Chapter 9

Back to the Future: Work and Energy

- **Introduce Kinetic Energy**
- **Introduce Work**
- **Introduce Potential Energy**
- **Using Conservation of Energy to Make Predictions**

Kinetic Energy and Work from Newton's Second Law

Starting with the 1D constant-acceleration result of Problem 2-2 (b) (and hence constant force),

$$v^2 = v_0{}^2 + 2a_0(x-x_0)$$

we can derive what we mean by "work" and "kinetic energy" and how this equation is essentially a relationship between kinetic energy and work. (Everything comes from Newton's second law!)

If we substitute $a_0 = F_0/m$ (note from Chapter 2 that acceleration a_0 and force F_0 are x-components

and not magnitudes here): $$v^2 = v_0{}^2 + 2\frac{F_0}{m}(x-x_0)$$

Now multiply both sides of the equation by m/2 in order to isolate the F_0 $(x-x_0)$ term:

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_0{}^2 + \tfrac{1}{2}\not{m}\not{2}\frac{F_0}{\not{m}}(x-x_0)$$

If we now change notation to refer to the initial velocity $v_0 = v_i$, the final velocity $v = v_f$, the final position $x = x_f$, and the initial position $x_0 = x_i$:

$$\tfrac{1}{2}mv_f^2 = \tfrac{1}{2}mv_i^2 + F_0 (x_f - x_i)$$

This cries out for us to define

"kinetic energy" $\boxed{K \equiv \dfrac{1}{2}mv^2}$ **(we will see that this is good for 3D velocities, too)**

"work" $\boxed{W \equiv F (x_f - x_i) \equiv F \Delta x}$ **(only for constant force in 1D)**

Thus $\tfrac{1}{2}mv_f^2 = \tfrac{1}{2}mv_i^2 + F_0 (x_f - x_i)$ becomes $K_f = K_i + W$ or

"work-energy relation" $\boxed{K_f - K_i = W}$ **(we will see that this is completely general)**

The names are motivated by the idea that when we somehow do "work" on the system, it gains energy of motion or "kinetic energy." We have "work in" = "kinetic energy out."

Generalization to 3D: The previous boxed expressions are correct for each of the three dimensions. We can get a total 3D kinetic energy from the sum

$$K = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2 \quad \text{or}$$

$$K = \frac{1}{2}mv^2 \quad \text{with} \quad v^2 = v_x^2 + v_y^2 + v_z^2$$
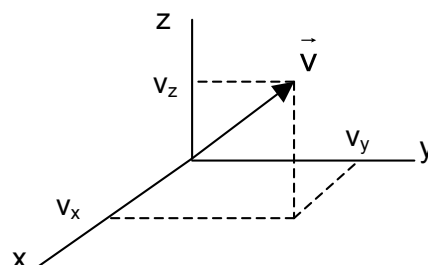
We can get a total work from the sum (in terms of the 3D force and its components)

$$\boxed{W = F_x\,(x_f - x_i) + F_y\,(y_f - y_i) + F_z\,(z_f - z_i) \quad \text{by } \mathbf{constant}\ \vec{F} = (F_x, F_y, F_z)}$$

The two above sums are therefore related by the same equation as before

$$K_f - K_i = W \qquad (\text{ work-energy relation in all 3D})$$

Work due to Force at an angle θ: Consider a mass m in straight-line motion along the x axis. Suppose $\vec{F}$ is constant (so all components are constant). We get the following picture at any point along the path
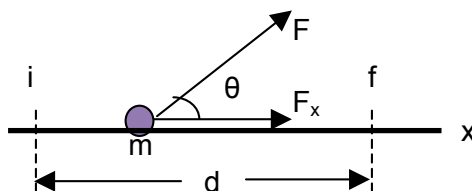
where the force is at an angle θ with the x axis, so that $F_x = F\cos\theta$ now with magnitude $F \equiv |\vec{F}|$. If we go a positive distance, $x_f - x_i = d$, we have $W = F_x\,(x_f - x_i) = F\cos\theta\, d$ from the 1D version of the above work formula. In fact, for any straight-line motion of a body going a distance d,

$$\boxed{W\ (\text{constant force, straight path, with θ between them}) = Fd\cos\theta}$$

Comments:

1) The figure on the right shows this formula can be used for any straight-line motion covering a distance d, where θ is the angle between the constant force $\vec{F}$ and the direction of the motion. We could define our x-axis along any straight line and get the same answer: $W = F_x\,(x_f - x_i) = F\cos\theta\, d$ using the new x axis.

2) Notice we get zero work for forces perpendicular to the motion because $\theta = \pi/2 \Rightarrow \cos\theta = 0$. We can understand this from the beginning discussion we had, since there's no acceleration (and hence no change in velocity and hence no change in kinetic energy) if there's no force component along the motion.
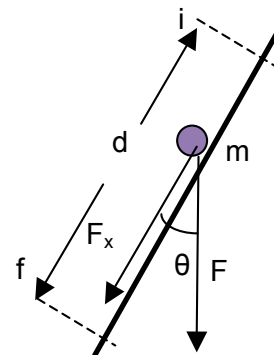
3) Notice that if the angle between the force and the displacement is 180° (i.e., they are in opposite directions), then the work done by the force is negative.

4) **When should you NOT use W = Fd cosθ ?** Don't use this if the force changes direction and/or magnitude, and/or if the path is not straight.

For a little preview of what to do if the force is varying (see cycle 2 for details), we first introduce the definition of the "dot product" between two vectors $\vec{A}$ and $\vec{B}$ as $\vec{A} \cdot \vec{B} \equiv A\,B\,\cos\theta$ with magnitudes A and B, and where θ is the angle between them. In terms of the dot product, $W = F\,d\,\cos\theta = \vec{F} \cdot \vec{d}$. If the force is not constant along the path, this is true only for an infinitesimal displacement: $\Delta W = \vec{F} \cdot \Delta\vec{r} \equiv F\,\Delta r\,\cos\theta$ and we then have to add up all the little pieces (the limit of the sum of many small pieces gives us ... the integral!) We get

$$W = \int \vec{F} \cdot d\vec{r}$$

More later in the next cycle; you don't have to know any of this stuff now.

Work and Kinetic Energy: 1D Prediction Examples

We can use the Work-Energy Relation

$$\Delta K \equiv K_f - K_i = W \text{ (for all forces working on m as it moves from } \vec{r}_i \text{ to } \vec{r}_f \text{)}$$

for which we calculate the work according to

$$W = F_x (x_f - x_i) = F\,d\,\cos\theta$$

to make predictions or otherwise understand **1D** motion for **constant** forces acting parallel or anti-parallel to the motion.

Units of Energy are J=Joules in SI and ft-lb in American units
• Kinetic energy, work (and potential energy – see later) all have same units
• 1 Newton × 1 meter = 1 Joule; 1 lb × 1 foot = (surprise) 1 ft-lb.

Positive work 1D example: A 200 kg pile driver falls 2.5 m (starting from rest).   What is its kinetic energy at the end of the fall?

The work done by gravity is (the figure shows us that θ =0)   m

$$W_{\text{gravity on m}} = F\,d\,\cos\theta\big|_{\theta=0} = +mgd = +200(9.8)(2.5) = +4900 \text{ J}$$

$$\text{Work} - \text{Energy} \Rightarrow K_f - \underset{\underset{0}{\downarrow}}{K_i} = W_{\text{gravity on m}} = +4900 \text{ J}$$

Therefore $\boxed{K_f = \ 4900 \text{ J}}$ The positive work (W > 0) done here by gravity goes into 4900

Joules of final kinetic energy. (We could have used old methods to find the kinetic energy, but the above helps us get more familiar with the powerful work-energy relationship.)
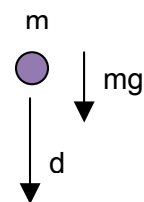
Negative work 1D example: A 10-lb ball is thrown upwards with an initial speed 10 ft/s . What is the work done by gravity up until the ball reaches its maximum height?

(Again, this is simple to analyze with our old methods, but let's use ΔK to directly find the work done by gravity all the way up to the top of the trajectory where $v_f = 0$!)

 Remember we divide weights in lb by 32 to get the mass in slugs. A weight of 10 lb means a mass of 10/32 slugs, so

$$K_i \ = \ \frac{1}{2}mv_i^2 \ = \ \frac{1}{2}\left(\frac{10}{32}\right)(10)^2 \ = \ 15.6 \ \text{ft-lbs}$$

$$K_f \ = \ 0$$
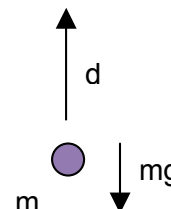
Therefore

$$W_{\text{gravity on m}} \ = \ 0 \ - \ 15.6 \qquad \text{or}$$

$$\boxed{W_{\text{gravity on m}} \ = \ - \ 15.6 \ \ \text{ft-lbs} \ < \ 0}$$

Comments:

• The negative sign agrees with $W = F\, d\, \cos\theta$ since the force is in the opposite direction to the displacement which means $\theta = \pi$ or $\cos\theta = -1$, so that $W = - Fd \ < 0$.

• You could solve for d by setting $W = F\, d\, \cos\theta = - 15.6$ with $F = 10$, $\cos\theta = -1$. So we get

$$d \ = 1.56 \text{ ft}$$

You could have found d with our old formula $v_f^2 \ - \ v_i^2 \ = \ 2a\,(x_f - x_i)$ in which $a = - 32$, etc. - the equation that was the starting point of this chapter!

Implications of negative work: W < 0 means that the system loses kinetic energy according to the work-energy relation:

$$\Delta K \ \equiv \ K_f - K_i \ = \ W \ < \ 0.$$

We say that work is done "against" gravity by the system if it comes out negative.

We say that work is done "by" gravity on the system if it comes out positive.

In the above example, the system uses up kinetic energy to work against gravity. **Gravity fought against the motion in the example and the kinetic energy has now been converted to some kind of stored energy – we return to this later.**

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Problem 9-1 *Before we get to more detailed work-energy problems, consider a block of wood moving along a horizontal, rough, and fixed table surface. Its initial motion might have come, for instance, from an initial shove. Now consider an external force being applied to the block: EXPLAIN whether the following statements are true or false:*
  (a) *The work done by the normal force of support on the block due to the table depends on the direction of motion.*
  (b) *The work done by gravity on this horizontally moving block is always zero.*
  (c) *The work done by the table frictional force on the block is always positive.*
  (d) *The sign on the work done by the table frictional force on the block depends on the direction of the motion.*
  (e) *The work done by the external force always increases the kinetic energy.*
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Problem 9-2 *With the brakes fully applied, a 1500-kg automobile decelerates at the constant rate of 8.0 m/s^2 from the initial speed of 90 km/h all the way to rest.*
  (a) *What is the constant braking force in magnitude and direction (positive or negative) acting on the automobile, if the direction of the initial motion is defined as positive?*
  (b) *What is the change in the kinetic energy of the automobile?*
  (c) *From (b) what must be the work done by the braking force in bringing the automobile to a stop from 90 km/h? What does the sign on your answer mean?*
  (d) *Find the total distance the automobile goes before it comes to a stop.*
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Zero work 1D example: A man lifts up a 30 kg box, at **constant speed, to** a vertical distance of 1.5 m. What is the total work done on the box?

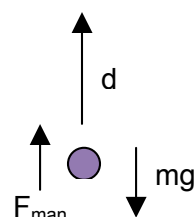The two forces, F_{man} and mg, acting on the box are shown in the figure. The magnitudes are related by $F_{man} = mg$ by the second law with zero acceleration. In $W = F d \cos\theta$, we have $\theta = 0$ for the work by the man and $\theta = \pi$ for the work by gravity:

$$W_{by\ man\ on\ m} = F_{man} d \cos\theta\big|_{\theta=0} = + mgd = 30(9.8)(1.5) = 441\ J$$

$$W_{by\ gravity\ on\ m} = F_{grav} d \cos\theta\big|_{\theta=\pi} = - mg\ d = - 441\ J$$

Thus $\boxed{W_{total} = 441 - 441 = 0}$

Comments:
  • The zero total work checks with $\Delta K \equiv K_f - K_i = 0$. It also checks with calculating the total work all at once, since $F_{total} = 0$ for constant velocity.
  • We can now think of three different ways the total work can vanish:

          1) if $F_{total} = 0$, or

          2) if d = 0 , or

          3) if $\vec{F}_{total} \perp \vec{d}$ (i.e., if its $\theta = \pi/2$) – if the total force is perpendicular to the path

The above example also helps us to think of the work done by the man as now being stored in gravitational "potential" energy (aha! a new name that's very helpful – see what's next).

Getting Potential Energy from Work: Constant Gravity Example

If the mass m moves from y_i to y_f, the work done by gravity on m is most conveniently calculated by the coordinate form of the work formula, $W = F_y (y_f - y_i)$ with $F_y = -mg$, if up is the positive y direction:

$$W_{\text{by gravity on m}} = - mg (y_f - y_i) = - (mgy_f - mgy_i)$$

This is independent of whether $y_i < y_f$ or $y_i > y_f$. We have rearranged the right-hand-side to motivate the following definition

$$\boxed{W_{\text{gravity on m}} = - \left[U_g(y_f) - U_g(y_i) \right]}$$

in terms of a new function we call the "potential energy" for constant gravity

$$\boxed{U_g(y) \equiv mg\, y}$$

Conservative Forces Lead to Potential Energy Functions

Constant gravity is a simple example of forces that have "potential energy" functions U associated with them. We rewrite the above two boxed equations for the general case of conservative forces:

$$\boxed{W \text{ (by "conservative forces")} = - \Delta U}$$

with

$$\boxed{\Delta U \equiv U_f - U_i}$$

What does this all mean?

- If ΔU is positive, we have negative work. The negative work by a force means that, instead of saying that the force does work on our system, we say our system does work against it - and positive energy has gone somewhere.

- Such positive energy is said to have been stored (through an increased "potential" energy), and we can recover it just by going back from y_f to y_i .

- If that positive energy is recoverable, we have a "conservative" force and gravity is just such a force. With such forces we could return to the starting point and end up with "zero work," like in our earlier example.

- A key feature is that the work done thus only depends on the end points i and f. It is independent of the path taken from the beginning point to the ending point. See later.

- What forces are "non-conservative?" Well, for example, friction! See the next problem.

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Problem 9-3 *Consider sliding friction. Explain without a detailed calculation*
    *(a) why the work done **by** the friction for a body sliding on a fixed surface is always negative*
        *(i.e., so the work done **against** friction – to overcome friction – is always positive), and*
    (b) why this means that the friction force is not conservative. You now can go back to Problem
        9-1 and compare your answer to two of the true/false questions.

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Conservative Plus Non-Conservative Work-Energy Relation

Separate the total work into work done by conservative forces plus the rest of the work (i.e., the work done by the non-conservative forces):

$$W = W_c + W_{nc}$$

where

$$W_c \equiv \text{work by all the conservative forces} = -\Delta U$$
$$W_{nc} \equiv \text{work by all the non-conservative forces}$$

Thus

$$W = -\Delta U + W_{nc}$$

Thus the general work-energy relation $\Delta K = W$ now reads $\Delta K = -\Delta U + W_{nc}$ or

$$\Delta K + \Delta U = W_{nc}$$ rewritten equivalently as $$K_f + U_f = K_i + U_i + W_{nc}$$

remembering $\Delta K \equiv K_f - K_i$ and $\Delta U \equiv U_f - U_i$,

Comment 1: Note that if there are lots of particles in a system, K is the sum of all the particles' kinetic energies. Similarly, U and W_{nc} would also be sums of the particles' potential energies and non-conservative work, respectively. If there are additional conservative and non-conservative forces, they will also contribute to U and W_{nc}, respectively.

Comment 2: we might add in an external work $W_{external}$ like that done by the man in the example on p. 9-5, where $W_{external}$ might or might not be conservative but we often don't need to know which it is!

Conservation of Mechanical Energy

Now for the biggie. If there are **only** conservative forces, we have a ***conservation law***:

CONSERVATION OF MECHANICAL ENERGY:
$$\Delta K + \Delta U = 0$$ or rewritten as $$K_f + U_f = K_i + U_i$$
provided all forces are conservative ($W_{nc} = 0$)

Comments:

• The total mechanical energy K + U is constant for all time if there are no non-conservative forces. We often just say "total energy" instead of total mechanical energy.

• This gives us short cuts to problem solving (especially in jumping to the future without having to bother with the details in between). It's good old Doc (Emmett) Brown in the Back to the Future movies!

• And even the simple short cuts are simpler than we might first guess. Come with us now on the next page as we journey through the Earth's gravitational force but only close to its surface.

The Heights of the Gravitational Potential Energy $U_g = mgy$

A wonderful thing about the gravitational potential energy is that it is independent of the horizontal position – it depends only on the altitude y you're at. No matter how many horizontal zigs and zags the path is in moving from (x_i, y_i, z_i) to (x_f, y_f, z_f), the work done by gravity is

$$- \Delta U_g = - (U_{g, f} - U_{g, i}) = - (mg\, y_f - mg\, y_i).$$

independent of the changes in x and z.

Indeed, the work done by gravity in moving horizontally is ... zero. Recall when we move perpendicularly to any force, that force does no work on us.

Example:

Consider three identical balls of mass m sliding from rest down inclined frictionless boards of various lengths, as shown in the figure to the right. Each board has been angled so that its ball rolls down from the same height H in each case. What is the kinetic energy at the bottom of each board? Thus which ball is going fastest when it hits the bottom of its board?
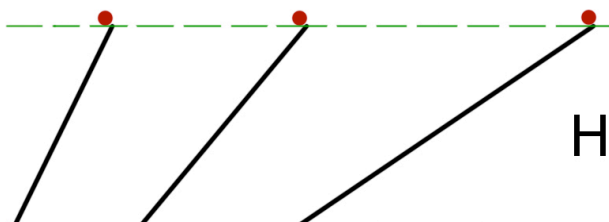
Solution: There are two forces acting on each ball: Uniform gravity and the normal surface force. The normal force does no work; a force perpendicular to the motion along each board yields W =0.

Gravity is conservative so we can now have conservation of mechanical energy $K + U_g$:

$$(K + U_g)_{top} = (K + U_g)_{bottom}$$

With $K_{top} = 0$, we solve for K_{bottom} :

$$K_{bottom} = U_{g, top} - U_{g, bottom} = mgy_{top} - mgy_{bottom} = mg(y_{top} - y_{bottom}) = mg\, H$$

All balls have the same kinetic energy at the bottom! They're all going the same speed!

Comment: But using $\Delta K = W$, doesn't our old formula $W = Fd \cos\theta$ tell us that the work (and hence the kinetic energy) is larger for a longer board (a longer d)? No, because $d \cos \theta = H$ (see the figure) and $F = mg$ – end of story.

We lie. We do want to say this another way. An increase in d is exactly cancelled by the decrease in $F\cos\theta$, which is the component of gravity along the board.
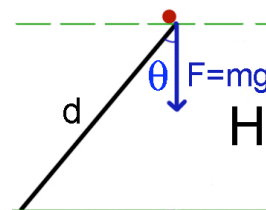
A problem is given on the next page.

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Problem 9-4

A cable car with mass 2000 kg is moving on a track up Nob Hill in San Francisco.

(a)   What is the change in the gravitational potential energy if the cable car has moved 60 m higher in altitude (vertical elevation) from its lowest point? Does it matter what you call the origin of your vertical coordinate? Does it matter whether the cable car has moved horizontally at the same time (i.e., does it matter whether the new position is not directly above the original point)?

Suppose that the car is ascending at 13 km/h along the track when the cable breaks at an elevation of 30 m above the lowest point. The car slows down and eventually is sent racing back down the track to its lowest point!

(b)   Use the change in the gravitational potential energy to find the speed in km/h that the car attains at that lowest point. We ignore friction. Why is this answer independent of the mass of the car?

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Problem 9-5   Food for Thought

a) Describe very briefly how you might have to generalize the work-energy theorem for a situation where you are pushing hard against a wall, the wall is not moving, yet you get tired! You're definitely using up energy. Where did that energy come from and where did it go? Hint: Although you can give a simpler answer, look up "ATP depletion" for a fancy answer.

b) Explain very briefly once and for all what the sign on W means. That is, give examples where W > 0 and W < 0, and explain what their signs mean.

c) Explain very briefly the difference between the two formulas, $W = \Delta K$ and $W = Fd\cos\theta$. Hint: one equation explains how to calculate something and one predicts what happens during some motion!

d) A quick question that we'll answer in cycle two but you can anticipate here: Suppose an applied force, in one dimension, is not constant but instead is a function of x, given by $F(x)$ (the x-component as usual). In general, the work performed by this force in going from 0 to x is NOT given by $F(x) x$. Explain why $F(x) x$ is not true, for the general case of non-constant force, to your kid brother, who knows the formula $W = F_x d$. Can you guess the general form of the answer, using calculus?

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