Chapter 10

Back to The Future II: Force and Momentum

- **Momentum Definition (really, linear momentum) and Impulse**
- **Rate of Change of Momentum: A General form of Newton's Second Law**
- **All 1D**
- **Conservation of Momentum: Predictions**

Momentum Motivation

We notice in 1D that the second law can be rewritten as

$$F = ma = m\frac{dv}{dt} = \frac{d}{dt}(mv) \quad \text{if} \quad m = \text{constant}$$

This motivates the definition

$$\boxed{p = mv} \equiv \text{momentum in 1D}$$

so that

$$\boxed{F = \frac{dp}{dt}} \equiv \text{Newton's second law in terms of momentum in 1D}$$

In 3D, it is easy to see the above generalizes to

$$\boxed{\vec{p} = m\vec{v}} \equiv \text{momentum in 3D}$$

so that

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt}} \equiv \text{Newton's second law in terms of momentum in 3D}$$

Comments:

- There is no new unit for momentum (just N • s or kg·m/s).
- The $\vec{v}$, $\vec{a}$, $\vec{p}$ quantities may also be called linear velocity, linear acceleration, linear momentum (to distinguish from angular velocity, angular acceleration, angular momentum – see later).
- This is really the **more general form of Newton's second law** and it is good even when m is **not** constant. If m is not constant, then

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt} = m\vec{a} + \frac{dm}{dt}\vec{v}$$

The new $\frac{dm}{dt}\vec{v}$ term will play a role in systems whose mass is changing in time (like a rocket burning fuel – see later). BUT we assume m is constant everywhere until we say otherwise!

Back To The Future II : Conservation of Momentum

So right now, it's Doc (Emmett) Brown time again. We are about to get another tool with which we can predict the future.

Suppose $\vec{F} = 0$, then $\vec{F} = \dfrac{d\vec{p}}{dt} = 0$ so

$$\boxed{\text{If } \vec{F} = 0, \text{ then } \vec{p} = \text{constant in time}}$$

- Well, this is obvious for a single particle since we know zero force means zero acceleration means constant velocity.
- But it generalizes to a rich theorem for a general system of particles, blocks, cars, planets, ...

The theorem as a formula:

$$\boxed{\text{If } \vec{F}_{net} = 0, \text{ then } \vec{p}_{total} = \text{constant in time}}$$

The theorem in words:

If there is no external net force ($\vec{F}_{net} = 0$) on the system, then the system's total momentum $\vec{p}_{total}$ is constant in time – no matter how much internal interaction there is!

To quickly understand this theorem, let's prove it for two particles interacting with each other (there's a force on each due to the other) but with nobody else. (A general proof is described in the next chapter.)

First, each particle obeys Newton's second law:

$$\vec{F}_{2\text{ on }1} = \frac{d\vec{p}_1}{dt} \quad \text{and} \quad \vec{F}_{1\text{ on }2} = \frac{d\vec{p}_2}{dt}$$

Therefore, the net (total) force can be written as

$$\vec{F}_{2\text{ on }1} + \vec{F}_{1\text{ on }2} = 0 \quad \Rightarrow \quad \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0$$

But by Newton's third law ($\vec{F}_{2\text{ on }1} = - \vec{F}_{1\text{ on }2}$)

$$\vec{F}_{2\text{ on }1} + \vec{F}_{1\text{ on }2} = 0 \quad \Rightarrow \quad \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0$$

Finally, since the total momentum is $\vec{p}_{total} = \vec{p}_1 + \vec{p}_2$,

$$\frac{d\vec{p}_{total}}{dt} = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 \quad \Rightarrow \quad \vec{p}_{total} \text{ is constant in time}$$

Q.E.D.

Conservation of Momentum: 1D Examples

Let us consider two simple but relevant situations. We consider **two bodies restricted to 1D motion**, but with no external forces along that direction. An example is two cars colliding on a frictionless straight track. With no external horizontal forces, we must have the total 1D momentum along the track, $p_1 + p_2$, constant for all time, no matter how complicated the interaction is between the two cars. **BELOW, WE USE X-COMPONENTS FOR MOMENTA AND VELOCITIES SO THEY CAN BE POSITIVE OR NEGATIVE (NOT MAGNITUDES NOW).**

1) Systems becoming "unstuck" (e.g., disintegrations or fission):

Consider the two bodies sitting next to each other and assume they somehow push away from each other (e.g., stick a big compressed and massless spring between the bodies and let it go!)

Before: (1)(2) resting $\Rightarrow$ $p_1 = p_2 = 0$ so certainly $p_1 + p_2 = 0$ initially

no outside forces

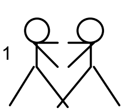
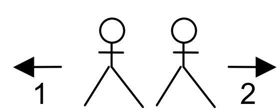
so the total momentum must be zero afterward, too:

After: ← (1) (2) → $p_1 + p_2 = 0 \Rightarrow m_1v_1 + m_2v_2 = 0$ (1D, positive to the right)

$$\Rightarrow m_2v_2 = -m_1v_1 \Rightarrow \boxed{\frac{v_2}{v_1} = -\frac{m_1}{m_2}}$$ (minus sign $\Rightarrow$ moves opposite as expect)

Result: The final speeds vary in inverse relation to their masses – a conclusion you know from pushing hard on some huge person while you're ice skating or roller skating! (If they're really huge, you might not only recoil away fast, you might decide to go running away even faster!)

Example: A 20-kg girl (m_1 below) and a 30-kg girl (m_2) are standing initially at rest in the middle of an ice-covered pond. One girl shoves the other (what did you think? We were going to put a cherry bomb between them?), and they slide apart such that the 20-kg girl has a final speed of 2 m/s, moving to the left as shown so $v_1 = -2 < 0$ (remember these are x-components). Assuming that the girls slide on the ice with negligible friction, determine the final speed of the 30-kg girl.

Before: m_1  m_2 **After:** 

$$\therefore\ p_{tot}(\text{initial}) = 0 \text{ which means that } p_{tot} = 0 \text{ forever!}$$

The above result $\frac{v_2}{v_1} = -\frac{m_1}{m_2}$, applies here, where the heavier girl is body 2:

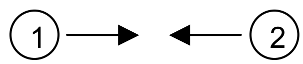
Thus $v_2 = -\frac{m_1}{m_2}v_1 = -\frac{20}{30}(-2) \Rightarrow v_2 = +\frac{40}{30} = 1.33\ \text{m/s} < 2\ \text{m/s}$ as expected..

2) Systems becoming "stuck" (e.g., fusion)

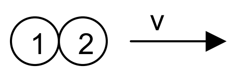
Now consider two bodies moving straight at each other. They hit and stick together into a single blob. With no external forces, the momentum of the blob must be equal to the sum of the original momenta, $\vec{p}_1 + \vec{p}_2$, and, since the original momentum vectors lie on the same line, the final blob's momentum must be in that same direction, too. (Conservation of momentum tells us that the vectors $\vec{p}_{final} = \vec{p}_{initial}$ are equal, both in magnitude AND direction.)

Since all the momenta lie on a straight line, we have a simple 1D problem again (we look at 2D collisions later), and we can use 1D momentum conservation before and after to get an equation for the final speed of the blob. Referring to the right direction as the positive 1D axis:

Before: $p_{total}^{initial} = m_1 v_1 + m_2 v_2$



After: $p_{total}^{final} = (m_1 + m_2)\, v$



These momenta must be equal: $m_1 v_1 + m_2 v_2 = (m_1 + m_2)\, v$

Solve for v of the blob: $\boxed{v = \dfrac{m_1 v_1 + m_2 v_2}{m_1 + m_2}}$

Consider the case where body 2 is initially at rest: $v_2 = 0$ so that

$$\Rightarrow \quad v = \frac{m_1}{m_1 + m_2} v_1$$

Since $\dfrac{m_1}{m_1 + m_2} < 1 \quad \Rightarrow \quad v < v_1$.

The inequality $v < v_1$ is what you might guess for this case – the blob should have a lower final speed than the first body's initial speed.

3) Kinetic energy digression

We discuss below that, while momentum **is** conserved in the above examples, kinetic energy **is not** conserved in those examples:

<u>I) Systems becoming UNSTUCK:</u> There was no kinetic energy at the beginning but afterwards we have some! Where does it come from? It always has to come from some new source of energy. Ultimately, the skaters' muscles, or, really, food! Or a hidden coiled spring!

<u>II) Systems becoming STUCK:</u> We can show that kinetic energy is always lost in this "sticking." For convenience, look at an example where $v_2 = 0$, and ... well, let's let you do it in the following problem. The problem following that gives you practice for both of these cases.

****

Problem 10-1

a) *In the "fusion" scenario on the previous page, let* $v_2 = 0$. *Now show that the final kinetic energy is always less than the initial kinetic energy in this collision. Hint:again use* $\frac{m_1}{m_1 + m_2} < 1$

(As we alluded above, we call this a "completely inelastic" collision, in comparison to other collisions where less kinetic energy is lost or even an "elastic" collision which is defined to be a collision where the overall kinetic energy is conserved. See the next cycle for a discussion of elastic collisions.)

b) Where did the extra kinetic energy go? This is a qualitative kind of question, meaning you just discuss it with words and no calculation.

****

****

Problem 10-2

A father (of mass 70 kg) faces his young daughter (of mass 20 kg) on a level roller rink and they're both on roller skates (so when they are moving, we will assume they are rolling such that they move with practically no friction).

a) Dad pushes his daughter and she rolls backward at a speed of 2.5 m/s. Determine the speed of her dad and the direction of his motion relative to hers.

b) Now instead, suppose Dad is roller-skating toward his daughter with a speed of 2.0 m/s, and she is roller-skating toward him in the opposite direction. They grab each other gingerly to avoid getting hurt and hold on to each other. How fast was his daughter going originally if afterwards, when they are holding on to each other, they have completely stopped moving with respect to the roller rink?

c) Discuss qualitatively what might be changed in both of the above situations if dad and daughter are wearing regular shoes instead of roller skates? In (a), they're standing next to each other initially and in (b) they're running at each other.

****

An old, famous poem:

Gin a body meet a body comin' thro' the rye, gin a body kiss a body - need a body cry? well, that's a complicated meeting. But if a body meet another body on an ice rink or on a roller rink such that the external forces are zero, then that's much easier!

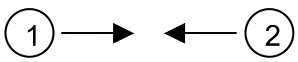
"Impulse J" ≡ Change in Momentum

We define **impulse J** as simply the momentum change **a given body** undergoes in an interaction, like a collision. So in the full 3D world, between some initial time and some final time, we have

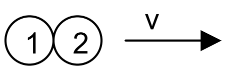
$$\boxed{\text{Impulse } \vec{J} \equiv \Delta\vec{p} = \vec{p}(\text{final}) - \vec{p}(\text{initial})}$$

Impulse is an especially appropriate name for the momentum change during a collision (the name suggests a quick change, doesn't it?). We often just write impulse $\Delta\vec{p}$ and forgo using $\vec{J}$.

Example: From our "fusion" scenario, we now look at the initial and final momenta of each body 1 and body 2, instead of their total momentum:

Before: 

$p_1^{\text{initial}} = m_1 v_1$, $p_2^{\text{initial}} = m_2 v_2$ (here, $v_2 < 0$)

After: 

$p_1^{\text{final}} = m_1 v$, $p_2^{\text{final}} = m_2 v$

Impulse: Body 1: $\Delta p_1 = m_1 v - m_1 v_1$

Body 2: $\Delta p_2 = m_2 v - m_2 v_2$

But we're not done. We can show that $\Delta p_1 + \Delta p_2 = 0$ (i.e., $\Delta p_1 = - \Delta p_2$) :

$$\Delta p_1 + \Delta p_2 = m_1 v - m_1 v_1 + m_2 v - m_2 v_2$$

$$= (m_1 + m_2)\, v - m_2 v_2 - m_1 v_1 \quad \text{by rearrangement}$$

$$= 0 \quad \text{by momentum conservation (p. 10-4)}$$

(Alternatively, you can see the cancellation between by substituting $v = \dfrac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$ into the second line.)

Therefore, from momentum conservation it follows that

> **When two bodies collide, their impulses are equal and opposite.**

(But when their masses are very different, their changes in velocity – and the pain they feel – can be very different, even though their impulses have the same magnitude!)

The Relation of Impulse to the Force that Caused It!

Recall the more powerful form of the second law directly relates the force acting on a body to the derivative of the body's momentum:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

We can integrate this to find the change in momentum – the impulse – over some time interval. Multiply both sides by the time differential dt:

$$\vec{F}\,dt = d\vec{p}$$

Integrate both sides from some initial state and time to a final state and time:

$$\int_{t_i}^{t_f} \vec{F}\,dt = \int_i^f d\vec{p} = \vec{p}_f - \vec{p}_i$$

Therefore

$$\boxed{\Delta\vec{p} = \int \vec{F}\,dt = \text{Impulse } \vec{J}}$$

Comments:

1) Thus we know something about the force acting on the body, if we know how the body's momentum has changed.

2) In mathematics, we have $A_{average} \equiv \dfrac{\int_0^T A\,dt}{T}$, an expression for the time average of any quantity A over a time T.

   Therefore: $\Delta\vec{p} = \vec{F}_{average}\,\Delta t$ with $\vec{F}_{average} \equiv \dfrac{\int \vec{F}\,dt}{\Delta t}$

3) If $\vec{F}$ is constant in time, $\int \vec{F}\,dt = \vec{F}\int dt = \vec{F}\,\Delta t$, so that

$$\Delta\vec{p} = \vec{F}\,\Delta t$$

   which you might have written down immediately given $\vec{F} = \dfrac{d\vec{p}}{dt}$

The upshot of all this is that we may not know the detailed forces, but we often can easily determine the initial and final momenta! That is, we may know the impulse. Thus if we know the time interval over which the force acts, we know its average value!

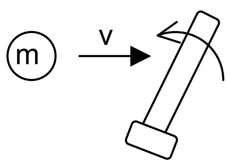
See the example on the next page.

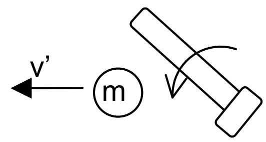
Example:

A pitched baseball of mass 145 g (= 0.145 kg) crossed the plate recently at a speed of 90 MPH (90 MPH = 0.447 x 90 = 40.2 m/s).

It was hit by a Mr. Hafner and the ball left his bat at 135 MPH (0.447 x 135 = 60.3 m/s), coming straight back along the line it came in on (straight back at the pitcher). We have a 1D collision.

If the total time of collision with the bat was about 0.5 ms (millisecond), then what was the average force on the ball during this time?

before: 

after: 

ASSUME 1D - collinear

Note that $v' < 0$ since the ball moves left after the collision (again, velocities are x-components)

$$\Delta p = mv' - mv = m(v' - v) = 0.145\,(-60.3 - 40.2) = -14.6 \text{ kg m/s}$$

$$F_{average} \cong \frac{\Delta p}{\Delta t} = \frac{-14.6}{5 \times 10^{-4}} = -29{,}000 \text{ N}$$

(The impulse and the average force are in the negative x-direction.)

Compare the magnitude of this average collision force to the force on the ball due to gravity:

$$|F_{average}| \gg F_{weight} = mg = 0.145(9.8) = 1.4 \text{ N}$$

20,000 times bigger!

****

Problem 10-3

A 1.0-kg medicine ball drops vertically hitting the floor with a speed of 20 m/s. It rebounds vertically upwards with a speed of 10 m/s immediately after leaving the floor.

a) *What impulse acts on the ball during contact?*

b) *If the ball is in contact for 0.025 s, what is the average force exerted on the floor during this collision?*

c) *How does your answer (b) compare with the weight of the medicine ball?*

d) *How would your answer to (b) change if the medicine ball just went 'splat' and didn't bounce at all?*

****

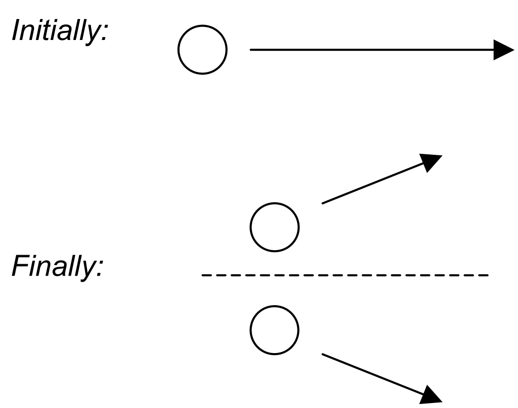
****

Problem 10-4 Quick while we've got some momentum:

a) Explain using examples what the difference is between elastic and inelastic collisions.

b) Describe a collision example where the "impulse" is zero.

c) In the second cycle, we will look at momentum conservation in 2D. For example, consider an explosion of a body into two fragments. The process looks like the following:



This is an isolated system that must have the same total momentum vector afterwards it had before the explosion. But how could the above happen? There was no initial momentum in the vertical direction and now we see that the final momenta of the fragments have vertical components. They are no longer along the original momentum direction. Can you explain or at least guess why momentum can still be conserved when the fragments go off at angles as shown?

****