PHYS 121: Cycle 2 Review Sheet, Part 1

February 19, 2024

Vectors:

We did a quick review of vectors in class. In this course the math level is such that we *assume* that you have seen vectors before specifically:

- Vectors are quantities that have both magnitude (length) and direction. Scalars have magnitude only. We indicate a vector by putting a little arrow on top: $\vec{A}$.
- You should know what we mean very precisely when we speak of the following vectors: (1) The position vector, (2) the velocity vector, and (3) the acceleration vector.
- Remember, we define *speed* as the *magnitude* of the velocity vector. This means speed is never negative.
- The magnitude of $\vec{A}$ is given by $|\vec{A}|$ or simply A. This can be confusing, and you have to infer what is going on from the context.
- Make sure that you understand *graphically* what we mean by the magnitude and direction of a vector. Remember that neither of these depends on the where the vector is placed on the graph. For vectors constrained to a two-dimensional plane, the direction is generally represented by the angle of the vector θ measured counter-clockwise from the horizontal. In other words, you should be able to look at a graphically drawn vector $\vec{A}$ and you should be able to infer the values of magnitude, A, and direction, θ.
- Make sure you can *graphically* add and subtract vectors: the so called "tail to head" rule. Also make sure that you understand what it means to multiply a vectors by a scalar number. Note that we can multiply a vector times a scalar but we can never add a vector and a scalar.
- In 2-D, be ready to switch from vector magnitude and direction to components and vice versa:

  Given the magnitude A and the direction θ find the components: A_x and A_y:

  $$A_x = A\cos\theta$$
  $$A_y = A\sin\theta$$

  Given the components A_x and A_y find the magnitude A and the direction ω:

  $$|\vec{A}| = A = \sqrt{A_x^2 + A_y^2}$$
  $$\tan\theta = \frac{A_y}{A_x}$$

Vector Representation in Cartesian Components:

Recall that we can represent any vector as a sum of three **Cartesian Components:**

$$\vec{A} = A_x\,\hat{\boldsymbol{\imath}} + A_y\,\hat{\boldsymbol{\jmath}} + A_z\,\hat{\boldsymbol{k}}$$

Note that the scalar terms A_x, A_y, and A_z are called **components**. In other words, A_x is called the "x-component of the vector $\vec{A}$. For Physics 121, Cycle 2, we will generally restrict ourselves to vectors constrained within the xy-plane:

$$\vec{A} = A_x\,\hat{\boldsymbol{\imath}} + A_y\,\hat{\boldsymbol{\jmath}}$$

Remember that the **Cartesian Unit Vectors** called "i-hat", "j-hat", etc., are *not* numbers or variables or parameters of any kind. They are **basis vectors** which have unity (=1) magnitude and which point in the three orthogonal directions indicated by the definition of a Cartesian coordinate system. In other words, unit vectors do not have a "value" – they only have a direction. They tell you where the things "points".

Note that Cartesian unit vectors remain **fixed** regardless of the motion of bodies in a given problem. In particular unit vectors are constant and do not change with time, which allows you to calculate derivatives term-by-term:

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\,\hat{\boldsymbol{\imath}} + \frac{dA_y}{dt}\,\hat{\boldsymbol{\jmath}} + \frac{dA_y}{dt}\,\hat{\boldsymbol{k}}$$

Cartesian Coordinates and the Position Vector:

Consider a position of a particle constrained to the $z = 0$ plane. Once we put down a coordinate system, the position of that particle can be represented by two plane **Cartesian coordinates**, (x, y)

We can use *Vector Representation* to define the position of the particle:

$$\vec{r} \equiv x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}}$$

This is called the **position vector**. Graphically, the tail of the vector sits at the origin, and the arrow head sits on the point particle. Here x, and y are the Cartesian *coordinates*, which also happen to be the *Cartesian components* of the position vector.

Note that in general, the particle is moving in time, so that $\vec{r}$, x, and y are implicitly function of time:

$$\vec{r}(t) \equiv x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}}$$

Don't forget, Cartesian unit vectors do *not* change with time.

We can take a derivative with respect to time to get the **velocity vector**:

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\,\hat{\boldsymbol{\jmath}} = v_x\,\hat{\boldsymbol{\imath}} + v_y\,\hat{\boldsymbol{\jmath}}$$

Likewise for the **acceleration vector:**

$$\vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\,\hat{\boldsymbol{\imath}} + \frac{dv_y}{dt}\,\hat{\boldsymbol{\jmath}} = a_x\,\hat{\boldsymbol{\imath}} + a_y\,\hat{\boldsymbol{\jmath}}$$

Also don't forget these handy relationships:

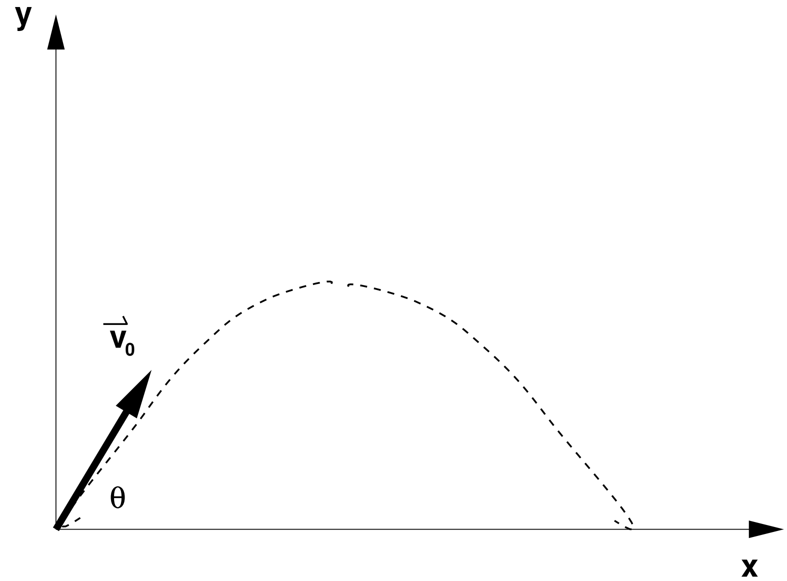
$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} = 1$$
$$\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}} = 1$$
$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} = 0$$

These lead directly to a very handy way of calculating dot products of vectors from their components:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y$$

Projectile Motion Kinematics:

General Projectile Motion Problems – Draw a Picture:



Given Initial Parameters for position: x_0 and y_0, velocity vector: $\vec{v}_0$:

Initial Parameters for velocity:

$$v_{x0} = v_0 \cos\theta$$
$$v_{y0} = v_0 \sin\theta$$

Break projectile motion into two components:

Horizontal (X-component) motion: **Constant Velocity:**

Acceleration	$a_x = 0$
Velocity:	$v_x = v_{x0}$
Position:	$x = x_0 + v_{x0}t$

Vertical (Y-component) motion: **Constant Acceleration: (Free Fall)**

Acceleration	$a_y = -g$
Velocity:	$v_y = v_{y0} - gt$
Position:	$y = y_0 + v_{y0}t - \frac{1}{2}gt^2$

Note that with the above (1-D) descriptions for the motion of projectiles, we can easily write down expressions in unit-vector notation for the position, velocity, and acceleration of a projectile:

Projectile Position Vector as a function of time:

$$\vec{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}}$$

$$\vec{r}(t) = (x_0 + v_{x0}t)\,\hat{\boldsymbol{\imath}} + \left(y_0 + v_{y0}t - \frac{1}{2}gt^2\right)\hat{\boldsymbol{\jmath}}$$

Projectile Velocity Vector as a function of time:

$$\vec{v}(t) = v_x(t)\hat{\boldsymbol{\imath}} + v_y(t)\hat{\boldsymbol{\jmath}}$$

$$\vec{v}(t) = (v_{x0})\,\hat{\boldsymbol{\imath}} + (v_{y0} - gt)\,\hat{\boldsymbol{\jmath}}$$

Projectile Acceleration Vector as a function of time:

$$\vec{a}(t) = a_x(t)\hat{\boldsymbol{\imath}} + a_y(t)\hat{\boldsymbol{\jmath}}$$

$$\vec{a}(t) = -g\hat{\boldsymbol{\jmath}}$$

Strategies for solving problems with Projectile Motion:

1. Draw a careful diagram of motion. State problem carefully in terms of your diagram. Choose an appropriate coordinate system.

2. State carefully the initial conditions, x_0, y_0, v_{x0}, v_{y0}.

3. If you are given numerical values for various initial parameters, then write down equivalent variable symbols and then use *only the symbols* and not the numbers until the very end of the problem. In other words, if you are told that the initial velocity is 22 meters per second write down $v_0 = 22$m/s and the use the variable v_0 to do your work.

4. Look for symmetries, and for circumstances where the parameters of motion, x, y, v_x, v_y are zero or are known.

5. Try solving for the time, t, in one component (usually y), and then try substituting this into your equation for the other component.

Tangential vs. Centripetal Acceleration:

In general, when we have motion on *any* smooth curved path we can calculate the total acceleration vector by breaking the acceleration in *centripetal* and *tangential* components:

$$\vec{a}_{tot} = \vec{a}_c + \vec{a}_v$$

- The **centripetal component of acceleration** is a vector that points towards to the circle at the center of the "radius of curvature". This means that $\vec{a}_c$ is always *perpendicular* to the path of the object. Remember, $\vec{a}_c$ always points "into the circle" and the magnitude is given by $a_c = \frac{v^2}{r}$ where r is the length of the radius of curvature. In other words, for a given speed v the centripetal acceleration increases with decreased r. In other words the "sharper the curve" the greater the centripetal acceleration. On a circular path, the terms centripetal acceleration and radial acceleration are synonymous. In other words, $a_c \equiv a_{rad}$.

- The **tangential component of acceleration** is a vector that always points *tangential* to the path and therefore aligned with the *velocity* of the object. The vector $\vec{a}_v$ has a magnitude that is given by:

  $$|\vec{a}_v| = \frac{d|\vec{v}|}{dt}$$

  In other words the tangential component of acceleration does not depend on the direction of the velocity at all, only upon the rate of change of the speed. If the speed is constant then the tangential acceleration is zero. If the speed is *increasing* with time then the tangential acceleration vector points *forward* along the direction of motion. If the speed is *decreasing* then the tangential acceleration vector points *backwards* in the opposite direction relative to the motion.

Example:

1. Speed is constant, path is straight, then $\vec{v}$ is constant and $\vec{a} = \vec{a}_{tot} = \vec{a}_c + \vec{a}_v = 0 + 0 = 0$
2. Speed is constant, path is curved to the left, then v is constant and $\vec{a} = \vec{a}_{tot} = \vec{a}_c + \vec{a}_v$ where $a_c = \frac{v^2}{r}$ and $a_v = 0$.
3. Speed is increasing, path is straight, then v is not constant and $\vec{a} = \vec{a}_{tot} = \vec{a}_c + \vec{a}_v$ where $a_v = \frac{dv}{dt}$ and $a_c = 0$.
4. Speed is decreasing, path is curved to the right, then v is not constant and $\vec{a} = \vec{a}_{tot} = \vec{a}_c + \vec{a}_v$ where $|a_v| = \frac{dv}{dt}$ (pointed in direction opposite to the velocity) and $a_c = \frac{v^2}{r}$ pointed toward center of circle to right.

Universal Gravity:

Newton showed that there is an attractive force that is between any two bodies that is proportional to their mass and inversely proportional to the distance squared. He called this the force of Universal Gravity:

$$F_{UG} = \frac{GMm}{r^2}$$

Note that the force of *Universal Gravity* is a non-contact force. Note also that it is a *known* force in that given the masses, you know the force. Also, you might notice that "Weight" and "Universal Gravity" are two forms of the same thing (right)? We'll do more with this in Cycle 2.

Note that the force of Universal Gravity is always **attractive**. This means that the force vector on one body always points toward the other body.