CHAOS

Period Doubling

Revised August 2021

Final experiment in PHYS 123

Learning Objective:

During this lab, you will explore a physical system that exhibits a period-doubling path to chaos, along with its universality, and you will observe fully chaotic behavior.

Logistics:

You will carry out the experiment and fill out the worksheet as you go, uploading it to Canvas at the end of the lab and completing the entire work and requirement for this last lab. No additional write-up or time beyond Wednesday's session is needed.

A. Introduction

In the experiment you will study the forced oscillations of a damped nonlinear *RLC* circuit (resistor-inductor-capacitor combination). The "capacitor" will actually be a rectifier diode because its capacitance depends on the voltage providing the key nonlinearity needed to study chaos. Nonlinearity here means that the voltage across the capacitor will not be a simple linear function of charge, in contrast to a typical condenser.

The connection between this system and our mechanics studies is that the differential equation describing the circuit has the same form as the mechanical nonlinear oscillator such as the Duffing model. The Duffing equation describes the nonlinear circuit in the present experiment

A Duffing circuit allows you to observe the phenomenon of "period doubling" as the circuit is driven with sine waves of increasing peak-to-peak voltage (twice the sinusoid amplitude). You will measure the first few critical drive voltages at which period doubling occurs, and view the onset of chaos. From the measurements, you repeat Feigenbaum's discovery corresponding to the geometrically decreasing input voltage intervals between period doublings (i.e., there is exponential decrease for each subsequent interval), and also the geometrically decreasing maximum output-voltage splittings defined just before the doublings. The universal factor of decrease of the input voltage intervals is the first Feigenbaum number δ = 4.669… The universal factor of decrease in the maximum output splittings is the second Feigenbaum number α = 2.502...

B. Apparatus

Our complete apparatus for carrying out the electrical Duffing experiment consists of three components: 1) an inductor-capacitor-resistor **electrical LCR circuit box**, which is to be driven into nonlinear oscillations (a diode capacitor is used to provide the nonlinear capacitance – such circuits have resonant frequencies Ω just as linear circuits do); 2) a voltage generator with a variable drive frequency (i.e., a **frequency generator**) providing the $V_0 \cos\omega t$ input (driving) voltage source for the electrical circuit (after it is tuned to the resonant frequency – i.e., we tune to $\omega = \Omega$). The frequency generator also allows us to increase V_0 and observe period-doubling transitions; 3) a **dual-channel oscilloscope** is used to monitor the drive voltage (the input voltage is fed into Channel 1) and the capacitive/diode voltage (the output voltage is fed into Channel 2) simultaneously. Coaxial cables are used to connect the three components.

C. Theory

We refer to the class lecture notes.

D. Experimental Procedure

D.1 Preset Apparatus

We will set up the apparatus to help you get started. This is an experiment for you to focus on the nonlinear period doubling phenomenon and not on the details of the electronics. In our mechanics course, we are not in a position to spend time on the details of oscilloscopes and electrical circuits. Therefore, we have preset the components in the following ways (customized for each RLC circuit) and you will only need to change a small number of dials as you go. See below (if you change other dials, the experiment may be different from what you want! ☺)

Frequency generator (FG)

You will be using a LeCroy Waveform Generator. When we begin operation, you'll first turn the FG on by pushing the button way over at the lower left corner AND push the CH2 green input button on the lower right corner (or else nothing gets out). **We are hopeful that you've read through this whole write-up before starting.** Now go to the vertical set of buttons just to the right of the little monitor screen, and push the top button until "CH2 Waveform" shows up left and lower on the screen (it will represent a change from "CH1 Waveform" if that is what is showing up initially). Then push the second button "Freq" to begin the steps to set the frequency to the value written in black on the side of your circuit box (it's the "resonance frequency" for your box) and will be somewhere between 20kHz and 90kHz; ***record this on your Worksheet.*** To set the frequency you need to familiarize yourself with the big turning knob at the upper right corner AND the left ◁ and right ▷ arrows right under it to pick which digit you are changing. Play with those two to see how they work. (The first setting should be the ones digit so dial clockwise to get up to 10 kHz and then push the left arrow to be able to change the tens digit and thus dial faster. Later, you'll push the right arrow to concentrate on the ones, tenths, hundredths, and thousandths.) Now after you've set the frequency, the only thing you need to do for the rest of the experiment is to change the input voltage by first pushing the "Ampl" button just below the "Freq" button. You'll change the digits with the two left and right arrow buttons just like in the frequency case. As for the other FG controls, the sinusoid mode is set and nothing else should be touched. The coaxial cable will be already attached to the appropriate output, and the only thing you need to do with the FG is adjust the voltage amplitude for the rest of the experiment.

Oscilloscope (OS)

i) Top row – Starting on the left, **push the OS power button** after the FG is on. The TIME/DIV setting (here, and below, look for a small dotted ridge on the outer knob – the inner knob will be "off" and stay "off") will be set somewhere between 1 and 50 μs/cm on the screen (one cm corresponds to the size of one grid unit), with a period of 1 up to 50 μs corresponding to a frequency range (f=1/T) a range from 1 Megahertz down to 20 kilohertz. ***Record the TIME/DIV in μs.***

The SWP VAR should be off. The horizontal position ⟺ already will be close to what you want for the first operation step, **but can be adjusted later to help you in measuring the voltage splits off the grid.** The MODE should be AUTO. The LEVEL should be preset with no adjustment needed. The SOURCE will be preset to INT.

ii) Middle row – Starting on the left, the intensity and focus knobs should already be preset to give the best brightness and least blur, respectively. The first VOLTS/DIV will be preset to give you a good horizontal scale (the x-axis when we go to the x-y display – see later) in volts/cm for your input voltage patterns (CHANNEL 1 patterns; we have preset it for your particular RLC box), but you can change this to zoom in on patterns – just remember you have then changed the voltage scale! You can adjust the vertical position here by the next knob $\Updownarrow$ but it should be already preset for your first operational step. The MODE should be preset to ALT and not changed. The next knob $\Updownarrow$ is for the vertical position of the output voltage (CHANNEL 2) coming from the RLC box and should also already be preset **but also can be adjusted by you later for convenience in measurements.** Next is the VOLTS/DIV outer knob, which is preset with the right scale for your circuit. (Again, the inner knob should always be off.) ***Record the VOLTS/DIV FOR BOTH CH1 AND CH2.***

iii) Bottom row - Starting on the left, the first coaxial connection (CH1) will already be in place and come from the FG, the input voltage. The next three vertical switches should be "set and forget" at, respectively, AC, at CH2 (the INT TRIG, but CH1 works, too), and at AC. The CH2 coaxial connection will already be in place and will be connected through a clamp probe with alligator clips to measure the output voltage on the nonlinear diode.

RLC Circuit Box

The coaxial connector will have a "T" connection with one end connected to the FG, picking up the input voltage, and the other end feeding the CH1 input of the OS. The probe clamp will be attached between the inductor (attached to the center of the box) and the diode (attached to the side of the box), with its grounding alligator clip attached to the box wall. This will pick up the nonlinear output voltage response measured on CH2 of the OS. Be careful that it does not come loose – if it does, get help from us if you don't know where to reattach it. Finally, note that there is an unimportant fine-tuning resistance (potentiometer) knob on the box for a small adjustment of the input voltage - but your FG voltage control is all you need so *ignore the box knob*.

D.2 Observe Low Input and Output Voltages and Measure Their Frequency

Knock on wood, we will have all the settings (frequency and initial amplitude on FG; TIME/DIV, VOLTS/DIV, POSITIONS, LEVELS on OS) and all cables correctly in place. After you've turned on the FG and then the OS, you should see both the input voltage (upper wave form in time) and the output voltage dropped across the diode (lower wave form in time). We have initially turned down your FG voltage knob low enough so you are not close to period doubling.

In your worksheet, ***sketch the different input and output voltage wave forms*** (do you see what has happened to our nice sinusoid after the diode has filtered it, even at the low voltage?). Next determine the frequency of the input sinusoid (e.g., use the horizontal $\Leftrightarrow$ and the CH1 $\Updownarrow$ knobs to align the wave form so you can more easily count the grid steps on the OS (for more accuracy), and count the number N (it can and should include a fraction if needed) of periods T that fit into some number n (you choose, up to 10) of OS grid divisions Δt. The latter gives you the total time $\tau = n\Delta t$ in μs (Δt is shown on your TIME/DIV dial in μs). Thus:

$$NT = \tau = n \text{ times (TIME/DIV)}$$

from which we can solve for T. What do you get for $f = 1/T$? *It should agree with what the FG says it is.* (**Learning to estimate carefully the fractions in the periods here will help you later in estimating the second Feigenbaum number!)** Notice also that the output wave form has the same frequency but a distorted shape, as we said, along with a phase change. ***Sketch this comparison.***

D.3 Observe Period Doubling!

A) As Time Series

The output voltage time series you have been looking at can now be shown to exhibit period doubling if we increase the FG input voltage (very gently! **You need to learn how to increase the ones digit on the FG and as soon as something happens, drop back one and use the ⊳ right button to change the tenths digit, etc.**). First, you need to increase this voltage sufficiently to get past some spurious pattern changes (see Section B below). (By the way, you can compare your - more accurate - FG digital readout to the peak-to-peak voltages you see on the oscilloscope monitor for the input – they should agree!) After getting past the initial anomaly, you continue to gently increase the input voltage and you will see how the pulse heights change and, as V_0 goes through the first critical amplitude, the pulses will break into an alternating pattern of high and low peaks. This transition is the first period-doubling. Continue to increase V_0 and observe further doubling transitions. Note that different peaks may appear "on top of each other" because the oscilloscope repeats the wave form many times; a period doubling has occurred whenever the wave form appears to have a new, similar copy superimposed right on top of it ("splittings"). ***Describe very briefly what you observe qualitatively.***

But this time series approach to viewing period doubling makes measurements difficult, and we have a better way to quantify period doubling. The method is described next.

B) As Parametric Plots ("Loops")

First, return the FG amplitude voltage amplitude knob back down to its lowest value of 1V. Now use the OS x-y mode (**turn the TIME/DIV knob clockwise all the way to this mode**), which plots the output voltage (CH2) on the vertical y-axis against the input voltage (CH1) on the horizontal x-axis. We will now look at the "loops" in such parametric plots.

Recall that in a linear problem, where both input and output have the same frequency, we would get a (perhaps tilted) elliptical loop, in general, for $V_{in} = V_0\cos(\omega t+\phi)$ and $V_{out} = A\cos(\omega t+\theta)$. But now we have a nonlinear problem, where the frequencies may differ, and recall also from our numerical Duffing simulations that period doubling first shows itself as a single loop splitting into two loops and then into four loops, etc. This is what we are going to look for here.

But one more difference. Recall that the diode chops off the current and voltage so that we only see their positive values, say. Moreover, the diode has complicated voltage dependence and does not lead to a single smooth nonlinear function at the lower voltages. Thus we will see is a folded-over distorted ellipse **with the bottom flattened.** We need to slowly increase the FG V_0 until we get past the folding transition (the anomaly referred to before) and reach the smoother voltage dependence of the diode. We can finally see "loop-splitting!" You should see at least three clear loop splittings where, at the third splitting, it is fuzzy and the limitations on the instruments we have available stop us from the precision needed to go much further. (But don't be afraid to try squinting for another splitting!) ***Describe very briefly what you observe qualitatively, including what chaos looks like!***

D.4 Observe Geometrical Decrease in the Transition Values $V_0(n)$ for Period Doubling: First Feigenbaum Number

Going back to lower voltages, you should read the peak-to-peak voltage $2V_0$ (the horizontal x-axis = CH1 = input voltage) right off the digital readout on your FG monitor in the x-y mode plots. (You can also check it approximately by looking at the overall width of the loop figure, which will be close to the width of the flat bottom. Move around with the horizontal ⟺ and the CH1 ⇳ to help you with this comparison.) Next, we define the first clear loop splitting as a transition from period one to period two, which occurs at what we call $V_0(1)$, its transition voltage (this is like R_1 in our logistics discussion). The peak-to-peak transition FG voltage is actually $2V_0(1)$, so we will use $2V_0$ in our discussions, but no matter. The factor of 2 cancels out in the Feigenbaum calculations.

Go on now to measure the FG drive voltage $2V_0(2)$ at which the next doubling occurs (now period 4). This is on the verge of creating 4 loops. Continue to $2V_0(3)$, which is on the verge of creating 8 loops. Now the lack of precision is our enemy. We see that chaos is just a tiny voltage away. While you don't have as much precision, you can guesstimate $2V_0(4)$, the transition to period 16 (and 16 loops). In any case, you should qualitatively see the first hallmark of Feigenbaum's universality: *The geometric decrease of the differences between $V_0(n)$ and $V_0(n+1)$.*

With errors resulting from the lack of accuracy in observing exactly when the period doubling occurs and from the fact that we really should look at larger n, we can't expect very accurate Feigenbaum numbers, but we might get close: Compute the first number, using the notation of the logistics discussion, in which you are free to use the peak-to-peak voltages $2V_0$ instead of V_0 since as we said the factor of two cancels out:

$\delta(2) = [2V_0(2) - 2V_0(1)]/[2V_0(3) - 2V_0(2)]$

Again, because of the limitations don't worry if your answer is off, but in fact you might be pleasantly surprised here!

We see that the doubling to 32 is difficult to distinguish from all-out CHAOS! As noted, you could play with the "fuzziness" beyond $V_0(3)$ to estimate $V_0(4)$, and another estimate of the first Feigenbaum number ***Record your $2V_0$s and find your δs.***

D.5 Observe Geometrical Decrease in the Maximum Output Voltages Splittings at Period-Doubling: Second Feigenbaum Number

Recall the bifurcation plot in the logistics lectures where we saw the maximum splittings $\varepsilon(n)$ just before the next transition value R_{n+1}. Here, too, we can look at the maximum splitting of a pair of loops, just before they split into two new pairs. That is, we can watch how the *vertical split* in the x-y plot grows from zero at the initial period doubling voltage to a maximum just before the next period doubling. **(Now we must estimate the voltage on the oscilloscope directly; we don't have a digital readout here.)** Watch the vertical output voltage gap grow between the tops of the loops after they split apart in the first period doubling, and measure the maximum *vertical* gap achieved *just before* each loop itself splits at the next period doubling. (Don't worry about the fact that the peaks have shifted apart horizontally, too.) This vertical voltage measurement $\Delta V_{out}(n) \equiv \varepsilon(n)$ is therefore also right before a period doubling transition where we were measuring the horizontal voltage $V_0(n+1)$.

Specifically, revisit the drive voltage $V_0(2)$ at which the second doubling occurs and just before this, measure what is the maximum vertical difference of the two loops that grew out of one loop at $V_0(1)$. This is $\varepsilon(1)$. Now you can follow the split of either of the two loops (you may have to choose a lower one and its splits), continue to $V_0(3)$, and find the maximum split occurring just before $V_0(3)$. This is $\varepsilon(2)$. Despite the lack of precision, freely guesstimate $\varepsilon(3)$. You can try for $\varepsilon(4)$ but don't worry if it's too uncertain. What we can observe is Feigenbaum's universality at work again: *The geometric decrease of the differences between $\varepsilon(n)$ and $\varepsilon(n+1)$.*

With the warning that we have bigger errors from visually estimating with the oscilloscope screen the voltage splits described before, we can now get a very crude estimate of our second Feigenbaum number:

$$\alpha(2) = \varepsilon(1)/\varepsilon(2)$$

and very roughly guesstimate $\alpha(3)$ from your $\varepsilon(3)$. (You can try for $\alpha(4)$ if you have time.) ***Record your ε's and find your α's.***

D.6 Discussions and Errors

Finally, decide how you want to assess the errors in this experiment. One way is to estimate the uncertainty of exactly when splitting occurs for your voltage readings. (Perhaps repeat and see how different your FG reading is in the second or third decimal place.) Then you can look at limits for your Feigenbaum numbers by using the uncertainties to see how big δ would be if $V_0(2)$ were **increased** by your error estimate (e.g., 5.1 volts instead of 5.0) and $V_0(1)$ and $V_0(3)$ were **both smaller.** Then do the reverse to find how small δ could be. ***Use this "bracketing" in your worksheet to estimate the $\pm$ error in δ.*** Your error in the By adding or subtracting the bigger error you have for the vertical voltage splits, **estimate the $\pm$ *error in* α** by simply maximizing the numerator and minimizing the denominator, and vice versa. We should also remember the effect of having a diode with complicated voltage dependence has on testing period doubling theory. What do you think causes the distorted looping behavior you observed at lower voltages? (Hint: what complications would we have if there were, for example, two maxima instead of one in the logistics function! Feigenbaum's universality relies on a single smooth hump! ***Discuss briefly some of the limitations in this experiment.***