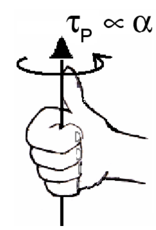
Chapter 14

Moments of Inertia and Angular Momentum

- **Rotational Form of Newton's Second Law**
- **Moments of Inertia**
- **Angular Momentum and Conservation of Angular Momentum**

The Torque Relation to Angular Acceleration

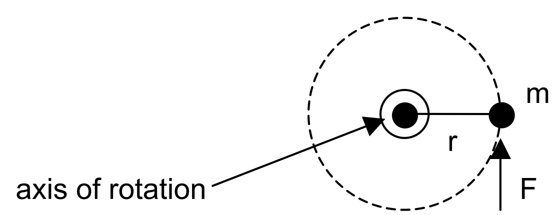
- In words, recall nonzero torque about a certain axis causes angular acceleration about that axis. So, if there's no torque, there's no angular acceleration. So, if you see uniform angular velocity, you know there's no torque. (We should say net torque all the time, but we won't always. So sue us!)
- In formula, recall we promised to show that a net torque $\vec{\tau}_P$ around a pivot axis P causes angular acceleration $\vec{\alpha}$ through **Newton's rotational second law**: $\boxed{\vec{\tau}_P = I_P\vec{\alpha}}$, where the proportionality constant I_P is the moment of inertia The vectors $\vec{\tau}_P$, $\vec{\alpha}$ point along the pivoting axis P using the right-hand rule with the fingers curled around in the direction of the rotation, as shown:



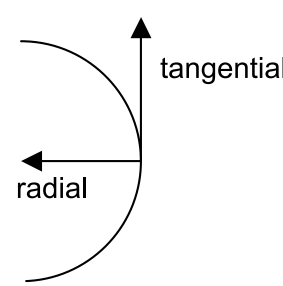
- Because the axis is often fixed in direction, we almost always only need the equation for the single component of torque along that fixed axis equation: $\boxed{\tau_P = I_P\alpha}$ where τ_P and α are **positive for CCW** (counter-clockwise) rotations and **negative for CW** rotations. It's the single-component (i.e., the "1D" version) we'll prove below.
- We should strictly remind ourselves that nonzero $\vec{\tau}_P$ (and thus nonzero $\vec{\alpha}$) implies that the rotation rate will be accelerating and not uniform (i.e., $\vec{\omega}$ is not constant).

Proof of $\tau_P = I_P\alpha$ for simple examples:

Example 1 – <u>We wish to derive the torque equation for a single point mass m that is free to rotate in a circle</u> at the end of a very light rod in the plane of the page, and around the rotation axis (shown pointing out of the page in the figure on the right), which is perpendicular to the page. Note that r is the perpendicular distance (the shortest distance) to the axis.



A force with magnitude F is applied to the mass m such that the force vector is tangential to the circular path to which the mass m is restricted. So F is parallel to the velocity vector of m. Recall what we refer to as the tangential and radial directions follows the definition of tangential and radial acceleration directions in Ch. 12. These directions are shown on the right.



The derivation of the torque equation for example 1 follows on the next page.

Derivation – Recall that the torque due to F around P can be calculated from the general formula

$$\tau_P \;=\; \pm\; F\,R\sin\theta \quad \text{for} \quad \begin{pmatrix} \text{CCW} \\ \text{CW} \end{pmatrix}$$

With $\theta = 90^o$ from the tangential direction of the force, the rotation angularly accelerated in the CCW direction, and R = r, we calculate for our first example

$$\tau_P \;=\; +\,F\,r$$

(There is no torque due to the rod tension since $\theta = 0^o$ for that force. And there's no gravity considered.) We can find F from Newton's second law applied to m in the tangential direction, recalling from Ch.12 that $a_{tangential} = \alpha r$ for angular acceleration α of m around P:

$$F \;=\; m\,a_{tangential} \;=\; m\,(\alpha r)$$

(To be consistent with our torque convention, an angular acceleration in the CCW direction corresponds to $\alpha > 0$.) Therefore the torque relation $\tau_P = +\,F\,r$ becomes

$$\tau_P \;=\; +\,m\,(r\alpha)\,r \;=\; m\,r^2\,\alpha$$

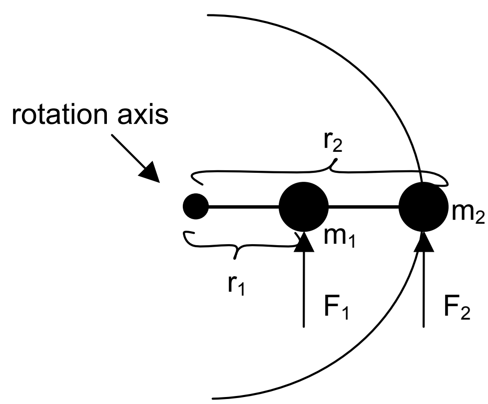
We get the promised result: **Newton's rotational second law** for example 1:

$$\boxed{\tau_P \;=\; I_P\,\alpha}$$

with the **moment of inertia** (sometimes called the **"rotational inertia"**)

$$\boxed{I_P \;=\; mr^2}$$ for a point mass m a distance r from the axis

Example 2 – Next we wish to derive the torque equation for two point masses m_1 and m_2 attached along the same rigid rod but at the different radii r_1 and r_2. Again the system is free to rotate in the plane of the page, as shown on the right, around the same rotation axis as in example 1.



If the masses m_1 and m_2 are accelerated tangentially, there must be a tangential force on each of them, by the second law. Call their magnitudes, F_1 and F_2 , respectively, such that each force vector is tangential to the circular path its mass follows, and tending to push it in a CCW direction. This is just like example 1, but for two masses.

The derivation of the torque equation for example 2 follows on the next page.

Derivation – Again using the general formula $\tau_P = + F R \sin\theta$ for each CCW torque, $\theta = 90^o$ and $R = r_1$ and $R = r_2$ for forces F_1 and F_2 , respectively, we calculate for our second example

$$\tau^P_{net} = + F_1 r_1 + F_2 r_2$$

(Again, there is no torque due to the rod tension since $\theta = 0^0$ for that force.) From the second law applied in the tangential direction, $F_1 = m_1 a_1^{tangential} = m_1(\alpha r_1)$ and $F_2 = m_2 a_2^{tangential} = m_2(\alpha r_2)$ in terms of α, as we did in example 1 . Therefore the net torque becomes

$$\tau^P_{net} = + m_1 (\alpha r_1) r_1 + m_2 (\alpha r_2) r_2 = (m_1 r_1^2 + m_2 r_2^2) \alpha$$

We get the promised form again: **Newton's rotational second law** for example 2:

$$\boxed{\tau^P_{net} = I_P \alpha}$$

with the **moment of inertia** now for the two point masses at the two different distances

$$\boxed{I_P = m_1 r_1^2 + m_2 r_2^2}$$

Proof of $\tau_P = I_P\alpha$ by generalizing the above:

We can comfortably generalize the above work to any number of point masses arbitrarily situated and rotating about some fixed axis P:

Rotational Version of Newton's Second Law

$$\boxed{\begin{array}{c} \tau^P_{net} = I_P^{total} \alpha \\ \text{where } I_P^{total} = \sum_i m_i r_i^2 \end{array}}$$

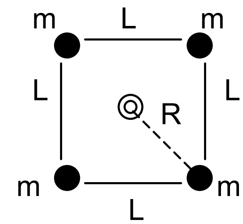
Comments:

- We see that the moment of inertia of any body relevant for rotations about an axis P is the **sum of $m_i r_i^2$ for each mass point m_i of the body**, where r_i is the perpendicular distance from the axis P
- For continuous bodies, the **sums will turn into integrations**. The integrals for many useful body shapes, like cylinders, will be easy. See the second cycle.

Examples of Moments of Inertia

As we look at some examples of moments of inertia, just keep thinking mass times r-squared. **The golden rule for moments of inertia is that the total moment of inertia of a system is the sum of the moments of inertia of its pieces.** This follows from the fact that the moment of inertia is just the sum of all the individual point mass moments of inertia. Of course, the moment of inertia for each piece must be calculated **with respect to the same axis P**.

Example 1: What is the moment of inertia of four equal point masses m laid out in a square pattern in the plane of the page, as shown, if the axis is perpendicular to the plane and goes through the center of the square?



Answer: Since R is the distance of each mass m from the axis, we have

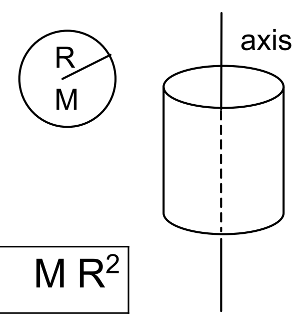
$$I = mR^2 + mR^2 + mR^2 + mR^2 = 4mR^2$$

Pythagoras comes along and tells us how to relate R to L

$$R^2 = \left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^2 \quad \Rightarrow \quad R^2 = \frac{L^2}{2}$$

$$\therefore \boxed{I = 2\,m\,L^2}$$

Example 2: What is the moment of inertia of a uniform thin hoop of mass M and arbitrary length, if the axis lies along the center axis of symmetry? Both a view on end and a perspective view from the side are shown on the right.



Answer: If it is truly thin, and that is the approximation we make, then all mass points are at the same R. We have

$$I = \sum_i m_i R^2 = (\sum_i m_i)\ R^2 = MR^2$$

$$\therefore \boxed{I_{\text{thin-walled hoop}}(\text{around symmetry axis}) = M R^2}$$

The answer of MR^2 is good for an arbitrary length of the hollow cylinder. You might think that the moment of inertia would increase as the length of the cylinder increases, and it does, because M grows with the length.

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Problem 14-1:

(a) *An oxygen molecule with total mass 5.3×10^{-26} kg, has a moment of inertia 1.9×10^{-46} $kg{\cdot}m^2$ (defined about an axis at the center of, and perpendicular to, the line joining the two oxygen atoms). Estimate, from this data, the distance between the two atoms (that is, between their two centers).*

(b) *Calculate the moment of inertia of a 68-cm-diameter bicycle wheel. The rim and tire together have a mass of 1.3 kg. (Why can the contributions to the moment due to the small hub and the thin spokes be ignored?)*

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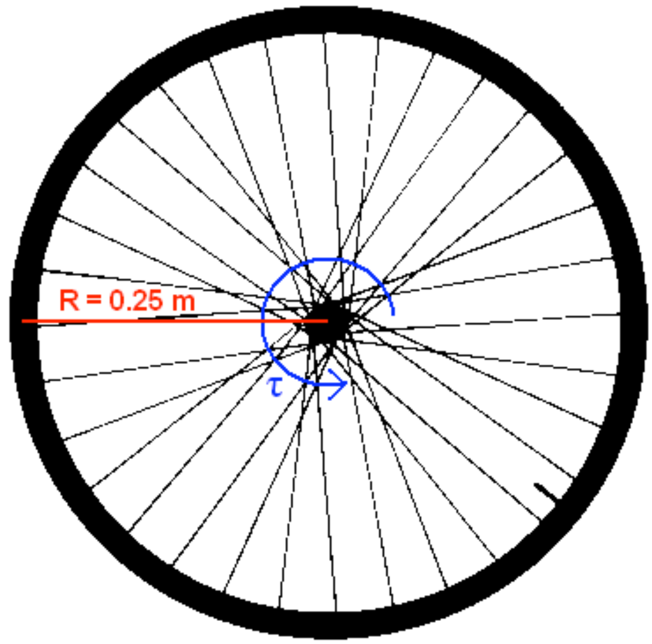
Problem 14-2:

*A **constant** torque of + 0.05 Newton-meters is applied around the axis of a free-spinning bicycle wheel of mass 4 kg, radius 0.25 meters as shown. The moment of inertia of the wheel is given to good approximation (see the previous problem) by* $I = MR^2$ *where* M *is the mass of the disk and* R *is the radius of the wheel.*



a) *What is the angular acceleration of the wheel?*

b) *Assuming the wheel starts at rest, how much time will it take for the wheel to be spun to an angular velocity of 20 radians/second?*

c) *A small piece of mud is stuck to the outside edge of the wheel. What is the centripetal acceleration on the piece of mud immediately after 10 seconds of torque have been applied?*

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Angular Momentum for Fixed-Axis Rotation

Review: Recall that we could motivate momentum p, and momentum conservation as well, by rewriting (stick with 1D for now) the linear second law as

$$F = ma = m\frac{dv}{dt} = \frac{d(mv)}{dt}$$

assuming m is constant. Thus, the product mv ≡ momentum arises:

$$\boxed{F = \frac{dp}{dt}} \quad \text{for} \quad \boxed{p \equiv mv} \quad \text{(units kg} \cdot \text{m/s)}$$

Thus, if there's no force, we have **conservation of momentum**

$$\boxed{F = 0 \quad \Rightarrow \quad p = \text{constant in time}}$$

Similarly, we can rewrite the rotational second law around a single fixed axis as

$$\tau = I\alpha = I\frac{d\omega}{dt} = \frac{d(I\omega)}{dt}$$

assuming the moment of inertia I is constant. Thus defining the product $I\omega$ ≡ angular momentum allows us to write the rotational second law in the following way:

$$\boxed{\tau = \frac{dL}{dt}} \quad \text{for} \quad \boxed{L \equiv I\omega} \quad \text{(units kg} \cdot \text{m}^2\text{/s)}$$

Thus, if there's no torque, we have **conservation of angular momentum**

$$\boxed{\tau = 0 \;\Rightarrow\; L = \text{constant in time}}$$

We generalize this stuff to the general case in 3D on the next page. The general "angular momentum" vector for a pure rotation about a fixed axis is

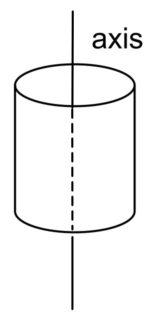
$$\boxed{\vec{L}_P = I_P\,\vec{\omega}}$$

and the rotational Newton's second law in 3D now can be written as

$$\boxed{\vec{\tau}_P = \frac{d\vec{L}_P}{dt}}$$

The direction of the angular momentum vector $\vec{L}$ is parallel* to the angular velocity $\vec{\omega}$, and hence it lies along the rotational axis and points in the direction given by the right-hand rule, and it's déjà vu all over again.

*** For most of our freshman situations $\vec{L}$ and $\vec{\omega}$ are parallel to each other. Just so you know, a rotation about a different axis other than the symmetry axis generally leaves us with $\vec{L}$ and $\vec{\omega}$ nonparallel. But we almost always consider spinning along the symmetry axis, like in the example immediately coming up.**



Example of angular momentum: The first figure on the right shows the hollow thin-walled cylinder we discussed earlier. It has mass m and radius r. Suppose it is spinning around its symmetry axis with ω such that it is rotating in the CCW direction as we look down on it – like in the right-hand-rule figure. Therefore the angular momentum around the spinning axis is

$$L = +I\,\omega = +mr^2\omega\,.$$

Comments about the conservation of angular momentum:

- We have said that for fixed-axes rotational motion, $\vec{L} = I\,\vec{\omega}$ would stay constant if there is no torque. We just need the net torque to be zero. Just as internal forces may be present (but must cancel out) in the conservation of linear momentum of a system, internal torques may be present (but must cancel out – see a proof in a later cycle) in the conservation of angular momentum. It is only the **net torque that must be zero.**
- In the original derivation of $\tau = \frac{dL}{dt}$, we assumed the moment of inertia was constant in time. While we don't have time to prove it, you **can use the angular momentum conservation law even when the moment of inertia is changing in time**. Just make sure the external net torque is zero. In fact, we will also use $F = \frac{dp}{dt}$ in cases where m is changing and that can be justified, too. Changes in the moment of inertia take place in some of the most famous examples of the conservation of angular momentum, like the problem below. For that problem, $L = I\,\omega$ is conserved even if $\frac{dI}{dt} \neq 0$!

Example of angular momentum and its conservation: A frictionless merry-go-round-like disk of small mass but of radius 2.0 m is rotating at an angular velocity ω = 1.25 rad/s with a bunch of kids riding it whose total mass is 400 kg. The kids are all sitting exactly on the edge of the disk, so the moment of inertia is $m_{total}r^2 = 400 \cdot 4\ kg \cdot m^2$. The kids are small ("point-like"!) and we ignore the merry-go-round mass.

What is the total angular momentum, with respect to the center pivot, of the rotating kids?

Answer: $$L = I\omega = 1600 \cdot 1.25 = \boxed{2000\ kg \cdot m^2/s}$$

Suppose the kids now crawl in toward the center until they're all (squished) at exactly radius 1 m. What now is the total angular momentum?

Answer: It is the **same** as above! The kids crawling corresponds only to internal forces and torques and with zero external torque around the pivot (the pivot's own force gives zero torque with respect to the center since its $R_P = 0$) the angular momentum is conserved.

What then is the merry-go-round's new angular velocity?

Answer: Conservation of angular momentum tells us that

$$I_i\ \omega_\iota = I_f\ \omega_f \ .$$

The final moment of inertia is now $I_f = m_{total}r_f^{\ 2} = 400 \cdot 1\ kg \cdot m^2$, and the initial angular momentum was found above, $I_i\ \omega_\iota = 2000\ kg \cdot m^2/s$. Thus

$$\omega_f = I_i\ \omega_\iota / I_f = 2000/400 = \boxed{5\ rad/s}$$

This is four times larger than the angular velocity at the beginning (1.25 rad/s) because the moment of inertia is four times smaller!
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Problem 14-3:
a) A woman stands on a frictionless platform that is rotating with a frequency of 1.2 rev/s; her arms are outstretched and she holds a weight in each hand. With her hands in this position the total moment of inertia of the woman, the weights, plus the platform adds up to 6.0 kg-m^2. If, by moving the weights, the woman decreases the total moment of inertia to 2.0 kg-m^2, what is the resulting rotation frequency of the platform? (Note that angular momentum about center is conserved because the net torque about that point is zero, even though the external force on the pivot of the platform from the floor is NOT zero.)

b) Think ahead about later cycle material and guess:

1) Which is bigger, the moment of inertia around the symmetry axis of a hollow cylinder or a solid cylinder (same mass, radius, and both are uniform in mass distribution)?

2) Which is bigger, the moment of inertia around an axis perpendicular to (and going through) the center of a uniform straight thin stick, or around an axis perpendicular to (and going through) its end?

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