

## Chapter 11+

### Revisit the Center of Mass

<b>Revisit:</b> <ul style="list-style-type: none"> <li>• CM in all three dimensions</li> <li>• Relation of the CM velocity to the total momentum</li> <li>• Relation of the CM acceleration to the net force</li> </ul>	<b>To-Do:</b> <ul style="list-style-type: none"> <li>• CM calculations for continuous bodies - CM for different shapes, and multiple bodies</li> <li>• Fixed CM in a three-body example with a canoe</li> </ul>
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### Revisit The Center Of Mass Definition

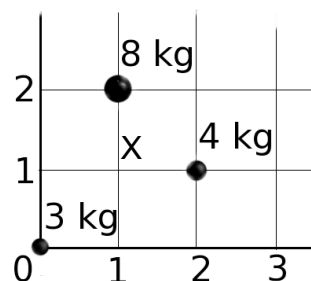
Recall the extension of the 1D definition of the center of mass to all three dimensions. In vector notation, for N bodies, it is

$$\vec{r}_{\text{CM}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + m_4 \vec{r}_4 + \dots}{m_1 + m_2 + m_3 + m_4 + \dots} = \frac{\sum_{i=1}^N m_i \vec{r}_i}{\sum_{i=1}^N m_i}$$

**A 2d cm example:** Three masses are laid out on a 2D x-y grid shown below with position units in meters. The CM components are found by adapting the above formula to 2D. The CM is calculated with one more significant digit than allowed (so we're bad) and shown on the grid as an X – close to where we might have guessed!

$$x_{\text{CM}} = \frac{3(0) + 8(1) + 4(2)}{15} = 1.1 \text{ m}$$

$$y_{\text{CM}} = \frac{3(0) + 8(2) + 4(1)}{15} = 1.3 \text{ m}$$



**A 3d CM example is found in the following problem:**

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**Problem 11-5** The Local Group of galaxies consists of our galaxy and its nearest neighbors. The mass of the most important members of the Local Group are as follows (in multiples of the mass of the Sun): our galaxy,  $2 \times 10^{11}$ ; the Andromeda galaxy,  $3 \times 10^{11}$ ; the Large Magellanic cloud,  $2.5 \times 10^{10}$ ; and NGC598,  $8 \times 10^9$ . Choosing our galaxy at the center, the x, y, z coordinates of these galaxies are, respectively, as follows (in thousands of light-years): (0, 0, 0); (1640, 290, 1440); (8.5, 56.7, -149); and (1830, 766, 1170). Find the coordinates of the center of mass of the Local Group (in thousands of light years). Treat all the galaxies as “points,” but in fact we will emphasize in this chapter that our answer will be the same if these galaxy positions were in fact the positions of their own CM’s!

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## CM Calculations

Let us consider ways of finding the CM for various objects

- **Continuous pieces:** Generally, for continuous objects, the sum in the CM goes over to an integration in the continuous limit, after we replace  $\sum \Delta m_i \vec{r}_i \rightarrow \int dm \vec{r}$

$$x_{CM} = \frac{\int x \, dm}{M}, \quad \text{or} \quad \vec{r}_{CM} = \frac{\int \vec{r} \, dm}{M},$$

but we **hardly ever do much integrating in freshman physics, relying instead on symmetry**. E.g., a uniform straight rod has its CM at its center.

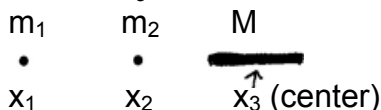
a) masochist: uniform rod has constant linear mass density  $\lambda = M/L$  so  $dm = \lambda dx$

$$x_{CM} = \frac{\int x \, dm}{M} = \frac{\int x \lambda \, dx}{M} = \lambda \frac{\int_0^L x \, dx}{M} = \lambda \frac{L^2/2}{M} = \frac{M}{L} \frac{L^2/2}{M} = \frac{1}{2} L$$

b) smarty-pants says there's a molecule at a distance  $d$  to the left of the center for every molecule at the distance  $d$  to the right so voilà

$$x_{CM} = \frac{1}{2} L$$

- **Hierarchy – if we know the CM of each of its pieces, then think of squishing each to a point at their respective CM, then find the overall CM of those points.** In the following example,  $M$  has its own CM at  $x_3$ . The other masses are points.



Hence the overall CM in the  $x$  dimension is  $x_{CM} = \frac{m_1 x_1 + m_2 x_2 + M x_3}{m_1 + m_2 + M}$

The official proof of this “CM squish theorem” is straightforward (divide a general body into a bunch of small pieces and then group some of them) so perhaps you’ll forgive us if we don’t give its details and so we just go on and assume it.

EXAMPLE: Noting that the CM of each side (mass  $M$ , which cancels out) is at its center for a cubical box with no top, we find

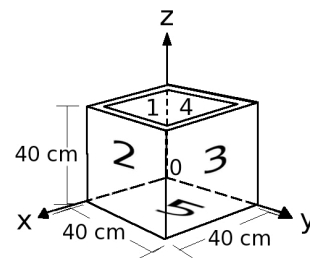
$$\vec{r}_{CM} = \frac{M\vec{r}_1 + M\vec{r}_2 + M\vec{r}_3 + M\vec{r}_4 + M\vec{r}_5}{5M}$$

The midpoints of the sides are, in cm,

$$\begin{aligned} \vec{r}_1 &= (20, 0, 20), \quad \vec{r}_2 = (40, 20, 20), \quad \vec{r}_3 = (20, 40, 20), \\ \vec{r}_4 &= (0, 20, 20), \quad \vec{r}_5 = (20, 20, 0) \end{aligned}$$

The individual  $x$ -,  $y$ -, and  $z$ -components of the CM are now plug-and-chug-able, and with one extra significant figure than allowed, we have

$$\therefore \vec{r}_{CM} = (20, 20, 16) \text{ cm}$$



## A Little Richer Example

As we noted earlier in Ch. 11, if there are no external forces (i.e., if the system is isolated), the CM velocity is constant. (To say it another way, the total momentum is constant if there's zero net external force, and the CM velocity is proportional to the total momentum.)

Now suppose the whole isolated system is initially at rest, then certainly the CM velocity is zero. Then, as time goes on, even if there are individual changes in position,  $\Delta x_i$ , they must be such that the CM stays fixed. **(The CM velocity must stay zero.)** That is, they must satisfy

$$\Delta x_{CM} = 0$$

Thus, since  $\Delta x_{CM} = \frac{m_1 \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3 + \dots}{m_1 + m_2 + m_3 + \dots}$ , we must have

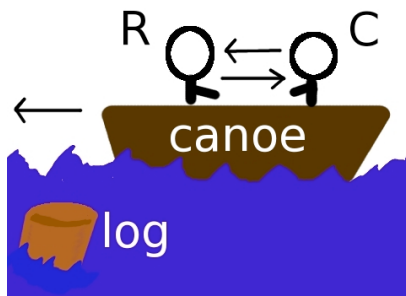
$$m_1 \Delta x_1 + m_2 \Delta x_2 + m_3 \Delta x_3 + \dots = 0$$

where the changes  $\Delta x_i$  are with respect to any fixed (inertial) origin, with respect to which the CM velocity is zero.

**If these weighted terms didn't all add up to zero, the CM would have to move** – and it can't! - essentially a consequence of the conservation of momentum. A little richer example can illustrate this 1D story:

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**Problem 11-6** Ricardo, with mass 80 kg, and Carmelita, who is known to be **lighter**, are enjoying Lake Merced at dusk in a 30-kg canoe. When the canoe is at rest in the placid water, they change seats, which are 3.0 m apart and symmetrically located with respect to the canoe's center. Ricardo notices that the canoe moved 40 cm relative to a submerged log and that he can calculate Carmelita's mass, despite the fact she has told him it is none of his business. So, what is her mass?



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RECALL: Ricardo, Carmelita, the canoe, and from previous homework, the shark, boat, galaxy all have size themselves! As we said before, the positions we associate with them are **their own** CM points!

### Revisit Newton's Second Law for the CM:

$$\mathbf{F}_{\text{NET}} = M \mathbf{a}_{\text{CM}} \text{ where } M \text{ is the Total Mass}$$

Recall the really great thing about the CM that we keep mentioning and we proved earlier in Ch. 11 and we put to good use, for example, in discussing wheel friction: **The motion of the CM point of a system moves like a single point particle whose mass is the total mass  $M$  of the system and whose acceleration is determined by the external net force through Newton's 2nd law.**

Let's rewrite the result in terms of the CM variables.

The total momentum is  $\vec{p} = \sum_{i=1}^N m_i \vec{v}_i = M \vec{v}_{\text{CM}}$  where  $\vec{v}_{\text{CM}} = \frac{d\vec{r}_{\text{CM}}}{dt}$  and  $M = \sum_{i=1}^N m_i$

and the CM second law is

$$\vec{F}_{\text{ext, net}} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(M \vec{v}_{\text{CM}}) = M \frac{d\vec{v}_{\text{CM}}}{dt} = M \vec{a}_{\text{CM}}$$

Comments:

- Again, when we are talking about cars, balls, boats, etc., as "points," the points are really the CM points, and it is sensible to draw free-body diagrams with forces all drawn as if they were applied at a single point, ignoring the size of the objects, if all we want is the overall motion! (For torque discussions, on the other hand, we need extended FBD showing where the forces really are applied.) This is another "squish" theorem: Just squish the body to its CM point and do simple point calculations.

- And again, if the net external force is zero, the total momentum is constant, and the CM moves uniformly forever! In terms of the individual velocities and masses, this means that the CM velocity given by

$$\vec{v}_{\text{CM}} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + m_4 \vec{v}_4 + \dots}{m_1 + m_2 + m_3 + m_4 + \dots} = \frac{\sum_{i=1}^N m_i \vec{v}_i}{\sum_{i=1}^N m_i} = \frac{\vec{p}}{M}$$

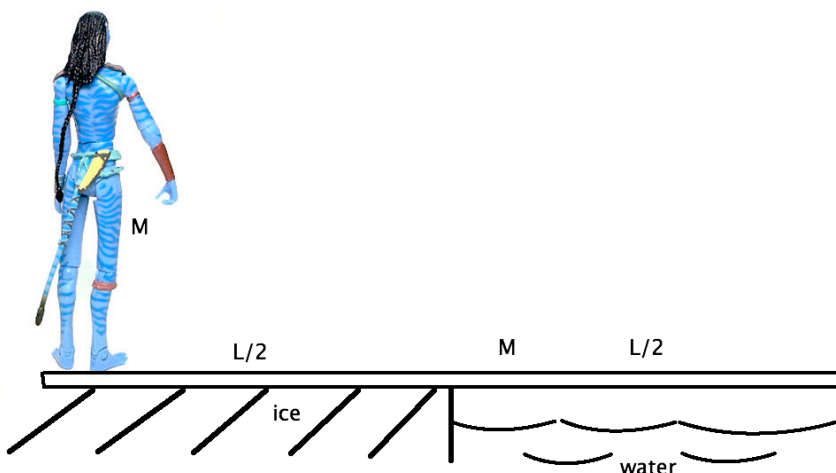
would be constant IF THE NET EXTERNAL FORCE IS ZERO. Then, for example, the CM velocity, which is  $v_{\text{CM}} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$  for the 1D collisions described in Ch.10+, would stay the same forever (or until an external force arose).

**See the next page for a problem.**

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**Problem 11-7**

Neytiri is standing on the end of a long thin uniform board of length  $L$ . Half of the board including the end where Neytiri is standing rests on very slippery ice with the other half over open water, as shown. Lucky for our calculations, the board and Neytiri have the same mass  $M$ ! ☺



- Find the  $x$  coordinate, relative to the ice, of the overall CM of Neytiri and the board. Define  $x = 0$  at her end, and  $x = L$  at the other end.
- If Neytiri walks all the way to the other end, will she fall into the water? Give your reasoning.
- Suppose Neytiri moves at a speed  $v$  **relative to the ice**, while she is walking to the right as described in part (b) from her original end position. What is the CM velocity?
- How fast is the board moving relative to the ice and in what direction, if Neytiri is moving at speed  $v$  to the right?

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