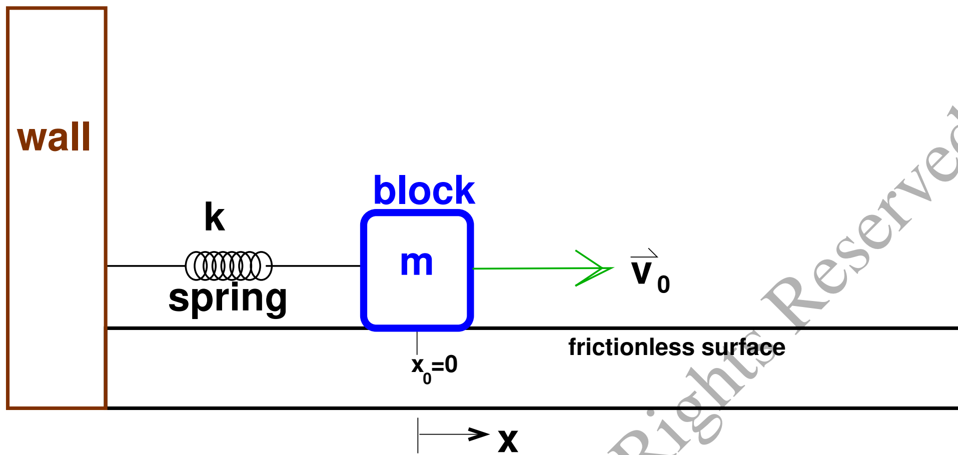
# PHYS 121: Solutions to Written Homework #10
## May 1, 2024

Note: this homework is worth **15 points** as follows:

| | | | |
|---|---|---|---|
| ___ | Problem 1 | 5 points | 5 pts total |
| ___ | Problem 2 | 5 Points | 5 pts total |
| ___ | Problem 3 | 5 points | 5 pts total |

**Problem 1: Simple Harmonic Motion**

Suppose a block of given mass $m$ is attached to a spring of given spring constant $k$ and released on a frictionless surface with an initial horizontal velocity given as $v_0$ at time $t = 0$. At time $t = 0$ the block is located at the equilibrium position corresponding to $x = 0$ as shown:




We showed in class that the Equation of Motion for a Simple Harmonic Oscillator is this:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

We also showed that the *general solution* for the Equation of Motion of a Simple Harmonic Oscillator is given by this equation:

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

Determine the particular values of $\omega$, $A$, and $B$ that correspond to this particular system. Give your answer in terms of the given parameters: $m$, $k$, and $v_0$. Explain how you know this. Hint: Consider the value of $x$ at time $t = 0$. Now consider the value of $v = \frac{dx}{dt}$ at time $t = 0$.

**SOLUTION TO PROBLEM 1:**

First we consider the general solution to the *Equation of Motion*. Suppose we write down the solution and the time derivatives:

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$
$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$
$$a(t) = \frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$$

If we substitute in expressions for $x(t)$ and $a(t)$ into our Equation of Motion we get:

$$[-A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)] + \left(\frac{k}{m}\right)[A\cos(\omega t) + B\sin(\omega t)] = 0$$
$$-A\left(\omega^2 - \frac{k}{m}\right)\cos(\omega t) - B\left(\omega^2 - \frac{k}{m}\right)\sin(\omega t) = 0$$

This can only always be true for all value of $A$, $B$, and $t$ if the term $\omega^2 - \frac{k}{m} = 0$:

$$\omega^2 - \frac{k}{m} = 0$$
$$\omega^2 = \frac{k}{m}$$
$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

Note that this solution for $\omega$ is part of the general solution for the oscillator. The oscillation frequency does not depend on initial parameters, only the mass and spring constant.

To get $A$ and $B$ we consider what is happening at time $t = 0$ first with position:

$$x(t) = A\cos(\omega t) + B\sin(\omega t) = 0$$
$$x(t=0) = A\cos(0) + B\sin(0) = 0$$
$$A + 0 = 0$$
$$\boxed{A = 0}$$

Now we look at the velocity:

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t) + B\omega\cos(\omega t) = v_0$$

$$v(t=0) = -A\omega\sin(0) + B\omega\cos(0) = v_0$$
$$0 + B\omega = v_0$$

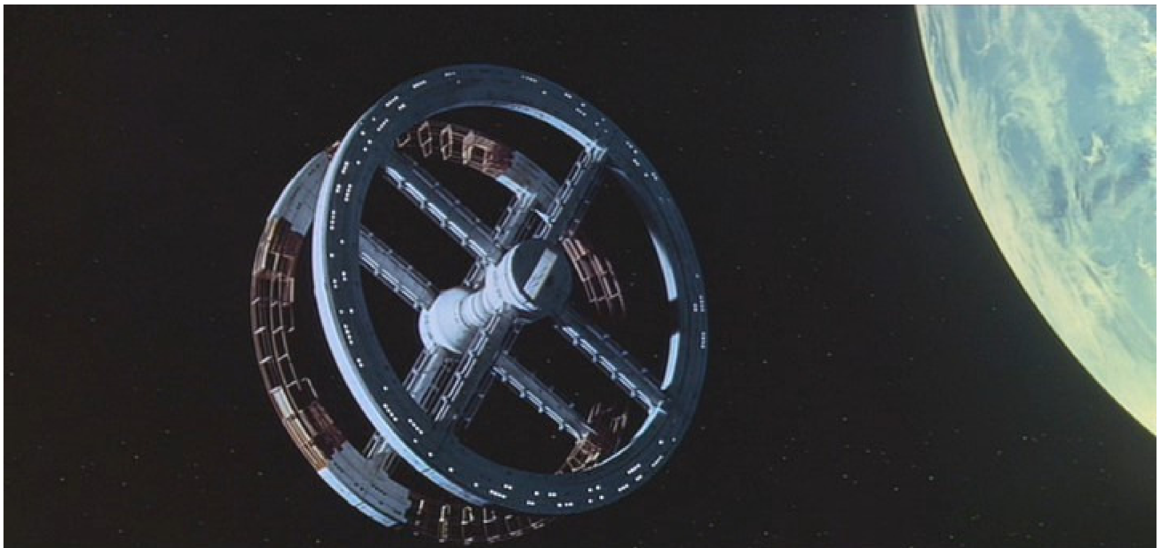
$$\boxed{B = \frac{v_0}{\omega}}$$

So the "complete and particular solution" is:

$$\boxed{x(t) = \left(\frac{v_0}{\omega}\right)\sin(\omega t) \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}}$$

**Problem 2: Space Station**.

Suppose a space-station is designed in s shape of a torus such as this one depicted in Stanley Kubrick's "2001: A space Odyssey" below:



Suppose the space station has a diameter of 175 meters. At what rotational rate should the station be spun so that occupants standing on the inside surface of the outer wall (maximum distance from the center) experience earth-normal artificial gravity? Give your answer in terms of rotations per minute.

Suppose that inside the space station there is a transport cart for transport around the rim that can reach a maximum speed of 25.0 meters per second. Using such a cart, how could you tell which way the space station from spinning from inside without looking out any windows? Be quite specific about what you experience when you drive around with the cart in one direction or the other.

**Solution to Problem 2:**

**Part (a):**

Okay the space station is spinning at some value of rotational speed $\omega$. A person standing at the rim of the station will experience centripetal acceleration according to:

$$a_c = \omega^2 R$$

This is the acceleration *inward* that will result in a fictitious force (artificial gravity) *outward*. We want to set this equal to the acceleration due to gravity on Earth:

$$\omega^2 R = g$$

$$\omega = \sqrt{\frac{g}{R}}$$

Two last complication are that we are given the diameter of the wheel $D$=175 meters, which is twice the radius ($R = D/2$) and we want a rotational frequency which is the angular frequency divided by two pi: ($\omega = 2\pi f$):

$$2\pi f = \sqrt{\frac{g}{D/2}}$$

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{2g}{D}}}$$

Now we plug the numbers in:

$$f = \frac{1}{2\pi}\sqrt{\frac{2(9.81 \text{ m/s}^2)}{(175 \text{ m})}}$$

$$\boxed{f = 0.053 \text{ rotations per second}}$$

Doing a unit conversion here:

$$f = 0.053 \text{ rotations per second} \left(\frac{1 \text{ minute}}{60 \text{ seconds}}\right)$$

$$\boxed{f = 3.20 \text{ rotations per minute}}$$

**(Solution continues next page...)**

**Part (b):**

So the idea here is that by changing our speed (with the cart) we can change the centripetal acceleration. The translational speed of a person standing on the inside surface is just given by the rolling constraint (uniform circular motion):

$$v_{rim} = \omega R$$
$$v_{rim} = \left(\sqrt{\frac{g}{R}}\right) R$$
$$v_{rim} = \sqrt{Rg}$$
$$v_{rim} = \sqrt{Dg/2}$$
$$v_{rim} = \sqrt{(175 \text{ m})(9.81 \text{ m/s}^2)/2}$$
$$v_{rim} = 29.3 \text{ meters/second}$$

So if we have a cart that can travel relative to the rim-wall of the space-station at $V_{cart}$=25 meters per second, then this can have a large impact on the centripetal acceleration of the occupant. For example, if the cart is drive in the "spinward" direction (so that the net speed is larger) then the centripetal acceleration will be significantly larger:

$$a_c = \frac{v_{net}^2}{R} = \frac{(v_{rim} + V_{cart})^2}{R}$$
$$a_c = \frac{2(v_{rim} + V_{cart})^2}{D}$$
$$a_c = 2\frac{[(29.3 \text{ m/s}) + (25.0 \text{ m/s})]^2}{(175.0 \text{ m})}$$
$$a_c = 33.7 \text{ m/s}^2$$

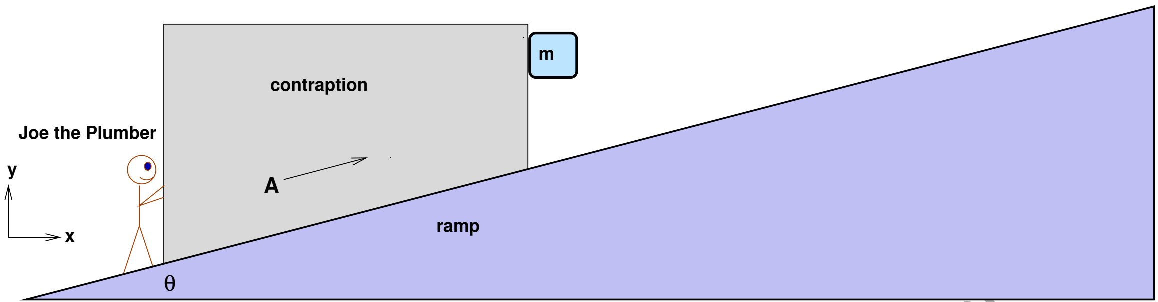
That's more than three G's of artificial gravity, you'd really notice that. Likewise if you travel in the "anti-spinward" direction, you net speed will be much lower, and so the artificial gravity will be much less than one G.

$$a_c = \frac{v_{net}^2}{R} = \frac{(v_{rim} - V_{cart})^2}{R}$$
$$a_c = \frac{2(v_{rim} - V_{cart})^2}{D}$$
$$a_c = 2\frac{[(29.3 \text{ m/s}) - (25.0 \text{ m/s})]^2}{(175.0 \text{ m})}$$
$$a_c = 0.21 \text{ m/s}^2$$

That's much less than a tenth of a G. You'd feel near-weightless.

In terms of your experience, then, you **feel much heavier if you drive the cart spinward** and **you feel much lighter if you drive the cart anti-spinward.**

**Problem 3: Joe the Plumber** (from prior year's final exam):




Joe the plumber pushes a contraption up a ramp at a given small angle $\theta$ as shown at a given constant acceleration $\vec{A}$. A small box of mass $m$ is in contact with the vertical surface of the contraption as shown.

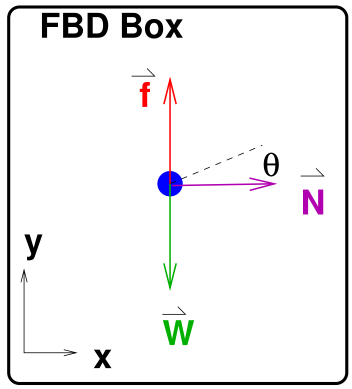
**a)** Assume that there is *static friction* between the small box and the contraption. The coefficient of static friction between the box and the contraption is $\mu$. Calculate the direction and the magnitude of *all forces* on the small box in terms of the given parameters. Explain your work.

**b)** Suppose the small box contains a mini-aquarium with water and fish. Assuming that the system is in equilibrium, with what angle will the surface of the water inside the aquarium make with respect to the horizontal? Explain how you know this.

**Solution to Problem 3: (Joe the Plumber)**

**Part (a):**

We are *only interested in the box*. All kinds of details about Joe and the contraption are not relevant. What are the forces on the box? We write down a Free Body Diagram:




Note that in this problem we are **given a coordinate system** which is up and down (vertical/horizontal) so we are obligated to work in this coordinate system. Happily we can easily break the acceleration into components in each Cartesian coordinate: note that by construction we are given the acceleration in

$$a_x = A\cos\theta$$
$$a_y = A\sin\theta$$

Now we apply **Newton's Second Law** one component at a time. Starting with the horizontal component:

$$F_x = ma_x$$
$$\boxed{N = mA\cos\theta}$$

Now the vertical component:

$$F_y = ma_y$$
$$f - W = mA\sin\theta$$
$$f = mA\sin\theta + W$$
$$f = mA\sin\theta + mg$$
$$\boxed{f = m\left(A\sin\theta + g\right)}$$

Pretty simple. Note that the coefficient of friction is a red herring here.

**Part (b):**

This is a problem in **Non-inertia reference frames**. The frame inside the box is accelerated. This has the effect of changing the effective gravity:

$$\vec{g}_{eff} = \vec{g}_{real} - \vec{A}$$

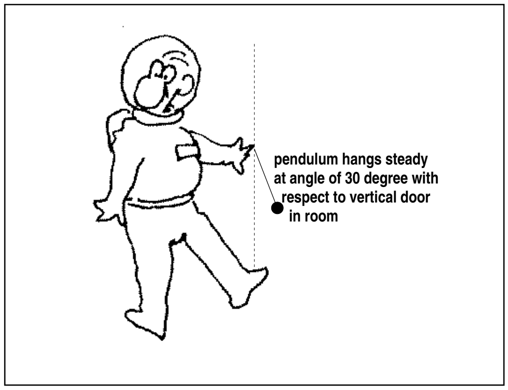
Component-wise:

$$\vec{g}_{eff} = -g\hat{\boldsymbol{\jmath}} - (A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}})$$
$$\vec{g}_{eff} = -A_x\hat{\boldsymbol{\imath}} - (g + A_y)\hat{\boldsymbol{\jmath}}$$

Since the effective gravity has both horizontal and vertical components, it is "tipped" by an angle we will call phi:

$$\tan\phi = \frac{A_x}{g + A_y}$$
$$\tan\phi = \frac{A\cos\theta}{g + A\sin\theta}$$
$$\boxed{\phi = \arctan\left(\frac{A\cos\theta}{g + A\sin\theta}\right)}$$

**Problem 4:**




After a disorienting and ill-advised episode involving mind-altering substances, Ringo inexplicably finds himself in a locked room with no windows. The room appears normal in every respect except that the whole room appears to be tilted at an angle of 30 degrees relative to level. Ringo determines this tilt by holding up a small pendulum relative to the door-frame that is aligned with the room and measuring with his protractor that the pendulum hangs steady at 30 degrees relative to the door frame. There is a loud vibrating fan in the room that makes it impossible for Ringo to get any noise or vibration from outside the room. Ringo concludes that *either* he is in in a tilted room, *or else* he is in a room that is moving in a special way to give the illusion that the room is tilting.

**Part (a)**: Ringo is trying to guess what is going on. First, he assumes that the room is not moving at all but is simply tilted. Draw a simple sketch indicating how the room looks from the outside (just draw a box and indicate the tilt angle).

**Part (b)**: Now Ringo is wondering if his observations can be explained as a result of the motion of the room rather than a tilt. Suppose we assume that the room is not really tilted but is actually moving horizontally with constant velocity. Can Ringo's observations be explained under this assumption? Explain your answer.

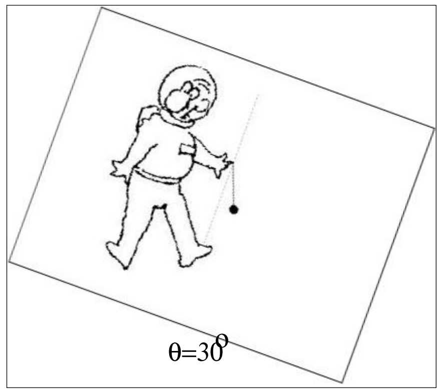
**Part (c)**: Now assume that the room is not really tilted but is actually accelerating with constant acceleration in the horizontal direction. Can Ringo's observations be explained under this assumption? If so, what is the magnitude of this acceleration? Hint: work this problem in the inertial frame where you can use Newton's second law. Consider the forces and acceleration on Ringo's pendulum. Draw a sketch that shows the room and the pendulum as it would appear to an outside observer.

**Part (d)**: Suppose in fact that Ringo is not on the surface of the Earth but has been kidnapped by space aliens so that the whole room is really on an accelerating space ship in deep space. Draw a sketch to indicate the direction that space ship is accelerating relative to the room. What the is magnitude of the acceleration?

**Part (e)**: Is there any physics experiment or any other test that Ringo can do to verify that he is in an accelerating spaceship in deep space vs. sitting in a room on Earth? If not, why not?

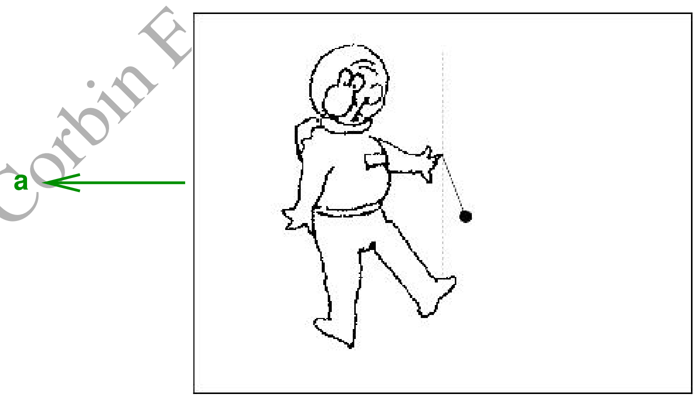
**Solution to Problem 4:**

**Part a)** This is not so hard. Just tilt the drawing by 30 degrees:




**Part b)** No, moving the room at a constant velocity *cannot* explain what Ringo observes. A frame at constant velocity is an *inertial* frame and Newton's Laws work the same in all inertial frames. Specifically, there is no "fictitious force" or any other effect that Ringo would notice relative to the room at rest. The apparent tilt cannot be explained by constant velocity.

**Part c)** Yes, at least *qualitatively*, what Ringo sees can be explained if the entire room is undergoing a constant horizontal acceleration to the *left*. In this case, there must be a net force on the room and also a net force on any object that is moving with the room. Specifically, if we consider the "pendulum" it has a net acceleration and thus a net force is on it. Here is the Free Body Diagram (appropriate to describe the forces as seen by an observer standing outside the room in an inertial frame).






We can see that there are two forces on the pendulum bob, Weight and Tension. We can work Newtons Second Law to determine the value of the acceleration:

$$F_y = ma_y$$

$$T\cos\theta - W = 0$$

$$T = \frac{mg}{\cos\theta}$$

and

$$F_x = ma_x$$
$$T\sin\theta = ma_x$$
$$\left(\frac{mg}{\cos\theta}\right)\sin\theta = ma_x$$
$$a_x = g\tan\theta$$

In other words the angle that Ringo measures is determined by the horizontal acceleration. If Ringo's room is moving with an acceleration of 0.58 G's to the left, he can explain his angle.

There are two "worries" here, however:

First, there is the practice problem that it is nearly impossible to build a practical machine or vehicle that can deliver a sustained uniform acceleration for any appreciable time. The fastest and most powerful trains and race cars would have a difficult time sustaining such accelerations for times even as long as a minute . So all Ringo needs to do is wait a little while to see if this scenario can be ruled out.

Second, the horizontal acceleration of the room results in a horizontal "fictitious force" that is gravity-like which adds to the real gravity going down. But the downward (real) gravity is constant so the total magnitude of the gravity is greater than what we experience normally on Earth. Specifically:

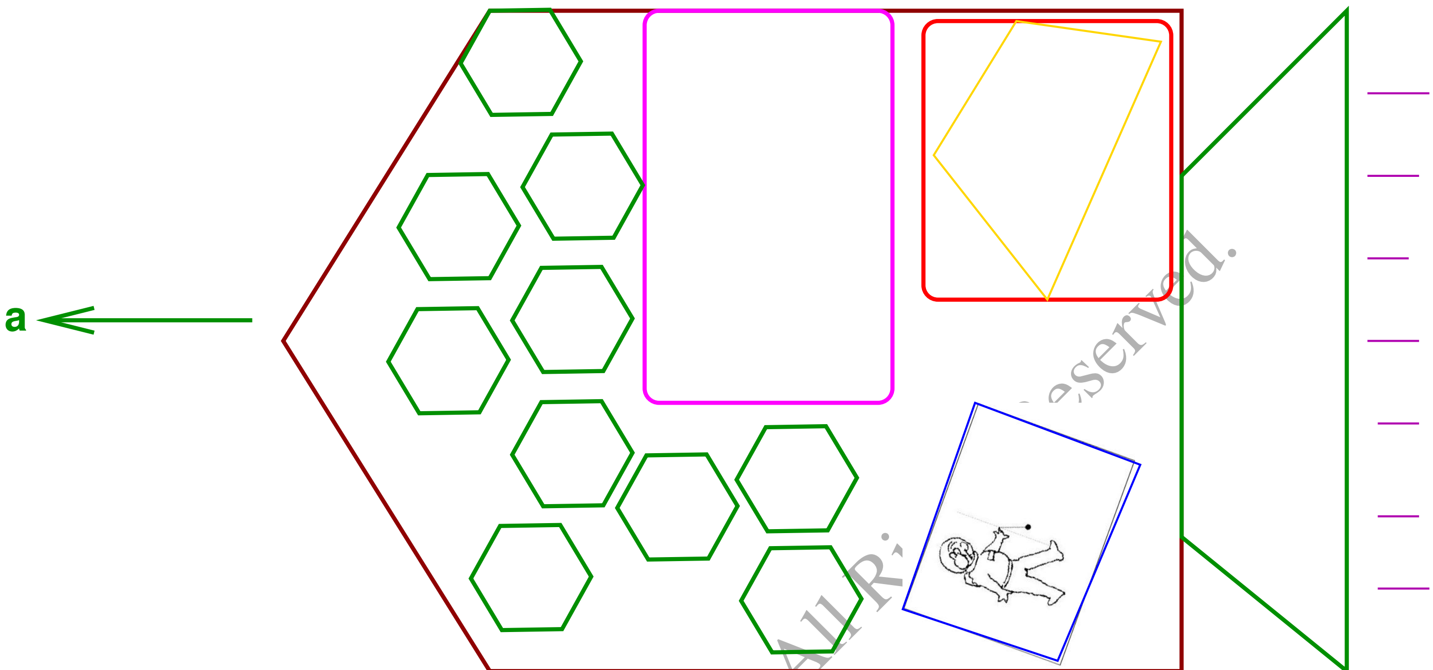
$$\vec{g}_{\text{effective}} = \vec{g}_{real} + \vec{g}_{fic}$$
$$\vec{g}_{\text{effective}} = \vec{g}_{real} - \vec{a}$$
$$\vec{g}_{\text{effective}} = -g\hat{j} - a_x\hat{i}$$

so

$$|g_{\text{effective}}| = \sqrt{g^2 + a_x^2}$$
$$g_{\text{effective}} = \sqrt{[1 + (0.58)^2]}g$$
$$g_{\text{effective}} = (1.154)g$$

In other words, in this scenario, the magnitude of the "apparent gravity" would be about 15% larger than what we normally experience on Earth. A sudden increase of 15% in the strength of gravity would certainly be noticeable by any clear-headed person. But I bet that that such an increase applied very gradually while a person is sleeping would not be immediately obvious to the person when that person wakes. Ringo just might attribute his sensation of "feeling heavier" to an after-effect of his libations.

**Part d)** It is possible to explain Ringo's observations if we assume that the room is arbitrarily tipped with respect to the acceleration of the spacer ship. Here's one example of this:




To match perfectly the gravity on earth $a = g$ as shown.

**Part e)** Einstein's Principle of Equivalence tells us: There is *no* experiment that can be done that can tell the different between uniform acceleration and a uniform gravitational field. Once you restrict yourself to local measurements (not allowed to look out the window) this is an iron-clad premise of Einstein's theory of gravity.

**Homework Problem 5.**

**This problem is entirely optional!**

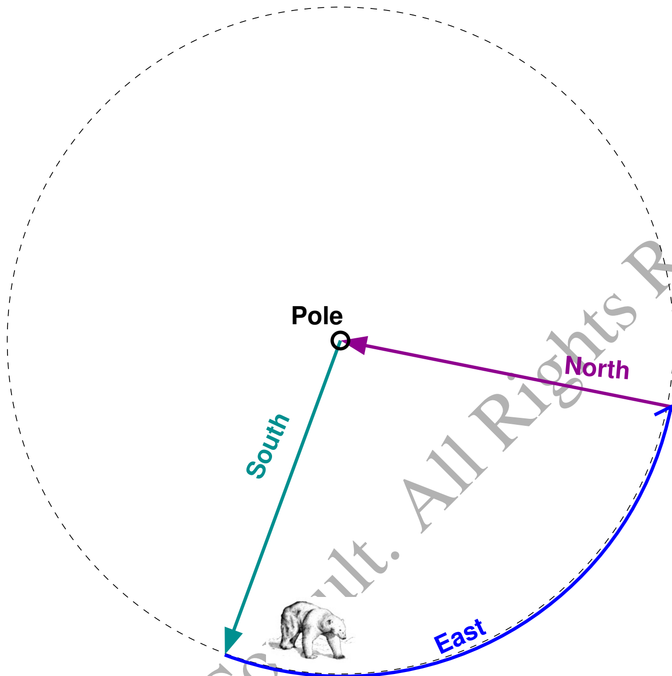
A bear travels 100 miles due south (precisely). Then he travels 100 miles due east. Then he travels 100 miles due north, to arrive at the exact same location from where he started. ***What color is the bear?***

(This problem is demonstrates the effects of "curved space" on geometric objects, such as triangles). Based on this example, describe how – in principle – a space traveler might be able to measure the ***curvature of space due to gravity*** by traveling and very carefully measuring distances and angles. Here, imagine that somehow the practical constraints on traveling and communicating over great cosmological distances could be easily overcome.

**Solution to Problem 5:**

Obviously the bear is white since the only place this can happen is the north pole.

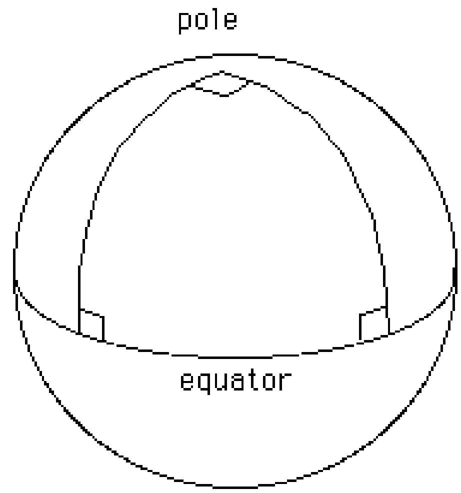
This problem demonstrates two issues regarding non-Euclidean geometry. First, at the pole, the definition of East or West corresponds to a path that is not "straight" but circular. Thus the definition of objects such as triangle become distorted.






Here you can see that in defining notions of North and South near the pole, we end up with East-West lines running in circular paths instead of appearing "straight".

However, a deeper probe of notion curvature of the surface of the Earth shows that this will have an impact on geometry even if we restrict our definition of straight lines to the "shortest possible path between two points" (a so-called "geodesic" or a segment of a "great circle") For example, a proper triangle can be defined by traveling from the north pole to the equator, along the equator some distance and back to the north pole. For this path each segment represents the shortest distance between two points. Nonetheless, the triangle that results (with three right angle at each point summing 270 degrees) is clearly not Euclidean (where we expect a sum of 180 degrees for the three angles).




This is relevant to the question of the curvature of the Universe as a whole. If a person could in principle travel a long distance on a space ship, out to Galaxy X, then to Galaxy Y and then home again at each vertex you could measure the angle and sum the three angles. A total of greater than exactly 180 degrees would represent positive curvature (closed, sphere-like). A total of less than 180 degrees would represent negative curvature (open, saddle-like). A total of exactly 180 degree corresponds to a flat Universe.

Before the late 1990's scientists believed that Geometry (governed by density) would determine the fate of the physical universe. High density means gravity wins, the Universe is closed, and we get a Big Crunch. Low density means gravity loses, the Universe is open, and we get the Big Freeze.

However, in the late 1990's astrophysicists discovered that the expansion of the Universe is accelerating. This implies the existence of Dark Energy. At present most evidence suggests a very-nearly Flat Universe but the unknown nature of Dark Energy means that we cannot (yet) use this fact to predict the ultimate fate of the physical Universe.