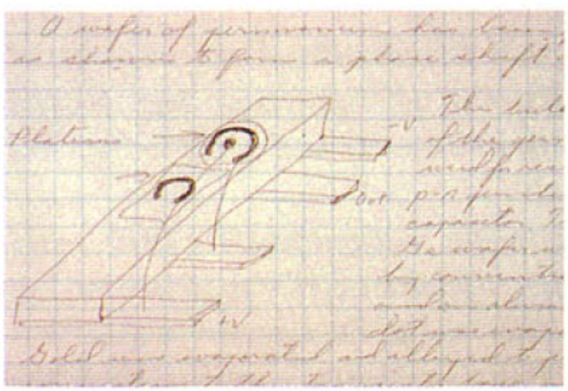
Physics 122 Spring 2025

Dielectrics

Homework 6

Due: Sunday March 23th, 11.59PM



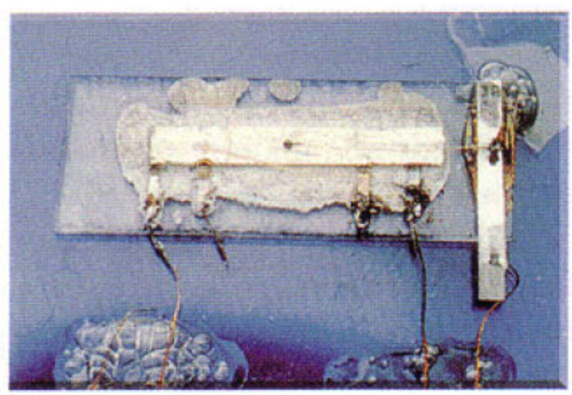


Figure 1: (Left) Sketch from the lab notebook of Jack Kilby dated July 24, 1958. Here he describes his “monolithic idea” that circuit elements such as resistors, capacitors, distributed capacitors and transistors—if all made from the same material—could be included in a single chip. (Right) Kilby’s first realization of his idea, the world’s first integrated circuit. The images are from Kilby’s Physics Nobel Prize lecture in 2000.

Guidelines

1. Please write legibly. If necessary solve the problem first on scratch paper. Then write a cogent and legible solution based on your scratch work. Learning to make a clear and logical presentation of your work is an invaluable asset, well worth the trouble of having to write it twice.

2. Please begin each problem on a new page just as we have done with the questions.

3. Parts of problems marked with an asterisk (if any) are only to be read. No response is needed.

How to submit homework: Submit your homework in PDF format via Canvas. You will find a link to submit under Assignments. Please submit before the deadline. Deadlines are enforced and late homework may not be graded. If you have compelling reasons to miss the deadline please contact the instructor preferably before the deadline. In order to generate a PDF of your homework you should scan it. If you do not have access to a scanner there are a number of effective and free scanner apps available for phones. AdobeScan is one that many students have found worked well for them in previous courses. The other option is to take pictures of your homework and convert them to PDFs using your computer. Finally if you work with a tablet you can of course easily save your work as a PDF.

1. Dielectric interface. The x-y plane is the interface between two essentially infinite dielectric slabs. Each dielectric is uniformly polarized. $\mathbf{P} = P_1\hat{\mathbf{k}}$ for $z < 0$ and $\mathbf{P} = P_2\hat{\mathbf{k}}$ for $z > 0$.

(a) Determine σ, the surface density of bound charge at the interface. Give your answer in terms of P_1 and P_2.

(b) Assume that the dielectrics are both linear. The one below the x-y plane has susceptibility χ_1; the one above, χ_2. Determine $\mathbf{E}_1$ and $\mathbf{E}_2$ the electric fields inside the two dielectrics. Give your answer in terms of $P_1, P_2, \chi_1, \chi_2, \epsilon_0$ and $\hat{\mathbf{k}}$.

(c) What is the total charge density σ_{tot} of the interface? Give your answer in terms of P_1, P_2, χ_1, χ_2 and ϵ_0.

(d) Now assume that there is no free charge at the interface. Determine P_2 in terms of P_1, χ_1 and χ_2.

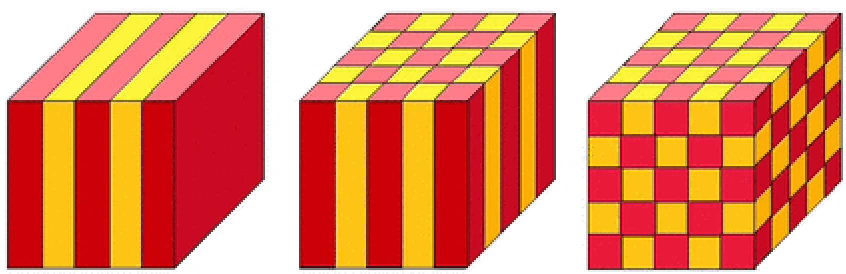


Figure 2: Note: This figure is not directly related to the problem. However the kind of interface analysis carried out here is relevant to understanding the kinds of photonic structures shown here. Advances in fabrication now make it possible to make complex structures with features on the micron scale. The figure shows schematics of photonic crystals. At left slabs of two dielectrics alternate to make a one-dimensional crystal. Two and three dimensional structures are shown center and right. Figure from *Photonic Crystals: Molding the Flow of Light* by Joannopoulos *et al.* Although this problem focussed on static fields, the type of reasoning used here is also needed to analyze the flow of light through photonic crystals. It is expected that photonic crystals will be useful for optical communications technology.

2. Charged dielectric sphere. A dielectric sphere of radius a has free charge Q uniformly distributed over its interior.

(a) What is the total charge of the sphere?

(b) In what direction do you expect the electric field to point?

(c) Use Gauss's law to determine the electric field external to the sphere for $r > a$ (where r denotes distance from the the center of the sphere).

(d) In the interior of the sphere let us assume that the electric field is given by

$$\mathbf{E} = E\hat{\mathbf{r}}. \tag{1}$$

Here $\hat{\mathbf{r}}$ is a unit vector that points directly away from the center of the sphere. E might vary with r, the distance from the center.

(i) Assume the dielectric is linear with susceptibility χ. What is the polarization inside the sphere? Give your answer in terms of $E, \hat{\mathbf{r}}, \epsilon_0$ and χ.

(ii) Consider a Gaussian surface of radius $r < a$. How much bound charge does the surface contain? Give your answer in terms of E, r, ϵ_0 and χ.

(iii) How much free charge does the Gaussian surface of part (ii) contain? Give your answer in terms of Q, r and a.

(iv) Now apply Gauss's law to the surface to determine E in terms of Q, r, a, ϵ_0 and χ. Thus we have now determined the electric field both inside and outside the dielectric sphere.

(v) Now determine the polarization $\mathbf{P}$ inside the sphere. Give your answer in terms of Q, r, a, ϵ_0 and χ. [*Hint:* use the results of parts (i) and (iv)].

(vi) What is the polarization $\mathbf{P}$ outside the sphere?

(vii) What is the surface density σ_b of bound charge on the surface of the dielectric sphere? Give your answer in terms of Q, ϵ_0, a and χ.

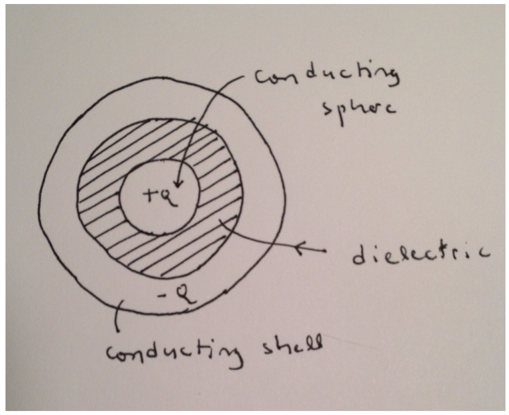


Figure 3: The radius of the conducting sphere is a. The inner radius of the conducting shell is b; the outer radius, c.

3. Spherical capacitor. A conducting sphere of radius a is surrounded by a conducting shell (inner radius b, outer radius c). The sphere has a charge Q; the shell, a charge $-Q$. The space between the sphere and the shell is filled with a linear dielectric with susceptibility χ. Assume the dielectric contains no free charge.

(a) What is the total charge of the system (sphere, shell and dielectric)? What is the electric field external to the shell (i.e. for $r > c$ where r denotes distance from the center). Justify your answer.

(b) What is the electric field in the interior of the conducting sphere? Where does the charge Q reside?

(c) What is the electric field within the shell (for $b < r < c$)?

(d) Where does the charge on the shell reside? On the inner surface? The outer surface? Over both? Justify your answer.

(e) Now assume that the electric field in the dielectric region is

$$\mathbf{E} = E\hat{\mathbf{r}} \tag{2}$$

where $\hat{\mathbf{r}}$ is a unit vector that points radially outward, away from the center. E might depend on r.

(i) Determine the polarization $\mathbf{P}$ inside the dielectric. Give your answer in terms of E, χ, ϵ_0 and $\hat{\mathbf{r}}$.

(ii) Consider a spherical Gaussian surface of radius r with $a < r < b$. How much bound charge does it enclose? Give your answer in terms of χ, E, r and ϵ_0.

(iii) How much total charge does the Gaussian surface enclose? Give your answer in terms of χ, E, r, ϵ_0 and Q.

(iv) Use Gauss's law to determine the electric field E. Give your answer in terms of χ, ϵ_0, r and Q.

(v) Determine the potential difference $V_2 - V_1$ where V_2 is the potential of the conducting shell and V_1 is the potential of the conducting sphere.

(f) *Capacitance.* A capacitor is a device that consists of two conducting "electrodes". Usually one electrode has a positive charge; the other has an equal and opposite negative charge. The capacitance of the capacitor is defined as $C = Q/V$ where Q is the charge on the positive electrode and $V = V_+ - V_-$ is the potential difference between the two electrodes.

(i) Calculate the capacitance of the spherical capacitor analyzed in parts (a)-(e). Give your answer in terms of a, b, ϵ_0 and χ.

*(ii) You should find that $C \propto (1+\chi)$. In the absence of a dielectric filling $\chi = 0$. Thus filling the space between the electrodes with dielectric enhances the capacitance by a factor of $1+\chi$.

4. Capacitor and dielectrics A parallel plate capacitor with plate area A and plate separation d is connected to a battery of voltage V_0. It is initially empty and then a dielectric slab of thickness t (where $t < d$) and dielectric constant κ (or χ) is inserted between the plates such that it covers the entire plate area but doesn't completely fill the space between the plates.

1. Find the capacitance of the capacitor when it is empty.
2. Find the capacitance of the capacitor after the dielectric is inserted.
3. If the battery remains connected while the dielectric is inserted, by what factor does the energy stored in the capacitor change? Check your results for $t = d$ and $t = 0$ to see if your factor has the expected behavior.