

Chapter 4+

Gravity Force: Kepler's Laws and Circular Orbits, KE and PE

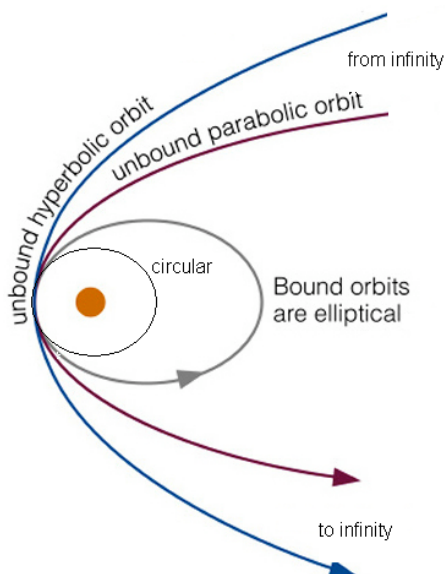
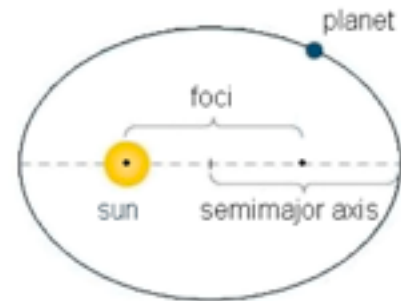
Revisit: <ul style="list-style-type: none"> Kepler's laws 	To-Do: <ul style="list-style-type: none"> Detail Kepler's laws for circular orbits Kinetic and potential energy for gravitational orbits
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More on Kepler's Three Laws

Recall that one of Newton's great achievements was to show that Kepler's laws could be derived from his gravitational force put into his second law. Recall also how we have bailed out and said that the general derivation for elliptical orbits would have to wait for your next mechanics course. **But we can almost trivially analyze these laws at work for circular orbits and that's our task in this section.**

FIRST LAW: All planets move in elliptical orbits, with the sun at one focus.

The planets are "bound" to the sun, which means that they can never escape the sun's gravitational force. Their maximum distance from the sun (their aphelion) never goes to infinity. Remember that planets (but not plutoids) actually have nearly circular orbits.



"Unbound" objects will escape to infinity and their orbits are hyperbolas in general. There is the interesting unbound case, where the object is just barely able to escape; this object has a parabolic orbit.

See the figure (which pertains to orbits for Newton's gravity) where we notice that the objects come in from infinity and escape to infinity for the unbound cases. The bound cases, including the circular orbit are shown as well.

SECOND LAW: The line connecting the planet to the Sun sweeps out equal areas in equal times. Although it isn't obvious, the proof of Kepler's second law is based on angular momentum conservation. While we don't present this proof in this course, we will verify Kepler's second law for the **circular orbit** below.

THIRD LAW: The square of the period, T (its "year!") of any planet is proportional to the cube of the semimajor axis, a, of the elliptical orbit: $T^2 \propto a^3$ We'll verify this, too, with the **circular orbit** in a problem after which we'll check it for our planets (which follow practically circular orbits).

Below we emphasize with bold font where the discussion is limited to a **circle**.

The Circular Orbit Case:

It's useful and easy to work out the details of a **circular** orbit. We thus have centripetal acceleration, which must be caused by the inward gravitational force (inward along the radial direction):

$$F_{\text{radial}}^{\text{gravity}} = - \frac{GMm}{r^2} = ma_{\text{centripetal}} = m \left(- \frac{v^2}{r} \right)$$

$$\Rightarrow v^2 = \frac{GM}{r} \quad \text{CIRCULAR ORBIT!!}$$

So if we know r we know v (or vice versa) for a **circular** gravitational orbit.

A) Check: (1) Kepler's second law, and (2) its correspondence to angular momentum conservation. This is all really trivial for a **circle** for which r is a constant, and hence, from $v^2 = GM/r$ for a **circular** orbit, v is constant; hence $\omega = v/r$ for **circular** motion is constant, too!

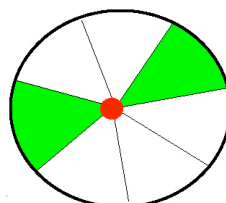
- 1) Consider the area of a sector swept out by the straight line connecting the planet to the center (sun's position), where the line has moved through an angle θ . This area is the fraction $\theta/2\pi$ of the total area of the **circle**:

$$\text{fractional area} = \frac{\theta}{2\pi} \pi r^2 = \frac{1}{2} r^2 \theta$$

and, to talk about the area swept out per unit time, consider the time rate of change of this fractional area, which is given by

$$\frac{d}{dt}(\text{fractional area}) = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega = \text{constant in time}$$

which is certainly constant because both r and ω are constant. With this constant rate of change, any two areas swept out in equal times must be equal, like for example, the shaded sectors in the figure.



- 2) The angular momentum for a mass m going in a **circular** orbit is $I\omega = mr^2\omega$, and this is constant because the gravitational force produces no torque on the planet. Both the angular momentum and the rate of change of the area swept out are proportional to the same quantity, $r^2\omega$. So, if one is constant, then the other is, too.

B) Check Kepler's third law by doing the following problem:

Problem 4-4

a) Check Kepler's third law for the **circular** orbit. That is to say, apply Newton's second law to a planet orbiting the sun with a **circular** orbit and use that information to demonstrate that the **period** of rotation is related to the **semimajor axis** of the orbit.

b) Find the proportionality constants in Kepler's third law for three different planets in our solar system from the data for their periods and their semimajor axes.

Potential Energy for Newton's Gravitational Force

Recall Newton's gravitational force on a mass m a distance r from another mass M at

the origin points radially inward: $F_{\text{radial}}^{\text{gravity}} = -\frac{GMm}{r^2}$.

Suppose that m travels from, say, $r = r_1$ to $r = r_2$. To calculate the work done by gravity on m during this motion, we need only consider the radial direction because the force only depends on r .^{*} BUT we do have to chop up the radial path into N little steps Δr_i ($i = 1, 2, \dots, N$) because the force is changing with r and is NOT constant. We find

$$W_{\text{on } m}^{\text{gravity}} \cong \sum_{i=1}^N F(r_i) \Delta r_i$$

To do this more and more accurately, we take the limit of infinitesimal steps ($N \rightarrow \infty$) which brings us to ... an integral! We have

$$W_{\text{on } m}^{\text{gravity}} = \int_{r_1}^{r_2} F \, dr = - \int_{r_1}^{r_2} \frac{GMm}{r^2} \, dr = \frac{GMm}{r_2} - \frac{GMm}{r_1}$$

Lo and behold, this only depends on the end points and we have a conservative force situation (which is not a surprise since the constant gravity approximation gave us a conservative force). Recall for a conservative force, its associated work is given simply by the negative of a change in its potential energy: $W = -\Delta U$. So from the above we have

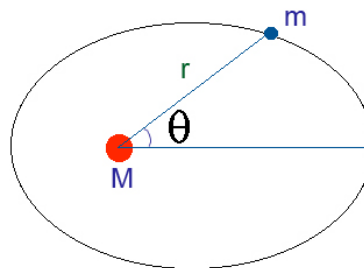
$$\boxed{U(\text{gravity}) = -\frac{GMm}{r}} \quad \text{for general Newtonian gravity}$$

So we have another formula we can really put to good use in energy discussions.

^{*}This is entirely analogous to the fact that the work done against constant gravity depends only on the changes in the vertical position and not any changes in the horizontal position.

Two comments about U(gravity):

- 1) Note that U(gravity) of m only depends on r and not the angle θ (see the figure). While the direction of the gravitational force keeps changing so that it always points toward M, its magnitude only depends on r.
- 2) Note also that the force is just the negative of the derivative of the potential energy:



$$F(\text{gravity}) = - \frac{d}{dr} U(\text{gravity}) = - \frac{d}{dr} \left(- \frac{GMm}{r} \right) = - \frac{GMm}{r^2}$$

This is a general story that we will come back to later in Chapter 8+: We derive our potential energies from doing negative integrations over the forces, and therefore we can go backwards and derive our forces by taking the negative derivatives of the potential energy! For one dimension s (such as x for a spring, or y for uniform gravity, or r for

general gravity): $U(s) \Leftrightarrow F(s) = - \frac{dU}{ds}$ See you later, alligator!

Total Energy for Newton's Gravitational Force

We saw above that Newton's gravitational force is conservative. If it is the only force present, the **total mechanical energy E is conserved** for a body of mass m subjected to the gravity of another mass M (which we consider fixed in position so we ignore its kinetic energy):

$$E = KE + PE = \frac{1}{2}mv^2 + U(r) = \frac{1}{2}mv^2 - \frac{GMm}{r} \quad \text{is constant!}$$

Therefore we obtain the same value of E for any two positions a and b on any general orbit :

$$E = \frac{1}{2}mv_b^2 - \frac{GMm}{r_b} = \frac{1}{2}mv_a^2 - \frac{GMm}{r_a}$$

Problem 4-5

a) Let's revisit the circular orbit analysis, which gave us v^2 as a function of r, and which will allow us to find E in terms of r only. Substitute this into the total energy and combine terms to show that

$$E = - \frac{GMm}{2r} \quad \text{for a circle only}$$

b) We have found the circle is an example of an orbit for which $E < 0$. But suppose you found $E = 0$ or $E > 0$ for some other orbit. (And remember the total energy is conserved, which means E is constant throughout a given orbit.) Guess in anticipation of the third cycle, 1) what the sign on the energy for gravitational orbits means with respect to whether the orbit is bound or unbound, and 2) which of the following orbits are bound or unbound: ellipse, parabola, hyperbola (note that a circle is just a special ellipse), and 3) thus what do you guess are the signs on the energy E for elliptical, parabolic, and hyperbolic orbits?
