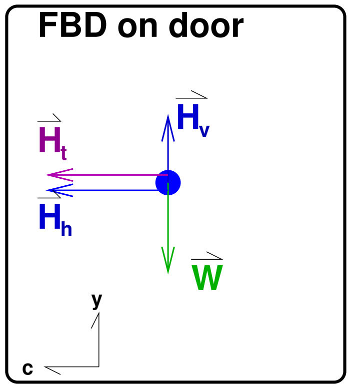
$$\mu_s (F_{app} - \mu_k mg) \geq 4mg$$
$$\mu_s F_{app} \geq \mu_s \mu_k mg + 4mg$$
$$\mu_s F_{app} \geq (\mu_s \mu_k + 4) mg$$
$$\boxed{F_{app} \geq \frac{(\mu_s \mu_k + 4) mg}{\mu_s}}$$

or

$$\boxed{F_{app} \geq \left(\mu_k + \frac{4}{\mu_s} \right) mg}$$

Solution to Problem 5: 40 points total

We start with **Newton's Second Law:** and write down the **Free Body Diagram** for the door treating it as a point mass:



For **Vertical Forces** we apply **Newton's Second Law:**

$$F_y = ma_y$$
$$H_v - W = ma_y$$

We know the vertical acceleration is zero:

$$H_v - W = 0$$

$$H_v = W$$

$$\boxed{H_v = mg}$$

For **Horizontal Forces** we apply **Newton's Second Law:**

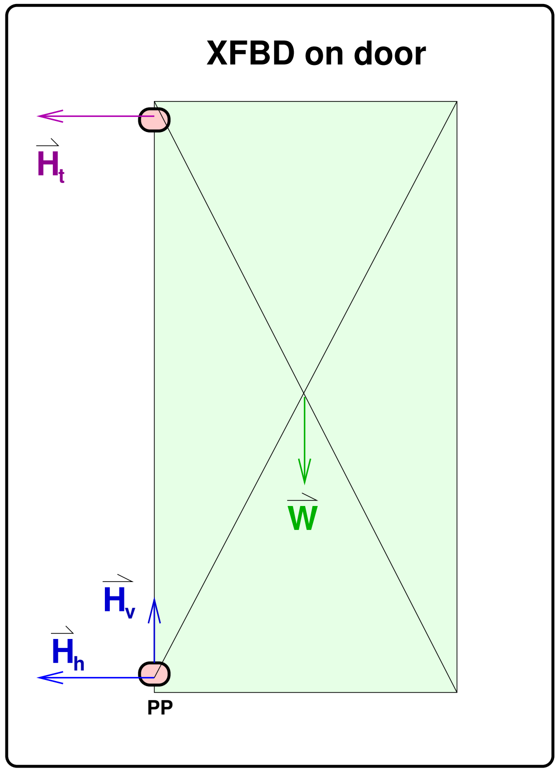
$$F_c = ma_c$$
$$H_h + H_t = ma_c$$

Since the door is swinging at rotational speed Ω we know that there is a *centripetal acceleration* on the door. Treating the position of the door as a "point mass" at the center mass out at a radius of $r = w/2$:

$$H_h + H_t = mr\Omega^2$$
$$H_h + H_t = mw\Omega^2/2$$

This is as far as we can go with translational motion because we have three unknowns but only two equations. So we move on to rotational motion:

Now we deal with the rotational motion: We consider the **Extended Free Body Diagram** for the door:



We consider the torque about the pivot point defined at the lower hinge. In this case, the torques due to the lower hinge go to zero. We write down **Newton's Second Law in Rotational Form:**

$$\tau_{net} = I\alpha$$

We note that the door is not not rotating about the given pivot point in the plane of the door.

$$\tau_{net} = 0$$

$$\tau_{H_t} + \tau_W = 0$$

Using the convention that counter-clockwise is positive and noting that the torque due to the weight is given by $\tau_W = r_{perp}W = (\mathrm{w}/2)mg$ then[4]

$$hH_t - \left(\frac{\mathrm{w}}{2}\right)mg = 0$$

$$\boxed{H_t = \frac{\mathrm{w}mg}{2h}}$$

So now we plug this in to get the horizontal component of the lower hinge force:

$$H_h + \frac{\mathrm{w}mg}{2h} = m\mathrm{w}\Omega^2/2$$

[4]To clarify. The weight force is down. The radial vector runs from the hinge to the center of the door. Since the Weight is down (vertical) the perpendicular component corresponds to the horizontal component of the radial vector. This (by construction) is the width of the door divided by two.

$$\boxed{H_t = \frac{wm\Omega^2}{2} - \frac{wmg}{2h}}$$

Solution to Problem 6:

Part (a) – : We apply the concept of **Universal Gravity:**

$$F_{ug}(r) = \frac{Gm_1m_2}{r^2}$$

Here, m_1, the mass of the earth that is within your radius and m_2 is your own mass. We note that the mass within your radius is just the fractional volume times the earth's mass:

$$m_1 = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}M$$

$$m_1 = \frac{r^3}{R^3}M$$

Therefore:

$$F_{ug}(r) = \frac{G\left(\frac{r^3}{R^3}M\right)m}{r^2}$$

$$F_{ug}(r) = \frac{GMmr}{R^3}$$

$$\boxed{F_{ug}(r) = \left(\frac{GMm}{R^3}\right)r}$$

In other words, the force is linearly proportional to the radius r.

Part (b) – : We start with **Newton's Second Law:**

$$F_r = ma_r$$

$$-F_{ug} = m\frac{d^2r}{dr^2}$$

$$-\left(\frac{GMm}{R^3}\right)r = m\frac{d^2r}{dt^2}$$

$$\boxed{\frac{d^2r}{dt^2} + \left(\frac{GM}{R^3}\right)r = 0}$$

This is the **Equation of Motion**. It looks Quite Familiar.

continues next page...

Part (c) – :

This looks very similar to the Equation of Motion for the Simple Harmonic Oscillator:

$$\frac{d^2r}{dt^2} + \omega^2 x = 0$$

Where the oscillation frequency is given by the coefficient of the second term:

$$\omega^2 = \frac{GM}{R^3}$$

So the **solution** should look oscillatory:

$$r(t) = A\cos(\omega t + \phi)$$

In this particular problem $r(t = 0) = R$ so $A = R$ and $\phi = 0$:

$$r(t) = R\cos(\omega t)$$

$$\boxed{r(t) = R\cos\left[\left(\sqrt{\frac{GM}{R^3}}\right) t\right]}$$

Part (d) – :

During the entire trip you are in **Free-Fall**. That is to say that the only force on you is the force of **gravity**. Since you are falling you experience the sensation of **"weightlessness"** the same as if you were in free-fall at the surface of the earth and/or floating in deep space. The fact that gravity changes magnitude and direction does not change the sensation of being weightless. It only changes the kinematics of your motion. You always **feel weightless** because the effective gravity is always zero in your (accelerating) reference frame.

Since you cannot infer your kinematic speed as you pass the earth's center, and since you sense "weightlessness" you **cannot** determine when you have reached the center point (unless you have timed your decent quite carefully). This is a manifestation of the Einsteins **Principle of Equivalence** which is the foundation of the theory of General Relativity.

Solution to Problem 7:

We break this problem into two distinct sub-parts:

- **Sub-part (1): The Collusion** The collision between the clay blob and the rod that imparts motion to the rod. For this part we will use **Conservation of Angular Momentum**, and
- **Sub-part (2): The Swing-Up** The swing-up of the rod from a vertical orientation to an horizontal orientation. For this we will use **Conservation of Mechanical Energy** where we will need to contend with the energy of both the rod and the blob.

Since we want the minimum value of the speed of the blob v_i just before it hits the rod, we will want to **work backwards**. So we start with **Sub-part (2)** where the goal is to find the value of the rotational speed of the bar-blob system that is required to move the bar up to the ceiling.

Sub-part (2): Swing Up: Find rotational speed required to get bar-bob system to the ceiling. Because the only forces on the system are Weight and the "Hinge Force" (which does no work here) we are free to use **Conservation of Mechanical Energy**. The "Before" corresponds to immediately after the collision with the bar in the vertical orientation. The "After" corresponds the bar swinging up to the ceiling in the horizontal orientation. We have to contend with both the potential energy and the kinetic energy of both the blob and the rod:

$$E_{tot} = E'_{tot}$$
$$E_B + E_R = E'_B + E'_R$$
$$U_B + K_B + U_R + K_R = U'_B + K'_B + U'_R + K'_R$$
$$mgy_B + \frac{1}{2}mv_B^2 + mgy_R + \frac{1}{2}I\omega_R^2 = mgy'_B + \frac{1}{2}mv_B'^2 + mgy'_R + \frac{1}{2}I\omega_R'^2$$

So now we have to contend with each of these terms. We start by choosing an appropriate vertical coordinate system for the potential energy. Here we choose the center of rotation as the point corresponding to $y = 0$. With this choice we get $y_B = -L$, $y_R = -\frac{L}{2}$, $y'_B = 0$, and $y'_R = 0$. We also note that if the bar just (barely) comes in contact with the ceiling this corresponds to zero final velocity, so $v'_B = 0$ and $\omega'_R = 0$:

$$-mgL + \frac{1}{2}mv_B^2 - \frac{mgL}{2} + \frac{1}{2}I\omega_R^2 = 0 + 0 + 0 + 0$$
$$-2mgL + mv_B^2 - mgL + I\omega_R^2 = 0$$
$$-2mgL + mv_B^2 - mgL + \left(\frac{1}{3}mL^2\right)\omega_R^2 = 0$$

Here we note that since the blob is stuck to the bar, we can use the **Rolling Constraint** that tells us $v_B = L\omega_R$ so:

$$-2mgL + m(\omega_R L)^2 - mgL + \left(\frac{1}{3}mL^2\right)\omega_R^2 = 0$$
$$m(\omega_R L)^2 + \left(\frac{1}{3}mL^2\right)\omega_R^2 = 3mgL$$

$$\omega_R^2 L + \left(\frac{1}{3}\right)\omega_R^2 L = 3g$$
$$\left(\frac{4}{3}\right)\omega_R^2 L = 3g$$
$$\omega_R^2 = \left(\frac{9}{4}\right)\frac{g}{L}$$
$$\boxed{\omega_R = \left(\frac{3}{2}\right)\sqrt{\frac{g}{L}} \qquad \text{Answer to Sub-part 2}}$$

Sub-part (1): The Collision. We use **Conservation of Angular Momentum**. We can do this despite the fact that we have a completely unknown hinge force because if we chose the pivot point (axis of rotation) at the hinge then there is no net torque on the system that includes the clay blob and the rod (ignoring the effects of gravity on the clay blob before impact). We define the direction of positive angular momentum in the conventional way: an out of the page positive vector corresponds to counter-clockwise. The "before" and "after" correspond to immediately before and after the collision between the bullet and the rod, before the rod has moved and appreciable distance:

$$\vec{\mathcal{L}}_{tot} = \vec{\mathcal{L}}'_{tot}$$
$$\vec{\mathcal{L}}_{blob} + \vec{\mathcal{L}}_{rod} = \vec{\mathcal{L}}'_{blob} + \vec{\mathcal{L}}'_{rod}$$
$$\vec{r}\times\vec{p} + 0 = \vec{r}\times\vec{p}' + I\vec{\omega}$$

Okay the angular momentum of the blob before the collision is given by $\vec{r}\times\vec{p}$ so the magnitude of this is given by $(L)(mv_0)\sin\theta$. We know that by construction v' for the blob is perpendicular to the rod.

$$Lmv_i\sin\theta = Lmv_B + I\omega_R$$
$$Lmv_i\sin\theta = Lmv_B + \left(\frac{1}{3}mL^2\right)\omega_R$$
$$v_i\sin\theta = v_B + \left(\frac{1}{3}\right)L\omega_R$$
$$v_i\sin\theta = \omega_R L + \left(\frac{1}{3}\right)L\omega_R$$
$$v_i\sin\theta = \left(\frac{4}{3}\right)L\omega_R$$
$$v_i = \left(\frac{4}{3\sin\theta}\right)L\omega_R$$

Now we plug in our answer from **Sub-part 2** for ω_R to get:

$$v_i = \left(\frac{4}{3\sin\theta}\right)L\left(\frac{3}{2}\right)\sqrt{\frac{g}{L}}$$
$$\boxed{v_i = \left(\frac{1}{2\sin\theta}\right)\sqrt{gL}}$$

Solution to Problem 8:

Part (a) – 15 points: In equilibrium puck has centripetal acceleration whilst large block has zero acceleration.

Newton's Second Law: applied to Puck, centripetal coordinate:

$$F_c = ma_c$$

$$T = \frac{mv^2}{r}$$

Newton's Second Law: applied to Block, vertical coordinate:

$$F_y = Ma_y$$

$$T - W_M = 0$$

$$T = W_M = Mg$$

Putting these together for one value of the tension T:

$$\frac{mv^2}{r} = Mg$$

$$\boxed{v = \sqrt{\frac{Mgr}{m}}}$$

Part (b) – 15 points: Since the puck is undergoing a purely radial force, there is zero torque on puck as calcualted about the hole: Therefore since the system is **isolated to external torques** we can say that **Angular Momentum is Conserved**:

$$L_{tot} = L'_{tot}$$

$$mvr = mv'r'$$

$$\boxed{v' = \frac{vr}{r'}}$$

Part (c) – 10 points: The easiest way to get this is to use the **Classical Work-Energy Theorem**

$$W_{tot} = \Delta K$$

In this problem the work is done by pulling on the large block. This does not significantly change the velocity of the block but it does change the speed of the puck in accordance with Part (b). So we use this result to calculate the work:

$$\mathcal{W}_{tot} = K' - K$$

$$\mathcal{W}_{tot} = \frac{1}{2}mv'^2 - \frac{1}{2}mv^2$$

$$\mathcal{W}_{tot} = \frac{1}{2}m\left(\frac{vr}{r'}\right)^2 - \frac{1}{2}mv^2$$

$$\mathcal{W}_{tot} = \frac{1}{2}mv^2\left(\frac{r}{r'}\right)^2 - \frac{1}{2}mv^2$$

$$\boxed{\mathcal{W}_{tot} = \frac{1}{2}mv^2\left[\left(\frac{r}{r'}\right)^2 - 1\right]}$$

Solution to Problem 9:

Part (a)– (10 points):

We notice that the geometry of the situation allows us to apply simple trigonometry to write down an expression of the vertical position of the box relative to the center point of the radius of curvature of the sphere:

$$y(\theta) = -R\cos(\theta)$$

Now since there is no friction, Weight is conservative, and Normal does no work, we can apply **Conservation of Energy**. In this case, the "before" corresponds to moving with velocity v_0 on the flat surface and the "after" corresponds to arriving at the maximum angle θ_{max}:

$$\text{“}BEFORE'' = \text{“}AFTER''$$
$$E_{tot} = E'_{tot}$$
$$U_{tot} + K_{tot} = U'_{tot} + K'_{tot}$$
$$mgy + \frac{1}{2}mv^2 = mgy' + \frac{1}{2}mv'^2$$

Note that if we put the coordinate system at the center of the pivot point, the initial y-position is $-R$. Note also that the final velocity at the maximum angle will be zero:

$$mg(-R) + \frac{1}{2}mv_0^2 = mgy' + 0$$

We use our expression for y in terms of the angle. Here the angle is the maximum angle:

$$-mgR + \frac{1}{2}mv_0^2 = mg[-R\cos(\theta_{max})]$$

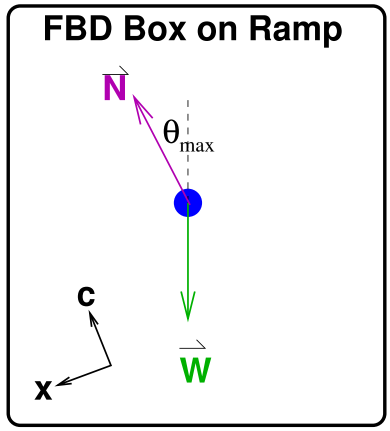
Solving for θ_{max}:

$$mg[-R\cos(\theta_{max})] = -mgR + \frac{1}{2}mv_0^2$$
$$\cos(\theta_{max}) = 1 + \left(\frac{1}{-mgR}\right)\left(\frac{1}{2}mv_0^2\right)$$
$$\cos(\theta_{max}) = 1 - \frac{v_0^2}{2gR}$$
$$\boxed{\theta_{max} = \cos^{-1}\left(1 - \frac{v_0^2}{2gR}\right)}$$

Solution continues next page....

Part (b)– (10 points):

Here we have a situation that is very similar to our old problem with the block on the ramp. The FBD looks like:



Here we define the direction of acceleration (back down the ramp) as "x" and (for later use) we define the direction of centripetal acceleration as "c".

We use Newton's Second Law in the "x" coordinate:

$$F_x = ma_x$$

Here we define the direction of acceleration (back down the ramp) as "x" and (for later use) we define the direction of centripetal acceleration as "c".

We use Newton's Second Law in the "x" coordinate:

$$F_x = ma_x$$

The only force is the component of Weight:

$$W\sin(\theta_{max}) = ma_x$$

Solving for the acceleration:

$$mg\sin(\theta_{max}) = ma_x$$

$$g\sin(\theta_{max}) = a_x$$

$$\boxed{a_x = g\sin(\theta_{max})}$$

This is the same answer we have derived in the past for a frictionless ramp. We need to use our answers from Part (a) to get an answer in terms of the original parameters:

$$\boxed{a_x = g\sin\left[\cos^{-1}\left(1 - \frac{v_0^2}{2gR}\right)\right]}$$

This can be simplified further using trig identities but it's not really worth it.

Solution continues next page....

Part (c)– (10 points):

In general the total acceleration contains both centripetal and tangential components:

$$\vec{a}_{tot} = \vec{a}_{centripetal} + \vec{a}_{tangential}$$

In terms of the coordinates we defined in the FBD:

$$\vec{a}_{tot} = \vec{a}_c + \vec{a}_x$$

For the box on the curved ramp at some arbitrary angle, we can calculate the magnitude of both terms. The expression for the tangential acceleration from Part (b) is fine. We need to modify what we did for Part (a) to get the speed for any angle:

$$U_{tot} + K_{tot} = U'_{tot} + K'_{tot}$$
$$mgy + \frac{1}{2}mv^2 = mgy' + \frac{1}{2}mv'^2$$
$$mg(-R) + \frac{1}{2}mv_0^2 = mg[-R\cos(\theta)] + \frac{1}{2}mv'^2$$

Solving for v' in terms of θ:

$$1 - \frac{v_0^2}{2gR} = \cos(\theta) - \frac{v'^2}{2gR}$$
$$\frac{v'^2}{2gR} = \frac{v_0^2}{2gR} + \cos(\theta) - 1$$
$$v'^2 = v_0^2 - 2gR[1 - \cos(\theta)]$$

So this gives us the expression for the centripetal acceleration:

$$a_c = \frac{v'^2}{R} = \frac{v_0^2 - 2gR(1-\cos\theta)}{R}$$
$$a_c = \frac{v_0^2}{R} - 2g(1-\cos\theta)$$

In other words, the centripetal acceleration on the curved section starts out with a maximum value and decreases to zero at the maximum angle. In contrast the tangential acceleration starts out zero and increases. An expression for the total acceleration, then is:

$$\vec{a}_{tot} = \vec{a}_c + \vec{a}_x$$
$$\vec{a}_{tot} = a_c\hat{\boldsymbol{c}} + a_x\hat{\boldsymbol{\imath}}$$
$$\boxed{\vec{a}_{tot} = \left[\frac{v_0^2}{R} - 2g(1-\cos\theta)\right]\hat{\boldsymbol{c}} + g\sin(\theta)\hat{\boldsymbol{\imath}}}$$

where "c" and "x" correspond to centripetal and tangential coordinates.[5]

Solution continues next page....

[5]Here I define $\hat{\boldsymbol{x}}$ as a unit vector that is akin to $\hat{\boldsymbol{\imath}}$ or $\hat{\boldsymbol{v}}$ or $\hat{\boldsymbol{t}}$. Any of these coordinate systems works just as well here. I chose "c-hat" and "x-hat" because these are two directions that correspond to the (positive) direction of acceleration in each of these components.

Part (d)– (10 points):

To figure out what is going in in Bill's frame, we consider the effects of the acceleration and tipping. Further, the box experiences both centripetal and tangential acceleration.

To consider the effect of tipping the box, we note that in Bill's frame, the tipping breaks the real gravity (toward the center of the earth) into two components, one projected onto the "up-and-down" as seen in the box (the centripetal component) and one directed "side-to-side" as seen from the frame of the box (the tangential component): A little thought brings us quickly to the notion that the components depends on sines and cosines of the tip angle:

$$\vec{W} = -mg\hat{\boldsymbol{\jmath}} = -mg\cos\theta\,\hat{\boldsymbol{c}} + mg\sin\theta\,\hat{\boldsymbol{\imath}}$$

To consider the "effective gravity" inside the box as experience by Bill:

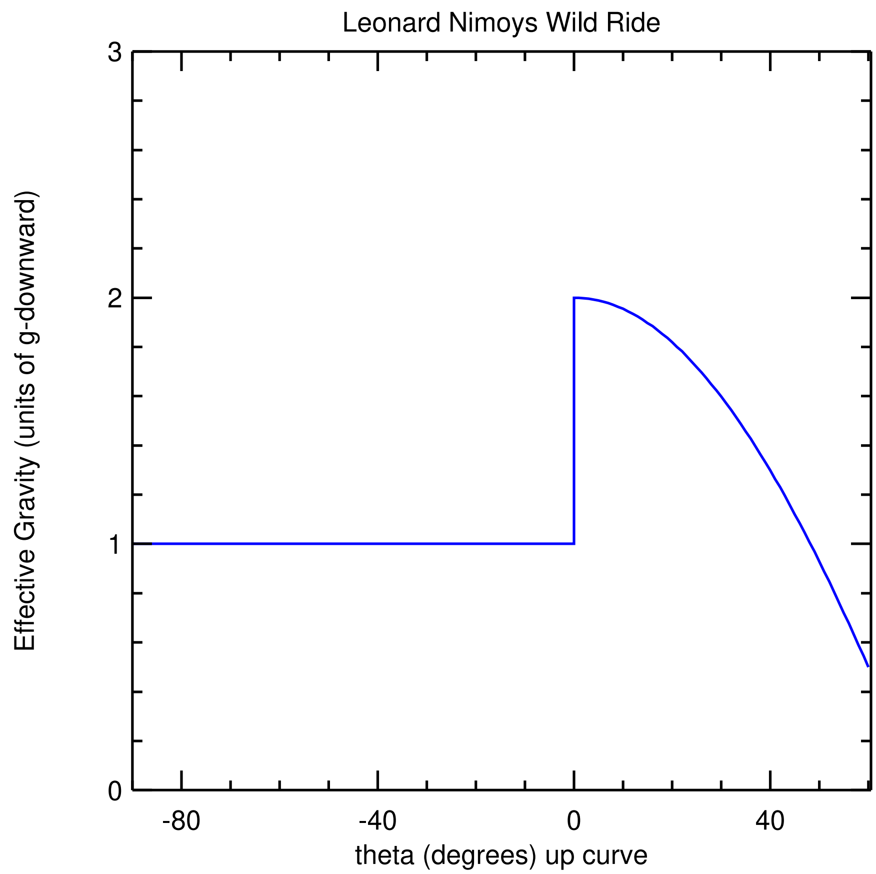
$$\vec{W}_{eff} = \vec{W}_{real} + \vec{W}_{fict}$$
$$\vec{W}_{eff} = \vec{W} - m\vec{a}_{tot}$$
$$\vec{W}_{eff} = (W_c - ma_c)\hat{\boldsymbol{c}} + (W_x - ma_x)\hat{\boldsymbol{\imath}}$$
$$\vec{W}_{eff} = (-mg\cos\theta\ - ma_c)\hat{\boldsymbol{c}} + (mg\sin\theta - a_x)\hat{\boldsymbol{\imath}}$$
$$\vec{W}_{eff} = \left\{-mg\cos\theta - m\left[\frac{v_0^2}{R} - 2g(1-\cos\theta)\right]\right\}\hat{\boldsymbol{c}} + (mg\sin\theta - mg\sin\theta)\hat{\boldsymbol{\imath}}$$
$$\vec{W}_{eff} = \left(-mg\cos\theta - m\frac{v_0^2}{R} + 2mg - 2mg\cos\theta\right)\hat{\boldsymbol{c}} + 0\,\hat{\boldsymbol{\imath}}$$
$$\vec{W}_{eff} = \left(-m\frac{v_0^2}{R} + 2mg - 3mg\cos\theta\right)\hat{\boldsymbol{c}} + 0\,\hat{\boldsymbol{\imath}}$$
$$\vec{W}_{eff} = m\left[(2-3\cos\theta)g - \frac{v_0^2}{R}\right]\hat{\boldsymbol{c}} + 0\,\hat{\boldsymbol{\imath}}$$

Note that surprising result that the magnitude of the tangential component is always zero.

In this problem, the *effective gravity* in the "side-to-side" direction (tangential) as experienced by Bill, gives rise to a fictitious Weight force $W_{eff} = mg_{eff}$ that is *exactly* the same as the perpendicular component of (real) Weight when the box is tipped. But remember, the fictitious force applies in the *opposite* direction relative to the acceleration, and therefore it applies in the opposite direction relative to the perpendicular weight component. In other words, **as Bill experiences it, because the effective gravity due to tangential deceleration exactly cancels out the perpendicular component of weight as the box is tipped, Bill does not feel any tip at all.** None. In this situation, with the boxed closed and no windows to the outside, Bill can do no experiment that will tell him that he is in a tipped box as he heads up the curved ramp. If Bill drops a penny, he will see the penny fall "straight down" to his floor, and this is true regardless of where on the ramp and what angle he happens to be at. It's amazing!

Note, however, that Bill still knows something weird is going on. In particular, as soon as he "hits the curve" the force of "gravity" will suddenly feel increased, and then will fall off, and then will increase again. In other words, Bill might predict that he is in some elevator that is accelerating upward and then decelerating. But he will not feel the "tip".

Suppose, for example, that we chose an initial speed such that $\frac{v_0^2}{R} = g$ and therefore $\frac{v_0^2}{2gR} = \frac{1}{2}$ corresponding to $\theta_{max} = 60$ degrees. We can plot the amplitude of the effective weight as a function of theta:



Here we see that at the onset of 'hitting the curve' the effective gravity jumps sharply upward (due to centripetal acceleration) but then falls off to "half-a-Gee" when the box gets to the turn-around point. So clearly Bill will know that something "non-inertial" is going on with his frame of reference, but he will not have any indication that the box he is in is being tipped.

Solution to Problem 10: 40 points total

Part (a): 10 points Because $\omega >> \Omega$ we can assume that all of the angular momentum is in the spinning wheel:

$$L = I_{wheel}\omega$$
$$L = \frac{1}{2}MR^2\omega$$

According to the right hand rule, $\vec{L}$ points in the $-x$ direction as shown above:

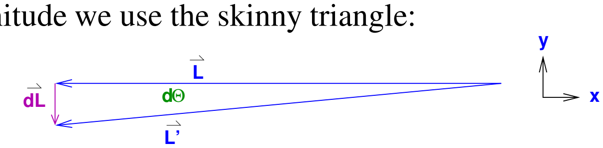
$$\vec{L} = \frac{1}{2}MR^2\omega\,(-\hat{\boldsymbol{\imath}})$$
$$\boxed{\vec{L} = -\frac{1}{2}MR^2\omega\,\hat{\boldsymbol{\imath}}}$$

Part (b): 10 points The turntable **forces** the angular momentum vector to rotate at a fixed rate Ω. To do this, a torque must be applied:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

Since we are rotating only, the angular speed of the gyroscope does not change. Therefore $\vec{\tau}$ must be perpendicular to $\vec{L}$ as shown in the figure above. We know that $\vec{\tau}$ must point in the $-y$ direction in order to make $\vec{L}$ turn clockwise as seen from above.

To work on the magnitude we use the skinny triangle:



$$\tau = \frac{dL}{dt}$$
$$\tau = \frac{Ld\Theta}{dt}$$
$$\tau = L\Omega$$
$$\tau = \frac{1}{2}MR^2\omega\Omega$$

In other words, given L and Ω, then τ must have this value in order that Ω rotate at this frequency. This relationship remains true, regardless of what causes the torque.

To write out a vector expression, we note that $\vec{\tau}$ rotates in the x-y plane. In other words, we have the standard "cosine" and "sine" oscillation in each coordinate, with the $t = 0$ direction corresponding to the $-y$ direction:

$$\vec{\tau}(t) = \frac{1}{2}MR^2\omega\Omega(\sin\Omega t\,\hat{\boldsymbol{\imath}} - \cos\,\Omega t\,\hat{\boldsymbol{\jmath}})$$

At the instant shown, corresponding to $t = 0$ then we get:

$$\boxed{\vec{\tau}(t=0) = -\frac{1}{2}MR^2\omega\Omega\,\hat{\boldsymbol{\jmath}}}$$

Part (c): 10 points In order to achieve this torque, new forces must be applied to the axle by the pylons. To get a torque in the $-y$ direction requires forces going down at Pylon "1" and up at Pylon "2". Since each pylon must also apply upward forces to hold the gyroscope in place, the effect is that the total force on the gyro due to Pylon "1" is decreased, while the net force on Pylon "2" must be increased. This means that $\boxed{\text{the net force on Pylon "2" is greater than the net force on Pylon "1"}}$.

Assuming a symmetric arrangement, we can calculate these forces:

$$\begin{aligned}
\tau_{tot} &= \tau_1 + \tau_2 \\
\tau_{tot} &= -F_1\ell + F_2\ell \\
\tau_{tot} &= -\left(\frac{1}{2}mg - F\right)\ell + \left(\frac{1}{2}mg + F\right)\ell \\
\tau_{tot} &= 2F\ell \\
F &= \frac{\tau_{tot}}{2\ell} \\
F &= \frac{MR^2\omega\Omega}{4\ell}
\end{aligned}$$

Therefore the net force on Pylon "1" is:

$$\boxed{\vec{F}_1 = \left(\frac{Mg}{2} - \frac{MR^2\omega\Omega}{4\ell}\right)\,\hat{\boldsymbol{k}}}$$

Likewise for Pylon "2":

$$\boxed{\vec{F}_2 = \left(\frac{Mg}{2} + \frac{MR^2\omega\Omega}{4\ell}\right)\,\hat{\boldsymbol{k}}}$$

Part (d): 10 points Given that we have an *ideal bearing* and we are *maintaining* a fixed angular rotation of the turntable Ω, the required forces to apply the torque to the gyroscope via the two pylons are **vertical forces** which are doing no work, because nothing is moving vertically. Furthermore, we see that the gyroscope neither speeds up nor slows down. This tells us by the **Work-Energy Relation** that no Work is being done here. Zero work, no change in kinetic energy. Therefore $\boxed{\textbf{zero power}}$ is required to maintain the turntable speed.

One way to see this is that the arms of the turntable only apply vertical forces, not horizontal ones. In other words, there are no horizontal forces anywhere here that would result in a torque applied over an angle or a force over a distance. Since the forces on the gyroscope are applied in a direction that is perpendicular to any motion of the gyroscope, no work is done. No work, no power.