## PHYS 121: Second Hour Exam (Exam #2)

**March 22, 2024**

Do not open this exam until instructed to do so.

**Important: Only these vector forces allowed on your Free-Body-Diagrams:**

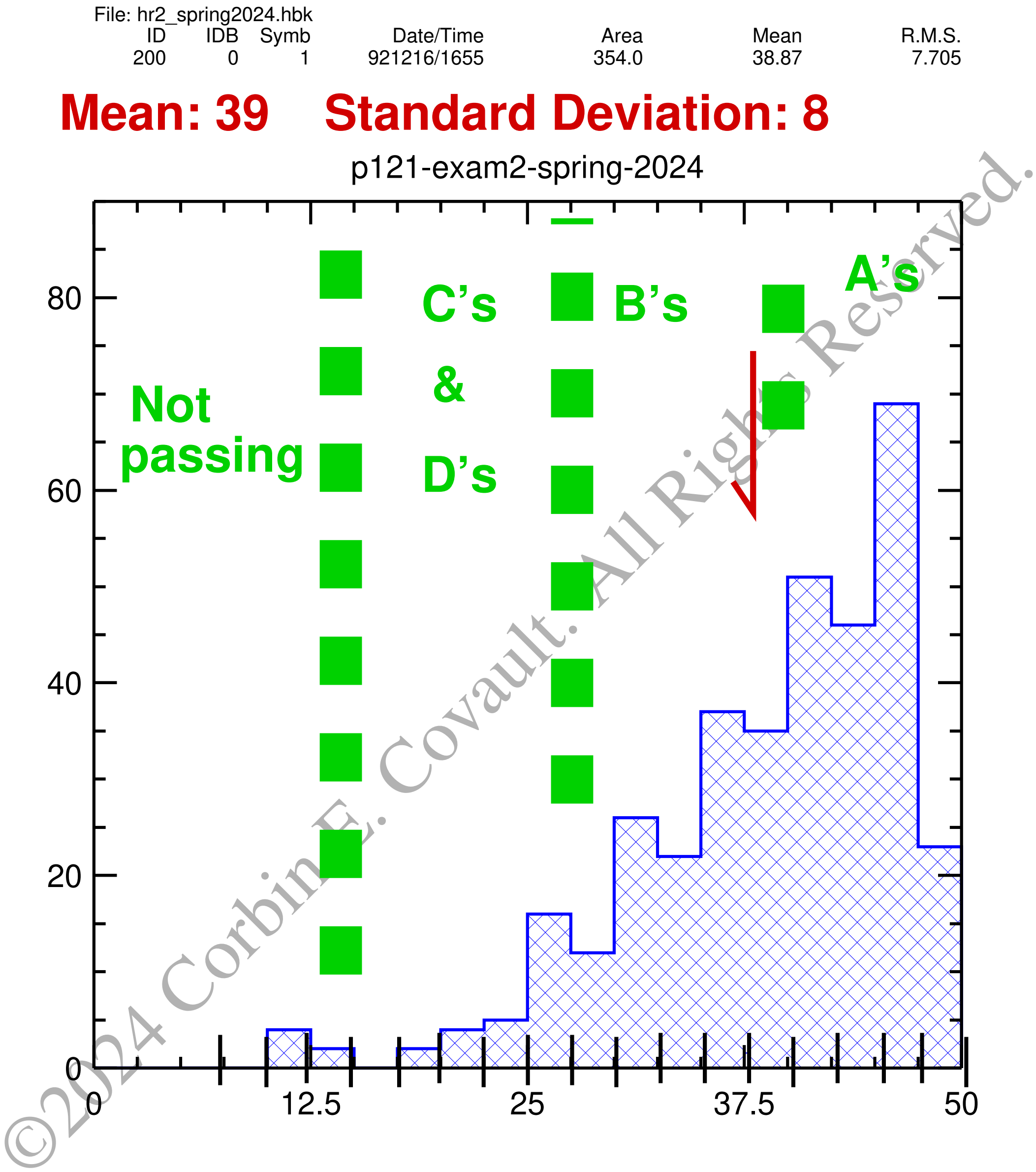
- $\vec{W}$ = Weight Force
- $\vec{N}$ = Normal Force
- $\vec{T}$ = Tension Force
- $\vec{f}$ = Friction Force
- $\vec{F}_{app}$ = Applied force
- $\vec{F}_{sp}$ = Spring force
- $\vec{F}_{ug}$ = Force of Universal Gravity
- $\vec{H}$ = Hinge Force (force due to an axle or pin)

**Example Notation for force subscripts:**

- $\vec{T}$ = "The Tension Force."
- $\vec{W}_A$ = "The Weight Force *on* Body A."
- $\vec{N}_{BT}$ = "The Normal Force *on* Body B *due* to the Table."
- $f_{AB}$ = "The magnitude of the Friction Force *on* Body A *due* to Body B."
- $F_c$ = "The magnitude of the *net* force in the *centripetal* direction."
- $F_{Ax}$ = "The magnitude of the *net* force on Body A in the $x$-direction."

## Physics 121 Spring 2024:

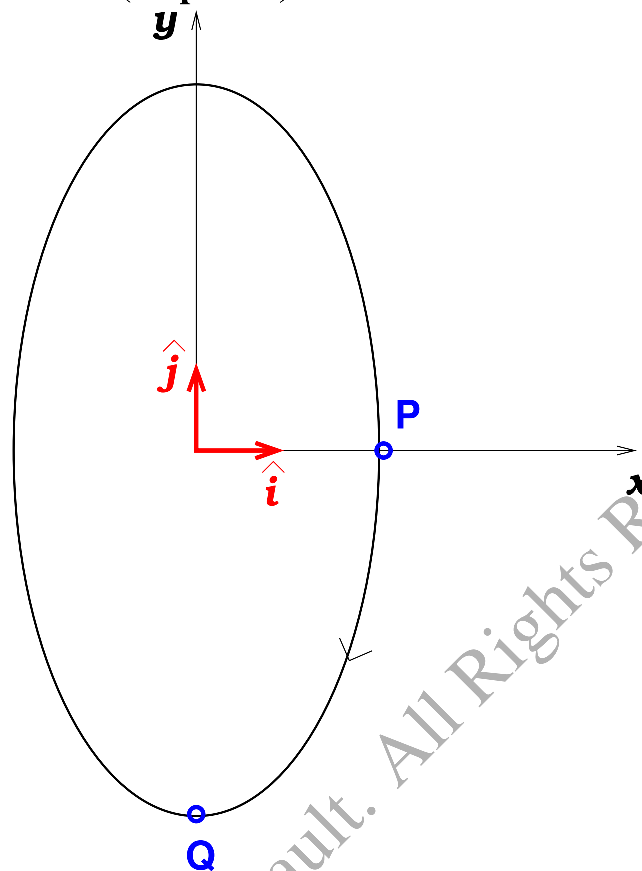
## Second Hour exam grade boundaries are VERY VERY VERY APPROXIMATE and subject to change





**41 and higher = "A"**
**28 thru 40 = "B"**
**20 thru 27 = "C"**
**13 thru 19 = "D"**

**Problem 1: An Oval Path (15 points)**






A particle of given mass $m$ moves clockwise around an oval path as shown above according to the following equation for the position vector as a function of time:

$$\vec{r}(t) = C\cos(\omega t)\,\hat{\imath} - 2C\sin(\omega t)\,\hat{\jmath}$$

where $C$ and $\omega$ are given positive parameters and $\hat{\imath}$ and $\hat{\jmath}$ are standard Cartesian unit vectors as shown.

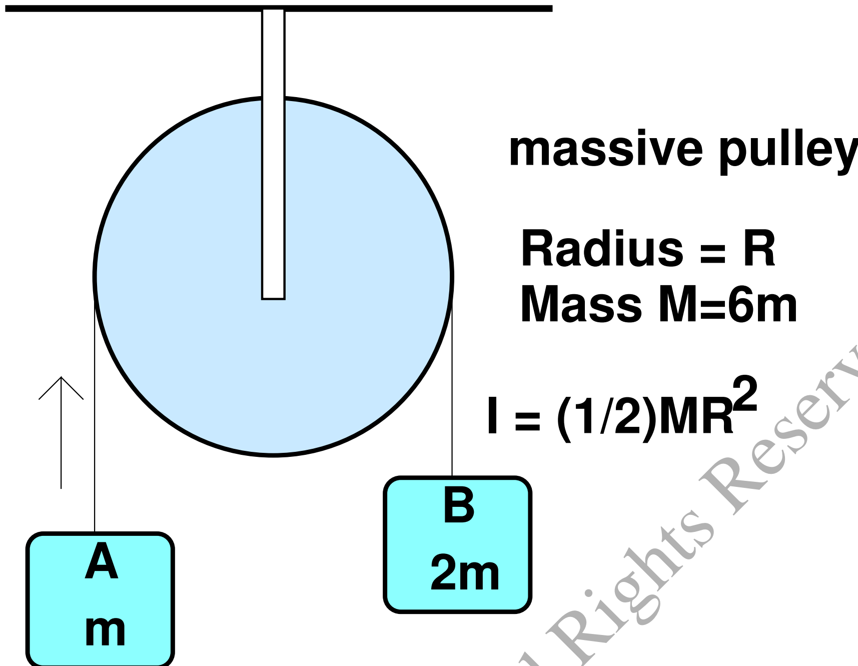
**Part (a) 5 pts:** – Point "**P**" corresponds to the position of the particle at time $t = 0$. What is the speed of the particle at point **P**? Give your answer in terms of the given parameters. Explain your work.

**Part (b) 5 pts:** – Write down a *vector expression* that gives the ***net force*** on the particle when it is at point **P** in terms of the given parameters and the unit vectors as indicated. Explain how you know this. Show your work.

**Part (c) 5 pts** – Point "**Q**" corresponds to the position of the particle at time $t = \frac{\pi}{2\omega}$. What is the **radius of curvature** when the particle is at point **Q** as shown? Give your answer in terms of the given parameters. Explain how you know this.

*Some possibly useful trigonometric facts:* $\sin(0) = 0$, $\cos(0) = 1$, $\sin(\frac{\pi}{2}) = 1$, and $\cos(\frac{\pi}{2}) = 0$.

**Problem 2: Atwood's Machine – (20 points)**






Two blocks are suspended by an ***massive pulley*** which is attached to the ceiling as shown above. The masses of the blocks are $m_A = m$ and $m_B = 2m$ and the mass of the pulley $M = 6m$ where $m$ is given. Assume that the string is ideal and turns the pulley without slipping or sliding. The radius of the pulley is given as $R$. Assume that the rotational inertia of the pulley is that of a uniform disk: $I = \frac{1}{2}MR^2$.

**Part (a) – (5 points):** Draw careful and properly labeled **Free-Body Diagrams (FBDs)** that shows *all* of the forces on each of the two blocks. Be sure to include a coordinate system for each FBD. Also, draw a careful and properly labeled **Extended Free-Body Diagram (XFBD)** for the massive pulley.

**Part (b) – (5 points):** Determine the magnitude of the **upward acceleration of Block A**. Give your answer in terms of the given parameters. Explain your work. ***Hint:*** First apply Newton's Second Law to each of the two blocks. Then apply Newton's Second Law for Rotations to the massive pulley. Note that because the pulley is massive, the tension on Block B is not equal to the tension on Block A. You will need to apply the Rolling Constraint. You are looking to arrive to *three equations with three unknowns*. Solve these for the acceleration.

**Part (c) – (5 points):** Determine the magnitude of the **force of tension on Block A**. Give your answer in terms of the given parameters. Explain your work.

**Part (d) – (5 points):** Determine the magnitude of the **force of tension on Block B**. Give your answer in terms of the given parameters. Explain your work.

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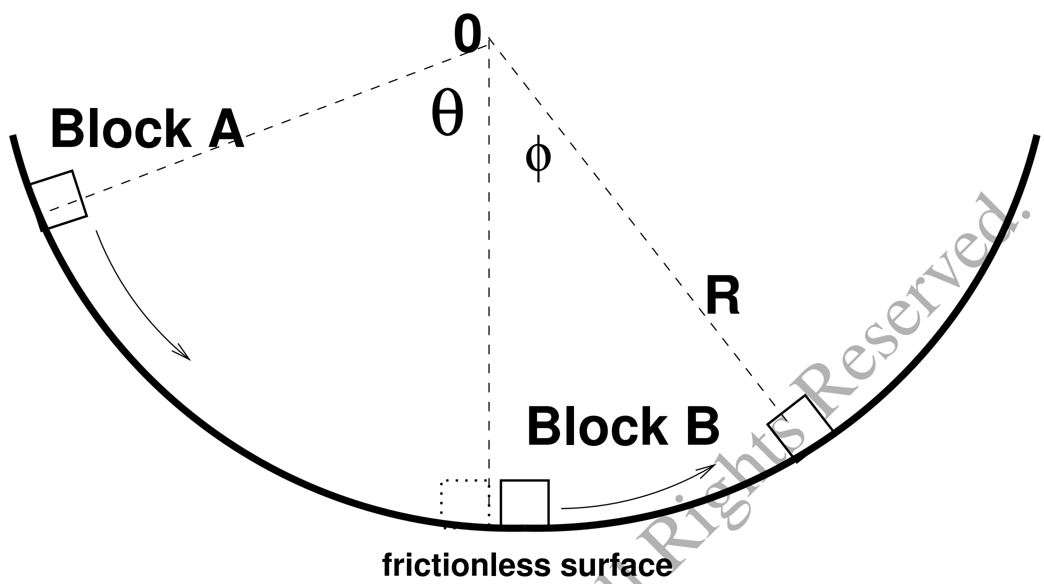
**Problem 3: Circular Bowl and Elastic Collision (15 points)**






Block A of given mass $m_a = M$ is placed on the inside surface of a circular bowl released at rest at the position corresponding to a given angle $\theta$ relative to the vertical as shown. After release, the block slides without friction, eventually colliding with Block B at the bottom of the bowl which has a given mass $m_b = 3M$. The collision is **totally elastic**. After the collision, Block B continues up the other side of the bowl to reach a maximum position corresponding to the angle $\phi$ as shown. Assume that the size of the blocks is small compared to the radius of the bowl.

**Part (a)** (5 points) – What is the magnitude of ***acceleration of Block A***? immediately after it has been released? Express your answer in terms of the given parameters only. Explain how you got your answer.

**Part (b)** (5 points) – What is the ***speed of Block A*** that is achieved just before it collides with Block B at the bottom of the bowl? Express your answer in terms of the given parameters only. Explain how you got your answer. Hint: If you choose the center of curvature of the bowl as the y=0 reference point (as shown), then the initial vertical position of Block A is $y_a = -R\cos\theta$.

**Part (c)** (5 points) – More difficult: Determine the angle $\phi$ corresponding to the maximum vertical position of the Block B after the collision. Give your answer in terms of the given parameters. Explain your work. Hint: First contend with the totally elastic collision.

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# Solution to Problem 1:

## Part (a): (5 points)

To get a handle on the speed, which is the **magnitude of the velocity** we want to write down the vector expression for velocity by taking the **time derivative of the position vector:**

$$\vec{v}(t) \equiv \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[C\cos(\omega t)\,\hat{\boldsymbol{\imath}} - 2C\sin(\omega t)\,\hat{\boldsymbol{\jmath}}\right]$$
$$\vec{v}(t) = -C\omega\sin(\omega t)\,\hat{\boldsymbol{\imath}} - 2C\omega\cos(\omega t)\,\hat{\boldsymbol{\jmath}} \tag{1}$$

Now we plug in $t = 0$. The sine term goes to zero, the cosine term goes to one:

$$\vec{v}(t) = -C\omega\sin(0)\,\hat{\boldsymbol{\imath}} - 2C\omega\cos(0)\,\hat{\boldsymbol{\jmath}}$$

$$\vec{v}(t) = -C\omega(0)\,\hat{\boldsymbol{\imath}} - 2C\omega(1)\,\hat{\boldsymbol{\jmath}}$$
$$\vec{v}_P = \vec{v}(t=0) = 0\hat{\boldsymbol{\imath}} - 2C\omega\hat{\boldsymbol{\jmath}}$$
$$\vec{v}_P = -2C\omega\hat{\boldsymbol{\jmath}}$$

This is a velocity vector that points along the negative y-direction with a magnitude of $2C\omega$, corresponding to the speed:

$$\text{Speed at P} \equiv |\vec{v}_P|$$
$$\boxed{\text{Speed at P} = 2C\omega}$$

**Part (b):** (5 points)

In this problem, we have no idea what the actual specific forces are on the particle, so trying to calculate the net force by totally up specific forces will not be helpful.

To get the net force we need to consider **Newton's Second Law** in vector form:

$$\vec{F}_{net} = m\vec{a}$$

We are given the mass so all we need to do is calculate the kinematic vector acceleration. To get the acceleration we just take the **time derivative of the velocity vector** which we calculated above (Equation 1):

$$\begin{aligned}\vec{a}(t) \equiv \frac{d\vec{v}}{dt} &= \frac{d}{dt}\left[-C\omega\sin(\omega t)\,\hat{\boldsymbol{\imath}} - 2C\omega\cos(\omega t)\,\hat{\boldsymbol{\jmath}}\right] \\ \vec{a}(t) &= -C\omega^2\cos(\omega t)\,\hat{\boldsymbol{\imath}} + 2C\omega^2\sin(\omega t)\,\hat{\boldsymbol{\jmath}} \qquad (2)\end{aligned}$$

Again, we plug in $t = 0$:

$$\begin{aligned}\vec{a}(t=0) &= -C\omega^2\cos(0)\,\hat{\boldsymbol{\imath}} + 2C\omega^2\sin(0)\,\hat{\boldsymbol{\jmath}} \\ \vec{a} &= -C\omega^2(1)\,\hat{\boldsymbol{\imath}} + 2C\omega^2(0)\,\hat{\boldsymbol{\jmath}} \\ \vec{a} &= -C\omega^2\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} \\ \vec{a} &= -C\omega^2\hat{\boldsymbol{\imath}}\end{aligned}$$

Finally we plug this into Newton's Second Law:

$$\vec{F}_{net} = m\vec{a}$$

$$\boxed{\vec{F}_{net} = -mC\omega^2\hat{\boldsymbol{\imath}}}$$

**Part (c):** (5 points)

The radius of curvature is kinematically connected to the **centripetal component of the acceleration**:

$$a_c = \frac{v^2}{R_c}$$

This means we need to find the acceleration and the velocity at Point $Q$. So we use our results from Parts (a) and (b), plugging in $t = \frac{\pi}{2\omega}$. Starting with velocity: [see Equation (1)]

$$\vec{v}(t) = -C\omega \sin(\omega t)\,\hat{\boldsymbol{\imath}} + -2C\omega \cos(\omega t)\,\hat{\boldsymbol{\jmath}}$$
$$\vec{v}(t = \frac{\pi}{2\omega}) = -C\omega \sin\left(\frac{\pi}{2}\right)\,\hat{\boldsymbol{\imath}} - 2C\omega \cos\left(\frac{\pi}{2\omega}\right)\,\hat{\boldsymbol{\jmath}}$$
$$\vec{v}_Q = -C\omega(1)\,\hat{\boldsymbol{\imath}} - 2C\omega(0)\,\hat{\boldsymbol{\jmath}}$$
$$\vec{v}_Q = -C\omega\,\hat{\boldsymbol{\imath}}$$

$$v_Q \equiv |\vec{v}_Q| = C\omega$$

Likewise, we calculate the acceleration using Equation (2) from Part (b):

$$\vec{a}(t) = -C\omega^2 \cos(\omega t)\,\hat{\boldsymbol{\imath}} + 2C\omega^2 \sin(\omega t)\,\hat{\boldsymbol{\jmath}}$$
$$\vec{a}(t = \frac{\pi}{2\omega}) = -C\omega^2 \cos\left(\frac{\pi}{2}\right)\,\hat{\boldsymbol{\imath}} + 2C\omega^2 \sin\left(\frac{\pi}{2}\right)\,\hat{\boldsymbol{\jmath}}$$
$$\vec{a}_Q = C\omega^2(0)\,\hat{\boldsymbol{\imath}} + 2C\omega^2(1)\,\hat{\boldsymbol{\jmath}}$$
$$\vec{a}_Q = 2C\omega^2\,\hat{\boldsymbol{\jmath}}$$

We note that the **acceleration** at point $Q$ points **perpendicular to the velocity** at point $Q$. In other words if we write down the expression for total acceleration:

$$\vec{a}_{tot} = \vec{a}_c + \vec{a}_v$$

at point $Q$, we see that the acceleration points only along the y-direction. This is the centripetal direction. Therefore **the acceleration must be purely centripetal**:

$$\vec{a} = \vec{a}_c$$

$$a_c = |\vec{a}| = 2C\omega^2$$

We then use the expression for **centripetal acceleration** along with our calculation of the speed to get the **radius of curvature**:

$$a_c = \frac{v^2}{R_c}$$

Note that this equation is an expression given in terms of magnitudes (only) and not vectors. We re-arrange to get the Radius of Curvature:

$$R_c = \frac{v_Q^2}{a_c}$$

Then we plug in our results for the speed and magnitude of the centripetal acceleration at point Q:

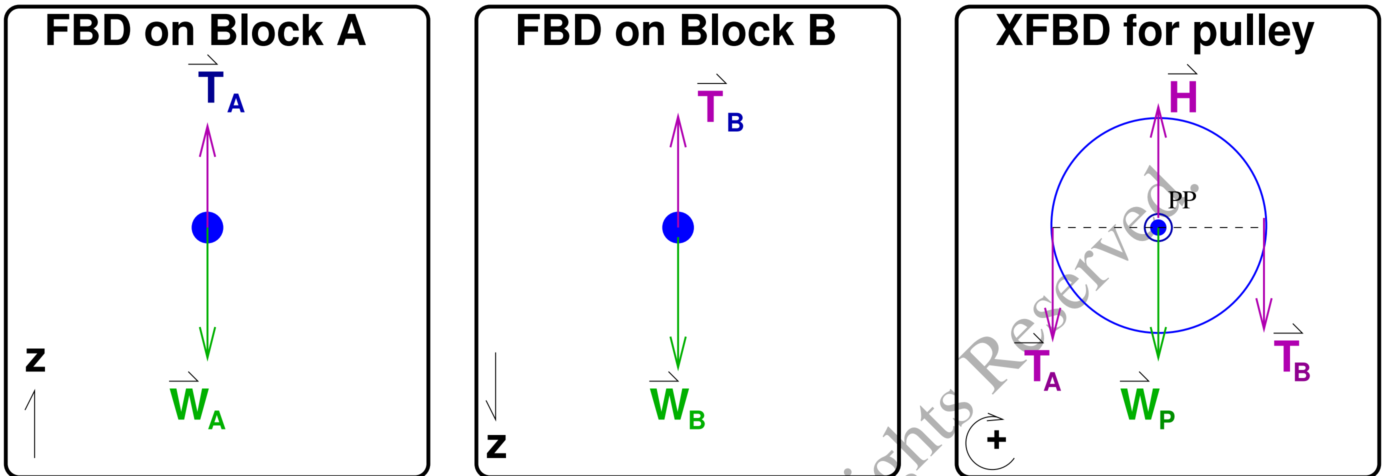
$$R_c = \frac{(C\omega)^2}{2C\omega^2}$$

$$R_c = \frac{C^2\omega^2}{2C\omega^2}$$

$$\boxed{R_c = \frac{C}{2}}$$

# Solution to Problem 2: (20 points)

## Problem 2: Part (a) – (5 points):




First we consider the FBDs for the two blocks: The only forces on each block are **Weight** and **Tension** appropriately labeled. Note that because the pulley is *massive* we cannot assume that the tension is the same everywhere on the string. So we have to define two different tensions, for example, $TA$ and $T_B$ (or alternatively $T_1$ and $T_2$).

Each FBD needs a coordinate system. We note that there is a **kinematic constraint** that applies to the two blocks: *they accelerate at the same magnitude*. But (because of the pulley) they accelerate in opposite directions. So we define a single "coordinate" corresponding to the "$z$-direction" where $z$ points down for Block A (this is the direction that Block A accelerates because it is heavier than block B) and up for Block B.[1] Now, by definition, whatever the acceleration of Block A, this is the same as the acceleration of Block B:[2] so the **kinematic constraint** is written as:

$$a \equiv a_{Az} = a_{Bz}$$

For the **Extended FBD of the pulley** we need to include **both Tension forces**, the **Weight** force and a **Hinge** force corresponding to the force applied (upward, by Newton's Second Law) to keep the pulley in place. The XFBD should also have a **coordinate system** indicating the direction of positive rotation (here counter-clockwise) and (ideally) a clearly delineated **pivot-point**.



[1]Note to grader: the student may of course define this direction as "x" or "y" as well as "z".

[2]Note to graders: An alternate but acceptable approach is to define the same coordinate system for each block and then to argue that the **kinematic constraint** is that $a_{Az} = -a_{Bz}$. As long as the student keeps correct track of the negative signs, the results will be the same.

**Problem 2: Part (b) – (5 points):**

We follow the "recipe". We have already considered the kinematic constraints on the acceleration of each block.

First, we apply **Newton's Second Law:** to Block A: (note $z$-positive is ***up***)

$$F_{Az} = m_A a_{Az}$$
$$T_A - W_A = ma$$
$$T_A - mg = ma \tag{1}$$

We **stop**. One equation, two unknowns: $T_A$ and $a$.

Next apply **Newton's Second Law:** Block B: (note $z$-positive is ***up***)

$$F_{Bz} = m_B a_{Bz}$$
$$W_B - T_B = 2ma$$
$$2mg - T_B = 2ma \tag{2}$$

We **stop**. One equation, two unknowns: $T_B$ and $a$.

We have **Two equations and Three unknowns:** $T_A$, $T_B$ and $a$. So we are not done.

Since we have a ***massive pulley*** and since this pulley accelerates with the string, we need to consider the torques on the pulley. We use the **Extended Free Body Diagram** where we define rotation **counter-clockwise as positive** which matches our definition of positive acceleration of the bodies on the string.

We now apply **Newton's Second Law for Rotations:**

$$\tau_{net} = I\alpha$$
$$\tau_{\vec{W}} + \tau_{\vec{H}} + \tau_{\vec{T_A}} + \tau_{\vec{T_B}} = I\alpha$$

We use the **Definition of Torque** and the XFBD to evaluate this term-by-term:

$$\tau_{\vec{F}} \equiv rF\sin\phi$$

The Hinge force and the Weight are both applied at the Pivot Point so the radial vector is of zero length and so both of these forces contribute zero torque. The strings each apply tension at a radius $R$ and at right angles to the radial vector. Therefore:

$$-RT_A + RT_B = I\alpha$$

Note that in accordance with our coordinate system, the torque due to $T_B$ is positive while the torque due to $T_A$ is negative.

$$RT_B - RT_A = I$$
$$RT_B - RT_A = \frac{1}{2}m_p R^2\alpha$$
$$RT_B - RT_A = \frac{1}{2}(6m)R^2\alpha$$
$$RT_B - RT_A = 3mR^2\alpha$$

Finally need one more piece which is the connection between the angular acceleration of the pulley and the linear acceleration of the rope. Since the rope rids on the outside of the pulley without slipping, we can use the **Rolling Constraint**:

$$a = \alpha R$$

Solving for $\alpha$ and plugging this in we get:

$$RT_B - RT_A = 3mR^2 \left(\frac{a}{R}\right)$$

The $R$s all cancel:

$$T_B - T_A = 3ma$$

$$T_B - T_A = 3ma \qquad (3)$$

This, happily, gives us **Three equations** (1), (2), and (3), and three unknowns: $T_A$, $T_B$, and $a$:

So **the rest is algebra:**

Use (1) to solve for $T_A$

$$T_A = mg + ma$$

Use (2) to solve for $T_B$

$$T_B = 2mg - 2ma$$

Plug these into equation (3):

$$(2mg - 2ma) - (mg + ma) = 3ma$$

Simplify and solve for the acceleration:

$$(2mg - 2mg) + (-2ma - ma) = 3ma$$

$$mg + 3ma = 3ma$$

$$6ma = mg$$

$$\boxed{a = \frac{g}{6}}$$

(Problem 2 solution continues next page...)

**Problem 2: Part (c) – (5 points):**

To get the Tension on Block A we just use this result and plug this into Equation (1)

$$T_A - mg = m\left(\frac{g}{6}\right)$$
$$T_A = \frac{mg}{6} + mg$$
$$\boxed{T_A = \frac{7mg}{6}}$$

**Problem 2: Part (d) – (5 points):**

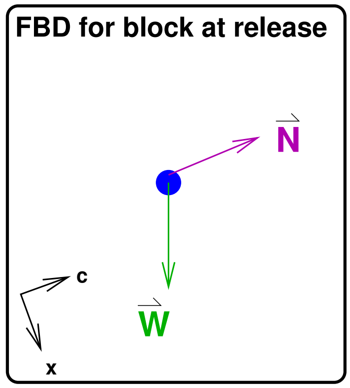
To get the Tension on Block B we plug our result for the acceleration into Equation (2):

$$T_B = 2mg - 2ma$$
$$T_B = 2mg - 2m\left(\frac{g}{6}\right)$$
$$T_B = 2mg - \frac{mg}{3}$$
$$T_B = \frac{6mg}{3} - \frac{mg}{3}$$
$$\boxed{T_B = \frac{5mg}{3}}$$

So as we expect, the tension on Block A is a little larger than the weight $mg$, the tension on Block B is a little more than the weight $2mg$.

# Solution to Problem 3: (15 points)

## Problem 3: Part (a) – (5 points):




The figure above shows the **Free-Body Diagram** at the release point. There are two forces: Weight and Normal. We choose a **tilted coordinate system** so that one of the coordinates (here called "x") points in the direction of tangential acceleration, and the other coordinate (here called "c") points toward the center of the circular path corresponding to the direction of centripetal acceleration. We note that at the position of release the **speed is zero** and therefore the centripetal component of the acceleration which is given by $a_c = v^2/R$ is also zero.

Since we are looking for the acceleration, we better apply **Newton's Second Law**. We note that the acceleration is is aligned with the the "x" coordinate direction:

$$F_x = ma_x$$

Looking at the Free-Body Diagram we see that we get the sine component of Weight in the tangential direction:

$$W \sin\theta = ma_x$$

$$mg \sin\theta = ma_x$$

$$\boxed{a_x = g \sin\theta}$$

**Part (b):** (5 points)

There are two forces on the block from the point of release to just before the collision: *Weight* and *Normal*. We know that **Weight is Conservative**, and that **the Normal force does no work in this problem.** Therefore we have **met the conditions** to apply **Conservation of Energy**.

We define the "Before" as the block at the release point shown in the figure. We define the "After" as the block at the bottom of the bowl.

We write down **Conservation of Energy:** in symbolic form:

$$E_{tot} = E'_{tot}$$
$$U + K = U' + K'$$
$$mgy + \frac{1}{2}mv^2 = mgy' + \frac{1}{2}mv'^2$$

We want to choose an appropriate zero-point reference for the y-coordinate of the block. As suggested, we choose y=0 at the central location of the sphere. in this case: $y = y_a = -R\cos\theta$ and $y' = -R$. We set the initial speed $v = 0$ and solve for the velocity of the block $v' = v_a$ before the collision:

$$mg(-R\cos\theta) + 0 = mg(-R) + \frac{1}{2}mv_a^2$$
$$\frac{1}{2}mv_a^2 = mg(R - R\cos\theta)$$
$$v_a^2 = 2gR(1 - \cos\theta)$$
$$\boxed{v_a = \sqrt{2gR(1 - \cos\theta)}}$$

**Part (c):** (5 points)

**Note: this is a challenging compound problem!**

First we need to contend with the Totally Elastic Collision. There are several acceptable ways to do this. First, we can use "Master Equations" for 1-D Totally Elastic Collisions, starting with **Conservation of Linear Momentum**[3]

$$P_a + P_b = P'_a + P'_b$$
$$m_a v_a + m_b v_b = m_a v'_a + m_b v'_b$$

The initial velocity for Block A comes from Part (b) so for now we treat this as a given parameter.[4] The initial velocity of Block B is zero:

$$mv_a = mv'_a + 3mv'_b \tag{1}$$

Next we use the 2nd-master equation:

$$v_a - v_b = v'_b - v'_a$$

$$v_a = v'_b - v'_a \tag{2}$$

We want $v'_b$ so we rearrange Equation (2) for $v'_a$:

$$v'_a = v'_b - v_a$$

and plug this into Equation (1):

$$mv_a = m(v'_b - v_a) + 3mv'_b$$
$$v_a = (v'_b - v_a) + 3v'_b$$
$$2v_a = 4v'_b$$
$$\boxed{v'_b = \frac{v_a}{2}}$$

***Alternatively,*** we can use "Mr. Covault's 4-box method" where we calculate the speed of the center-of-mass:

$$V_{CM} = \frac{m_A v_A + m_B v_B}{m_A + m_B}$$
$$V_{CM} = \frac{M v_a + 0}{m + 3m}$$
$$V_{CM} = \frac{v_a}{4}$$

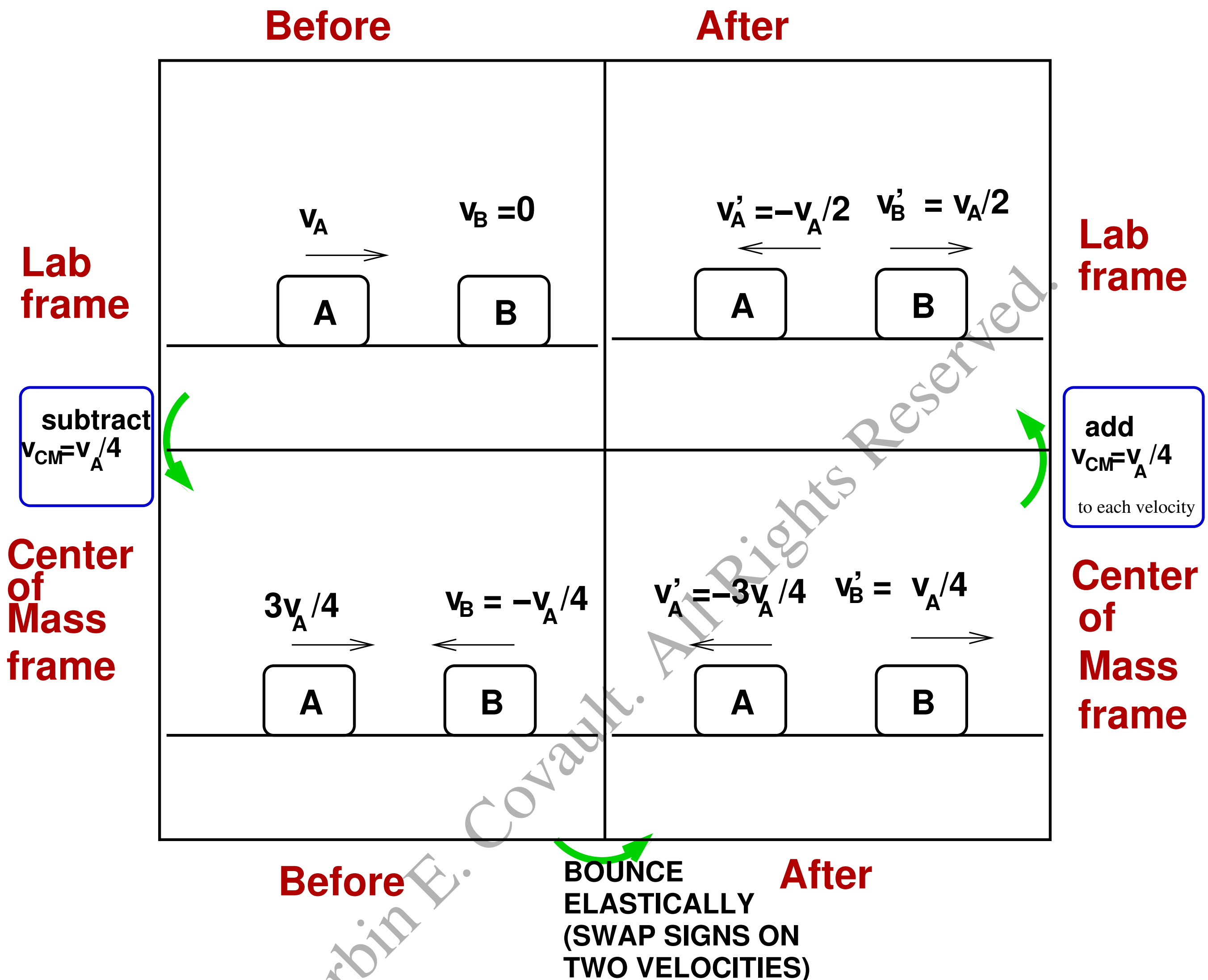
That this is the same answer to Part (a) is the obvious expectation (and a quick shortcut). ***On the next page*** is the application of the 4-box method for this particular scenario. The results are as follows:

$$\boxed{v'_b = \frac{v_a}{2}}$$

[3] So *technically* we can argue that (just barely) before, during and after the collision, the system of the two-blocks is isolated because the blocks are moving on a frictionless (nearly) horizontal surface.

[4] This is allowed as long we remember to substitute before we get to the final answer.

**Solution to Problem 3 Part (c) continues....**



**Solution to Problem 3 Part (c) continues....**

Finally, we want to ask to what angle we come up to on the right side. Now we once again use Conservation of Mechanical Energy as we did in Part (b) but just reverse ("Before" = just after the collision, and "After" = max position where the final velocity is zero. Same basic algebra to get relation between speed and angle:

$$2gR(1-\cos\phi) = (v_b')^2$$

$$2gR(1-\cos\phi) = \left(\frac{v_a}{2}\right)^2$$

$$\cos\phi - 1 = \frac{\cos\theta - 1}{4}$$

$$\cos\phi - 1 = \frac{\cos\theta}{4} - \frac{1}{4}$$

$$\cos\phi = \frac{\cos\theta}{4} + \frac{3}{4}$$

$$\boxed{\phi = \arccos\left(\frac{\cos\theta + 3}{4}\right)}$$

This answer looks weird but it makes sense. Block B has twice the mass and half the speed of Block A after the collision so it has have the kinetic energy. This means that it will travel up the bowl a vertical distance that is half as far as as Block A dropped.