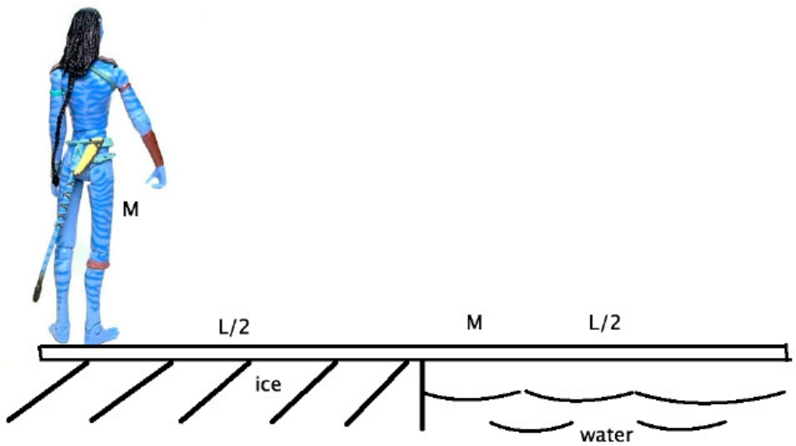
PHYS 121: Solutions to Written Homework #08
April 12, 2024

Note: this homework is worth **15 points** as follows:

___	Problem 1	5 points	5 pts total
___	Problem 2	5 Points	5 pts total
___	Problem 3	5 points	5 pts total

Problem 1: This is an Important Conceptual Question

Problem 11-7
Neytiri is standing on the end of a long thin uniform board of length L. *Half of the board including the end where Neytiri is standing rests on very slippery ice with the other half over open water, as shown. Lucky for our calculations, the board and Neytiri have the same mass* M *!* ☺



a) Find the x *coordinate, relative to the ice, of the overall CM of Neytiri and the board. Define* x = 0 *at her end, and* x = L *at the other end.*

b) If Neytiri walks all the way to the other end, will she fall into the water? Give your reasoning.

*c) Suppose Neytiri moves at a speed v **relative to the ice**, while she is walking to the right as described in part (b) from her original end position. What is the CM velocity?*

d) How fast is the board moving relative to the ice and in what direction, if Neytiri is moving at speed v to the right?

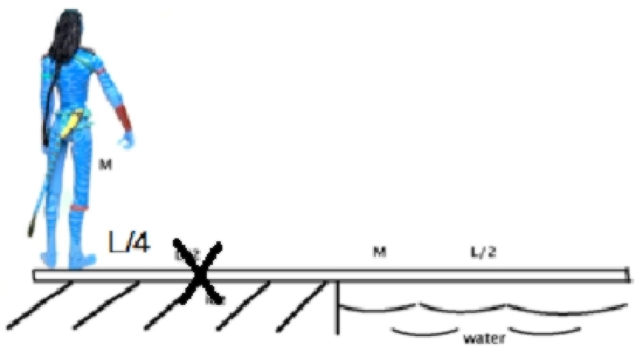
See next page for important hints on this problem.

Hints on Problem 1: Ask yourself these questions:

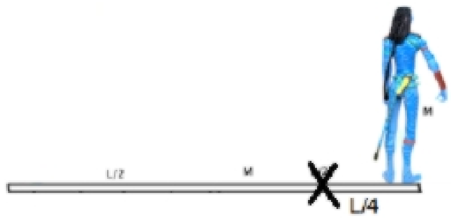
- What is the **system** here? (getting the right answer depends on this choice!)
- Is the system **isolated**? (yes, it is.)
- Does the system **remain isolated**? (yes, it does.)
- Before Neytiri moves at all, what is the x-position of the board? (Hint: these two answers are incorrect: $x_{board} = 0$ and $x_{board} = L$.)
- Before Neytiri moves at all, what is x_{cm} corresponding to the position of the Center-of-Mass of the system? (Hint: use the Definition of Center-of-Mass.)
- Before Neytiri moves at all, what is v_{cm} corresponding to the velocity of the Center-of-Mass of the system?
- If the system remains **isolated** what does this tell you about how the velocity of the center-of-mass of the system will change?
- What does this tell you about the how the position of the center-of-mass will change?
- Consider this question. If Neytiri steps one centimeter to the right, how far will the board move and in what direction? How do you know this must be true? (Yes, the board moves! The board **must** move to ensure that your answers to the above questions remain true.)

11-7)

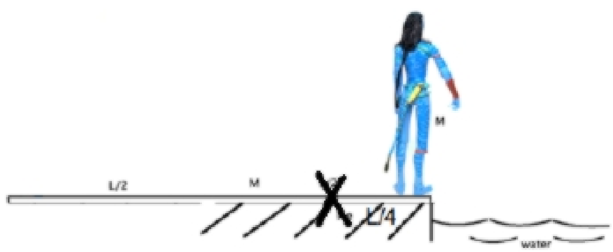
a) $x_{CM} = \dfrac{M \cdot 0 + M\dfrac{L}{2}}{M+M} = \dfrac{M\dfrac{L}{2}}{2M} = \dfrac{L}{4}$ **so the X marks the CM below:**



b) When Neytiri is at the other end, the CM is now at the X shown below:



BUT this CM must be at the same place as before since the CM is not moving (Neytiri was originally standing still). So relative to the edge of the ice, we have:



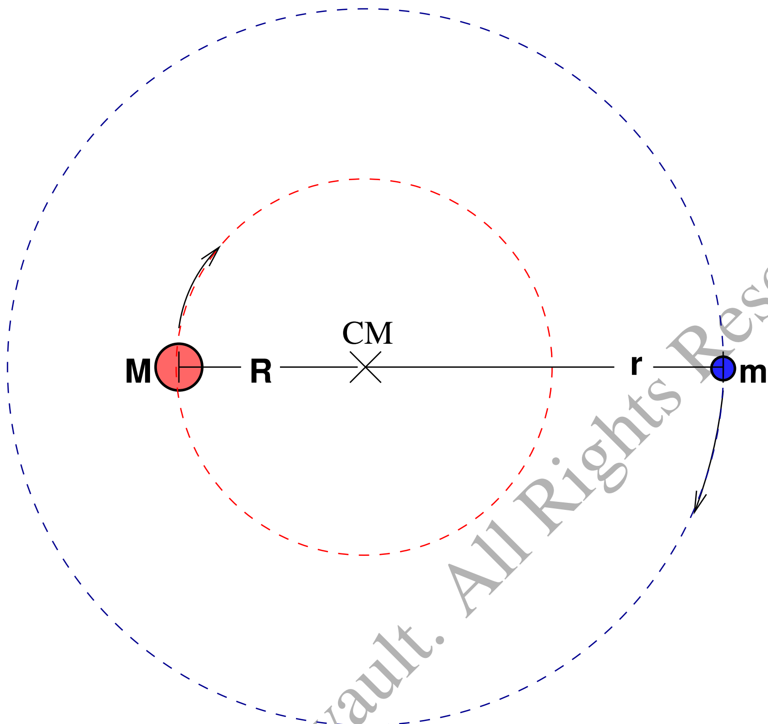
So we can say she's on the verge but the board CM is left of the edge so it won't tip in.

c) With no horizontal forces and no initial motion, the CM must always stay put, as we have already asserted above. Thus $v_{CM} = 0$

d) Well, if $v_{CM} = 0$, then we must have $v_{CM} = \dfrac{M \bullet v + M v_{board}}{M + M} = 0$ which demands that $v_{board} = -v$ so the board moves to the left with speed v.

Problem 2: Universal Gravity (from a previous final exam):

Consider two stars, each in its own *circular* orbit around a common Center-of-Mass (CM) point as show:



Suppose the masses of the two stars M and m are *given*. The radii of each of the two orbits is indicated by R and r. Assume that the radius r is *given* but the radius R is *not* given.

Calculate the magnitude of the linear velocity v_m of the star with mass m (on the right) in terms of the two given masses and the radius r. Explain your work.

Important: For this problem, you need to set up a step-by-step process and explain each step. You will be graded on both your answer and on your explanation of your method. You must explain your work in terms of Physics Concepts.

Hint: It may be helpful to *define* the (fixed!) Center-of-mass point as the origin (corresponding to "position zero") with the right star at a positive position and the left star at a negative position, and then use the Definition of Center-of-Mass to calculate the ratio R/r in terms of the given masses. Then use this result together with Newton's Second Law to find an expression for the velocity of the particular star.

SOLUTION TO PROBLEM 2:

To get a handle on the (unknown) radius for the large (left) star we use the **Definition of the Center-of-Mass**:

$$x_{cm} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Since the net external force on the system of two stars is zero, we know that the position of the Center-of-Mass will remain fixed. We can define this position as "zero" and at the instant shown, we can assign the position of the right star as positive and the position of the left star as negative:

$$0 = \frac{M(-R) + mr}{M + m}$$
$$M(-R) + mr = 0$$
$$mr = MR$$
$$\frac{R}{r} = \frac{m}{M}$$
$$R = \frac{mr}{M}$$

This gives us R in terms of given parameters. Now we apply Newton's Second Law to the right star with mass m which is moving with Uniform Circular Motion on a circle of radius r:

$$F_{net} = ma_c$$
$$F_{UG} = \frac{mv_m^2}{r}$$

We apply Newton's Law of Universal Gravity where the distance between the star $\mathcal{R} = (r + R)$:

$$\frac{GMm}{\mathcal{R}^2} = \frac{mv_m^2}{r}$$
$$\frac{GMm}{(r+R)^2} = \frac{mv_m^2}{r}$$

We substitute in for R from our previous result, and ***the rest is algebra*** to solve for velocity:

$$\frac{GM}{\left[r + \left(\frac{mr}{M}\right)\right]^2} = \frac{v_m^2}{r}$$
$$\frac{GM}{\left[r\left(1 + \frac{m}{M}\right)\right]^2} = \frac{v_m^2}{r}$$
$$\frac{GM}{r^2\left[1 + \left(\frac{m}{M}\right)\right]^2} = \frac{v_m^2}{r}$$
$$\frac{GM}{r\left[1 + \left(\frac{m}{M}\right)\right]^2} = v_m^2$$

$$v_m^2 = \frac{GM}{r\left[1+\left(\frac{m}{M}\right)\right]^2}$$

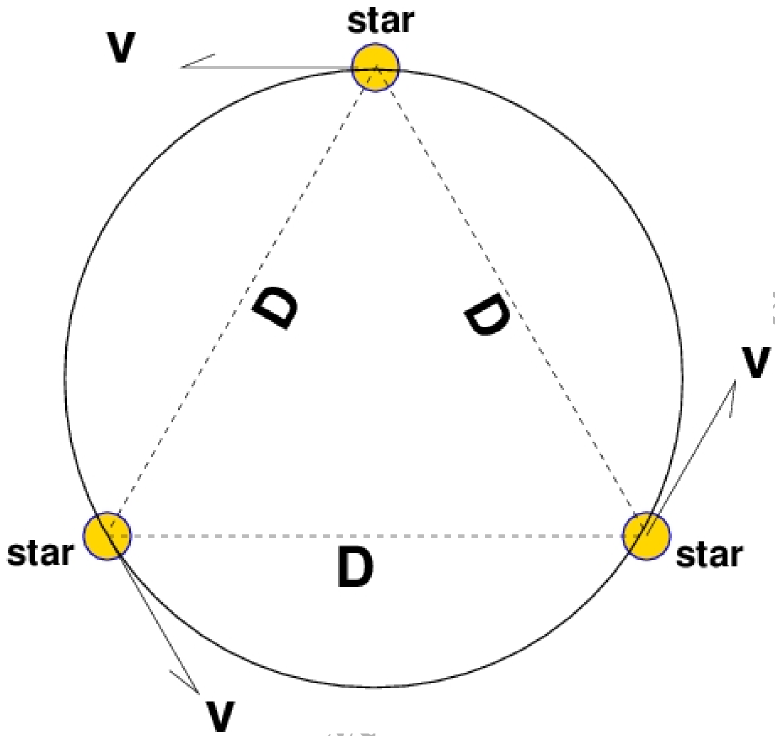
$$\boxed{v_m = \sqrt{\frac{GM}{r\left[1+\left(\frac{m}{M}\right)\right]^2}}}$$

or

$$\boxed{v_m = \frac{1}{1+\left(\frac{m}{M}\right)} \cdot \sqrt{\frac{GM}{r}}}$$

Problem 3: A Special Triple Star System

This problem was on the final exam for Fall 2012. You need to apply Newton's Second Law.



Consider a very special arrangement of three stars place in orbit about each other as shown in the figure above. Each star has given mass m and each star is positioned on the three corners of an equilateral triangle so that the distance between any two stars is given as D – the length of one side of the triangle. All three stars travel along a circular orbit of radius $= \frac{\sqrt{3}D}{3}$ in accordance with geometry.

Calculate the *speed* v of any of the three stars in orbit. Given your answer in terms of the given parameters. Explain your work. Some possibly useful relations:

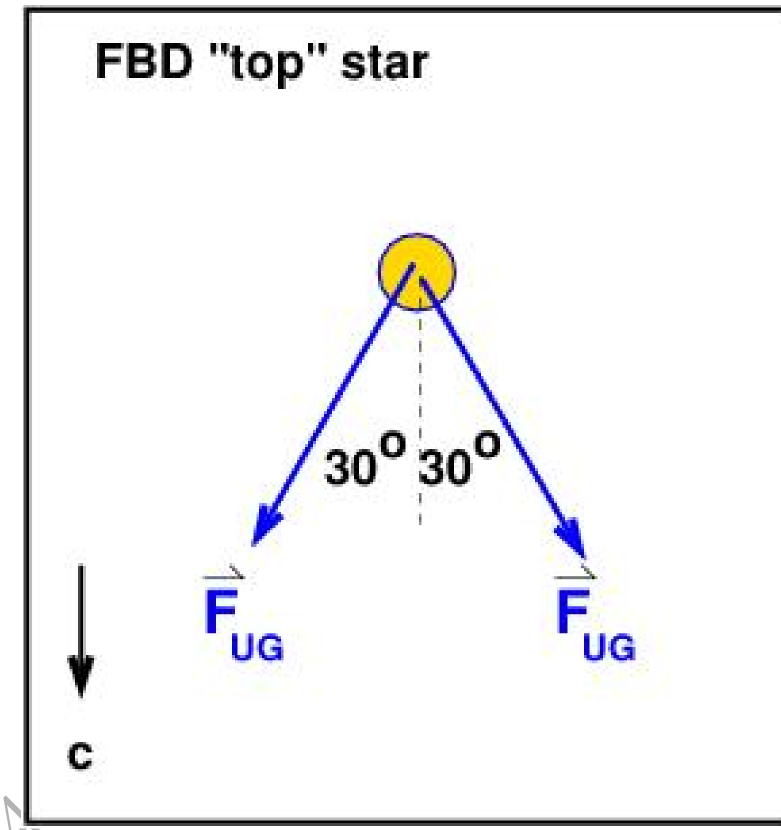
- $\sin 30° = \frac{1}{2}$
- $\cos 30° = \frac{\sqrt{3}}{2}$
- $\sin 60° = \frac{\sqrt{3}}{2}$
- $\cos 60° = \frac{1}{2}$

Solution to Problem 3:

Since we are in deep space, the only force that applies is **Universal Gravity** which is an attractive radial force between any two masses in accordance with:

$$F_{UG} = \frac{GmM}{r^2}$$

where m and M are the masses of the two bodies and r corresponds to the distance between bodies. Since there are three stars, each of the two stars feels the attraction due to the other two stars. For example, if we consider the **Free Body Diagram** of the "top" star in the figure it looks like this:



By **symmetry** we argue that the net force due to each of the two stars is a force on the top star that is directed toward the center of the orbit – in other words in the **centripetal** direction. If we write down **Newton's Second Law** for the centripetal component, we get:

$$F_{left}\cos\theta + F_{right}\cos\theta = ma_c$$

Here $F_{left} = F_{right} = F_{UG} = \frac{Gm^2}{D^2}$ corresponding to the force of Universal Gravity from either the left bottom or right bottom stars. The term $\cos\theta = \cos(30°) = \frac{\sqrt{3}}{2}$. Therefore:

$$\frac{Gm^2}{D^2}\left(\frac{\sqrt{3}}{2}\right) + \frac{Gm^2}{D^2}\left(\frac{\sqrt{3}}{2}\right) = ma_c$$

$$\frac{\sqrt{3}Gm^2}{D^2} = ma_c$$

$$\frac{\sqrt{3}Gm}{D^2} = a_c$$

$$a_c = \frac{\sqrt{3}Gm}{D^2}$$

We use the fact that we know the expression for the centripetal acceleration $a_c = \frac{v^2}{R}$ but in this case the geometry of the triangle tells us that the radius of the orbit is given by $R = \frac{\sqrt{3}D}{3}$ and so:

$$\frac{v^2}{\left(\frac{\sqrt{3}D}{3}\right)} = \frac{\sqrt{3}Gm}{D^2}$$

Solving for the speed:

$$v^2 = \left(\frac{\sqrt{3}D}{3}\right)\frac{\sqrt{3}Gm}{D^2}$$

The two $\sqrt{3}$ terms cancel the 3 in the bottom and we are left with

$$v^2 = \frac{Gm}{D}$$

$$\boxed{v = \sqrt{\frac{Gm}{D}}}$$

So this is a surprisingly simple answer for this symmetrical and unrealistic system.