Chapter 15++
Waves are Oscillations that Move

Revisit:	To-Do:
• Oscillations and oscillation parameters	• Physical examples • Traveling wave harmonic representation and its parameters • Wave velocity • Superposition: standing waves • Boundary conditions for standing waves

Examples:	**Motion**	**Restoring force ("cause")**
1) sound waves	**air is pushed along the direction of the wave "longitudinal"**	molecular collisions carry signal and restore equilibrium
2) electromagnetic waves	**electric and magnetic fields oscillate transverse to the wave**	Faraday and Ampere Laws of electric and magnetic fields – see next semester
3) water waves	**water particles on surface go in circles**	surface tension, gravity
4) string waves	**the string motion is transverse to the wave**	tension
5) seismic waves	**both longitudinal and transverse vibrations**	Earth's elastic crust

Waves → "something shaking that moves from place to place"

Starting with examples of well-known waves, such as sound, light, and water, we take the sinusoidal oscillations representation such as

$$y(t) = A\cos(\omega t + \phi) \quad \text{about} \quad y = 0$$

and generalize it to traveling and standing waves in 1D.

Imagine setting up a steady wave that travels out along a taut string by continuously shaking one end (ignoring damping)

string wave at a time t_1 $\Rightarrow y(x,t_1)$

The tension at each point leads to simple harmonic motion in time up and down at each point. But the interaction between different parts of the string means that for a given time, the string oscillates in space, too. In fact, the picture on the previous page suggests that! We expect the form to be

$$y(x,t) = A\cos(\omega t - kx + \phi)$$

means oscillates in space analogous to oscillating in time

means "right-going (see later)

k is NOT spring constant now – see later

Features of the $y(x,t) = A\cos(\omega t - kx + \phi)$ sinusoid:

1) This wave form comes from solving a "wave equation" that is derived from a good old second-law F = ma analysis on the string. We show you the wave equation later, and it would not be hard for you to see how the above y(x,t) satisfies it.

2) We have the familiar interpretation of A as amplitude (maximum displacement) and ϕ as phase (which can be used to adjust where our humps are for a given x,t).

3) Time period T such that $y(x,t) = y(x,t+T)$ or that $\omega T = 2\pi$ so

$$\boxed{\omega = \frac{2\pi}{T}} \Rightarrow \boxed{\omega = 2\pi f \text{ for } f = \frac{1}{T}}$$

ω = # wiggles in radians / unit time

4) Space period $\lambda \equiv$ "wavelength" such that $y(x,t) = y(x+\lambda,t)$ or that $k\lambda = 2\pi$ so

$$\boxed{k = \frac{2\pi}{\lambda}}$$

k = # wiggles in radians / unit length

AGAIN, k is NOT the spring constant now – sorry for the ambiguity !!!
This k is called the "WAVE NUMBER"

5) To prove that it is going to the right, look at the positive hump in the wave corresponding to $\omega t - kx + \phi = 0$ (where, therefore, y = A). As t marches on, then –kx must change, too, to keep $\omega t - kx + \phi = 0$, and x must becoming more positive. Thus the hump must be moving to the right!

6) So $y(x,t) = A\cos(\omega t + kx + \phi)$ must be going to the left by similar reasoning!

7) The speed of the wave (the humps) is

$$\boxed{v = \omega/k} \quad \text{or} \quad v = \frac{2\pi f}{2\pi/\lambda} \Rightarrow \boxed{v = f\lambda}$$

Proof: Follow the motion of any part of the wave, which you can define by $\omega t - kx + \phi = C$ for some constant C and then solve for x:

$$x = \frac{1}{k}(\omega t + \phi - C) \Rightarrow \frac{dx}{dt} = \frac{\omega}{k} > 0 \text{ (to right)}$$

alternatively, $\omega t + kx + \phi = C \Rightarrow \frac{dx}{dt} = -\frac{\omega}{k} < 0$, to left

8) Some people use the notation for waves as $y(x,t) = A\sin(kx \pm \omega t + \theta)$ for right-going (-) and left-going (+). We can turn their answer in ours just by relating our phase in terms of theirs. For instance,

$A\sin(kx \pm \omega t + \theta) = A\cos(kx \pm \omega t + \phi)$ with $\phi = \theta - \pi/2$

Example using sine instead of cosine:

$$y(x,t) = 0.48\sin(5.6x + 84t) \text{ in SI units .}$$

For this representation: First, note that their phase is $\theta = 0$,

$A = 0.48$ m, $k = 5.6\ \text{m}^{-1}$, $\lambda = \frac{2\pi}{k} = 1.1$ m,

$\omega = 84\ \text{s}^{-1}$, $f = \frac{\omega}{2\pi} = 13.4$ Hz, $v = \omega/k = 15$ m/s,

direction is to the left ($kx + \omega t$ form), $\phi = -\pi/2$ to make the $\cos(\dots + \phi)$ form give the above form

Also, $\frac{dy}{dt} = \underbrace{84(0.48)}_{40.3\text{ m/s}} \cos(5.6x + 84t)$ $\Rightarrow$ max transverse velocity

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Problem 15-7 Suppose the function y(x,t) = 44.0 cos(-9.0x + 2.0t + π/8) describes a wave on a long string. (SI units)

a) What are the amplitude, wave number, wavelength, angular frequency, frequency, wave speed, and direction of propagation?

b) Find the smallest positive time when the wave has a positive transverse maximum at x = 0. (HINT: The cosine has its maximum positive value of +1 when its argument is 2πn, for n = any positive or negative integer n, or zero.)

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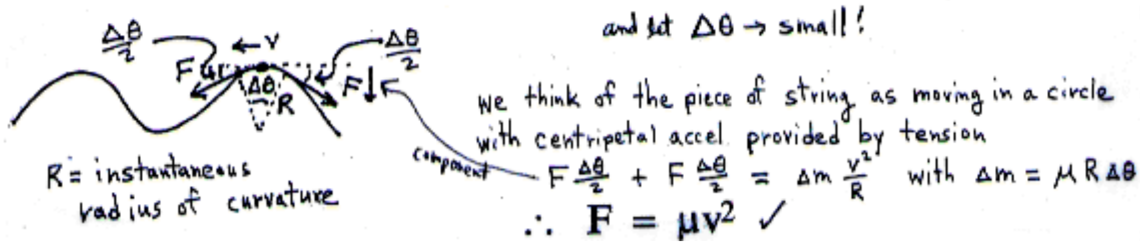
VELOCITY OF WAVE ON A STRING RELATED TO TENSION AND MASS DENSITY

Now, let's relate the speed v of the waves to the property of the "something" that's wiggling (just like we related the frequency of a spring to the spring constant and the mass, $\omega = \sqrt{k_{spring} / m}$)

FORMULA: $\boxed{v = \sqrt{\frac{F}{\mu}}}$

where F is the string tension, and μ is the linear mass density

PROOF: While you're not responsible for this material, you can understand it: go to reference frame moving along with wave (say, right-going) so the piece of the string at one of its humps (i.e., peaks) looks like it is moving left (opposite to the original motion of the wave) with speed v. In fact, it looks like its going in a circle with the inward components of the tension on both sides of the piece causing the centripetal acceleration inward by the second law:



Let's look at another kind of wave – one we can talk about because it lets us talk!

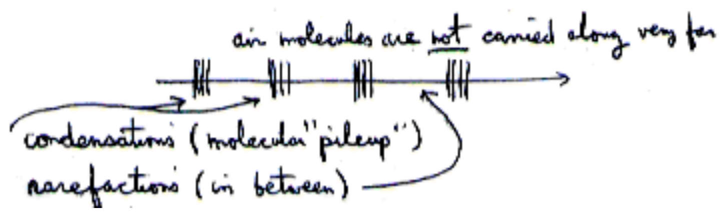
SOUND WAVES

v = 331 m/s = 1080 ft/s at STP air (0° C, 1atm, ρ = 1.29 kg / m^2)

v = 343 m/s = 1130 ft/s at NTP air (20° C, 1atm, ρ = 1.21 kg / m^2)

What is shaking? The density/pressure fluctuates in the direction of the wave propagation (longitudinal).

Particle picture:



From the form for the string:

$$v = \sqrt{\frac{\text{tension}}{\text{linear density}}}$$

we look for the general form

$$v = \sqrt{\frac{\text{restoring "force"}}{\text{mass "density"}}}$$

In fact, for sound waves in any **solid/liquid/gas**

$$v = \sqrt{\frac{\text{bulk modulus}}{\text{volume density}}} = \sqrt{\frac{B}{\rho}}$$

The elasticity – the bulk modulus quantity B – is the restoring force producing waves traveling along solids – another sound wave! We don't officially prove it, but it does make sense from our spring experience: the square root of "stiffness" divided by density.

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Problem 15-8 Two similar problems involving different kinds of waves:
a) A wire 15.0 m long and having a mass of 100 g is stretched under a tension of 200 N. If two disturbances, separated in time by 0.030 s, are generated one at each end of the wire, where will the disturbances meet, as measured in meters from the end where the later disturbance originated?
b) A person sees a heavy stone strike concrete pavement. A moment later two sounds are heard from the impact: one travels in the air and the other in the concrete, and they are 1.5 s apart in time. Which is heard first? How far away did the impact occur? Use $\rho_{concrete} = 2300\ kg/m^3$, $B_{concrete} = 20 \times 10^9\ N/m^2$, and the NTP speed of sound in air.

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Superposition of Waves

To study waves on strings, in air tubes, etc. we need first to show how to combine traveling waves and get standing waves. So first find out what happens when two waves are put on top of each other. Consider two separate waves $y_1(x,t)$, $y_2(x,t)$ that propagate separately along a string. In fact, they can coexist, yielding the net overall disturbance y(x,t):

$$y(x,t) = y_1(x,t) + y_2(x,t) \quad \text{(the superposition principle)}$$

The wave equation that all these waves satisfy is found below (and you could check that our previous sines and cosines satisfy it). It is seen to be linear so the **sum of two solutions is another solution.**

The physics wave equation is $\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}$ which, in words, means "wiggling in time" must be balanced by "wiggling in space." We won't derive this but, like the discussion of the wave speed v on the previous page, it comes from considering the second law f = ma for a piece of a string where the net force comes from the difference of tension on each end of the piece – with the difference leading to a second derivative with respect to x, and the acceleration gives a second derivative with respect to t! All of this stuff is advanced material, but we thought you might like to hear a little bit about it. You know enough to understand the details, if you had time and need to do so, but you don't need to in this course, so to heck with it.

Standing Waves

If we attempt to continuously produce a right-traveling wave and send it down a string that is tied down (which means y = 0) somewhere (at x = L, say), a reflected traveling wave coming back to the left is created. The net wave anywhere is the sum of the "incident" wave plus the "reflected" wave, and they cancel each other so that, indeed, y = 0, always, at that end, x = L. To get that cancellation, their amplitudes must be equal.

Let us see this. Consider a simple superposition of the two travel waves,

$$y_1(x,t) = A\sin(kx - \omega t)$$

travels to the right

and

$$y_2(x,t) = A\sin(kx + \omega t)$$

travels to the left

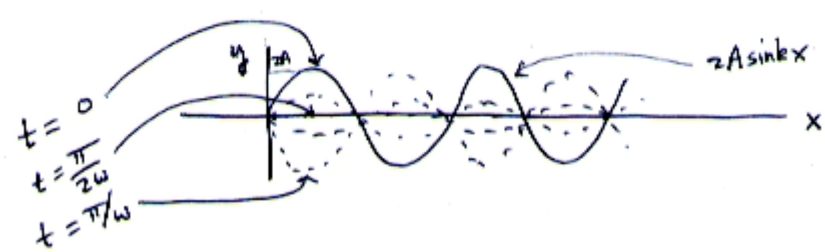
Their sum is

$$y = A[\sin(kx - \omega t) + \sin(kx + \omega t)]$$

$$= A\left[\sin kx \cos\omega t - \cancel{\cos kx \sin\omega t} + \sin kx \cos\omega t + \cancel{\cos kx \sin\omega t}\right]$$

So $$y(x,t) = 2A \sin kx \cos\omega t$$

A wave of this form is a "standing wave" and looks like:



a) <u>crests (we have also called them humps or peaks before)</u> stay at the same place but oscillate in size and sign

b) <u>nodes</u> stay at the same place

Standing waves can be considered to be a sum of two traveling waves, and, vice versa, traveling waves can be considered to be a sum of two standing waves.

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Problem 15-9 A quickie! Use a trig identity to rewrite the following sum of the two standing waves: C(coskx)(cosωt) - C(sinkx)(sinωt). Your result is a traveling wave. In what direction is it going?

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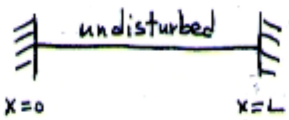
Now let's discuss standing waves where we find that the end conditions force the wavelengths to fit in just right:

STRING TIED DOWN AT BOTH ENDS



The pictures shows a string at rest, but even when it is disturbed, it still must have y = 0 at the ends x = 0, L . We use this to derive the wavelengths below (we just solve by pictures rather than finding the k for which the sines vanish; we get the same answers.)

a) simplest sinusoid is the FUNDAMENTAL or 1st harmonic

$$L = \frac{\lambda}{2} \Rightarrow \lambda = 2L \Rightarrow k = \frac{2\pi}{2L} = \frac{\pi}{L}$$

$$\text{so } \omega = kv = \frac{\pi}{L}v, \text{ where } v = \sqrt{\frac{F}{\mu}}$$

The standing wave form is thus $y_1 = A \sin\frac{\pi}{L}x \cos(\omega t + \phi)$ with ω as given.

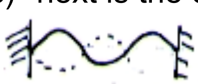
Note that here and below, all y_n vanish at x = 0, L.

b) next simplest is the 2nd harmonic

$$L = \lambda \Rightarrow k = \frac{2\pi}{L} , \omega = \frac{2\pi}{L}v$$

$$y_2 = A \sin\frac{2\pi}{L}x \cos(\omega t + \phi)$$

c) next is the 3rd harmonic



$$L = \lambda + \frac{\lambda}{2} \Rightarrow \lambda = \frac{2}{3}L \Rightarrow k = \frac{2\pi}{2L/3} = \frac{3\pi}{L}$$

$$\omega = \frac{3\pi}{L}v$$

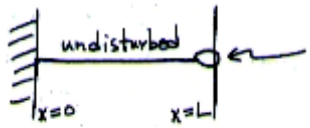
$$y_3 = A \sin\frac{3\pi x}{L} \cos(\omega t + \phi)$$

etc.

in general, $\lambda = \frac{2L}{n}$

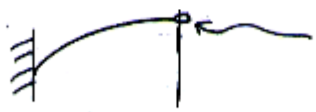
$(L = n\frac{\lambda}{2}, n = 1, 2, 3, \dots)$ n^{th} harmonics!

STRING TIED AT ONE END, "FREE" AT OTHER



An example of a "free" end is a string tied to a frictionless loop on vertical rod. We again use this picture to find the wavelengths:

a) simplest sinusoid FUNDAMENTAL or 1st harmonic



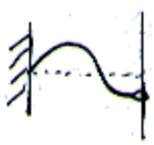
must come in, perpendicular to rod since no friction

$$L = \frac{1}{4}\lambda \Rightarrow \lambda = 4L \Rightarrow k = \frac{2\pi}{4L} = \frac{\pi}{2L}$$

$$\omega = \frac{\pi}{2L} v$$

$$y_1 = A \sin\frac{\pi x}{2L} \cos(\omega t + \phi)$$

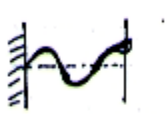
b) 2nd harmonic



$$L = \frac{1}{4}\lambda + \frac{\lambda}{2} \Rightarrow \lambda = \frac{4}{3}L \Rightarrow k = \frac{2\pi}{4L/3}$$

etc. for y_2

c) 3rd harmonic



$$L = \frac{1}{4}\lambda + \lambda \Rightarrow \lambda = \frac{4}{5}L$$

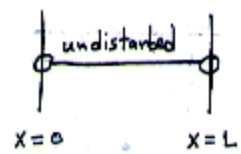
etc.

In general, $\lambda = \dfrac{4L}{2n-1}$ $\quad n = 1, 2, \ldots$

$$\left(L = \frac{1}{4}\lambda + (n-1)\frac{\lambda}{2} \right)$$

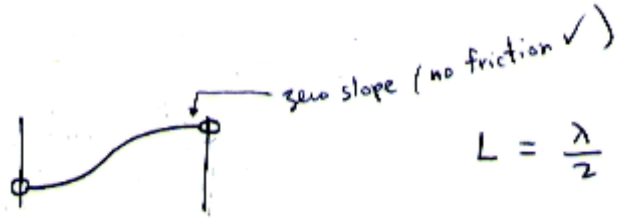
STRING “FREE” AT BOTH ENDS

A little bit odd, but still possible. You just need to start the vibrations shown below by pulling up on one part while you pull down on the other! (Otherwise the string will just slide up or down.)



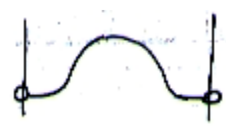
Imagine frictionless loops on both ends (we always ignore gravity – the waves are again due to string tension!)

a) simplest sinusoid



$$L = \frac{\lambda}{2}$$

b) 2^{nd} harmonic



$$L = \frac{\lambda}{2} + \frac{\lambda}{2}$$

c) 3^{rd} harmonic

$$L = \frac{\lambda}{2} + \lambda$$

etc.

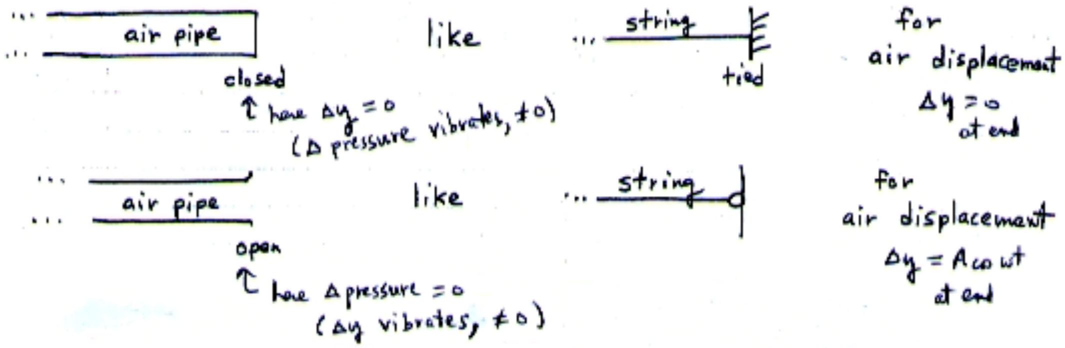
$\therefore$ n^{th} harmonic $\Rightarrow \lambda = \frac{2L}{n}$

Just like closed on both ends!

$\left(L = n\frac{\lambda}{2} \right)$ $n = 1, 2, \ldots$

SOUND PIPES

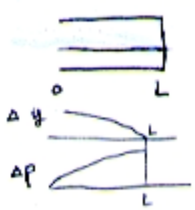
If Δy is the longitudinal displacement of air



It is reversed if we are talking about pressure changes Δp! Zero displacement of air against a closed end means maximum change in pressure. Maximum displacement of air at an open end is where there is hardly any change in pressure.

Example: What is the length of a closed-end organ pipe for middle C?

Answer: First, take the temperature T = 20° C → v = 343 m/s. Next, we know middle C = 264 Hz = f. We have the basic speed-wavelength-frequency relation, $v = \lambda f$ (recall this is from $\omega = kv$ or $2\pi f = \frac{2\pi}{\lambda} v$). So we know the wavelength λ. So finally we know the length:



$$\Rightarrow L = \frac{1}{4}\lambda \Rightarrow L = \frac{1}{4}\frac{v}{f} = 0.32 \text{ m}$$

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Problem 15-10

a) Find the three lowest natural resonant frequencies of an organ pipe 1.5 m long when the speed of sound is 343 m/s if (i) both ends of the pipe are open and (ii) one end is closed and the other is open.

b) Calculate the resonant frequency of the column of air in the outer ear of a human being, which is about 2.5 cm long. (use v = 343 m/s)

Professional singers develop a very strong "singers' formant" at 3000 Hz. It may be that this is why operatic tenor or soprano voices are so exciting – literally! That is, compare this frequency to your answer in (b).

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