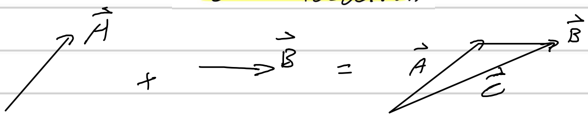
Vectors and 2D kinematics

Recall: Vectors have both magnitude and direction

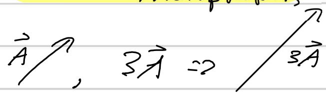
Length = "magnitude" $|\vec{A}|$

Vector Addition

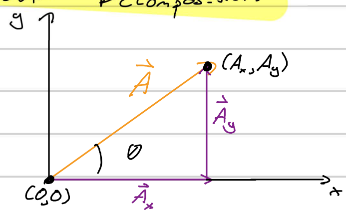


$\vec{C} = \vec{A} + \vec{B}$

Scalar Multiplication



Vector Decomposition



Vectors are defined by numbers!

$|\vec{A}| = \text{magnitude} = A$ (no → symbol)

Direction $= \theta$ (some reference angle)

Or we can break into components!

$$\vec{A} = \vec{A}_x + \vec{A}_y$$

Components

$|\vec{A}_x| = A_x = A\cos\theta$

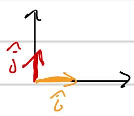
$|\vec{A}_y| = A_y = A\sin\theta$

You can find the components of a vector if given magnitude and direction

Reverse Direction: { To find A if A_x, A_y given: $A = \sqrt{A_x^2 + A_y^2}$ Wow!

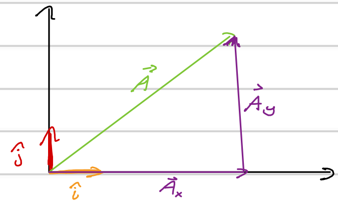
To find θ if A_x, A_y given: $\theta = \arctan\left(\frac{A_y}{A_x}\right)$ Crazy

Cartesian Basis Unit Vector



Don't use $\langle\ \rangle$ Notation in this class!

Unit Vector = length of One!



Means...

- $\vec{A}_x = A_x\hat{\imath}$
- $\vec{A}_y = A_y\hat{\jmath}$

$\therefore$ $\boxed{A = A_x\hat{\imath} + A_y\hat{\jmath}}$

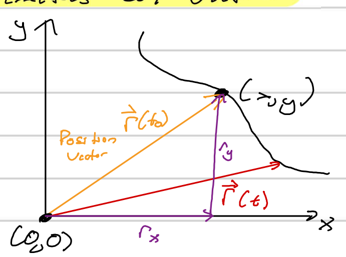
Vector Representation

Example: $\vec{C} = \vec{A} + \vec{B}$

$\begin{cases} \vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} \\ \vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} \end{cases}$ $\therefore$ $\vec{C} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath}$

Path Integrals will Come Later

Kinematics w/ Vectors



$\vec{r} = \vec{r}_x + \vec{r}_y = r_x\hat{\imath} + r_y\hat{\jmath}$

Position Vectors — a vector whose components are its own coordinates

$\vec{r} \equiv x\hat{\imath} + y\hat{\jmath}$ => $\vec{v} = \frac{d\vec{r}}{dt}$ => $\vec{a} = \frac{d\vec{v}}{dt}$

$\vec{v} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}$

$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$

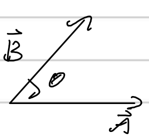
$\vec{a} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath}$

$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$

Example: $\vec{r}(t) = (Bt^2)\hat{\imath} + (Ce^{-t/\tau})\hat{\jmath}$ τ, B, C Given Constants   Find Velocity and accel

$\vec{v}(t) = (2Bt)\hat{\imath} - \left(\frac{C}{\tau}e^{-t/\tau}\right)\hat{\imath}$ and $\vec{a}(t) = (2B)\hat{\imath} - \left(\frac{C}{\tau^2}e^{-t/\tau}\right)\hat{\imath}$

Dot Products!



$C = \vec{A}\cdot\vec{B} = AB\cos\theta$

$\hat{\imath}\cdot\hat{\imath} = 1$, $\hat{\imath}\cdot\hat{\jmath} = 0$

Using Decomposition

$\therefore$ $C = (A_x\hat{\imath} + A_y\hat{\jmath})\cdot(B_x\hat{\imath} + B_y\hat{\jmath})$

FOIL $= A_xB_x(1) + A_xB_y(0) + A_yB_x(0) + A_yB_y(1)$

$$\boxed{C = A_xB_x + A_yB_y}$$

Example: Related to HW4

Define: $\vec{r}(t) = \underbrace{R\cos(\omega t)}_{x(t)}\hat{i} + \underbrace{R\sin(\omega t)}_{y(t)}\hat{j}$, R, ω given

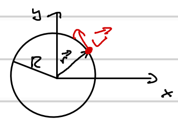
Take Derivatives of Functions of time!

$|\vec{r}| = \sqrt{x^2+y^2}$ → $|\vec{r}| = \sqrt{(R\cos(\omega t))^2 + (R\sin(\omega t))^2}$

Determine Magnitude

$= \sqrt{R^2(\cos^2(\omega t) + \sin^2(\omega t))} = \sqrt{R^2} = \boxed{R}$

Suppose



Speed $= |\vec{v}| = \sqrt{v_x^2 + v_y^2}$

$\vec{v} = \frac{d\vec{r}}{dt} = -\omega R\sin(\omega t) + \omega R\cos(\omega t)$

$\therefore |\vec{v}| = \sqrt{R^2\omega^2(\sin^2(\omega t) + \cos^2(\omega t))} = \boxed{R\omega}$

Omg! its the really constant

$\vec{a} = -R\omega^2\cos(\omega t) - R\omega^2\sin(\omega t)$

$|\vec{a}| = \sqrt{R^2\omega^4(\cos^2(\omega t) + \sin^2(\omega t))}$

$= \sqrt{R^2\omega^4} = R\omega^2 = \frac{v^2}{R}$

$\boxed{a = \frac{v^2}{R} = a_c}$

woah! Its centripetal acceleration

Projectile Motion

x-direction:

no air resistance → velocity is constant

$$\boxed{\begin{aligned} a_x &= 0 \\ v_x &= \text{Const} = v_{x_0} \\ x &= x_0 + v_{x_0} t \end{aligned}}$$

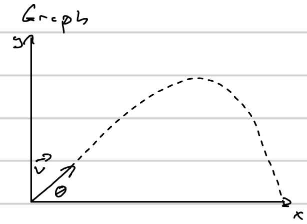
Two sets of equations to describe an objects motion

y-direction:

Free Fall

$$\boxed{\begin{aligned} a_y &= -g \\ v_y &= v_{y_0} - gt \\ y &= y_0 + v_{y_0} t - \tfrac{1}{2}gt^2 \end{aligned}}$$

Graph



we say: $v_{x_0} = v_0\cos\theta$ and $v_{y_0} = v_0\sin\theta$

$\therefore \vec{r} = x\hat{i} + y\hat{j}$

Motion of Projectile in vector form

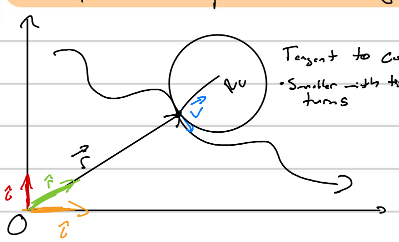
Identical to relations shown above

$\vec{r}(t) = (x_0 + v_{x_0}t)\hat{i} + (y_0 + v_{y_0}t - \frac{1}{2}gt^2)\hat{j}$

$\vec{v}(t) = (v_{x_0})\hat{i} + (v_{y_0} - gt)\hat{j}$

$\vec{a}(t) = 0\hat{i} - g\hat{j}$

Example: A particle on a given path



Tangent to curve

• Smaller with tighter turns

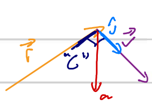
$\vec{r} = x\hat{i} + y\hat{j}$

Unit vector in the r direction

$$\boxed{\begin{aligned} \vec{r} &= r\,\hat{r} \\ \vec{v} &= v\,\hat{v} \end{aligned}}$$

where $r = |\vec{r}|$ and $|\hat{r}| = 1$ r is mag, $\hat{r}$ is dir

where $v = |\vec{v}|$ and $|\hat{v}| = 1$ v is mag, $\hat{v}$ is dir



$\vec{a} = a_x\hat{i} + a_y\hat{j}$

$\vec{a} = \underbrace{a_v\hat{v}}_{\text{Tangential Component}} + \underbrace{a_c\hat{c}}_{\text{Centripetal Component}}$

$\hat{c}$ is $\perp$ to $\hat{v}$

magnitude

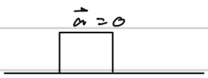
$a_v = \frac{d}{dt}(v)$ where v is the speed

$a_c = \frac{v^2}{R_c}$

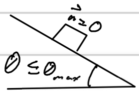
Both depend on speed

R_c is radius of Curvature

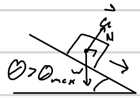
Forces at Angles



$F_{Net} = 0$



$F_{Net} = 0$



$F_{Net} > 0$

θ_{max} is the maximum angle required for the body to remain at rest.

Steps

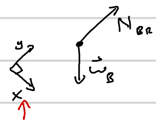
1. FBD
2. Coordinate System
3. Kinematic Constraint
4. N2L
   - 4.1 Decompose Forces

Only decompose forces Not aligned with the chosen coordinate System!

Decomposing Necessary forces into their necessary components is very important

Given: m, θ; find (i) a and (ii) all forces

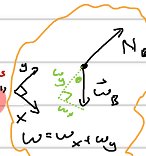
FBD Block



avoids having to break net force into components

"tipped coordinate"

Kinematic Constraint
$a_x = a$
$a_y = 0$

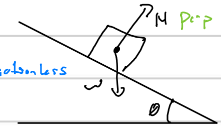


$w = w_x + w_y$

$\cos\theta = \frac{w_y}{w}$

$w_y = w\cos\theta$

$w_x = w\sin\theta$



frictionless

N perp to surface

N2L $F_{Net} = ma$ Y-dir

$F_y = ma_y$ ← Normal force and y component of the weight force

$N - w_y = ma_y$

$N - w\cos\theta = 0$

$N - mg\cos\theta = 0$

$\boxed{N = mg\cos\theta}$

N2L $F_{Net} = ma$ x-dir

$w_x = ma_x$ ← only w_x in x dir

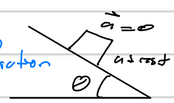
$w\sin\theta = ma$

$mg\sin\theta = ma \Rightarrow \boxed{a = g\sin\theta}$

Small $\theta \rightarrow$ Small a

Incr. $\theta \rightarrow a$ approaching g

Suppose there is friction (Static)



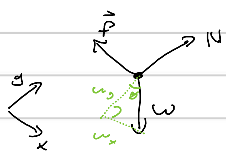
friction

$\vec{a} = 0$ at rest

Given m, θ; Find (i) all forces

$a = 0$ as at rest

FBD Block



$a_x = 0 \quad a_y = 0$

N2L

x: $F_x = ma_x$

$w_x - f = 0$

$w_x = w\sin\theta = mg\sin\theta$

$mg\sin\theta - f = 0$

$\boxed{f = mg\sin\theta}$

y: $F_y = ma_y$

$N - w_y = 0$

$w_y = w\cos\theta = mg\cos\theta$

$\boxed{N = mg\cos\theta}$

Suppose there is Kinetic Friction

Given μ_K, m, θ Find (i) a (ii) all forces

N2L

y: $F_y = ma_y$

Same as the bit to the left

$\boxed{N = mg\cos\theta}$

$f = \mu_K N$

$\boxed{f = \mu_K mg\cos\theta}$

$f = \mu_K N$

x: $w_x - f = ma_x$

$mg\sin\theta - f = ma$

$mg\sin\theta - \mu_K N = ma$

$N = mg\cos\theta$

$mg\sin\theta - \mu_K mg\cos\theta = ma$

$\cancel{m}g(\sin\theta - \mu_K\cos\theta) = \cancel{m}a$

$\boxed{a = g(\sin\theta - \mu_K\cos\theta)}$