More on Rotations

Summary of Important Collisions

Torque: $\tau_p = rF\sin\theta$

N2L Rot: $\tau_{Net} = I\alpha$

R.C: $v = \omega r$ and $a = \alpha r$

Values for Rotational Inertia (I)

Textbook/Labs may call it moment of Inertia

$I = C m R^2$ for some constant C

Common Values for I:

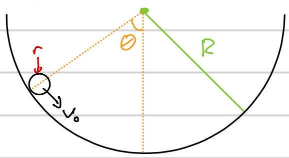
$I_{Hoop} = mR^2$ $\quad I_{Hollow\ Sphere} = \frac{2}{3}mR^2$

$I_{Disk} = \frac{1}{2}mR^2$ $\quad I_{Rod\ CM} = \frac{1}{12}ml^2$

$I_{Solid\ Sphere} = \frac{2}{5}mR^2$ $\quad I_{Rod\ End} = \frac{1}{3}ml^2$

Good to have on cheatsheet to reference easily

Example: Hollow ball rolling in bowl



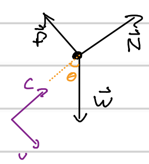
Given

- m, r, v_0, θ of ball, no slip/slide
- R of Bowl
- $I = \frac{2}{3}mr^2$

Find Normal Force, Friction force, a

Assume $R \gg r$

FBD Ball



N2L c-coord centripetal

$F_c = ma_c$

$N - W\cos\theta = m\frac{v_0^2}{R}$ radius of bowl as motion dictated by the bowl

$N = mg\cos\theta + \frac{mv_0^2}{R}$

$\boxed{N = m(g\cos\theta + \frac{v_0^2}{R})}$

N2L v-coord tangential

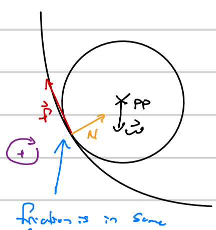
$F_v = ma_v$

$W\sin\theta - f = ma_v$

Stop!

Now Continued with Rotational

xFBD Ball



Friction is in same direction as coordinate system so torque is positive

N2L Rot

$\tau_{Net} = I\alpha$

$\tau_w + \tau_N + \tau_p = I\alpha$

$0 + 0 + rf\sin(90°) = I\alpha$

$\cancel{r}f = \frac{2}{3}mr^{\cancel{2}}\alpha \qquad \alpha = \frac{a_v}{r}$ R.C

$f = \frac{2}{3}mr\left(\frac{a_v}{r}\right)$

Stop! $f = \frac{2}{3}ma_v$

① $\tau_w: 0w\sin\theta = 0$

② $\tau_N: rN\sin(180°) = 0$

Checking G in torque

force is positive for torque if it is aligned with coordinate system.

Two Equations, Two Unknowns

① $mg\sin\theta - f = ma_v$

② $f = \frac{2}{3}ma_v$

$\cancel{m}g\sin\theta - \frac{2}{3}\cancel{m}a_v = \cancel{m}a_v$

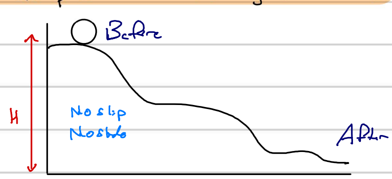
$g\sin\theta = \frac{5}{3}a_v$

$\boxed{a_v = \frac{3}{5}g\sin\theta}$

$f = \frac{2}{3}m(\frac{3}{5}g\sin\theta)$

$\boxed{f = \frac{2}{5}mg\sin\theta}$

Example: Ball rolling on slide



Given: Release @ rest

$m, R, I = \frac{2}{5}mR^2$ of ball

H of ball initially

Translational Kinetic: $K_T = \frac{1}{2}mv^2$

Rotational Kinetic: $K_R = \frac{1}{2}I\omega^2$

$\therefore K_{Tot} = K_T + K_R = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

Friction does "no work" @ point of contact

$E_{Tot} = E'_{Tot}$

$U + K_T + K_R = U' + K'_T + K'_R$

$mgy + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgy' + \frac{1}{2}v'^2 + \frac{1}{2}I\omega'^2$

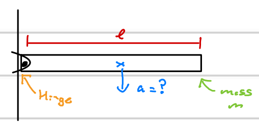
$y = H, y' = 0, v = 0, \omega = 0, \quad \omega = \frac{v}{r}$ R.C

$mgH + 0 + 0 = 0 + \frac{1}{2}mv_p^2 + \frac{1}{2}(\frac{2}{5}mR^2)(\frac{v_T}{r})^2$

$\cancel{m}gH = \frac{1}{2}\cancel{m}v_p^2 + \frac{1}{5}\cancel{m}v_T^2 \qquad v_T = v_p$

$gH = \frac{7}{10}v_p^2 \rightarrow \boxed{v_p = \sqrt{\frac{10}{7}gH}}$

Rod on Hinge

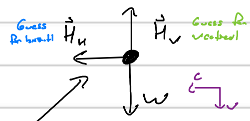


① Find Acceleration of CoM @ release. Rod starts @ rest!

Not worth spending a lot of time determining direction of H components

CoM is moving on circular path so use v/c-coordinates.

FBD Rod



Hinge Force is completely unknown! Just assign two arbitrary Forces to represent it. One is vertical one is horizontal.

If you misguess direction, answer is negated. Just note that and move on to get positive back.

N2L Horizontal c-coord

$F_c = m a_c$

$H_H = m \frac{v^2}{"R"}$ @ rest so $v=0$

$\therefore H_H = 0$ woah!

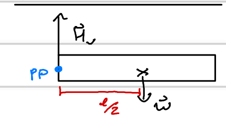
N2L Vertical v-coord

$F_v = m a_v$

$w - H_v = m a_v$

$mg - H_v = m a_v$ Stop!

x FBD Rod



② Neglect H_H bc it is 0 from N2L

N2L Rot

$\tau_{Net} = I \alpha$ Known for Rod

$\tau_v + \tau_w = \frac{1}{3} m \ell^2 \alpha$

$0 + \frac{\ell}{2} mg (\sin 90°) = \frac{1}{3} m \ell^2 \alpha$

$\frac{1}{2} \cancel{\ell} \cancel{m} g = \frac{1}{3} \cancel{m} \cancel{\ell^2} \left(\frac{a_v}{\cancel{\ell}/2}\right)$

$\frac{1}{2} g = \frac{1}{3} 2 a_v$

$\boxed{a_v = \frac{3}{4} g}$

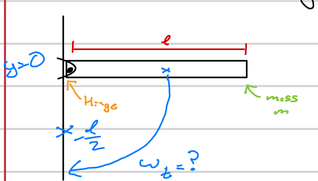
R.C. $a = \alpha r$

$\alpha = \frac{a}{r}$

$r = \frac{\ell}{2}$

r is radial distance from pp to CoM here!

Another Rod on Hinge



Hinge Force does no work be its @ point of contact. No Force applied over a distance

You will still get the right answer if you use a different reference point!

CofME Choose End of Rod as Reference, no translational K!

Conditions: W consts conservative and Hinge does no work @ point of contact!

Before = After

$E_{Tot} = E'_{Tot}$

$U + K_T + K_R = U' + K'_T + K'_R$

$U + 0 + K_{Rod} = U' + 0 + K'_{Rod}$

$mgy + 0 + \frac{1}{2} I \omega^2 = mgy' + 0 + \frac{1}{2} I \omega'^2$

$0 + 0 + 0 = mg\left(-\frac{\ell}{2}\right) + \frac{1}{2}\left(\frac{1}{3} m \ell^2\right)\omega_f^2$

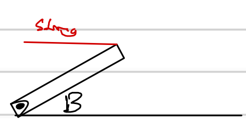
Algebra!

$\frac{1}{2} \cancel{m} g \cancel{\ell} = \frac{1}{6} \cancel{m} \ell^{\cancel{2}} \omega_f^2$

$\boxed{\omega_f = \sqrt{\frac{3g}{\ell}}}$

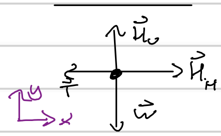
If you choose a different Reference Point, It's different!

Hinge with a String



$a = 0,\ \alpha = 0$

FBD Rod



At rest! so H_x and the must hold rod in place

N2L y-dir

$F_y = m \cancel{a_y}$ 0

$H_v - w = 0$

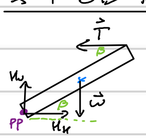
$\boxed{H_v = mg}$

N2L x-dir

$F_x = m \cancel{a_x}$ 0

$H_H - T = 0$ Stop!

x FBD Rod



$\tau_H = 0$

$\tau_T = \ell T \sin\beta$

$\tau_w = \left(\frac{\ell}{2}\right) mg \sin(90 - \beta)$

$\tau_w = \frac{\ell}{2} mg \cos\beta$ Trig!

Statics

Static Friction

$f_s \le \mu_s N$

Constrained

Kinetic Friction

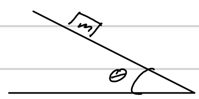
$f_k = \mu_k N$

Equality

For Statics, $a = 0$ and $\alpha = 0$

Use N2L to find N $\rightarrow$ N2L to find f_s $\rightarrow$ Apply constraint

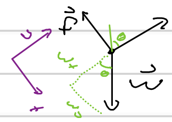
Example



Given: m, θ, μ_s

Find θ_{max}

FBD Block



N2L y-dir

$F_y = m a_y^{\,0}$

$N - w\cos\theta = 0$

$N = mg\cos\theta$

N2L x-dir

$F_x = m a_x^{\,0}$

$w\sin\theta - f_s = 0$

$f_s = mg\sin\theta$

Example shows $f_s \le \mu_s N$ is not useless! Good for determining maximums!

Apply Constraint: $f_s \le \mu_s N$

No an Equal Sign!

$mg\sin\theta \le \mu_s\, mg\cos\theta$

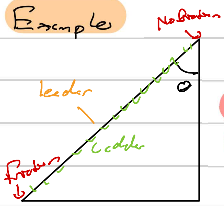
$\frac{\sin\theta}{\cos\theta} \le \mu_s$

$\tan\theta \le \mu_s$

$\theta \le \arctan \mu_s$

$\therefore$ $\boxed{\theta_{max} = \arctan(\mu_s)}$

Example

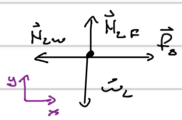


Given

m, θ, L

Find all forces

FBD Ladder



N2L y-dir

$F_y = m a_y^{\,0}$

$N_{LF} - W = 0$

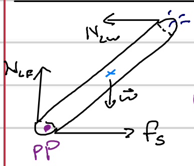
$\boxed{\begin{array}{l} N_{LF} = mg \quad (1) \\ W = mg \quad (2) \end{array}}$

N2L x-dir

$F_x = m a_x^{\,0}$

$f_s - N_{LW} = 0$ Stop!

FBD Ladder



N2L Rot

$\tau_{Net} = I\alpha^{\,0}$

$\tau_{N_{LF}} + \tau_{f_s} + \tau_w + \tau_{N_{LW}} = 0$

$0 + 0 - \frac{L}{2} w\sin\theta + L N_{LW}\sin(90-\theta) = 0$

$-\frac{1}{2}\cancel{L} mg\sin\theta + \cancel{L} N_{LW}\cos\theta = 0$

$N_{LW}\cos\theta = \frac{1}{2} mg\sin\theta$

$\boxed{N_{LW} = \frac{1}{2} mg\tan\theta}$ (3)

$f_s - N_{LW} = 0$

$f_s = N_{LW}$

$\boxed{f_s = \frac{1}{2} mg\tan\theta}$ (4)

Place PP where it kills most forces

Recall: $\sin(90-\theta) = \cos\theta$

$\cos(90-\theta) = \sin\theta$

Center of Mass and Matter Distribution

CoM: $\vec{r} \equiv \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots m_n \vec{r}_n}{M}$

$x_{cm} \equiv \frac{m_1 x_1 + m_2 x_2 + \dots m_n x_n}{M}$

$a_{cm} \equiv \frac{d}{dt}(\vec{v}_{cm}) = \frac{m_1 a_1 + m_2 a_2 + \dots m_n a_n}{M}$

$M \equiv m_1 + m_2 + \dots m_n$

$v_{cm} \equiv \frac{d}{dt}(\vec{x}_{cm}) = \frac{m_1 v_1 + m_2 v_2 + \dots m_n v_n}{M}$

Good definitions to have in your back pocket

Continuous Distribution of Matter

$$\boxed{\begin{array}{l} \vec{r}_{cm} \equiv \frac{1}{M}\sum_{i=1}^{n} m_i r_i \\ \vec{r}_{cm} \equiv \frac{1}{M}\int_{volume} r\, dm \end{array}}$$

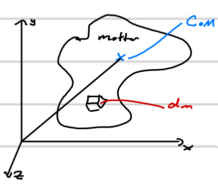
where $M = \sum_{i=1}^{n} m_i$

Uniform Density

$\rho = \frac{m}{V}$

Not Uniform Density

$\rho = \frac{dm}{dV} \rightarrow dm = \rho\, dV$



$x_{cm} = \frac{1}{M}\int_{volume} x\, dm$

$= \frac{1}{M}\int x\rho\, dV$

$= \frac{1}{M}\iiint x\rho\, dx\,dy\,dz$

Triple Integral DN

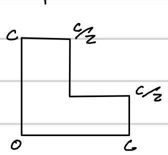
Calculus 3 CoM

$\vec{x}_{cm} \equiv \frac{1}{M}\iiint x\rho\, dx\,dy\,dz$

$\vec{y}_{cm} \equiv \frac{1}{M}\iiint y\rho\, dx\,dy\,dz$

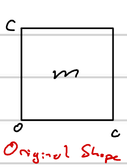
$\vec{z}_{cm} \equiv \frac{1}{M}\iiint z\rho\, dx\,dy\,dz$

Example: A cut-out Square from a Square



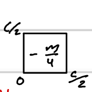
Find x_{cm}?

=



Original Shape

+



Negative Area of the Square

$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m\left(\frac{c}{2}\right) + \left(-\frac{m}{4}\right)\left(\frac{c}{4}\right)}{m + \left(-\frac{m}{4}\right)}$

$= \frac{\frac{mc}{2} - \frac{mc}{16}}{\frac{3m}{4}} = \frac{2\cancel{m}c - \frac{\cancel{m}c}{4}}{3\cancel{m}} = \frac{\frac{7}{4}c}{3} = \boxed{\frac{7c}{12}}$

Recall: $\vec{F}_{\substack{Net\\Sys\\Ext}} = m\vec{a}_{cm}$

Special Cases

① If $\vec{F}_{\substack{Net\\Sys\\Ext}} = 0$ means $\vec{a}_{cm} = 0$, $\vec{v}_{cm} = \text{constant}$

then $\vec{P}_{Tot} = m\vec{v}_{cm} = \text{constant}$ CofLM

This means system is isolated

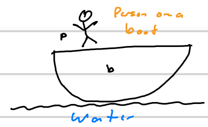
Individual Forces don't need to be 0, just need to cancel

② If $\vec{v}_{cm} = 0$ means $P_{Tot} = 0$

$V_{cm} = 0 \rightarrow$ CoM is not changing!

Then $\vec{r}_{cm} = \text{constant}$ in CoMRF

Recall: Elastic $\rightarrow K = K'$



$x_{cm} = \frac{m_p x_p + m_b x_b}{m_p + m_b}$

x_{cm} is constant so if you move one way, the boat will move one way to keep x_{cm} constant

Moving to CoMRF (center of mass reference frame)

$v' = v - v_{cm}$

velocity in CoMRF; CoMRF in "Lab" Frame; CoM in "Lab" Frame

Used in 4 box Method!

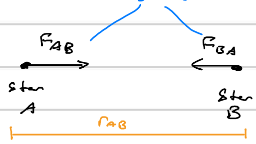
Universal Gravity Revisited

$F_{UG} = \frac{GMm}{r^2}$

"attractive"

distance of separation

Example

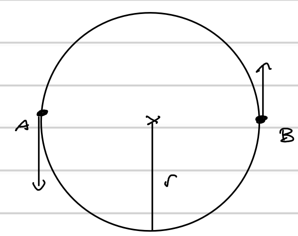


$F_{AB} = F_{BA} = F_{UG} = \frac{GM_AM_B}{r_{AB}^2}$

$a_A = a_B = a$ is true!

$a = \frac{v^2}{r}$

Find v_A, v_B



N2L

$F_{Net} = m\,a$

$\frac{GMm}{r^2} = m\frac{v_a^2}{r}$ ← radius; r^2 ← distance of separation

Let $m_A = m_B = m$

$\frac{GM^2}{(2r)^2} = m\frac{v^2}{r}$

$\frac{GM^2}{4r^2} = \frac{mv^2}{r}$

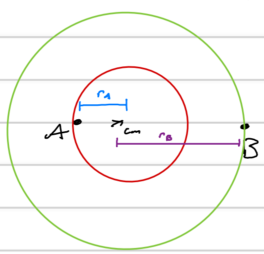
$\frac{GM}{4r} = v^2$

$\boxed{v = \sqrt{\frac{GM}{4r}}}$

Both stars move w/ the same velocity

$v_A = v_B = v$

Example: $m_A > m_B$



Find $v_A, v_B, \omega_A, \omega_B$

$x_{cm} = \frac{m_A x_A + m_B x_B}{m_A + m_B} = 0$

$= \frac{m_A(r_A) + m_B(r_B)}{m_A + m_B} = 0$ Distance between

$m_A r_A + m_B r_B = 0$

$\frac{r_A}{r_B} = \frac{m_B}{m_A}$

N2L $F_{Net} = m_A$

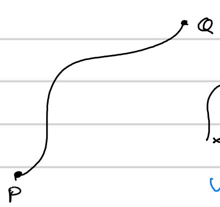
$\frac{GM_AM_B}{(r_A + r_B)^2} = m_A\frac{v_A^2}{r_A}$ radius

Replace A with B for star

$\frac{GM_Br_A}{(r_A + r_B)^2} = v_A^2 \rightarrow \boxed{v_A = \sqrt{\frac{GM_Br_A}{(r_A + r_B)^2}}}$

R.C $v = \omega r \rightarrow \omega = \frac{v}{r}$

Path Integrals and Conservative Forces



Work Definition

$W_P = \int_P^Q \vec{F}\cdot d\vec{r}$

$\int_{x_p}^{x_Q} F_x dx + \int_{y_p}^{y_Q} F_y dy + \int_{z_p}^{z_Q} F_z dz$

Work is Conservative, Friction is not

Conservative Forces

① Force is conservative if work done by that force from P to Q is independent of path taken

② The work done by that force around a closed loop is zero

Example

Given $F = Axy\,\hat{i} + Bz\,\hat{j}$, is conservative?

$\int_a^b Axy\,dx + \int_a^b Bz\,dy$

$\frac{1}{2}Ax^2y\Big|_a^b + Bzy\Big|_a^b$

$W_f \neq 0$ so not conservative

Example

$F = \int Ax\,\hat{i} + \int By^3\,\hat{j} + \int Ce^{-bz}\,\hat{k}$

$\frac{x^2}{2}\Big| \quad \frac{y^4}{4}\Big| \quad -be^{-bz}\Big|$

Bridge Concepts

Newton's Laws: 1st, N2L, N3L

Conservative Laws: CofME, CofLM, CofAM

Classical Work Energy Theorem

$W_T = W_{F_1} + W_{F_2} + W_3 \ldots = \Delta K$

$W_T = \int F_{Net}\,dr = \Delta K$

Momentum Impulse Relation

$F_{Net} = \frac{dP}{dt}$

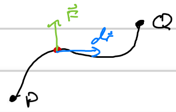
$\int F_{Net}\,dt = \Delta p$

Spring Force

$F_{sp} = Kx$ $\quad \int Kx\,dx$

$\rightarrow \frac{1}{2}Kx^2$

$W_{\vec{F}}(P \to Q) = -\int_{\text{Path}} \vec{F} \cdot d\vec{r}$



$\vec{F} = F_x \hat{\imath} + F_y \hat{\jmath}$

$d\vec{r} = (dx)\hat{\imath} + (dy)\hat{\jmath}$

Add z components if needed

$W_{\vec{F}}(P \to Q) = -\int_{x_P}^{x_Q} F_x dx + -\int_{y_P}^{y_Q} F_y dy$

Recall Rotational Inertia

$I_{\text{hoop}} = mR^2$

$I_{\text{Disk}} = \frac{1}{2} m R^2$

$I_{\text{Solid Sphere}} = \frac{2}{5} m R^2$

$I_{\text{Hollow Sphere}} = \frac{2}{3} m R^2$

$I_{\text{Rod @ Cent}} = \frac{1}{12} m l^2$

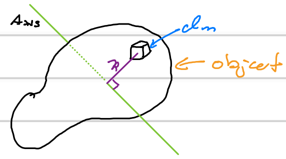
$I_{\text{Rod @ End}} = \frac{1}{3} m l^2$

Definition of Rotational Inertia

$I \equiv \iiint R^2 dm$

R - distance from mass element to axis of rotation (Perp. to axis!)

dm - mass element



$\to I = \int_x \int_y \int_z R^2 \rho \, dz \, dy \, dx$

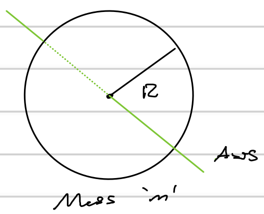
Contend with dm

$dm \to \therefore dm = \rho dV$

mass density: $\rho = \frac{M}{V}$ ← mass, ← volume

e.g. $dV = dx\,dy\,dz$

Example: A hoop



$I = \iiint R^2 dm$

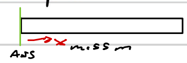
Here, $R = R$

$\therefore I_{\text{hoop}} = R^2 \underbrace{\iiint dm}$

mass integrated over all space

$\to \boxed{I_{\text{hoop}} = mR^2}$

Example: Rod about end



$I_{\text{end rod}} = \iiint R^2 dm$

Here $R = x$

$dm = \rho dV$

$V_{\text{rod}} = lA$ $A =$ cross sectional area of rod

$dV = A\,dx$

$\rho = \frac{m}{lA}$ def of density

$\to dm = \left(\frac{m}{lA}\right)(A\,dx) = \frac{m}{l} dx$ (ρ, dV)

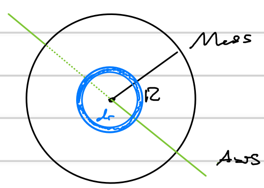
$I_{\text{end rod}} = \int_{x=0}^{x=l} x^2 \frac{m}{l} dx$

$= \frac{m}{l} \int_0^l x^2 dx$

$= \frac{1}{3}\left(\frac{m}{l}\right)(x^3)\Big|_0^l$

$\boxed{I_{\text{end Rod}} = \frac{1}{3} m l^2}$

Example: A disk



Here $R = r$

$dm = \rho dV$

$V_{\text{disk}} = A_{\text{disk}} T = \pi r^2 T$ Thickness T

$dV = dA_{\text{annulus}} T = 2\pi r (dr) T$

$\rho = \frac{m}{V} = \frac{m}{\pi r^2 T}$

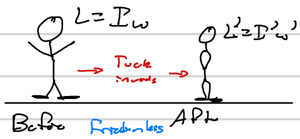
$\boxed{I_{\text{disk}} = \frac{1}{2} m R^2}$

Use integration and inertia def to solve

Angular Momentum

$L \equiv I\omega$

Example: Spinning Figure Skater



Conservation of Angular Momentum

Condition: "Isolated" "CoAM"

$\to \tau_{\text{Net Sys Ext}} = 0$

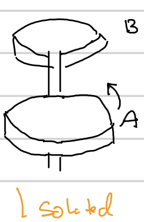
Check Torques! Not Forces

Individual Forces can be ≠ 0 but must sum to 0

Then: Before = After

$L_{\text{Tot}} = L'_{\text{Tot}}$

Example: Disk Dropped on a turntable



Given

m_A, m_B, R_A, R_B

$\omega_A = \omega_0$ $\omega_B = 0$

Friction must be zero so CoAM ok!

Find $\omega_f' = \omega_A = \omega_B$ after disk gets dropped

Before = After

$L_{\text{Tot}} = L'_{\text{Tot}}$

$L_A + L_B = L_A' + L_B'$

$I_A \omega_A + I_B \omega_B = I_A \omega_A' + I_B \omega_B'$

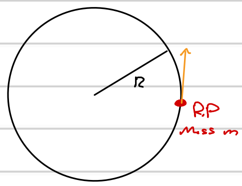
$I_A \omega_A + 0 = I_A \omega_f + I_B \omega_f$

$\omega_f = \frac{I_A \omega_A}{I_A + I_B} = \frac{\frac{1}{2} m_B R_B^2}{\frac{1}{2} m_B R_B^2 + \frac{1}{2} m_A R_A^2} \omega_A$

$\boxed{\omega_f = \frac{\omega_A m_B R_B^2}{m_B R_B^2 + m_A R_A^2}}$

Angular Momentum of a point Particle

P.P. UCM



Recall: $I \equiv \int_{\text{All space}} R^2 \, dm$

$= mR^2$ for P.P. UCM

Particle in Circular Motion

$I_{PP} \equiv mR^2$

$L_{UCM} = I\omega = mR^2\omega$ R.C $v = \omega r$

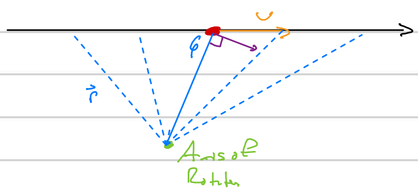
$= mvR$

Particle in Circular Motion

$L_{UCM} = mvR$

Example: P.P. moving on straight line

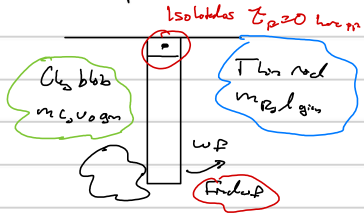
$L_{PP} = mv_\perp r = mvr\sin\phi = mvr_\perp$



$r_\perp = b$ is "impact parameter"

Example: Clay Blob

Isolated as $\tau_P = 0$ here



Before = After

$L_{tot} = L_{tot}'$ L_{UCM}

$L_c + L_R = L_c' + L_R'$

$m_c v_0 l \sin(90)° + 0 = m_c l^2 \omega_f + \frac{1}{3} m_r l^2 \omega_f$

$m_c v_0 l = \omega_f (m_c l^2 + \frac{1}{3} m_r l^2)$

$$\boxed{\omega_f = \frac{m_c v_0 l}{m_c l^2 + \frac{1}{3} m_r l^2}}$$