Chapter 10++

More on Momentum
BACK TO THE FUTURE II++
More Predictions with Momentum Conservation

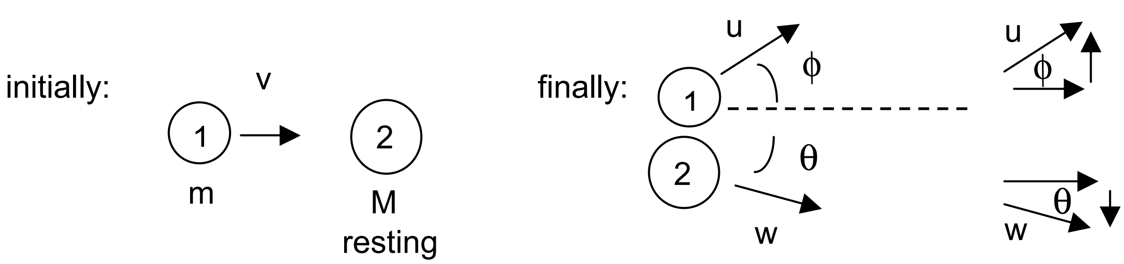
Revisit:	To-Do:
• Momentum conservation • Elastic collisions	• 2D elastic collisions (especially for equal masses) • Rocket motion through fuel burning

By the way, why do we always restrict ourselves to two-body collisions? Because collisions among three or more bodies taking place at exactly the same instant are rare and rarely relevant to things happening around us!

Something New: 2D Collisions that are Elastic (and not just those inelastic things that go splat)

Finally, we consider a two-body collision that is NOT head-on and therefore the bodies may sail off at angles with respect to the original projectile.

Elastic two-body collisional example: (recall that elastic means that kinetic energy is conserved during the collision). An initial body of mass m moves in and strikes another body M. It is not a head-on collision and hence we now have motion going off at angles afterwards. **Note**: the magnitudes (v, u, w) of the velocities are used here (i.e., the speeds). We thus put the signs of the velocities in by hand as needed:



1) Momentum conservation – momentum is a vector and we see in the above reaction that the initial momentum vector of 1 must add up to the sum of the two momentum vectors of 1 and 2 afterwards. A vector addition triangle could be used, or, as we do below, the x- and y-components (i.e., the horizontal and vertical components) can be separately dealt with and are separately "conserved." Note that the vector addition triangle helps us understand that all three momenta (2's initial momentum is zero) must lie in the same plane so our picture is right!

Before "conserving" our horizontal and vertical components of momentum, notice that the little golden triangles above help us find the components of the final velocities. Conservation of horizontal momentum yields

$$mv + 0 = mu\cos\phi + Mw\cos\theta$$

Conservation of vertical momentum (note the minus sign for 2's final component):

$$0 \quad + \quad 0 \; = \; mu\sin\phi \; - \; Mw\sin\theta$$

2) kinetic energy conservation (kinetic energy is NOT a vector – **don't use any cosine or sine factors anywhere here!**):

$$\frac{1}{2}mv^2 + 0 \; = \; \frac{1}{2}mu^2 \; + \; \frac{1}{2}Mw^2$$

So given v (and m and M), we have three equations with four unknowns (u, w, ϕ, θ). **We typically are also given θ, and then we can solve for u, w, ϕ.** For an example that is easier to analyze, we turn to the equal-mass case next.

2D Equal-Mass Elastic Scattering

The case of equal masses M=m is useful because "equal" starts with E and that rhymes with P or B and those spell "pool" or "billiards." Those games involve equal-mass balls! ☺ E also rhymes with G for gas, and applies, for example, to collisions of gas made up of a single kind of molecule. While E rhymes with T, there's no trouble here – it's kind of easy. See below.

First Method: The mass m cancels out and the two momentum-conservation equations reduce to

$$v \; = \; u\cos\phi + w\cos\theta, \qquad 0 \; = \; u\sin\phi - w\sin\theta \qquad \textbf{(1)}$$

and the kinetic energy conservation (besides m, also cancel the ½ factor) reduces to

$$v^2 \; = \; u^2 + w^2 \qquad \textbf{(2)}$$

Now rewrite the top two equations in (1) so as to isolate the w (and θ) terms

$$v - u\cos\phi = w\cos\theta, \qquad u\sin\phi = w\sin\theta \qquad \textbf{(3)}$$

Then square both sides of each of these equations in (3)

$$(v - u\cos\phi)^2 = (w\cos\theta)^2, \qquad (u\sin\phi)^2 = (w\sin\theta)^2 \qquad \textbf{(4)}$$

Next add the two equations in (4) to each other, rearrange and use $\cos^2\theta + \sin^2\theta = 1$ to eliminate θ:

$$(v - u\cos\phi)^2 + (u\sin\phi)^2 = (w\cos\theta)^2 + (w\sin\theta)^2 = w^2(\cos^2\theta + \sin^2\theta) = w^2 \qquad \textbf{(5)}$$

Multiplying out the left-hand-side of (5), and using $\cos^2\phi + \sin^2\phi = 1$, we get

$$v^2 - 2\,u\,v\cos\phi + u^2 = w^2 \qquad \textbf{(6)}$$

Insert (6) into (2) to get rid of w, and we do simple algebra to get

$$2u^2 = 2\,u\,v\cos\phi$$

Discard the u = 0 solution which leads back to a head-on collision (u = 0 means body 1 comes to rest and 2 moves ahead just like in billiards). We look at the other solution in order to understand colliding and going off at angles:

$$\boxed{u = v\cos\phi} \quad \textbf{(7)}$$

We can stick this into (2) to get

$$\boxed{w = v\sin\phi} \quad \textbf{(8)}$$

But on the other hand we can stick both (7) and (8) into the second equation in (3) and show that

$$\cos\phi = \sin\theta \quad \textbf{(9)}$$

(9) means $\phi = \pi/2 - \theta$, or

$$\boxed{\phi + \theta = \frac{\pi}{2}} \quad \textbf{(10)}$$

For equal masses, they always scatter at right angles to each other!

Second Method: see the following problem:

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Problem 10-8 We were going to show you the power of vectors and dot products, but what the heck, why not let you do it! You are about to derive (10) super fast!

Use $\vec{v} = \vec{u} + \vec{w}$ instead of (1) and square it (that is, take the dot product of both sides with themselves: $\vec{v}\bullet\vec{v} = (\vec{u} + \vec{w})\bullet(\vec{u} + \vec{w})$). Then use the dot product definition (in particular, the dot product of a vector with itself being just its magnitude squared), and define ψ as the angle between $\vec{u}$ and $\vec{w}$. By comparing your result with (2), show that you must have

$$\cos\psi = 0 \quad \text{or} \quad \psi = \pi/2$$

But $\psi = \phi + \theta$! **Q.E.D.**

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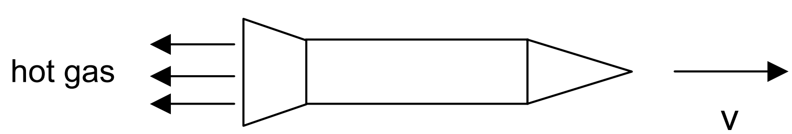
Finally, let's look at an example where the mass of an object changes with time.

VARIABLE MASS EXAMPLE

ROCKET MOTION



Gas velocity is backwards (we use velocity magnitudes (speeds) for the rocket and the gas, so as always we need to put in the minus signs explicitly if the 1D velocity is backwards. Call u the gas speed and assume it is constant relative to the rocket.

1) We are interested in relating the rocket speed v(t) (relative to the stars) to the rocket mass M(t) at time t. We ignore gravity.

2) At time t + dt, M has changed to M+ dM, with a loss dM in mass, dM < 0. This is consistent the fact that the gas with positive mass –dM > 0 was ejected out the back. The speed of that gas is v - u relative to the stars (the gas goes at - u with respect to spaceship and the spaceship goes +v with respect to inertial frame of stars).

3) Now with no external forces, the momentum of the whole system (relative to the stars) must always be constant. In particular, the momentum of the gas of mass -dM must be equal and opposite to the change in the momentum of the rocket when M went to M+dM. (Forget the previous gas ejected; it balanced the previous changes in rocket momentum.)

In particular, the following momenta must be equal to each other:

Initial: (time t) rocket momentum = Mv

Final: (time t+dt) rocket+gas momentum = $(M+dM)(v+dv) + m_{gas}(v-u)$ (with $m_{gas} = -dM$)

Setting them equal:

$$Mv = (M+dM)(v+dv) - dM(v-u)$$

or

$$Mv = Mv + Mdv + dM\,v + dMdv - dM\,v + dMu$$

and notice that the Mv and dM v terms cancel out.

Notice also that you can ignore the second-order term dM dv relative to the first-order terms (the ratio of the second-order term to any of the first-order terms vanishes in the calculus limit).

Therefore $0 = Mdv + udM \Rightarrow dv = -u\,dM/M$

and integrate to get

$$\boxed{v - v_0 = u \ln (M_0/M)} \quad \text{(define } v = v_0 \text{ when } M = M_0\text{)}$$

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Problem 10-9

a) If a rocket, initially at rest, is to attain a terminal velocity (relative to the stars and when the fuel is all burnt up) of a magnitude that happens to equal the magnitude of the exhaust velocity (which is fixed relative to the rocket), what fraction of the initial mass must be fuel? (Hint: examine the situation when we have reached v = u.)

b) Can a rocket go faster than this terminal velocity? What is its limit?

c) Describe the motion of the overall original system CM, starting from the initial moment (when the rocket first began to blast off) to the attained final velocity in (a).

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