Chapter 14+

Revisit Moments of Inertia and Angular Momentum

Revisit:	To-Do:
• Moments of inertia • Rotational second law • Rotational kinetic energy, work, and angular momentum	• List of rotational variables • New moments of inertia • Pulleys with mass

Rotational Formulas Analogous to Linear Formulas

Let's gather together our rotational variables now that we have them all (thanks to Ch. 12+). We have $\tau = I\,\alpha$ as the rotational second-law analog of the linear second law, $F = ma$, and we saw rotational analogs of the work-energy equation, the kinetic energy formula, and linear momentum. We show the specific analogs of each variable in the table found on the right. For 1D, we have $x \Rightarrow \theta$, $v \Rightarrow \omega$, $a \Rightarrow \alpha$, but this does not mean $x = \theta$, etc. They are not even the same units! It just means x and θ play analogous roles in equations for linear and rotational motion, respectively.

Linear Motion Variable	**Rotational Analog**
x	θ
v	ω
a	α
F	τ
m	I
$K = \frac{1}{2} mv^2$	$K = \frac{1}{2} I\,\omega^2$
$W = F\,\Delta x$	$W = \tau\Delta\theta$
$p = mv$	$L = I\,\omega$

Moreover, you often can generate the analogous equations just by substituting $x \to \theta$, $v \to \omega$, $a \to \alpha$, $F \to \tau$, $m \to I$, … For example, the linear work-energy equation $W = F\,\Delta x = \Delta K$, with $K = \frac{1}{2}mv^2$, is changed to $W = \tau\,\Delta\theta = \Delta K$, with $K = \frac{1}{2}I\,\omega^2$, both having the same units of energy (Joules). But when we go from $p = mv$ to angular momentum $L = I\omega$, linear momentum and angular momentum have different units and separate conservation laws. For instance, the total linear momentum is always zero for a rotating cylinder whose axis is fixed in place (and thus it is not a useful number) because for every little piece with linear momentum there is another piece on the other side of the symmetry axis with equal and opposite linear momentum canceling it. But its total angular momentum is not zero (and therefore it <u>is</u> a useful number – see our various discussions and problems on angular momentum).

Anyway, back to the old meat grinder …

The Moment of Inertia of a Uniform Solid Cylinder (or a disk)

For many applications, especially a solid pulley, we need the moment of inertia for a uniform solid cylinder about its symmetry axis (the axis parallel to the cylinder and through its center). We often refer to moments of inertia relative to the axis of symmetry the CM moments of inertia, I_{CM} .

Recall a hollow uniform cylinder has $I_{CM} = MR^2$ for its symmetry axis. (All the mass is out at radius R.) We can use this to help us derive the solid cylinder result as follows.

To find the moment of inertia of a solid cylinder around its symmetry axis, think of it as a limiting sum of a stack of concentric hoops with differential mass dm (each with differential $dI = dm\, r^2$), going over into an integral:*

$$I = \sum_i \Delta m_i r_i^2 \rightarrow \int dm\, r^2$$

Now dm = ρ dV for mass volume density ρ and volume of the little hoop dV = 2πr dr L where the radius of the hoop is r and its small width is dr. So we can rewrite the integral as

$$I = \int dm\, r^2 = \rho 2\pi L \int_0^R r\, dr\, r^2$$

We now need to do the integration

$$\int_0^R r^3\, dr = \left.\frac{r^4}{4}\right|_0^R = \frac{R^4}{4} - 0 = \frac{R^4}{4}$$

With mass volume density for the uniform solid cylinder equal to $\rho = M/(\pi R^2 L)$, we gather the pieces together and obtain

$$I = \frac{M}{\cancel{\pi} R^2 \cancel{L}} 2\cancel{\pi}\cancel{L} \frac{R^4}{4} = \frac{1}{2} MR^2$$

Therefore,

$$\boxed{I_{CM} = \frac{1}{2} M R^2} \left\{ \begin{array}{l} \text{for a uniform SOLID cylinder, radius R, mass M, and the axis along} \\ \text{the center (i.e., the CM moment of inertia) and parallel to the cylinder} \end{array} \right.$$

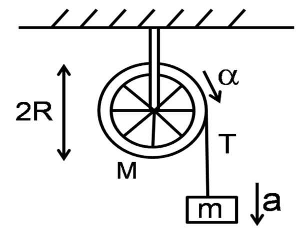
This moment of inertia is especially handy for short solid cylinders, i.e., disks, and we'll use this later for solid pulleys.

* Here and later in this chapter we show you how to do some of the integrals in these calculations **but this is not a math class** and these are just shown to you because we think you can understand them fairly easily, we think you'll like to see how they work out, and it may help motivate your math class.

Pulleys with Mass

We revisit pulleys and now add the effect of their mass through their moment of inertia. If we include the pulley inertia, there will be **changes in the tension for ropes around pulleys with mass**, as this inertia "eats up" part of the force, and there will be generally less acceleration of the rest of the system for a given set of external forces.

Consider the **hoop** pulley example shown where the weight mg falls with acceleration a as the pulley with mass M unwinds. There is tension T in the rope. We need two equations to determine a and T, and we will get them from the second law for m and the rotational second law for the pulley.



The second law for the vertical motion of m is

$$mg - T = ma$$

For the rotational second law, we naturally use the pulley rotation axis for our torque calculation, and the hoop moment of inertia MR^2 for this pulley. There is no torque due to the weight of the pulley or the support post around the rotation axis since those forces are directed at the pulley center (the torque angle $\theta = 0$ for both) and the torque due to the rope is simply +TR (we will use CW as + for convenience here – this is just like the freedom to choose the positive x-axis to point to the left!).

The rotational second law $\tau_{net} = I\alpha$ reads:

$$+TR = MR^2\alpha \quad \Rightarrow \quad T = MR\alpha$$

where we also define α as positive if the angle is accelerated in the CW direction.

But 1) the tangential acceleration of the pulley is the **same as the acceleration of the mass**, $a_{tan} = a$, and 2) it is recalled from Ch. 12 that the tangential acceleration of the rim of the pulley is **related to the angular acceleration** by $a_{tan} = \alpha R$. Thus $a = \alpha R$ and we simply have

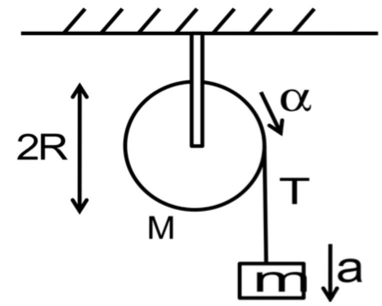
$$T = Ma$$

Thus we stick T = Ma into mg – T = ma, and solve for a, and then T. We get a constant acceleration and tension:

$$\boxed{a = \frac{m}{m + M}g < g} \quad \text{and} \quad \boxed{T = \frac{M}{m+M}mg}$$

The massive pulley has eaten up some of the gravity force on m, since $\frac{M}{m+M} < 1$.

Also, note that a different moment of inertia will change all of this. See the upcoming problem, where, since uniform disks are good approximations for pulleys, we use $\frac{1}{2}MR^2$ for a more appropriate pulley moment of inertia.



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Problem 14-4

a) Following the text example, find the tension T *and the acceleration* a *downward of the mass* m *unwinding from the pulley configuration as shown for a* ***solid uniform cylindrical pulley*** *of radius* R *and mass* M*.*

b) Suppose the system starts from rest. You see from (a) that the acceleration is constant. Find the kinetic energy of m *and the rotational kinetic energy of* M *after the mass* m *has fallen a distance* d*, in terms of* m, M, a*, and* d*. BUT do this by using the constant acceleration formulas, where you first find the velocity of* m *and the angular velocity of the pulley when the mass* m *has fallen a distance* d*.*

c) Now substitute for a *from your answer in (a) and get the kinetic energy of* m *and the rotational kinetic energy of* M *after the mass* m *has fallen a distance* d*, in terms of* m, M, g*, and* d*.*

d) During the time interval considered in (b), how much work does gravity do on m*, how much work does the tension force do on the pulley (recall the work-torque relation from p. 12-16), and how much work does the tension force do on* m*? How does the total work done compare with the total final kinetic energy in (b)?*

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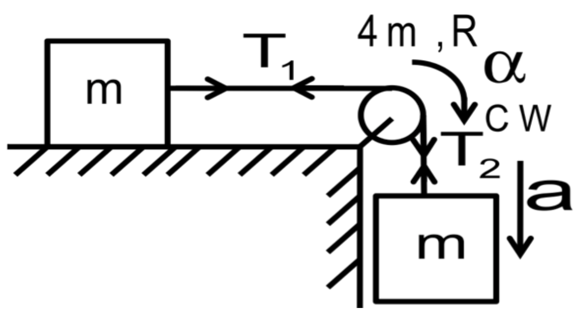
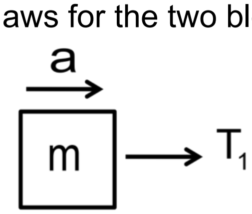
Add Pulley Mass to Other Block Examples

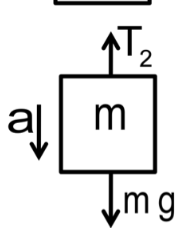
Looking at the figure found below for **frictionless** motion, we see three unknowns: a, T_1, and T_2. These three unknowns can be found from the three Newton's second laws for the **translation of the two masses** and for the **rotation of the pulley**. See the three equations below:



Drawing only the parts of the FBD's we need, the translational (as opposed to rotational) second laws for the two blocks are

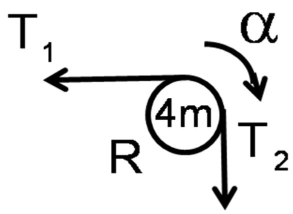


$$\Rightarrow \boxed{T_1 = ma}$$



$$\Rightarrow \boxed{mg - T_2 = ma}$$

Calling CW + and drawing only the part of the XFBD generating nonzero torques (the pivot force and pulley weight are needed for force balance but generate zero torque around the pulley pivot), we have the rotational second law for the pulley:



$$\Rightarrow\ T_2R - T_1R = I\alpha = \frac{1}{2}\left(4mR^2\right)\alpha = 2mR^2\,(a/R)$$

or $\boxed{T_2 - T_1 = 2ma}$

where we have used our new solid disk moment of inertia (½ mass radius^2) and the angular acceleration – tangential linear acceleration relation a = αR for circular motion, from Ch. 12 (and we refer the reader back to the remarks on p. 14-10).

To solve these fast, just **add all three equations together** (the three left-hand sides add up to mg with all the tensions cancelling!), giving

$$mg = 4ma \quad \text{or} \quad a = \tfrac{1}{4}\,g$$

(we could have guessed this from an FBD for the overall system)

Then just shove this back into the first two equations to get the tensions.

Answers: $\boxed{a = \frac{1}{4}g \qquad T_1 = \frac{1}{4}mg \qquad T_2 = \frac{3}{4}mg}$

Notice how the tension changes from one side to the other of the pulley, owing to the pulley inertia, which eats up part of the tension force.

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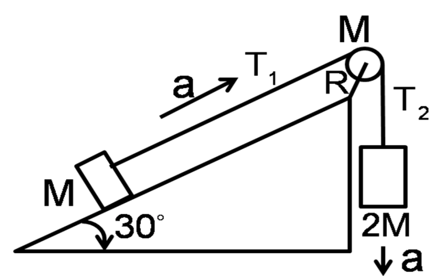
Problem 14-5 *A block of mass* M *on a* ***frictionless*** *30° inclined plane is connected to a hanging block of mass* 2M *by a string which passes over a* ***pulley of mass*** **M** ***and radius*** **R.** *Assuming that there is no slippage between the string and the pulley, find the linear acceleration a of the system and the tensions* T_1 *and* T_2 *in the parts of the string attached to the masses* M *and* 2M*, respectively.*



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Going beyond pulleys, we need yet more moments of inertia to help us write down the rotational second law and also to expand the usefulness of the energy and angular momentum conservation short cuts.

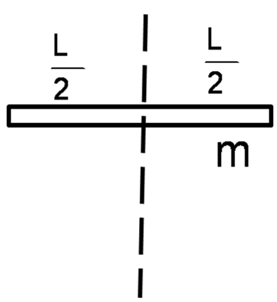
The CM Moment of Inertia of a Uniform Straight Thin Rod (a stick!)

We often need the moment of inertia of a thin stick around the axis through, and perpendicular to, its center. This is the CM moment of inertia for the stick or rod and its formula (to be derived below) is

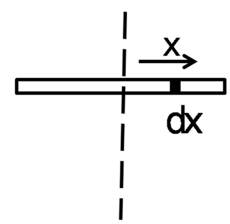
$$\boxed{I_{CM} = \frac{1}{12} M L^2} \quad \begin{cases} \text{for a uniform rod, length L, mass M, and the axis through and} \\ \text{perpendicular to the } \textbf{center} \text{ of the rod (i.e., the CM moment of inertia)} \end{cases}$$

This will be valuable for rotating propellers, etc. and it also applies to thin rectangular plates if the axis lies in the plane of the plate and goes through the plate CM and is perpendicular to one of the two edges of the plate.

Derivation: To find the moment of inertia of a thin rod around an axis perpendicular to its center, think of it as a limiting sum of increments over the length of the rod (say, along the x axis), each with differential mass dm (each with differential $dI = dm\, x^2$), going over into an integral:



$$I = \sum_i \Delta m_i x_i^2 \rightarrow \int dm\, x^2$$



The little increments of mass are $dm = \lambda\, dx$, for mass linear density λ (mass per unit length) for differential length dx. So we can rewrite the integral as

$$I = \int dm\, x^2 = \lambda \int_{-L/2}^{L/2} dx\ \ x^2$$

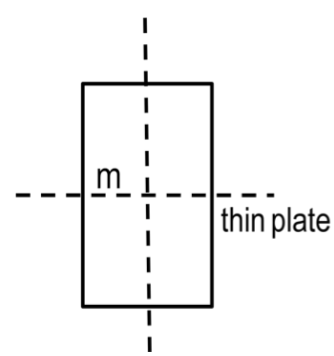
We now need to do the integration

$$\int_{-L/2}^{L/2} dx\, x^2 = \frac{1}{3}\left(\frac{L}{2}\right)^3 - \frac{1}{3}\left(-\frac{L}{2}\right)^3 = \frac{1}{3}\ \frac{2}{8}\ L^3$$

Since the mass linear density for the uniform rod is equal to $\lambda = M/L$, we combine everything to get

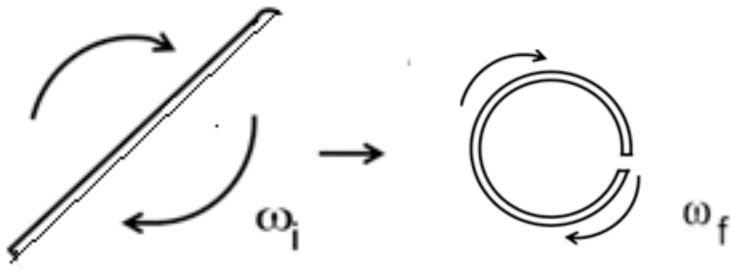
$$I = \frac{M}{L}\frac{1}{12}L^3 = \frac{1}{12}ML^2$$

as advertised. This is also applicable to a thin plate for either axis shown as dashed lines in the figure on the right.



Revisit Zero Net Torque Around Some Axis During Motion: Angular Momentum Conservation!

A snake does funny things after a girl does bad things: A slender snake of length ℓ has fallen asleep (*do they close their eyes?*) on an ice pond and is frozen stiff and straight as a rod. A mischievous girl comes along and spins the snake about an axis perpendicular to it and through its CM with an angular frequency ω_i. Then the snake wakes up, thaws out, and unstiffens (by internal forces!), forming a perfect circle, all the while spinning. Ignoring friction because of the smooth ice, what is the relation between the final angular frequency ω_f and the initial ω_i?



Again, to have a shortcut to this prediction, we ask what is conserved here? Think about whether energy, linear momentum, or angular momentum is conserved BEFORE you see the explanations below. **DON'T LOOK YET :) OK, now look:**

i) While the total linear momentum of the snake is conserved (there is no net external force (the vertical forces from the pond "normal force" support and gravity cancel locally everywhere), the CM velocity is always zero and this doesn't help us learn anything about the rotational motion before and after.

ii) The snake's internal forces are at work and thus the mechanical energy is not separately conserved. It can start from rest and move, without external energy, after all. We would have to include its food intake to get overall conservation!

iii) Recall that the total angular momentum is conserved if there is no net external torque. That is, we saw

$$\boxed{\tau = \frac{dL}{dt}} \quad \text{if } \tau \text{, L both calculated with respect to the same axis}$$

We see

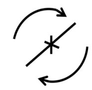
If $\tau = 0$, then L = constant.

For **our fixed-axes rotational motion,** $\boxed{L_P = I_P\,\omega}$ **would stay constant if there is no net (total) external torque around the axis P.** Just as internal forces are allowed in the conservation of linear momentum of a system, internal torques are allowed in the conservation of angular momentum.

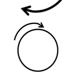
Let's get back to the snake.

Angular momentum of the snake around the CM axis (perpendicular to the snake) **IS** conserved because there is **no external torque around that axis** (or any other axis, for that matter). The CM won't move and the original spinning axis through it is perfect, so let's use it! As we just said, the angular momentum for fixed-axis (pure) rotation is given by $L_{CM} = I_{CM}\,\omega$, where I_{CM} is defined with respect to the CM rotation axis. We can use the fact that $I_{CM}\,\omega$ is constant in time to predict the final angular velocity ω_f:

Initially, a straight snake stick: $(I_{CM}\,\omega)_i = \frac{1}{12} M\,\ell^2\omega_i$



Finally, a snake hoop: $(I_{CM}\,\omega)_f = M r^2\omega_f$



with ℓ the snake length and r its radius when it forms a perfect circle. Hence $\ell = 2\pi r$. Conservation of angular momentum, $(I_{CM}\,\omega)_i = (I_{CM}\,\omega)_f$, gives us

$$\frac{1}{12} M\,\ell^2\omega_i = M r^2\omega_f \qquad \text{or} \qquad \omega_f = \frac{1}{12}\frac{M\,\ell^2}{M r^2}\omega_i = \frac{1}{12}\frac{(2\pi r)^2}{r^2}\omega_i$$

We find $\boxed{\omega_f = \frac{\pi^2}{3}\omega_i}$ The snake is spinning faster by more than a factor of three.

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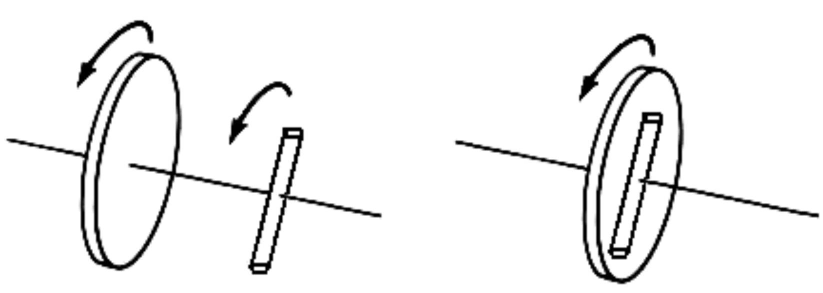
Problem 14-6

A disk and a rod are rotating without friction about their centers with ω_d = 800 rad/s and ω_r = 1200 rad/s in the same direction as shown in the figure (i), and both are perpendicular to their common rotation axis. The disk has mass 2.0 kg and radius 1.0 m. The rod has mass 1 kg and total length 1.5 m. A clutch mechanism (which generates no torque around the axis) brings them together and, after some slipping, they rotate together as shown in (ii), but still about their centers and still perpendicular to their common rotation axis.

(a) What is the final angular velocity of the pair?

(b) How much thermal energy is generated in this process?



(i) (ii)

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