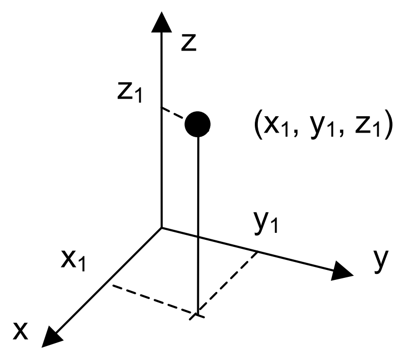
Chapter 3

Constant Force and Constant Acceleration in 2D, 3D

- **Extend Constant Acceleration to all Dimensions**
- **Simplest Gravity Trajectories in 2D (Starts off Horizontally)**

Let's prepare ourselves for 2D trajectories (the paths bodies follow under the influence of gravity):

ACCELERATION IN THREE DIMENSIONS



Recall 3D means we need three numbers with respect to some reference point to give our position. Thus we need three numbers each for velocity and acceleration, because they tell the rate of change of the position and the rate of change of the rate of change of the position in each of the three directions. For example, $a_x = d^2x/dt^2$ (i.e., $= dv_x / dt$) and this number tell us the acceleration in the x-direction. Also remember that each of these numbers is called a component. (e.g., v_x is the x-component of the velocity vector, y is the y-component of the position vector, a_z is the z-component of the acceleration vector.)

If each and every acceleration component (again, the acceleration in a given direction) is constant (each constant is possibly different from the other constants), we can treat the body's motion as the result of three separate 1D motions. That is, if all three acceleration components are constants in time, we have similar-looking equations for all three x, y, z coordinates. From Chapter 1, with different subscripts to distinguish the three directions, we get the velocity and position components given by

CONSTANT ACCELERATION FORMULAS FOR ALL COORDINATES

x coordinate:	**y coordinate:**	**z coordinate:**
$a_x = a_{0x}$ (constant)	$a_y = a_{0y}$ (constant)	$a_z = a_{0z}$ (constant)
$v_x = v_{0x} + a_{0x}t$	$v_y = v_{0y} + a_{0y}t$	$v_z = v_{0z} + a_{0z}t$
$x = x_0 + v_{0x}t + \frac{1}{2}a_{0x}t^2$	$y = y_0 + v_{0y}t + \frac{1}{2}a_{0y}t^2$	$z = z_0 + v_{0z}t + \frac{1}{2}a_{0z}t^2$

The above must have come from having constant $\vec{F}_0$ in $\vec{F} = m\vec{a}$ which implies we have constant acceleration $\vec{a}_0 = (a_{0x}\ ,\ a_{0y}\ ,\ a_{0z}\ )$. That is, if every force component is constant, then every acceleration component is constant (because it's proportional to its corresponding force component).

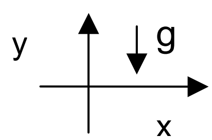
THE CONSTANT SURFACE GRAVITY ACCELERATION g
Supercalifragilisticexpialidocious Example Of Constant Acceleration

Close to the surface of the Earth (and we'll understand this better in Chapter 4), gravity vertically exerts a constant force "downward" such that there is a downward constant acceleration: $|a_{oy}| = g = 9.8 \text{ m/s}^2 = 32 \text{ ft/s}^2$ *

We use **g > 0 always**. If the y-axis is positive upward and gravity is downward, the vertical component of acceleration is a negative constant. We have

$$\boxed{a_{0y} = -g = -9.8 \text{ m/s}^2 = -32 \text{ ft/s}^2}$$

Our picture is usually:



* Most introductory texts use g = 32 ft/s^2 = 9.8 m/s^2 = 980 cm/s^2 in calculations. (For more accurate numbers, we need to know the latitude – see later discussions on "centripetal" effects due to the Earth's rotation.) In SI, one could even use g = 10 m/s^2, since it generates only a 2% error!

CONSTANT GRAVITY TRAJECTORY PROBLEMS
Air Friction Neglected for Most of this Course!!!

Quick 1D Trajectory Example (this really is just Chapter 2 stuff):

A stone is released from rest, 2 m above ground. How long does it take for the stone to hit the ground? (The neglect of air friction is not a bad approximation here.)

Solution: Make the usual choice of y for the vertical position (with up representing the positive y direction). You can choose other conventions – just stay consistent. In addition, let's choose the initial time t = 0 where the initial position is $y_0 = 0$ and the initial velocity is $v_0 = 0$. Then for the time t_{hit} that the stone hits the ground,

$$y(t_{hit}) = -\frac{1}{2} g t_{hit}^2 = -2 \text{ m}$$

we solve for t_{hit} (notice how you have to keep track of all the minus signs so that you don't make a mistake and get things like square roots of negative numbers)

$$t_{hit}^2 = \frac{2(-2)}{-g} \Rightarrow t_{hit} = \sqrt{\frac{-4 \text{ m}}{-9.8 \text{ m/s}^2}} = \sqrt{\frac{4 \text{ m}}{9.8 \text{ m/s}^2}} = 0.64 \text{ s}$$

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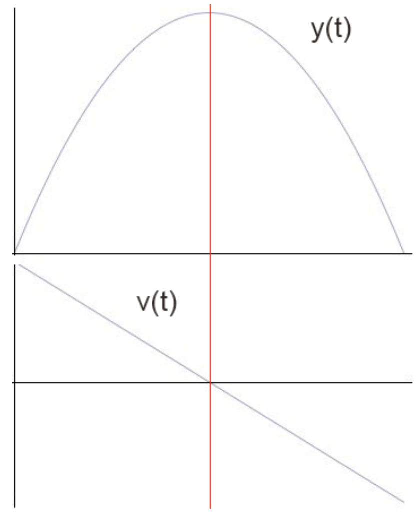

Problem 3-1 *To review your old high school stuff, suppose you throw a ball straight upward with an initial speed of 15 m/s. You eventually catch the ball when it comes back down. The plot of its position* y(t) i*s shown on the right and is a good old parabola symmetric around the vertical red line that goes through its peak. The velocity* v(t) = dy/dt, *which is of course the slope of* y(t), *is shown in the graph below the* y(t) *plot.*

a) How long will it take to get to the maximum height?
b) How high (relative to your hand) will the ball go?
c) How long does it take for the ball to fall from its maximum height back to your hand? Don't use symmetry but check your answer against (a). (For posterity, we prove the symmetry below; but don't use it, pretty please!)
d) What will the speed of the ball be when you later catch it in your hand? Again, don't use symmetry but compare your answer to the initial value.

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Symmetry Proof: We can prove this fast if we put our origin, **y=0, t=0, at the peak** so that times on the left of the middle **vertical** line are negative. Then, if we can show **y(-t) = +y(t)** and **v(-t) = - v(t)**, for any t, we are done! With y=0, t=0 at the peak, then $y(0) \equiv y_o = 0$, $v(0) \equiv v_o = 0$, and we have $y(t) = -1/2gt^2$ and $v(t) = -g\,t$. We immediately see y(-t) = y(+t) and v(-t) = - v(+t) . Q.E.D. (You already knew this for a parabola from math class so ... never mind! – in the words of the late Gilda Radner)

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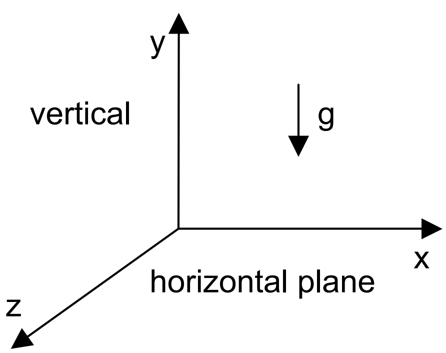
2D Trajectory Formulas:

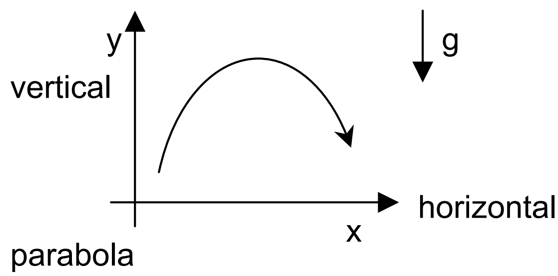
Now we add the horizontal directions to our motion in constant gravity in the negative y (vertical) direction. Therefore insert, $a_{0x} = a_{0z} = 0$, in addition to $a_{0y} = -g$ into our boxed equations of p. 3-1. The corresponding picture is

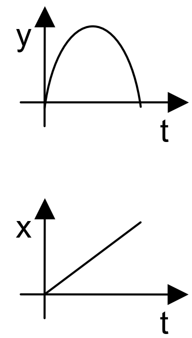


It is important to note that, for constant acceleration, a body's three position-components (x, y, z) change independently of each other. We'll see the implications of that over and over.

Forget z: Why do we just talk about 2D instead of 3D in much of our trajectory stuff? Well, motion will be in planes (the directions of the initial velocity and gravity determine a plane to which the motion will be restricted, so choose the x-y plane for convenience to be that plane and we will therefore only need x-y plots from now on!!

Parabolas for both y(t) and y(x) but straight lines for x(t): We are often interested in y(x) plots, which are what we see, to go along with our old time plots, y(t) and x(t). The general kinds of plots we get are parabolas for y(t) and straight lines for x(t). (This means that y is a quadratic function of t and x is a linear function of t.) We thus also get parabolas for y(x) because if we take the relation x(t) and then solve it for t in terms of x, and then substitute it into y(t), we get y as a quadratic function of x. The various plots look like:





The general formulas are

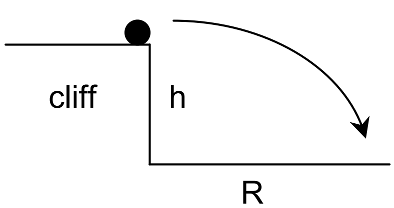
1) For zero acceleration in the x direction, we have the linear relation for x and v_x is constant:

$$x = x_0 + v_{0x}\, t \qquad \text{and} \qquad v_x = v_{0x} = \text{constant}$$

2) For acceleration – g in the y direction, we have the quadratic relation for y and the linear relation for v_y :

$$y = y_0 + v_{0y} t - \tfrac{1}{2} g\, t^2 \qquad \text{and} \qquad v_y = v_{0y} - g\, t$$

Simple 2D Trajectory Example: A stone is thrown with an initially horizontal velocity with magnitude (i.e., speed) 3.0 m/s from the edge of a cliff. So the vertical velocity is initially zero:



$v_{0x} = 3.0$ m/s, $v_{0y} = 0$

Choose the initial position of the stone to be $x_0 = y_0 = 0$ at $t_0 = 0$. Suppose it hits the ground at $t = 2.0$ s. The question is, **where does it hit**? That is, what are the h and R shown in the figure? With "up" defined as positive and h defined as positive, we know that $y = -h$ when the stone hits the ground:

Vertically: $y = -\frac{1}{2} g t^2 \Rightarrow -h = -4.9(2)^2$ so $h \cong 20$ m

Horizontally: $x = v_{0x} t = 3(2) = 6$ m so $R = 6$ m

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Problem 3-2

a) *Notice the irrelevance of v_{0x} (and x) to the h calculation in the above example. To highlight this irrelevance further, suppose someone in 2010 shoots marbles* ***horizontally*** *off of one of the tallest buildings in the world (in Hong Kong) that is presently under construction and will be 490 m high. She will shoot one marble at a horizontal velocity of 10 m/s and a second one at a horizontal velocity of 20 m/s. Calculate the time it takes each marble to reach the ground, neglecting as always air drag.*

b) *What would we need to do if the initial vertical component of the velocity was not zero? That is, if $v_{0y} \neq 0$? Consider the following situation: A strong-armed kid is sitting up in a tree, holding a small ball that is 5.0 m above the ground. She then throws this ball such that it takes 5.0 s from the instant she throws it until it hits the ground* ***directly below where she was sitting****. Ignore air drag, don't worry about hitting any branches, and use the approximation $g = 10.0$ m/s^2* ***(warning: we won't always use this approximation!).***

i) Using a y-axis that is positive vertically upwards relative to the ground (and whatever you want to use for the x-axis direction, right or left), what must have been the ball's initial horizontal velocity component and what must have been its initial vertical velocity component? You can set $y_0 = 0$ for the ball's starting point.

ii) We need to understand how we get the same answer in (i) independent of the choice of axis direction. Thus, please redo your calculation in (i) by using a y-axis that is ***positive downwards*** *relative to the ground. You can still set $y_0 = 0$ for the ball's starting point.*

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