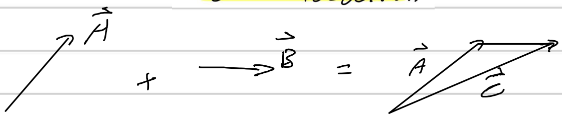
Vectors and 2D kinematics

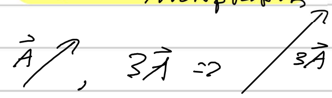
Recall: Vectors have both magnitude and direction — Length = "magnitude" $|\vec{A}|$

Vector Addition



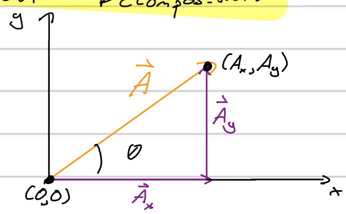
$\vec{C} = \vec{A} + \vec{B}$

Scalar Multiplication



$\vec{A}$, $3\vec{A}$ => $3\vec{A}$

Vector Decomposition



Vectors are defined by numbers!

$|\vec{A}|$ = magnitude = A (no → symbol)

Direction = θ (some reference angle)

Or we can break into components!

$$\vec{A} = \vec{A}_x + \vec{A}_y$$

Components

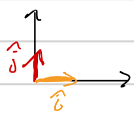
$|\vec{A}_x| = A_x = A\cos\theta$

$|\vec{A}_y| = A_y = A\sin\theta$

You can find the components of a vector if given magnitude and direction

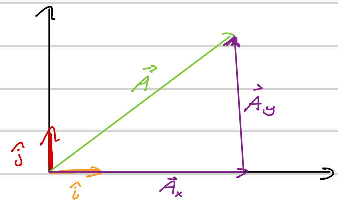
Reverse Direction:
- To find A if A_x, A_y given: $A = \sqrt{A_x^2 + A_y^2}$ Wow!
- To find θ if A_x, A_y given: $\theta = \arctan\left(\frac{A_y}{A_x}\right)$ Crazy

Cartesian Basis Unit Vector



Don't use $\langle\ \rangle$ Notation in this class!

Unit Vector = Length of One!



Means...
- $\vec{A}_x = A_x\hat{i}$
- $\vec{A}_y = A_y\hat{j}$

$\therefore$ $\boxed{A = A_x\hat{i} + A_y\hat{j}}$

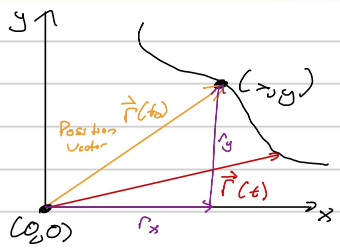
Vector Representation

Example: $\vec{C} = \vec{A} + \vec{B}$

$$\begin{cases} \vec{A} = A_x\hat{i} + A_y\hat{j} \\ \vec{B} = B_x\hat{i} + B_y\hat{j} \end{cases} \quad \therefore \quad \vec{C} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

Path Integrals will Come Later

Kinematics w/ Vectors



$\vec{r} = \vec{r}_x + \vec{r}_y = r_x\hat{i} + r_y\hat{j}$

Position Vectors — a vector whose components are its own coordinates

$\vec{r} \equiv x\hat{i} + y\hat{j}$ => $\vec{v} = \frac{d\vec{r}}{dt}$ => $\vec{a} = \frac{d\vec{v}}{dt}$

$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$

$\vec{v} = v_x\hat{i} + v_y\hat{j}$

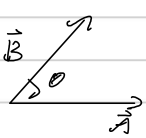
$\vec{a} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j}$

$\vec{a} = a_x\hat{i} + a_y\hat{j}$

Example: $\vec{r}(t) = (Bt^2)\hat{i} + (Ce^{-t/\tau})\hat{j}$ τ, B, C Given Constants Find Velocity and accel

$\vec{v}(t) = (2Bt)\hat{i} - \left(\frac{C}{\tau}e^{-t/\tau}\right)\hat{j}$ and $\vec{a}(t) = (2B)\hat{i} - \left(\frac{C}{\tau^2}e^{-t/\tau}\right)\hat{j}$

Dot Products!



$C = \vec{A}\cdot\vec{B} = AB\cos\theta$

$\hat{i}\cdot\hat{i} = 1$, $\hat{i}\cdot\hat{j} = 0$

Using Decomposition

$\therefore C = (A_x\hat{i} + A_y\hat{j})\cdot(B_x\hat{i} + B_y\hat{j})$

FOIL $= A_xB_x(1) + A_xB_y(0) + A_yB_x(0) + A_yB_y(1)$

$$\boxed{C = A_xB_x + A_yB_y}$$

Example: Related to HW4

Define: $\vec{r}(t) = \underbrace{R\cos(\omega t)}_{x(t)}\hat{i} + \underbrace{R\sin(\omega t)}_{y(t)}\hat{j}$, R, ω given

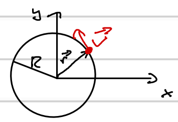
Take Derivatives of functions of time!

$|\vec{r}| = \sqrt{x^2+y^2}$ → $|\vec{r}| = \sqrt{(R\cos(\omega t))^2 + (R\sin(\omega t))^2}$

Determine Magnitude

$= \sqrt{R^2(\cos^2(\omega t) + \sin^2(\omega t))} = \sqrt{R^2} = \boxed{R}$

Suppose



Speed $= |\vec{v}| = \sqrt{v_x^2+v_y^2}$

$\vec{v} = \frac{d\vec{r}}{dt} = -\omega R\sin(\omega t) + \omega R\cos(\omega t)$

$\therefore |\vec{v}| = \sqrt{R^2\omega^2(\sin^2(\omega t) + \cos^2(\omega t))} = \boxed{R\omega}$

Omg! its the radius constant

$\vec{a} = -R\omega^2\cos(\omega t) - R\omega^2\sin(\omega t)$

$|\vec{a}| = \sqrt{R^2\omega^4(\cos^2(\omega t) + \sin^2(\omega t))}$

$= \sqrt{R^2\omega^4} = R\omega^2 = \frac{v^2}{R}$

$\boxed{a = \frac{v^2}{R} = a_c}$

woah! its centripital acceleration

Projectile Motion

x-direction:

no air resistance → velocity is constant

$$\boxed{\begin{aligned} a_x &= 0 \\ v_x &= \text{Const} = v_{x_0} \\ x &= x_0 + v_{x_0}t \end{aligned}}$$

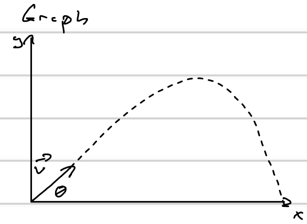
Two sets of equations to describe an objects motion

y-direction:

Free Fall

$$\boxed{\begin{aligned} a_y &= -g \\ v_y &= v_{y_0} - gt \\ y &= y_0 + v_{y_0}t - \tfrac{1}{2}gt^2 \end{aligned}}$$

Graph



we say: $v_{x_0} = v_0\cos\theta$ and $v_{y_0} = v_0\sin\theta$

$\therefore \vec{r} = x\hat{i} + y\hat{j}$

Motion of Projectile in vector form

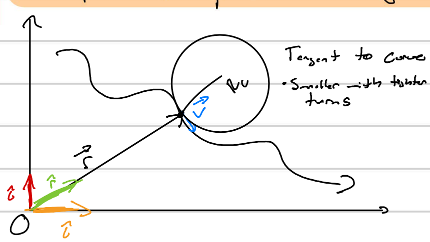
Identical to relations shown above

$\vec{r}(t) = (x_0 + v_{x_0}t)\hat{i} + (y_0 + v_{y_0}t - \frac{1}{2}gt^2)\hat{j}$

$\vec{v}(t) = (v_{x_0})\hat{i} + (v_{y_0} - gt)\hat{j}$

$\vec{a}(t) = 0\hat{i} - g\hat{j}$

Example: A particle on a given path



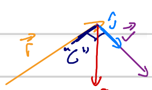
Tangent to curve
• Smaller with tighter turns

$\vec{r} = x\hat{i} + y\hat{j}$

Unit vector in the r direction

$\boxed{\begin{aligned} \vec{r} &= r\hat{r} \\ \vec{v} &= v\hat{v} \end{aligned}}$ where $r = |\vec{r}|$ and $|\hat{r}| = 1$ r is mag, $\hat{r}$ is dir

where $v = |\vec{v}|$ and $|\hat{v}| = 1$ v is mag, $\hat{v}$ is dir



$\vec{a} = a_x\hat{i} + a_y\hat{j}$

$\vec{a} = \underbrace{a_v\hat{v}}_{\text{Tangential Component}} + \underbrace{a_c\hat{c}}_{\text{Centripital Component}}$

R_c is radius of Curvature

$\hat{c}$ is $\perp$ to $\hat{v}$

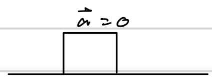
magnitude

$a_v = \frac{d}{dt}(v)$ where v is the speed

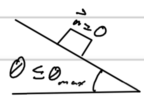
$a_c = \frac{v^2}{R_c}$

Both depend on speed

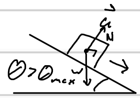
Forces at Angles



$F_{Net}=0$



$F_{Net}=0$



$F_{Net} > 0$

θ_{max} is the maximum angle required for the body to remain at rest.

Steps

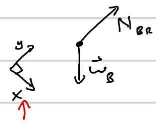
1. FBD
2. Coordinate System
3. Kinematic Constraint
4. N2L
   - 4.1 Decompose Forces

Only decompose forces Not aligned with the chosen coordinate System!

Decomposing Necessary forces into their necessary components is very important

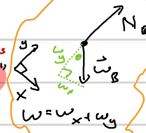
Given: m, θ; find (i) a and (ii) all forces

FBD Block

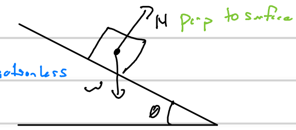


avoids having to break net force into components
"tipped coordinate"

Kinematic Constraint
$a_x = a$
$a_y = 0$



$\cos\theta = \frac{w_y}{w}$
$w_y = w\cos\theta$
$w_x = w\sin\theta$



N2L $F_{Net} = ma$ Y-dir

$F_y = ma_y$ ← Normal force and y component of the weight force

$N - w_y = ma_y$

$N - w\cos\theta = 0$

$N - mg\cos\theta = 0$

$\boxed{N = mg\cos\theta}$

N2L $F_{Net} = ma$ x-dir

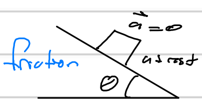
$w_x = ma_x$ ← only w_x in x-dir

$w\sin\theta = ma$

$mg\sin\theta = ma \Rightarrow \boxed{a = g\sin\theta}$

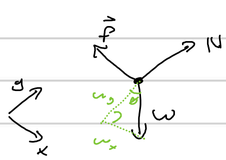
Small θ → Small a
Incr. θ → a approaching g

Suppose there is friction CSF me



Given m, θ; Find (i) all forces
$a=0$ as at rest

FBD Block



$a_x = 0 \quad a_y = 0$

N2L

x: $F_x = ma_x$

$w_x - f = 0$

$w_x = w\sin\theta = mg\sin\theta$

$mg\sin\theta - f = 0$

$\boxed{f = mg\sin\theta}$

y: $F_y = ma_y$

$N - w_y = 0$

$w_y = w\cos\theta = mg\cos\theta$

$\boxed{N = mg\cos\theta}$

Suppose there is Kinetic Friction

Given μ_K, m, θ Find (i) a
(ii) all forces

$f = \mu_K N$

N2L

y: $F_y = ma_y$
Same as the bit to the left

$\boxed{N = mg\cos\theta}$

$f = \mu_K N$

$\boxed{f = \mu_K mg\cos\theta}$

x: $w_x - f = ma_x$

$mg\sin\theta - f = ma$

$mg\sin\theta - \mu_K N = ma$

$N = mg\cos\theta$

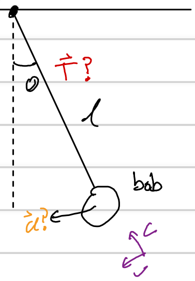
$mg\sin\theta - \mu_K mg\cos\theta = ma$

$\cancel{m}g(\sin\theta - \mu_K\cos\theta) = \cancel{m}a$

$\boxed{a = g(\sin\theta - \mu_K\cos\theta)}$

Pendulums

Jumping right in with an example



Circular Motion!
- c points radially inward
- v is tangential

Pendulum bob with mass m released at rest angle θ

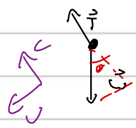
Q1 What is the tension on string?

Q2 What is the acceleration of bob immediately after its released?

Given mass m, θ, and length of string l

a_c and a_v must be considered

FBD Bob



N2L : c-coord

$F_c = m a_c$

$F_c = m\left(\frac{v^2}{R}\right) = m\left(\frac{0^2}{l}\right) = 0$

$T - W\cos\theta = 0$

$T - mg\cos\theta = 0 \rightarrow$ ① $\boxed{T = mg\cos\theta}$

N2L : v-coord

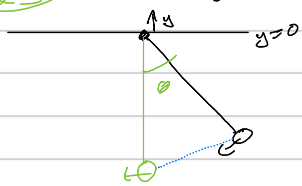
$F_v = m a_v$

$W\sin\theta = m a_v$

$mg\sin\theta = m a_v$

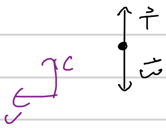
② $\boxed{a_v = g\sin\theta}$

Q3 What is the Tension on string when bob at lowest position?



Still using coordinates as path still circular

FBD Bob



N2L c-coord

$F_c = m a_c$

$T - w = m\left(\frac{v^2}{l}\right)$

$T - mg = m\left(\frac{v^2}{l}\right)$ stop!

v is unknown here!

$T - mg = \frac{m v_e^2}{l}$ and $v' = \sqrt{2gl(1-\cos\theta)}$

Let $v' = v_e$

$T = mg + \frac{m\left(2gl(1-\cos\theta)\right)}{l}$

$T = mg + 2mg(1-\cos\theta)$

$T = mg\left(1 + 2(1-\cos\theta)\right)$

$T = mg\left(1 + 2 - 2\cos\theta\right) \rightarrow \boxed{T = mg(3 - 2\cos\theta)}$

Find v

CofME $\vec{w}$ conservative

$\vec{T}$ does no work as $\perp$ to path

Before = After

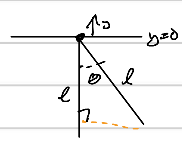
"Before" = release

"After" = lowest position

Conditions met!

$E_{Tot} = E'_{Tot}$

$U + K = U' + K'$



$mgy + \frac{1}{2}mv^2 = mgy' + \frac{1}{2}mv'^2$

$y = -l\cos\theta$, $y' = -l$

$v = 0$, $v' = ?$

$mg(-l\cos\theta) + \frac{1}{2}m(0)^2 = mg(-l) + \frac{1}{2}mv'^2$

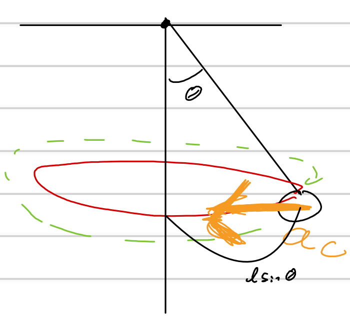
$-\cancel{m}gl\cos\theta = -\cancel{m}gl + \frac{1}{2}\cancel{m}v'^2$

$gl - gl\cos\theta = \frac{1}{2}v'^2$

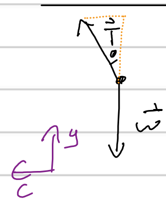
$v'^2 = 2gl(1-\cos\theta)$

$\therefore \boxed{v' = \sqrt{2gl(1-\cos\theta)}}$

Conical Pendulums



FBD Bob



N2L y-coord

$F_y = m a_y$ $a_y = 0$

$T\cos\theta - W = 0$

$\boxed{T = \frac{mg}{\cos\theta}}$

N2L c-coord

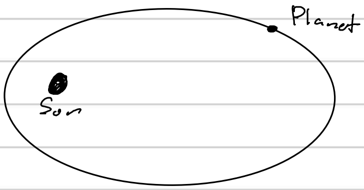
$F_c = m a_c$

$T\sin\theta = m\frac{v^2}{l\sin\theta}$

$\frac{mg\sin\theta}{\cos\theta} = \frac{mv^2}{l\sin\theta}$

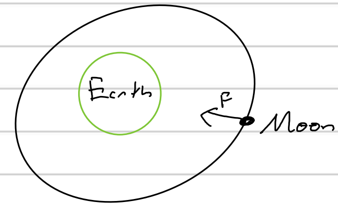
$mg\tan\theta = \frac{mv^2}{l\sin\theta}$, solve for v

Planets!



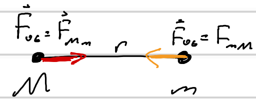
Kepler's 3rd Law (K3L)

$$\frac{P^2}{R^3} = \frac{(\text{Period})^2}{(\text{"radius"})^3} = \text{constant?}$$



Universal Gravity

$$F_{UG} = \frac{G \cdot M \cdot m}{r^2}$$

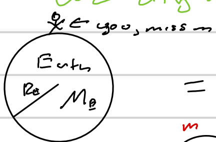


G = Gravitational Constant

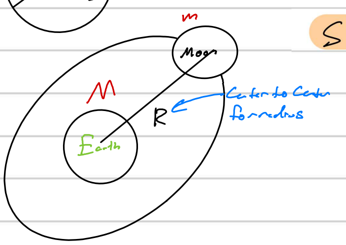
$= 6.67 \times 10^{-11}\ N \cdot m^2 \cdot kg^{-2}$ ← Super small

↑ we only notice it b/c of earth's size

$$F_{UG} = \frac{G(1)(1)}{1} = G$$ ← barely anything at all



$$\Rightarrow F_{UG} = \frac{G M m}{R_E^2} = w = mg \rightarrow \boxed{g_E = \frac{G M_E}{R_E^2}}$$ ← where "g" comes from!



Situation $M \gg m$

Orbit ≈ Circular

(1) what are the forces on the moon

N2L Moon

$F_{moon} = m a_c$

$$\frac{G M \cancel{m}}{R^2} = \cancel{m} \frac{v^2}{R}$$

Solve for $\frac{P^2}{R^3}$

$\rightarrow$ Period $\frac{2\pi R}{P} = v$ — $\frac{\text{distance}}{\text{time}}$

$$\frac{GM}{R^2} = \frac{4\pi^2 R^2}{P^2 R}$$

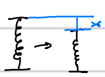
$$GM = 4\pi^2 \left(\frac{R^3}{P^2}\right) \rightarrow \boxed{\frac{P^2}{R^3} = \frac{4\pi^2}{GM}}$$ ← K3L back!

Force Menu

Symbol	what	Known?	Conservative?	Contact?
$\vec{w}$	weight	$w = mg$	Yes	No
$\vec{N}$	Normal Push	No	No	Yes
$\vec{T}$	Tension Pull	No	No	Yes
$\vec{f}$	Friction	Sometimes	Absolutely No!	Yes ← Known if Kinetic and given μ_K
$\vec{F}_{UG}$	Universal Gravity	Yes	Yes	No
$\vec{F}_{sp}$				

Hooke and Springs

"Spring"



Compressed or Stretched

x is the change in length

$F_{sp} = K|x|$ ← x is displacement doesn't care about actual spring length

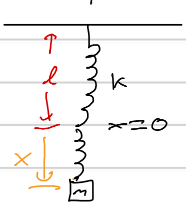
Force points in the direction opposite to x

$F_{sp} = K|x|$ — Larger K equates to increased force

K = spring constant — with equal displacement compared to small K

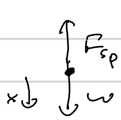
x = displacement relative to a relaxed spring

Example



what is x s.t. mass m in equilibrium? → a=0

FBD



N2L x

$F_x = m a \rightarrow 0$

$w - F_{sp} = 0$

$mg - Kx = 0$

$$\boxed{x = \frac{mg}{K}}$$

$w = mg$

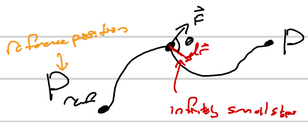
$F_{sp} =$

You can only assume a=0 when mass or object is in equilibrium

Definition of Potential Energy

$$U_{\vec F}(P) - U_{ref}(P_{ref}) = -\int_{P_{ref}}^{P} \vec F \cdot d\vec r$$

Path integral



$\vec F \cdot d\vec r = F dr \cos\theta$

Change w/ time

W.K.: $W_{cons} = -\Delta U_F$

work done by a Conservative force

Computing the integral

Weight w/ coord P_y } for weight

Pot. Energy

$$U_{\vec w}(y) - U_{ref}(y_{ref}) = -\int \vec w \cdot d\vec r$$

$\rightarrow \vec w \cdot d\vec r = mg\, dy \cos(180°)$

$= -mg\, dy$

mg of weight; $\vec w$; $d\vec r$; y_{ref}

Angle btwn. $\vec w$ and $d\vec r$ is 180°

$$U_{\vec w}(y) - U_{ref}(\cancel{y_{ref}=0})^{0} = -\int_{y=0}^{y'=y} -(mg\, dy)$$

$d\vec r$ mag is dy

$$\boxed{U_{\vec w}(y) = mgy}$$

Just an easy integral

Set ref to 0 in PHYS121, not always true in PHYS122

Calculating for Spring Force

Spring force against motion

$$U_{sp}(x) - U_{ref}(x_{ref}) = -\int \vec F_{sp} \cdot d\vec r$$

x_{ref} $d\vec r$ F_{sp} x

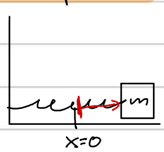
$\vec F_{sp} \cdot d\vec r = Kx\, dx \cos(180°)$

$|\vec F_{sp}|$ $= -Kx\, dx$ $|d\vec r|$

$$U_{sp}(x) - \cancel{U_{ref}(x=0)}^{0} = -\int_{x=0}^{x'=x} -(Kx'dx')$$

$$\boxed{U_{sp}(x) = \tfrac{1}{2}Kx^2}$$

Example



Release at rest at $x = x_i$. Find max speed

Given: K, m, x_i

Consider C of ME when asked to find speed

All conservative or no work done

"Before" = "After"

$U_{sp} + K = U'_{sp} + K'$

$\tfrac{1}{2}Kx^2 + \tfrac{1}{2}mv^2 = \tfrac{1}{2}Kx'^2 + \tfrac{1}{2}mv'^2$

$x = x_i,\ v = 0,\ x' = 0\ \ v' = v_f?$

$\tfrac{1}{2}Kx_i^2 + 0 = 0 + \tfrac{1}{2}mv_f^2$

$\therefore \boxed{v_f = x_i\sqrt{\tfrac{K}{m}}}$

Computing for Universal Gravity

$$U_{UG}(r) - U_{ref}(r_{ref}) = -\int \vec F_{UG} \cdot d\vec r$$

$d\vec r$ $\vec F_{UG}$ r_{ref} r

$$\vec F_{UG} \cdot d\vec r = \frac{GMm}{r^2} dr \cos 180°$$

$$= -\frac{GMm}{r^2} dr$$

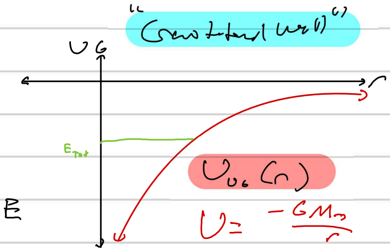
F_{UG} goes to 0 when $r = \infty$

can't be 0, be divided by 0

$$U_{UG}(r) - \cancel{U_{ref}(r_{ref}=\infty)}^{0} = -\int_{r=\infty}^{r'=r} -\left(\frac{GMm}{r^2}\right) dr$$

$$\boxed{U_{UG}(r) = -\frac{GMm}{r}}$$

"Gravitational well"



$U_{UG}(r)$

$U = -\frac{GMm}{r}$

Conservative Forces

$\vec w$ Weight $U_{\vec w} = mgy$

$\vec F_{sp}$ Spring Force $U_{sp} = \tfrac{1}{2}Kx^2$

$\vec F_{UG}$ Universal Gravity $U_{UG} = -\frac{GMm}{r}$

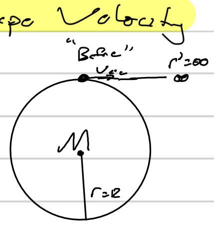
C of ME

$E_{Tot} = E'_{Tot}$

$U + K = U' + K'$ ← $U^{(')}$ can be a sum of different potential Energies

Escape Velocity

← Ignoring Air resistance, high K sends an object out of atmosphere as $\frac{1}{r^2}$ decreases



After: $v' = 0$

C of ME

$U + K = U' + K'$

$-\frac{GMm}{r} + \tfrac{1}{2}mv^2 = -\frac{GMm}{r'} + \tfrac{1}{2}mv'^2$

$-\frac{GMm}{R} + \tfrac{1}{2}mv_{esc}^2 = 0 + 0$

Algebra $\boxed{V_{esc} = \sqrt{\frac{2GM}{r}}}$ $\approx 1.2 \times 10^4$ m/s or ~7 miles/sec

Thrown with some velocity V_{esc}. Initial radius = R. Masses of object not needed. Planet mass M. Thrown Knowing the following is 0 and $r' = \infty$. Find the escape velocity.

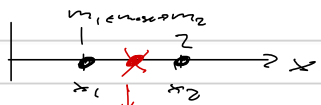
More on Systems

N2L for Systems

$$F_{\substack{Net\\Sys\\Ext}} = M_{sys}\, a_{sys}$$

works for systems of n objects and directions only with CoM

Center of Mass (CoM)



$$X_{AVE} = \frac{x_1 + x_2}{2}$$

CoM is not an actual thing, just a concept

Suppose $m_1 \neq m_2$, find weighted Avg

2pts: $$\boxed{X_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$ (X_{cm}: com) weighted avg equation

N pts: $$X_{cm} = \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{m_1 + m_2 + \dots + m_n}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + \dots + m_n y_n}{m_1 + m_2 + \dots + m_n}$$

2 coordinates with n bodies with given mass and positions

Time derivatives of CoM

$$\frac{dx_{cm}}{dt} = V_{cm} = \frac{m_1 V_1 + m_2 V_2}{m_1 + m_2}$$

derivative of a sum is the sum of derivatives

$$\frac{dv_{cm}}{dt} = a_{cm} = \frac{m_1 a_1 + m_2 a_2}{m_1 + m_2}$$

Can be expanded to N points

N2L System

$$\vec{F}_{\substack{Net\\Sys\\Ext}} = M_{sys}\, \vec{a}_{cm}$$

vector because it applies to any number of dimensions

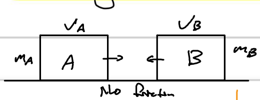
if $F_{\substack{Sys\\Net\\Ext}} = 0 \Rightarrow$ "Isolated" $\Rightarrow$ $a_{cm} = 0$

$V_{cm} = \text{Constant}$ "conserved"

$M_{sys} \cdot V_{cm} = P_{Tot}$ CofLM Momentum!

Inertial reference frame is defined to be at rest or moving with constant velocity

Totally Elastic Collisions



$K = K'$ is what totally elastic means

$$\therefore \tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B v_B^2 = \tfrac{1}{2} m_A v_A'^2 + \tfrac{1}{2} m_B v_B'^2$$

CofK

Isolated ⇒ $P_{Tot} = P_{Tot}'$ 1eq 2unknowns CofLM

$$m_A V_A + m_B V_B = m_A V_A' + m_B V_B'$$

Annoying and a waste of time to solve this quadratically

Use this instead!

Bob Brown Master Eq

$$\boxed{V_A - V_B = V_B' - V_A'}$$

Only true in
- Totally Elastic Collisions
- 1-D Situations

Must be true!!!

Center of Mass Reference Frame

$$V_{cm} = \frac{m_A V_A + m_B V_B}{m_A + m_B} = \dots$$

Constant!

can be calculated if m_A, m_B, V_A, and V_B given

Such as that above!

Choose coordinate System that moves with center of mass (CoM) of System!

Relate to CoM Reference Frame V_{cm} is 0 → $V_{cm} = 0$

Example: $m_A = m_B = m$

$V_A = V_0 \quad V_B = 0$

① $m v_0 = m v_A' + m v_B'$

② $v_0 = v_B' - v_A'$

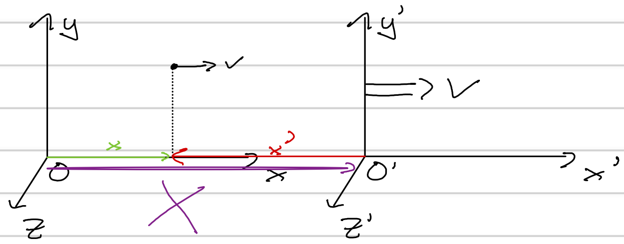
$v_B' = v_0 + v_A'$ into ①

$m v_0 = m v_A' + m(v_0 + v_A')$

$2 v_A' = 0 \Rightarrow v_A' = 0$

② → $V_B' = V_0$

Reference Frames



$x' = x - X$

$x' = x - (X_0 + Vt)$

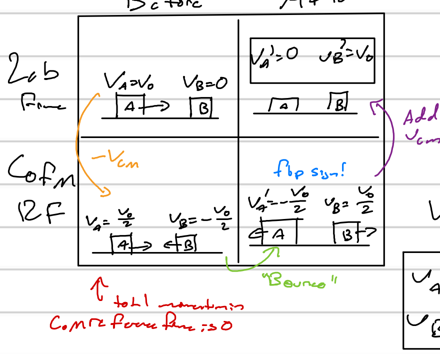
$\boxed{v' = v - V}$

∴ $X = X_0 + Vt$ constant velocity

- Lab = stationary still
- Center-of-mass Ref. Frame

$V_{CoM\,RF} = V_{Lab} - V_{cm}$

Coulouts 4-box method



$V_{cm} = \frac{m_A v_A + m_B v_B}{m_A + m_B}$

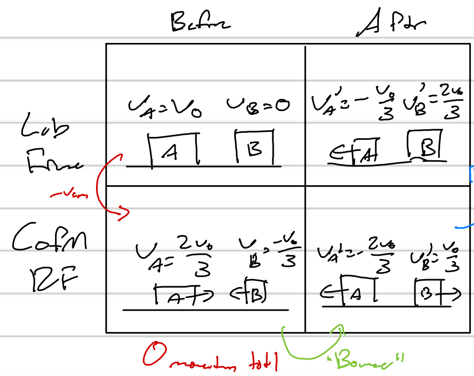
$= \frac{m_A V_0}{m_A + m_B}$ $m_A = m_B = m$

$V_{cm} = \frac{m v_0}{m + m} = \frac{v_0}{2}$

$\boxed{v_A' = 0 \quad v_B' = V_0}$

Example: $m_A = m$ $m_B = 2m$

$v_A = V_0$ $v_B = 0$

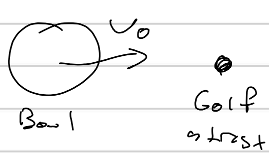


$V_{cm} = \frac{m v_0 + 0}{m + 2m}$

$V_{cm} = \frac{v_0}{3}$

$\boxed{v_A' = -\frac{V_0}{3} \quad v_B' = \frac{2v_0}{3}}$

Example: Totally elastic with rocky guns



Estimate V_{cm}!

Let Bowl mass = m_B

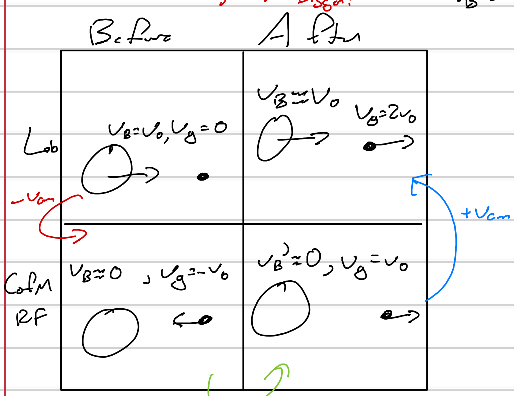
Let Golf mass = m_G

Much Bigger! ⟶ $m_B >> m_G$

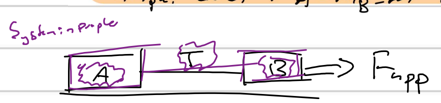
$v_{cm} = \frac{m_B v_0 + 0}{m_B + m_G}$ ← assume negligible terms

$\approx \frac{m_B v_0}{m_B + 0} = v_0$

If one mass much greater, assume it dominates V_{cm}



Example: Cars $m_A = m_B = m$ and F_{app} ← F_{app} always given in 101 / result of using rope



Frictionless

Find Tension

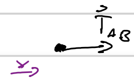
Kinematic Constant

accel? $a_A = a_B = a$

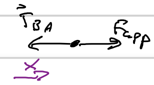
Shortcut!

FBD A

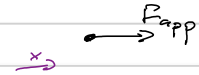
Ignore vertical forces



FBD B



FBD System



N2L x-dir System (A+B)

$F_{x\,sys} = m_{sys}\, a_{sys}$

$F_{app} = 2m\, a$

∴ $a = \frac{F_{app}}{2m}$

N2L A x-dir

$F_x = m_A a_A$

$T_A = m\left(\frac{F_{app}}{2m}\right)$ ← Calculated already

∴ $\boxed{T = \frac{F_{app}}{m}}$

Example n number of blocks

No friction

Given: m, N, F_{app}. Find T_i

$a_1 = a_2 = a = a_{sys} = ?$

FBD System ← all blocks

$\longrightarrow F_{app}$

N2L x-dir Sys

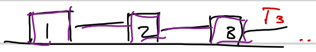
$F_{xsys} = M_{sys} a_{sys}$

$F_{app} = (Nm) a$

$a = \frac{F_{app}}{Nm}$

To find T_i, define a system where the boundary of the system is at the point where that force is applied

Suppose $i = 3$



New system = leftmost i blocks

FBD new system

$\longrightarrow T_i$

only force on FBD of system, oh well! ☺

N2L new system

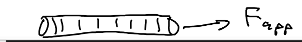
$F_{sys} = M_{sys} a_{sys}$

$T_i = (im) \frac{F_{app}}{Nm}$

$$\therefore \boxed{T_i = \frac{i}{N} F_{app}}$$

Example: Massive Rope

Massive Rope { length ℓ, mass m }, F_{app} given



Friction less

$T(x)$: Tension in rope a distance x from left end

FBD Rope

$\longrightarrow F_{app}$

N2L Rope System

$F_{xsys} = m_{sys} a$

$F_{app} = m a$

$a = \frac{F_{app}}{m}$

New System $T(x)$ applied to left side up to x

FBD

$\longrightarrow T(x)$

$$\therefore \boxed{T(x) = \left(\frac{x}{\ell}\right) F_{app}}$$

N2L new system

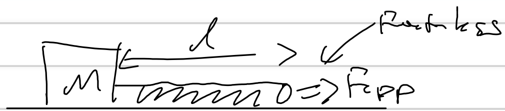
$F_{xsys} = m_{sys} a$

$T(x) = \left(\frac{x}{\ell} m\right) a$

$T(x) = \left(\frac{x}{\ell} m\right)\left(\frac{F_{app}}{m}\right)$

ratio of Rope times tension

Example: Massive rope pulls block



Block M mass

Rope m mass

F_{app}, ℓ

Find $T(x)$

N2L whole

$F_{sys} = M_{sys} a$

$F_{app} = (M+m) a_{sys}$

$a_{sys} = \frac{F_{app}}{M+m}$

New System = Boundary @ x

FBD

$\longrightarrow T(x)$

N2L new system

$F_{new\,sys} = M_{sys} a_{sys}$ ← $M + \frac{x}{\ell} m$

$T(x) = \left(\left(\frac{x}{\ell}\right) m + M\right)\left(\frac{F_{app}}{M+m}\right)$

simplify

$T(x) = \left(\frac{xm}{\ell} + M\right)\frac{F_{app}}{M+m}$

$$\therefore \boxed{T(x) = \left(\frac{M + \frac{x}{\ell} m}{M+m}\right) F_{app}}$$

Example: Fun

moving up @ constant velocity

m_s, m_p, m_b all given

what is Tension in rope?

System defined this way to include everything

FBD System

$\uparrow 4\vec{T}$

$\downarrow W_{sys}$

4 contact points

$\uparrow y$

N2L System

$F_{sys} = m_{sys} a_{sys}$ (0, constant v)

$4\vec{T} - M_{sys} g = 0$

$\vec{T} = \frac{M_{sys} g}{4}$

$$\boxed{\vec{T} = \frac{(m_s + m_p + m_b) g}{4}}$$

Rotations!

Translational Motion	Rotational Motion
Kinematics: x, v, a	Kinematics: θ, ω, α
Dynamics: $F_{Net} = ma$	Dynamics: $\tau_{Net} = I\alpha$
N2L	N2L Rot

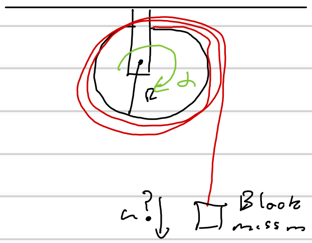
Review of Motion Components

New Force

$\vec{H} =$ "Hinge Force"

axles, cranks, pin

Example: Massive Pulley



Given

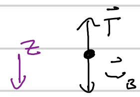
Find accel of block

- Radius of Pulley
- Block of Mass m
- Pulley Mass is 4m
  - ↳ Pulley Mass = m_p
- $I_p = \frac{1}{2} m_p R^2$

$a = ?$ $\alpha = ?$

↳ given for now!

FBD Block



N2L Block

$F_{Bz} = m_B a_{Bz}$

$w_B - T = ma$

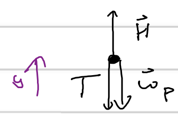
$mg - T = ma$ Stop!

Always assume string/rope is not slipping or sliding on the Pulley

Pulley has $a = 0$ so a Force must be pointing upward to hold the pulley

Hinges/Axels apply force in any direction needed to satisfy N2L

FBD Pulley



N2L Pulley

$F_{Py} = m_P a_{Py}$ ($a_{Py} = 0$)

$H - T - w_P = 0$

$H - 4mg - T = 0$ Stop!

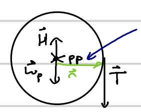
Two equations w/ Too many unknowns

1. $mg - T = ma$
2. $H - 4mg - T = 0$

Always worth a try to solve problem w/o rotational motion

- Weight Force always @ CoM
- Coordinate System is lined up w/ rotation

Extended FBD (XFBD) Pulley ← No longer a point-like object



N2L Rot Pulley

$\tau_{Net} = I_p \alpha$

$\tau_w + \tau_H + \tau_T = (\frac{1}{2} m_p R^2)\alpha$

$0 + 0 + RT\sin 90 = \frac{1}{2} m_p R^2 \alpha$

0 distance from PP for W and H so $\tau = 0$

$\therefore RT = \frac{1}{2}(4m)R^2 \alpha$

Pulley $\alpha = \frac{a}{R}$ Rolling Constraint

R.C is acceptable

$RT = 2mR^2(\frac{a}{R})$

$\therefore T = 2ma$

Recall: $mg - 2ma = ma$

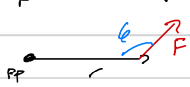
$mg = 2ma + ma$

$g = a(2+1)$

$$\boxed{a = \frac{1}{3}g}$$

Definition of Torque

$\tau_F \equiv rF\sin\theta$



Important Equation!

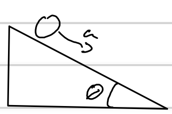
Rolling Constant

$a = \alpha r$, $v = \omega r$

Tangential acceleration, rotational, radius from pp

Very Helpful for HW6 #6

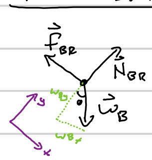
Example: Ball on Ramp



Given

- $\theta, R, m, I = \frac{2}{5} m R^2$
- Rolls w/ no slip or slide

FBD ball



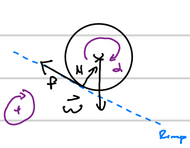
N2L x-coord

$F_{Bx} = m_B a_{Bx}$

$mg\sin\theta - f = ma$

Stop!

XFBD ball



N2L Rot ball

$\tau_{net} = I\alpha$

$\tau_w + \tau_N + \tau_f = (\frac{2}{5} m R^2)\alpha$

$0 + RN(\sin 180°) + Rf = \frac{2}{5} m R^2 \alpha$

$\alpha = \frac{a}{R}$ R.C

$Rf = \frac{2}{5} m R^2 \frac{a}{R}$

$f = \frac{2}{5} ma$

↳ $mg\sin\theta - f = ma$

$\cancel{m}g\sin\theta - \frac{2}{5}\cancel{m}a = \cancel{m}a$

$g\sin\theta = \frac{7}{5}a$

$$\boxed{a = \frac{5}{7} g\sin\theta}$$

Review Lecture

Topics

2D Kinematics

Vectors

- $\vec{A}$: $|\vec{A}|$ is magnitude $= A$
- Notation: $\vec{A} = A_x\hat{i} + A_y\hat{j}$
- Projectile Motion, UCM
- Centripetal vs. Tangential Acceleration

↳ $\vec{a} = \vec{a}_x + \vec{a}_y$

$= \vec{a}_v + \vec{a}_c$

$a_v = \frac{d}{dt}(\text{speed})$

$a_c = \frac{v^2}{R_c}$ for R_c - radius of curvature

Conservative Forces

Know Formula

All Topics Outlined on the Cheat Sheets

Dynamics

Forces at Angles

- Tipped Coordinate System in direction of Acceleration

New Forces

Both Conservative

- Universal Gravity $F_{UG} = \frac{GMm}{R^2}$ → RWB's might Be there
- Spring Forces $F_{sp} = k|x|$ → 4-box Method

x is displacement

Systems of Multiple Bodies

- Massive Rope
- Massive Pulley ⇐ Rotational Motion

Systems

N2L

$F_{Net,\,Sys,\,Ext} = M_{sys}a_{cm}$

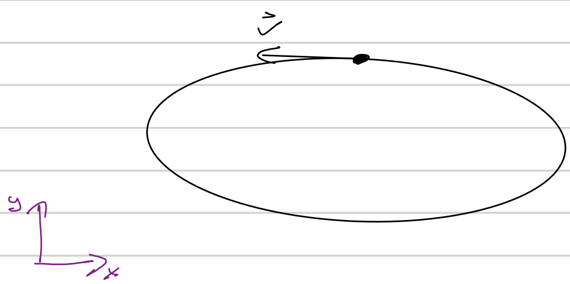
CoM apply and calculate

Elastic Collisions

$K_{Tot} = K'_{Tot}$

- CoM Frame vs. Lab Frame

First Exam Topics all Fair Game



$\vec{v} = -v\hat{i}$

speed → slows

$\frac{d(\text{speed})}{dt} = \text{negative}$

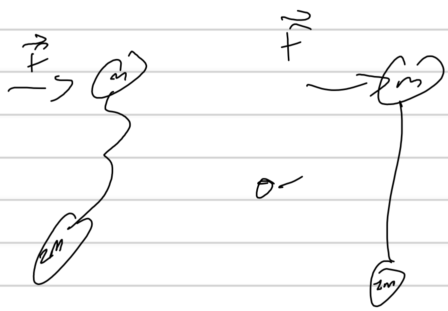
$\vec{a}_v = +P\hat{i}$

$\vec{a}_c = -Q\hat{j}$

$\boxed{P\hat{i} - Q\hat{j}}$

B

$\text{Impulse} \equiv \int F\,dt \equiv \Delta p = F_{avg}\cdot\Delta t$



More on Rotations

Summary of Important Collisions

Torque: $\tau_P = rF\sin\theta$

N2L Rot: $\tau_{Net} = I\alpha$

R.C: $v = \omega r$ and $a = \alpha r$

Values for Rotational Inertia (I)

Textbook / Labs may call it moment of Inertia

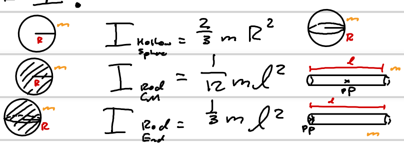
$I = C m R^2$, for some const C

Common Values for I:

$I_{Hoop} = mR^2$ $\quad$ $I_{Hollow\ Sphere} = \frac{2}{3}mR^2$

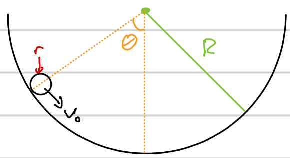
$I_{Disk} = \frac{1}{2}mR^2$ $\quad$ $I_{Rod\ CM} = \frac{1}{12}m\ell^2$

$I_{Solid\ Sphere} = \frac{2}{5}mR^2$ $\quad$ $I_{Rod\ End} = \frac{1}{3}m\ell^2$



Good to have on Cheatsheet to reference easily

Example: Hollow ball rolling in bowl



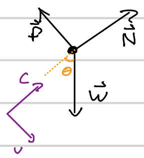
Given

- m, r, v_0, θ of ball, no slip/slide
- R of Bowl
- $I = \frac{2}{3}mr^2$

Find Normal Force, Friction force, a

Assume $R \gg r$

FBD Ball



N2L c-coord centripetal

$F_c = ma_c$

$N - W\cos\theta = m\frac{v_0^2}{R}$ radius of bowl as motion defined by the bowl

$N = mg\cos\theta + \frac{mv_0^2}{R}$

$\boxed{N = m\left(g\cos\theta + \frac{v_0^2}{R}\right)}$

N2L v-coord tangential

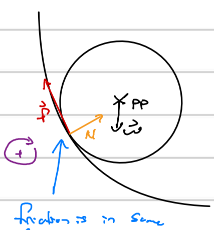
$F_v = ma_v$

$W\sin\theta - f = ma_v$

Stop!

Now Contd with Rotational

xFBD Ball



friction is in same direction as coordinate sys so torque is positive

N2L Rot

$\tau_{Net} = I\alpha$

$\tau_W + \tau_N + \tau_P = I\alpha$

$0 + 0 + rf\sin(90°) = I\alpha$

$\cancel{r}f = \frac{2}{3}mr^{\cancel{2}}\alpha$ $\quad$ $\alpha = \frac{a_v}{r}$ R.C

$f = \frac{2}{3}mr\left(\frac{a_v}{r}\right)$

Stop! $f = \frac{2}{3}ma_v$

① $\tau_W: 0W\sin\theta = 0$

② $\tau_N: rN\sin(180°) = 0$

Checking θ in torque

force is positive for torque if it is aligned with coordinate system.

Two Equations, Two Unknowns

① $mg\sin\theta - f = ma_v$

② $f = \frac{2}{3}ma_v$

$\cancel{m}g\sin\theta - \frac{2}{3}\cancel{m}a_v = \cancel{m}a_v$

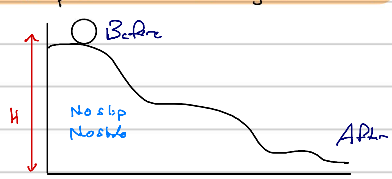
$g\sin\theta = \frac{5}{3}a_v$

$\boxed{a_v = \frac{3}{5}g\sin\theta}$

$f = \frac{2}{3}m\left(\frac{3}{5}g\sin\theta\right)$

$\boxed{f = \frac{2}{5}mg\sin\theta}$

Example: Ball rolling on slide



Given: Release @ rest

$m, R, I = \frac{2}{5}mR^2$ of ball

H of ball initially

Translational Kinetic: $K_T = \frac{1}{2}mv^2$

Rotational Kinetic: $K_R = \frac{1}{2}I\omega^2$

$\therefore K_{Tot} = K_T + K_R = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

Friction does "no work" @ point of contact

$E_{Tot} = E'_{Tot}$

$U + K_T + K_R = U' + K_T' + K_R'$

$mgy + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgy' + \frac{1}{2}v'^2 + \frac{1}{2}I\omega'^2$

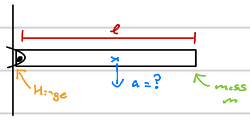
$y = H,\ y' = 0,\ v = 0,\ \omega = 0,\ \omega = \frac{v}{r}$ R.C

$mgH + 0 + 0 = 0 + \frac{1}{2}mv_p^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v_T}{r}\right)^2$

$\cancel{m}gH = \frac{1}{2}\cancel{m}v_p^2 + \frac{1}{5}\cancel{m}v_T^2$ $\quad$ $v_T = v_P$

$gH = \frac{7}{10}v_0^2 \rightarrow \boxed{v_P = \sqrt{\frac{10}{7}gH}}$

Rod on Hinge

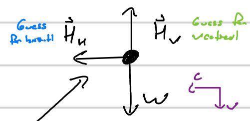


① Find Acceleration of CoM @ release. Rod starts @ rest!

Not worth spending a lot of time determining direction of H components

CoM is moving on circular path so use v/c-coordinates.

FBD Rod



Hinge Force is completely unknown! Just assign two orbiting forces to represent it. One is vertical one is horizontal

If you mis-guess direction, answer is negated. Just note that and move on to get positive back.

N2L Horizontal c-coord

$F_c = m a_c$

$H_H = m \frac{v^2}{"R"}$ @ rest so $v = 0$

$\therefore H_H = 0$ woah!

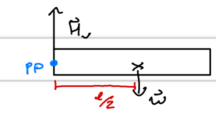
N2L Vertical v-coord

$F_v = m a_v$

$W - H_v = m a_v$

$mg - H_v = m a_v$ Stop!

x FBD Rod



$\circlearrowleft +$ Neglect H_H b/c it is 0 from N2L

N2L Rot

$\tau_{Net} = I\alpha$ Known for Rod

$\tau_{H} + \tau_W = \frac{1}{3} m \ell^2 \alpha$

$0 + \frac{\ell}{2} mg(\sin 90°) = \frac{1}{3} m \ell^2 \alpha$

$\frac{1}{2} \cancel{\ell} \cancel{m} g = \frac{1}{3} \cancel{m} \ell^{\cancel{2}} \left(\frac{a_v}{\cancel{\ell}/2}\right)$

$\frac{1}{2} g = \frac{1}{3} 2 a_v$

$\boxed{a_v = \frac{3}{4} g}$

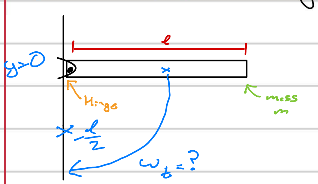
R.C. $a = \alpha r$

$\alpha = \frac{a}{r}$

$r = \frac{\ell}{2}$

r is radial distance from PP to CoM here!

Another Rod on Hinge



Hinge Force does no work be its @ point of contact. No Force applied over a distance

CofME Choose End of Rod as Reference, no translational K!

Conditions: Wnc = 0 Conserved and Hinge does no work @ point of contact!

Before = After

$E_{Tot} = E'_{Tot}$

$U + K_T + K_R = U' + K'_T + K'_R$

$U + 0 + K_{Rod} = U' + 0 + K'_{Rod}$

$mgy + 0 + \frac{1}{2} I \omega^2 = mgy' + 0 + \frac{1}{2} I \omega'^2$

$0 + 0 + 0 = mg\left(-\frac{\ell}{2}\right) + \frac{1}{2}\left(\frac{1}{3} m \ell^2\right)\omega_f^2$

Algebra!

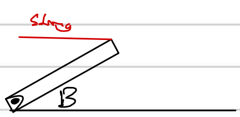
$\frac{1}{2} \cancel{m} g \cancel{\ell} = \frac{1}{6} \cancel{m} \ell^{\cancel{2}} \omega_f^2$

$\boxed{\omega_f = \sqrt{\frac{3g}{\ell}}}$

If you choose a different Reference Point, It's different!

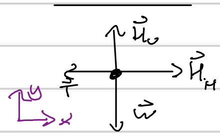
You will still get the right answer if you use a different reference point!

Hinge with a string



$a = 0, \alpha = 0$

FBD Rod



At rest! so H_H and the must hold rod in place

N2L y-dir

$F_y = m \cancel{a_y}$ 0

$H_v - W = 0$

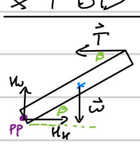
$\boxed{H_v = mg}$

N2L x-dir

$F_x = m \cancel{a_x}$ 0

$H_H - T = 0$ Stop!

x FBD Rod



$\tau_H = 0$

$\tau_T = \ell T \sin\beta$

$\tau_W = \left(\frac{\ell}{2}\right) mg \sin(90 - \beta)$

$\tau_W = \frac{\ell}{2} mg \cos\beta$ Trig!

Adofi

Statics

Static Friction

$f_s \le \mu_s N$

Constrained

Kinetic Friction

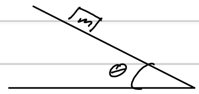
$f_k = \mu_k N$

Equality

For Statics, $a=0$ and $\alpha=0$

Use N2L to find N → N2L to find f_s → Apply constraint

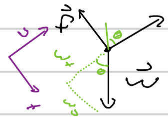
Example



Given: m, θ, μ_s

Find θ_{max}

FBD Block



N2L y-dir

$F_y = m\cancel{a_y}^{\,0}$

$N - W\cos\theta = 0$

$N = mg\cos\theta$

N2L x-dir

$F_x = m\cancel{a_x}^{\,0}$

$W\sin\theta - f_s = 0$

$f_s = mg\sin\theta$

Example shows $f_s \le \mu_s N$ is not useless! Good for determining maximums!

No an Equal Sign!

Apply Constraint: $f_s \le \mu_s N$

$mg\sin\theta \le \mu_s\, mg\cos\theta$

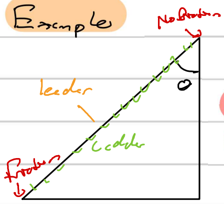
$\frac{\sin\theta}{\cos\theta} \le \mu_s$

$\tan\theta \le \mu_s$

$\theta \le \arctan\mu_s$

$\therefore\ \boxed{\theta_{max} = \arctan(\mu_s)}$

Example

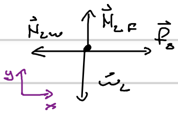


Given

m, θ, L

Find all forces

FBD Ladder



N2L y-dir

$F_y = m\cancel{a_y}^{\,0}$

$N_{LF} - W = 0$

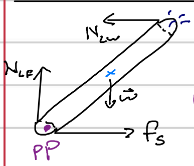
$\boxed{\begin{aligned} N_{LF} &= mg \quad (1) \\ W &= mg \quad (2) \end{aligned}}$

N2L x-dir

$F_x = m\cancel{a_x}^{\,0}$

$f_s - N_{LW} = 0$ Stop!

→FBD Ladder



Place PP where it kills most forces

N2L Rot

$\tau_{Net} = I\cancel{\alpha}^{\,0}$

$\tau_{N_{LF}} + \tau_{f_s} + \tau_w + \tau_{N_{LW}} = 0$

$0 + 0 - \frac{L}{2}W\sin\theta + LN_{LW}\sin(90-\theta) = 0$

$-\frac{1}{2}\cancel{L}mg\sin\theta + \cancel{L}N_{LW}\cos\theta = 0$

$N_{LW}\cos\theta = \frac{1}{2}mg\sin\theta$

$\boxed{N_{LW} = \frac{1}{2}mg\tan\theta}$ (3)

Recall: $\sin(90-\theta) = \cos\theta$

$\cos(90-\theta) = \sin\theta$

$f_s - N_{LW} = 0$

$f_s = N_{LW}$

$\boxed{f_s = \frac{1}{2}mg\tan\theta}$ (4)