PHYS 121: Cycle 2 Review Sheet, Part 2

March 4, 2024

Totally elastic collisions:

In a *totally elastic* collision between two bodies A and B, by definition we know that the total kinetic energy in the bodies is conserved:

$$\tfrac{1}{2}m_a v_a^2 + \tfrac{1}{2}m_b v_b^2 = \tfrac{1}{2}m_a v\,_a'^2 + \tfrac{1}{2}m_b v\,_b'^2$$

If we have an elastic collision in an isolated system we can combine this expression with the Conservation of Linear momentum (here in full vector form):

$$m_a\vec{v}_a + m_b\vec{v}_b = m_a\vec{v}\,'_a + m_b\vec{v}\,'_b$$

In general, if the system is **isolated** (zero net external force) and the collision is **totally elastic** we can use this vector equation in combination with Conservation of Kinetic Energy of to solve for the two unknowns: $\vec{v}\,'_a$ and $\vec{v}\,'_b$.

Special Case: 1-D Elastic Collsions 'Master Equations'

If we further constraint the collision to **one-dimension**, it is possible to show that for an isolated system with a totally elastic collision, conservation of linear momentum and conservation of kinetic energy can be functionally re-written in the form of two **"master equations"** as follows

$$m_a v_a + m_b v_b = m_a v\,'_a + m_b v\,'_b \tag{1}$$

$$v_a - v_b = v\,'_b - v\,'_a \tag{2}$$

These two equations can be used to solve **any** 1-D collision problem without resorting to quadratic equations. In other words, for 1-D elastic collisions, the momentum is conserved and the *relative* velocity between the two blocks remains the same magnitude but changes sign. As a rule these master equations represent "two equations with two unknowns." where we solve for the unknowns. $v\,'_a$ and $v\,'_b$.

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Non-conservative forces and non-recoverable forms of Energy:

Sometimes the Law of Conservation of Energy appears to be violated. If a wad of clay is thrown to the floor, it sticks there. The potential and kinetic energy associated with the wad of clay goes to zero. Apparently, energy is not conserved. However, if we think that the Law of Conservation of Energy might be a good physical law then we must explain where the energy went. There are several forms of energy that are "non-recoverable" or "non-conservative" which is to say that once the energy is in this form, it is difficult (but not impossible) to convert the energy back into a form where Work can be done. Some forms of "non-recoverable" energy include:

mechanical deformation: When the bumper of a car crumples on impact or when a blob of clay flattens as it hits the floor, energy goes into the work required to force the body into a new shape.

light, sound and other waves: Light is a form of energy that is generally non-recoverable. (If you are very clever with solar cells, for example, you can get some of the energy back as useful work – but you cannot get all of it). Likewise sound and other mechanical waves are a mix of potential and kinetic energy.

heat: This is the ultimate form of all non-recoverable energy. When the dust settles and everything in the system stops moving, then the initial energy of the system has been transformed into heat.

Note: Friction generally converts energy from recoverable to non-recoverable forms. (Rub your hands together!) Friction is *not* a form of energy, (it is a force) but as a result of friction, energy is "lost" to non-recoverable forms, usually heat. As a rule, where there is friction, there is heat.

Some general things you should know about energy:

By now, you should have a good handle on what we mean by energy and work and the various different forms of energy. You should be able to distinguish between recoverable and non-recoverable forms of energy, and you should be able to describe how energy changes from one form to another in various situations. Examples include: a falling brick, a bouncing ball, a swinging pendulum, a body bouncing on a spring. In each of these cases, you should be able to describe how conservation of energy works. You should be able to describe how the energy moves from one form to another, in the case where energy is lost to non-recoverable form, you should be able to describe where the energy goes.

You should be able to work 1-dimensional collisions and explosions applying either the conservation of linear momentum or the conservation of energy or both. Be sure you understand for each problem whether or not you can treat the bodies as an "isolated system". You should be able to recognize both perfectly elastic and perfectly inelastic collisions.

The Center-of-Mass of a System (vectors):

Suppose we have a system of N particles each with a given position in terms of a set of position vectors $\vec{r}_1$ to $\vec{r}_N$. If we have a total of N masses in the system with given masses m_1, m_2, m_3,....m_N, then we define the center of mass of the entire system as the "weighted average" position of all of the particles:

$$\vec{r}_{cm} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_n\vec{r}_N}{M}$$

where $M = m_1 + m_2 + m_3 + ... + m_N$, the total mass of the system.

Alternately, we can write this down simply using the "series sum" symbol:

$$\vec{r}_{cm} \equiv \sum_{i=1}^{N} \frac{m_i\vec{r}_i}{M}$$

Note that for a system of particles with positions given in 2-D and 3-D we can just extend the idea to other coordinate, e.g.,

$$x_{cm} \equiv \frac{m_1x_1 + m_2x_2 + m_3x_3 + ... + m_nx_N}{M}$$

$$y_{cm} \equiv \frac{m_1y_1 + m_2y_2 + m_3y_3 + ... + m_ny_N}{M}$$

$$z_{cm} \equiv \frac{m_1z_1 + m_2z_2 + m_3z_3 + ... + m_nz_N}{M}$$

Note that we apply simple kinematic relationships here. For example, the velocity of the center-of-mass can be determined by taking the time derivative:

$$\vec{v}_{cm} = \frac{d}{dt}\left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_n\vec{r}_N}{M}\right)$$

$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_2\vec{v}_2 + m_3\vec{v}_3 + ... + m_nv_N}{M}$$

Likewise:

$$\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + ... + m_n\vec{a}_N}{M}$$

The Center-of-Mass as a point representation the entire system:

Once you know the center of mass point, you can talk about the dynamics of the entire system. Again, we restrict ourselves to one dimension and consider the total momentum of the system:

$$\vec{P}_{tot} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + ... + m_n\vec{v}_N$$

Using our definition of center-of-mass, we see that:

$$\vec{P}_{tot} = M\vec{v}_{cm}$$

In other words, if we know the velocity of the center-of-mass, we can immediately calculate the linear momentum of the entire system. We can treat the center-of-mass point as the "kinematic position" of the entire system.

Newton's Second Law Applied to a System: (vector)

We take a time-derivative of the last expression above and we get back something quite powerful:

$$\vec{F}_{ext} = M\vec{a}_{cm}$$

Here $\vec{F}_{ext}$ means the **external net force** on the entire system. In other words, the (vector) sum of all of the forces on all of the bodies that result from outside the system.

This works because all *internal forces* (that is to say forces between any two bodies within the system) must cancel each other out in any overall sum due to Newton's Third Law. In other words, when we consider the motion of the center-of-mass point, we need only consider external forces applied to the system and we can ignore *all* internal forces between bodies within the system completely. We can ignore these internal forces no matter how complicated they are.

Note that we can immediately use this equation to infer Conservation of Linear Momentum. If the net external force is zero, then by Newton's Second Law Applied to a System, it must be true that the the acceleration of the center-of-mass of the system is also zero, and therefore the velocity of the center-of-mass of the system must be constant as a function of time, and therefore the total linear momentum of the system is also constant, and therefore conserved.

Kinetic Energy Applied to a System:

Another equation holds for the kinetic energy of a system of masses:

$$K_{tot} = K_{int} + \frac{1}{2}Mv_{CM}^2$$

Since kinetic energy depends on the square of the velocity and is a scalar quantity (not a vector) the internal kinetic energy does not "cancel out".

Physics in the Center-of-Mass Reference Frame:

Once you know the center of mass point, you can talk about the dynamics of the entire system:

$$\vec{P_{tot}} = M\vec{v}_{cm}$$

Remember that the Laws of physics allow us to work any problem in any inertial reference frame we want. Sometimes it is very convenient to work some problems in the Center-of-Mass Reference Frame. Two very convenient things that happens in the Center-of-Mass frame:

$$\vec{P_{tot}} = 0$$

$$K_{tot} = K_{int}$$

Furthermore in the Center-of-Mass Reference Frame some symmetries of the problem become more obvious. This is especially true in collisions.

The trick is learning how to change from one reference frame to another. Suppose the center of mass velocity of the system in the lab frame is given by $\vec{v}_{cm}$. Then to convert to the center of mass frame, *subtract* this velocity from the velocity of each particle in the system. Then work the problem and get the new particle velocities (still in the center of mass system). Finally go back to the lab frame by *adding* $\vec{v}_{cm}$ to the velocity of each particle.

Some problems we have looked at this way include

- For any *totally inelastic* collision, the final velocity in the center of mass system is zero.
- For a *totally elastic* collision between two equal mass balls, the collision looks completely symmetric in the center of mass system.
- For a *totally elastic* collision between two objects where one of the objects is much more massive than the other, the center of mass of is nearly equally to the center of mass of the massive object. We showed in class how this leads to the surprising result when a massive object moving at speed v_0 that collides elastically with a low-mass object at rest, the final velocity of the light object is nearly $2v_0$.

Mr. Covault's Four-boxes method for two-body collisions:

In class we introduced the so called "Four Boxes" method for solving collisions. The key here is to **move from the Lab Frame to the center-of-mass (CM) frame.** The steps are:

- Calculate v_{cm}, the velocity of the center of mass for the system. Use this value for the remainder of the problem.
- Draw four boxes that show "Before" and "After" in each of the Lab and Center-of-Mass frames.
- Write down the initial conditions in the "Before" "Lab Frame" box.
- Move to the "Before CM Frame" box by *subtracting* v_{cm} from the velocity of each object.
- Go to the "After CM Frame" box by "bouncing" symmetrically. All you do is change positive velocities to negative velocities and vice versa. Swap signs.
- Go to the "After Lab Frame" box by *adding* v_{cm} from the velocity of each object. Careful with signs here.