Chapter 12++

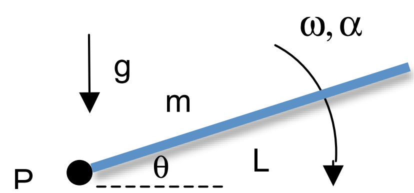
Revisit Circular Motion

Revisit:	To-Do:
• Angular variables • Second laws for radial and tangential acceleration • Circular motion • CM 2nd Law with F_{net}	• Vertical circular motion in gravity • Complete the rotating rod story by combining the force, energy conservation, and torque equations • Tension within rotating rods

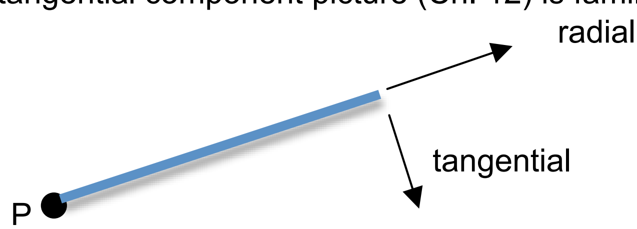
Vertical Circular Motion in Constant Gravity

Rod Swinging Vertically on a Pivot

We started the story of the falling rod or stick pivoted on one end in Ch. 9++ and continued it in Ch. 11++ and we will complete it here. Note that the results also apply to a falling trap door, because the moment of inertia for a thin door viewed "on edge" has the same form as that for a thin rod.



I) **Review:** The radial and tangential components of the **force on the rod due to the pivot** are related by the **translational second law** to the angular velocity ω and the angular acceleration α (with the above picture, ω and α have been defined to be positive in the **CW direction**, for convenience). By now, we hope that the radial and tangential component picture (Ch. 12) is familiar to the reader:



The radial and tangential components of the translational second law $\vec{F}_{net} = M\vec{a}_{CM}$ have been found in the Ch. 11++ text and the problem solution, respectively:

$$\boxed{F_{P,\,radial} - mg\sin\theta = -m\omega^2 L/2} \quad \textbf{A} \qquad \text{(p. 11-17)}$$

and

$$\boxed{F_{P,\,tangential} + mg\cos\theta = +m\alpha L/2} \quad \text{Problem 11-8a}$$

(continued)

In summary, if we knew the angular velocity ω and the angular acceleration α, for a given θ, the force vector due to the pivot on the rod would be completely determined by these equations. And by the way, in the design of doors and bridges and ladders, etc., we need to know these forces for personal and structural safety. The degree to which materials can withstand and bear up under weights and other forces is vital to engineering design. *This is the alpha and omega of our physics study!* ☺

II) **Review:** The angular velocity $\omega(\theta)$ can be predicted if we know an earlier angular velocity $\omega_0(\theta_0)$ from **the conservation of energy**. In Ch. 9++, we used conservation of energy to show that

$$\boxed{\omega^2 = \omega_0^2 + 3\frac{g}{L}(\sin\theta_0 - \sin\theta)} \qquad \textbf{B}$$

We could now insert this expression into $F_{P,\text{ radial}} - mg\sin\theta = -m\omega^2 L/2$, but there's no lovely simplification in doing so, so let's wait to look at special cases.

III) The missing piece: the angular acceleration α from the rotational second law. What remains is to show how to find α at a given θ through $\tau_{\text{net},P} = I_P\alpha$, the rotational second law form for a rotation of the rod around its end pivot point P. The first ingredient is $I_P = \frac{1}{3}mL^2$ for the rod around the axis perpendicular to and through its end (which we already used in Ch. 9++ to calculate the rotational kinetic energy). The torque due to the rod's weight is $+mg\frac{L}{2}\cos\theta$ because 1) we now use the convention "CW is positive," 2) the CM is at L/2, and 3) $F_\perp = mg\cos\theta$ (or, if you prefer, you can use the equivalent torque formula, $FR\sin(\frac{\pi}{2}+\theta) = mg\frac{L}{2}\cos\theta$, which gives the same answer). The rotational second law, $\tau_{\text{net},P} = I_P\alpha$, now reads

$$+mg\frac{L}{2}\cos\theta = +\frac{1}{3}mL^2\alpha$$

or, simplifying and rearranging,

$$\boxed{\alpha = \frac{3}{2}\frac{g}{L}\cos\theta} \qquad \textbf{C}$$

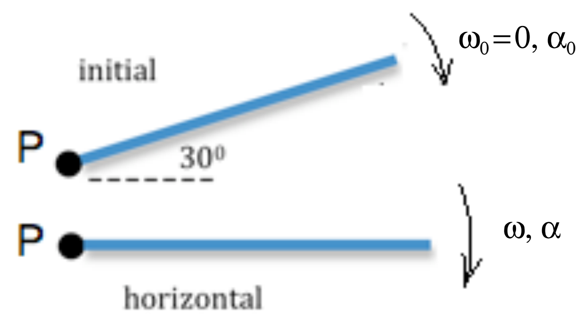
Inserting this into $F_{P,\text{ tangential}} + mg\cos\theta = +m\alpha L/2$, we have

$$F_{P,\text{ tangential}} = -mg\cos\theta + m\frac{3}{2}(\frac{g}{\cancel{L}}\cos\theta)\cancel{L}/2, \qquad \text{or}$$

$$\boxed{F_{P,\text{ tangential}} = -\frac{1}{4}mg\cos\theta} \qquad \textbf{D}$$

(example on next page)

Example: Suppose a thin uniform rod of mass m and length L has its frictionless hinge pivoted on a horizontal axis. It is lifted up to the angle of 30° with respect to the horizontal plane and is released from rest, rotating downward due to gravity. As a result, it swings down through $\theta=0$ having some nonzero angular velocity at that horizontal position. Notice from our general formula **C** obtained from the rotational second law that



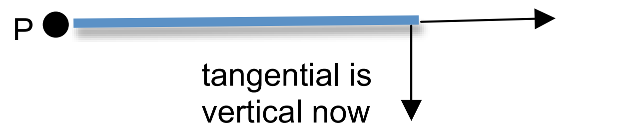
$$\alpha_0 = \frac{3\sqrt{3}}{4}\frac{g}{L} \text{ at the initial } 30^o \text{ position}$$

and

$$\alpha = \frac{3}{2}\frac{g}{L} \text{ at the horizontal position.}$$

What are the forces due to the pivot on the rod at the horizontal position ($\theta=0$), after falling from rest at the initial angle of 30^0?

1) First, let's go after $F_{P,\text{ radial}}$ where we need ω^2. Note from the figure on the right, this is the horizontal force on the rod due to the pivot.



With $\omega_0 = 0$ at $\theta_0 = 30^o$, the energy result **B**, $\omega^2 = \omega_0^2 + 3\frac{g}{L}(\sin\theta_0 - \sin\theta)$, reduces

to: $$\omega^2 = -3\frac{g}{L}(0 - \sin 30^o) = +\frac{3}{2}\frac{g}{L}$$

So formula **A,** $$F_{P,\text{ radial}} - mg\sin\theta = -m\,\omega^2\,L/2$$

turns into $$F_{P,\text{ radial}} - 0 = -m\frac{3}{2}\frac{g}{\cancel{L}}\cancel{L}/2 \,,$$ or

$$\boxed{F_{P,\text{ radial}} = -\frac{3}{4}mg \quad \text{at } \theta = 0 \qquad \text{(this is only true for the fall from rest at } 30^0\text{)}}$$

Indeed, the pivot has to pull back to the left on the rod as it swings down (to make it go in a circle!) This component, - ¾mg, is **horizontal and points to the left**, like a good centripetal force should for the rod in a horizontal position.

(continuing next to the tangential force component)

2) Now for $F_{P,\ tangential}$, which, at θ = 0, is a vertical component (see the previous figure). Plugging into formula **D,** $F_{P,\ tangential} = -\frac{1}{4} mg \cos\theta$, we get

$$\boxed{F_{P,\ tangential} = -\frac{1}{4} mg \quad \text{at } \theta = 0 \text{ (but this is independent of the initial conditions)}}$$

From the minus sign (down was positive), the pivot thus pushes **vertically up** on the rod in reaction to its weight, at θ = 0. But notice it is NOT $-\frac{1}{2} mg$ as it would be if the rod were supported on both ends and wasn't rotating down. **The interesting fact is that the pivot supports less than half the weight when the rod is swinging down!!**

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Problem 12-8

a) To feel better about being able to derive things from the basics without just plugging into our general formulas, derive the instantaneous angular acceleration result $\alpha = \frac{3}{2}\frac{g}{L}$ of a rod, **just as it passes through the horizontal position**, starting with the rotational second law at that position.

b) If $\alpha = \frac{3}{2}\frac{g}{L}$ is the instantaneous angular acceleration of a rod just as it passes through the horizontal position, then what is the linear acceleration downward (i.e., the tangential acceleration) of a point on the rod a distance r from the pivot?

c) Consider the following values r = L, L/2, and 2/3 L in your formula from (b). Discuss anything interesting you find. There is a neat demonstration related to this problem utilizing pennies on a meter stick – see your lecture!

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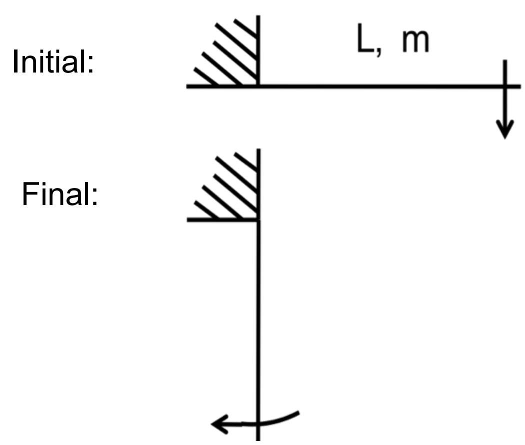
Problem 12-9 A thin rod of uniform mass m and length L is released from **rest** at the horizontal position, as shown, with its left end pivoted at a frictionless hinge attached to a ceiling corner. It falls due to gravity but is constrained by the pivot as it falls.



a) Using formulas **A, B,** and **D,** of pp. 12-18 and 12-19, find, **as the rod passes through the downward vertical position**, the vertical ($F_{P,radial} \equiv F_V$) and horizontal ($F_{P,tangential} \equiv F_H$) components of the force exerted on the rod by the pivot in terms of m, g, or nothing! ☺, , also shown.

b) As a separate exercise, derive F_V and F_H from "first principles:" namely, the translational second law, the rotational second law, and conservation of energy.

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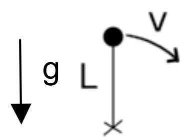
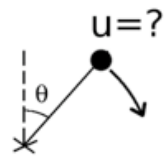
Vertical Circular Motion in Constant Gravity

We're going on a diet right now and we're shrinking all our masses down to points so we won't need any moments of inertia in this section. We instead want to concentrate on combining energy and centripetal motion.

Example: A **small** mass m is being swung in a vertical circle at the end of a massless rod of length L. The other end of the rod is connected to a frictionless pivot. Suppose the speed of the mass at the top of the circle is v.

What is its speed $u(\theta)$ when the rod makes an angle θ with the vertical?

Beginning: 

Later: 

Answer: A statement about the frictionless pivot is a clue to use conservation of energy. The pivot force does no work. Thus K + U at the top is equal to K + U anywhere else (in particular, at angle θ) where U = U(constant gravity) = mgy:

$$\left.\begin{aligned} K_i + U_i &= \frac{1}{2} m v^2 + m g y_{top} \\ K_f + U_f &= \frac{1}{2} m u^2 + m g y(\theta) \end{aligned}\right\} \quad \frac{1}{2} m v^2 + m g y_{top} = \frac{1}{2} m u^2 + m g y(\theta)$$

Set the origin y=0 at the pivot (wherever you choose your origin to be, it wouldn't matter – the difference in y value would cancel out on both sides and you'd get the same answer). So $y_{top} = L$, and $y(\theta) = L\cos\theta$ from a golden triangle. With the mass m canceling out, gathering L terms and multiplying by 2, we have the answer for the speed $u(\theta)$:

$$\boxed{u^2 = v^2 + 2 g L (1 - \cos\theta)}$$

After a quick digression to remind you of the equivalent, and perhaps your preferred way of writing energy conservation, we pose another question – about the rod's force on m as a function of θ - which we can answer now with the help of the expression for $u(\theta)$.

Digression: Recall the old story about how we may replace $K_i + U_i = K_f + U_f$ by $\Delta K = -\Delta U$, the equivalent form of energy conservation. (Δ means the change made in going from the initial state to the final state). We first say

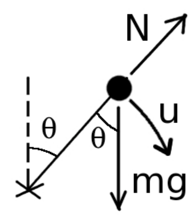
$$\Delta K = -\Delta U = -m\,g\,\Delta y$$

Then we use $\Delta y = y_f - y_i = L\cos\theta - L$ *plus*

$$\Delta K = \frac{1}{2}\,m\,u^2 - \frac{1}{2}\,m\,v^2$$

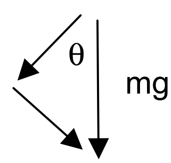
and we recover the previous result. It is really the same calculation but whatever helps you make the fewest errors, go ahead and use it.

As we said, we want to answer the following question: How does the magnitude and sign of the radial force on m due to the rod in the previous example change as a function of θ as the rod swings around?



Answer: Use the second law along the radial direction to get an equation for the outward radial force N on m due to the rod. (Calling the outward axis positive, the radial component of the force on m due to the rod is defined as +N. If we find N < 0, then this means the rod is pulling inward on m). The circular motion means the radial acceleration is centripetal and inward (so its component is negative). For the previous convention (velocity u at the angle θ), we find, from the FBD shown, the second law along the radial direction is

$$N - m\,g\cos\theta = -m\,\frac{u^2}{L}$$



with a golden triangle calculation yielding $-mg\cos\theta$, the radial component, for the weight.

Also, since the centripetal acceleration involves u^2, we can use the energy conservation result for u^2 at θ, and relate it to the original velocity v:

$$N - m\,g\cos\theta = -m\,\frac{v^2 + 2\,g\,L\,(1 - \cos\theta)}{L}$$

Solving for N:

$$N = m\,g\cos\theta - m\frac{v^2}{L} - 2\,mg\,(1 - \cos\theta)$$

or, finally,

$$\boxed{N = m\,g(3\cos\theta - 2) - m\frac{v^2}{L}}$$

(comments on the next page)

Comments:

- If N > 0, we have **compression** (the rod pushes up on m)

- If N < 0, we have **tension** (the rod pulls in on m)

- Indeed, with $N = m\,g(3\cos\theta - 2) - mv^2 / L$, we get the expected result that the rod will have to pull inward on the mass if it is going really fast (e.g., N < 0 in the above expression for large v).

- And, for small θ (i.e., near the top), we get the expected result that the rod would have to push out to hold the mass up, if it's hardly moving (e.g., N > 0 in the above for smaller v and smaller θ.

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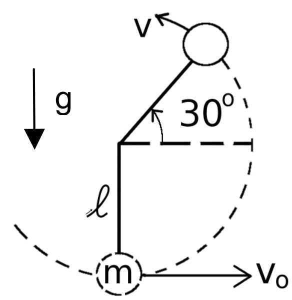
Problem 12-10

A **small** mass m attached by a very light metal rod of length ℓ to a frictionless pivot moves under constant gravity in a vertical circle. At the bottom of the circle suppose it has a speed given by $v_0 = 2\sqrt{(g\ell)}$ (this is a contrived choice in terms of the given parameters in order to simplify the problem calculation!!)



a) What is and is not conserved in this problem? Why and why not?

b) What is the speed v of m when it is 30° above the horizontal, as shown, in terms of g and ℓ?

c) What is the tension or compression (which is it?) in the rod when m is in the 30° position, in terms of m and g?

d) In general, is a rod swinging around like this always under tension, or can it be "under compression" at some point as it swings around? What does the question about tension and compression depend upon? (Note that it would always be under tension at the bottom – why?)

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