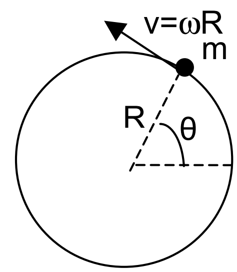
# Chapter 12+

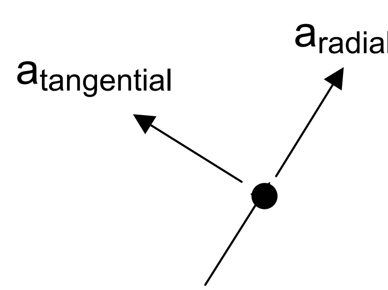
## Revisit Angular Variables and Circular Motion

| Revisit: | To-Do: |
| --- | --- |
| • Angular variables<br>• Radial and tangential acceleration<br>• Circular motion | • Second laws for radial and tangential acceleration<br>• Unit vector representations<br>• Rolling wheel analysis<br>• More angular motion formulas: work and kinetic energy |

### A Mass Going in a Circle with Both Radial and Tangential Acceleration




A small mass m is attached to a light rod pivoted on the other end and goes in a circular path with arbitrary angular velocity $\omega(t)$. (For now, we don't know what's producing this motion.) Given $\omega(t)$, we can find the acceleration $\alpha(t)$ through $\alpha = \frac{d\omega}{dt}$ . Thus recall from earlier in Ch. 12 that we now know the total acceleration of the mass, $\vec{a}_{total} = \vec{a}_{radial} + \vec{a}_{tangential}$ where:




- The radial acceleration $\vec{a}_{radial}$ points inward along the rod with radial component (calling outward positive):

  $$a_{radial} = -\frac{v^2}{R} = -\omega^2 R$$

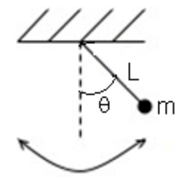
  with speed $v = \omega R$. (In the equation and the picture on the right, $a_{radial}$ refers to the radial component, and thus $a_{radial} < 0$.)

- The tangential acceleration $\vec{a}_{tangential}$ points along the tangent to the circle with tangential component (defined as positive if CCW):

  $$a_{tangential} = \alpha R$$

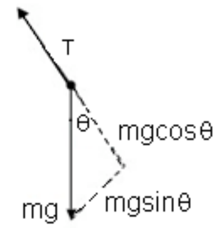
and the total force components $\vec{F}_{total} = \vec{F}_{radial} + \vec{F}_{tangential}$ are found from the second law by trivially multiplying the above stuff by m:

- The net radial component of the force is $\boxed{F_{radial} = -m\frac{v^2}{R} = -m\omega^2 R}$
- The net tangential component of the force is $\boxed{F_{tangential} = m\alpha R}$




**Example:** A **pendulum bob** follows a circular arc and thus the mass m at its end has **both radial and tangential acceleration components** as it swings back and forth. (Each component varies and even vanishes at times – see below.)

Ignoring friction, the only forces that have these components are gravity and the tension in the rod (this tension is purely radial). The rod has negligible mass. The FBD for m including a golden triangle decomposition for the weight of gravity is shown on the right and tells us:




- The net radial force component (if the radially outward direction is positive):

$$F_{radial} = +mg\cos\theta - T$$

- The net tangential force component (if the CCW direction is positive):

$$F_{tangential} = -mg\sin\theta$$

Insert the above forces into the earlier second-law results to obtain equations for $\omega$ and $\alpha$:

$$\boxed{ma_{radial} = +mg\cos\theta - T \quad \text{with} \quad a_{radial} = -v^2/L = -\omega^2 L}$$

$$\boxed{ma_{tangential} = -mg\sin\theta \quad \text{with} \quad a_{tangential} = +\alpha L}$$

Questions we can now answer using the above expressions:

a) At what points does $\vec{a}_{radial}$ have the biggest or smallest magnitude?

When v is biggest or smallest (zero), which is at the bottom or outermost part of the pendulum's swing, respectively. Notice that T must therefore equal (cancel) $mg\cos\theta$ at the outermost "peak" of the swing.

b) At what points does $\vec{a}_{tangential}$ have the biggest or smallest magnitude?

When $\theta$ is biggest or smallest (zero), which is at the outermost or bottom part of the pendulum's swing, respectively. Notice that this corresponds to when the tangential component of gravity is biggest (most negative) or smallest (zero).

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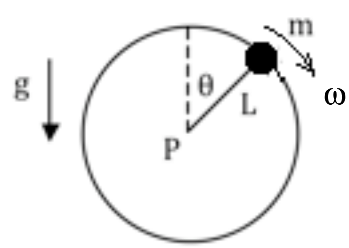


***Problem 12-4** A ball of mass* m *and negligible radius is attached to the end of a light rod (i.e., of negligible mass) of length* L *and the other end is attached to a center pivot* P*. The rod is forced by a motor at the center to swing about the pivot such that the ball rotates in a vertical circle with **constant angular velocity** $\omega$, in the CW direction as shown, while immersed in the Earth's uniform surface gravity.*

*(a) Draw an FBD for the general orientation defined by the angle $\theta$ the rod makes with the vertical as shown in the figure. There are three forces: The outward radial force* $F_r$ *on* m *due to the rod, the CCW tangential force* $F_t$ *imposed by the motor through the rod on* m *to offset gravity and enforce uniform motion, and the weight* mg *of the ball.*

*(b) Find the two second laws for the radial and tangential directions, respectively, which are analogous to the boxed equations above.*

*(c) Over one full cycle, when do* $F_r$ *and* $F_t$ *take on their largest magnitudes? Can they ever vanish? (Hint: the vanishing of* $F_r$ *depends on how big $\omega$ is.)*

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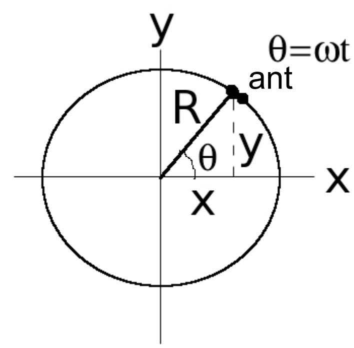
## Revisit Circular Motion With Unit Vectors

Let's introduce new unit vectors, which make 2D and 3D vectors so much easier to write down and work with! If we are willing to invest a little time here, someday we'll be thankful we did!

**Revisit that dizzy ant:** Recall, from p. 15-8, the ant in CCW circular motion, with constant ω and initial position x(0) = R, y(0) = 0, has a 2D position given by $x(t) = R\cos(\omega t)$, $y(t) = R\sin(\omega t)$ which indeed traces out a circle as it should.




In old homework (Problem 15-4) we essentially examined the position vector and its derivatives, the velocity and acceleration vectors. But we now write them in two ways: An old way followed by a new way:

$$\vec{r}(t) = (R\cos(\omega t),\ R\sin(\omega t)) = R\cos(\omega t)\,\hat{x} + R\sin(\omega t)\,\hat{y}$$

$$\vec{v}(t) = (-\omega R\sin(\omega t),\ \omega R\cos(\omega t)) = -\omega R\sin(\omega t)\,\hat{x} + \omega R\cos(\omega t)\,\hat{y}$$

$$\vec{a}(t) = (-\omega^2 R\cos(\omega t),\ -\omega^2 R\sin(\omega t)) = -\omega^2 R\cos(\omega t)\,\hat{x} - \omega^2 R\sin(\omega t)\,\hat{y}$$

What in the world are these "unit vectors" $\hat{x}, \hat{y}, \hat{z}$ ? Well, they are another way, a really useful way, of writing vectors and you may like using them, but you won't have to, in most of what we do. Here's an explanation:

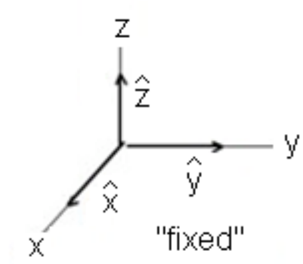
### A Little Unit-Vector Primer

Suppose the magnitude of a vector is always fixed to be unity (one), so it is dimensionless (has no units). Then let us use a special notation for it:

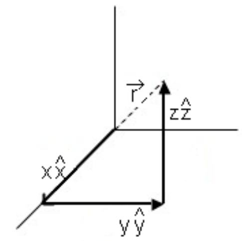
$$\boxed{\text{If we always have } |\vec{V}| = 1 \text{, define it as a unit vector } \hat{V}}$$

Sets of unit vectors are useful for expressing other vectors in terms of them, and to specify given directions. There are different names used for the rectangular (i.e., Cartesian or xyz guys) unit vectors. Some people call them $\hat{i}, \hat{j}, \hat{k}$ ; we like $\hat{x}, \hat{y}, \hat{z}$ . Whatever they are called, they are fixed in direction, and so the rectangular unit vectors are constants and so any derivative of them is zero.

$$\hat{x} = (1,0,0) \equiv \hat{i}$$
$$\hat{y} = (0,1,0) \equiv \hat{j}$$
$$\hat{z} = (0,0,1) \equiv \hat{k}$$




(Other unit vectors can be defined that are not constant; for example, we could use a radial unit vector $\hat{r}$ that changes as the position vector swings around!)

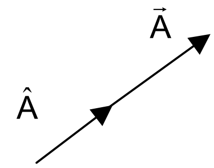



For the position vector, there is therefore a new way of writing its Cartesian coordinates along with the old:

$$\vec{r} = (x, y, z) = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

where x is the component along the $\hat{x}$ direction, etc. An arbitrary vector can be described in both ways now, too, and also in terms of its own magnitude and unit vector:

$$\vec{A} = (A_x, A_y, A_z) = A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z}$$




Another neat thing is to define the unit vector $\hat{A}$ in the direction of $\vec{A}$, so that

$$\vec{A} = A\,\hat{A}$$

where $A$ is the magnitude of $\vec{A}$ (i.e., $A$ gives the proper units and magnitude for $\vec{A}$, since $\hat{A}$ has no dimension and has unit length). So, by the way, $\vec{A}$ could be shorter than $\hat{A}$.

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***Problem 12-5*** *Recall the definition of a dot product of two vectors:* $\vec{A} \bullet \vec{B} = AB\cos\theta_{AB}$ *where* $\theta_{AB}$ *is the angle between* $\vec{A}$ and $\vec{B}$ *, and* $A$ and $B$ *are their respective magnitudes. Thus we have the following big bunch of dot products between all the Cartesian unit vectors:*

$$\hat{x}\cdot\hat{x} = 1,\ \hat{x}\cdot\hat{y} = 0,\ \hat{x}\cdot\hat{z} = 0,\ \hat{y}\cdot\hat{x} = 0,\ \hat{y}\cdot\hat{y} = 1,\ \hat{y}\cdot\hat{z} = 0,\ \hat{z}\cdot\hat{x} = 0,\ \hat{z}\cdot\hat{y} = 0,\ \hat{z}\cdot\hat{z} = 1$$

*The complete set of dot products, for all combinations of the unit vectors, is consistent with the facts that* $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$ *(the dot product is "commutative") and that* $\hat{x}$, $\hat{y}$, and $\hat{z}$ *are "orthonormal" (they are orthogonal to each other and their magnitudes are defined to equal one).*

*a) Please verify a few of these; namely, check that* $\hat{x}\cdot\hat{x} = 1,\ \hat{x}\cdot\hat{y} = 0,\ \hat{x}\cdot\hat{z} = 0$ *follow from the dot product formula* $\vec{A} \bullet \vec{B} = AB\cos\theta_{AB}$.

*b) Using the unit vector "orthonormality" and the Cartesian representations of the two vectors,*

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}, \quad \vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z},$$

*show that*

$$\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

*Remember this? We worked hard to work this into our work on work and homework on that work!*
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**Hey you! Keep in mind for later reference two helpful remarks about the dot product:**

**1) The dot product of a vector with itself gives its magnitude squared:** $\vec{A}\bullet\vec{A} = A^2$. **Therefore the magnitude of any vector is just** $\sqrt{\vec{A}\bullet\vec{A}}$

**2) The dot product of two orthogonal vectors is zero:** $\vec{A}\bullet\vec{B} = 0$ **if** $\vec{A} \perp \vec{B}$

**End Of A Little Unit-Vector Primer**

## Use Unit-Vectors to Represent Radial and Tangential Acceleration

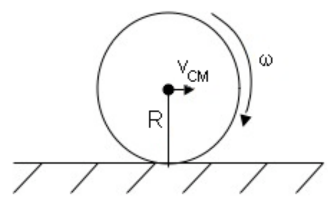
**Re-revisit that dizzy ant:**

Go back to the ant in circular motion, with position x(0) = R, y(0) = 0, but now for an arbitrary $\theta(t)$. Using the unit vector representation, the ant position is now given by

$$\vec{r}(t) = R\cos\theta(t)\,\hat{x} + R\sin\theta(t)\,\hat{y}$$

We can now calculate $\vec{v}(t) = \frac{d\vec{r}}{dt}$ and $\vec{a}(t) = \frac{d\vec{v}}{dt}$, and then pick out the radial and tangential parts, but let's not spoil the fun. Let's let you do it in a problem, following a brief discussion of rolling.

### Rolling: A Moving Rotational Axis




Consider a wheel with radius R, CM velocity $v_{CM}$ , and angular velocity $\omega$ around its CM. If it is rolling along the ground without slipping, then we will quickly prove that $\omega$ is related to $v_{CM}$ and that $\alpha = d\omega / dt$ is related to $a_{CM}$ in familiar ways:

$$\boxed{v_{CM} = \omega R \quad \text{(IF IT IS ROLLING WITHOUT SLIPPING!)}}$$

so (by taking a derivative)

$$\boxed{a_{CM} = \alpha R \quad \text{(IF IT IS ROLLING WITHOUT SLIPPING!)}}$$

The reason that these are not trivially obvious is that before we talked about the velocity of the edge of the wheel relative to a fixed center, and now it is the velocity of the center relative to the road. The neat thing is that $\omega$ is the angular velocity and $\alpha$ is the angular acceleration around the center for both uses of these formulas!

PROOF:

1) the distance s the wheel CM goes on the road is also the total arc length $R\theta$ over which the wheel rolls and touches the ground ($s=R\theta$)

2) just take a lousy derivative of s: $v_{CM} = ds/dt = R\, d\theta/dt = R\omega$, or $v_{CM} = R\omega$.

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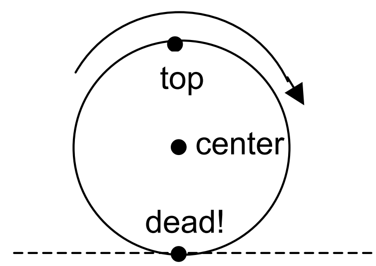
***Problem 12-6***

a) *Find the ant's velocity representation* $\vec{v}(t)$*, in terms of unit vectors and* $\omega(t) = \frac{d\theta}{dt}$*, by taking the time derivative of* $\vec{r}(t)$ *given above (remember any derivative of the unit vectors is zero).*

b) *Find the ant's (total) acceleration* $\vec{a}(t)$ *by taking the time derivative of your velocity representation from (a). Noting that* $\alpha(t) = \frac{d\omega}{dt}$*, identify the tangential part of* $\vec{a}(t)$*. What's left is the radial part. Now check that the radial part lies parallel to the* $\vec{r}(t)$ *direction and that the tangential part lies parallel to the velocity direction* $\vec{v}(t)$ *(That is, you will find that your radial part is proportional to* $\vec{r}(t)$ *and your tangential part is proportional to* $\vec{v}(t)$*.)*

c) *Shifting gears to the rolling situation, having nothing to do with the above ant and nothing to do with parts (a) and (b), let's talk about another ant. This ant has just been squashed by a tire, where the tire is rolling along with CM* $v_{CM}$ *. The tire ran over it! Just before it got squashed, it looked up at another ant riding on the axis of the tire. How fast does the doomed ant see the "center ant" is going?*




d) *Yet another ant is riding on the edge of the tire and is right up at the top of the tire when our poor ant is flattened. How fast does the doomed ant see the "top ant" is going?*

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**More Angular Motion Formulas: Work and Kinetic Energy**

When bodies move in circles, we can find convenient work and kinetic energy formulas in terms of torque and moments of inertia, respectively, so let's do it:

**Rotational Work Formula in Terms of Torque:**

We expect to do work if we applied a torque to a system and rotated it through some angle. Indeed, we will see a work formula involving the torque and the angle! Consider the ubiquitous small particle with **mass m, moving in a circle**. Recall the rotational formula from p. 12-8 for **constant angular acceleration**

$$\omega^2 - \omega_0^2 = 2\alpha_0(\theta - \theta_0) \equiv 2\alpha_0\Delta\theta$$

If we multiply both sides of the above equation by $r^2$ , we can identify $\omega^2 r^2 = v^2$, which is true for circular motion, and we obtain

$$v^2 - v_0^2 = 2\alpha_0 r^2\Delta\theta$$

Multiplying both sides by 1/2 m (everything is analogous to the linear case, p. 9-1), we get

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = m\,\alpha_0\,r^2\Delta\theta = (mr^2\alpha_0)\Delta\theta$$

But the left-hand-side is the difference in kinetic energy and the right-hand-side is equal to torque times angular displacement (since $\tau = I\,\alpha_0$ with $I = m\,r^2$ for this simple case). Thus the angular version of the work-energy theorem **for constant torque** reads:

$$\boxed{W = \tau_P\Delta\theta = \Delta K_{\text{rotation around P}} \text{ for constant } \tau_P \text{ over angle } \Delta\theta}$$

As we found for the rotational version of the second law, this is true for an extended body made up of a bunch of particles. See below for an example of its use. Adding up tiny steps as in Ch. 9+, we can generalize this further to non-constant torque using an integral:

$$\boxed{W = \int_{\Delta\theta} \tau_P\, d\theta = \Delta K_{\text{rotation around P}} \text{ for non-constant } \tau_P \text{ over angle } \Delta\theta}$$

Why did we bother telling you this when we're not even going to use it in an example? Well, we thought you might wonder about the analogous thing to $W = \int_{\Delta x} F dx$.

**Rotational Kinetic Energy Formula in Terms of Moment of Inertia and Angular Velocity:**

We derive next a handy formula for the kinetic energy of a rotating body in terms of the rotation angular velocity $\omega$. Suppose we have a system in rotation with angular velocity $\omega$ about a fixed axis. Then every mass increment $m_i$ has its speed given by $v_i = \omega r_i$ where $r_i$ is the perpendicular distance from $m_i$ to the axis. That is, $r_i$ is the radius of the circle that each $m_i$ is moving around on.

The kinetic energy for this rotating system is the sum over all these increments:

$$K(\text{rotation only}) = \sum_i \tfrac{1}{2}m_i v_i^2\Big|_{v_i=\omega r_i} = \sum_i \tfrac{1}{2}m_i\omega^2 r_i^2$$

and pulling out of the sum the common $\omega^2$ and ½ factors:

$$K(\text{rotation only}) = \tfrac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

Thus the kinetic energy for pure rotation around some axis P is

$$\boxed{K_{rotation} = \frac{1}{2} I_P\ \omega^2}$$

where the moment of inertia must be calculated with respect to the rotational axis.

**Comment:** We can use this formula for the kinetic energy due to pure rotation when we need to include rotational kinetic energy in our work-energy and conservation-of-energy calculations. See the example below.

**Example of work and kinetic energy for rotational motion:** A woman is running along side of a small frictionless merry-go-round-like disk of small mass but of radius 2.0 m, and she is pushing tangentially on its edge with a constant force of 1.0 N. Suppose a bunch of kids whose total mass is 400 kg are all sitting around the disk, and near the edge of the disk. If the woman starts pushing when the disk was at rest and she keeps pushing until it has rotated 100 times, how much work has she done and what final angular velocity is attained by the disk?

Answer: 100 revolutions = $200\pi$ radians = $\Delta\theta$, and the torque she applies is $\tau = 1 \cdot 2$ N·m for a force at ninety degrees to the radius (so $\sin\theta = 1$ in the torque formula). Thus

$$W = \tau\,\Delta\theta = 2 \cdot 200\pi = \boxed{1260 \text{ J}}$$

Since $K_i = 0$, $W = K_f$. Also, the moment of inertia is $m_{total} r^2 = 400 \cdot 4$ kg·m$^2$, because all the kids are on the edge and they are small ("point-like"!), and we ignore the merry-go-round mass:

$$W = 1260 = \frac{1}{2} I\ \omega^2 = \frac{1}{2} \cdot 1600\omega^2$$

Solving for $\omega$: $\omega = \boxed{1.25 \text{ rad/s}} \Rightarrow v = \omega r = 2.5 \text{ m/s} = 5.6 \text{ MPH}$

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***Problem 12-7*** *A merry-go-round has a mass of 1600 kg and a moment of inertia of* $5.2 \times 10^5$ kg m$^2$

a) *How small could you construct this merry-go-round to be, such that it still has its moment of inertia the same? That is, what is the smallest overall radius of the merry-go-round for a moment of inertia fixed at* $5.2 \times 10^5$ kg m$^2$*?*
b) *Ignoring friction, how much work is needed to accelerate the merry-go-round from rest to a rotation rate of one revolution for every 8.0 s?*
c) *If the torque applied to accelerate the merry-go-round in part (b) is constant, and equal to 1000 N m, how many times did the merry-go-round go around in getting to the final speed, starting from rest?*

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