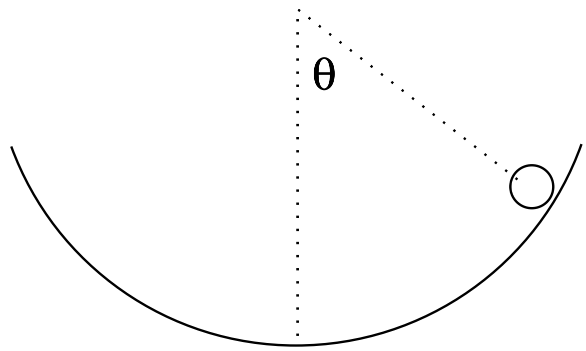
PHYS 121: Solutions to Written Homework #07
April 8, 2024

Note: this homework is worth **15 points** as follows:

___	Problem 1	5 points	5 pts total
___	Problem 2	5 Points	5 pts total
___	Problem 3	5 points	5 pts total

Problem 1: A Final Exam Problem From 2011:



A round marble with a given mass m and given radius r is placed inside a smooth spherical bowl as shown. The radius of curvature for the inside of the bowl is given as R and the marble is released at rest at a position corresponding to a given angle θ_0 relative to the the vertical radius line as shown. Assume $r << R$. Assume that θ_0 is small enough so that the marble rolls without slipping.

a) Draw a Free Body Diagram (FBD) indicating the forces on the marble at the point it is released. Be sure to include a proper coordinate system that is aligned with the direction of acceleration.

b) Draw an Extended Free Body Diagram (XFBD) indicating the forces on the marble at the point it is released. Be sure to indicate a "pivot point" and also indicate the direction of positive torque.

c) Calculate the magnitude and direction of the torque about the center of the marble that is applied to the marble at the point instantly after it is released. Give you answer in terms of the given parameters. Explain your work.

d) Calculate the magnitude of the translational acceleration of the marble at the point instantly after it is released? Give you answer in terms of the given parameters. Explain your work.

e) Use Conservation of Energy to determine the linear speed of the marble when it is at the bottom of the bowl. Don't forget that you must include both linear and rotational kinetic energy. Give you answer in terms of the given parameters. Explain your work.

f) What is the magnitude of the Normal Force on the marble when it is at the bottom of the bowl? Give you answer in terms of the given parameters. Explain your work. Hint: $N = mg$ is the wrong answer.

Solution to Problem 1:

Our general approach to all of these kinds of **rigid body motion** problems is to follow this **recipe**:

1. Write down a **regular FBD** for the body in question.
2. Apply **Newton's Second Law** in each relevant coordinate. See how far this gets us.
3. Write down an **XFBD** for the same body. Put the same forces on it. and pick a pivot point for calculating torques.
4. **Apply Newton's Second Law for torques:**
5. If applicable, apply geometric constraints, such as the **Rolling Constraint**
6. If required, solve for N equations with N unknowns to find the quantities we want to calculate in terms of the given parameters only.

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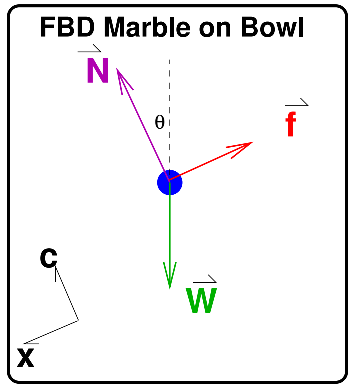
Part (a)

We start with a regular **Free Body Diagram** showing all of the forces on solid round marble which when it is located at angle θ



We note in particular that the force of friction must point back up the surface of the bowl. We choose a **tipped coordinate system** so that one of the coordinate (here labeled the "x-coordinate") points in the direction of acceleration. Note that so-called 'rolling friction' is really just **static friction** and we **cannot** know the value of this force without applying Newtons Laws.[1]

So we continue on our recipe with **Newton's Second Law:** applied in the "x-coordinate"

$$F_x = ma_x$$
$$W\sin(\theta) - f = ma_x$$
$$mg\sin(\theta) - f = ma_x \tag{1}$$

So this is as far as we can go. We have two unknowns, f and a_x, but only one equation.

As an *aside* it's pretty clear that solving for Newton's Second Law in the "centripetal" coordinate does not get you much:

$$F_c = ma_c$$
$$N - W\cos\theta = ma_c$$
$$N - W\cos\theta = m\frac{v^2}{r}$$
$$N - W\cos\theta = 0$$
$$N = W\cos\theta$$
$$N = mg\cos\theta$$

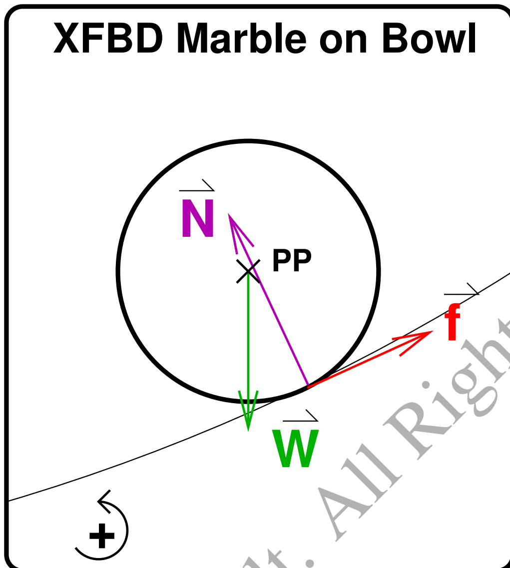
Solution continues next page...

[1]Never *ever* use this equation for static friction: $f_s = \mu_s N$. ***This equation is never applicable.***

Now you know the Normal force on the marble, but this does not help you determine the acceleration in the x-direction.

So we must continue on to make an **Extended Free Body Diagram**:

Part (b)



Note that the exact same forces that were on our FBD are now on our XFBD. Note that we have selected a "Pivot Point" corresponding to the axis of rotation of the solid spherical marble, and note that we have choose a definition of "positive angular acceleration" corresponding to counter-clock-Wise (CCW) so that as the marble rolls, the acceleration in both linear and angular terms is positive.

So we continue on our recipe with **Newton's Second Law for Rotations:**

$$\tau_{net} = I\alpha$$

$$\tau_N + \tau_W + \tau_f = I\alpha$$

To calculate the torques, we use the **definition of torque:**

$$\vec{\tau}_{\vec{F}} \equiv \vec{r} \times \vec{F} \qquad \rightarrow \qquad \tau_F = rF\sin(\phi)$$

where phi is the angle between the force vector and the position vector (measured from the pivot point to the point where the force is applied).

We see by inspection that the torque due to Weight is zero (since the force is applied at the pivot point so the position vector has zero length), and we see also that the Normal force results in zero torque (because the position vector is parallel to the force vector). The only force that contributes to the torque is friction:

$$\tau_f = I\alpha$$

$$Rf = I\alpha$$
$$Rf = (\frac{2}{5}mR^2)\alpha$$

At this point, since we want linear acceleration, not angular acceleration, we apply the **Rolling Constraint** that says $a_x = \alpha R$:

$$Rf = \left(\frac{2}{5}mR^2\right)\left(\frac{a_x}{R}\right)$$
$$f = \frac{2}{5}ma_x \tag{2}$$

Okay, now we have two equations and two unknowns. We plug (2) into (1) and solve for acceleration:

$$mg\sin(\theta) - \frac{2}{5}ma_x = ma_x$$
$$ma_x + \frac{2}{5}ma_x = mg\sin(\theta)$$
$$a_x + \frac{2}{5}a_x = g\sin(\theta)$$
$$a_x\left(1 + \frac{2}{5}\right) = g\sin(\theta)$$
$$\boxed{a_x = \left(\frac{5}{7}\right)g\sin(\theta)} \qquad \boxed{\textbf{Part (d)}}$$

Now that we have the acceleration, we can calculate the torque:

$$\tau_{net} = I\alpha$$
$$\tau_{net} = \left(\frac{2}{5}mR^2\right)\left(\frac{a_x}{R}\right)$$
$$\tau_{net} = \left(\frac{2}{5}mR^2\right)\left(\frac{5g\sin(\theta)}{7R}\right)$$
$$\boxed{\tau_{net} = \left(\frac{2}{7}\right)mRg\sin(\theta)} \qquad \boxed{\textbf{Part (c)}}$$

Solution continues next page...

To get the speed at the bottom of the bowl we use **Conservation of Energy:** We are allowed to do this because the forces are either conservative (like weight) or do no work. Note that friction does not do work because it is **static friction**.

$$E_{tot} = E'_{tot}$$
$$U + K_{tot} = U' + K'_{tot}$$
$$mgy + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgy' + \frac{1}{2}mv'^2 + \frac{1}{2}I\omega'^2$$

Now in this case, we see from trigonometry that $y = h \equiv R(1 - \cos\theta)$ and $v = 0$ and $\omega = 0$ since we release at rest. The final position $y' = 0$.

$$mgh + 0 + 0 = 0 + \frac{1}{2}mv'^2 + \frac{1}{2}I\omega'^2$$

We apply the **Rolling Constraint:** $v' = \omega' R$:

$$mgh + 0 + 0 = 0 + \frac{1}{2}mv'^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v'}{R}\right)^2$$

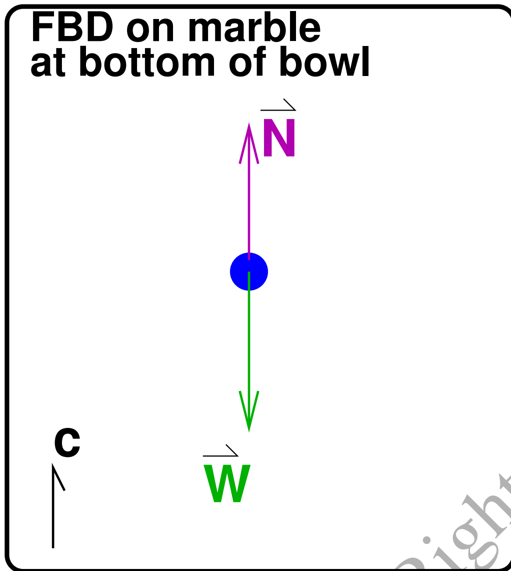
Solving for v':

$$\frac{1}{2}mv'^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v'}{R}\right)^2 = mgh$$
$$v'^2\left[\frac{1}{2} + \frac{1}{2}\left(\frac{2}{5}\right)\right] = gh$$
$$v'^2\left(\frac{1}{2} + \frac{1}{5}\right) = gh$$
$$v'^2\left(\frac{7}{10}\right) = gh$$
$$v'^2 = \left(\frac{10}{7}\right)gh$$
$$v' = \sqrt{\left(\frac{10}{7}\right)gh}$$

$$\boxed{v' = \sqrt{\left(\frac{10}{7}\right)gR(1 - \cos\theta)}}$$ **Part (e)**

Solution continues next page...

Finally, to get the magnitude of the **Normal Force** we re-consider the free-body diagram for the marble that the bottom of the bowl:



This is a simple problem in **Newton's Second Law** with centripetal acceleration:

$$F_c = ma_c$$

$$N - W = m\frac{v'^2}{R}$$

$$N = m\frac{v'^2}{R} + W$$

$$N = m\frac{\left(\frac{10}{7}\right) gR(1-\cos\theta)}{R} + mg$$

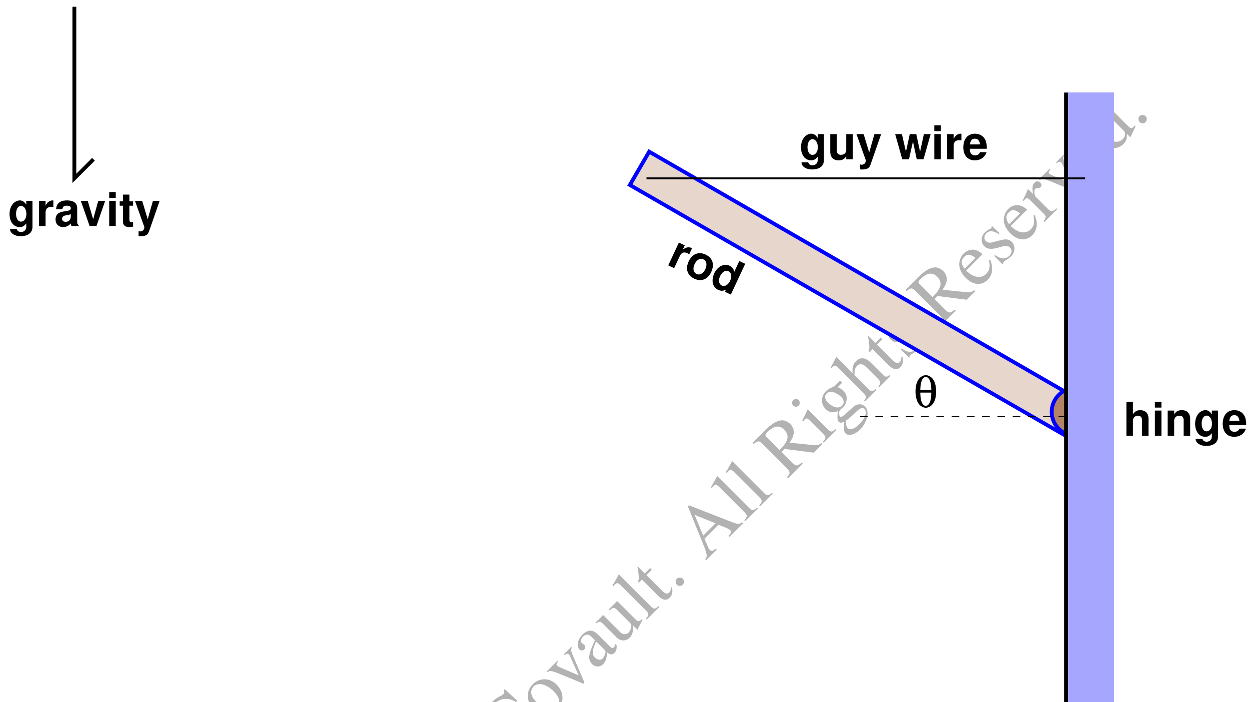
$$N = mg\left(\frac{10}{7}\right)(1-\cos\theta) + mg$$

$$\boxed{N = mg\left(\frac{17}{7} - \frac{10}{7}\cos\theta\right)}$$ **Part (f)**

Note that this answer converges to $N = mg$ in the limit that theta goes to zero, which is just what we expect

Problem 2: From a Previous Exam:

A thin rod of given mass m is attached to an hinge so that it can swing vertically. The length of the rod is given as ℓ and the rotational inertial of the rod about one end is $I = \frac{1}{3}m\ell^2$. The rod is held in place at a given angle θ with respect to the horizontal by an guy wire which we represent as an ideal string which connects the far end of the bar to the wall as shown.



Part (a) – Draw a careful and properly labeled Free-Body Diagram (FBD) that shows *all* of the forces on the thin rod. Also, draw a careful and properly labeled Extended Free-Body Diagram (XFBD) for the thin rod. In your diagrams, the unknown *hinge force* should be represented by two components, H_H and H_V, corresponding to the *horizontal* and *vertical* components respectively.

Part (b) – Calculate the magnitudes of both H_V and H_H. Express your answers in terms of the given parameters. Explain. Hint: for this part use a coordinate system that is lined up with the Hinge Force components (*not* tilted). Another hint: $\sin(90^\circ - \theta) = \sin(\frac{\pi}{2} - \theta) = \cos\theta$.

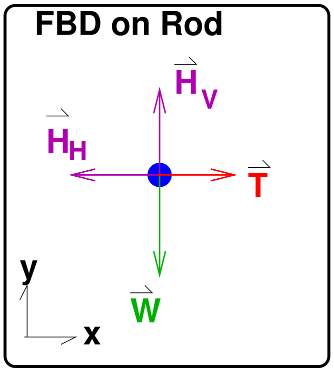
Part (c) – Suppose the guy wire is cut so that the rod is suddenly free to begin to rotate on the hinge. What is the magnitude of the acceleration of the center-of-mass point of the rod just at the instant after the wire is cut? Express your answer in term of the given parameters. Explain your work. Hint: for this part use a coordinate system that is *tilted* so that it is aligned with the instantaneous direction of acceleration.

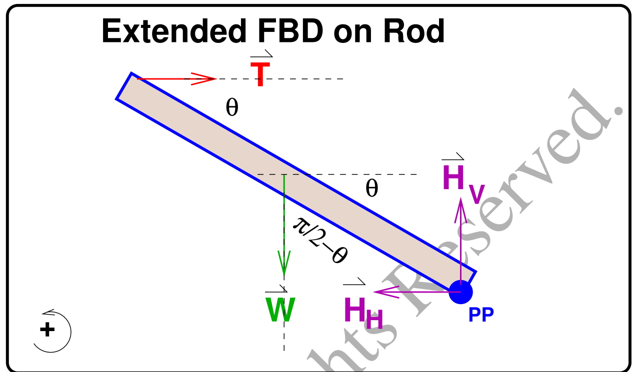
Part (d) – After the wire is cut, the rod swings down and collides flush with the wall. What is ω_{hit}, the angular speed of the rod at the instant just before it hits the wall? Express your answer in term of the given parameters. Explain your work.

Solution to Problem 2:

Part (a):

Here is the **Free Body Diagram** and the **Extended Free Body Diagram** for the thin bar on the hinge supported by the horizontal guy wire:





The forces on the bar are:

- W – The **Weight** of the bar (downward).
- T – The **Tension** on the bar due to the guy wire (horizontal to the right).
- H_H – The horizontal **hinge force** which we will guess is to the left.
- H_V – The vertical **hinge force** which we will guess is upward.

For the Extended Free Body Diagram, these same four forces are applied at the following locations:

- W – The **Weight** is applied at the **Center-of-Mass Point**.
- T – The **Tension** is applied at the upper left end of the rod.
- H_H – The horizontal **hinge force** is applied at **pivot point** on the hinge.
- H_V – The vertical **hinge force** is applied at **pivot point** on the hinge.

Part (b):

To determine the Hinge Force components H_V and H_H, we apply **Newton's Second Law** to the rod: Starting with the vertical component:

$$F_y = ma_y$$
$$H_V - W = 0$$
$$\boxed{H_V = mg}$$

Now the Horizontal component:

$$F_x = ma_x$$
$$T - H_H = 0$$
$$H_H = T$$

Unfortunately, we don't know the tension, so we are not done. We apply **Newton's Second Law for Rotations** to the static rigid body of the rod.

$$\tau_{net} = I\alpha$$
$$\tau_{net} = 0$$
$$\tau_{\vec{T}} + \tau_{\vec{W}} + \tau_{\vec{H}_V} + \tau_{\vec{H}_H} = 0$$

We inspect the *Extended Free Body Diagram* to infer each of the torques in accordance with the **definition of torque:**

$$\tau_{\vec{F}} \equiv |\vec{r} \times \vec{F}| = rF\sin\phi$$

where $\vec{r}$ is the radial vector from the axis of rotation (pivot point) to the point of application of the force and ϕ is the angle between vectors $\vec{r}$ and $\vec{F}$. We see that for the selected pivot point, the torque due to each component of the hinge force is **zero**. So we just calculate the torque from the Weight and Tension noting that both of these have different values for ϕ. In this case we arbitrarily defined positive rotation counter-clockwise:

$$\tau_{\vec{T}} + \tau_{\vec{W}} + 0 + 0 = 0$$
$$-T\ell\sin\theta + (mg)(\ell/2)(\sin\pi/2 - \theta) + 0 + 0 = 0$$
$$-T\ell\sin\theta + \frac{mg\ell\cos\theta}{2} = 0$$
$$-2T\sin\theta + mg\cos\theta = 0$$
$$T - \frac{mg\cos\theta}{2\sin\theta} = 0$$
$$T = \frac{1}{2}(mg\cot\theta)$$

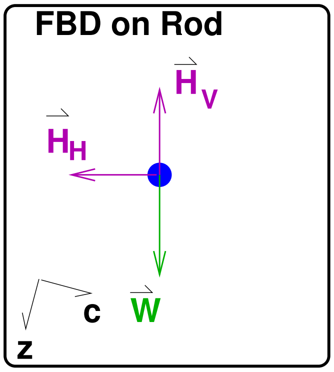
So we plug this in to get H_H:

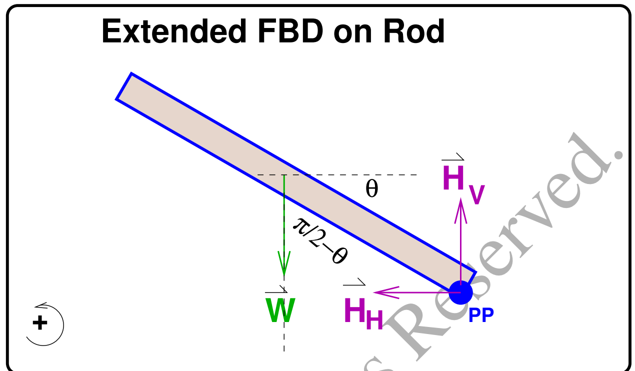
$$\boxed{H_H = \frac{1}{2}(mg\cot\theta)}$$

Solution continues next page....

Part (c):

Now we slightly reconfigure our diagrams to select coordinates appropriate to the motion of the bar. Since the wire is cut, there is no tension. We define the positive direction of acceleration as z and the positive direction of angular acceleration **counter-clockwise**





To get the angular acceleration, we consider **Newton's Second Law for Rotations:**

$$\tau_{net} = I\alpha$$

$$\tau_{\vec{W}} + \tau_{\vec{H}_V} + \tau_{\vec{H}_H} = I\alpha$$

Only the **Weight** contributes to the torque:

$$(mg)(\ell/2)(\sin\pi - \theta) = (1/3)(m\ell^2)\alpha$$

$$\frac{mg\ell\cos\theta}{2} = \frac{m\ell^2}{3}\alpha$$

And we put in the **Rolling Constraint** where $a = R\alpha$ and now R is the radius from the center-of-mass to the pivot point: $R = \ell/2$:

$$\frac{mg\ell\cos\theta}{2} = \frac{m\ell^2}{3}\frac{a_z}{(\ell/2)}$$

Algebra:

$$\frac{mg\ell\cos\theta}{2} = \frac{2m\ell a_z}{3}$$

$$\boxed{a_{cm} = a_z = \frac{3g\cos\theta}{4}}$$

Solution continues next page....

Part (d):

Since we are asking for speed, we consider **Conservation of Energy**. But first we have to **check the condition:** The forces on the rod are (1) Weight (conservative) and the Hinge Forces which **do no work**. So meet the condition and we are all set. The "BEFORE" corresponds to release at angle θ and the "AFTER" corresponds to hitting the wall:

$$\text{"BEFORE"} = \text{"AFTER"}$$
$$E_{tot} = E'_{tot}$$
$$U_W + K_{tot} = U'_W + K'_{tot}$$
$$U_W + K_{translational} + K_{rotational} = U'_W + K'_{translational} + K'_{rotational}$$

We note that in the case that we chose the hinge as the pivot point, we model the motion as **purely rotational**.

$$U_W + 0 + K_{rotational} = U'_W + 0 + K'_{rotational}$$
$$mgy_{cm} + \frac{1}{2}I\omega^2 = mgy'_{cm} + \frac{1}{2}I\omega'^2$$

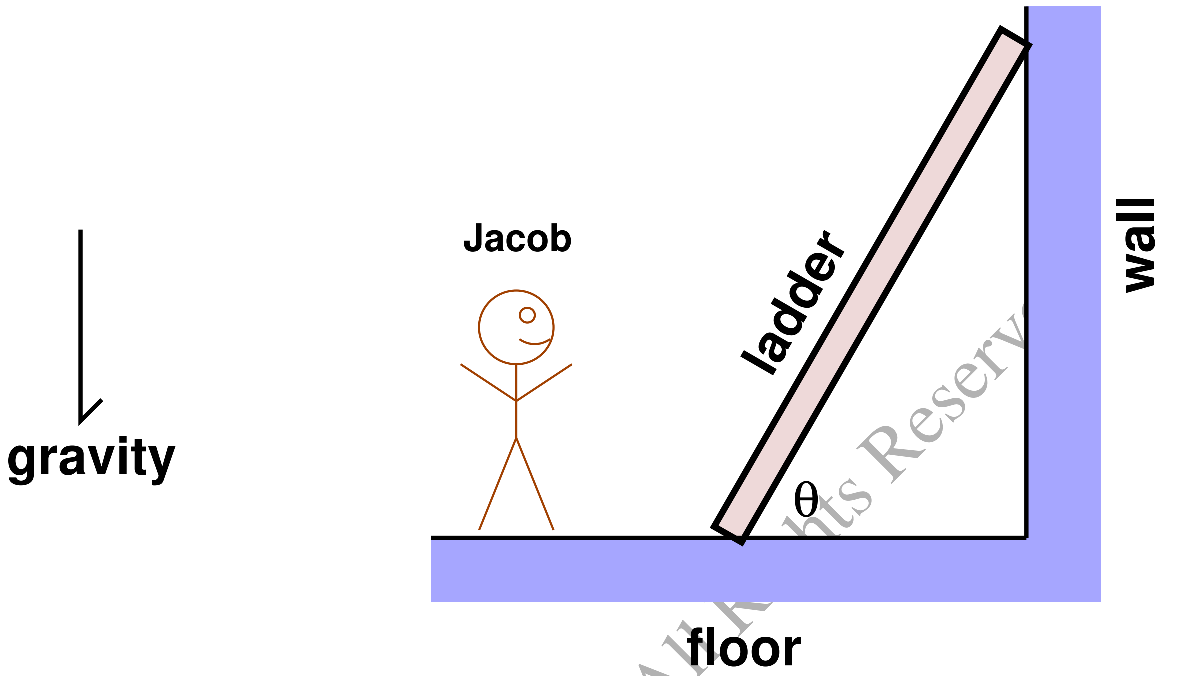
We choose an appropriate vertical coordinate for potential energy with $y = 0$ at the pivot point. Then $y_{cm} = (\ell/2)\sin\theta$ and $y' = -(\ell/2)$: Initial angular speed is zero. Plug these in along with our expression for $I = (1/3)m\ell^2$ and solve for the unknown angular speed:

$$mg(\ell/2)\sin\theta + 0 = mg(-\ell/2) + \frac{1}{2}\left(\frac{1}{3}m\ell^2\right)\omega'^2$$

Algebra:

$$\frac{1}{2}\left(\frac{1}{3}m\ell^2\right)\omega'^2 = mg(\ell/2)\sin\theta + mg(\ell/2)$$
$$\omega'^2 = \frac{3g(1+\sin\theta)}{\ell}$$
$$\boxed{\omega' = \sqrt{\frac{3g(1+\sin\theta)}{\ell}}}$$

Problem 3: Jacob's Ladder



A ladder of given length L and mass M leans ***at rest*** against the wall at a given angle θ as shown. Assume that the surface of the *wall* is **frictionless**, but assume that the *floor* applies both a **Normal** force *and* an **Friction** force to the ladder.

Part (a) – (5 points): Draw a **Free Body Diagram** and also an **Extended Free Body Diagram** for the ladder. Be sure to label each of your forces appropriately.

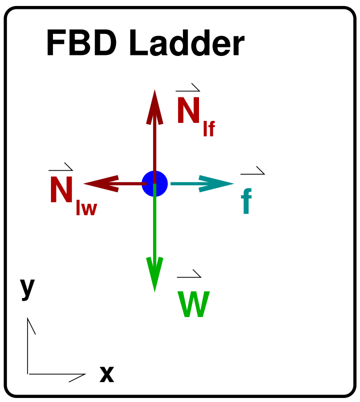
Part (b) – (15 points): Calculate the **magnitude** of *each and every force* on the ladder. Present your answers in terms of given parameters. Explain your work.

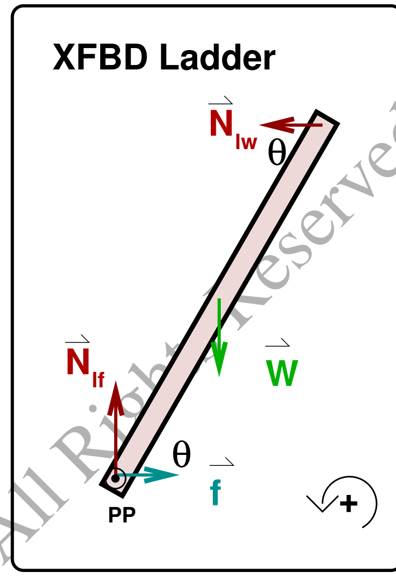
Part (c) – (10 points): Suppose the coefficients of **kinetic friction** and **static friction** are *given* as μ_k and μ_s respectively. What is the ***constraint*** that applies to the value of $\boldsymbol{\theta}$ that ensures that the ladder stays *at rest* and does **not** slip and fall down? Explain your work.

Solution to Problem 3:

Part (a) – (5 points):

We start with the **Free Body Diagrams** and the **Extended Free Body Diagram**





There are **four forces** on the ladder:

1. **Weight** labeled $\vec{W}$ pointed ***down***.
2. **Friction** labeled $\vec{f}$ pointed ***to the right***.
3. The **Normal Force** on the ladder due to the floor, here labeled $\vec{N}_{lf}$ pointed ***up***.
4. The **Normal Force** on the ladder due to the wall, here labeled $\vec{N}_{lw}$ pointed ***to the left***.

The **XFBD** has these four forces drawn on the ladder as shown.

Part (b) – (15 points):

We apply **Newton's Second Law** to the ladder. Since the ladder is ***not moving at all*** this is a statics problem, so ***all accelerations are zero:***

Newton's Second Law: Y-direction:

$$F_y = ma_y$$
$$N_{lf} - W = 0$$
$$N_{lf} - mg = 0$$

$$\boxed{N_{lf} = W = mg} \tag{3}$$

That's two out of four forces calculated.

Newton's Second Law: X-direction:

$$F_x = ma_x$$

$$f - N_{lw} = 0 \tag{4}$$

Two unknowns, f and N_{lw}.

Newton's Second Law for Rotation:

$$\tau_{net} = I\alpha$$
$$\tau_W + \tau_f + \tau_{N_{lf}} + \tau_{N_{lw}} = 0$$

We use the **Definition of Torque** to calculate the terms:

$$\tau_F \equiv rF\sin\phi$$

Since we chose the Pivot Point at the bottom of the ladder, the torque due to the friction and the vertical Normal force is zero. For the Weight force, $r = L/2$, $F = mg$ and $\sin\phi = \sin(00^\circ - \theta) = \cos\theta$. For the Normal force due to the wall, $r = L$, $F = N_{lw}$ and $\sin\phi = \sin\theta$. Plugging these in and noting that the torque due to the Weight is negative:

$$-\frac{mgL}{2}\cos\theta + N_{lw}L\sin\theta = 0$$
$$N_{lw}\sin\theta = \frac{mg}{2}\cos\theta$$

$$\boxed{N_{lw} = \frac{mg}{2}\cot\theta} \tag{5}$$

Finally we plug Equation (6) into Equation (5) to get the friction:

$$\boxed{f = \frac{mg}{2}\cot\theta} \tag{6}$$

Part (c) – (10 points):

The requirement that the ladder not slip and fall corresponds to the **constraint on static friction**:

$$f_s \leq \mu_s N$$

In this case, the friction is the force labeled f and the normal force that matters is the one at the point of contact, N_{lf}. To apply the constraint we plug in our results from Part (b) for both forces:

$$f \leq \mu_s N_{lf}$$

$$\frac{mg}{2}\cot\theta \leq \mu_s mg$$

$$\boxed{\cot\theta \leq 2\mu_s}$$

When we take reciprocal of both sides, we need to ***flip the inequality***:

$$\boxed{\tan\theta \geq \frac{1}{2\mu_s}}$$

$$\boxed{\theta \geq \arctan\left(\frac{1}{2\mu_s}\right)}$$

or

$$\boxed{\text{The \textbf{minimum} value of } \theta \text{ is } \arctan\left(\frac{1}{2\mu_s}\right)}$$