Chapter 13

Torque*

- **How Shall a Force be Applied to Start a Rotation?**
- **The Torque Formula**
- **Systems in Equilibrium: "Statics"**

Glimpse of Newton's rotational second law:

Q: Review: What causes linear acceleration in, say, the x direction?

A: Any net x-component F_x of force, through **Newton's linear second law**: $\boxed{F_x = ma_x}$

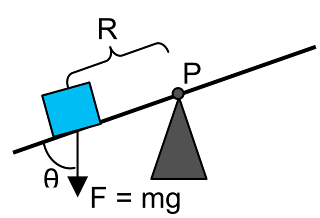
Q: Preview: What causes angular acceleration α around a particular axis?

A: Any net "torque τ_P" around that axis, through **Newton's rotational second law**: $\boxed{\tau_P = I_P\alpha}$

where the proportionality constant I_P is the moment of inertia. We denote by P the axis pivot point. We give the formula for torque and a simple example below – see Ch. 14 for a proof of all this stuff.

Details and remarks:

- The magnitude of the torque is given by $\boxed{|\tau_P| = F R_P \sin\theta}$ where F is the magnitude of the force causing the rotation, R_P is the distance from the axis of rotation (pivot point) to the point where F is applied, and θ is the angle between the direction of the force and the direction along R_P. In the teeter-totter figure (we ignore the teeter-totter mass), we consider the mass m stuck on the board and part of the system. F is the weight mg causing the teeter-totter to angularly accelerate around the pivot point P, the torque magnitude is $|\tau_P| = mgR\sin\theta$. (The axis is perpendicular to the page and pierces the point P.)



- The teeter-totter also offers us a simple example of the moment of inertia. The small mass m has $I_P = mR^2$. In general, the moment of inertia is a sum of terms like this, one for each bit of mass (e.g., yielding an integral for a continuous body) – see Ch. 14.
- We can already see that to get a larger angular acceleration α, we want 1) F larger, 2) R larger, 3) θ larger, or 4) smaller I_P. (With the teeter-totter, making R larger made the torque larger, but it increased I_P more, since it means moving the mass m out farther.)
- More insights, including a discussion of the direction of torque are coming up. As for a calculation, we'll have sompun' happun' in a *moment*.

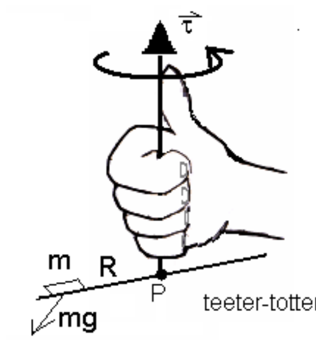
***This is not pronounced torkay like in porkay peeg, but rather like tork in dork.**

Torque $\vec{\tau}$ is a vector with a direction closely related to that of $\vec{\omega}$ and $\vec{\alpha}$:
(Warning: we will often leave the P understood and just call it $\vec{\tau}$ instead of $\vec{\tau}_P$.)

- The direction of the torque vector $\vec{\tau}$ is along the axis about which the force will tend **to start or accelerate** the rotation of the system and in the direction indicated by the right-hand rule (see below and review Ch. 12). We will henceforth refer to the “torque rotating the system” around its axis, but remember that a nonzero torque means there’s an angular acceleration (i.e., the rotation cannot be uniform).

- This is exactly the same direction used to define $\vec{\alpha}$ since $\vec{\tau}$ is proportional to $\vec{\alpha}$, and in fact it’s also the same as for $\vec{\omega}$ if $\vec{\omega}$’s axis is fixed. That is, since $\vec{\alpha} = \dfrac{d\vec{\omega}}{dt}$, we see that $\vec{\alpha}$ is parallel to $\vec{\omega}$, provided only the magnitude $\omega \equiv |\vec{\omega}|$ is changing, and not $\vec{\omega}$’s direction.

- As an example, lay the teeter-totter on its side so that the axis around which it will rotate is pointing vertically, as in the figure on the right. Curling your fingers of your right hand in the direction of the rotation caused by the gravitational torque, you get your thumb pointing along that rotational axis and in the direction of the torque.



- If your eye is above your thumb and you look down at it and at the teeter-totter, as shown in the figure on the left, you see the system rotating in the **counter-clockwise direction** (CCW). We show the point where the axis pierces the paper as well as the CCW rotation in the figure for the next bullet.

- The above **CCW** rotation corresponds to the torque vector $\vec{\tau}$ having a **positive** component along the axis denoted by the big fat arrow. This positive torque component is given by

$$\tau = +F R \sin\theta \quad \text{for} \quad \text{CCW}$$

(If you like using the right-hand screw to help you with the right-hand rule, the right-hand screw will advance out of the page when you turn it CCW, which is exactly the torque’s vector direction: ⊙)

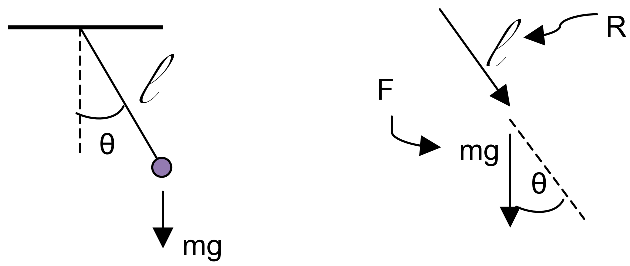
- The **CW** rotation corresponds to the torque vector $\vec{\tau}$ having a **negative** component along the axis. This negative torque component is given by

$$\tau = -\ F R \sin\theta \quad \text{for} \quad \text{CW}$$

(The right-hand screw will advance into the page when you turn it CW, which is exactly the torque’s vector direction here: ⊗)

A numerical torque example:

A small 0.75-kg ball is attached to a 1.25-m massless rod and hung from a pivot. When the resulting pendulum is swinging and is at 30^{O} from the vertical, what is the magnitude and direction of the torque due to the ball weight, at that instant, about the pivot?



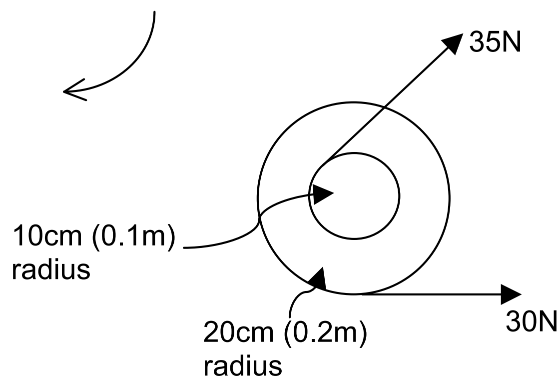
1) The <u>magnitude</u> of the torque due to gravity about the pivot is

$$|\tau| = mg\,\ell\,\sin\theta = (0.75)\,9.8\,(1.25)\,\sin 30^{O} = 4.6\text{ N-m}$$

We call the units of torque, N-m or Newton-meters, and not J or Joules, even though the torque dimensions are the same as work and energy. We distinguish the units in this way because the concept and equations are different – don't ever put torque = energy! ☹

2) The direction of motion of the pendulum ⇒ clockwise rotation (looking down on it) so that the torque points into the page and has the negative component

$$\tau = -\ 4.6\text{ N-m}$$

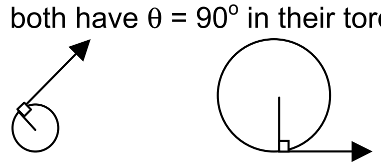


A net torque example: What is the total torque about the center for the following spool, which has two applied forces?

Answer: Since both torques are around the same pivot and have the same axis, we can simply add up their components according to

$$\tau = \pm\ F\,R\sin\theta \quad \text{for} \begin{pmatrix}\text{CCW}\\ \text{CW}\end{pmatrix}$$

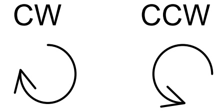
The two forces 35N and 30N both have $\theta = 90^{o}$ in their torque formula – see the two figures:



and their respective rotational sense is CW and CCW, respectively. The net torque is therefore positive (CCW) because the smaller force has a larger R:

$$\tau = -35(0.1) + 30(0.2) = +2.5\text{ N-m}$$

CW CCW Total is CCW despite 30 < 35



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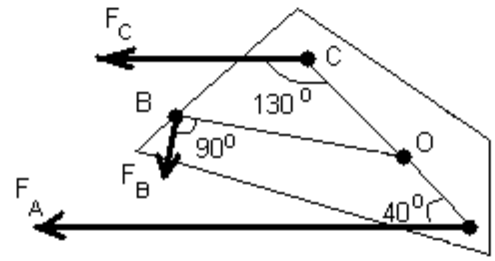
Problem 13-1



The body shown can pivot around an axis through O*. The axis is perpendicular to the paper. Three forces act on it perpendicular to the axis and in the directions* shown on the figure. Thus each of their torques do lie along the axis. The force magnitudes and application points are as follows:*

F_A = 35 N *at point* A*, 2.0 m from* O *,*
F_B = 7 N *at point* B*, 4.5 m from* O *,*
and
F_C = 21 N *at point* C*, 3.2 m from* O*.*

Give the magnitude and direction of the individual torques. What are the magnitude and direction of the total resultant torque about O*? Which way will the body rotate?*

* ***Note a great thing about the*** $\sin\theta$ ***factor in the torque calculation. You get the same answer if you use*** $\pi - \theta$ ***(the supplement to*** θ***) since*** $\sin(\pi - \theta) = \sin\theta$***. So don't worry about that ambiguity in finding the angle between*** $\vec{F}$ ***and*** $\vec{R}$***.***

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"Statics"

Before going on to analyze angular accelerations due to nonzero torques, let us consider systems that are not moving at all! The fact that nothing is moving means that ***both*** the net torque, and the net force must vanish:

If everything is always at rest, then we must have $\vec{a} = 0$, or

$$\boxed{\vec{F}_{net} = 0}$$

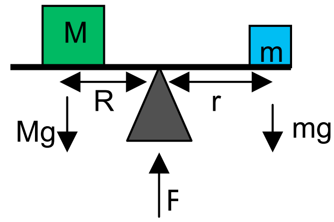
and also we must have no rotation at any time: $\vec{\omega} = 0$ always, which means $\vec{\alpha} = 0$ always, and by the second-rotational law:

$$\boxed{\vec{\tau}_{net} = 0} \quad \text{(always and about any point!)}$$

To help us write down zero net torque equations we generalize the free-body diagram to an **extended free-body diagram** that shows not only what forces are present but WHERE they are applied – see the next page for an example.

And remember the torque around an axis P due to a force with magnitude F applied at a point a perpendicular distance R_P from the axis (we put back in reference to the pivot axis):

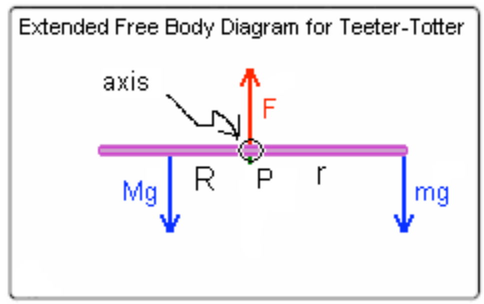
$$\boxed{\tau_P = \pm F R_P \sin\theta} \quad \text{for CCW (+) or CW (-) torque}$$



Trivial statics example: Consider two different weights balanced horizontally as shown on a light teeter-totter (i.e., we neglect the teeter-totter's weight, but it doesn't matter for a center pivot – see later). We consider the weights and the board all one system, since we assume the ***weights are stuck to the board***. (Note! Later on in our learning we will consider these masses not stuck and we may have to consider the possibility they separate from the board! Then we need to discuss the contact forces between the weights and the board.)

Question: Given R and r, what must be the ratio of M/m?

Guess the answer: From your experience in pre-school, you know the weight Mg must be larger than the weight mg for R < r. And you can guess that they're in inverse relation to their distance: M/m = r/R. We will see below that torque balance gives exactly this.



Calculate the answer: The pivot axis is shown in the extended FBD (**XFBD**), **which is an FBD but with the external force application points shown**, along with the forces. For the equilibrium condition in statics, we demand that $\tau_{net}(P) = 0$. Thus the individual torque contributions to the net torque will cancel out (they <u>must be</u> calculated around the same pivot axis P!):

$$\tau_{net}(P) = \tau_{Mg}(P) + \tau_F(P) + \tau_{mg}(P) = 0$$

Since $M\vec{g}$ is perpendicular to $\vec{R}$ and wants to rotate the system CCW:

$$\tau_{Mg}(P) = +\, Mg\, R \sin 90^o = +\, MgR$$

Since $m\vec{g}$ is perpendicular to $\vec{r}$ and wants to rotate the system CW:

$$\tau_{mg}(P) = -\, mg\, r \sin 90^o = -\, mgr$$

Since $\vec{F}$ is applied right at the pivot, it can't rotate the system at all. The math shows this since $R_F = 0$ in the torque formula giving a zero torque:

$$\tau_F(P) = F \cdot 0 = 0$$

Thus $\tau_{net}(P) = \tau_{Mg}(P) + \tau_F(P) + \tau_{mg}(P) = +\, MgR + 0 - mgr = 0$

Therefore $\quad MgR = mgr$ or $\boxed{\dfrac{M}{m} = \dfrac{r}{R}}$

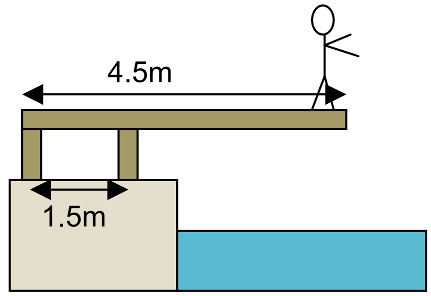
There is also vertical force balance which determines F, which we don't need here:

$$0 = F_{net} = F - mg - Mg \;\Rightarrow\; F = mg + Mg$$

but we did need the torque equation in order to find a relation between M and m.

Another statics example:

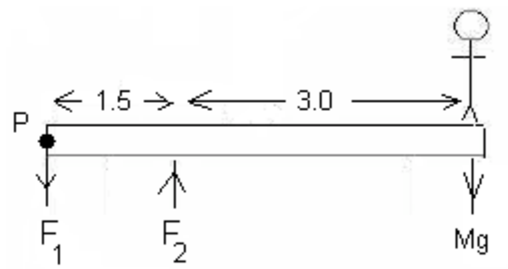
A diver of weight Mg = 580 N stands at the end of a 4.5-m diving board of negligible weight. The board is attached by two pedestals 1.5 m apart, as shown in the figure. We again think of the diver as part of the board (or "stuck" to it).



a) Find the values of the vertical forces exerted by the pedestals on the board-diver system.

b) What are the directions of the forces? That is, tell whether each pedestal is under tension or compression!

Answer: Look at the XFBD on the right where we can easily guess the directions of the external pedestal forces but don't worry. If, in some problem, you guessed wrong, your math will tell you! You will find that the F_i in question would come out negative indicating the opposite direction and all would be OK!



We want to calculate F_1 and F_2, so we have two unknowns, so we need two equations. They will come from our statics conditions $\vec{F}_{net} = 0$, $\vec{\tau}_{net}(\text{any axis}) = 0$. We can get one equation from force balance and one equation by demanding that the sum of the torques around any axis vanishes.

Vertical force balance (second-law for statics):

$$F_2 - F_1 - 580 = 0 \quad \Rightarrow \quad \boxed{F_1 = F_2 - 580}$$

Torque balance around the left end of the diving board (pivot P shown in the figure above):

$$\tau_{net}(P) = \tau_{F_1}(P) + \tau_{F_2}(P) + \tau_{Mg}(P) \overset{\text{must}}{=} 0$$

Ingredients in the above: Since F_1 is applied right at the pivot, we have a zero torque:

$$\tau_{F_1}(P) = F_1 \cdot 0 = 0$$

Since F_2 is perpendicular to the board and wants to rotate the system CCW around P:

$$\tau_{F_2}(P) = +F_2\ 1.5 \sin 90^\circ = +1.5\, F_2$$

Since $M\vec{g}$ is perpendicular to the board and wants to rotate the system CW around P:

$$\tau_{Mg}(P) = -\ Mg\ 4.5 \sin 90^\circ = -580\ (4.5)$$

Thus $$\tau_{net}(P) = \tau_{F_1}(P) + \tau_{F_2}(P) + \tau_{Mg}(P) = 0 + 1.5\, F_2 - 580\ (4.5) \overset{\text{must}}{=} 0$$

Solving this for F_2 and then using the boxed equation above for F_1:

$$\boxed{F_2 = +1740\ \text{N}} \quad \Rightarrow \quad \boxed{F_1 = 1160\ \text{N}}$$

Indeed, the right pedestal is under compression (the up direction was correct for F_2), and the left pedestal is under tension (the down direction was correct for F_1).

Important comments on statics calculations:

- You must always use t**he same pivot axis P for each torque contribution** in putting together the net torque expression due to a set of forces $\{F_i\}$:

$$\tau_{net}(P) = \tau_{F_1}(P) + \tau_{F_2}(P) + \tau_{F_3}(P) + ...$$

- For statics, you can choose **any pivot axis point P for the whole equation** $\vec{\tau}_{net}(P) = 0$. Just remember, as noted in the previous bullet, each torque contribution must then be calculated for **that same point P**.

- Your **choice of P can make your life easier**! In the last example, we chose the left end for P so that there was no contribution from the torque due to F_1 and therefore we got a net torque equation involving only F_2.

- In fact, suppose we chose the contact point at F_2 for our pivot axis P for the torque calculation. This is also **nice because the torque due to F_2 is zero**, and we get an equation for F_1 alone (again all the forces are perpendicular to their respective $\vec{R}$'s so all the θ's are 90° in the torque calculations):

$$\tau_{net}(\text{P at } F_2 \text{ contact point}) = + F_1\,(1.5) + 0 - 580\,(3.0) = 0$$

$$\Rightarrow\; F_1 = +1160 \text{ N}$$

- Recall we used **force balance as one of our equations** instead of a second torque balance equation (such as the one we just analyzed to get F_1). The force balance was easier to write down.

- Always remember that the pivot point P represents an **axis piercing the page and perpendicular to the page.**

- If you are heavily into masochism, you can take your answers and check that the net torque vanishes relative to any old point **in a given statics problem** – say, your nose.

- Every time you calculate torque you must **confess what axis (point P) you are choosing** to your priest, minister, imam, rabbi, or physics teacher.

- Finally, you might be wondering what we do if the teeter-totter or diving board had weight? As we'll officially prove later, the torque due to any object in uniform gravity is found fantastically easily! It is calculated simply by assuming the **object's entire weight is applied at the object's CM.** This is another uniform gravity "squish" result. See the next cycle.

See the next page for a problem for practicing this stuff.

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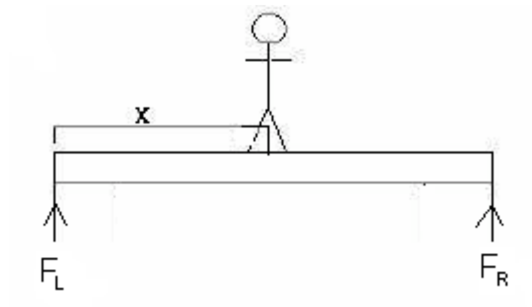
Problem 13-2: *A person with a mass of* M *stands a distance* x *from the left end of a very light board, length* L*, which is supported at its two ends by two columns. The upward forces by the two columns can be labeled* F_L *and* F_R*, respectively.*



a) *Draw an XFBD (i.e., an extended FBD) for the person-board system. Remember only the external forces should appear in the XFBD. The contact force between the person's feet and the board is an internal force since we are considering the person as part of the system.*

b) *Just bravely guess what values you think* F_L *and* F_R *will have in the following three cases: if the person is standing on the left end* (x=0)*, in the middle* (x=L/2)*, or on the right end* (x=L).

c) *Find two equations, one gotten by balancing forces* ($F_{net} = 0$) *and another gotten by balancing torques* ($\tau_{net}(P) = 0$)*, for the specific choice of pivot point* P *for which you calculate the torques* ***to be the left end of the board****, in terms of the two unknowns,* F_L *and* F_R*, and also* Mg, x, *and* L.

d) *Solve your two equations for the two unknowns,* F_L *and* F_R.

e) *Verify that your answers to (d) also yield torque balance* $\tau_{net}(P) = 0$ *for an axis point* P *at the right end.*

f) *Consider the three cases* x = 0, L/2, L *for your answer in (d) and compare to your guesses in (b). Did they agree?*

In fact, if your answers are right, they'll make the torque zero with respect to any pivot axis! Prove that by considering all the axes in the world (JUST KIDDING!!!)

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Problem 13-3 is on the next page.

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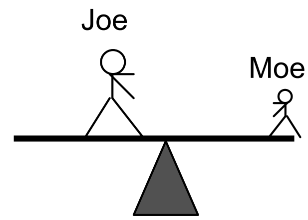
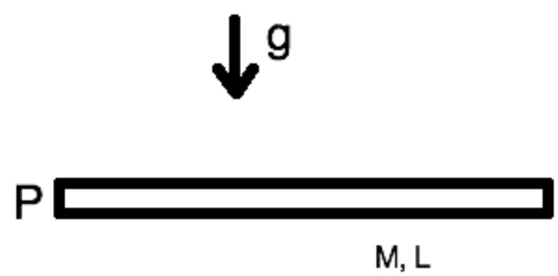
Problem 13-3 Quickies!

a) Joe and Moe are balancing on a teeter-totter. Joe is heavier, so he needs to move in toward the center, for them to balance as shown. What is neat is that if they balance while the teeter-totter is level, they will balance when the teeter-totter is at any angle! Use the torque formula to convince yourself that, if Moe and Joe's torques cancel when the teeter-totter is level, they will cancel at any angle of the teeter-totter.



b) In anticipation of the next cycle, take a guess as to what the torque around the point P *is due to the weight of the diving board, if the board is uniform, it is level (horizontal), and it has mass* M *and length* L. *Hint: It is not* MgL !



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