Solution to Problem 1:

Part (a): (10 points)

This is a problem in **One-Dimensional Kinematics.** We are given the position as a function of time. To get the velocity, **by definition** we take the ***derivative of the position with respect to time***:

$$y = A + Bt - Ct^2 + Dt^3$$

$$v(t) \equiv \frac{dy}{dt} = B - 2Ct + 3Dt^2$$

Plugging in for $t = 0$ we get:

$$\boxed{v(t = 0) = B}$$

Part (b): (10 points)

We continue to apply **One-Dimensional Kinematics. By definition** we can calculate the acceleration by taking the ***derivative of the velocity with respect to time***:

$$a(t) \equiv \frac{dv}{dt} = -2C + 6Dt$$

Plugging in for $t = T$ we get:

$$\boxed{a(t = T) = -2C + 6DT}$$

Part (c): (10 points)

If the ***net force is zero*** then we know by **Newton's Second Law** that the acceleration must also be zero:

$$F_{net} = ma = 0$$
$$a(t = t_n) = 0$$

We use our result from **Part (b)**: to express the acceleration at time $t = t_n$:

$$a(t = t_n) = -2C + 6Dt_n = 0$$

Solving for t_n:

$$-2C + 6Dt_n = 0$$
$$6Dt_n = 2C$$
$$t_n = \frac{2C}{6D}$$
$$\boxed{t_n = \frac{C}{3D}}$$

Solution to Problem 1 Continued...

Part (d): (10 points)

We consider the **Effective Weight as calculated in the non-inertial reference frame of the elevator.** If your mass is m then the Effective Weight is given by the (vector) sum of the Real Weight plus the Fictitious Weight:

$$\vec{W}_{eff} = \vec{W}_{real} + \vec{W}_{fict}$$
$$\vec{W}_{eff} = -mg\hat{\boldsymbol{\jmath}} - m\vec{a}$$

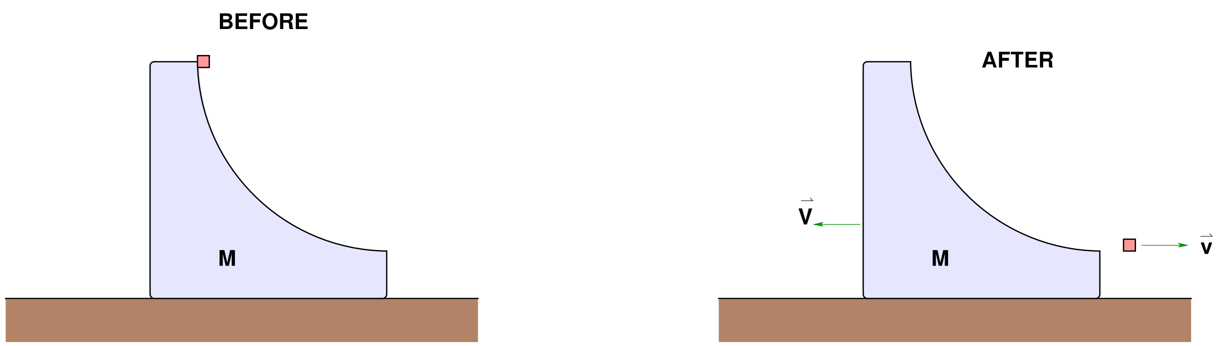
Using our result from Part (b):

$$\vec{W}_{eff} = -mg\hat{\boldsymbol{\jmath}} - m(-2C + 6Dt)\hat{\boldsymbol{\jmath}}$$

$$\vec{W}_{eff} = -m(g - 2C + 6Dt)\hat{\boldsymbol{\jmath}}$$
$$W_{eff} = m(g - 2C + 6Dt)$$

At time $t- = 0$ the magnitude of the Effective Weight is $m(g - 2C)$ (less than mg) but increasing ***linearly with time*** to a value $m(g - 2C + 6DT)$ (greater than mg). In other words, during the interval, you start out feeling $\boxed{\textit{\textbf{lighter than normal}}}$ but as time progresses, you steadily feel $\boxed{\text{increasingly heavy}}$.

Solution to Problem 2:



The key is to see this problem as an **inverse collision** where both (horizontal) momentum and energy are conserved.

In the **horizontal direction** we can say that momentum is conserved. There are no net horizontal forces on the system.

$$
\begin{aligned}
P_{tot} &= P'_{tot} \\
p_M + p_m &= p'_M + p'_m \\
0 + 0 &= MV + mv \\
V &= -v\frac{m}{M}
\end{aligned}
$$

Solution continues next page....

Next we consider **Conservation of Energy**: which works fine since the only forces here are Weight (conservative) and Normal (which here does no work).

$$E_{tot} = E'_{tot}$$
$$U + K = U' + K'$$
$$mgy + 0 = mgy' + \frac{1}{2}mv^2 + \frac{1}{2}MV^2$$
$$mgR + 0 = 0 + \frac{1}{2}mv^2 + \frac{1}{2}MV^2$$
$$mgR = \frac{1}{2}mv^2 + \frac{1}{2}MV^2$$
$$mgR = \frac{1}{2}mv^2 + \frac{1}{2}M\left(-v\frac{m}{M}\right)^2$$
$$v^2 + \frac{M}{m}\left(-v\frac{m}{M}\right)^2 = 2gR$$
$$v^2 + v^2\left(\frac{m}{M}\right) = 2gR$$
$$v^2\left[1 + \left(\frac{m}{M}\right)\right] = 2gR$$
$$v^2 = \frac{2gR}{1 + \left(\frac{m}{M}\right)}$$
$$\boxed{v = \sqrt{\frac{2gR}{1 + \left(\frac{m}{M}\right)}}}$$

Note to Grader: Students who fail completely to contend with the motion of the large block have made a **major conceptual error** which will have a large impact on their ability to solve the problem using the physics concepts we want them to use.

Some students argue that they can use Conservation of Energy to calculate the velocity of the small block **relative** to the large block. This, while clever, is conceptually incorrect. The large block does not represent an "inertial reference frame" – indeed while the small block is sliding down, the large block is *accelerating* to the left.

The only correct way to use Conservation of Energy is to consider the kinetic energy of both the large block and the small block at the same time.

Solution to Problem 3:

Part (a): (5 points)

The "tee" is a ***system*** that is made from two different thin rods. By **symmetry** we know that the left (horizontal) rod has a center-of-mass point located at a distance $x_L = L/2$ corresponding to half-way along the rod. Likewise, the center-of-mass point for the right (vertical) rod is located at the right end of the left rod $x_R = L$. We use the **Definition of the Center-of-Mass** which tells is that x_{cm} is calculated as the *weighted average position* of each component (treating each rod as a point-mass):

$$x_{cm} = \frac{m_L x_L + m_R x_R}{m_R + m_L}$$

$$x_{cm} = \frac{M(L/2) + ML}{2M}$$

$$x_{cm} = \frac{3L/2}{2}$$

$$x_{cm} = \left(\frac{3}{4}\right) L$$

Therefore, since $x_{cm} = KL$ we see immediately that

$$\boxed{K = \frac{3}{4}}$$

Guidelines for grader:

- **5 pts:** Correct answer includes **Definition of Center-of-mass**.
- **3 pts:** Correct answer but missing definition.
- **3 pts:** Correct set up but incorrect algebra or other manipulation.
- **2 pts:** Failure to contend with both rods.
- **at most 1 pts:** Some other incorrect physics ideas.

Solution continues next page...

Part (b): (5 points)

We are asked to determine a *numerical* constant C where $I = CMR^2$ for the tee.[1] The total rotational inertial of the system of two rods must be equal to the **sum of that for each rod**,[2] as calculated about the hinge pivot point:

$$I_{PP} = I_L + I_R$$

Here I_L is just that for a thin rod about one end: $I_L = \frac{1}{3}ML^2$. For the right rod, however, we need to use the **Parallel Axis Theorem** since the center of mass is displaced from the pivot-point:

$$I_R = I_{cm} + MD^2 = \frac{1}{12}ML^2 + ML^2$$

Altogether then:

$$I_{PP} = \frac{1}{3}ML^2 + \left(\frac{1}{12}ML^2 + ML^2\right)$$

$$I_{PP} = \left(\frac{4}{12}\right)ML^2 + \left(\frac{13}{12}\right)ML^2)$$

$$I_{PP} = \left(\frac{17}{12}\right)ML^2$$

Therefore, since $I_{pp} = CML^2$ we see that

$$\boxed{C = \frac{17}{12}}$$

Guidelines for grader:

- **5 pts:** Correct answer includes both **Sum** and **Parallel Axis Theorem** concepts.
- **3 pts:** Correct answer but missing explanations.
- **3 pts:** Correct set up but incorrect algebra or other manipulation.
- **2 pts:** Failure to contend with both rods properly.
- **at most 1 pts:** Some other incorrect physics ideas.

Solution continues next page...

[1] Several students took a disastrous path of applying Newton's Second Law to Part (b) here. When we ask for a "numerical constant" we are literally looking for a number. Applying Newton's Second Law does not yield the value of rotational inertia here, but even if it did, the "answer" comes out in terms of symbolic parameters such as the mass, length and so forth. Also a "five point" part tells you that the work for this must be short and quick.

[2] To be clear, we have a fundamental definition of the rotational inertia as a integral, which of course is just a sum. In this problem, we have two bodies move together as one system. So the moment of inertia, which is an integral of the sum of the two bodies, is also the sum of the integrals. In other words, we just **add** the rotational inertia as calculate for each separate body about the given pivot point.

Part (c): (12 points)

Here we use **Newton's Second Law in Rotational Form**.

$$\tau_{net} = I\alpha$$

The only forces on the "tee" are the (unknown) Hinge force and the force of Weight. So we need to find these two torques:

$$\tau_{\vec{H}} + \tau_{\vec{W}} = I\alpha$$

To calculate the two torques we use the **Definition of Torque:**

$$\tau_{\vec{F}} \equiv rF\sin\phi$$

where r is the distance from the pivot point to the location where the force is applied.

For the Hinge force, since this is applied at zero distance from the pivot point, *the torque due to the Hinge force is zero*.

The force of Weight is applied to the "tee" at the position of the center-of-mass point in the downward (perpendicular) direction. We the distance to the center-of-mass is $r = KL$ So the torque due to the Weight force is just the weight force $W_{tee} = M_{tee}g = 2Mg$ and so:

$$0 + (KL)(2Mg)\sin(90°) = I\alpha$$
$$2KLMg = I\alpha$$

Solving for α and plugging in for the rotational inertia:

$$\alpha = \frac{2KLMg}{I} = \frac{2KLMg}{CML^2}$$

$$\boxed{\alpha = \left(\frac{K}{C}\right)\frac{2g}{L}}$$

Plugging in our results from parts (a) and (b):

$$\alpha = \left(\frac{3/4}{17/12}\right)\frac{2g}{L} = \left(\frac{3}{4}\right)\left(\frac{12}{17}\right)\frac{2g}{L} = \left(\frac{3}{1}\right)\left(\frac{3}{17}\right)\frac{2g}{L}$$

$$\boxed{\alpha = \left(\frac{18}{17}\right)\frac{g}{L}}$$

Guidelines for grader:

- **12 pts:** Correct answer includes explanation and correct calculation of the torque due to the Weight.
- **11 pts:** Correct answer except assigning Weight = Mg (incorrect) instead of $2Mg$ (correct).
- **10 pts:** Correct answer except force applied at $(L/2)$ instead of Center-of-Mass point.
- **6 to 9 pts:** Basic concept and setup correct but incorrect method.
- **at most 3 pts:** Some other incorrect physics ideas.

Solution continues next page...

Part (d): (12 points)

Since we want the *angular speed* a good concept to consider is **Conservation of Mechanical Energy**. First however, we must consider the ***Condition for Conservation of Energy*** which is that all of the forces on the system are *either* Conservative *or* do no work. The two forces on the "tee" are Weight (which is Conservative) and the Hinge Force (which is *not* Conservative *but* does no work in this problem since the force is not applied over any finite distance). So we have ***met the Condition*** and we apply the Conservation Law:

$$“BEFORE'' = “AFTER''$$

$$E_{tot} = E'_{tot}$$

$$U + K = U' + K'$$

$$mgy + \frac{1}{2}I\omega^2 = mgy' + \frac{1}{2}I\omega'^2$$

Here y represents the position of ***the center-of-mass of the system***. We can chose any zero-point that we wish. For simplicity we will chose $y = 0$ as the hinge point. In the "Before" case we start at $y = 0$ and we end at $y' = (-KL)$. Note that here $m = M_{sys} = 2M$.

For kinetic energy, since the entire system rotates about the hinge, and since we have already chosen the hinge as the *reference point* for calculating the rotational inertia, the proper definition of kinetic energy as being ***purely rotational***:[3]
$K_{tot} = K_{rot} = \frac{1}{2}I\omega^2$. in the "Before" case, release at rest, $\omega = 0$ and in the "After" case $\omega' = \omega_{max}$ which is what we want to find. Putting this all in:

$$(2M)g(0) + \frac{1}{2}I(0)^2 = (2M)g(-KL) + \frac{1}{2}I\omega_{max}^2$$

$$0 + 0 = -2mgKL + \frac{1}{2}I\omega_{max}^2$$

Solving for ω_{max} and plugging in the rotational inertia, the rest is algebra:

$$\frac{1}{2}I\omega_{max}^2 = 2MgKL$$

$$I\omega_{max}^2 = 4MgKL$$

[3]In general, the total kinetic energy is a sum of translational and rotational kinetic energy, and in this case, one could argue that the "tee" is both translating and rotating. This would be the correct argument if we had calculated the rotational inertia I_{tot} at a reference point that corresponds to a position on the "tee" that is moving translationally, such as the center-of-mass point. But in this case – as in the case of any rotational motion about a fixed hinge or axis – if we define the reference point at the hinge, then – by definition – this point of the body is not moving translationally, and so the body has zero translational kinetic energy and the motion is correctly described as "purely rotational" in terms of kinetic energy. In other words $K_{tot} = K_{rot} = \frac{1}{2}I_{pp}\omega^2$ only and the translational kinetic energy $\frac{1}{2}mv^2$ is set to zero. Note that in principle, we are *free* to choose a different reference point – such as the center-of-mass point of the "tee" – and in this case we would have both non-zero rotational and translational kinetic energy. But in this case we would need to use the value for the rotational inertia that corresponds to the reference point, and we would need to calculate this, since $I_{cm} \neq I_{pp}$. Choosing to include translational kinetic energy without also specifying the reference point and the associated rotational inertia as calculated about that reference point is incorrect.

$$\omega_{max}^2 = \frac{4MgKL}{I}$$

$$\omega_{max}^2 = \frac{4MgKL}{CML^2}$$

$$\omega_{max}^2 = \left(\frac{K}{C}\right)\frac{4g}{L}$$

$$\boxed{\omega_{max} = \sqrt{\left(\frac{K}{C}\right)\frac{4g}{L}}}$$

Plugging in K and C as before:

$$\omega_{max} = \sqrt{\left(\frac{9}{17}\right)\frac{4g}{L}}$$

$$\omega_{max} = \sqrt{\left(\frac{36}{17}\right)\frac{g}{L}}$$

$$\boxed{\omega_{max} = 6\sqrt{\frac{g}{17L}}}$$

Guidelines for grader:

- **12 pts:** Correct answer includes explanations.
- **11 pts:** Correct answer but incorrectly used M instead of $2M$ for the system mass.
- **10 pts:** Correct answer but forgot to explicitly check whether we have **met the Conditions** for application of Conservation of Energy.
- **10 pts:** Correct answer but incorrectly setting $y = y_{cm}$ equal to either length of rod or half-length of rod instead of position of center-of-mass of rod.
- **at most 9 pts:** Incorrect calculation of total kinetic energy (e.g., non-zero translational kinetic energy, etc.)
- **at most 6 pts:** Correct concept and setup up but incorrect method.
- **at most 2 pts:** Some other incorrect physics idea.

Solution continues next page...

Part (e): (6 points)

To get the linear speed at Point P we apply the **Rolling Constraint:**

$$v = \omega r$$

Here ω is just ω_{max} that we calculated in Part (d). The value of "r" is the distance ***from* the pivot point *to* Point P**. This distance, call it r_P is given by applying the Pythagorean Theorem to the "tee":

$$r_P = \sqrt{L^2 + (L/2)^2} = \sqrt{L^2 + \frac{L^2}{4}} = \sqrt{\frac{5L^2}{4}} = \left(\frac{\sqrt{5}}{2}\right) L$$

And so:

$$v_p = (\omega_{max})(r_P)$$

$$v_p = \left[\sqrt{\left(\frac{K}{C}\right)\frac{4g}{L}}\right]\left[\left(\frac{\sqrt{5}}{2}\right) L\right]$$

$$v_p = \frac{1}{2}\sqrt{\frac{(K)(4g)(5)(L^2)}{(C)(L)}}$$

$$v_p = \boxed{\sqrt{\frac{5KgL}{C}}}$$

$$v_p = \sqrt{5\left(\frac{3}{4}\right)\left(\frac{12}{17}\right) gL}$$

$$v_p = \sqrt{\left(\frac{36}{4}\right)\left(\frac{5}{17}\right) gL}$$

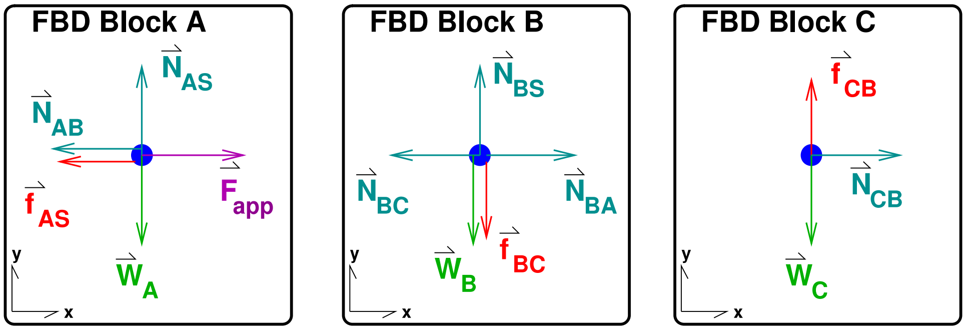
$$\boxed{v_p = 3\sqrt{\frac{5gL}{17}}}$$

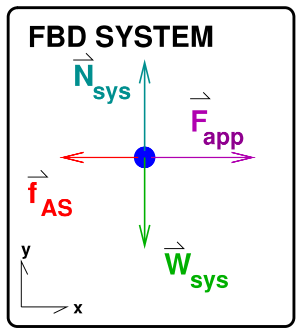
Guidelines for grader:

- **6 pts:** Correct answer includes both **Rolling Constraint** and **correct radial distance** concepts explained.
- **3 pts:** Correct concept of Rolling constraint but incorrectly used either the length of the rod or the half-length of the rod as the radial distance instead of applying the Pythagorean theorem.
- **at most 1 pts:** Some other incorrect physics ideas.

Solution to Problem 4:

Part (a): (10 points)





For the three blocks:

- There are five forces on Block A:
  - $\vec{W}_A$ The *Weight* force on Block A (downward),
  - $\vec{F}_{app}$ The *Applied* Force (rightward),
  - $\vec{N}_{AS}$ The *Normal* Force on Block A due to the horizontal surface (upward)
  - $\vec{f}_{AS}$ The *Friction* Force on Block A due to the horizontal surface (leftward), and
  - $\vec{N}_{AB}$ The *Normal* Force on Block A due to Block B (leftward).
- There are five forces on Block B:
  - $\vec{W}_B$ The *Weight* force on Block B (downward)
  - $\vec{N}_{BS}$ The *Normal* Force on Block B due to the horizontal surface (upward),
  - $\vec{N}_{BA}$ The *Normal* Force on Block B due to Block A (rightward),
  - $\vec{N}_{BC}$ The *Normal* Force on Block B due to Block C (leftward), and
  - $\vec{f}_{BC}$ The *Friction* Force on Block B due to Block C (downward).
- There are three forces on Block C:
  - $\vec{W}_C$ The *Weight* force on Block C (downward),

- $\vec{N}_{CB}$ The *Normal* Force on Block C due to Block B (rightward), and
- $\vec{f}_{CB}$ The *Friction* Force on Block C due to Block A (upward).

For the system as a whole:

- There are four (external) forces on the system as a whole:
  - $\vec{W}_{sys}$ The *Weight* force on the system where $W_{sys} = W_A + W_B + W_C$,
  - $\vec{N}_{sys}$ The *Normal* force on the system where $N_{sys} = N_AS + N_BS$,
  - $\vec{F}_{app}$ The *Applied* Force, and
  - $\vec{f}_{AS}$ The *Friction* Force on Block A due to the horizontal surface

Note to grader: The vertical forces on Block B and the vertical forces on the System as a whole are of no consequences to this problem and can be missing and/or incorrect with no consequence to the grading.

Part (b): (5 points)

We apply **Newton's Second Law** to Block A, first in the vertical direction:

$$F_{Ax} = m_A a_y$$
$$N_{AS} - W_A = 0$$
$$N_{AS} = W_A$$
$$N_{AS} = mg$$

And now we just apply the special rule for **Kinetic Friction**: $f_k = \mu_k N$

$$f_{AS} = \mu_k N_{AS}$$
$$\boxed{f_{AS} = \mu_k mg \qquad \text{leftward}}$$

Part (c): (5 points)

To get the acceleration we apply **Newton's Second Law** to the **entire system** in the vertical direction, using the FBD as show:

$$F_{sys,x} = M_{sys} a_x$$

$$F_{app} - f_{AS} = (m_A + m_B + m_C)a_x$$
$$F_{app} - \mu_k mg = (m + 2m + m)a_x$$
$$F_{app} - \mu_k mg = 4ma_x$$
$$a_x = \frac{F_{app} - \mu_k mg}{4m}$$

Finally, we note that since the three blocks move together, the acceleration any block, including Block B, is the same as the acceleration of the entire system:

$$\boxed{a_B = a_x = \frac{F_{app} - \mu_k mg}{4m}}$$

Part (d): (5 points)

We want N_{BA} There are at least two ways to do this.

Option 1: Most Elegant: We apply **Newton's Second Law** to the **(sub)-system** that is ***Block B and Block C together*** in the horizontal direction: In this case there is only one external force on this (sub)-system, and that's the one we want:

$$F_{sysBC,x} = M_{sysBC}\, a_x$$
$$N_{BA} = (3m)\left(\frac{F_{app} - \mu_k mg}{4m}\right)$$
$$\boxed{N_{BA} = \frac{3(F_{app} - \mu_k mg)}{4} \qquad \text{rightward}}$$

Option 2: We apply **Newton's Second Law** to Block A in the horizontal direction:

$$F_{Ax} = m_A a_X$$
$$F_{app} - f_{AB} - N_{AB} = (m)\left(\frac{F_{app} - \mu_k mg}{4m}\right)$$
$$F_{app} - \mu_k mg - N_{AB} = \frac{F_{app} - \mu_k mg}{4}$$
$$N_{AB} = (F_{app} - \mu_k mg) - \frac{F_{app} - \mu_k mg}{4}$$
$$N_{AB} = \frac{3(F_{app} - \mu_k mg)}{4}$$

Then we apply Newton's Third Law which tells us $N_{BA} = N_{AB}$:

$$\boxed{N_{BA} = \frac{3(F_{app} - \mu_k mg)}{4} \qquad \text{rightward}}$$

Part (e): (5 points)

We want N_{BC}. The easiest way to get this is to consider the forces on Block C. We apply **Newton's Second Law** to Block C in the horizontal direction:

$$F_{Cx} = m_C a_x$$
$$N_{CB} = (m)\left(\frac{F_{app} - \mu_k mg}{4m}\right)$$
$$N_{CB} = \frac{F_{app} - \mu_k mg}{4}$$

And then **Newton's Third Law** tells us:

$$N_{CB} = N_{BC}$$

So

$$\boxed{N_{BC} = \frac{F_{app} - \mu_k mg}{4} \qquad \text{leftward}}$$

Part (f): (5 points)

We apply **Newton's Second Law** to Block C in the vertical direction:

$$F_{Cy} = m_C a_y$$
$$F_{Cy} = 0$$
$$f_{CB} - W_C = 0$$
$$f_{CB} - mg = 0$$
$$\boxed{f_{CB} = mg \qquad \text{upward}}$$

Part (g): (5 points)

We apply the **Constraint for Static Friction:** $f_k \leq \mu_s N$. Here we apply this to the forces between Block B and Block C:

$$f_{BC} \leq \mu_s N_{CB}$$

Plugging in results from Parts (e) and (f):

$$mg \leq \mu_s \left(\frac{F_{app} - \mu_k mg}{4} \right)$$

Re-arranging:

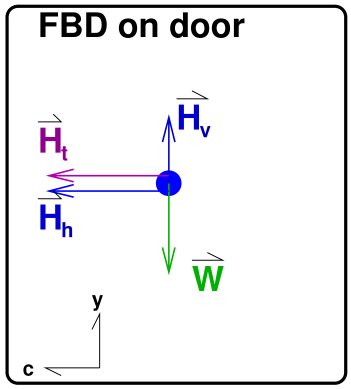
$$\mu_s (F_{app} - \mu_k mg) \geq 4mg$$
$$\mu_s F_{app} \geq \mu_s \mu_k mg + 4mg$$
$$\mu_s F_{app} \geq (\mu_s \mu_k + 4) mg$$
$$\boxed{F_{app} \geq \frac{(\mu_s \mu_k + 4) mg}{\mu_s}}$$

or

$$\boxed{F_{app} \geq \left(\mu_k + \frac{4}{\mu_s} \right) mg}$$

Solution to Problem 5: 40 points total

We start with **Newton's Second Law:** and write down the **Free Body Diagram** for the door treating it as a point mass:



For **Vertical Forces** we apply **Newton's Second Law:**

$$F_y = ma_y$$
$$H_v - W = ma_y$$

We know the vertical acceleration is zero:

$$H_v - W = 0$$

$$H_v = W$$
$$\boxed{H_v = mg}$$

For **Horizontal Forces** we apply **Newton's Second Law:**

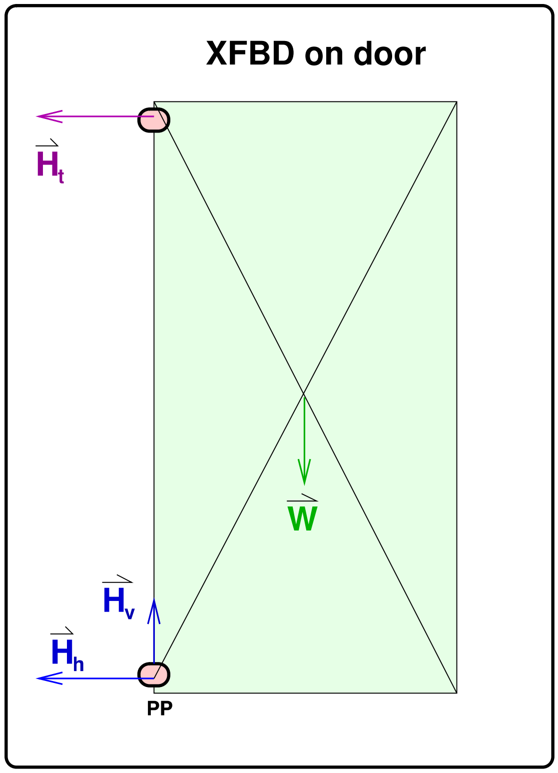
$$F_c = ma_c$$
$$H_h + H_t = ma_c$$

Since the door is swinging at rotational speed Ω we know that there is a *centripetal acceleration* on the door. Treating the position of the door as a "point mass" at the center mass out at a radius of $r = w/2$:

$$H_h + H_t = mr\Omega^2$$
$$H_h + H_t = m\mathrm{w}\Omega^2/2$$

This is as far as we can go with translational motion because we have three unknowns but only two equations. So we move on to rotational motion:

Now we deal with the rotational motion: We consider the **Extended Free Body Diagram** for the door:



We consider the torque about the pivot point defined at the lower hinge. In this case, the torques due to the lower hinge go to zero. We write down **Newton's Second Law in Rotational Form:**

$$\tau_{net} = I\alpha$$

We note that the door is not not rotating about the given pivot point in the plane of the door.

$$\tau_{net} = 0$$

$$\tau_{H_t} + \tau_W = 0$$

Using the convention that counter-clockwise is positive and noting that the torque due to the weight is given by $\tau_W = r_{perp}W = (w/2)mg$ then[4]

$$hH_t - \left(\frac{w}{2}\right) mg = 0$$

$$\boxed{H_t = \frac{wmg}{2h}}$$

So now we plug this in to get the horizontal component of the lower hinge force:

$$H_h + \frac{wmg}{2h} = mw\Omega^2/2$$

[4]To clarify. The weight force is down. The radial vector runs from the hinge to the center of the door. Since the Weight is down (vertical) the perpendicular component corresponds to the horizontal component of the radial vector. This (by construction) is the width of the door divided by two.

$$\boxed{H_t = \frac{wm\Omega^2}{2} - \frac{wmg}{2h}}$$

Solution to Problem 6:

Part (a) – : We apply the concept of **Universal Gravity:**

$$F_{ug}(r) = \frac{Gm_1m_2}{r^2}$$

Here, m_1, the mass of the earth that is within your radius and m_2 is your own mass. We note that the mass within your radius is just the fractional volume times the earth's mass:

$$m_1 = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}M$$

$$m_1 = \frac{r^3}{R^3}M$$

Therefore:

$$F_{ug}(r) = \frac{G\left(\frac{r^3}{R^3}M\right)m}{r^2}$$

$$F_{ug}(r) = \frac{GMmr}{R^3}$$

$$\boxed{F_{ug}(r) = \left(\frac{GMm}{R^3}\right)r}$$

In other words, the force is linearly proportional to the radius r.

Part (b) – : We start with **Newton's Second Law:**

$$F_r = ma_r$$

$$-F_{ug} = m\frac{d^2r}{dr^2}$$

$$-\left(\frac{GMm}{R^3}\right)r = m\frac{d^2r}{dt^2}$$

$$\boxed{\frac{d^2r}{dt^2} + \left(\frac{GM}{R^3}\right)r = 0}$$

This is the **Equation of Motion**. It looks Quite Familiar.

continues next page...

Part (c) – :

This looks very similar to the Equation of Motion for the Simple Harmonic Oscillator:

$$\frac{d^2r}{dt^2} + \omega^2 x = 0$$

Where the oscillation frequency is given by the coefficient of the second term:

$$\omega^2 = \frac{GM}{R^3}$$

So the **solution** should look oscillatory:

$$r(t) = A\cos(\omega t + \phi)$$

In this particular problem $r(t = 0) = R$ so $A = R$ and $\phi = 0$:

$$r(t) = R\cos(\omega t)$$

$$\boxed{r(t) = R\cos\left[\left(\sqrt{\frac{GM}{R^3}}\right) t\right]}$$

Part (d) – :

During the entire trip you are in **Free-Fall**. That is to say that the only force on you is the force of **gravity**. Since you are falling you experience the sensation of **"weightlessness"** the same as if you were in free-fall at the surface of the earth and/or floating in deep space. The fact that gravity changes magnitude and direction does not change the sensation of being weightless. It only changes the kinematics of your motion. You always **feel weightless** because the effective gravity is always zero in your (accelerating) reference frame.

Since you cannot infer your kinematic speed as you pass the earth's center, and since you sense "weightlessness" you **cannot** determine when you have reached the center point (unless you have timed your decent quite carefully). This is a manifestation of the Einsteins **Principle of Equivalence** which is the foundation of the theory of General Relativity.

Solution to Problem 7:

We break this problem into two distinct sub-parts:

- **Sub-part (1): The Collusion** The collision between the clay blob and the rod that imparts motion to the rod. For this part we will use **Conservation of Angular Momentum**, and
- **Sub-part (2): The Swing-Up** The swing-up of the rod from a vertical orientation to an horizontal orientation. For this we will use **Conservation of Mechanical Energy** where we will need to contend with the energy of both the rod and the blob.

Since we want the minimum value of the speed of the blob v_i just before it hits the rod, we will want to **work backwards**. So we start with **Sub-part (2)** where the goal is to find the value of the rotational speed of the bar-blob system that is required to move the bar up to the ceiling.

Sub-part (2): Swing Up: Find rotational speed required to get bar-bob system to the ceiling. Because the only forces on the system are Weight and the "Hinge Force" (which does no work here) we are free to use **Conservation of Mechanical Energy**. The "Before" corresponds to immediately after the collision with the bar in the vertical orientation. The "After" corresponds the bar swinging up to the ceiling in the horizontal orientation. We have to contend with both the potential energy and the kinetic energy of both the blob and the rod:

$$\begin{aligned}
E_{tot} &= E'_{tot} \\
E_B + E_R &= E'_B + E'_R \\
U_B + K_B + U_R + K_R &= U'_B + K'_B + U'_R + K'_R \\
mgy_B + \frac{1}{2}mv_B^2 + mgy_R + \frac{1}{2}I\omega_R^2 &= mgy'_B + \frac{1}{2}mv_B'^2 + mgy'_R + \frac{1}{2}I\omega_R'^2
\end{aligned}$$

So now we have to contend with each of these terms. We start by choosing an appropriate vertical coordinate system for the potential energy. Here we choose the center of rotation as the point corresponding to $y = 0$. With this choice we get $y_B = -L$, $y_R = -\frac{L}{2}$, $y'_B = 0$, and $y'_R = 0$. We also note that if the bar just (barely) comes in contact with the ceiling this corresponds to zero final velocity, so $v'_B = 0$ and $\omega'_R = 0$:

$$\begin{aligned}
-mgL + \frac{1}{2}mv_B^2 - \frac{mgL}{2} + \frac{1}{2}I\omega_R^2 &= 0 + 0 + 0 + 0 \\
-2mgL + mv_B^2 - mgL + I\omega_R^2 &= 0 \\
-2mgL + mv_B^2 - mgL + \left(\frac{1}{3}mL^2\right)\omega_R^2 &= 0
\end{aligned}$$

Here we note that since the blob is stuck to the bar, we can use the **Rolling Constraint** that tells us $v_B = L\omega_R$ so:

$$\begin{aligned}
-2mgL + m(\omega_R L)^2 - mgL + \left(\frac{1}{3}mL^2\right)\omega_R^2 &= 0 \\
m(\omega_R L)^2 + \left(\frac{1}{3}mL^2\right)\omega_R^2 &= 3mgL
\end{aligned}$$

$$\omega_R^2 L + \left(\frac{1}{3}\right)\omega_R^2 L = 3g$$
$$\left(\frac{4}{3}\right)\omega_R^2 L = 3g$$
$$\omega_R^2 = \left(\frac{9}{4}\right)\frac{g}{L}$$
$$\boxed{\omega_R = \left(\frac{3}{2}\right)\sqrt{\frac{g}{L}} \qquad \text{Answer to Sub-part 2}}$$

Sub-part (1): The Collision. We use **Conservation of Angular Momentum**. We can do this despite the fact that we have a completely unknown hinge force because if we chose the pivot point (axis of rotation) at the hinge then there is no net torque on the system that includes the clay blob and the rod (ignoring the effects of gravity on the clay blob before impact). We define the direction of positive angular momentum in the conventional way: an out of the page positive vector corresponds to counter-clockwise. The "before" and "after" correspond to immediately before and after the collision between the bullet and the rod, before the rod has moved and appreciable distance:

$$\vec{\mathcal{L}}_{tot} = \vec{\mathcal{L}}'_{tot}$$
$$\vec{\mathcal{L}}_{blob} + \vec{\mathcal{L}}_{rod} = \vec{\mathcal{L}}'_{blob} + \vec{\mathcal{L}}'_{rod}$$
$$\vec{r}\times\vec{p} + 0 = \vec{r}\times\vec{p}' + I\vec{\omega}$$

Okay the angular momentum of the blob before the collision is given by $\vec{r}\times\vec{p}$ so the magnitude of this is given by $(L)(mv_0)\sin\theta$. We know that by construction v' for the blob is perpendicular to the rod.

$$Lmv_i\sin\theta = Lmv_B + I\omega_R$$
$$Lmv_i\sin\theta = Lmv_B + \left(\frac{1}{3}mL^2\right)\omega_R$$
$$v_i\sin\theta = v_B + \left(\frac{1}{3}\right)L\omega_R$$
$$v_i\sin\theta = \omega_R L + \left(\frac{1}{3}\right)L\omega_R$$
$$v_i\sin\theta = \left(\frac{4}{3}\right)L\omega_R$$
$$v_i = \left(\frac{4}{3\sin\theta}\right)L\omega_R$$

Now we plug in our answer from **Sub-part 2** for ω_R to get:

$$v_i = \left(\frac{4}{3\sin\theta}\right)L\left(\frac{3}{2}\right)\sqrt{\frac{g}{L}}$$
$$\boxed{v_i = \left(\frac{1}{2\sin\theta}\right)\sqrt{gL}}$$

Solution to Problem 8:

Part (a) – 15 points: In equilibrium puck has centripetal acceleration whilst large block has zero acceleration.

Newton's Second Law: applied to Puck, centripetal coordinate:

$$F_c = ma_c$$

$$T = \frac{mv^2}{r}$$

Newton's Second Law: applied to Block, vertical coordinate:

$$F_y = Ma_y$$

$$T - W_M = 0$$

$$T = W_M = Mg$$

Putting these together for one value of the tension T:

$$\frac{mv^2}{r} = Mg$$

$$\boxed{v = \sqrt{\frac{Mgr}{m}}}$$

Part (b) – 15 points: Since the puck is undergoing a purely radial force, there is zero torque on puck as calcualted about the hole: Therefore since the system is **isolated to external torques** we can say that **Angular Momentum is Conserved**:

$$L_{tot} = L'_{tot}$$

$$mvr = mv'r'$$

$$\boxed{v' = \frac{vr}{r'}}$$

Part (c) – 10 points: The easiest way to get this is to use the **Classical Work-Energy Theorem**

$$W_{tot} = \Delta K$$

In this problem the work is done by pulling on the large block. This does not significantly change the velocity of the block but it does change the speed of the puck in accordance with Part (b). So we use this result to calculate the work:

$$\mathcal{W}_{tot} = K' - K$$

$$\mathcal{W}_{tot} = \frac{1}{2}mv'^2 - \frac{1}{2}mv^2$$

$$\mathcal{W}_{tot} = \frac{1}{2}m\left(\frac{vr}{r'}\right)^2 - \frac{1}{2}mv^2$$

$$\mathcal{W}_{tot} = \frac{1}{2}mv^2\left(\frac{r}{r'}\right)^2 - \frac{1}{2}mv^2$$

$$\boxed{\mathcal{W}_{tot} = \frac{1}{2}mv^2\left[\left(\frac{r}{r'}\right)^2 - 1\right]}$$

Solution to Problem 9:

Part (a)– (10 points):

We notice that the geometry of the situation allows us to apply simple trigonometry to write down an expression of the vertical position of the box relative to the center point of the radius of curvature of the sphere:

$$y(\theta) = -R\cos(\theta)$$

Now since there is no friction, Weight is conservative, and Normal does no work, we can apply **Conservation of Energy**. In this case, the "before" corresponds to moving with velocity v_0 on the flat surface and the "after" corresponds to arriving at the maximum angle θ_{max}:

$$\text{“}BEFORE'' = \text{“}AFTER''$$
$$E_{tot} = E'_{tot}$$
$$U_{tot} + K_{tot} = U'_{tot} + K'_{tot}$$
$$mgy + \frac{1}{2}mv^2 = mgy' + \frac{1}{2}mv'^2$$

Note that if we put the coordinate system at the center of the pivot point, the initial y-position is $-R$. Note also that the final velocity at the maximum angle will be zero:

$$mg(-R) + \frac{1}{2}mv_0^2 = mgy' + 0$$

We use our expression for y in terms of the angle. Here the angle is the maximum angle:

$$-mgR + \frac{1}{2}mv_0^2 = mg[-R\cos(\theta_{max})]$$

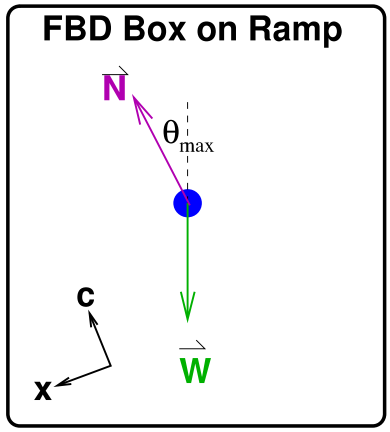
Solving for θ_{max}:

$$mg[-R\cos(\theta_{max})] = -mgR + \frac{1}{2}mv_0^2$$
$$\cos(\theta_{max}) = 1 + \left(\frac{1}{-mgR}\right)\left(\frac{1}{2}mv_0^2\right)$$
$$\cos(\theta_{max}) = 1 - \frac{v_0^2}{2gR}$$
$$\boxed{\theta_{max} = \cos^{-1}\left(1 - \frac{v_0^2}{2gR}\right)}$$

Solution continues next page....

Part (b)– (10 points):

Here we have a situation that is very similar to our old problem with the block on the ramp. The FBD looks like:



Here we define the direction of acceleration (back down the ramp) as "x" and (for later use) we define the direction of centripetal acceleration as "c".

We use Newton's Second Law in the "x" coordinate:

$$F_x = ma_x$$

Here we define the direction of acceleration (back down the ramp) as "x" and (for later use) we define the direction of centripetal acceleration as "c".

We use Newton's Second Law in the "x" coordinate:

$$F_x = ma_x$$

The only force is the component of Weight:

$$W\sin(\theta_{max}) = ma_x$$

Solving for the acceleration:

$$mg\sin(\theta_{max}) = ma_x$$

$$g\sin(\theta_{max}) = a_x$$

$$\boxed{a_x = g\sin(\theta_{max})}$$

This is the same answer we have derived in the past for a frictionless ramp. We need to use our answers from Part (a) to get an answer in terms of the original parameters:

$$\boxed{a_x = g\sin\left[\cos^{-1}\left(1 - \frac{v_0^2}{2gR}\right)\right]}$$

This can be simplified further using trig identities but it's not really worth it.

Solution continues next page....

Part (c)– (10 points):

In general the total acceleration contains both centripetal and tangential components:

$$\vec{a}_{tot} = \vec{a}_{centripetal} + \vec{a}_{tangential}$$

In terms of the coordinates we defined in the FBD:

$$\vec{a}_{tot} = \vec{a}_c + \vec{a}_x$$

For the box on the curved ramp at some arbitrary angle, we can calculate the magnitude of both terms. The expression for the tangential acceleration from Part (b) is fine. We need to modify what we did for Part (a) to get the speed for any angle:

$$U_{tot} + K_{tot} = U'_{tot} + K'_{tot}$$
$$mgy + \frac{1}{2}mv^2 = mgy' + \frac{1}{2}mv'^2$$
$$mg(-R) + \frac{1}{2}mv_0^2 = mg[-R\cos(\theta)] + \frac{1}{2}mv'^2$$

Solving for v' in terms of θ:

$$1 - \frac{v_0^2}{2gR} = \cos(\theta) - \frac{v'^2}{2gR}$$
$$\frac{v'^2}{2gR} = \frac{v_0^2}{2gR} + \cos(\theta) - 1$$
$$v'^2 = v_0^2 - 2gR[1 - \cos(\theta)]$$

So this gives us the expression for the centripetal acceleration:

$$a_c = \frac{v'^2}{R} = \frac{v_0^2 - 2gR(1 - \cos\theta)}{R}$$
$$a_c = \frac{v_0^2}{R} - 2g(1 - \cos\theta)$$

In other words, the centripetal acceleration on the curved section starts out with a maximum value and decreases to zero at the maximum angle. In contrast the tangential acceleration starts out zero and increases. An expression for the total acceleration, then is:

$$\vec{a}_{tot} = \vec{a}_c + \vec{a}_x$$
$$\vec{a}_{tot} = a_c\hat{\boldsymbol{c}} + a_x\hat{\boldsymbol{\imath}}$$
$$\boxed{\vec{a}_{tot} = \left[\frac{v_0^2}{R} - 2g(1 - \cos\theta)\right]\hat{\boldsymbol{c}} + g\sin(\theta)\hat{\boldsymbol{\imath}}}$$

where "c" and "x" correspond to centripetal and tangential coordinates.[5]

Solution continues next page....

[5] Here I define $\hat{\boldsymbol{x}}$ as a unit vector that is akin to $\hat{\boldsymbol{\imath}}$ or $\hat{\boldsymbol{v}}$ or $\hat{\boldsymbol{t}}$. Any of these coordinate systems works just as well here. I chose "c-hat" and "x-hat" because these are two directions that correspond to the (positive) direction of acceleration in each of these components.

Part (d)– (10 points):

To figure out what is going in in Bill's frame, we consider the effects of the acceleration and tipping. Further, the box experiences both centripetal and tangential acceleration.

To consider the effect of tipping the box, we note that in Bill's frame, the tipping breaks the real gravity (toward the center of the earth) into two components, one projected onto the "up-and-down" as seen in the box (the centripetal component) and one directed "side-to-side" as seen from the frame of the box (the tangential component): A little thought brings us quickly to the notion that the components depends on sines and cosines of the tip angle:

$$\vec{W} = -mg\hat{\boldsymbol{\jmath}} = -mg\cos\theta\,\hat{\boldsymbol{c}} + mg\sin\theta\,\hat{\boldsymbol{\imath}}$$

To consider the "effective gravity" inside the box as experience by Bill:

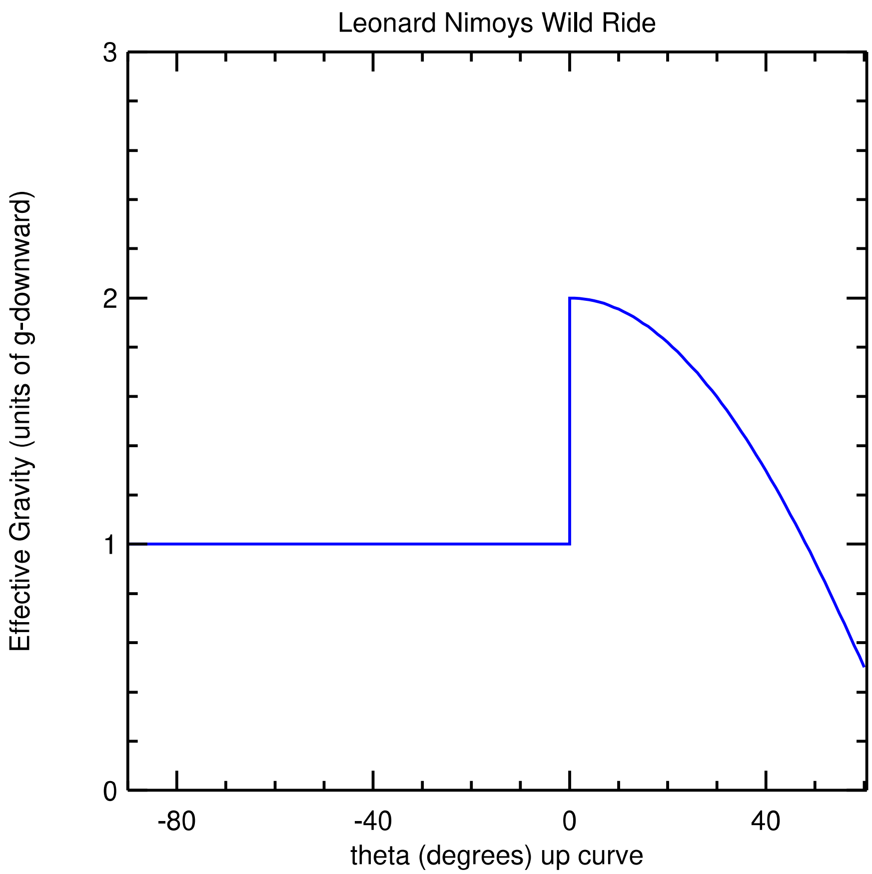
$$\vec{W}_{eff} = \vec{W}_{real} + \vec{W}_{fict}$$
$$\vec{W}_{eff} = \vec{W} - m\vec{a}_{tot}$$
$$\vec{W}_{eff} = (W_c - ma_c)\hat{\boldsymbol{c}} + (W_x - ma_x)\hat{\boldsymbol{\imath}}$$
$$\vec{W}_{eff} = (-mg\cos\theta\ - ma_c)\hat{\boldsymbol{c}} + (mg\sin\theta - a_x)\hat{\boldsymbol{\imath}}$$
$$\vec{W}_{eff} = \left\{-mg\cos\theta - m\left[\frac{v_0^2}{R} - 2g(1-\cos\theta)\right]\right\}\hat{\boldsymbol{c}} + (mg\sin\theta - mg\sin\theta)\hat{\boldsymbol{\imath}}$$
$$\vec{W}_{eff} = \left(-mg\cos\theta - m\frac{v_0^2}{R} + 2mg - 2mg\cos\theta\right)\hat{\boldsymbol{c}} + 0\,\hat{\boldsymbol{\imath}}$$
$$\vec{W}_{eff} = \left(-m\frac{v_0^2}{R} + 2mg - 3mg\cos\theta\right)\hat{\boldsymbol{c}} + 0\,\hat{\boldsymbol{\imath}}$$
$$\vec{W}_{eff} = m\left[(2-3\cos\theta)g - \frac{v_0^2}{R}\right]\hat{\boldsymbol{c}} + 0\,\hat{\boldsymbol{\imath}}$$

Note that surprising result that the magnitude of the tangential component is always zero.

In this problem, the *effective gravity* in the "side-to-side" direction (tangential) as experienced by Bill, gives rise to a fictitious Weight force $W_{eff} = mg_{eff}$ that is *exactly* the same as the perpendicular component of (real) Weight when the box is tipped. But remember, the fictitious force applies in the *opposite* direction relative to the acceleration, and therefore it applies in the opposite direction relative to the perpendicular weight component. In other words, **as Bill experiences it, because the effective gravity due to tangential deceleration exactly cancels out the perpendicular component of weight as the box is tipped, Bill does not feel any tip at all.** None. In this situation, with the boxed closed and no windows to the outside, Bill can do no experiment that will tell him that he is in a tipped box as he heads up the curved ramp. If Bill drops a penny, he will see the penny fall "straight down" to his floor, and this is true regardless of where on the ramp and what angle he happens to be at. It's amazing!

Note, however, that Bill still knows something weird is going on. In particular, as soon as he "hits the curve" the force of "gravity" will suddenly feel increased, and then will fall off, and then will increase again. In other words, Bill might predict that he is in some elevator that is accelerating upward and then decelerating. But he will not feel the "tip".

Suppose, for example, that we chose an initial speed such that $\frac{v_0^2}{R} = g$ and therefore $\frac{v_0^2}{2gR} = \frac{1}{2}$ corresponding to $\theta_{max} = 60$ degrees. We can plot the amplitude of the effective weight as a function of theta:



Here we see that at the onset of 'hitting the curve' the effective gravity jumps sharply upward (due to centripetal acceleration) but then falls off to "half-a-Gee" when the box gets to the turn-around point. So clearly Bill will know that something "non-inertial" is going on with his frame of reference, but he will not have any indication that the box he is in is being tipped.

Solution to Problem 10: 40 points total

Part (a): 10 points Because $\omega >> \Omega$ we can assume that all of the angular momentum is in the spinning wheel:

$$L = I_{wheel}\omega$$
$$L = \frac{1}{2}MR^2\omega$$

According to the right hand rule, $\vec{L}$ points in the $-x$ direction as shown above:

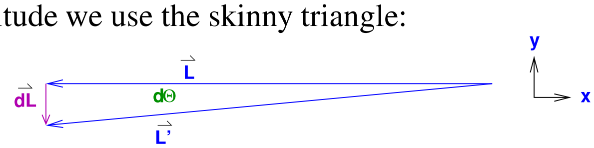
$$\vec{L} = \frac{1}{2}MR^2\omega\,(-\hat{\imath})$$
$$\boxed{\vec{L} = -\frac{1}{2}MR^2\omega\,\hat{\imath}}$$

Part (b): 10 points The turntable **forces** the angular momentum vector to rotate at a fixed rate Ω. To do this, a torque must be applied:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

Since we are rotating only, the angular speed of the gyroscope does not change. Therefore $\vec{\tau}$ must be perpendicular to $\vec{L}$ as shown in the figure above. We know that $\vec{\tau}$ must point in the $-y$ direction in order to make $\vec{L}$ turn clockwise as seen from above.

To work on the magnitude we use the skinny triangle:



$$\tau = \frac{dL}{dt}$$
$$\tau = \frac{Ld\Theta}{dt}$$
$$\tau = L\Omega$$
$$\tau = \frac{1}{2}MR^2\omega\Omega$$

In other words, given L and Ω, then τ must have this value in order that Ω rotate at this frequency. This relationship remains true, regardless of what causes the torque.

To write out a vector expression, we note that $\vec{\tau}$ rotates in the x-y plane. In other words, we have the standard "cosine" and "sine" oscillation in each coordinate, with the $t = 0$ direction corresponding to the $-y$ direction:

$$\vec{\tau}(t) = \frac{1}{2}MR^2\omega\Omega(\sin\Omega t\,\hat{\boldsymbol{\imath}} - \cos\,\Omega t\,\hat{\boldsymbol{\jmath}})$$

At the instant shown, corresponding to $t = 0$ then we get:

$$\boxed{\vec{\tau}(t=0) = -\frac{1}{2}MR^2\omega\Omega\,\hat{\boldsymbol{\jmath}}}$$

Part (c): 10 points In order to achieve this torque, new forces must be applied to the axle by the pylons. To get a torque in the $-y$ direction requires forces going down at Pylon "1" and up at Pylon "2". Since each pylon must also apply upward forces to hold the gyroscope in place, the effect is that the total force on the gyro due to Pylon "1" is decreased, while the net force on Pylon "2" must be increased. This means that the net force on Pylon "2" is greater than the net force on Pylon "1".

Assuming a symmetric arrangement, we can calculate these forces:

$$\tau_{tot} = \tau_1 + \tau_2$$
$$\tau_{tot} = -F_1\ell + F_2\ell$$
$$\tau_{tot} = -\left(\frac{1}{2}mg - F\right)\ell + \left(\frac{1}{2}mg + F\right)\ell$$
$$\tau_{tot} = 2F\ell$$
$$F = \frac{\tau_{tot}}{2\ell}$$
$$F = \frac{MR^2\omega\Omega}{4\ell}$$

Therefore the net force on Pylon "1" is:

$$\boxed{\vec{F}_1 = \left(\frac{Mg}{2} - \frac{MR^2\omega\Omega}{4\ell}\right)\hat{\boldsymbol{k}}}$$

Likewise for Pylon "2":

$$\boxed{\vec{F}_2 = \left(\frac{Mg}{2} + \frac{MR^2\omega\Omega}{4\ell}\right)\hat{\boldsymbol{k}}}$$

Part (d): 10 points Given that we have an *ideal bearing* and we are *maintaining* a fixed angular rotation of the turntable Ω, the required forces to apply the torque to the gyroscope via the two pylons are **vertical forces** which are doing no work, because nothing is moving vertically. Furthermore, we see that the gyroscope neither speeds up nor slows down. This tells us by the **Work-Energy Relation** that no Work is being done here. Zero work, no change in kinetic energy. Therefore **zero power** is required to maintain the turntable speed.

One way to see this is that the arms of the turntable only apply vertical forces, not horizontal ones. In other words, there are no horizontal forces anywhere here that would result in a torque applied over an angle or a force over a distance. Since the forces on the gyroscope are applied in a direction that is perpendicular to any motion of the gyroscope, no work is done. No work, no power.