Chapter 9+

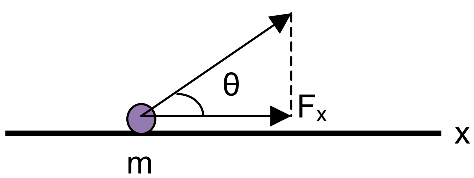
Revisit Work, Kinetic Energy, and Potential Energy

Revisit:	To-Do:
• Work-energy relation • Constant force at an angle • Separate work into conserved and nonconserved: $K_i + U_i + W_{net}^{nc} = K_f + U_f$	• Non-constant forces • Spring potential energy • Dot products and general work integral for 3D

BACK TO THE FUTURE: The Sequel!

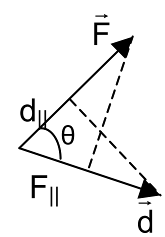
Revisit Work-Energy Relations

We recall, for any distance d along a straight line, and for constant $\vec{F}$ (magnitude F):



the work ON the mass m is W = Fd cos θ where F and d are positive magnitudes, θ = angle between $\vec{F}$ and the direction we go (a straight line). W is negative whenever θ > π/2.

COMPLETELY EQUIVALENTLY:



We can say $W = F_{||}\, d$ where the component of $\vec{F}$ along d is $F_{||} = F\cos\theta$ which can be negative.

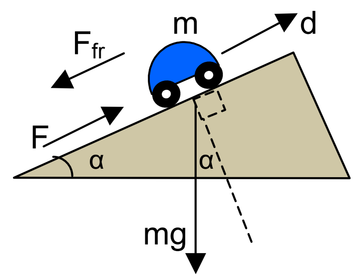
Or we can say $W = F\, d_{||}$ where the component of $\vec{d}$ along F is $d_{||} = d\cos\theta$ which can be negative.

Summarizing

$$W = F\, d \cos\theta = F_{||}\, d = F\, d_{||}$$

(motion must be in a straight line and the force must be constant during the whole motion for these (equivalent) formulas to be used)

Example: Suppose we push a car **uphill** a distance d and we characterize the friction by some effective coefficient of friction μ_k (this friction takes into account the wheel bearing friction, tire slippage, etc.). Suppose further that the pushing force is parallel to the hill and has magnitude F, the car has mass m, and the hill angle is α. What is the work that has been done by each of the forces acting on the car?



Answer: First recall that since the gravity force mg points downward, its component along the hill is - mg sinα (if uphill is called +) and its component normal to the hill is - mg cosα (if out of the hill is called +). Using these components, we can quickly do our calculations by one or another of the different forms in the boxed equation above.

1) By F: $W_{F\text{ on }m} = +Fd > 0$

(check: $W = F\,d\cos\theta = F_{||}\,d = F\,d_{||}$ all are easy, with $\theta = 0$)

2) By gravity: $W_{\text{grav on }m} = -\,mgd\sin\alpha < 0$

(check: $F_{grav}\,d\cos\theta$, with $\theta = \dfrac{\pi}{2} + \alpha$, but faster is $F_{grav,||}\,d$ with $F_{grav,||} = -\,mg\sin\alpha$)

3) By N: $W_{N\text{ on }m} = 0$ (of course, normal forces never do work)

(check: $F_N\,d\cos\theta$, with $\theta = \dfrac{\pi}{2}$, or $F_{N,||} = 0$ or $d_{||\text{ to }N} = 0$ - all are easy)

4) By friction: $W_{\text{friction on }m} = -\mu_k mg\cos\alpha\,d < 0$

(check: $F_{fr}\,d\cos\theta$, with $\theta = \pi$ and $F_{fr} = \mu_k mg\cos\alpha$ or $F_{||,\,fr} = -\,\mu_k mg\cos\alpha$ in $F_{||}\,d$)

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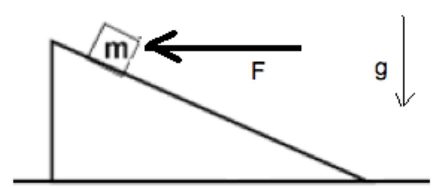
Problem 9-6 *A 5.0-kg block is pushed 3.0 m up a rough 37° inclined plane (the incline cannot move) by a **horizontal** force of 75 N. If the initial speed of the block is 2.2 m/s up the plane and there is a constant coefficient of kinetic friction μ_k = 0.3 opposing the motion, calculate*



(a) the initial kinetic energy of the block.

*(b) the work done **by** the 75-N horizontal force.*

*(c) the work done **by** gravity. What does the sign on your answer mean?*

(d) the work done by the normal force due to the inclined plane on the block. Why did we not bother to boldface "by" here?

*(e) the work done **by** the friction force. What does the sign on your answer mean?*

(f) the final kinetic energy of the block.

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Non-Constant F !

Now let's go to a variable force situation. We'll use only **one-dimension** (say the x-dimension) to learn this new thing most easily. We consider F as the component along the positive x-direction ($F_x \equiv F$ as we drop the x subscript for convenience) and not the magnitude now. Most importantly, F can vary with x:

$$F = F(x)$$

Consider a tiny (differential) path length Δx over which F is constant! And then we can add up all the little changes from the work-energy relation. On the left side of the equation we get the total change in the kinetic energy and on the right side we get a sum that goes over to an integral in the limit of infinitesimal steps dx:

$$K_f - K_i = \sum_{\substack{\text{steps} \\ j}} \Delta\left(\frac{1}{2}mv_j^2\right) = \sum_{\substack{\text{steps} \\ j}} F_j \Delta x_j \ \underset{\substack{\text{limit of} \\ \text{small steps}}}{\longrightarrow} \int_{x_i}^{x_f} F(x)\,dx$$

That is,

$$\boxed{\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \int_{x_i}^{x_f} F(x)\,dx \ , \quad \text{one dimension } (x)}$$

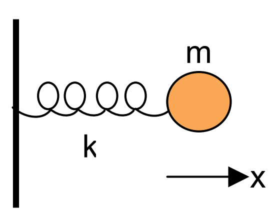
So, we **still** have the old work-kinetic energy conversion form: $\boxed{K_f - K_i = W}$, but now with the more general work expression

$$\boxed{W = \int_{x_i}^{x_f} F(x)\,dx \ = \text{ work due to } F(x) \text{ acting } \textbf{on} \text{ } m \text{ while going from } x_i \text{ to } x_f \text{ , in 1D}}$$

Spring force and potential energy: The most famous non-constant force example is the spring force we've discussed in Ch. 15.

Recall, if $F_{spring}(x)$ = force **by** the spring **on** m , for spring constant k, and if x=0 is the spring equilibrium position, we have Hooke's Law:

$$\boxed{F_{spring} = -kx}$$



Thus the work done **by** the spring **on** m is

$$W_{spring} = \int_{x_i}^{x_f} F_{spring}(x)\,dx = \int_{x_i}^{x_f} (-kx)\;dx = -\frac{1}{2}k\,x^2\Big|_{x_i}^{x_f} = -\left[\frac{1}{2}k\,x_f^2 - \frac{1}{2}k\,x_i^2\right]$$

We see that the work done by a spring on m depends only on the end points and we've got ourselves another **conservative force** and hence we've got ourselves another potential energy function:

For the spring, $\boxed{U_{spring}(x) \equiv \frac{1}{2}k\,x^2}$

so that

$$\boxed{W_{spring} = -\left[U_{spring}(x_f) - U_{spring}(x_i)\right] = -\Delta U_{spring}}$$

(We add this to any other work performed on the mass, the total of which will change its kinetic energy according to $K_f - K_i = W_{total}$)

Example: A spring with a mass on the end of it has a force constant of 3.5 x 10^4 N/m. The spring is initially relaxed at $x_i = 0$. Suppose **you** come along and slowly compress the spring by 0.10 m, so that the mass is still at rest at the end of the compression.

a) How much work did the spring do on m during this compression? What does the sign on your answer mean? Answer: With $x_i = 0$, we have

$$W_{spring} = -\left(\frac{1}{2}\,k\,x_f^2 - 0\right) = -\frac{1}{2}\,3.5 \times 10^4\,(0.1)^2 = -175\text{J} < 0$$

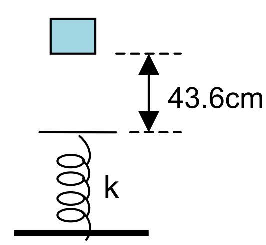
b) How much work did you do? Answer:

$$W_{you} = -W_{spring} \;\; (\text{because } F_{you} = +k\,x\,) = +175\text{J} > 0$$

(someone had to transfer energy to the spring, as expected from $W_{spring} < 0$ and that someone was you!)

Another Example: The spring potential energy can be used in the mechanical energy conservation law just as the constant gravity potential energy was. Let's have some fun with a combination of gravity and a spring in an energy conservation situation:

A 2.14-kg block is dropped from rest at a height of 43.6 cm above the top of a spring with force constant k = 18.6 N/cm, as shown.



Ignoring heating and any other friction losses, what is the lowest distance the block drops to, before it begins to be pushed back up by the spring? The analysis is on the next page.

Analysis: All forces (gravity and spring) are conservative since we ignore any heat from the collision! So we have conservation of energy

$$K + U = \text{constant} \text{ or, equivalently, } \Delta K = -\Delta U$$

But the kinetic energy is zero at the beginning and at the end

$$K_i = K_f = 0 \text{ so certainly } \Delta K = 0$$

and

$$\Delta U = 0 \Rightarrow \Delta U_{spring} + \Delta U_{gravity} = 0$$

If the spring is compressed a distance x,

$$\Delta U_{spring} + \Delta U_{gravity} = mg\Delta y + \frac{1}{2}kx^2 = 0$$

where Δy is the total displacement downward the block falls

$$\Delta y = -\left(43.6\text{cm} + x\right) \quad \text{(Many of us forget the extra x !)}$$

We end up with a quadratic equation for x (comes out nicely in units of cm after an overall unit of Newton cancels out on each side)

$$\frac{1}{2}\left(18.6\right)x^2 = +2.14\left(9.8\right)\left(43.6 + x\right) \Rightarrow 9.3x^2 - 20.97x - 914.4 = 0$$

$$\Rightarrow x = \frac{20.97 \pm \sqrt{439.7 + 34016}}{18.6}$$

The + sign yields $\boxed{x = 11.1\,\text{cm}}$ where the $-$ sign corresponds to the point above the spring where x < 0 and we also have $\Delta U = 0$. There's another interesting question to ask about this situation and we'll ask it in the context of a related problem:

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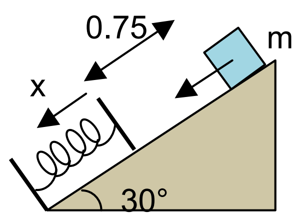
Problem 9-7 *A 1.0-kg block starts at rest and slides a distance of 0.75 m down a frictionless 30° incline where it hits a spring whose spring constant is 9.8 N/m.*



a) How much farther does the block slide down the incline, before it comes to a (momentary) stop?

b) Where does the block achieve maximum speed during this slide? (Hint: for a quick ____ solution here, think of what the acceleration must be at that point.)

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Work In All Dimensions (a good summary page!)

How do things change for more dimensions? Well, it is easy. For each of the three dimensions, the kinetic energy associated with motion in that dimension is changed according to the work done by the net force component in that direction:

$$\frac{1}{2}mv_{xf}^2 - \frac{1}{2}mv_{xi}^2 = \int F_x\,dx$$

$$\frac{1}{2}mv_{yf}^2 - \frac{1}{2}mv_{yi}^2 = \int F_y\,dy$$

$$\frac{1}{2}mv_{zf}^2 - \frac{1}{2}mv_{zi}^2 = \int F_z\,dz$$

where the general total (net) force is intended here: $\vec{F} = (F_x , F_y , F_z) = m\vec{a}$ is the net force **on m.** Now just add 'em all up, with the total velocity-squared $v^2 = v_x^2 + v_y^2 + v_z^2$

$$\therefore\ \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \int\left(F_x dx + F_y dy + F_z dz\right) = W_{\text{net on object}}$$

This is the full 3-D version, but still saying essentially the same thing: Any change in kinetic energy of an object is due to the net work done **on** the object

$$\boxed{\Delta K = K_f - K_i = W_{net}}$$

so $W_{net} > 0$ means $\Delta K > 0$ (and we'll now leave understood that we mean "on object" when we refer to the work W). In general, we have both conservative and non-conservative (e.g. friction) forces. Then the net work has two contributions

$$\boxed{W_{net} = W_{net}^{c} + W_{net}^{nc}}$$

where the conservative forces make contributions to the Republican Party, er, to the work that can be written in terms of a net sum of potential-energy differences

$$\boxed{W_{net}^{c} = -\left[U_f^{net} - U_i^{net}\right]}$$

and we still have the integral to do for the non-conservative part (it depends liberally on the path taken by the non-conservatives!)

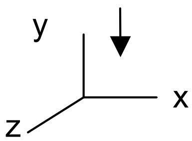
$$\boxed{W_{net}^{nc} = \int_i^f (F_x^{nc}\,dx + F_y^{nc}\,dy + F_z^{nc}\,dz)}$$

- this is where friction comes in (for which it is negative)
- we also may include the work due to external forces here

Our general **mechanical energy conservation law** now reads

$$\boxed{K_f - K_i = -\left(U_f - U_i\right) + W_{net}^{nc}}$$ or, what is the same thing, $$\boxed{K_i + U_i + W_{net}^{nc} = K_f + U_f}$$

Check constant gravity: Recall that we have been using gravitational potential energy $U_{gravity}$ = mgy in the vertical y variable alone, even when we move in all three x,y,z directions. With the above three-dimensional stuff, we can check that that is right. We specialize to gravity without friction or drag:



$\Rightarrow\ F_y = -mg,\ F_x = F_z = 0$

$\Rightarrow\ \Delta K = \int F_y\, dy = -mg(y_f - y_i) = -\Delta U$

This is now good in all 3 dimensions.

Work and the Dot Product

Finally, if we go back to our original work discussion for a force at an angle, we might very well have added up all the little pieces of work in a different way getting

$$W = \int_i^f F\ dr \cos\theta_{r,F}$$

for an angle $\theta_{r,F}$ between $d\vec{r}$ and $\vec{F}$. By comparison with the previous page, it must be that

$$F\ dr\ \cos\theta_{r,F} = F_x\ dx + F_y\ dy + F_z\ dz$$

and the first form cries out for us to invent the "dot product" (of course, it has already been invented!). Define the dot product (the scalar dot product of two vectors)

$$\vec{A} \bullet \vec{B} \equiv A\,B\cos\theta_{A,B}$$

for magnitudes A, B and the angle $\theta_{A,B}$ between $\vec{A}$ and $\vec{B}$

and the comparison implies that we must have two alternative, and equivalent, forms for calculating the dot product:

$$\boxed{\vec{A} \bullet \vec{B} = A\,B\cos\theta_{A,B} = A_xB_x + A_yB_y + A_zB_z}$$

where we make sure we understand that the magnitudes are related to the components by

$$\boxed{A = \sqrt{A_x^2 + A_y^2 + A_z^2} \text{ and } B = \sqrt{B_x^2 + B_y^2 + B_z^2}}$$

Anyway, we now have the **most general representation for work**

$$\boxed{W = \int_i^f \vec{F} \bullet d\vec{r}}$$

Physics is sure a lot of work!

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Problem 9-8 *For purely math practice in using the dot product, find the angle between the vectors **A** = (6, -4, 8) and **B** = (8, 2, -7), which are the (x, y, z) components, by demanding that the two ways of calculating the dot product give the same answer.*

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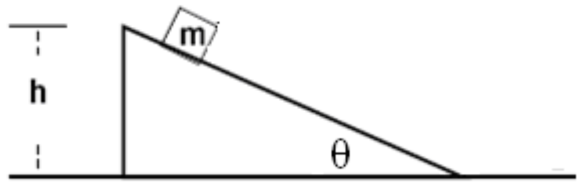
Problem 9-9



Consider the arrangement above: A block of mass m *is placed on a fixed ramp inclined at the angle* θ *shown. The block is placed at a height* h *above the surface of the table. The block is released from rest and then moves down the surface of the ramp without friction to the bottom end of the ramp. The block doesn't flip over.*

a) Draw a free body diagram to indicate all of the forces on the block.

b) Find the work done by the gravitational force on the block in the following three different ways (but the answers should be the same, of course):

i) Use the gravitational potential energy:

$$W_{grav} = -\left[U_f^{grav} - U_i^{grav}\right]$$

ii) Use the component form with y *as the vertical axis:*

$$W_{grav} = \int_i^f (F_x\,dx + F_y\,dy + F_z\,dz)$$

Hint: Only one component (F_x, F_y, F_z) *is nonzero.*

*iii) Use the dot product form involving the angle between the path and the force and integrate along the ramp path (*r *is the distance along the ramp):*

$$W_{grav} = \int_i^f F\,dr\cos\theta_{r,F}$$

Hint: $\int_i^f dr = R$ *for a path integration down the ramp of length* R

We wanted to give you this simple situation to really focus on these different forms for work and how they (must) give you the same answer.

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