# Chapter 3++

# More on 2D Trajectories

| Revisit: | Learn: |
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| • Center of Mass when there are no external net forces<br>• 2D trajectories (constant a)—with constant gravity | • How to use CM and 2D trajectory equations to locate pieces when a projectile breaks into pieces<br>• 2D intersections with relative velocities |

### 2D Trajectories — CM & "Break-Up"

We review a really great fact about the CM, namely that **the motion of the CM point** of a system moves like **a single point particle** whose **mass is the total mass M** of the system and whose **acceleration is determined by the external net force through Newton's 2nd law:**

$$\boxed{\vec{F}_{\text{ext, net}} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(M\,\vec{v}_{CM}) = M\,\frac{d\vec{v}_{CM}}{dt} = M\,\vec{a}_{CM}}$$

- Recall that, when we are talking about cars, balls, boats, etc., as "points," the points are really the CM points, and it is sensible to draw free-body diagrams with forces all drawn as if they were applied at a single point, ignoring the size of the objects, if all we want is the overall motion!
- Recall, this is why we could quickly predict that the net (friction) force direction on a wheel had to be in the same direction as the acceleration of the center of the wheel.
- Recall we could predict that, if the net external force is zero, the total momentum is constant, and the CM moves uniformly forever!

And now for something we haven't officially highlighted:

**Universal g for the CM in Free Fall:** That is, we can use the above stuff to also prove that we may treat the effect of uniform gravity on a body as if it were all squished into its CM point!!

Proof: Consider a system of N particles in uniform gravity:

$$\vec{F}_{\text{gravity,net}} = \sum_{1}^{N} m_i\vec{g} = \left(\sum_{1}^{N} m_i\right)\vec{g} = M\vec{g}$$

and, **if** this is the **only** external force,

$$M\vec{a}_{CM} = \vec{F}^{\text{external}}_{\text{net}} = \vec{F}_{\text{gravity,net}} = M\vec{g} \Rightarrow \boxed{\vec{a}_{CM} = \vec{g}}\ !$$

**Thus the CM of any body thrown into the air would follow the trajectory that we studied earlier (without air drag) for single points.**

**Example:** A ball of mass m with a firecracker of negligible mass inside of it is thrown straight up in the air from the point x = y = 0 with an initial velocity $v_0$ (at t=0). The firecracker goes off and the ball gets blown into two pieces $m_1$ and $m_2$ **before** it gets to the top of its trajectory ($m = m_1 + m_2$).

Some time after the explosion, suppose the $m_1$ piece has a position, at some time t, given by the coordinates $x_1(t)$ and $y_1(t)$ (you could calculate these coordinates if you knew the position and velocity at the time of the explosion but for now let's just say we know them). What then are the coordinates $x_2(t)$ and $y_2(t)$ of the $m_2$ piece at time t? Because we know what the CM is doing and we know what the other piece is doing, we can use the CM equation to solve for $x_2(t)$ and $y_2(t)$.

SOLUTION: We know the CM will continue its simple constant gravity trajectory, which is given by our early 1D stuff: $x_{CM}(t) = 0$ and $y_{CM}(t) = v_0t - ½\, gt^2$, because the ball is thrown straight up.

Thus we now know $x_2(t)$ and $y_2(t)$ since the combined positions of the two pieces yield the CM position by

**Solve for $x_2(t)$:**

$$x_{CM}(t) = 0 = \frac{m_1x_1(t) + m_2x_2(t)}{m_1 + m_2} \Rightarrow m_1x_1(t) + m_2x_2(t) = 0$$

$$\Rightarrow \quad x_2(t) = -\frac{m_1}{m_2}x_1(t)$$

**Solve for $y_2(t)$:**

$$y_{CM}(t) = v_0t - \tfrac{1}{2}gt^2 = \frac{m_1y_1(t) + m_2y_2(t)}{m_1 + m_2}$$

$$\Rightarrow y_2(t) = \frac{m_1+m_2}{m_2}(v_0t - \tfrac{1}{2}gt^2) - \frac{m_1}{m_2}y_1(t)$$

But there is an interesting wrinkle. The CM moves along its simple constant gravity trajectory **only** as long as there are no other external forces acting upon it. If either piece hits the ground first, what happens? The CM trajectory abruptly changes! **That is, hitting the ground represents an additional external force and the CM motion is changed from a purely constant gravity trajectory –** the piece hitting the ground certainly has had its trajectory changed. **Thus the other piece has a trajectory that is no longer given by the above formulas,** although it is easy to follow its path just by using our good old single particle trajectory stuff, given some initial data.

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**Problem 3-6** A projectile of mass 9.6 kg is shot off from the ground with an initial velocity of 12.4 m/s at an angle of 54° above the horizontal.

a) Where does the projectile hit the ground again (the terrain is completely horizontal for this trajectory) and how long after the launch?

Suppose that instead, and at some time after its launch, an explosion splits the projectile into two pieces. One piece, of mass 6.5 kg, is observed at 1.42 s after the launch at a height of 5.9 m and a horizontal distance of 13.6 m from the launch point.

b) Find the location of the second fragment (essentially the rest of the mass) at that same time. (Did it hit the ground?)

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## 2D Intersections with Relative Velocities

**Example:** Ursula throws a ball at 22.0 m/s and at a 63.0° angle above the horizontal. Jason rides a bike past Ursula at 5.0 m/s just when (say, at t=0) she releases the ball.

a) What is the ball's trajectory as seen by Ursula? Well, for Ursula, the ball is a projectile that follows a parabolic trajectory. The components of the initial (t=0) ball velocity relative to Ursula are

$$v_{0_x} = v_0\cos\theta = (22.0\text{ m/s})\cos(63.0^\circ) = 10.0\text{ m/s}$$

$$v_{0_y} = v_0\sin\theta = (22.0\text{ m/s})\sin(63.0^\circ) = 19.6\text{ m/s}$$

The x- and y-equations of motion for the ball's position at time t relative to Ursula are

$$x_{B/U} = x_0 + v_{0_x}t = 10.0\,t \quad \text{m}$$

$$y_{B/U} = y_0 + v_{0_y}t - \frac{1}{2}g\,t^2 = 19.6\,t - 4.90\,t^2 \quad \text{m}$$

where we chose the ball's initial position to be at the origin x = y = 0 .

From simple calculations we've done more times than we care to remember, we can show that the ball reaches a maximum height of 19.6 m at t = 2.0 s and hits the ground at 40 m at t = 4.0 s.

b) What is the ball's trajectory as seen by Jason? Well, we want to figure out the ball's **position** relative to **him**. Since Ursula moves with a constant velocity (equal to - 5 m/s) relative to Jason, her x position relative to Jason's is $x_{U/J} = -5\,t$ m/s . Her y position is always the same as his and for convenience we take this to be the same as the ball's initial y position: y = 0.

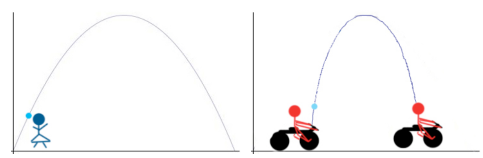
So in terms of Ursula's coordinates the ball's 2D velocity components relative to Jason are

$$v_{x,\,B/J} = v_{x,\,B/U} + v_{x,\,U/J} = v_{x,\,B/U} - 5.0$$
$$v_{y,\,B/J} = v_{y,\,B/U} + v_{y,\,U/J} = v_{x,\,B/U} + 0$$

and the ball's 2D position components relative to Jason are

$$x_{B/J} = x_{B/U} + x_{U/J} = 10.0\,t - 5.0\,t = 5.0\,t$$
$$y_{B/J} = y_{B/U} + y_{U/J} = (19.6\,t - 4.90\,t^2) + 0 = 19.6\,t - 4.90\,t^2$$

c) Notice above that $x_{B/J} < x_{B/U}$ and $y_{B/J} = y_{B/U}$ . This means that Jason sees (right figure) the ball follow a narrower trajectory than the trajectory that Ursula sees (left figure).



d) What happens when Jason travels at a faster speed, jogging at 10 m/s? Because Jason's horizontal motion is the same as the ball's ($V_x = v_x = 10.0$ m/s in Ursula's frame), he doesn't see the ball moving either right or left. Instead, Jason sees the ball move *vertically* up and down in frame J. . In the y-direction Jason still sees the ball go up to the same height that Ursula sees it go up to; for Jason, the ball goes straight up, reaches a height of 19.6 m, and falls straight back down.

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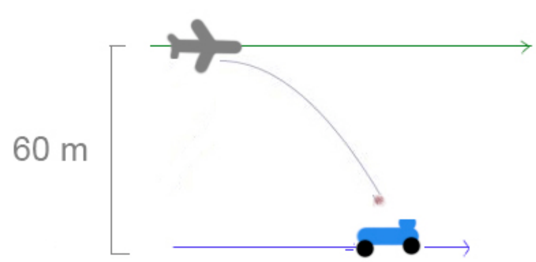
**Problem 3-7**

In your job, you are designing stunts for a new James Bond movie. In one scene the writers want to drop a package from a very low flying plane onto a moving convertible sports car. The car is really speeding along a horizontal road at 40 m/s and is being chased by the plane, which is catching up to it by flying at 50 m/s with respect to the roadway below. The plane is flying at a steady altitude of 60 m.




If the actor holding the package is viewing the car through a carefully aimed telescope to determine when the package should be dropped, at what angle with respect to the horizontal should the car be (as seen from the plane) when the package is released? The package is released by simply nudging it out the plane's side exit.

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