Chapter 12

Forces Causing Curved Motion

- **A Force Must be Applied to Change Direction**
- **Coordinates, Angles, Angular Velocity, and Angular Acceleration**
- **Centripetal Acceleration and Tangential Acceleration Along a Curve**

Acceleration in General

The instantaneous 3D acceleration vector is given by the derivative of the 3D velocity vector

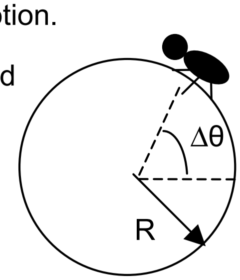
$$\vec{a} = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t} \equiv \frac{d\vec{v}}{dt}$$

Newton's second law tells us there must be a force whenever $\vec{a}$ is nonzero. When is $\vec{a}$ not zero? $\vec{a}$ is not zero if there is 1) **a change in magnitude** $v \equiv |\vec{v}|$, 2) **a change in direction** of $\vec{v}$, 3) or **both**. A change in magnitude means $d|\vec{v}| \neq 0$. A change in direction means $d\vec{v}$ is not parallel to $\vec{v}$. Both (1) and (2) can and often do occur at the same time.

Up to now we've been talking only about a change in the magnitude of the velocity – linear motion. In this chapter we include the important case of a change in the direction of the velocity. And we focus almost entirely on the most famous steady change in direction called the circle!

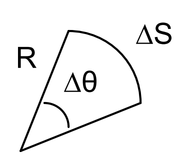
Angular Velocity in Circular Motion

The most interesting, useful, and important thing to study in changing direction is the persistent changing of direction involved in circular motion.* Even if you can go at a constant speed around a circle, you still have to exert a persistent force to make this happen. We'll get to what force is needed later, but we first got to get used to the new variables we use for circular motion.



Angular position on a circle – the angle θ: Suppose we see an ant move around a ring with radius R (a circle embedded in a 2D plane). We can describe the ant's position by the angle $\Delta\theta$ through which it has moved. Then the distance along the ring it has moved is the arc length $\Delta S = R\Delta\theta$ where $\Delta\theta$ must be in radians:

$$\Delta\theta\,(\text{in radians}) = 2\pi \frac{\Delta\theta(\text{in degrees})}{360}$$



as you recall from nursery school.

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* **Learning about circles is actually very powerful! That is, any point on any path can be considered as having a "radius of curvature." This means we can think of the neighborhood of that path as part of a circle with some radius. So what we learn about circles will help you some day think about more general paths, but don't worry about this for our class.**

Angular velocity on a circle – the time-derivative $\omega = \frac{d\theta}{dt}$: To describe the ant's motion on the ring, we want to talk about how θ is changing in time. As with an instantaneous velocity and acceleration, we define the (instantaneous) angular velocity as the time derivative:

$$\omega = \lim_{\Delta t \to 0} \frac{\theta(t+\Delta t) - \theta(t)}{\Delta t} = \frac{d\theta}{dt}$$

$$\Rightarrow \boxed{\omega = \frac{d\theta}{dt} \equiv \text{angular velocity}}$$ This is usually in units of rad/s **.**

The ant's speed v around the ring is the rate of change of the arc length $R\theta$ for fixed R ($dR/dt = 0$):

$$v = \frac{d}{dt}(R\theta) = R\frac{d\theta}{dt} = R\omega$$

$$\Rightarrow \boxed{v = \omega R \text{ for a given radius } R}$$

Uniform motion around a circle: This implies v is constant (say, v_0); hence, so is ω (say, ω_0). A simple integration of angular velocity can be done: $d\theta/dt = \omega_0$ leads to $\theta = \omega_0 t + C$ where the constant C is the initial angle θ_0 at $t = 0$:

$$\frac{d\theta}{dt} = \omega_0 \Rightarrow \boxed{\theta = \omega_0 t + \theta_0}$$

Note the similarity of this to the result $x = v_0 t + x_0$ for uniform <u>linear</u> velocity. The constant speed is given by

$$v_0 = \omega_0 R \text{ for uniform motion}$$

Average motion around a circle: They're easy, but for completeness let's list the following time averages for the ant if it goes from angle θ_1 at t_1 to angle θ_2 at t_2:

$$\omega_{\text{average}} = \frac{\text{angular interval}}{\text{time interval}} = \frac{\theta_2 - \theta_1}{t_2 - t_1}$$

and

$$v_{\text{average}} = \frac{\text{distance around ring}}{\text{time interval}} = \frac{R(\theta_2 - \theta_1)}{t_2 - t_1}$$

The latter is equal to $\omega_{\text{average}} R$ as expected.

Relationship of angular velocity to "cycle frequency:" We like to talk about the thing that hurts ... er, we mean the thing that comes in Hertz, which is abbreviated Hz. That is, we often like to use f, the cycle frequency such that

$$\boxed{\omega = 2\pi f} \quad \text{or} \quad \boxed{f = \frac{\omega}{2\pi}}$$

where f is given in cycles per second (Hz) and indeed multiplying that by 2π gives the radians/sec. Sometimes, people refer to ω as the "angular frequency," in view of the above relationship, and also because of the connection to oscillations (see Ch. 15). Here, ω and f are magnitudes (> 0).

Names of angular variables versus linear variables: We have been talking about angular velocity ω versus (linear) velocity v. But we'll usually just say "velocity" for v. Similarly in what's coming up later in this chapter and Ch. 14, we'll be referring to angular acceleration α versus (linear) acceleration a, and angular momentum L versus (linear) momentum p. We'll often drop the "linear" adjective. By the way, we often refer to frequency f, dropping the "cycle" adjective.

Period T for uniform motion around a circle: If the ant were going at constant speed (i.e., constant angular velocity), the frequency f cycles/second is constant, too, and thus we can find the seconds per cycle or <u>period T</u> for the ant to go completely around the ring:

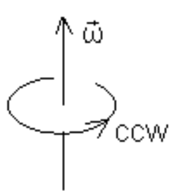
$$\boxed{T = \frac{1}{f} = \frac{2\pi}{\omega} \quad \text{for uniform motion}}$$

Again, ω and f are magnitudes (> 0).

Angular Velocity is a Vector

We should really refer to the angular velocity as a vector $\vec{\omega}$ because it has both a direction and a magnitude associated with it. The direction of this vector lies along the axis around which the ant circulates or "rotates." (We can think of the ant rotating around the ring.) The magnitude of this vector, usually written as ω, is the angular speed in radians/sec.

Right-hand rule: But which way does $\vec{\omega}$ really point along the rotational axis? It points according to the right-hand rule* where if you wrap your right hand around the axis such that the tips of your fingers are pointing the way the rotation flows around the axis, your thumb points in the correct direction to define $\vec{\omega}$.



So for the ant going in a counterclockwise (CCW) direction around the ring (i.e., CCW as seen from above in the diagram shown on the left), $\vec{\omega}$ lies along the symmetry axis of the ring and points upward in agreement with the right-hand rule.

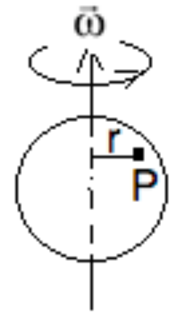
By the way, this picture refers to a particular instant in time if $\vec{\omega}$ were changing in time (and both its direction and magnitude may change with time, in more general situations).

*An alternative way to describe the right-hand rule is our favorite: lay a right-handed screw (all the screws in our hardware store are right handed!) along the axis line and turn it in the same sense as the rotation. It will always move in the direction of $\vec{\omega}$, no matter which way you lay it.

Angular Velocity for a Body Rotating Around an Axis

Suppose a body is spinning uniformly around an axis – we can use the Earth as a perfect example to illustrate this! This uniform motion means every point on or inside the Earth is rotating around the North-South-Pole axis with the same angular velocity $\vec{\omega}$.

The Earth's constant angular velocity $\vec{\omega}$ lies on the line connecting the poles and points toward the North Pole. The rotation indicated by the right-hand rule agrees with what we know: The Earth spins from West to East – in a CCW direction looking back down at the Earth from above the North Pole.



Consider the point P shown in the figure. This point is going in a circle with radius r and speed $v = \omega r$, where ω is the magnitude of $\vec{\omega}$. Every point on or inside the Earth is going in its own circle around that axis with radius given by the shortest distance to the axis (the length of the line drawn perpendicular to the axis from the point to the axis – see the line drawn to the point P as an example).

To repeat the above stuff, all parts of the Earth move with the same ω radians/sec around the same axis. But the linear speed v changes depending on how far you are from the earth's rotational axis, according to $v = \omega R_{\perp}$ where $R_{\perp}$ is the perpendicular distance (the shortest distance from wherever you are to the rotational axis) or equivalently the radius of the circle that you are moving in. (e.g., for point P, $R_{\perp} = r$, and for the equator, $R_{\perp} = R$, the Earth's radius). Sure enough, the people on the equator are spinning the fastest.

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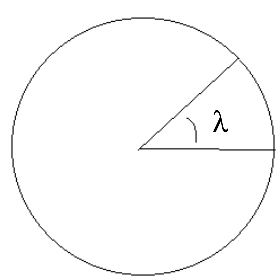
Problem 12-1

(a) *Given that the angular velocity of the earth about its axis is 7.27x10^{-5} rad/s, prove that one complete rotation of the earth takes (Surprise! Surprise!) 24 hours.*

(b) *Given that the linear speed due to the earth's rotation of a point on the equator is 463 m/s, find the earth's radius.*

(c) *What is Cleveland's linear speed in m/s due to the earth's rotation? You need to look up Cleveland's latitude λ, then draw a picture to help you calculate $R_{\perp}^{\text{Cleveland}}$, given your answer in (b) and the latitude λ.*



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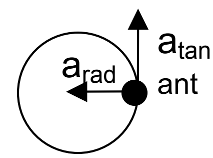
You're now entering an accelerated track in your education – a circular track that takes you back to our original question at the beginning of this chapter! So circle this date as the day you circle your wagons and get ready for more circular arguments and going in vicious circles until you suffer from circles under your eyes.

Acceleration for Circular Motion

Go back to our ant going around the circular ring. The ant's acceleration can be quite generally written in terms of two perpendicular components at any instant in time:

$$\boxed{\vec{a}_{total} = \vec{a}_{radial} + \vec{a}_{tangential}}$$



As shown in the figure, the component $\vec{a}_{radial}$ always points radially inward toward the center of the circle. The component $\vec{a}_{tangential}$ always points along the tangent to the circle and perpendicular to $\vec{a}_{radial}$. As the ant moves around, these components are defined to rotate around with it.

Before we examine each of these components in detail, we preview them, and make a comment:

- **$\vec{a}_{radial}$ produces changes in the ant's direction but <u>not</u> its speed.**

  a_{radial} is the famous "centripetal acceleration $\frac{v^2}{R}$" for a circle (see below)

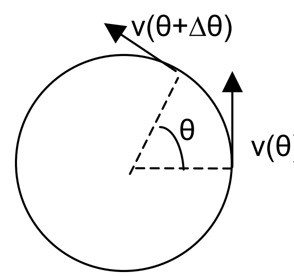
- **$\vec{a}_{tangential}$ produces changes in the ant's speed but <u>not</u> its direction.**

  $a_{tangential}$ is αR for a circle where α is the "angular acceleration" (see later)

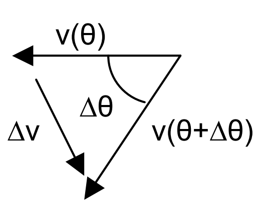
- Even in uniform circular motion where $\vec{a}_{tangential} = 0$, there is still acceleration: $\vec{a}_{radial} \neq 0$ and in fact $\vec{a}_{radial}$ is never zero for circular motion.

Details:

1) To find $\vec{a}_{radial}$, we will calculate $\frac{d\vec{v}}{dt}$ for constant speed $|\vec{v}| \equiv v$.

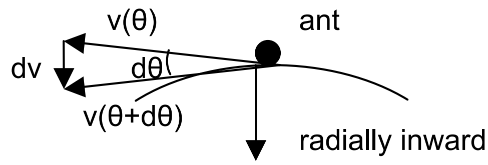


Look at the ant's velocity $\vec{v}(\theta)$ as a function of the angle θ giving its location. Consider $\vec{v}(\theta)$ changing to $\vec{v}(\theta+\Delta\theta)$ for $\theta \rightarrow \theta+\Delta\theta$ in time Δt , as depicted in the figure on the right.



Define $\Delta\vec{v} = \vec{v}(\theta+\Delta\theta) - \vec{v}(\theta)$, which corresponds to the vector addition diagram shown on the left (think: $\vec{v}(\theta + \Delta\theta) = \vec{v}(\theta) + \Delta\vec{v}$, which is equivalent, but makes it a little easier to visualize the vector addition). We assume $|\vec{v}|$ is constant: $\vec{v}$ changes only its direction but <u>not</u> its magnitude in time Δt .

As $\Delta t \to dt$ (and hence $\Delta\theta \to d\theta$), the diagram looks more like:



We can find both the direction and the magnitude of the acceleration from the above diagram:

a) $d\vec{v}$ points downward parallel to the **radially inward** direction (toward the center of the circle). Remember $|\vec{v}| \equiv v$ is constant: $\vec{v}$ changes only its direction and not its magnitude in time Δt.

b) The arc length formula ($\Delta s = r\,\Delta\theta$) extended to a "radius v" and subtended angle $d\theta$ gives $|d\vec{v}| = v\,d\theta$. Substitute this into the magnitude of the acceleration and recall $\frac{d\theta}{dt} \equiv \omega$, which is the angular velocity. We find

$$|\vec{a}_{radial}| = \frac{|d\vec{v}|}{dt} = \frac{v d\theta}{dt} = v\frac{d\theta}{dt} = v\omega$$

Also recall $v = \omega R$ on the circle, or, equivalently, $\omega = v / R$, and substitute this and – hooray! – we get the famous v^2/R formula:

$$|\vec{a}_{radial}| = v\omega = \frac{v^2}{R} \quad \text{(which is the same as } \omega^2 R\text{, too)}$$

Therefore, quite generally:

> The $\vec{a}_{radial}$ part of $\vec{a}_{total}$
> - has magnitude $\frac{v^2}{R}$ ($= v\omega = \omega^2 R$, since $v = \omega R$)
> - has direction radially inward ("centripetal" ≡ "center seeking")

Example: What is the centripetal acceleration for an ultracentrifuge for small test tubes, which whirls them in a circle of radius 10cm at 1000 revolutions per second?

Answer: First, convert f into ω rad/s, and then compute the radial acceleration:

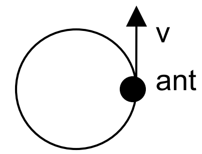
$$f = 1000 \text{ cps} \equiv 1000 \text{ Hz} = \frac{1}{T} \Rightarrow \omega = 2\pi\,(1000)$$

$$\Rightarrow a_{radial} = \omega^2 R = [2\pi(1000)]^2\,(0.1) = 3.9 \times 10^6 \text{ m/s}^2$$

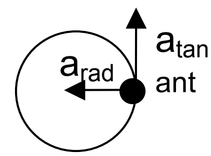
Notice this is huge compared with gravity where" one gee" = g = 9.8 m/s^2. See a later problem for practice with centripetal acceleration.

2) To find $\vec{a}_{tangential}$, we now consider the possibility that the ant's speed v changes:

We explain both the direction and the magnitude of the acceleration as follows:



a) This part of the ant's acceleration is like that for straight-line motion; it lies along the velocity vector $\vec{v}$ of the ant. See the figure on the right for a picture of $\vec{v}$, and obviously $\vec{v}$ is purely <u>tangential.</u>



This part of the acceleration would be parallel to the velocity vector if the ant was speeding up, and it would be opposite to the velocity vector if the ant was slowing down. Thus this acceleration direction is along the <u>tangent</u> to the circle as shown on the second figure on the right, and it explains why we call label it as $\vec{a}_{tangential}$

b) For the magnitude, we calculate $\frac{dv}{dt}$, starting with $v = \omega R$:

$$\frac{dv}{dt} = \frac{d}{dt}(\omega R) = \frac{d\omega}{dt}R = \alpha R$$

where we define the "angular acceleration"

$$\boxed{\alpha \equiv \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \equiv \text{angular acceleration}}$$

Therefore quite generally:

The $\vec{a}_{tangential}$ part of $\vec{a}_{total}$ • has direction along the velocity vector $\vec{v}$ • has magnitude $\lvert\alpha R\rvert$ ($\alpha > 0$ speeds up, $\alpha < 0$ slows down)

Comments:

- The angular acceleration α of a rotating body, like ω, is the same for all points in or on the body, as you might expect since $\alpha = \frac{d\omega}{dt}$. Taking the derivative of $v = \omega R_\perp$, we find $a_{tangential} = \frac{dv}{dt} = \frac{d\omega}{dt} R_\perp$ and thus $\boxed{a_{tangential} = \alpha R_\perp}$: This does depend on where you are on the rotating body, like v does.
- Later in cycle 2 we look at a pendulum example where we can discuss the roles of both radial and tangential components in $\vec{a}_{total} = \vec{a}_{radial} + \vec{a}_{tangential}$ and the forces causing them. For now it is enough to lay out constant angular acceleration formulas which are relevant and useful and have a familiar look to them - see the next page.

Constant angular acceleration formulas and their linear analogies: Suppose α is constant. Then we have analogous formulas to the linear 1D acceleration case of constant a. We show the two columns of angular and linear variables below:

Angular	Linear
$\alpha_0 = \text{constant} = \dfrac{d\omega}{dt}$	$a_0 = \text{constant} = \dfrac{dv}{dt}$
$\omega = \omega_0 + \alpha_0 t = \dfrac{d\theta}{dt}$	$v = v_0 + a_0 t = \dfrac{dx}{dt}$
$\theta = \theta_0 + \omega_0 t + \dfrac{1}{2}\alpha_0 t^2$	$x = x_0 + v_0 t + \dfrac{1}{2}a_0 t^2$
$\omega^2 - \omega_0^2 = 2\alpha_0(\theta - \theta_0)$	$v^2 - v_0^2 = 2a_0(x - x_0)$

Note that the last angular formula $\omega^2 - \omega_0^2 = 2\alpha_0(\theta - \theta_0)$ was derived by eliminating the time t in the same way we did the analogous linear formula $v^2 - v_0^2 = 2a_0(x - x_0)$.

Please do not use these formulas unless either the angular acceleration is really constant or you are estimating the motion by sticking an average value for the acceleration in them.

Examples:

1) $\alpha_0 = 0$: second hand of a clock: $\omega = \dfrac{2\pi}{60\text{ s}} = 0.105\text{ rad/s}$
2) $\alpha_0 \neq 0$: see the second part of the next problem.

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Problem 12-2 *Practice with centripetal and angular acceleration*

We can show why we don't have observable radial acceleration effects even though the Earth is spinning and we can comment quickly on the tangential acceleration we experience.

a) Specifically, show that Cleveland's centripetal (i.e., radial) acceleration due to the earth's rotation is negligible in comparison with the acceleration we feel due to gravity. Put your answer in "gees" for a direct comparison.

b) By the way, what is our angular acceleration in Cleveland? (Hoohah!) And therefore what is our tangential acceleration?

Now consider an example where we just focus on the angular variables with a nonzero angular acceleration. An old phonograph turntable is turning initially at 33 1/3 rpm. When the power to the turntable is turned off, the turntable slows down at a constant rate of 0.20 rad/s^2.

c) How many seconds elapse before the turntable stops?

d) How many revolutions will the turntable make before stopping?

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Problem 12-3

Now let's anticipate something we'll talk about in the second cycle. You can guess it right now with a high probability you'll get it right but don't spend too much time.

i) First, if a ball of mass m *is tied to the end of a rigid rod of length* L *and you grab the other end and you swing it around with constant angular velocity* ω *in a horizontal circle, what tension MUST be present in the rod. Forget about gravity (it would just pull vertically anyway) and assume the rod is so light that the tension will be the same throughout (see later cycles for understanding how the rod's mass changes things).*

ii) Second, instead, swing the ball in a vertical circle and DO include gravity, but still keep constant ω*. Now the tension is different at different points along the circular path the ball is taking. Why? When is it the largest and when is it the smallest? No formulas are asked for, just an answer in words, but with those words you are essentially solving Newton's second law along the radial direction for circular motion, taking into account two forces.*

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