Chapter 15+

Revisit Oscillations and Simple Harmonic Motion

Revisit:	To-Do:
• Oscillations • Simple harmonic motion	• Pendulum oscillations • Derive the parallel axis theorem for moments of inertia and apply it to physical pendulum

The Most Famous Second-Order Differential Equation
Its Famous Sinusoidal Solution
Simple Harmonic Oscillator

- Our world is stable - Lots of restoring forces stabilize many systems against small disturbances (e.g., atoms in a solid, springs, swings, tree branches, our standing position, etc. etc.)

- Restoring forces are generally proportional to the displacements X, at least for small displacements, and act opposite to the direction of the displacement.

Thus practically all over the world we meet up with the same equation we solved on p. 15-3:

$$\boxed{\ddot{X} = -C\,X}$$

with general solution

$$\boxed{X = A\cos(\omega t + \phi)}$$

where

$$\boxed{\omega = \sqrt{C}}$$

Comments:

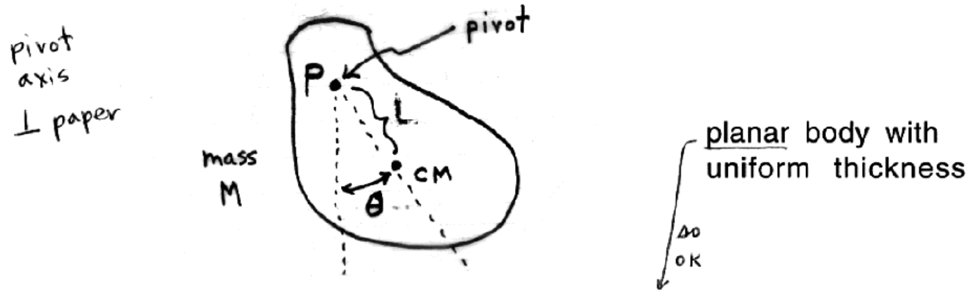
- Recall that our spring discussions had X = x and C = k/m

- Many physical systems undergo simple harmonic motion (small jiggling) with eventual damping out (see later in this chapter)

- Think of C as $C = \dfrac{\text{effective spring constant}}{\text{effective mass}}$ where you picture the restoring force as if it were like a spring

Physical Pendulum

The second most famous oscillator after the spring is the pendulum with small swinging motion. This oscillates due to its gravitational restoring force. The following is an outline of its rotational second law analysis where a special "parallel-axis" theorem for finding its moment of inertia is used and will be derived on p. 15-12.



torque equation:

$$\tau_P = I_P\,\alpha$$

$\tau_P \rightarrow -M\,g\,L\sin\theta$ ↳ CW ✓

$\alpha \rightarrow +\ddot{\theta}$ (CCW +)

$I_P \rightarrow I_{CM} + M\,L^2$ {parallel axis theorem}

$$\ddot{\theta} = -\frac{M\,g\,L}{I_P}\sin\theta$$

small angular displacement: $\sin\theta \cong \theta$ since $\theta << 1$

$$\ddot{\theta} \cong -\frac{M\,g\,L}{I_P}\theta$$

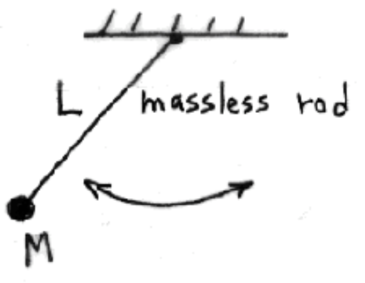
What we are saying about small angular displacements is that SHM is a good approximation for this system if the angle in radians is small compared to 1 radian. In fact, suppose it is even as big as $\theta=1/2$ of a radian (almost 30 degrees), then $\sin(\theta=1/2) = 0.479 \cong 0.5$ so pretty good!!

So we have a general PHYSICAL PENDULUM frequency: $\boxed{\omega = \sqrt{\frac{MgL}{I_P}}}$ with the corresponding period $T = \frac{1}{f} = \frac{2\pi}{\omega}$ so that $\boxed{T = 2\pi\sqrt{\frac{I_P}{MgL}}}$

The fact that T is independent of θ is a big deal. You have probably heard about how the pendulum period is the same if you start off with a bigger push as with a little one; that corresponds to this independence of θ. (But we can't push too hard: remember we have to keep θ rather less than a radian or so.)

Simple Pendulum

The most famous physical pendulum is the one with a small mass on the end of a very light rod.



(ignore size of M)

$$\therefore \; I_p = M L^2$$

so $\omega = \sqrt{\dfrac{MgL}{ML^2}} = \sqrt{\dfrac{g}{L}}$ and $T = \dfrac{1}{f} = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{L}{g}}$

$$\boxed{\text{FAMOUS SIMPLE PENDULUM RESULTS: } \omega = \sqrt{\frac{g}{L}} \text{ or } T = 2\pi\sqrt{\frac{L}{g}}}$$

- Notice that M has cancelled out and all you need to know is L! The increase with M of the moment of inertia is offset by the increase in torque, so the frequency or period depends only on the length, something you may already know.

- Again, everything is independent of θ, as long as θ is small.

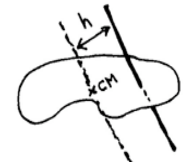
- Numerical example: A mass suspended from a parachute descending at constant velocity can be regarded as a simple pendulum. What is the frequency f of pendulum oscillations of a human body suspended 7 m below a parachute?

  Answer:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} = \frac{1}{2\pi}\sqrt{\frac{9.8}{7}} = 0.19 \text{ Hz} \Rightarrow T \cong 5 \text{ s}$$

But what are we going to do about more general physical pendula? In particular, how do we calculate I_p for more complicated bodies? Well, a great tool is the theorem we mentioned on p. 15-10 and we explain on the next page.

Parallel-Axis Theorem

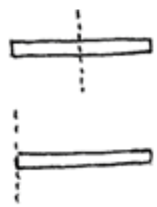


Now for a theorem that will save us from doing more integrals in many cases:

$$I \quad = \quad I_{CM} \quad + \quad M \quad h^2$$

I	I_{CM}	M	h^2
moment of inertia about **any given axis**	moment of inertia about the **axis through the CM which is parallel to the given axis**	total mass	h = **distance between the two || axes**

A) Check the theorem: Use rod moments of inertia. Recall we derived the moment of inertia for the axis through the center of a rod and we also considered what it is for an axis through the end in a clicker question. Here essentially is a proof of the latter:

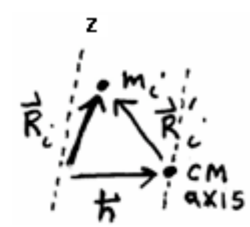


$\Rightarrow \; I_{CM} = 1/12\ M\ L^2$

$\Rightarrow \; I_{end} = 1/3\ M\ L^2$

Check: $I_{end} = \frac{1}{12} M L^2 + M \underset{\uparrow h}{\left(\frac{L}{2}\right)^2} = \frac{1}{3} M L^2$ ✓

B) Proof of the theorem: Start with the following pictures, which can be most easily understood by thinking of the rotational axis as along the z axis (and parallel to the other axis going through the CM). Then think of an x-y **plane perpendicular to those axes** going through each particle mass m_i where $\vec{R}_i$ is the position vector of m_i in that plane relative to the rotational axis, $\vec{R}'_i$ is the position vector of m_i in that plane relative to the parallel axis going through the CM, and $\vec{h}$ is their difference $\vec{h} \equiv \vec{R}_i - \vec{R}'_i$, and therefore the distance vector between and perpendicular to the two parallel axes. The desired moment of inertia is



$$I = \sum_i m_i R_i^2 = \sum m_i \vec{R}_i \cdot \vec{R}_i = \sum m_i (\vec{R}'_i + \vec{h}) \cdot (\vec{R}'_i + \vec{h}) = \underbrace{\sum m_i R_i'^2}_{I_{CM}} + 2\vec{h} \cdot \underbrace{\sum m_i \vec{R}'_i}_{0 \;\hookrightarrow \text{defn. of CM}} + h^2 \underbrace{\sum m_i}_{M}$$

$\vec{R}_i = \vec{h} + \vec{R}'_i$ in plane $\perp$ both axes

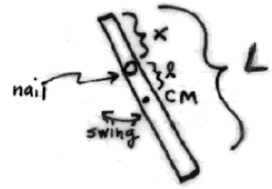
dot products!

✓ Q.E.D.

(A cute way to say the middle sum (after the last equal sign) is zero: The position of the CM if the CM is at the origin must of course be zero!)

C) Example of using the theorem in an un-simple pendulum

ignore friction



meter stick hung on nail through hole drilled in stick a distance x < 50 cm from top.

parallel axis theorem

$$I_P = \frac{1}{12} M L^2 + M\Big(\underbrace{\frac{L}{2} - x}_{\ell}\Big)^2$$

L = 100 cm = 1 m

Since the "pendulum length" $\ell = \frac{L}{2} - x$ here,

$$T = 2\pi\sqrt{\frac{I_P}{Mg\ell}} = 2\pi\sqrt{\frac{\frac{1}{12}\cancel{M}L^2 + \cancel{M}\left(\frac{L}{2} - x\right)^2}{\cancel{M}g\left(\frac{L}{2} - x\right)}}$$

$L = 1$, $\frac{L}{2} = 0.5$

e.g. x = 30 cm $\Rightarrow T = 2\pi\sqrt{\dfrac{\frac{1}{12} + (0.2)^2}{9.80(0.2)}}$ = 1.58 s

(don't mix cm and m !)

e.g. x = 50 cm $\Rightarrow T = \infty$ (no equilibrium point)

(no restoring force)

D) Practice using the theorem:

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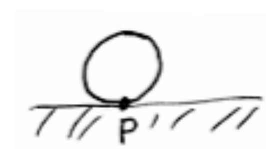
Problem 15-5 *Use the parallel-axis theorem to find*

*a) the moment of inertia of a meter stick, with mass 0.56 kg, about an axis perpendicular to the stick and located **at the 20-cm mark**.*

b) the moment of inertia of a uniform solid cylindrical wheel, radius R *and mass* M*, **around its contact point*** **P** *with the ground (the axis is perpendicular to the profile of the wheel and parallel to the wheel's axis).*



c) the moment of inertia of the S-profile object shown in the figure, where it comes from a uniform hoop, mass M *and radius* R*, that gets cut in half and shifted so that the two half hoops are laid side by side. The axis is perpendicular to the profile **at the point*** **P** *as shown.*

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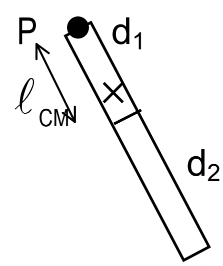
Problem 15-6

This is connected to resonance phenomena, because you are driving your legs with your muscles when you walk, but you are most comfortable doing so when you resonate with your natural period of swinging your leg!

a) Go outside with a friend and take a walk. Have her or him help you estimate, when you are walking naturally, how long it takes in seconds for you to move your left leg, say, through one full cycle.

b) Next calculate the natural period (in seconds) of the swinging motion of a human leg in the following way. Treat the leg as a rigid physical pendulum with axis at the hip joint. Pretend that the mass distribution of the leg can be approximated as two rods joined rigidly end to end. The upper rod (thigh) has a weight of 15 lb and a length of 17 in. The lower rod (shin plus foot) has a weight of 9 lb and a length of 18 in. (Ignore the thickness of the rods in calculating the moment of inertia.) Now the neat thing is that your answer should be roughly the same as the natural swinging motion of your leg when you walk at a normal rate! Compare!



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