Chapter 4

Force due to Gravity

- **Famous Newtonian Gravitational Force Giving Us Weight**
- **Famous Constant Gravity Approximation at Earth's Surface**
- **Famous Kepler's Laws for Orbital Motion**

Force of Gravity

In addition to his three laws, Newton also developed a theory for the force due to gravity that he could then use in his second law. He found that the magnitude (absolute value) of the force of gravity between any two bodies is given by the famous inverse-square law relation: $|\vec{F}_{grav}| = G\frac{Mm}{r^2}$,

where:

- M and m are the respective masses of the two objects
- r is the distance between the two objects
- G is a constant known as the Universal Gravitational Constant with a value of $6.67\text{x}10^{-11}$ Nm^2/kg^2 (or $\text{m}^3/(\text{s}^2\ \text{kg})$).

The direction of the force on either mass is TOWARD the other mass; hence, this is an "attractive" force between the two objects. The presence of the r^2 in the denominator is the inverse-square proportionality.

Constant Gravity Approximation at the Earth's Surface

Now what about the force of gravity holding us, or any mass m, to the surface of the Earth? Note that a wonderful thing is that **we can calculate the force caused by the Earth on us as if all the Earth's mass were concentrated into a point at its center.** (To prove this it is most straightforward to use the techniques introduced in the study of electric fields – see Gauss's law in your introduction to electromagnetism!)

Moreover, we observe that, at the surface of the Earth, the radius of the earth is huge compared to the changes in elevation we encounter in our daily lives. Therefore, if we elevate or drop a distance d of even a few miles, we can just neglect d in Newton's gravitational force. **This is the famous constant force approximation for gravity at the surface of the earth!** See the next page.

$R_{earth} \cong 4000$ miles

$r = R_e + d$ but $d << R_e$

so $r \cong R_e$

and $G\frac{Mm}{r^2} \cong G\frac{Mm}{R_e^2}$

Introducing the Famous Constant Gravity Acceleration "g" at the Earth's Surface: Define $\left|\vec{a}_{surface}\right|$ as the magnitude of the acceleration of a mass m at the surface. [As a vector, the gravitational force on m and hence the acceleration of m, points in toward the center of the Earth (the inward radial direction), according to Newton's second law.] From that law and the previous discussion of the smallness of our local elevation,

$$\left|\vec{F}_{surface\ gravity}\right| = \frac{GM_e m}{(R_e + d)^2} \cong \frac{GM_e m}{(R_e)^2} = m\left|\vec{a}_{surface}\right|$$

The m cancels out* giving

$$\left|\vec{a}_{surface}\right| = \frac{GM_e}{R_e^2} \equiv g$$

and so we have officially defined the constant acceleration coming from the constant force as the label g:

$$\boxed{g \equiv \frac{GM_e}{R_e^2} > 0}$$

in terms of which $\left|\vec{F}_{surface\ gravity}\right| = m\left|\vec{a}_{surface}\right| \cong m\frac{GM_e}{R_e^2} \equiv mg$.

Of course, this is the same g we saw in Chapter 3, where we looked at constant gravity trajectories.

We thus have the familiar weight "W" due to uniform gravity at the surface of the Earth:

$$\boxed{F_{\substack{surface\\gravity}} \equiv W = mg}$$

* Note that the above cancellation of m only occurs if the m in F=ma and the m in GMm/r^2 are the same (called the inertial mass and gravitational mass, respectively). This equivalence has been verified by experiment to one part in a trillion (10^{12}).

So the wonderful thing is, ignoring air friction (which we often do in this part of the mechanics course, even when it is bad to do so!), gravitational acceleration is the same for all bodies, independent of their mass and size!

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Problem 4-1 *The radius of the Earth is 6.37 x 10^6 m. The mass of the Earth is 5.974 × 10^{24} kg.*

a) *What is the gravitational acceleration (magnitude and direction) of a 0.00456789034567 kg object at the Earth's surface?*
b) *What is the gravitational acceleration (magnitude and direction) of a 10^5 kg object at the Earth's surface?*
c) *The International Space Station (ISS) orbits at an altitude approximately 380 km above the surface of the Earth. What is the acceleration due to gravity at that distance from the center of the Earth?*

[As long as your radial distance is at least as big as the radius of the Earth (your distance is $r = R_e + d$ *with* $d \geq 0$*) you can continue to calculate the force due to the Earth at those locations as if all the Earth's (huge) mass were at its center.]*

Note, in Chapter 12, we'll learn how to relate the ISS gravitational acceleration to the ISS velocity since this is circular motion (the well-known "centripetal" discussion).

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The Role of Mass Density in Gravity

The mass density ρ of a 3D body is its mass per unit volume. And if the mass is distributed uniformly, ρ is constant. And any uniform solid sphere with radius R and total mass M has the constant mass density

$$\rho = \frac{M}{\left(\frac{4\pi R^3}{3}\right)}$$

Because $\rho \frac{4\pi R^3}{3} = M$, we have $M \propto R^3$ if ρ is constant.

Indeed, as an approximation, we usually assume the Earth to be a sphere with constant mass density.

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Problem 4-2

a) *Calculate the approximate density of the Earth in kg/m^3 and compare it to that of water.*
b) *Find the gravitational acceleration in m/s^2 at the surface of a planet that has the same mass as Earth, but twice the (uniform) density.*

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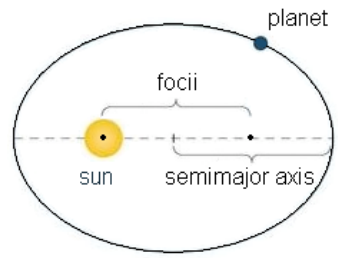
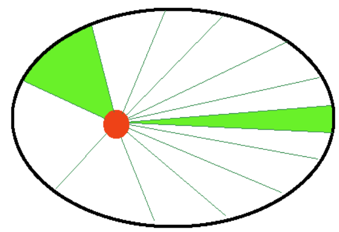
Kepler's Laws – The Basics

A great achievement of Newton was to show that Kepler's laws could be derived from putting his gravitational force into his second law. The proof of these laws for general elliptical orbits will have to wait for your next course in mechanics – sorry – but we'll prove this for circular orbits in Ch.4+.

Kepler's laws are (and see comments in the problem):



1st LAW: **All planets move in elliptical orbits, with the Sun at one focus (see first diagram).**



2nd LAW: **The line connecting the planet to the Sun sweeps out equal areas for equal lengths of time (such as the two equal shaded areas in the second figure).**

3rd LAW: **The square of the period, T (the "year!") of any planet is proportional to the cube of the semimajor axis, a, of the elliptical orbit (see first diagram): $T^2 \propto a^3$**

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Problem 4-3

a) *In discussing the First law, we have drawn the above elliptical pictures for planetary orbits. But, except for Pluto and plutoids, what is misleading about such pictures for the actual planetary orbits around our Sun?*
b) *In discussing the Second law in terms of the second figure above, we indicated that the planet traverses the arc lengths of the two shaded sectors in equal lengths of time. Use this example to explain why a planet must travel faster when it is closer to the Sun.*
c) *In the Third law, the proportionality constant is independent of the planet's mass (or density or radius). This is because the Third law is ultimately derived from Newton's second law, and, in the second law, the planet's mass cancels out between both sides of the equation. Show this.*
d) *An asteroid going around the Sun may be going so fast it comes in from far away ("infinity") and swings around the Sun as a focal point and then zooms off to infinity again, never to return. This is an "unbound" orbit. A planet, on the other hand, never leaves, and orbits around forever. This is a "bound" orbit. These Newtonian orbits are described by "conic section" curves. Use your favorite way of searching or looking up conic sections and decide which conic section curve describes a bound orbit and which describes an unbound orbit. (Actually, there is one special curve for the barely unbound orbit. Which do you think it is?)*

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