# PHYS 121: Cycle 3 Review Sheet, Part 1

**April 9, 2024**

## Calculating The Path Integral by Cartesian Components

The term $\vec{F} \cdot d\vec{r}$ means that we are taking the dot product of these two vectors. Remember, the dot product of any two vectors, $\vec{A}$ and $\vec{B}$ is given by:

$$\vec{A} \cdot \vec{B} = AB \cos\theta$$

The dot product can also be calculated in terms of given **Cartesian Compoenents:**

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

So this means that we can calculate the Work by *breaking the Path Integral into two or three* regular *1-D integrals:*

$$W_F = \int_Q^P \vec{F} \cdot d\vec{r}$$

$$= \int_{x_Q}^{x_P} F_x dx + \int_{y_Q}^{y_P} F_y dy \qquad \text{2-D}$$

or

$$= \int_{x_Q}^{x_P} F_x dx + \int_{y_Q}^{y_P} F_y dy + \int_{z_Q}^{z_P} F_z dz \qquad \text{3-D}$$

## Rotational Inertia

Rotational Inertia and the Moment of Inertia are two different ways to say the same thing.

### Calculating Rotational Inertia

We have outlined the method of calculating the Rotational Inertia (Moment of Inertia) by calculating the volume integral:

$$I = \int \int \int \mathcal{R}^2 dm$$

where $dm$ corresponds to the mass in a little volume of material for a continuous solid. Here $\mathcal{R}$ is the distance *from* the axis of rotation chosen *to* the mass element $dm$. When the axis of rotation threads the center-of-mass point of the body, this generally leads to a value for the rotational inertia of the form:

$$I = CMR^2$$

where $C$ is some constant. Here are the particular values of the rotational inertia that you should have "in your pocket"

$$I_{hoop} = MR^2$$

$$I_{disk} = \frac{1}{2}MR^2$$

$$I_{hollowsphere} = \frac{2}{3}MR^2$$

$$I_{solidsphere} = \frac{2}{5}MR^2$$

$$I_{rod-at-CofM} = \frac{1}{12}ML^2$$

$$I_{rod-at-end} = \frac{1}{3}ML^2$$

The $R$ above refers to either the *radius* of the disk or sphere. The $L$ refers to the end-to-end *length* of the rod.

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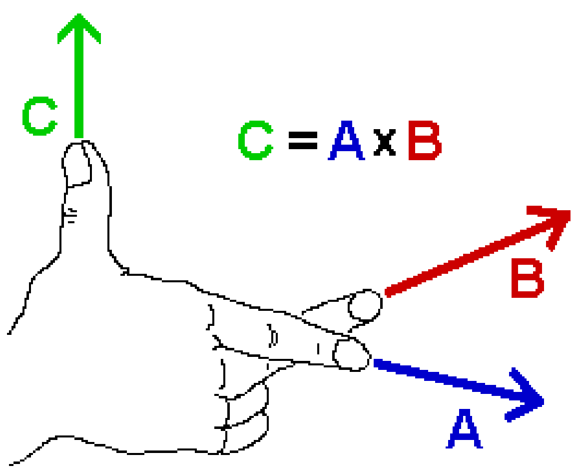
## Vectors: Two Different "Right Hand Rules" for Rotational Motion

For Cycle 3 we emphasize vector operations give us a more compact definition of torque as a vector quantity. The torque due to a given force can be **defined** as a *vector cross-product*:

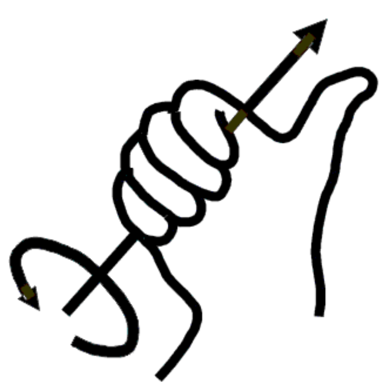
$$\vec{\tau}_{\vec{F}} \equiv \vec{r} \times \vec{F}$$

where $\vec{r}$ is the *position vector* from the "pivot point" to the location where the force is applied to the body.

The **vector cross-product** tells us that **torque is a vector** and that torque points in a direction that is *perpendicular to both the vector $\vec{r}$ and the vector $\vec{F}$*. The requirement for perpendicularity is paramount and generally constrains the direction of the vector to a single axis. The specific along that axis that corresponds to the positive vector direction is set by convention according to the **Right Hand Rule for Vector Cross-Products**:




Note that as a rotational vector, torque behaves in accordance with the **Curly-Fingered Right Hand Rule**. Specifically, if you curl your right hand fingers and stick your thumb in the direction of the torque, your fingers will curl in the direction of "positive twist".



## Review: The Rolling Constraint

A general and powerful **constraint** exists between the angular motion of a rigid body that is rotating about a fixed axis and the linear motion corresponding to some mass point embedded in that rigid body. This constaints is called the **Rolling Constraint:**

$$v = r\omega$$

$$a = r\alpha$$

It's worth emphasizing that $r$ is the distance from the chosen rotational axis to the point embedded in the rigid body. It's also worth emphasizing that this is a relationship between *scalars*, not vectors. Generally speaking, $\vec{v}$ and $\vec{\omega}$ do not point in the same direction for a spinning body.

## Review: Conservation of Angular Momentum:

By analogy we can construct the quantity of angular momentum:

$$L = I\omega$$

When the net torque on a system is zero, we call that system **isolated** in the context of torques. We can then apply **Conservation of Angular Momentum:**

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} = 0$$

$$\vec{L} = \text{ a constant}$$

In this case, we can write a "before" and "after" equation:

$$\vec{L}_{tot} = \vec{L}'_{tot}$$

Note that in some situations, it is possible for a system to be isolated in terms of torque and not forces (or vice versa). This especially happens if we have a **hinge** or a axle in the problem. As a rule, we have no idea at all about the direction or magnitude of the hinge force. But if we choose the hinge as the pivot point, then the hinge applies no torque to the system.

## Vector Angular momentum

The law of conservation of angular momentum says that in an isolated system, the total angular momentum of all bodies is the same at all times:

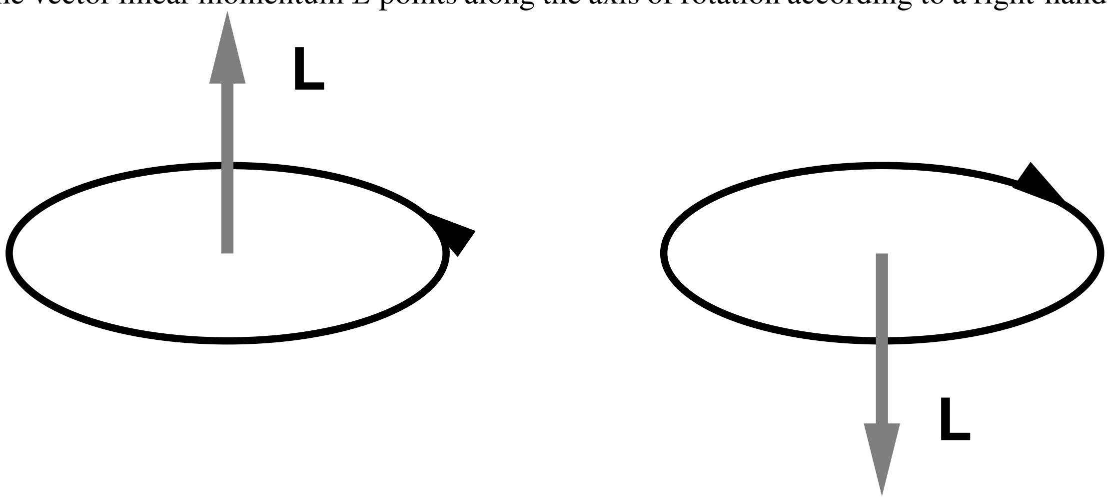
$$BEFORE = AFTER$$

$$L_{tot} = L'_{tot}$$

We note that, like linear momentum, angular momentum is a *vector* quantity. In other words, we can write:

$$\vec{L} = I\vec{\omega}$$

The vector linear momentum $\vec{L}$ points along the axis of rotation according to a right-hand-rule:




Note in particular that the vector nature of the angular momentum can be important. Recall the demonstration where the instructor stands on a turn-table while holding the axle of a spinning bicycle wheel. When the wheel is flipped, the instructor rotates in the opposite direction.

## The Angular Momentum of a Point-Like Object::

We can calculate the angular momentum associated with a point mass relative to an arbitrary axis of rotation:

$$\vec{L}_{point} = \vec{r} \times \vec{p}$$

Here, $\vec{r}$ runs from the axis of rotation to the point mass. The cross-product means that perpendicular component of the velocity relative to the radius vector:

$$L = mv_{\perp}r = mvr_{\perp} = mvr\sin\phi$$

Where $\phi$ is the angle between the radius vector $\vec{r}$ and the velocity vector $\vec{v}$. Note that we can combine the sine term with either the velocity or the radius: $v\sin\phi = v_{\perp}$ means the perpendicular (tangential) component of the velocity. $r\sin\phi = r_{\perp}$ means the perpendicular (to the velocity) component of the radius.

Note that if the point mass is moving with **Uniform Circular Motion** about the axis of rotation, this is simply expressed:

$$L_{point-particle,UCM} = L_{pp} = pr = mvr$$

For motion on a circular path, $r$ is constant and $v =_{\perp}$ is also constant so *Angular Momentum is Conserved* for particles moving with Uniform Circular Motion. Note that for **U.C.M.** we can also apply the **Rolling Constraint"** to say that

$$L_{pp,UMC} = mvr = mr^2\omega$$

Here effectively $I_{pp,UCM} = mr^2$.

For motion due to a purely **radial force** (sometime called a "central force", or a "centripetal force"), since the force is always aligned with the radius vector, torque on the system is zero, and so *Angular Momentum is Conserved.*

For motion in a straight line with **constant velocity**, there is no force, so there is no torque and therefore, *Angular Momentum is Conserved.* In this case, $v$ is a constant and $r = r_{\perp}$ is also a constant. Sometimes we call $r_{\perp}$ the "impact parameter" (sometimes labeled "$b$") which corresponds to the distance of closest approach to the pivot point. In this case we can say that $L_{pp} = mvr_{\perp} = mvb$.

## Parallel Axis Theorem::

We already talked about the method for calculating the rotational inertial of a solid body by integration:

$$I = \int\int\int \mathcal{R}^2 dm$$

Here we are taking a triple integral over the volume of the body. In this case $r$ represents the distance between a mass element $dm$ and the chosen axis of rotation.

In the case where we have an object where we can calculate the rotational inertia about some axis that goes through the center-of-mass, we have a *shortcut* for calculating the rotation inertia about some other axis that is parallel to the original axis. This is called the **Parallel Axis Theorem:**

$$I = I_{CM} + MD^2$$

Here $D$ is the distance between the two parallel axes. In effect, this means that the rotational inertia of a body is the sum of the inertia of the body about it's own center of mass plus the inertia that would be calculated if the body were treated as a point particle.