

Appendix VI. Probability Distributions

Revised July 2, 2004

A. Introduction

You will encounter several probability distributions in the course of your experiments in physics. The most common are the *Gaussian distribution* (also known as the Bell curve or normal distribution), the *Poisson distribution* and the *exponential distribution*. While modern computer programs may greatly simplify the handling of these distributions, science and engineering students should have an understanding of their basic properties and of the problems to which each may apply. In this Appendix we present the *normalized* form of each distribution, e.g., the expression that gives unity when summed or integrated over the allowed range of the distribution.

B. Gaussian Distribution

The Gaussian distribution is the most common probability distribution in science. Repeated, independent measurements with random uncertainties of almost any quantity follow this distribution. For example, the numbers of heads and tails you are likely to find if you flip a coin many times is described by a Gaussian distribution. When a large physics class is given an exam, a Gaussian distribution may describe the grades for the class.

The Gaussian is characterized by two parameters; the mean, μ , and the standard deviation, σ . The normalized Gaussian is expressed as

$$P(x)dx = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx \quad (1)$$

The term ‘normalized’ refers to a scaling of the distribution function so that the area under the curve equals unity.

The function extends from $x = -\infty$ to $x = +\infty$. $P(x)dx$ gives the probability of a

data point lying in a range of width dx centered on x . The mean μ is just the average value of x . The standard deviation σ characterizes the width of the distribution; about 67% of the data points will lie within one standard deviation of the mean. The distribution function is perfectly symmetric about its mean. A distribution function may also be characterized by its width at half of its maximum value. For a Gaussian distribution function, it can be shown that the Full Width at Half Maximum, FWHM, $\Gamma = 2.353 \sigma$.

Figure 1 shows a Gaussian distribution calculated and plotted with *Origin* using $\mu = 50$ and $\sigma = 10$.

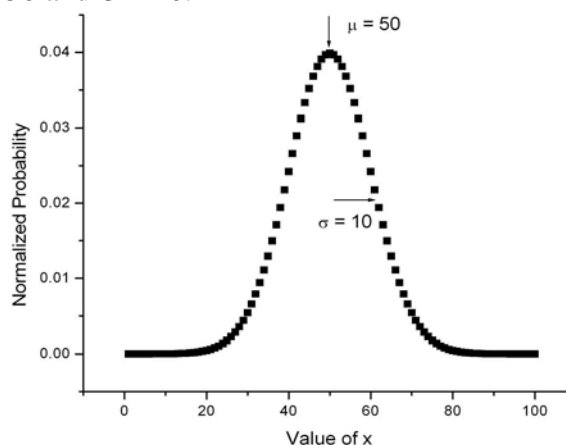


Figure 1: Gaussian Distribution with mean $\mu = 50$ and standard deviation $\sigma = 10$.

C. Poisson Distribution

Statistical processes are represented by the Poisson distribution. A statistical process is a process in which the probability of occurrence of an event in a given time interval is random. As an example, consider a counter that records the number of cosmic-ray particles detected in a given time interval. If the mean number of particles detected in the interval is μ , then the normalized probability of detecting n particles in that interval is

$$P(n) = \mu^n \frac{e^{-\mu}}{n!} \quad (2)$$

The Poisson distribution is characterized by a single parameter, the mean μ . The standard deviation is given by $\sigma = \mu^{1/2}$.

Note that the Poisson distribution is a discrete distribution. It is defined only for positive, integer values of n . However, the mean μ need not be integer although it must be positive. The range of the integer n is from 0 to ∞ . The Poisson distribution function is asymmetric about its mean. For large values of μ , the Poisson distribution is similar to a Gaussian distribution.

Figure 2 shows a Poisson distribution calculated using Excel. μ was set to 50 seconds, which means that σ should be 7.07 seconds. This plot looks similar to the Gaussian distribution function. Table 1 lists a selection of probability values on either side of the mean, showing the asymmetry of the Poisson distribution.

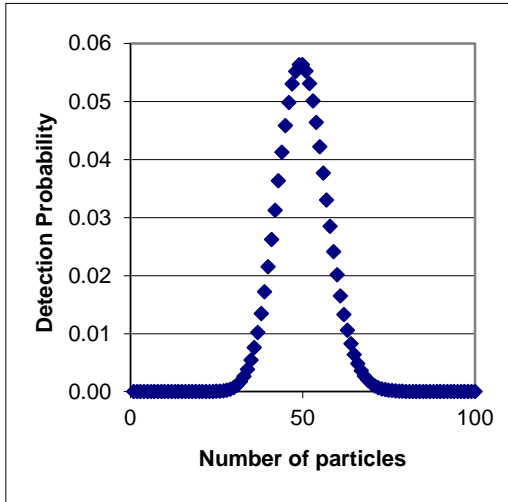


Figure 2: Poisson Distribution

45	4.5826E-02
46	4.9811E-02
47	5.2991E-02
48	5.5199E-02
49	5.6325E-02
50	5.6325E-02
51	5.5221E-02
52	5.3097E-02
53	5.0091E-02
54	4.6381E-02
55	4.2164E-02

Table 1: Selection of probability values from Poisson distribution shown in Fig. 2.

D. Lorentzian Distribution

The Lorentzian distribution is often appropriate for describing resonant behavior such as a mechanical or electronic oscillator. A resonance curve can be represented as a function of the driving frequency ω by the Lorentzian function

$$y = \frac{A}{\pi} \frac{\Gamma/2}{(\omega - \omega_r)^2 + (\Gamma/2)^2} \quad (3)$$

where ω_r is the resonant frequency, A is the area of the plot and Γ is the full width at half maximum. Figure 3 is a plot of a Lorentzian distribution for $\omega_r = 50$ and $\Gamma = 10$ (both in arbitrary units, perhaps Hz.)

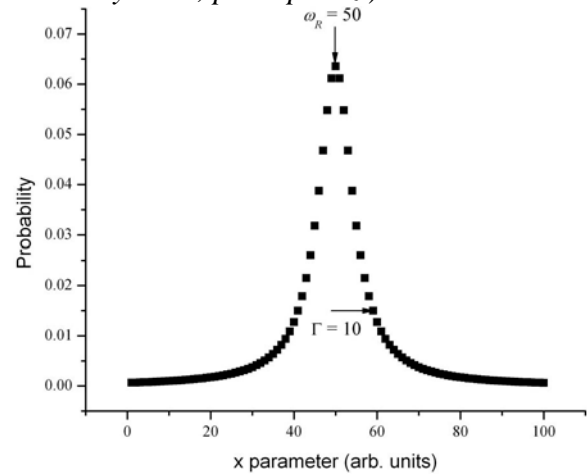


Figure 3: Lorentzian Distribution

The Lorentzian is pre-programmed as a fitting function in *Origin* although *Origin* uses different symbols from those in

Equation 3. *Origin* also adds an offset (or *constant background*) parameter to the Lorentzian expression. You can view the *Origin* fitting function and compare its symbols to those of Eq. 3 by clicking on ANALYSIS → FITTING → NONLINEAR CURVE and selecting the LORENTZ function.

E. Exponential Distribution

Closely related to the Poisson distribution is the exponential distribution. This distribution is most useful for studying as a function of time the number of decays produced from a decaying, finite source. Consider such a source comprised of N unstable particles (unstable nuclear states, or unpopped popcorn). After a time $t_{1/2}$, $N/2$ particles will decay, on the average. In another time interval $t_{1/2}$, half the remaining particles, or $N/4$ particles will decay. The time $t_{1/2}$ is referred to as the *half-life* of the state. It is often more convenient to refer to the mean lifetime, τ , of the state. The normalized probability function for the number of decays as a function of time is given in terms of the mean lifetime as

$$P(t)dt = \frac{1}{\tau} e^{-t/\tau} dt \quad (4)$$

This equation is a one-parameter distribution with mean value of the variable t equal to τ . The standard deviation is not a useful parameter for the exponential distribution. The range of the variable t is from 0 to ∞ . Figure 4 is a plot of the exponential distribution function made using *Origin* with a mean lifetime τ of 50 seconds.

Figure 4 Exponential Decay

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