Rotational Kinetic Energy

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1 Abstract

By using a *Roto-Dyne* wheel and the Conservation of Energy, I have derived the moment of inertia for a rotating object with 4 fixed point masses a constant distance from the pivot point. I created a Monte Carlo simulation in *Origin* to develop a working model to analyze data generated from a hanging mass falling from a string attached to an apparatus. My preliminary calculations generated a theoretical moment of inertia of $J = 0.045\ kg \cdot m^2$, which supported the result of the actual simulation of $I = 0.0450\ kg \cdot m^2$. I then used the simulation to analyze data generated from dropping a hanging mass off of a wheel with and without the point masses attached. These data were collected using the *Logger Pro* software. I calculated the moment of inertia with the point masses to be $I_1 = 0.0724\ kg \cdot m^2$ and $I_2 = 0.0338\ kg \cdot m^2$ for the wheel without the point masses. The difference of these values represents the experimental value of the moment of inertia for the mass loads, which I calculated to be $I_E = 0.0386 \pm 0.0005\ kg \cdot m^2$. I then predicted value for the moment of inertia for the 4 masses as $I_P = 0.0271 \pm 0.0003\ kg \cdot m^2$. These values do not agree as a result of large amounts of systematic error in my experiment.

2 Introduction and Theory

The Conservation of Energy can be used to derive the moment of inertia of a given object by utilizing rotational and transnational kinetic energy. The moment of inertia of a point object rotating in a circle can be calculated given a mass M and a radius R with the equation:

$$I = MR^2 \tag{1}$$

For non-point object's, the moment of inertia is dependent on the object's actual shape. In the case of a disk, this expression is:

$$I_{\text{disk}} = \frac{1}{2} M_{\text{disk}} R^2 \tag{2}$$

We are also interested in the moment of inertia for rings, which is:

$$I_{\text{ring}} = M_{\text{ring}} R^2 \tag{3}$$

The *Roto-Dyne* wheel used in this experiment is neither a disk or a ring, as it is not either hollow or completely filled in. The wheel shares characteristics of both objects, as seen in Figure 1. Despite this observation, we can safely assume that the moment of inertia lies between the expressions listed in Equations 2 and 3.

Our experiment contends with both transnational and rotational kinetic energy. The kinetic energy of an object with mass M and transnational speed v is:

$$K_T = \frac{1}{2}Mv^2 \tag{4}$$

For a rotating object with a given moment of inertia I and angular speed ω is a similar expression given by:

$$K_R = \frac{1}{2}I\omega^2 \tag{5}$$

We can define the $y = 0$ point as the plane level with the wheel seen in Figure 1. Assuming the hanging mass drops a distance y from this initial point, its potential energy, ΔU_W,is decreased as its kinetic energy, as seen in Equations 4 and 5, increases. The total sum of the energy can be compared to the total work done by friction in the following equation:

$$\Delta U_W + K_T + K_R = W_f \tag{6}$$

As explained in the experimental procedure, we added a small mass m to the string of the apparatus to overcome the W_f term. This mass, however, is negligible when compared to the hanging mass. Substituting Equations 4 and 5 into Equation 6 along with 0 work done by friction yields:

$$\Delta U_W + \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = 0 \tag{7}$$

We can let the positive y direction be defined as everything below the wheel. This allows us to substitute $-Mgy$ in for ΔU_W and $\omega = \frac{v}{r}$ via the rolling constraint. We can then derive a set of functionally equivalent equations:

$$\begin{aligned} 0 =& -Mgy + \frac{1}{2}Mv^2 + \frac{1}{2}I(\frac{v}{r})^2 && \text{Substitute Givens} && (8) \\ Mgy =& \frac{1}{2}[M + \frac{I}{r^2}]v^2 && \text{Rearrange} && (9) \\ gy =& \frac{1}{2}[1 + \frac{I}{Mr^2}]v^2 && \text{Solve for } gy && (10) \end{aligned}$$

Analyzing our data involves comparing v^2 against y in a plot. We can say y is the independent variable, so we must express Equation 10 as v^2 in terms of y:

$$v^2 = \frac{2gyMr^2}{Mr^2 + I} \tag{11}$$

It should be noted that we expect the line generated by a plot of v^2 vs. y to be roughly linear.

More functionally, we can solve Equation 10 for I, which can be used for further analysis in our experimental procedure. The algebra that leads up to this equation is unnecessary to present here, but it simplifies to the following:

$$I = Mr^2(\frac{2gy}{v^2} - 1) \tag{12}$$

For the Monte Carlo part of our experiment, we had to determine a value for Δt. To do this we need to emphasize the following relationship:

$$v = \frac{\Delta s}{\Delta t} \tag{13}$$

Where s is the value reported by *Logger Pro* based on when the gate is on and off.

We can then use Equation 13 in conjunction with Equation 10 to solve for Δt, yielding the following useful relationship:

$$\Delta t = \Delta s * \sqrt{\frac{1+I}{2gyMr^2}} \tag{14}$$

For the actual experiment, we had to contend with actual uncertainties in v^2, or $\delta_{v_t^2}$. We can use the following equation to calculate the uncertainty in velocity due to time in *Origin*. While this is related to any real error analysis, it is useful in determining the error bars for Figures 3 and 4. The equation is derived from squaring Equation 13 as follows:

$$\delta_{v_t^2} = \frac{\partial}{\partial \Delta t}(\frac{\Delta s^2}{\Delta t^2}) * \delta_{\Delta t} = 2\frac{\Delta s^2}{\Delta t^3}\delta_{\Delta t} = 2\frac{\Delta s^2}{\Delta t^3}\frac{\Delta s}{\Delta s} * \delta_{\Delta t} = 2\frac{v^3}{\Delta s} * \delta_{\Delta t} \tag{15}$$

This equation will not be used or referenced again in this paper, and serves only to illustrate our method for developing error bars for Figures 3 and 4.

3 Experimental Procedure

We started our experiment by creating a Monte Carlo Simulation to generate a model set of data. This first half of the experiment allowed us to create a random set of data for the apparatus shown in Figure 1. We then seeded random data using the number 0831, and also used the value $\Delta t = 0.002$ to run our simulation. To ensure that our data was in fact random as per the Gaussian Distribution, we ran the generator 4 times, yielding 4 different values each time. Knowing that our simulation was successful, we then moved on to the actual part of the experiment.

For the latter half of this experiment, we operated the actual apparatus in Figure 1. It is made up of a *Roto-Dyne* inertia wheel of given mass $M_r = 1.5kg$. Although the masses of the detachable weights were given to us, we measured them with the Lab's scale, yielding a combined weight of $0.9254 \pm 0.0001kg$. We chose this uncertainty because the scale is only accurate up to the tenths place in grams. This differed from the expected value of $0.225 \pm 0.002kg$ per weight, so we decided to use our measured value for all following calculations. We then measured the radius of the apparatus' wheel by measuring the diameter and then dividing by 2. This yielded a radius of $r = 0.209 \pm 0.001m$, which differed from the given radius of $r_{\text{given}} = 0.200 \pm 0.002m$. We then approximated the radial distance from the center of the Mass Loads to the center of the wheel as $k = 0.171 \pm 0.001m$ by subtracting 3 Load Radii of $r_{\text{load}} = 0.0126m$ from the total wheel radius. Note that we chose the uncertainty in our measurement because our meter stick is only accurate to one decimal place in millimeters.

After recording these measurements, we preceded onto the experiment involving no masses on the wheel itself. Before we recorded any data, we had to contend with the frictional force exerted by the string on the encoded pulley. To neglect the force of friction, we added $0.0009kg$ worth of paper clips to the hanging weight. This mass will *not* be included in any future calculations. We knew this mass was correct as the wheel moved at a constant velocity after a gentle push, as verified by *Logger Pro*. We then added the proper hanging mass of $M = 0.060kg$ to the string, and dropped it after winding up the wheel. Using *Logger Pro*, we recorded a good set of data and saved it for further use.

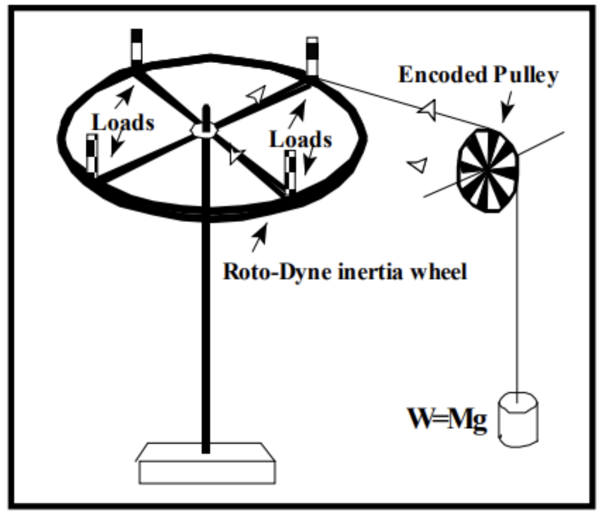

Figure 1: Experimental Apparatus

Finally, we repeated the above process, though this time we secured the 4 detachable masses on the wheel. Following the exact same procedure, we found that a paperclip mass of $0.0015kg$ was adequate, though this value will be neglected in further calculations. We then added the hanging mass of $M = 0.060kg$ back on the apparatus, and recorded data using *Logger Pro* once again.

We will only consider uncertainties in the radius for the apparatus, as well as the uncertainty in the summed mass of the detachable weights. It is safe to neglect the uncertainties in the other components of this system such as the Hanging Mass, as they make use of finely machined parts.

4 Results and Analysis

First we analyzed the Monte Carlo experiment to ensure our experimental design wasn't flawed. We had to first estimate a the moment of inertia for the wheel. As the wheel shares characteristics of both a ring and a disk, we can define a moment of Inertia J as:

$$
\begin{aligned}
J =&\frac{I_{disk} + I_{ring}}{2} \\
=&\frac{MR^2 + \frac{1}{2}MR^2}{2} \\
=&\frac{3}{4}MR^2
\end{aligned} \tag{16}
$$

Note that J is used above to avoid confusion in the simulation. It is derived by using Equations 2 and 3.

We can then solve for J using a given mass $M = 1.5kg$ and radius $R = 0.2m$. This yields the following:

$$J = \frac{3}{4} * (1.5kg) * (0.2m)^2 = 0.045\ kg \cdot m^2$$

We analyzed our simulation by plotting v^2 against y. While y was generated using an *Origin* script, we had to calculate v^2 using the following:

$$v^2 = (\frac{0.015}{\Delta t})^2 \tag{17}$$

With Δt be defined by Equation 14.

We needed to conduct analysis on Equation 17 which required finding the uncertainty in v^2. There is only one variable in this equation, so the uncertainty in v^2 can be represented by the following expression:

$$\delta_{v^2} = \sqrt{\delta^2_{v^2_{\Delta t}}} \tag{18}$$

Where we have to find the value $\delta_{v^2_{\Delta t}}$.

The uncertainty in velocity squared due to Δt can be calculated using the derivative method as follows:

$$
\begin{aligned}
\delta_{v^2_{\Delta t}} = \frac{\partial v^2}{\partial \Delta t} * \delta_{\Delta t} =&\frac{\partial}{\partial \Delta t}(\frac{0.015}{\Delta t})^2 * \delta_{\Delta t} \\
=&\frac{-2(0.015)^2}{(\Delta t)^3} * \delta_{\Delta t}
\end{aligned} \tag{19}
$$

The above equation can be plugged into Equation 18, Effectively taking the absolute value of the expression. Δt is evaluated dynamically, as it changes based on the inputted simulation. The given value for $\delta_{\Delta t}$ is 0.0002 sec, which we will plug into the expression, however:

$$\delta_{v^2} = \frac{2(0.015)^2}{(\Delta t)^3} * 0.0002 \tag{20}$$

We then plotted v^2 against y making use of the error bars produced from Equation 20. As seen in Figure 2, all of our simulated data points' error bars contain the best fit line generated from *Origin*. We expected this result because this data was purely a simulation, and any discrepancies would be a result of poor calculations, not experimental errors. The slope of the generated fit line, as seen in Figure 20, is $B = 0.993 \pm 0.002\frac{m}{s^2}$.

We can rewrite B, the slope of the line, in terms of y and v^2 to justify the substitution of B into Equation 12. This expression is $B = \frac{y}{v^2}$, whose substitution leads to the following expression:

$$I = Mr^2[\frac{2g}{B} - 1] \tag{21}$$

Plugging $g = 9.81\frac{m}{s^2}, M = 0.060\ kg, r = 0.2m, B = 0.993,\ \text{and}, \delta_B = 0.002$ into Equation 21 reveals the following value for I:

$$I = 0.06(0.2)^2[\frac{2(9.81)}{0.993} - 1] = 0.0450$$

To conduct error analysis on I, we should contend with uncertainties in both the radius r and in the slope B. For the simulation, however, we only need to contend with the uncertainty in B as the uncertainty in I due to r is assumed to be negligible. Therefore, the uncertainty in I can be represented by the following equation:

$$\delta_I = \sqrt{\delta_{I_B}^2} \tag{22}$$

We only need to contend with the uncertainty in B for the Monte Carlo simulation, the uncertainty in I is represented by effectively taking the absolute value of δ_{I_B}, as seen here:

$$\begin{aligned} \delta_{I_B} = \frac{\partial I}{\partial B} * \delta_B =& \frac{\partial}{\partial B}(Mr^2(\frac{2g}{B} - 1)) * \delta_B \\ =& \frac{-2gMr^2}{B^2} * \delta_B \end{aligned} \tag{23}$$

Using this Equation along with Equation 22 and plugging identical values as the preliminary moment of inertia calculation yields the value:

$$\delta_I = \sqrt{(\frac{-2(9.81)(0.06)(0.2)^2}{0.993^2} * 0.002)^2} = 0.000112$$

We can then express our value for I, rounded correctly with its uncertainty, as $I = 0.0450 \pm 0.0001\ kg \cdot m^2$. This value perfectly aligns with the calculated value derived from Equation 16 of $0.0450 kg \cdot m^2$. We can therefore conclude that our simulation and analysis procedure is correct, so we can now analyze the produced data from the actual trials.

Knowing that our method of analysis was correct, we then applied it to the actual data. First, we examined the trial with the 4 masses on the apparatus. As outlined in the procedure, we used a 0.06 kg hanging mass to conduct this test. We plotted the results produced from our software as a plot of v^2 against y. The outputted slope is $B_1 = 0.685 \pm 0.003\ \frac{m}{s^2}$, as verified by Figure 3.

We then proceeded to test without the masses on the wheel, while maintaining the same 0.06 kg mass as indicated in the procedure. We made a similar plot as that of the trial with mass, and reported the slope and its uncertainty. For this experiment, we recorded a slope of $B_2 = 1.410 \pm 0.003\frac{m}{s^2}$, as verified by Figure 4.

With two usable slopes, we then plugged them into Equation 21 to generate their corresponding values of I, appropriately named I_1 for slope B_1, and I_2 for slope B_2. As we operated the actual equipment, we used actual measurements as opposed to the theoretical ones used in the Monte Carlo simulation. This means that we used our measured radius of $r = 0.209m$ and hanging mass weight $M = 0.06kg$. Plugging the previously stated values of B into their respective equations with respect to both Equation 21 and given constants, we yielded the two following values for I:

$$\begin{aligned} I_1 =& 0.06(0.209)^2[\frac{2(9.81)}{0.685} - 1] = 0.0724 \ kg \cdot m^2 \\ I_2 =& 0.06(0.209)^2[\frac{2(9.81)}{1.410} - 1] = 0.0338 \ kg \cdot m^2 \end{aligned}$$

With the two values of I calculated, we then proceeded to calculate the experimental value of the moment inertia for the **mass loads alone**, as that is what we are ultimately interested in. The relationship is as follows:

$$I_E = I_1 - I_2 \tag{24}$$

We can substitute the values of I_1 and I_2 into Equation 24, yielding:

$$I_E = 0.724 - 0.338 = 0.0386 \ kg \cdot m^2$$

We must now contend with the uncertainties in I as a result of B and r. Substituting Equation 21 into Equation 24 with the appropriate values of B allows us to conduct error analysis. This substitution and following simplification is as follows:

$$\begin{aligned} I_E =& Mr^2[\frac{2g}{B_1} - 1] - Mr^2[\frac{2g}{B_2} - 1] \\ =& 2gMr^2[\frac{1}{B_1} - \frac{1}{B_2}] \end{aligned} \tag{25}$$

We can neglect uncertainties in M and g, as these values are either negligible in their own uncertainties or given constants, respectively. This means we can find the uncertainty in I_E, or δ_{I_E}, with the following expression:

$$\delta_{I_E} = \sqrt{\delta^2_{I_{E,B_1}} + \delta^2_{I_{E,B_2}} + \delta^2_{I_{E,r}}} \tag{26}$$

Making use of the derivative method and Equation 25, we can derive expressions for $\delta_{I_{E,B_1}}$, $\delta_{I_{E,B_2}}$, and $\delta_{I_{E,r}}$ as follows:

$$\delta_{I_{E,B_1}} = \frac{\partial I_E}{\partial B_1} * \delta_{B_1} = \frac{\partial}{\partial B_1}(2gMr^2[\frac{1}{B_1} - \frac{1}{B_2}]) * \delta_{B_1}$$
$$= \frac{-2gMr^2}{(B_1)^2} * \delta_{B_1} \tag{27}$$

$$\delta_{I_{E,B_2}} = \frac{\partial I_E}{\partial B_2} * \delta_{B_2} = \frac{\partial}{\partial B_2}(2gMr^2[\frac{1}{B_1} - \frac{1}{B_2}]) * \delta_{B_2}$$
$$= \frac{2gMr^2}{(B_2)^2} * \delta_{B_2} \tag{28}$$

$$\delta_{I_{E,r}} = \frac{\partial I_E}{\partial r} * \delta_r = \frac{\partial}{\partial r}(2gMr^2[\frac{1}{B_1} - \frac{1}{B_2}]) * \delta_r$$
$$= 4gMr[\frac{1}{B_1} - \frac{1}{B_2}] * \delta_r \tag{29}$$

We can then compute the values of these individual uncertainties based on given values. For the following calculations, we will use the same values as used in the calculation of the individual Inertia Calculations. In addition, we will use $\delta_{B_1} = 0.003$, $\delta_{B_2} = 0.003$, and $\delta_r = 0.001$ as discussed previously. Substituting appropriate values into Equations 27, 28, and 29 yields:

$$\delta_{I_{E,B_1}} = \frac{-2(9.81)(0.06)(.209)^2}{0.685^2} * .003 = -3.29 \times 10^{-4}$$

$$\delta_{I_{E,B_2}} = \frac{2(9.81)(0.06)(0.209)^2}{1.410^2} * 0.003 = 7.76 \times 10^{-5}$$

$$\delta_{I_{E,r}} = 4(9.81)(0.06)(0.209)[\frac{1}{0.685} - \frac{1}{1.410}] * 0.001 = 3.69 \times 10^{-4}$$

We can then plug these calculated values into Equation 26 to yield the following value of δ_{I_E}:

$$\delta_{I_E} = \sqrt{(-3.29 \times 10^{-4})^2 + (7.76 \times 10^{-5})^2 + (3.69 \times 10^{-4})^2} = 5 \times 10^{-4} = 0.0005\ kg \cdot m^2$$

Using this calculation, we can then express I_E with its uncertainty as $I_E = 0.0386 \pm 0.0005\ kg \cdot m^2$.

Now we must contend with the predicted moment of inertia for the **mass loads alone**. If we treat the Mass Loads as point particles of combined mass M at a distance k from the center of rotation, then we can say that their predicted moment of inertia is as follows:

$$I_P = Mk^2 \tag{30}$$

Where $M = 0.9254kg$ and $k = 0.171m$. We can solve this equation with the inputted values stated previously, yielding a predicted moment of inertia for the Load Masses of:

$$I_P = 0.9254(0.171)^2 = 0.0271\ kg \cdot m^2$$

We must now perform error analysis on I_P, where uncertainty is a result of both M and k. The expression for the uncertainty is very similar to the format of Equation 22, with the individual quantities being added in quadrature. The expression is as follows:

$$\delta_{I_P} = \sqrt{\delta^2_{I_P,M} + \delta^2_{I_P,k}} \tag{31}$$

Once again we will use the derivative method and Equation 30 to derive the components of this calculation:

$$\begin{aligned} \delta_{I_P,M} = \frac{\partial I_P}{\partial M} * \delta_M =& \frac{\partial}{\partial M}(Mk^2) * \delta_M \\ =& k^2 * \delta_M \end{aligned} \tag{32}$$

$$\begin{aligned} \delta_{I_P,k} = \frac{\partial I_P}{\partial k} * \delta_k =& \frac{\partial}{\partial k}(Mk^2) * \delta_k \\ =& 2Mk * \delta_k \end{aligned} \tag{33}$$

We can then numerically evaluate these expressions using the same values of M and k used to calculate I_P in Equation 30. We will also use $\delta_M = 0.0001\ kg$ and $\delta_k = 0.001\ m$ as discussed in the Procedure. Plugging these values into Equations 32 and 33 results in the following calculations:

$$\delta_{I_P,M} = 0.171^2 * 0.0001 = 2.92 \times 10^{-6}$$

$$\delta_{I_P,k} = 2(0.9254)(0.171) * 0.001 = 3.16 \times 10^{-4}$$

We can then plug these values into Equation 31 to yield the following value of δ_{I_P}:

$$\delta_{I_P} = \sqrt{(2.92 \times 10^{-6})^2 + (3.16 \times 10^{-4})^2} = 3.16 \times 10^-4 = 0.000316\ ks \cdot m^2$$

We can then combine our calculations that made use of Equations 30 and 31 to present I_P with its uncertainty as $I_P = 0.0271 \pm 0.0003\ kg \cdot m^2$.

Our values of I_E and I_P do not agree within their uncertainties, so we can conclude that a source of systematic error may have been present, or that one or more of our assumptions were incorrect moving through our testing and analysis process.

5 Conclusion

The expected value for the moment of inertia of the 4 Mass Loads was $I_P = 0.0271 \pm 0.0003\ kg \cdot m^2$. The measured value of the moment of inertia was $I_E = 0.0386 \pm 0.0005\ kg \cdot$

m^2. These two values do not agree within their uncertainties, with I_E being 23 standard deviations above I_P. The reason for this large error is most likely a result of systematic errors in our data collection. This hypothesis is supported by the fit lines in Figures 3 and 4. Some of the data points have error bars that do not capture the line of best fit. The data tends to be overestimated by the fit line, suggesting systematic error in our procedure. This could be a result of the *Roto-Dyne* wheel not being perfectly level resulting in artificial acceleration, or from a lack of proper adjustment of friction in our preliminary setup. Also, our estimate of k may have resulted in an incorrect value for I_P, though we can confidently say that our estimate was close enough to not alter our prediction too significantly. Finally, we may have treated the release of the hanging mass incorrectly, potentially starting from different points for the trials with and without mass, respectively. All of these factors came together to produce large, systematic error leading to the discrepancies in our results. Ultimately, we cannot develop an accurate representation for the moment of inertia regarding our apparatus, involving a system with 4 evenly spaced point masses.

5.1 Acknowledgements

I would like to thank Adi Mallik, CWRU Department of Physics, for his help in obtaining the experimental data, preparing the figures, and checking my calculations.

5.2 References

1. Driscoll, D., General Physics I: Mechanics Lab Manual, "Rotational Kinetic Energy," CWRU Bookstore, 2014.

6 Appendix

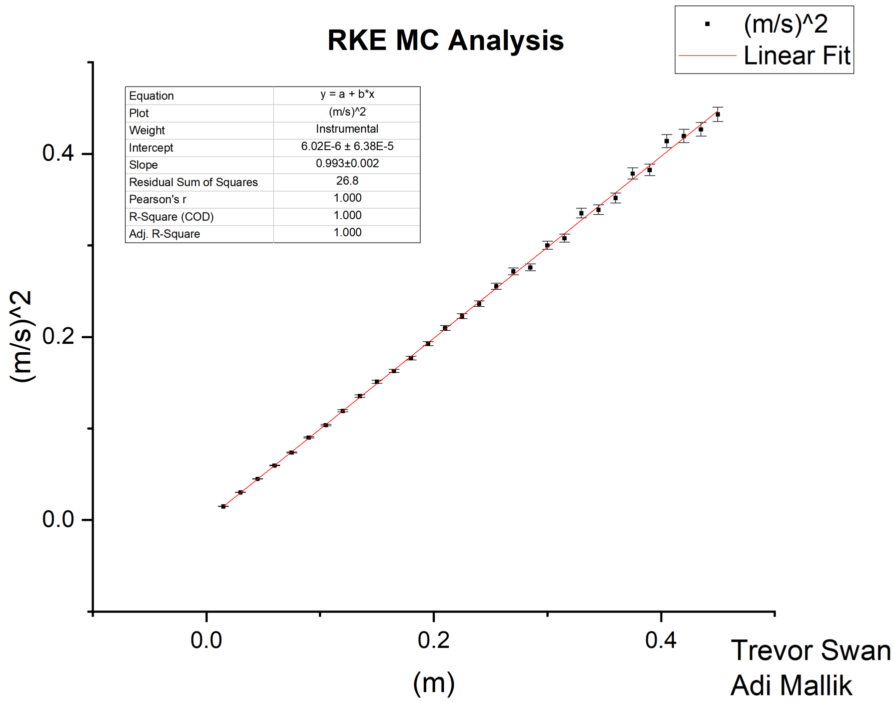

Figure 2: Monte Carlo Simulation

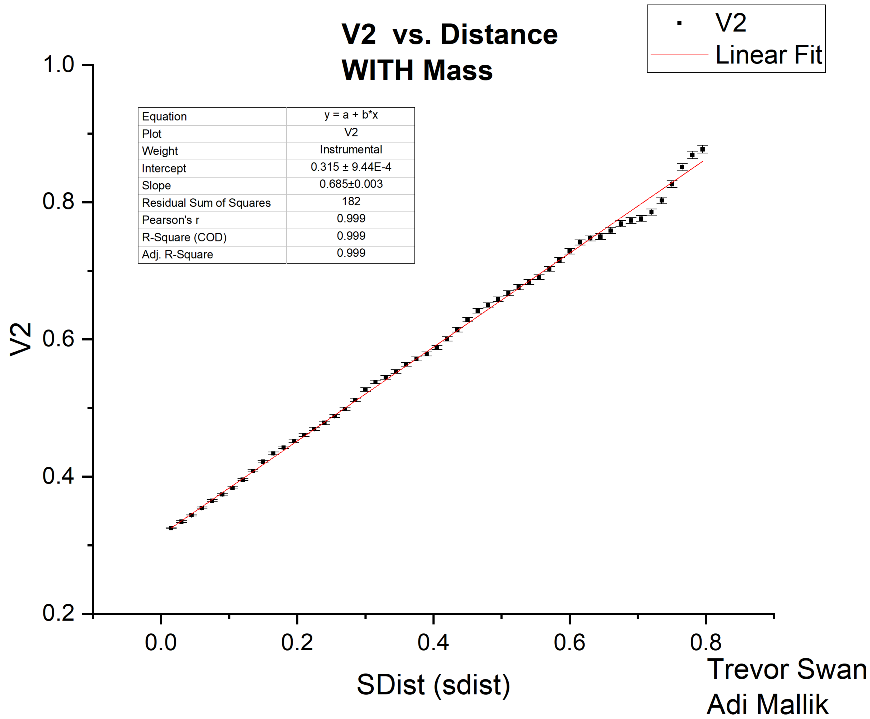

Figure 3: v^2 vs. y with mass

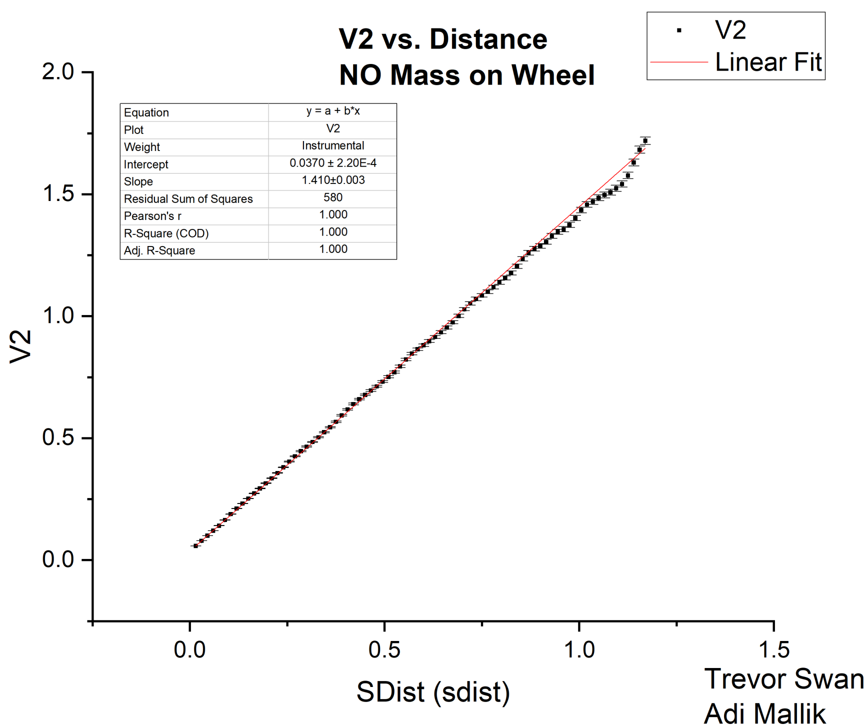

Figure 4: v^2 vs. y with no mass