PHYS 121: Solutions to Written Homework #09
April 16, 2024

Note: this homework is worth **15 points** as follows:

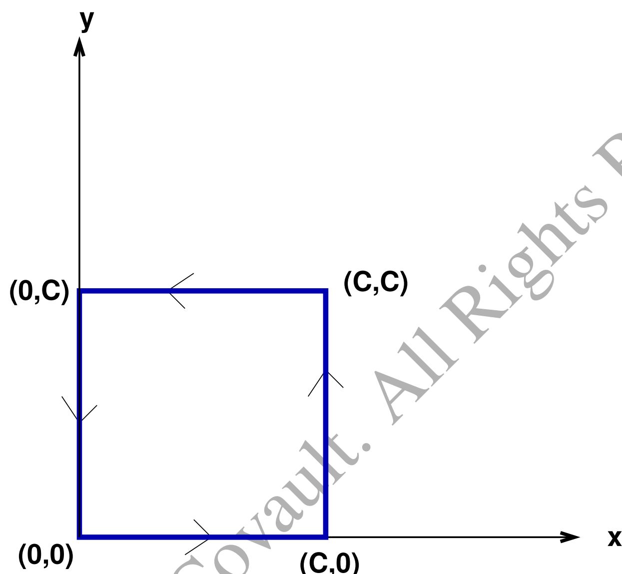
___	Problem 1	5 points	5 pts total
___	Problem 2	5 Points	5 pts total
___	Problem 3	5 points	5 pts total

Problem 1: The Path Integral

Consider a **new force** which is defined as a function of two-dimensional coordinates this way:

$$\vec{F} \equiv Ay\,\hat{\boldsymbol{\imath}} + Bx^4\,\hat{\boldsymbol{\jmath}}$$

Here x and y are standard Cartesian coordinates and A, and B are given real positive constants with appropriate units.



Consider the loop that goes around a square path as show in the figure. Each side of the square has a given length C.

Part a) What is the **Work** done by this specific force $\vec{F}$ for the closed path that goes around the loop in the *counter-clockwise* direction, starting from the origin $(0, 0)$? Important: You must *explicitly calculate* the path integral here. Explain your work. Give your answer in terms of the given parameters.

Part b) Based on your calculation from Part (a), answer this question: Is this force $\vec{F}$ Conservative or Not Conservative? Cite specific evidence and/or examples to support your claim either way. Explain your work completely here to get full credit.

Important: Problem 1 continues next page.....

Problem 1 continues .....

Possibly helpful hints for Problem 1: You **cannot** assume that just because the path is closed the integral is zero. You need to **calculate** the path integral explicitly here. Don't forget the Definition of Work on a path that goes from point P to point Q is given by:

$$\mathcal{W}_{\vec{F}}(P \to Q) \equiv \int_P^Q \vec{F} \cdot d\vec{r} = \int_{x_P}^{x_Q} F_x dx + \int_{y_P}^{y_Q} F_y dy$$

where F_x is the x-component of the force and F_y is the y-component of the force.

In order to calculate the Work done by the force around the square loop, you will need to calculate the Work done on *each* of the four "straight-line" pieces of the path. In other words, you need to break the closed-path integral into four regular 1-D integrals. For each "straight-line" piece you are holding one coordinate *fixed* at a *known value* and integrating with respect to the other coordinate.

SOLUTION TO PROBLEM 1:

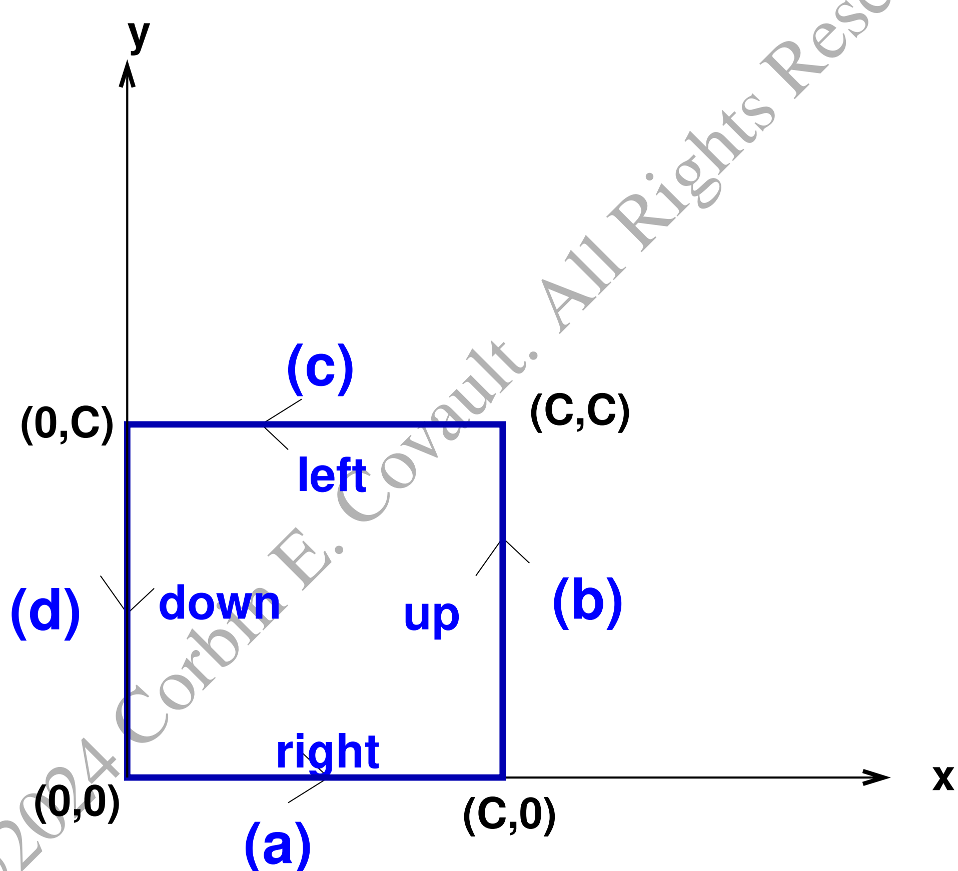
Part (a):

Okay the Work is defined as a **path integral** according to:

$$\text{Work}_{\vec{F}} \equiv \oint_{path} \vec{F} \cdot \vec{dr}'$$

where $\vec{dr}'$ is the differential distance moving along the path in the clock-wise direction.

Since the path is a square, we can break the complete path integral around the square loop into the *sum* of four path integrals, each along one side as shown in the figure below:



$$\text{Work}_{\vec{F}} \equiv \oint_{\text{path}} \vec{F} \cdot \vec{dr}' = \int_{\text{Path (a)}} \vec{F} \cdot \vec{dr}' + \int_{\text{Path (b)}} \vec{F} \cdot \vec{dr}' + \int_{\text{Path (c)}} \vec{F} \cdot \vec{dr}' + \int_{\text{Path (d)}} \vec{F} \cdot \vec{dr}'$$

We see that each of these paths is extremely easy to write down as a simple *regular integral* in a particular dimension. In particular the paths labeled (a) and (c) integrate with respect to "x" only, and paths (b) and (d) integrate with respect to "y" only. The dot-product means that we *only integrate the corresponding component of the force:*

$$\text{Work}_{\vec{F}} = \int_{\text{Path (a)}} F_x\,dx' + \int_{\text{Path (b)}} F_y\,dy' + \int_{\text{Path (c)}} F_x\,dx' + \int_{\text{Path (d)}} F_y\,dy'$$

Here the primes are attached to the particular variables that we are going to integrate over. This will be very helpful since sometimes we will be integrating over x while holding y constant and vice versa. Remember that each component of the force F_x and F_y is (in general) a function of both x and y. So we use the primes to indicate which variable is being integrated on each path. More explicitly:

$$\text{Work}_{\vec{F}} = \int_{\text{Path (a)}} F_x(x',y)\,dx' + \int_{\text{Path (b)}} F_y(x,y')\,dy' + \int_{\text{Path (c)}} F_x(x',y)\,dx' + \int_{\text{Path (d)}} F_y(x,y')\,dy'$$

We can look at the path to see the actual value of the variable that is not changing, corresponding to a "constant" value on that path. For Path (a) the value of y is held constant at $y = 0$. For Path (b) the value of x is held constant at $x = X$. And so forth around the square. And for each part of the path we also know exactly the limits and the values of the force function.

$$\text{Work}_{\vec{F}} = \int_{x'=0}^{x'=C} F_x(x', y=0)\,dx' + \int_{y'=0}^{y'=C} F_y(x=C, y')\,dy'$$
$$+ \int_{x'=C}^{x'=0} F_x(x', y=C)\,dx' + \int_{y'=C}^{y'=0} F_y(x=0, y')\,dy'$$

Now we look at the actual details of the force function. $\vec{F} = F_x\,\hat{\imath} + F_y\,\hat{\jmath}$. Since $F_x = F_x(x,y) \equiv Ay$ and $F_y = F_y(x,y) \equiv Bx^4$ then we can plug these values into the integral substituting for the "constant"(non-primed) variables that we already know for each path:

$$\text{Work}_{\vec{F}} = \int_{x'=0}^{x'=C} (0)\,dx' + \int_{y'=0}^{y'=C} (BC^4)\,dy' + \int_{x'=C}^{x'=0} (AC)\,dx' + \int_{y'=C}^{y'=0} (0)\,dy'$$

Notice that each of these corresponds to a *remarkably simple integral*. So we take the definite integral for each part, integrating the primed coordinates:

$$\text{Work}_{\vec{F}} = (0)\Big]_{x'=0}^{x'=C} + (BC^4)y'\Big]_{y'=0}^{y'=C} + (AC)x'\Big]_{x'=C}^{x'=0} + 0\Big]_{y'=C}^{y'=0}$$

$$\text{Work}_{\vec{F}} = 0 + (BC^4)C - (AC)C + 0$$

$$\text{Work}_{\vec{F}} = 0 + BC^5 - AC^2 + 0$$

$$\boxed{\text{Work}_{\vec{F}} = (BC^3 - A)C^2}$$

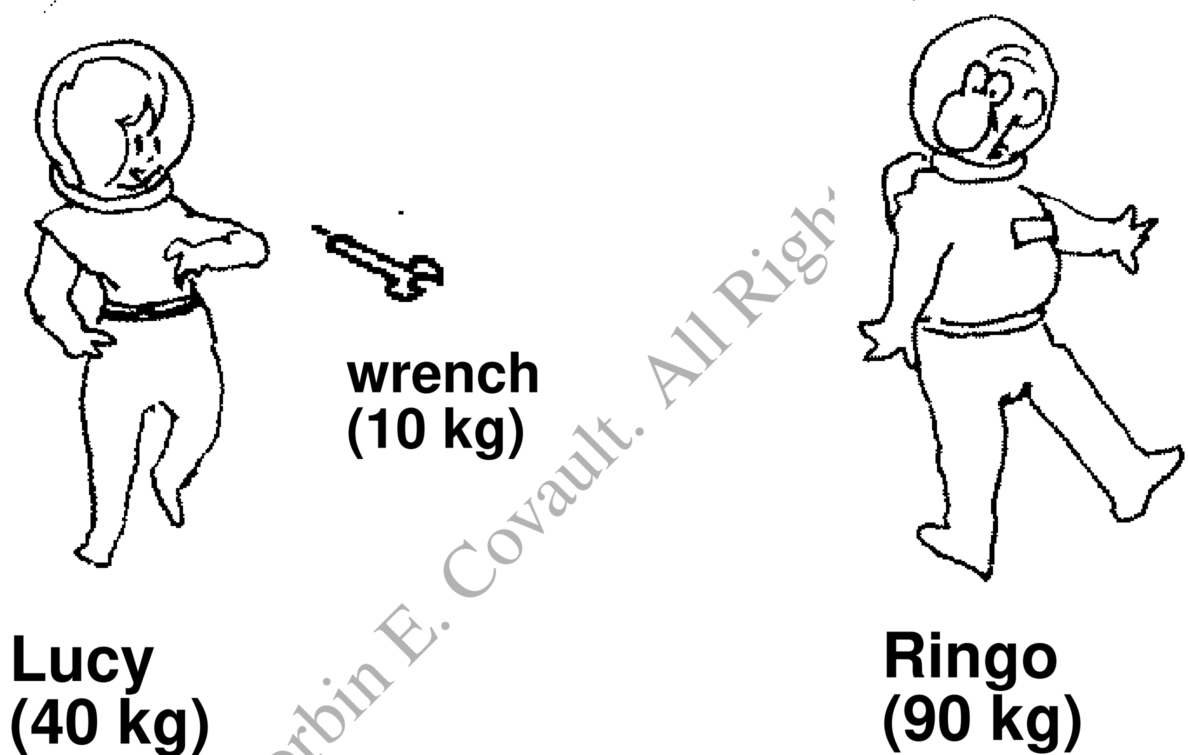
Part (b):

We know that **if a force is conservative, then the Work done over any closed path must be zero.** In this case, based on our answer for Part (a), we see that the Work done around the loop is *not zero* (unless we somehow specifically set $BC^3 = A$). So for general values of A, B and C, the force is not conservative.[1]

[1] Actually even if we define the force itself so that $BC^3 = A$ for one particular value of C corresponding to one particular path, so that the work for this one path is zero, the force itself is **Not Conservative** because the path integral is not zero for all possible values of C. In other words, even if we select values so that the integral around this particular path is zero, we know there are paths where the integral is non-zero.

Problem 2: Throw and Catch Revisited

Two astronauts, Lucy, and Ringo, are floating free in deep space, as shown. Each astronaut is defined to have zero velocity. Lucy has a mass m_L of 40 kg. She holds a wrench of mass m_w of 10 kg. Ringo has a mass m_R of 90 kg. Lucy throws a the wrench at Ringo. She throws the wrench so that it moves towards Ringo at a translational speed v_w of 5 meters/second. She also puts a rather good spin on the wrench, so that it rotates with an angular velocity ω_w of 16 radians per second. Ringo then catches the wrench. Assume that the wrench has a rotational inertia given as $I_{wrench} = 0.2$ kg·m^2 and assume that Lucy has a rotational inertia that is 80 times that of the wrench. Assume Ringo has a rotational inertia that is 125 times that of the wrench.



Important: Solve this problem **symbolically** first, then plug in numbers at the end.

a) What is Lucy's translational velocity after she throws the wrench?

b) What is Ringo's translational velocity after he catches the wrench?

c) What is Lucy's rotational velocity after she throws the wrench?

d) What is Ringo's rotational velocity after he catches the wrench?

SOLUTION TO PROBLEM 2 (from a previous exam):

Part a) Lucy throws a wrench. **This is an explosion which is physically the inverse of a totally inelastic collision**. The system is *isolated* so we have **met the Condition for the application of Conservation of Linear Momentum**.

We define the "Before" as before the wrench is thrown. We define the "After" as immediately after the wrench is throw. We define the *system* as **Lucy-and-Wrench**. Before Lucy throws the wrench, she is at zero velocity so therefore *the Lucy-and-Wrench system initially has zero total momentum:*

$$\text{“}BEFORE'' = \text{“}AFTER''$$

$$P_{tot} = P'_{tot}$$

$$0 = p'_L + p'_w$$

$$0 = m_L v'_L + m_w v'_w$$

$$\boxed{v'_L = -\frac{m_w}{m_L} v'_w}$$

Plugging in numbers:

$$v'_L = -\frac{(10kg)}{(40kg)}(5m/s)$$

$$\boxed{v'_L = -1.25 \text{ m/s}}$$

Part b) Ringo catches the wrench: The wrench and Ringo *stick* together, so this is by definition a *Totally Inelastic Collision*. Again, we *cannot* use Conservation of Energy because the forces involved are *not conservative*. Again we use **Conservation of Linear Momentum** but now *the system is now defined as wrench-and-Ringo* instead of wrench-and-Lucy.

$$\text{“}BEFORE'' = \text{“}AFTER''$$

$$P_{tot} = P'_{tot}$$

$$p_w + p_R = P'$$

$$m_w v_w + 0 = MV'$$

$$m_w v_w = (m_w + m_r)V'$$

$$\boxed{V' = \frac{m_w}{m_w + m_r} v_w}$$

$$V' = \frac{10kg}{10kg + 90kg}(5m/s)$$

$$\boxed{V' = 0.5 \text{ m/s}}$$

Part c)

For this part we need to contend with **Rotational Motion**. We have no idea what the torques or angular accelerations are on the wrench or Lucy so applying the rotational form of Newton's Second Law will not help us. But we do know the system is completely **isolated** so it **meets the condition Conservation of Angular Momentum** which we again apply[2] to the original *Lucy-and-wrench system*. As before, we argue that since Lucy is at rest when she throws the wrench the initial angular momentum has to be *zero*:

$$\text{“}BEFORE'' = \text{“}AFTER''$$

$$L_{tot} = L'_{tot}$$

$$0 = L'_L + L'_w$$

$$0 = I_L \omega'_L + I_w \omega'_w$$

$$\boxed{\omega'_L = -\frac{I_w}{I_L}\omega'_w}$$

Plugging in numbers:

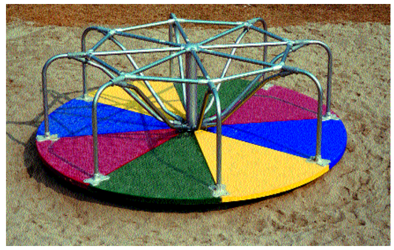
$$\omega'_L = -\left(\frac{1}{80}\right)(16 \text{ radians/s})$$

$$\boxed{\omega'_L = -0.2 \text{ radians/s}}$$

Part d) $\boxed{\text{You cannot use Conservation of Energy}}$ in this problem because throwing and catching of the wrench are **inelastic collisions**. Energy is only conserved in *totally elastic* collisions.Furthermore, the forces involved (grabbing wrenches) are **not Conservative**. You can only use Conservation of Mechanical Energy if you meet the **Conditions** that the forces are all either Conservative or that they *do zero work*. In this case, the force that Lucy applies to the wrench pushes it toward Ringo so she is clearly doing some non-zero work on the wrench.

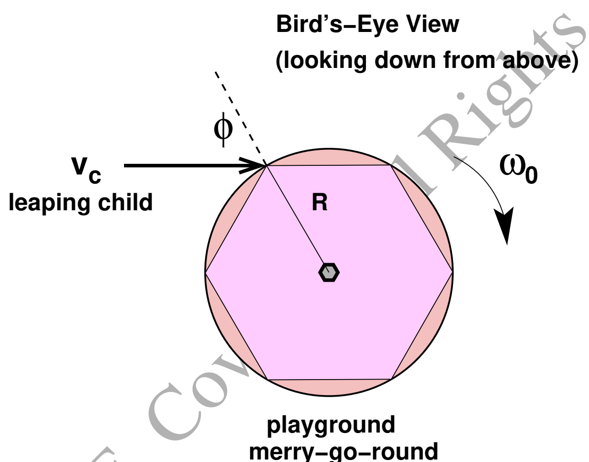
[2]We are completely glossing over a critical detail here, and that is the constraint that Lucy throw the wrench so that is moves in a straight line directly outward from her own center-of-mass. If Lucy throws the wrench on some other line, then in principle there is also angular momentum associated with the non-radial translational motion.

Problem 3 – From a Previous Exam



A playground merry-go-round.

A common recreational fixture in children's outdoor playgrounds is the "merry-go-round" which is generally a large circular platform mounted on a central bearing so that the platform can spin freely.



A "bird's-eye-view" of a merry-go-round is shown above. Assume that the bearing is ideal and that the total rotational inertia of the merry-go-round alone is given as I_m. Assume a small child with given mass m leaps from the ground with horizontal velocity given as v_c and then lands and sticks on the merry-go-round at a given distance R from the center. Assume that the child impacts the merry-go-round at a given angle ϕ relative to the radial direction as shown. Treat the child as a point-like-object. Ignore all vertical motion of the child. Assume the merry-go-round is initially moving with an angular speed of ω_0 in the clock-wise direction as shown.

Part (a) – What is the magnitude of the total angular momentum L_{tot} for the combined child-plus-merry-go-round system in the instant just before the child comes in contact with the merry-go-round as calculated for the pivot point corresponding to the central axle of the merry-go-round? Express your answer in terms of the given parameters. Explain your work.

Part (b) – Assume the merry-go-round is initially moving with a given angular speed of ω_0 but then comes up to angular speed ω_1 just after the child leaps on. Calculate the value of ω_1. Express your answer in terms of the given parameters. Explain your work.

Part (c) – Now assume that the child subsequently pushes straight off the merry-go-round so as to land just off the outside edge on the ground with precisely zero horizontal velocity. What is the angular speed ω_2 of the merry-go-round after the child performs this maneuver? Express your answer in terms of the given parameters. Explain your work.

Solution to Problem 3:

Part (a)–

- **Totally Inelastic Collision:** – Conservation of Mechanical Energy **cannot** be used.
- **System is not isolated to net force:** due to connection to axle to ground. Conservation of Linear Momentum **cannot** be used.

Part (b)–

The total angular momentum of the system that includes the child and the merry-go-round is just the sum of these:

$$\vec{L}_{tot} = \vec{L}_c + \vec{L}_m$$

As a **point-like-object**, the angular momentum of the child is given by:

$$\vec{L}_c = \vec{r} \times \vec{p}_c$$

where $\vec{r}$ is the radial vector from the pivot point to the location of the child. The magnitude can be inferred from the standard rule for cross-products:

$$L_c = rmv \sin\phi$$

Here ϕ is the angle between the radial vector and the direction of momentum as shown in the figure. The length of r is just R so:

$$L_c = mv_c R \sin\phi$$

The angular momentum of the merry-go-round is given by its rotation:

$$L_m = I_m \omega_0$$

Totaling up, then:

$$\vec{L}_{tot} = L_c + L_m$$

$$\boxed{L_{tot} = mv_c R \sin\phi + I_m \omega_0}$$

Solution continues next page....

Part (c)– :

Since the child lands on the merry-go-round and "sticks", this is a "totally inelastic collision". If we treat the airborne child and the merry-go-round as a system, we can argue that there are no external torques on the system, and therefore we can use **Conservation of Angular Momentum**: Defining the clockwise direction as "positive" we work just the magnitudes:

$$L_{tot} = L'_{tot}$$
$$L_c + L_m = L'_c + L'_m$$
$$L_c + L_m = L'_c + L'_m$$
$$mv_c R \sin\phi + I_m\omega_0 = mv'_c R + I_m\omega_1$$

We note that the child's angular momentum after landing on the merry-go-round is defined by his uniform-circular-motion, which is connected to the rotation rate of the merry-go-round by the **Rolling Constraint:** $v'_c = \omega_1 R$ and so:

$$mv_c R \sin\phi + I_m\omega_0 = m(\omega_1 R)R + I_m\omega_1$$
$$m\omega_1 R^2 + I_m\omega_1 = mv_c R \sin\phi + I_m\omega_0$$
$$\omega_1(mR^2 + I_m) = mv_c R \sin\phi + I_m\omega_0$$
$$\boxed{\omega_1 = \frac{mv_c R \sin\phi + I_m\omega_0}{mR^2 + I_m}}$$

Solution continues next page....

Part (d)– :

Now we have the **reverse inelastic collision** also called an *explosion*. Again, angular momentum is conserved:

$$L_{tot} = L'_{tot}$$
$$L_c + L_m = L'_c + L'_m$$

Since the child comes to rest, the child's angular momentum after jumping off is zero:

$$L_c + L_m = 0 + L'_m$$

We use our prior result for total angular momentum:

$$L'_m = mv_c R \sin\phi + I_m \omega_0$$
$$I_m \omega_2 = mv_c R \sin\phi + I_m \omega_0$$
$$\omega_2 = \frac{mv_c R \sin\phi + I_m \omega_0}{I_m}$$
$$\boxed{\omega_2 = \frac{mv_c R \sin\phi}{I_m} + \omega_0}$$