Chapter 13+

Revisit Torque

Revisit:	To-Do:
• Statics (equilibrium) • Torque formula	• Torque due to weight is simple • Different forms of the torque formula • Cross product

Revisit Statics

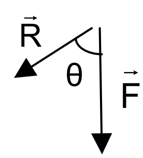
Recall that when nothing is moving throughout some time period, ***both*** the total (net) torque, and the total (net) force must vanish during that time:

$$\vec{F}_{net}^{ext} = 0$$
$$\tau_{net}^{ext} = 0 \text{ about any axis}$$

Comment: For the torque axis, remember we are often talking about axes perpendicular to the page in simple rotational examples. So we often say "torque about a ***point***." The point being the point where the axis pierces the page.

Revisit the Torque Formula and Direction

To calculate the torque around a given point (again, this means a given axis perpendicular to the drawing we make), assume the force $\vec{F}$ is applied at the location marked by the position vector $\vec{R}$ (defined from the rotation point to the point where the force is applied):



Torque magnitude:

$$|\tau| = FR\sin\theta$$

where θ = angle between $\vec{F}, \vec{R}$ and where F and R are the magnitudes of those vectors, respectively.

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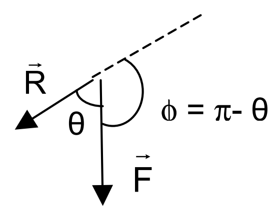
Notice the irrelevance of angle choice: You can choose the acute or the obtuse angle between R and F for calculating torque. This is because



$$\sin\phi = \sin(\pi - \theta) = \sin\theta\,)$$

By the way, this is not true for the work formula, $F\,d\cos\theta$, because $\cos(\pi - \theta) = -\cos\theta$.

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Torque sign:

As we have previously discussed, torque is, in general, a vector. In the simpler systems we discuss in freshman physics, we typically are looking at a rotating stick or cylinder or sphere with the axis of rotation perpendicular to the figure we draw on paper or the blackboard! Torque is then a vector pointing along that same axis and so we are reduced to talking about a single component (in one dimension) for the torque, and the question is simply the sign of that component. When we are pointing out of the paper (giving a counterclockwise rotation corresponding to a + sign) or into the paper (giving a clockwise rotation or a - sign). This sign follows the right-hand-rule (i.e., follow the direction that a right-hand screw travels). So for the present discussion we will often include the sign in τ according to

Torque magnitude and sign:

$$\boxed{\tau = \pm FR\sin\theta} \quad \text{where + for CCW and – for CW}$$

Torque due to Weight: Another Simple Squish Theorem

What do we do with the torque due to gravity on a board or a bridge or whatever?

The torque due to uniform gravity for any body, mass M, is simply calculated as if all the weight Mg were at the CM.

Proof: The total torque due to gravity acting on every particle making up the body is

$$\tau_{M,\ \text{constant gravity}} = \sum_1^N m_i g\,(\pm r_i \sin\theta_i)$$

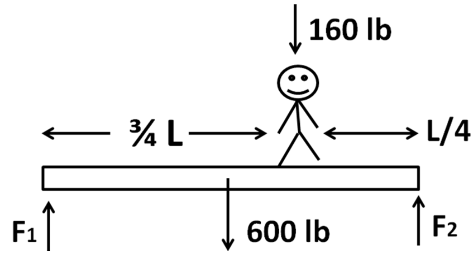
using the appropriate sign convention depending on whether each torque contribution is CCW (+) or CW (-). But $r_{i,\perp} = \pm r_i \sin\theta_i$, the projection of the particle position $\vec{r}_i$ perpendicular to the direction of gravity (the $\pm$ is included in the perpendicular projection since it changes sign depending on whether it points left or right). Thus

$$\tau_{M,\ \text{constant gravity}} = \sum_1^N m_i\, g\,(\pm r_i \sin\theta_i) = g\sum_1^N m_i\, r_{i,\perp} = g\,M\,r_{CM,\perp} = \pm Mg\, r_{CM} \sin\theta$$

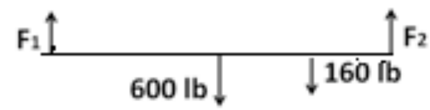
using the definition of the CM and defining θ as the angle between gravity and the position vector $\vec{r}_{CM}$(with magnitude r_{CM}):

This is the same as if M were squished to the CM point!

Example: A 160-lb man is walking across a level bridge and stops three-fourths of the way from one end. **The bridge is uniform and weighs 600 lb.** What are the values of the vertical forces exerted on each end of the bridge by its supports?



Answer: The torque reference point can be taken to be the right end (recall we must always define the point around which we are calculating the torques). The man's width is considered small relative to the other distances (L/4 which is the smallest of the other lengths). Referring to the XFBD below, the torque around the right end due to F_2 is zero since its "lever arm" is zero, and the rest of the force angles are 90^o. The bridge weight is treated as if it were applied to the center, so the net torque around the right end is: $-F_1 L + 600\frac{L}{2} + 160\frac{L}{4} = 0$



and L cancels out giving: $\boxed{F_1 = 340 \text{ lb}}$

Now use force balance as a quick way to get the other force: $F_1 + F_2 - 600 - 160 = 0$

and substituting F_1 gives: $\boxed{F_2 = 420 \text{ lb}}$

Comments:

1) Alternatively, and with more work, we could get F_2 by using the fact that the torque must be zero about another point (giving us another equation). Once we find two different equations, nothing new is gotten from looking at, say, the torque around a third point.

2) We could have solved this particular problem by forgetting about the bridge and then adding 300 lb to each support, because the bridge was symmetrically placed. In other situations it won't always be so symmetrical.

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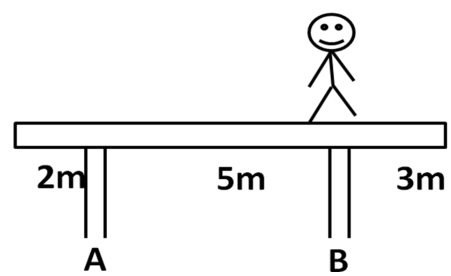
Problem 13-4 *A 10 m long uniform beam weighing* W *=500 N is supported on columns* A *and* B *as shown in the figure. The beam extends 3m to the right of* B *and 2m to the left of* A. *Just before going to Miami, Lebron James with a mass m of 102 kg was seen standing right above the* B *support.*

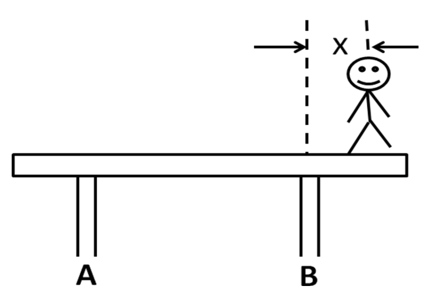


a) Draw an extended FBD for this situation

b) Guess what forces of support in N that each of the columns A *and* B *are providing, and explain your thought process.*

c) Derive the answers to (b) through a torque equation plus either a force balance equation OR a second torque equation.

d) To what distance x *on the right of* B *could Lebron have walked without tipping the beam over? Hint: Just at the tipping point, which column is just on the verge of no longer supporting anything?*



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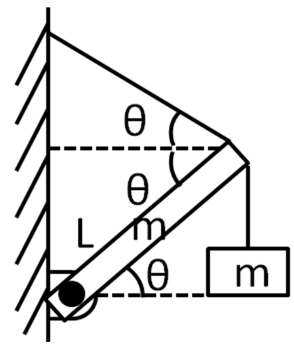
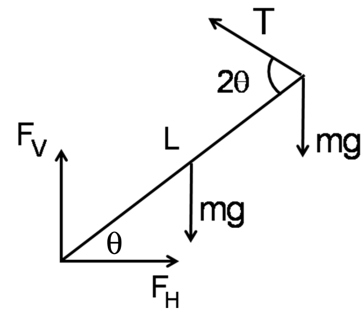
Practice Adding Torques for General Angles



EXAMPLE: Consider the illustrated boom with a load. For simplicity, assume the boom has the same mass as the load, and the triangle formed is a symmetric (isosceles) triangle as shown (though it would be easy to consider different masses and different angles, so this is no big deal). What is the tension T in the support cable and what are the horizontal (F_H) and vertical (F_V) support forces on the pivot?



An extended free-body force diagram is shown, where it is convenient to decompose the support force on the lower end of the boom into horizontal and vertical components. We want to calculate T, F_H, and F_V, so we need 3 equations, two of which we can get from the vanishing of the **sum of forces** in both vertical and horizontal directions and one from vanishing of the **net torque around ANY point**. Let's do the net torque equation first.

Vanishing of net torque around the lower end: We are free to calculate torques about the pivot; then F_H, F_v won't come into the equation since they produce zero torque there! Recall the torque due to the weight of the rod is calculated as if it were squished to its CM, and use a CCW torque as positive:

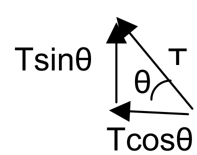
$$T\,L\,\sin 2\theta \; - \; mg\,L\sin(\pi/2-\theta) - mg\,L/2\,\sin(\pi/2-\theta) = 0$$

With $\sin 2\theta = 2\sin\theta\cos\theta$ and $\sin(\pi/2-\theta) = \cos\theta$: $\boxed{T = \dfrac{3mg}{4\sin\theta}}$

Noticing that L cancelled out since it was the only length in the problem.

To find F_H, F_V , we use the fact that the horizontal and vertical force separately balance. From its golden triangle, we obtain the tension components for that force balancing,

Horizontal: $F_H - T\cos\theta = 0 \quad \Rightarrow \quad \boxed{F_H = \dfrac{3}{4} mg\cot\theta}$



Vertical: $F_V + T\sin\theta - 2mg = 0 \quad \Rightarrow \quad \boxed{F_V = \dfrac{5}{4} mg}$

Another torque equation (a torque equation around a different point) could be used in place of one force equation, but that's usually more work.

A problem to practice with angles is next.

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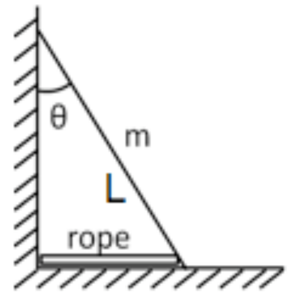
Problem 13-5 *The figure shows a uniform ladder of mass* m *and length* L *leaning up against a wall. The ladder makes an angle* θ *with the wall as shown. Since there is no appreciable friction anywhere on the wall or the floor, a horizontal rope is tied on one end to the bottom of the ladder and on the other to the wall, to keep the ladder in place.*



a) Draw an extended FBD.

b) Determine all forces on the ladder in terms of mg *and* θ *(but not* L *– it cancels out in your equations).*

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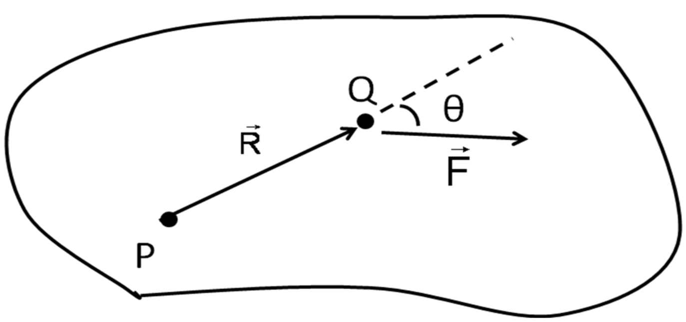
FOUR WAYS OF SAYING THE SAME THING

Remember how the work formula (for a **constant force over a straight line** path) could be written in four ways: $W = Fd\cos\theta = F_{||}d = Fd_{||} = \vec{F}\bullet\vec{d}$. Well, we can do the same thing for torque, if we include the other famous vector product (the vector cross product).

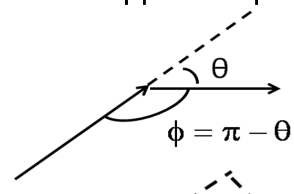
Consider a force $\vec{F}$ applied to a rigid body at point Q, which is positioned a vector distance $\vec{R}$ from a pivot point P:



The four equivalent ways of writing the torque formula for torque defined around the pivot point P (axis perpendicular to the above plane) are on the next page.

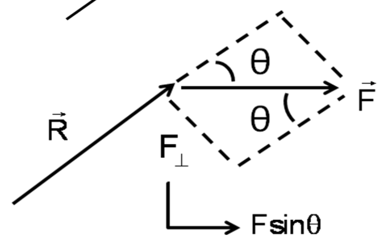
1) The magnitude of the torque about a point P due to a force $\vec{F}$ applied at point Q is:

$$\boxed{|\tau| = FR\sin\theta = FR\sin\phi}$$



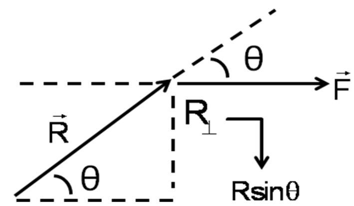
2) or equivalently (this might be faster to see)

$$\boxed{|\tau| = F_{\perp}R}$$



3) or equivalently (or this may be faster to see)

$$\boxed{|\tau| = FR_{\perp}}$$



4) or get both the magnitude and the sign (direction) from a vector **"cross product":**

For any two vectors $\vec{A}$, $\vec{B}$ and angle θ between them, the **cross product** $\vec{A} \times \vec{B}$ is defined to be a vector with

- magnitude $A\,B\sin\theta$
- direction perpendicular to the plane defined by $\vec{A}$, $\vec{B}$ and given by the right-hand rule (turn your hand from $\vec{A}$ to $\vec{B}$ and your thumb points in the direction – see the appendix)*

So our torque is $\boxed{\vec{\tau} = \vec{R} \times \vec{F}}$ which indeed has magnitude FRsinθ and a direction into the page for our rigid body picture on p.13-11, agreeing with a CW rotational axis.

The component along the axis perpendicular to the page is thus $\boxed{\tau = -FR\sin\theta}$ consistent with a minus sign convention for CW rotation.

In this course, we won't require you to master the **cross product** but you might find it helpful, and even neat, so go for it! The appendix is for your viewing pleasure (and for a small homework problem).

* Alternatively, the direction is the one toward which the right-hand screw advances when it is turned from $\vec{A}$ to $\vec{B}$. (Just as long as the screw is positioned perpendicular to the $\vec{A}$, $\vec{B}$ plane, it doesn't matter which way it points.)

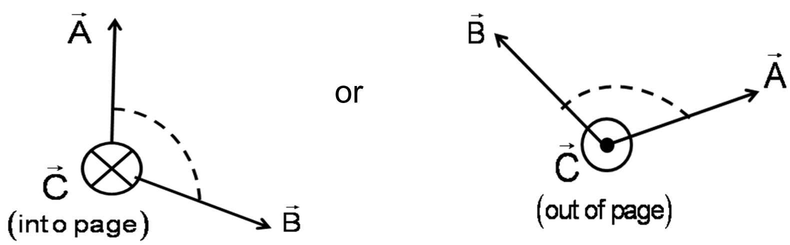
Appendix: Vector Cross Product

$$\vec{C} \equiv \vec{A} \times \vec{B} \equiv \text{vector cross product}$$

where $C \equiv |\vec{C}| \equiv AB\sin\theta \equiv |\vec{A}||\vec{B}|\sin\theta = AB_\perp = A_\perp B$ if θ is the angle between $\vec{A}, \vec{B}$. The direction of $\vec{C}$ is $\perp$ to the plane of $\vec{A}, \vec{B}$ using the right-hand rule to determine to which side it points. (Recall the right-hand rule: Lay the outside of your forefinger along $\vec{A}$ such that you can slide it toward $\vec{B}$ along the smaller angle between them. Then your thumb will point along $\vec{C}$ if you let it hang out.)

Properties (all of these follow from the above definition!)

1) Examples of $\vec{A} \times \vec{B}$ directions : (CW and CCW, respectively)

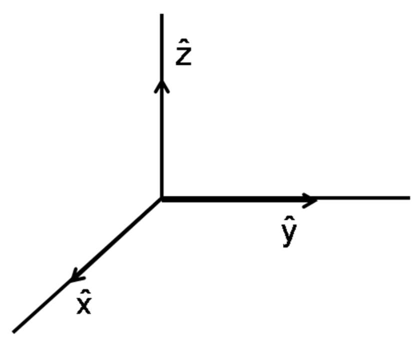


2) $\vec{A} \times \vec{A} = 0$

3) $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

4) Unit vector cross products

$$\begin{aligned}
&\hat{x} \times \hat{x} = 0, \quad \hat{y} \times \hat{y} = 0, \quad \hat{z} \times \hat{z} = 0 \\
&\hat{x} \times \hat{y} = +\hat{z}, \quad \hat{y} \times \hat{x} = -\hat{z} \\
&\hat{x} \times \hat{z} = -\hat{y}, \quad \hat{z} \times \hat{x} = +\hat{y} \\
&\hat{y} \times \hat{z} = +\hat{x}, \quad \hat{z} \times \hat{y} = -\hat{x}
\end{aligned}$$



5) $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

The above leads to the Cartesian formula for the cross product:

$$\boxed{\vec{A} \times \vec{B} = \hat{x}(A_y B_z - A_z B_y) + \hat{y}(A_z B_x - A_x B_z) + \hat{z}(A_x B_y - A_y B_x)}$$

The proof is on the next page.

PROOF: $\vec{A}\times\vec{B} = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\times(B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$

$$= 0 + A_xB_y\hat{z} + A_xB_z(-\hat{y}) + A_yB_x(-\hat{z}) + 0 + A_yB_z\hat{x} + A_zB_x\hat{y} + A_zB_y(-\hat{x}) + 0$$

using the unit vector identities and associativity that one can prove follows from the cross-product definition.

EXAMPLE:

$$\vec{A} = 2\hat{x} + 3\hat{y} + 2\hat{z} \qquad (\therefore A = \sqrt{4+9+4} = 4.1)$$
$$\vec{B} = 3\hat{x} + 4\hat{z} \qquad (\therefore B = \sqrt{9+16} = 5)$$

Cranking it out by hand:

$$\begin{aligned}\vec{A}\times\vec{B} &= 2\hat{x}\times3\hat{x} + 2\hat{x}\times4\hat{z} + 3\hat{y}\times3\hat{x} + 3\hat{y}\times4\hat{z} + 2\hat{z}\times3\hat{x} + 2\hat{z}\times4\hat{z}\\ &= 0 + 8\hat{y} + 9(-\hat{z}) + 12\hat{x} + 6\hat{y} + 0\\ &= 12\hat{x} - 2\hat{y} - 9\hat{z}\end{aligned}$$

Alternatively, plugging into the Cartesian formula yields the same answer:

$$\vec{A}\times\vec{B} = \hat{x}(12-0) + \hat{y}(6-8) + \hat{z}(0-9)) = 12\hat{x} - 2\hat{y} - 9\hat{z}$$

To get the angle between them, we can compare the magnitudes calculated in two different ways:

1) $\left|\vec{A}\times\vec{B}\right| = \sqrt{144+4+81} = 15.1$

2) $AB = 20.5$

Therefore $\sin\theta = \dfrac{\left|\vec{A}\times\vec{B}\right|}{AB} = \dfrac{15.1}{20.5} \Rightarrow \theta = 47^\circ$

But we confess that it is faster to use the dot product to get this same angle:

1) $\vec{A}\bullet\vec{B} = 6 + 0 + 8 = 14$

2) $AB = 20.5$

Therefore $\cos\theta = \dfrac{\vec{A}\bullet\vec{B}}{AB} = \dfrac{14}{20.5} \Rightarrow \theta = 47^\circ$

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Problem 13-6 *If* $\vec{A} = 6.0\,\hat{x} - 4.5\,\hat{y}$ *and* $\vec{B} = -3.5\,\hat{x} + 8.0\,\hat{y} + 2.0\,\hat{z}$, *determine the cross product* $\vec{A}\times\vec{B}$ *in terms of the Cartesian unit vectors, and the angle between* $\vec{A}$ *and* $\vec{B}$ *using the magnitude of that cross product.*

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