

Chapter 3+

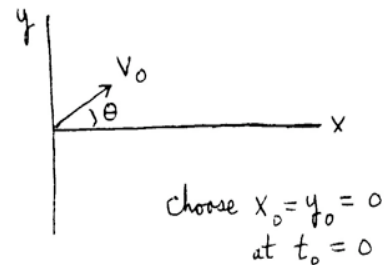
2D Constant-Gravity Trajectories: Maximum Range and Intersections

Revisit: <ul style="list-style-type: none"> • Constant acceleration in 2D • 2D trajectories—constant gravity force downward (y component) 	Learn: <ul style="list-style-type: none"> • Trajectories with an initial angle • How to calculate the maximum range of a trajectory • 2D intersections
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PROJECTILE, INITIAL ANGLE θ

Constant Gravity, No Air Drag:

Suppose the stone, or whatever, starts off from the x-y origin with a speed v_0 , but at an angle θ , initially, with respect to the x axis. Now both the vertical and horizontal velocity components are nonzero, initially.



Using a right triangle with v_0 as the hypotenuse and good old SOHCAHTOA, the x- and y-components are found to be

$$v_{0x} = v_0 \cos \theta \quad \text{and} \quad v_{0y} = v_0 \sin \theta$$

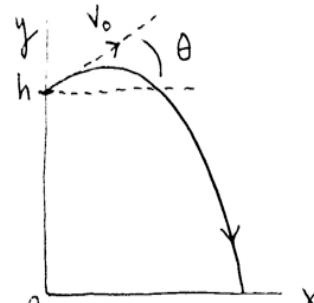
and so the position coordinates are given by

$$x = x_0 + v_0 \cos \theta t \quad \text{and} \quad y = y_0 + v_0 \sin \theta t - \frac{1}{2}gt^2$$

(If the projectile is aimed downward such that $\theta < 0$, then, indeed, $v_{0y} < 0$)

EXAMPLE WITH θ :

From a diving board 3 m above the water, a diver launches herself with an initial speed $v_0 = 2$ m/s at an angle of 60° above the horizontal. Determine the amount of time she is in the air.



Answer: This is a famous circumstance where only the vertical height matters and the x equation isn't needed.

We write down $y(t)$ and ask for the time $t = t_{\text{hit}}$ the diver hits the water.

Continued on the next page.

The vertical position is given by $y = y_0 + \underbrace{v_{0y}}_{v_0 \sin \theta} t - \frac{1}{2}gt^2$

At launch ($t = 0$), $y_0 = h$. At splashdown ($t = t_{\text{hit}}$), $y = 0$:

$$0 = h + v_0 \sin \theta t_{\text{hit}} - \frac{1}{2}g t_{\text{hit}}^2$$

or

$$0 = +3 + 2\left(\frac{\sqrt{3}}{2}\right) t_{\text{hit}} - 4.9 t_{\text{hit}}^2$$

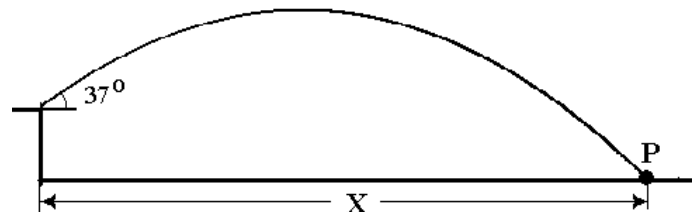
The quadratic solutions yield (for one significant digit)

$$t_{\text{hit}} = \begin{cases} 1 \text{ s} & \text{(the answer)} \\ -0.6 \text{ s} \end{cases}$$

The negative solution refers to the fact that, if we ran the trajectory back in time, and the diving board were out of the way, then this is the time when the diver would “return” to the water level!

Problem 3-3

A projectile is shot from the edge of a cliff 140 m above the ground level with an initial speed of 100 m/s at an angle of 37° with the horizontal, as shown.



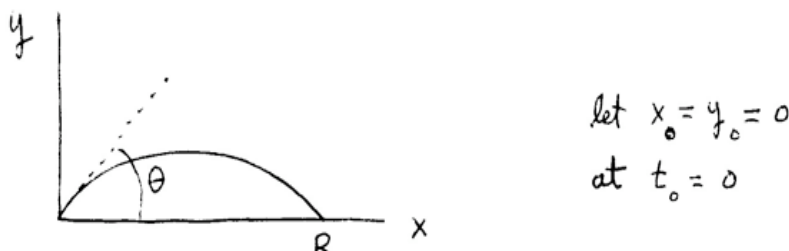
(a) Determine the time taken by the projectile to hit the point **P** on the ground level.

(b) Determine the range **X** of the projectile as measured from the base of the cliff.

GOOD OLD MAXIMUM RANGE (2D)

In studying 2D constant acceleration trajectories, we encounter the famous maximum range problem. (As always, the plane of the trajectory can be chosen to be the x-y plane.)

Range R: We consider a projectile shot off from ground level at an angle θ and define the range to be R (this is the horizontal distance traveled by the time the projectile returns to the ground).



1) x direction – we have the usual constant-velocity equation. It gives us the time that we reach the range R:

$$x = v_0 \cos\theta \, t$$

$$x = R \text{ at impact} \Rightarrow t = \frac{R}{v_0 \cos\theta}$$

2) y direction – we have the usual constant gravitational acceleration equation. It also gives us the time that we return to the ground level:

$$y = v_0 \sin\theta \, t - \frac{1}{2}gt^2$$

$$\text{And } y \text{ returns to zero at impact} \Rightarrow (v_0 \sin\theta - \frac{1}{2}gt)t = 0 \Rightarrow t = \frac{2v_0 \sin\theta}{g}$$

(The $t = 0$ solution checks with $y = 0$ at the beginning.)

3) So if we reach R at the same time we return to the ground level, these two times are the same, and setting them equal gives a relation between R, θ , and v_0

$$\text{We have } \frac{R}{v_0 \cos\theta} = \frac{2v_0 \sin\theta}{g} \text{ or } \boxed{R = \frac{v_0^2 \sin 2\theta}{g}} \text{ using } 2\sin\theta \cos\theta = \sin 2\theta$$

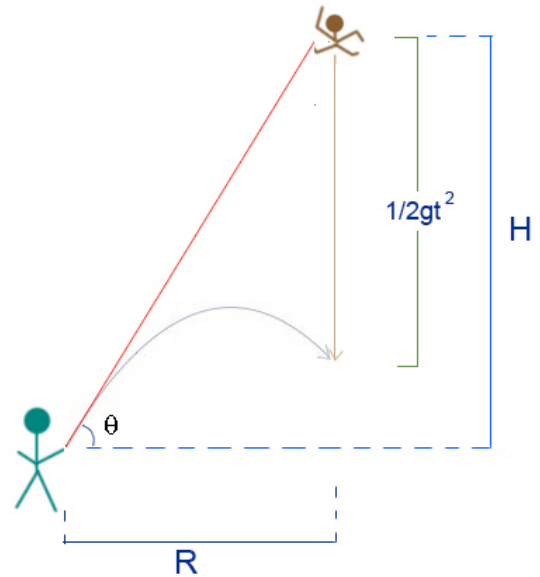
We get the famous result that R is a maximum at $\theta = \pi/4 = \boxed{45^\circ}$

Problem 3-4

Find a formula for the peak height H of the above projectile trajectory in terms of R and θ . (As a check: When R is a maximum, you should find $H = R/4$)

2D INTERSECTIONS

EXAMPLE: Suppose that a kid wants to throw a ball at a very bad monkey that is hanging from a tree branch located a long distance R away and a height H up from the kid's hand. The kid aims directly at the monkey and throws, but, at that instant in time, the branch breaks and the monkey falls straight down. This is another famous problem. Namely, the ball would not have hit the monkey if the tree branch had not broken off! But when the monkey begins to fall at the instant the kid lets go (say, $t=0$), the ball **always** hits the monkey. (The ball is also "falling" under gravity throughout its trajectory.)



SOLUTION: The **ball** position at time t is given by

$$x_B = v_{0x}t \quad \text{and} \quad y_B = v_{0y}t - \frac{1}{2}gt^2$$

for initial velocity components v_{0y} and v_{0x} and using the origin $x = y = 0$ to define the initial ball position.

Monkey position at time t : $x_M = R$ and $y_M = H - \frac{1}{2}gt^2$

They intersect when both x and y coordinates coincide at the same time:

$$\begin{aligned} x_M = x_B &\Rightarrow R = v_{0x}t \\ y_M = y_B &\Rightarrow H = v_{0y}t \end{aligned}$$

But this is nothing other than asking the kid to aim at the monkey! To prove this in gory detail, first note that these intersection equations give

$$\frac{H}{R} = \frac{v_{0y}}{v_{0x}} = \tan(\text{angle of the velocity vector})$$

Second:

$$\frac{H}{R} = \tan\theta$$

where θ is the angle of the red line shown above in the figure. Thus, by comparing these two H/R equations, we see the angle of the velocity vector must be the same as θ . The kid must aim at the original position of the monkey (independent of g !).

Problem 3-5 Sally and Mary are standing on top and at the edges of opposite walls of a canyon 100 m wide as shown. Sally throws a ball horizontally straight across at Mary and at a velocity of 10 m/s. **At that same instant**, Mary throws a ball from the same height, also horizontally aiming it at Sally. How fast must Mary initially throw her ball in order that the two balls collide? (Assume the canyon is deep enough so the balls don't hit bottom before colliding.) Draw a picture of the two trajectories.

