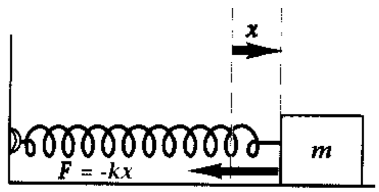
PHYS 121: Cycle 4 Review Sheet

April 22, 2024

Simple Harmonic Oscillators:



Review: Hooke's Law: We can demonstrate empirically that the force associated with displacement of a spring from equilibrium position $x = 0$ is given by:

$$|F_{spring}| = kx$$

Where k is called the "spring constant" and it depends upon the material of the spring. This is called Hooke's Law. Note that the direction of the force is in opposition to the displacement.

Review: Potential Energy in a Spring We have shown that the potential energy that we obtain when we do work against this spring as:

$$PE_{spring} = \frac{1}{2}kx^2$$

Motion of a Simple Harmonic Oscillator: For an simple harmonic oscillator that is represented by a mass on a spring, we can re-arrange Newton's Second Law:

$$\begin{aligned} F_{net} &= ma_x \\ -kx &= ma_x \\ \frac{d^2x}{dt^2} + \frac{k}{m}x &= 0 \\ \frac{d^2x}{dt^2} + \omega^2 x &= 0 \end{aligned}$$

This is called the **Equation of Motion** for the system. To solve this equation we make an "educated guess" that the correct form of the solution corresponds to sines and cosines:

$$x(t) = B\sin(\omega t) + C\cos(\omega t)$$

where

$$\omega = \sqrt{\frac{k}{m}}$$

We are free to choose values of the parameters B and C to match initial conditions (generally constraints on the initial position and velocity of the system). You should be able to determine values for B and C given initial position and velocities.

For example, suppose we release with the oscillator at $t = 0$ at rest with displacement x_0 then you should be able to show that:

$$C = x_0$$

$$B = 0$$

Likewise if the mass is given an initial velocity at $t = 0$ of v_0 at the position $x = 0$ then you should be able to show that:

$$C = 0$$

$$B = \frac{v_0}{\omega}$$

You should also know that we can use some trigonometry to re-write this solutions in terms of a single cosine function with two free parameters:

$$x(t) = A\cos(\omega t + \phi)$$

where A is the *Amplitude* of oscillation and ϕ is the *phase* of the oscillator. You should be able to switch back and forth between these two representations. For example, to get A and ϕ given B and C:

$$A = \sqrt{B^2 + C^2}$$

$$\tan\phi = \frac{-B}{C}$$

When we have motion defined in terms of the amplitude, it is pretty straightforward to show that the total energy of the oscillator (a constant) is proportional to the amplitude squared and the spring constant of the oscillator:

$$E_{tot} = U + K = \frac{1}{2}kA^2$$

We define the **Period** of the oscillator as the term:

$$P = \frac{2\pi}{\omega}$$

and the *frequency* of the oscillator as simply the inverse of the period:

$$f = \frac{1}{P} = \frac{\omega}{2\pi}$$

Obviously, the *angular frequency* and the frequency are simply related by a factor of 2π:

$$\omega = 2\pi f$$

Mechanical Waves:

Wave types:

Transverse Waves: These are waves where the displacement is perpendicular to the direction of motion. A good example is a pulse moving down a rope or a slinky.

Longitudinal Waves: These are waves where the displacement is parallel to the direction of motion. In this case, the medium is alternately compressed and rarefied. Sound waves are a good example.

Harmonic Waves:

These are waves that look like "sine waves" in both time and position. The general expression for a transverse harmonic wave that travels in the x direction is:

$$y(x,t) = A\cos\left(kx - \omega t + \phi\right)$$

Note that we can re-write this in terms of wavelength and period:

$$y(x,t) = A\cos\left(\frac{2\pi x}{\lambda} - \frac{2\pi t}{P} + \phi\right)$$

where A is the *Amplitude* of the wave, λ is the *wavelength*, P is the *Period*, and ϕ represents the *phase* of the wave.

Velocity of Harmonic Waves:

Note that the speed of the wave is given by:

$$v = \frac{\omega}{k}$$

Alternatively:

$$v = \lambda f$$

where λ is the wavelength and $f = 1/P$ is the frequency of the wave.

Reflecting Waves:

Waves carry both momentum and energy. When the wave hit the end of the medium, it will reflect. If energy is conserved, the reflected wave will have the same amplitude and velocity as the original wave.

Refracting Waves:

If a wave moves from one medium to another, the wave may be reflected, refracted, or both. A wave will refract when the velocity of the wave changes from one medium to the next. For example, light waves refract when they move from the medium of air to the medium of water. One by-product of refraction is that the path of wave can be deflected as it moves across the medium. Refraction in this manner is used to deflect the paths of light waves in lenses and other optical components.

Principle of Superposition:

When two waves encounter each other in a medium, they simply "add together" at each point in the medium.

When waves encounter each other the summing yields an effect we call *interference*. If one wave overlaps a second wave, and both waves are displacements in the same direction, then the increased amplitude is called *constructive interference*. If on the other hand the displacements are opposite, the waves cancel each other out at just the point of overlap, and we have *destructive interference*.

The property of interference is unique to waves. Specifically, particles do not "interfere". The just bounce off each other.

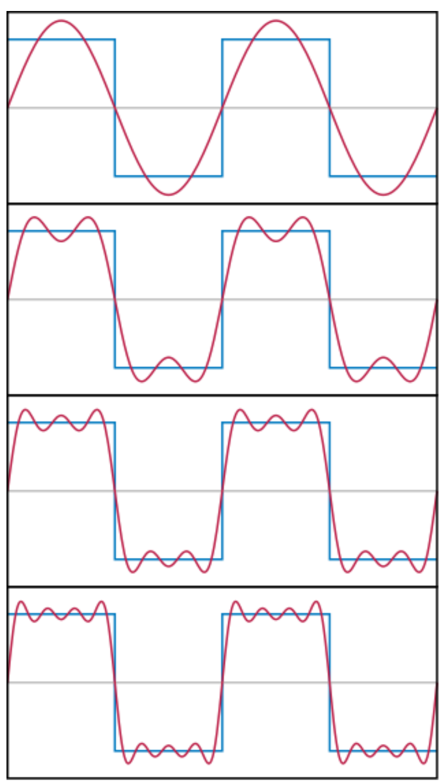
Spectral Analysis (Harmonic Analysis) and the Fourier Series:

One of of the major motivations for looking at Harmonic Waves is because Joseph Fourier showed that any periodic function can be represented as a sum of harmonic sines and cosines.

Fourier showed that any 2π-periodic function $f(x)$ can be represented using a series sum of sines and cosines:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right]$$

What this means in words is that (virtually) and function you want to come up with can be fabricated by adding up sums of sines and cosines with the right values for the coefficients a_n and b_n. Here is a graphical example of using the first four terms in the Fourier series to generate a "square wave":



The method for calculating these coefficients is beyond the scope of the course but is quite simple:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

We note that technically, the "Fourier Series" only applies to periodic (repeating) functions $f(x)$. However, there are straightforward methods to extend the method to any functional form (again beyond the scope of this course).

Standing Waves:

A classic example of the application of both Fourier's ideas and the concept of superposition is the idea of standing waves. If a medium is bounded or clamped on two ends, then we can get an effect called standing waves. A standing wave represents two harmonic waves, each the same frequency and wavelength, but traveling in opposite direction. The principle of superposition tells us in this case there are *nodes* in the medium. Nodes are locations in the medium where the effect of destructive interference results in the medium remaining motionless at these points. For a standing waves, the nodes must (at least) match the boundaries of the medium. This means that the only allowed waves are those where an integer number of half-wavelengths matches the extent of the medium.

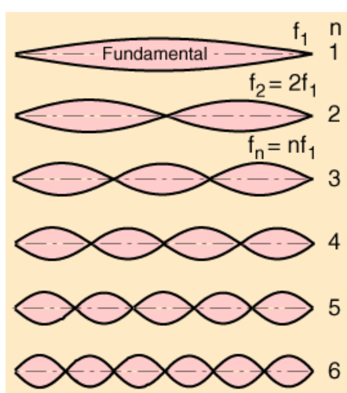
$$\lambda_n = \frac{2L}{n}$$

In the case that $n = 1$ we call this the *fundamental* wavelength, with higher values of n corresponding to higher "harmonics".

For example, a rope of length L tied at both ends will support standing waves of length $2L/n$. The frequencies that we can obtain from such waves are:

$$f_n = \frac{nv}{2L}$$

In other words, for higher harmonics corresponding to larger n and shorter wavelengths, we get higher frequencies:



The speed of a wave on a string or wire under tension is given by

$$v = \sqrt{\frac{T}{\mu}}$$

where T is the tension in the string and μ is the mass per unit length (also called the linear mass density).

Non-inertial Reference Frames:

We can get a clue about the connection between acceleration and gravity by considering how physics works in a uniformly accelerating (non-inertial) reference frame. If a frame is undergoing an uniform acceleration $\vec{A}$ then this has the effect of generating what appears to be a **fictitious force** in this frame. The fictitious force is not a real force; it is by-product of attempting to apply Newton's Second Law in a non-inertial reference frame where the acceleration is not invariant:

$$\vec{a}' = \vec{a} - \vec{A}$$

$$\vec{F}' = m\vec{a} - m\vec{A}$$

Here $m\vec{a}$ is the mass times the actual acceleration associated with the real force $\vec{F}$, while $-m\vec{A}$ is the *apparent* acceleration (as seen in the non-inertial reference frame) associated with the fictitious force:

$$\vec{F}' = \vec{F}_{\text{real}} + \vec{F}_{\text{fict}}$$

where

$$\vec{F}_{\text{fict}} = -m\vec{A}$$

Note that the fictitious force has these two key qualities:

- the fictitious force is *not* a contact force.
- the fictitious force that is experienced by a body is proportional to the mass.

This means that the fictitious force acts just like a *weight force*, essentially a **fictitious weight** where the fictitious weight force on any body of mass m is given by:

$$\vec{W}_{\text{fict}} = m\vec{g}_{\text{fict}} = -m\vec{A}$$

In other words, the fictitious weight acts in opposition to the direction of the acceleration of the frame. This fictitious weight appears in addition to any real weight that the body experiences.

Sometimes we use the word "effective gravity" to refer to the combined effects of real (Newtonian) gravity and the fictitious gravity that arises within an accelerating reference frame. We might say:

$$\vec{g}_{\text{eff}} = \vec{g}_{\text{real}} + \vec{g}_{\text{fict}}$$
$$\vec{g}_{\text{eff}} = \vec{g}_{\text{real}} - \vec{A}$$

In the case that we use a coordinate system with $\hat{\boldsymbol{\jmath}}$ pointed up, for example:

$$\vec{g}_{\text{eff}} = -g\,\hat{\boldsymbol{\jmath}} - \vec{A}$$

It's worth emphasizing this point: **Almost any problem can and should be worked in an inertial reference frame.** As a rule avoid fictitious forces, and work the problem in an inertial reference frame. You should only work a problem in an accelerating reference frame if you are very specifically interested in the apparent effects of fictitious forces as seen by an observer who is moving in that frame.

Newton's Worries about Universal Gravitation:

- **Newton's First Worry:** Is it really true that a spherical solid object, such as the Earth can be represented as a point mass at the center of the object? In other words is it really the case that the vector sum of all of the attractive parts of the solid sphere add up to exactly match the force that you would get from a point source at the center? **Answer:** Yes this works. Newton had to invent integral calculus to prove this to himself.

- **Newton's Second Worry:** How can we explain the fact that gravity is "action at a distance". For all of the other obvious forces, the force arises from *contact* between two bodies. How can gravity exert a force between two bodies that are physically separated by huge distances? **Answer:** Newton did not really solve this problem, but people who came shortly after him suggested the concept of a *gravitational field*. The idea of a field is to thing of a set of vectors, each one corresponding to every position is space, and each one indicating the force per unit test mass at that point due to all of the other masses in a system. This is the gravitational field. The field due to one mass is given by:

$$\vec{g} = -\frac{GM}{r^2}\hat{\boldsymbol{r}}$$

- **Newton's Third Worry:** The equality of inertial and gravitational mass. When we use the mass of an object for Newton's second law, we are describing the property of inertia of the object:

$$F = m_{inert} a$$

  where m_{inert} is called the *inertial mass* of the object. However, when we use the law of universal gravity to determine how something responds to gravity, we use the mass in a different way:

$$F = \frac{GMm_{grav}}{r^2}$$

  where m_{grav} is called the *gravitational mass*. Newton realized that these appear to be *equal* for any body:

$$m_{inert} = m_{grav} \qquad \text{for all bodies}$$

  The fact that they are equal explains why bodies fall at the same rate regardless of mass. However, Newton did not have an explanation for *why* these should be equal. Nor did Newton have an explanation as to how gravity can act as a force over large distances:

  "...I feign no hypothesis."

Einstein's Theory of Relativity:

Review: Einstein's theory of *General Relativity* was was published in 1916, exactly 100 years ago. General relativity deals with the more general case of accelerating reference frames, and is considerably more complex than Special Relativity. General relativity is based upon the **Principle of Equivalence**. General relativity deals with the physics of *gravity* and the relationship between matter and curved space.

Principle of Equivalence

Einstein's General Relativity is based on the following postulate:

> Experiments performed in a uniformly accelerating reference frame give *exactly* the same results as experiments carried out under the influence of a uniform gravitational field.

In other words, there is no experiment that can be done that could distinguish between being in a spaceship with an acceleration of 9.81 m/s^2 and sitting in a room on the surface of the earth where the acceleration due to gravity is 9.81 m/s^2.[1]

> In fact, we can say rather specifically that the effect of doing physics in an accelerated reference frame is that in that frame there appears to be a "gravitational field" that "pulls" in the direction *opposite* the acceleration and the magnitude of the field (g_{eff}) has a value that is equal to (but oppositely directed from) the acceleration of the frame.

The Curvature of Space-Time: Gravity Bends Light

Note that although the Equivalence Principle is a simple idea, Einstein's theory of General Relativity is not a simple theory. Einstein explains gravity as a phenomena that arises from the fact that matter actually distorts the geometry of space. In the presence of a massive body, space (and time) is curved. This curvature changes the straight-line paths of objects and light in the vicinity, causes them to be bent toward the massive object.

Note that the picture is complicated further by one of the fundamental equivalences from Einstein's Special Theory of Relativity, namely the interchangeability of matter and energy:

$$E = mc^2$$

What this means is that gravity arises not only in the presence of matter, it also arises in the presence of energy. And since gravity itself results in a localization of potential energy, gravity in turn also creates gravity. Thus gravity is an intrinsically "non-linear" phenomena.

[1] For sticklers: this is true for a "local" measurement – that is to say it is true in the approximation that the measurement is constrained to take place over a small volume of space. Without this constraint the postulate does not quite apply. For example, if you could make a measurement of gravity over a large distance of many miles you would in principle be able to determine that the direction of gravity changes to stay aligned with the slowly curving surface of the Earth.

The Einstein Field Equation for Curvature

Einstein worked out the fundamentals of how matter-energy results in the curvature of space. He was able to write down an expression:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Here $T_{\mu\nu}$ which represents the *stress-energy tensor* which is a matrix that describes the distribution of all matter and the interactions (energy) that contribute to curvature as well, and $G_{\mu\nu}$ is the curvature tensor. The two indices each refer to the four space-time coordinates (three spatial dimensions, one time dimension) and therefore the expression above is actually matrix shorthand for a set of sixteen coupled non-linear equations. Finding closed-form solutions to these equations is remarkably difficult, and in general they can only be solved iteratively using numerical methods and a computer.

Note that the term $\Lambda g_{\mu\nu}$ is an extra term that Einstein threw in, a so called "cosmological constant." For many years, it was assumed that the value of Λ was zero. Now it appears that Λ may be non-zero and this has major implications for the evolution of the universe as a whole.

Black Holes: Gravity Bends Light

If a massive object has sufficiently large mass and sufficiently small radius, the concentration of matter may be so large that the curvature of space becomes so severe that not even light can escape. This is called a *black hole*. This point of no return is called an event horizon – within which nothing, not even light, can escape.

We can calculate the mass and radius required to form a black hole by setting the escape velocity equal to that of the speed of light:

$$v_{esc} = \sqrt{\frac{2GM}{R}} = c$$

Rearranging:

$$R = \frac{2GM}{c^2}$$

where R is called the Schwartzchild Radius or event horizon – the radius within which nothing, not even light, can escape.

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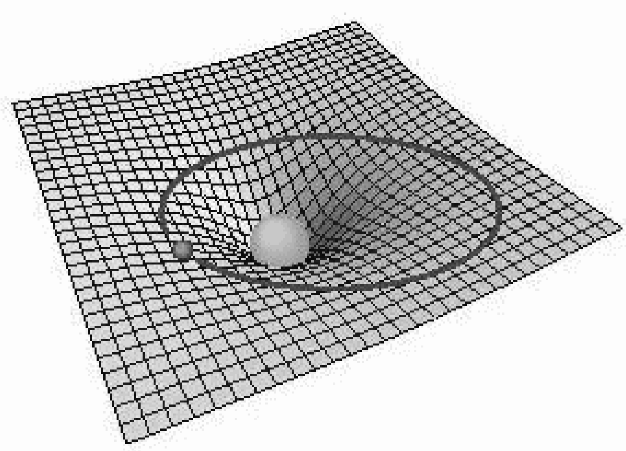
Einstein vs. Newton

Newton said gravity is a *force*. He "feigned no hypothesis" as to the cause of gravity. He was worried any puzzled about the fact that gravity acts over large distances with no apparent mechanism.

Einstein said that gravity is manifestation of of curved space itself. He explained gravity in term of the relationship between matter and space. He argued that the presence of matter distorts the curvature of space in much the same way as a heavy object places on a rubber sheet will distort the sheet.



Definition of Cosmology

Cosmology is the physics of the Universe as a whole. Everything.

Olbers' Paradox

Olbers assumed:

1. The Universe is infinite and homogeneous in space.
2. The Universe is infinite and homogeneous in time.

These assumptions were motivated based upon the *Copernican Principle* which states that the humanity (and thus the earth) are not located at some special or central place or time in the universe. This is a philosophical view.

Based on these two assumptions, Olbers showed that the night sky should not be dark. Rather the sky should be brilliant light with the brightness of the surface of a star (like the sun) in all directions. This follows from the argument that out from any line of sight, in an infinite universe that line of sight will eventually intersect the surface of a star.

The Hubble Law

In the 1920' Edwin Hubble made a series of observations of distant galaxies of stars. He showed that all distant galaxies appear to be receding from our own Milky Way galaxy at high speeds. Hubble showed that there is a linear relationship between the observed radial velocity of a galaxy away from us, and the distance to the galaxy:

$$v = Hd$$

Where v is the velocity of the galaxy, d is the distance to the galaxy, and H is a constant called the "Hubble Constant". This relationship is called Hubble's Law.

Hubble's observation implies that Olbers' second assumption is not correct. All of the galaxies appear to be moving from a single point starting between 10 and 20 billion years ago.

The Hot Big Bang

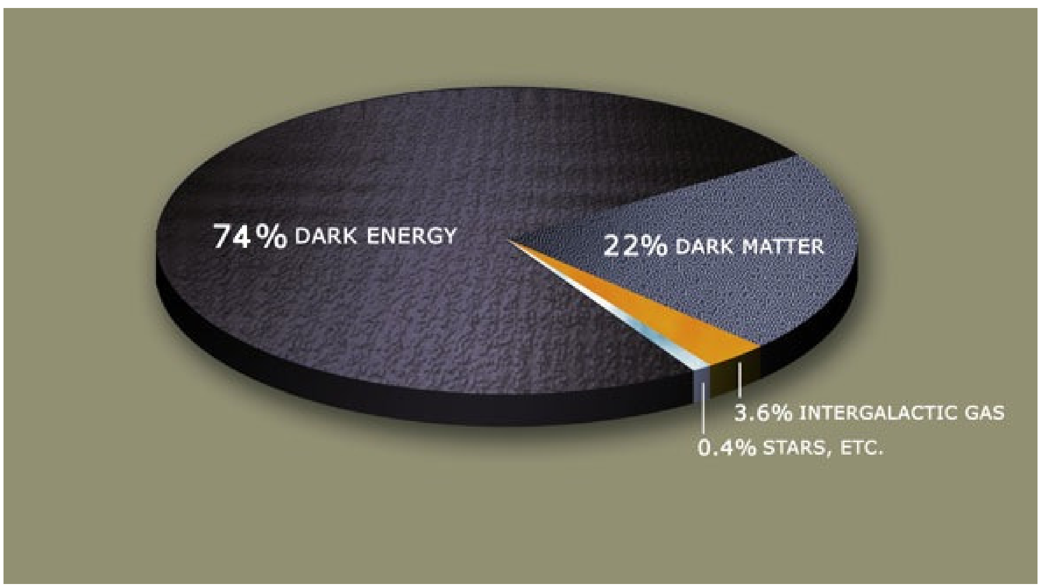
Hubble's Law can be explained if we say that the fabric of space is *expanding* with time. In this picture, the expansion of the universe starts very nearly 13.6 billion years ago with all of space, energy, and matter are concentrated at a single "point". This is the beginning of the physical universe, and the universe has been expanding, and cooling ever since.

This theory for the universe has been supported by many observations, including detection of the "cosmic microwave background radiation" which is a cooled echo of the residual energy of the Big Bang.

Gravity and the Fate of the Universe

During the late 20th Century, it seemed a final determination of the ultimate fate of the universe could be made depending upon either careful measurement of the density of the universe or measure of the curvature of the space-time and the Hubble constant. However, the picture is complicated by the identification of two major components of our Universe that are quite mysterious and very different from ordinary matter. Normal stuff like stars and galaxies and gas comprise just few percent of the matter in the universe. It appears that about 22 percent of the universe is made of something called "Dark Matter", a mysterious kind of matter that has mass but does not interact as a rule in any normal way with other matter. And furthermore, even stranger, it seems that about 74 percent of the universe is made of something called "Dark Energy", a strange effect that appears to be accelerating the expansion of the universe.

The absolute final word is not in, and will depend very much on unknown properties of the dark matter and the dark energy. But it is beginning to look more and more likely that we are living in an open flat universe where the expansion will gradually accelerate and the universe will end by diluting itself into increasingly cold and empty nothingness. The gas and dust and stars will get sucked into black holes. The black holes will decay into sub-atomic particles, which in turn will (most probably) decay into a neutrinos. In this scenario, if we wait long enough the entire universe will be filled with cold empty space populated by an ever-increasingly dilute collection of isolated of photons and cold neutrinos. A whole lot of Nothing going on. Forever. To get to this point will take a Very Long Time (probably much more than 10^{100} years, but we can't say for sure because it depends on what the Dark Energy really is and how it will evolve with time.



In short, all of modern science to date has explored a tiny fraction of the stuff that comprises the universe. The vast majority remains mysterious and unexplored.