# Chapter 9++

## More on Kinetic Energy and Potential Energy

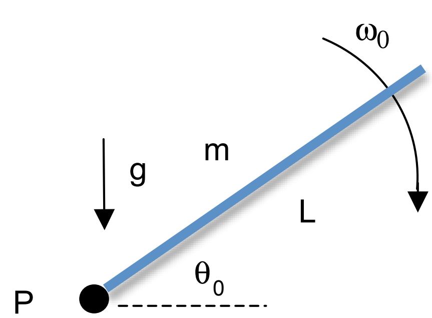
*BACK TO THE FUTURE I++*

*More Predictions with Energy Conservation*

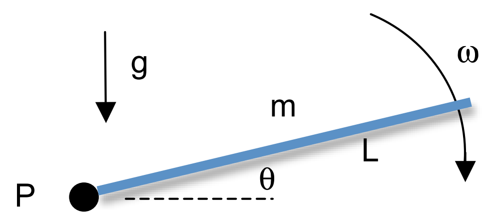
| Revisit: | To-Do: |
|---|---|
| • Kinetic energy for rotation<br>• Potential energy $M_{total}\, g\, y_{CM}$ for a body in constant gravity | • Kinetic energy for rolling motion<br>• How to calculate power – a quickie formula<br>• Appendices for<br>  • Showing kinetic energy has two parts<br>  • More moments of inertia - spheres |

**Rotation About a Fixed Axis: Example of a Rod**

A uniform rod or stick of mass m and length L has one end pivoted at a frictionless hinge. It can only rotate and its rotation axis is through P and perpendicular to the "page." The rod has been given an initial angular velocity $\omega_0$ at the initial angle $\theta_0$ with the horizontal as shown. Then it is allowed to swing downward under the influence of gravity and, as a result, swings down with increasing angular velocity. How can we predict the new $\omega$ at the later angle $\theta$, given $\omega_0$ at $\theta_0$?




Answer: To have a shortcut to this prediction we ask what is or is not conserved here? *Linear momentum* of the CM (or of the mass m) is **NOT** conserved because the pivot and gravity represent a nonzero net external force during the rotation (and they don't cancel each other in general); linear momentum is not useful for pure rotations anyway. *Angular momentum* of the CM (or of the individual masses) around the pivot is **NOT** conserved because gravity provides a nonzero torque around that pivot. But the phrase **"without friction" is a big hint:** *kinetic plus potential (total mechanical) energy* **IS** conserved since gravity is a conservative force and the pivot is frictionless.




So this gets us a quick path to a prediction: Equate the total energy before and after.

Recall that the entire kinetic energy for fixed-axis rotation is given by $K_P = \frac{1}{2} I_P \omega^2$ where $I_P$ is defined with respect to the rotation axis P. Thus $I_P$ here is the moment of inertia for a rod around its end: $I_P = I_P(\text{rod}) = 1/3\, mL^2$. Recall also that the potential energy in uniform gravity is as if all the mass m were concentrated at the CM: $U=mgy_{CM}$, and, while it is not new to us, we see the change in gravitational potential energy is found by following the change in the rod's CM height. The rod CM is originally at the height $\frac{L}{2}\sin\theta_0$ above the horizontal position. Its height is $\frac{L}{2}\sin\theta$ at any other angle $\theta$. Therefore potential energy change is mg times the change in height, or

$$\Delta U = m\, g\, \Delta y = mg\frac{L}{2}(\sin\theta - \sin\theta_0)$$

Since the change in the rotational kinetic energy is

$$\Delta K_P = \frac{1}{2}I_P\,\omega^2 - \frac{1}{2}I_P\,\omega_0^2 = \frac{1}{2}I_P(\omega^2 - \omega_0^2) = \frac{1}{6}mL^2(\omega^2 - \omega_0^2)\ ,$$

conservation of energy demands $\Delta K_P = -\Delta U$ in going from angular velocity $\omega_0$ to $\omega$. Thus

$$\Delta K_P = \frac{1}{6}mL^2(\omega^2 - \omega_0^2) = -\Delta U = -mg\frac{L}{2}(\sin\theta - \sin\theta_0)$$

or $\quad \omega^2 - \omega_0^2 = -g\frac{3}{L}(\sin\theta - \sin\theta_0)$ or, rearranging the minus signs,

$$\boxed{\omega^2 = \omega_0^2 + 3\frac{g}{L}(\sin\theta_0 - \sin\theta)}$$

Since $\theta$ decreases as the rod falls (and note that $\theta$ becomes negative as it goes below the horizontal axis), we can verify that $\omega^2 > \omega_0^2$ when we go from a higher elevation to a lower one.

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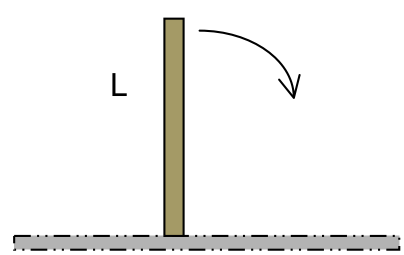
**Problem 9-10**

A uniform thin beam of length L and mass M is in a vertical position with its lower end on a rough surface that prevents this end from slipping. Suppose the beam is nudged so as to topple in the direction shown. Find the angular velocity (as a vector $\vec{\omega}$: magnitude and direction) of the beam, about its fixed end, just before impact in terms of g and L. You might like to derive the answer by conservation of energy, or you can use the result in the text for a quick escape and conserving your own energy! ☺

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## Rotations About a Moving Axis: Rolling Kinetic Energy

Any rigid body motion such as rolling, or flying through the air, or whatever, can be considered as a combination of translational plus rotational motion

**(See the proof in the appendix at the end of this chapter.)**

In particular, the total kinetic energy is

$$\boxed{K_{total} = K_{CM} + K_{internal}}$$

where the energy associated with ***CM motion*** (i.e., the translational motion) is

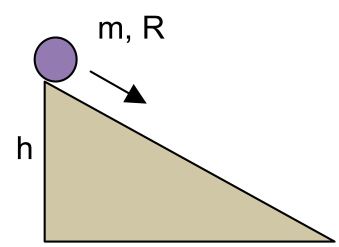
$$\boxed{K_{CM} = 1/2\, M\, v_{CM}^2}$$

and, for a rigid body, the internal motion can always be written in the rotational form

$$\boxed{K_{internal} = \tfrac{1}{2} I_{CM}\, \omega^2 \qquad \text{(rigid body)}}$$

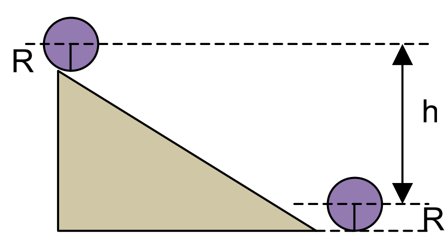
where the ***rotation is about the CM*** with angular velocity $\omega$

**Famous inclined plane example: using energy conservation to predict the speed of a ball or wheel rolling down an inclined plane.**




Consider a cylinder starting from rest and rolling down a straight slope without slipping. Notice the rotational axis is now moving (but stays perpendicular to the page). We use conservation of energy to predict how fast the cylinder is going at the bottom. Including the rotational KE, the decrease in PE is taken up by the increase in total KE: $\Delta K = - \Delta U$ (remember this is just $K_f - K_i = - (U_f - U_i)$, which is equivalent to $K_i + U_i = K_f + U_f$ , if you prefer!). With $U = mgy$, $y_f - y_i = - h$ * , and $K_i = 0$, we find from the above two KE contributions:

$$mgh = \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I_{CM} \omega^2$$




* Recall that the constant-gravity potential energy is easy to calculate! It is as if the total mass were concentrated at the CM point (also, note in the figure to the right that the CM point drops vertically the same distance h as the hill is high!).

BUT how are $\omega$ and $v_{CM}$ related? Recall from Ch. 12:

$$\boxed{v_{CM} = \omega R \qquad \text{(IF IT IS ROLLING WITHOUT SLIPPING!)}}$$

Getting back to the inclined plane, consider any kind of ball or cylinder, having radius R, mass m, and moment of inertia $\beta m R^2$ about its rolling axis (through its CM). (As we discuss in the appendix, the dimensionless constant $\beta$ would be equal to 2/5 for a solid uniform sphere, and 2/3 for a hollow sphere, for examples.) Starting from rest and rolling all the way down the incline, for a given $\beta$, the ball has a final speed v that can be predicted from the energy conservation equation:

$$mgh = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2 = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}\beta mR^2\left(\frac{v_{CM}}{R}\right)^2 = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}\beta mv_{CM}^2 \quad \text{or}$$

$$mgh = \frac{1}{2}(1+\beta)mv_{CM}^2 \Rightarrow v_{CM} = \sqrt{\frac{2gh}{1+\beta}}$$

Comments on this result? Well, do this problem!

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**Problem 9-11** We should not be the only ones suffering, er, having fun here:

(a) What happens when we change m for the above rolling object? Explain!

(b) What happens when we change R for the above rolling object? Explain!

(c) What happens when we change $\beta$ for the above rolling object? Explain!

(c') What happens when we keep on asking you more and more questions? Just ignore this! ☺

(d) How does the final speed for this rolling ball compare with the final speed for any object that slides without rotating (and without friction)?

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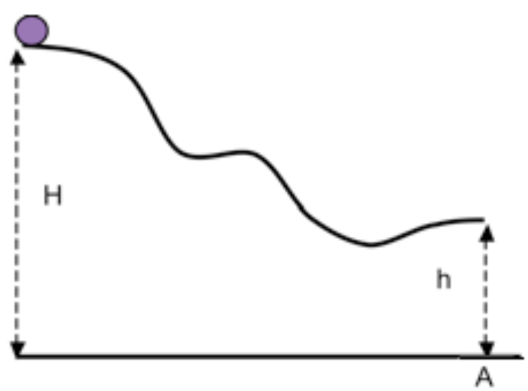
**Problem 9-12** A homogeneous sphere of mass m and radius r has the CM moment of inertia given by 2/5 $mr^2$, which has been derived in the appendix. It starts from rest at the upper end of the track shown, and rolls without slipping until it flies off the right-hand end. (Ignore the possibility that the ball might fly into the air before the end and not stay on the curvy track!)




(a) For a rolling sphere with no slipping, find the total kinetic energy in terms of m and the speed v of its center.

$$\text{HINT: } \frac{1}{2}I_{CM}\,\omega^2 = \frac{1}{2}\,\frac{2}{5}\;mR^2\frac{v^2}{R^2} = \frac{1}{5}mv^2$$

(b) If H=60.0 m and h=20.0 m and the track is horizontal at the right-hand end, determine the distance, to the right of point A, where the ball strikes the horizontal base line. (Notice how the mass cancels out again in all of this, as in the previous examples.)

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## Power and its Calculation

What about the fact that some sources of work can do the job much faster than others? As usual, we want to talk about the (time) rate of doing work and compare rates. We call the "work rate" or the "rate of energy output or input" the **power** or **power output** or **power input.**

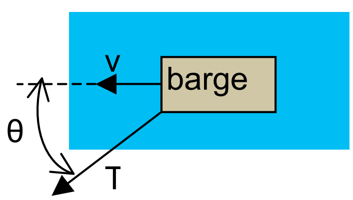
**Average Power and Units:** $P_{AV}$ = work done in a time interval/time interval. The SI unit is the Watt (W) = Joule/second. As for the exciting American units we know and love, 1 US HP = 550 ft lb/s = 746 W (1 W = 0.738 ft lb/s).

**Instantaneous Power:** $P = \dfrac{dW}{dt}$

1 dimension: Recall that $dW = F\,dx$ $\Rightarrow P = \dfrac{dW}{dt} = F\dfrac{dx}{dt} \Rightarrow \boxed{P = Fv \quad \text{1D}}$

3 dimensions: Recall $dW = F\,dr\cos\theta \Rightarrow P = \dfrac{dW}{dt} = F\dfrac{dr}{dt} \Rightarrow \boxed{P = Fv\cos\theta = \vec{F}\bullet\vec{v} \quad \text{3D}}$

**Example:** A horse walking along the shore pulling a barge through a rope with tension T in the rope




$$\Rightarrow \quad P = Tv\cos\theta$$

A little more interesting example resides in the following problem:

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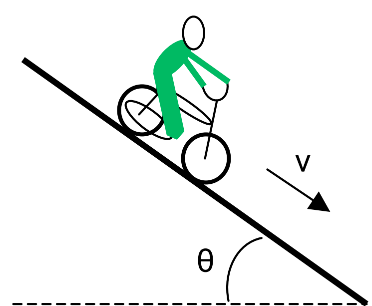
**Problem 9-13** A bicyclist is coasting down a hill at constant speed v. The bicycle and woman have a total mass m. The hill is inclined at angle θ.




a) What is the power generated by the force of gravity on the bicyclist?

b) What is the power generated by all the friction forces operating on the bicyclist?

c) What is the power generated by the normal force on the bicycle due to the hill?

d) Are your answers instantaneous power or average power?

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# Appendix A

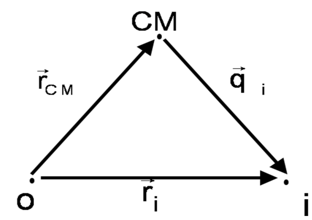
## Kinetic Energy when the CM is Moving

We add translational motion to any rotational motion - the general theorem to be discussed below is that any motion of a rigid body can be considered as a combination of translational plus rotational motion. (Recall that earlier we separated out the overall CM motion and we saw how it is related to the net external force.)

The total motion can be decomposed into the motion ***of*** the CM plus motion ***about*** the CM using the positions and velocities relative to the CM, as follows. Recall we have used $\vec{r}_i$ as the position of particle $i$ in these kinds of general discussions. Now define $\vec{q}_i$ as the position of that particle relative to the CM position:

$$\vec{r}_i = \vec{r}_{CM} + \vec{q}_i \quad \text{or} \quad \vec{q}_i = \vec{r}_i - \vec{r}_{CM}$$

which corresponds to the vector addition triangle shown:




Then we can talk about the velocity of the particle in terms of its velocity relative to the CM motion, just by taking the derivative of the above,

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \frac{d\vec{r}_{CM}}{dt} + \frac{d\vec{q}_i}{dt} \quad \text{but} \quad \frac{d\vec{r}_{CM}}{dt} = \vec{v}_{CM} \text{ and } \frac{d\vec{q}_i}{dt} \equiv \vec{u}_i$$

$$\Rightarrow \vec{v}_i = \vec{v}_{CM} + \vec{u}_i$$

This lets us derive the relation between the total momentum and the CM velocity very easily:

$$\vec{p} = \sum_{i=1}^{N} m_i \vec{v}_i = \sum_{i=1}^{N} m_i \vec{v}_{CM} + \sum_{i=1}^{N} m_i \vec{u}_i = \left(\sum_{i=1}^{N} m_i\right)\vec{v}_{CM} + \sum_{i=1}^{N} m_i \frac{d\vec{q}_i}{dt}$$

but $\sum_{i=1}^{N} m_i d\vec{q}_i = 0$ (changes in the individual motions cannot change the CM – remember the canoe problem as an example!) and $\sum_{i=1}^{N} m_i = M$. Thus, as far as momentum is concerned, there is no net extra "internal" momentum:

$$\vec{p} = M\vec{v}_{CM} = \vec{p}_{CM}$$

This is something we already knew. (Remember that we defined $\vec{p} \equiv \vec{p}_{total}$.)

But there **is** an "internal" contribution to kinetic energy. See the next page.

We now show that the total kinetic energy of a bunch of particles reduces to a sum of "CM kinetic energy" plus "internal kinetic energy."

Remembering our vector dot products stuff, we get

$$K_{total} = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i \, \vec{v}_i \bullet \vec{v}_i = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\vec{v}_{CM} + \vec{u}_i\right) \bullet \left(\vec{v}_{CM} + \vec{u}_i\right)$$

$$= \sum_{i=1}^{N} \frac{1}{2} m_i (v_{CM}^2 + 2\vec{u}_i \bullet \vec{v}_{CM} + u_{CM}^2) = (\sum_{i=1}^{N} \frac{1}{2} m_i)\, v_{CM}^2 + (\sum_{i=1}^{N} m_i \vec{u}_i) \bullet \vec{v}_{CM} + \sum_{i=1}^{N} \frac{1}{2} m_i u_i^2$$

As before, $\sum_{i=1}^{N} m_i \vec{u}_i = 0$ (the "canoe identity!") and $\sum_{i=1}^{N} m_i = M$. Therefore, as we advertised earlier,

$$\boxed{K_{total} = K_{CM} + K_{internal}}$$

with

$$\boxed{K_{CM} = \frac{1}{2} M \, v_{CM}^2}$$

and

$$\boxed{K_{internal} = \sum_{i=1}^{N} \frac{1}{2} m_i u_i^2}$$

If the system is a rigid body, then $K_{int}$ is due entirely to rotation about the CM

$$K_{internal}(\text{rigid body}) = \sum_i \tfrac{1}{2} m_i u_i^2 \Big|_{v_i = \omega r_i} = \frac{1}{2} (\sum_i m_i \, r_i^2)\, \omega^2$$

Therefore

$$\boxed{K_{internal}(\text{rigid body}) = \frac{1}{2} I_{CM} \omega^2}$$

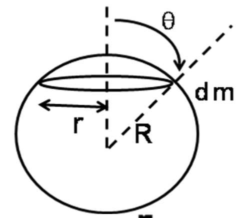
where the axis goes through the CM and the direction of the axis is determined by the situation (e.g., it is parallel to the surface for rolling).

# Appendix B

- **CM Moment of Inertia of a Uniform Solid Sphere**

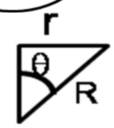
**Answer:** $\boxed{I_{CM} = \frac{2}{5} M R^2}$ $\left\{ \begin{array}{l} \text{for a uniform SOLID sphere, radius R, mass M, and} \\ \text{the axis through the center (i.e., the CM moment of inertia)} \end{array} \right.$

Proof (again just for your enjoyment – that is, enjoying the fact that you don't have to master it BUT you might be surprised at your ability to follow it):

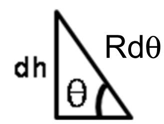



Think of as a stack of concentric solid discs, each of mass dm, as shown. Summing over the whole stack of little moments of inertia gives the integral as the limit of differential dm (each with differential $dI = ½\, dm\, r^2$):

$$I = \sum_i \frac{1}{2} \Delta m_i r_i^2 \rightarrow \int \frac{1}{2} dm\, r^2$$




But dm = ρ dV for mass volume density ρ and volume of the disc dV = π $r^2$ dh where the radius of the disc is r and its small height is dh. Also, r = Rsinθ from the little golden triangle on the right with the sphere radius R as the hypotenuse. Continuing to relate everything to θ, we have dh = Rdθsinθ from another golden triangle noting the hypotenus Rdθ is the little arc length subtended by dθ. So far we have gotten to




$$I = \int \frac{1}{2} dm\, r^2 = \frac{1}{2} \rho\pi \int (R\sin\theta)^2\, R\,d\theta \sin\theta\, (R\sin\theta)^2 = \frac{1}{2} \rho\pi R^5 \int_0^{\pi} \sin^5\theta\, d\theta$$

The mass density for the uniform solid sphere is $\rho = M / \frac{4}{3}\pi R^3$ and … now to do the integration!

Change variables from θ to x = cosθ (so dx = - sinθ dθ) to obtain

$$I = \frac{1}{2} \frac{M}{\frac{4}{3}\pi R^3} \pi R^5 \int_0^{\pi} \sin^5\theta\, d\theta = \frac{3}{8} MR^2 \int_{+1}^{-1} (1 - x^2)^2 (-dx) = \frac{3}{8} MR^2 \int_{-1}^{+1} (1 - 2x^2 + x^4)\, dx$$

$$= \frac{3}{8} MR^2 (2 - 2(2/3) + 2/5) = \frac{3}{8} MR^2 (16/15) = \frac{2}{5} MR^2 \quad \text{(as promised)}$$

- **CM Moment of Inertia of a Uniform Hollow Sphere**

**Answer:** $\boxed{I_{CM} = \frac{2}{3} M R^2}$ $\left\{ \begin{array}{l} \text{for a uniform HOLLOW sphere, radius R, mass M, and} \\ \text{the axis through the center (i.e., the CM moment of inertia)} \end{array} \right.$

No proof shown – someday when you've got nothing to do and nowhere to go ... try it! You can adapt the above kind of integration to a hollow sphere (think of a stack of concentric hoops). As we expect, it's bigger than for a solid sphere, since more mass is farther out for a given R.