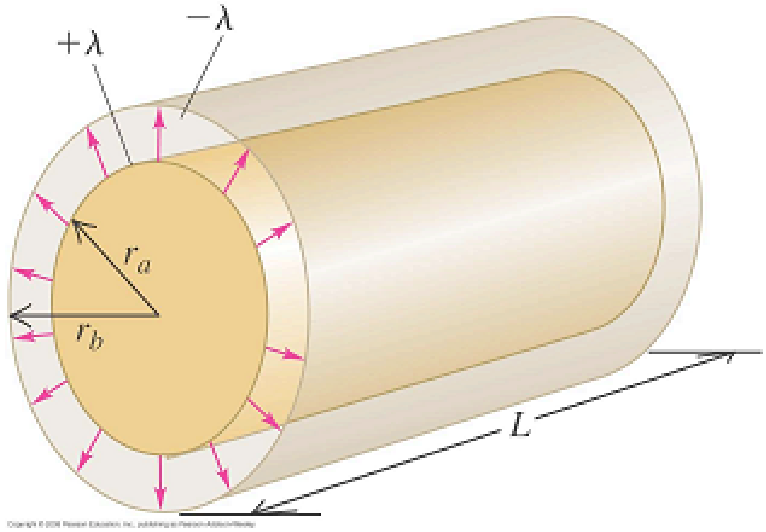


Problem 1: Cylindrical capacitor

A cylindrical capacitor consists of a conducting cylindrical core and a bigger thin cylindrical shell (also conducting) with a common axis. The radius of the inner core is r_a and of the outer shell r_b. For simplicity both are taken to be infinite in length ($L \to \infty$). The inner core has a charge per unit length λ; the outer has a charge per unit length $-\lambda$. [We use linear charge density so in a piece of capacitor of a given length, there is the same amount of charge in both plates. You can relate λ and σ as $\lambda = 2\pi r_a \sigma_a$ and $\lambda = 2\pi r_b \sigma_b$.]

(a) [25 pts] What is the electric field for

(i) $0 < r < r_a$ (inside the inner conducting core)?

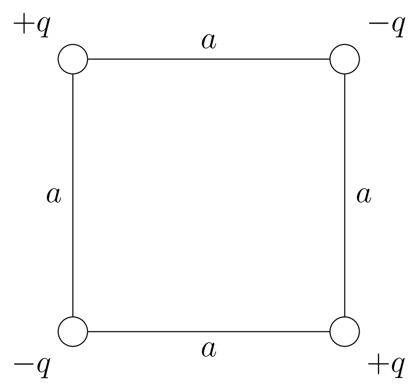
(ii) $r_a < r < r_b$ (between the core and the shell)?

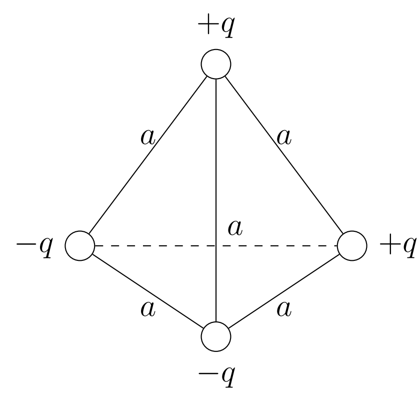
(iii) $r_b < r$ (outside the shell)?

Here, r denotes the distance from the axis of the capacitor.

(b) [20 pts] Calculate the potential difference $\Delta\phi = \phi_2 - \phi_1$ where ϕ_2 is the potential of the outer shell and ϕ_1 is the potential of the inner shell.

(c) [5 pts] Calculate $\mathcal{C}$, the capacitance per unit length of the cylindrical capacitor knowing that $\mathcal{C} = \lambda/V$ where $V \equiv \Delta\phi$ is the potential difference between the capacitor plates.





Problem 2: Electrostatic Potential Energy

Consider two systems, each containing four charges. Two of the charges are +q and two charges are -q.

System 1: Planar Square Configuration

The charges are arranged at the vertices of a square with side length 'a' in the xy-plane. The charge distribution is: +q, -q, +q, -q (alternating around the square).

System 2: Spatial Tetrahedron Configuration

The charges are arranged at the vertices of a regular tetrahedron with edge length 'a'. The charge distribution is: +q, -q, +q, -q (any arrangement is equivalent due to symmetry).

(a) [10pts] Calculate the total electrostatic potential (ϕ) for System 1 at the center of the square.

(b) [18pts] Calculate the total electrostatic potential energy (U) for System 1 (square).

(c) [18pts] Calculate the total electrostatic potential energy (U) for System 2 (tetrahedron).

(d) [4pts] Compare the electrostatic potential energies of System 1 and System 2. Which system has a lower potential energy (that will be the more stable arrangement)?

Solutions

Problem 1: Cylindrical Capacitor

(a) [25 pts] What is the electric field for

(i) $0 < r < r_a$ (inside the inner conducting core)?

$$E_{\text{inside core}} = 0$$

Rubric: 8' give partial credits 6' if they started with Gauss's law or the enclosed charge

(ii) $r_a < r < r_b$ (between the core and the shell)?

$$\vec{E}_{\text{between core shell}} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{n}$$

Rubric: 9' give partial credits 7' if they started with Gauss's law or the enclosed charge

(iii) $r_b < r$ (outside the shell)?

$$E_{\text{outside shell}} = 0$$

Rubric: 8' give partial credits 6' if they started with Gauss's law or the enclosed charge

(b) [20 pts] Calculate the potential difference $\Delta\phi = \phi_2 - \phi_1$ where ϕ_2 is the potential of the outer shell and ϕ_1 is the potential of the inner shell.

$$\Delta\phi = (-)\frac{\lambda}{2\pi\epsilon_0}\log\left(\frac{r_b}{r_a}\right)$$

Rubric: $\Delta\phi = \phi_2 - \phi_1$=-$\int_{r_a}^{r_b} E\, dr$(this is 15',-1 if it's not negative;-1 if ra rb is switched)
Substituting E (this is 3')
$\Delta\phi = (-)\frac{\lambda}{2\pi\epsilon_0}\log\left(\frac{r_b}{r_a}\right)$(answer is 2'. Don't punish twice for math errors)]

(c) [5 pts] Calculate $\mathcal{C}$, the capacitance per unit length of the cylindrical capacitor knowing that $\mathcal{C} = \lambda/V$ where $V \equiv \Delta\phi$ is the potential difference between the capacitor plates.

$$\mathcal{C} = \frac{2\pi\epsilon_0}{\log\left(\frac{r_b}{r_a}\right)}$$

Rubric: substituting answers from a and b to get the correct answer 5', don't take any points off if they got wrong answers from a and b

Problem 2: Electrostatic Potential Energy

(a) [10pts] Calculate the total electrostatic potential (ϕ) for System 1 at the center of the square.

The total electrostatic potential at the center of the square is given by:

$$\phi_{\text{center}} = 0$$

Due to the symmetry and alternating charge distribution, the potentials from each charge cancel out.
Rubric: getting the correct answer 8', the reasoning is 2'. Take 1 point off if missing the reasoning. Give partial credits 5'-8' if getting the wrong answer and some reasonings.

(b) [18pts] Calculate the total electrostatic potential energy (U) for System 1 (square).

The total electrostatic potential energy for System 1 is given by:

$$U_{\text{square}} = \frac{q^2}{4\pi\epsilon_0 a}\left(-4 + \sqrt{2}\right)$$

This is calculated by summing the potential energies between each pair of charges. Rubric: we use the formula for the electrostatic potential energy between two point charges

$$U = \frac{q_1 q_2}{4\pi\epsilon_0 r}$$

(this equation or anything equivalent is 5')
For the given system, the charges are arranged as follows:

$$-q(0,0); +q(a,0); -q(a,a); +q(0,a).$$

No need to take points off as long as they can build a reference system that is consistent $(2' points)$Between -q at (0,0) and +q at (a,0):

$$U_1 = \frac{q_1 q_2}{4\pi\epsilon_0 r} = -\frac{q^2}{4\pi\epsilon_0 a}$$

$(2' points)$ Between +q at (a,0) and -q at (a,a):

$$U_2 = \frac{q_1 q_2}{4\pi\epsilon_0 r} = -\frac{q^2}{4\pi\epsilon_0 a}$$

$(2' points)$ Between -q at (a,a) and +q at (0,a):

$$U_3 = \frac{q_1 q_2}{4\pi\epsilon_0 r} = -\frac{q^2}{4\pi\epsilon_0 a}$$

($2' points$) Between +q at (0,a) and -q at (0,0):

$$U_4 = \frac{q_1 q_2}{4\pi\epsilon_0 r} = -\frac{q^2}{4\pi\epsilon_0 a}$$

($2' points$) Between -q at (0,0) and -q at (a,a):

$$U_5 = \frac{q_1 q_2}{4\pi\epsilon_0 r} = \frac{q^2}{4\sqrt{2}\pi\epsilon_0 a}$$

($2' points$) Between +q at (a,0) and +q at (0,a):

$$U_6 = \frac{q_1 q_2}{4\pi\epsilon_0 r} = \frac{q^2}{4\sqrt{2}\pi\epsilon_0 a}$$

summing up all the potential energies(1 point)

$$U_{total} = U_1 + U_2 + U_3 + U_4 + U_5 + U_6 = \frac{q^2}{4\pi\epsilon_0 a}\left(-4 + \sqrt{2}\right)$$

Don't punish twice for math errors. Give enough credits if they use equivalent methods.

Common error: Using U=q*V gives partial credits that is no more than 14' in total. Give 5' for Understanding of U=qV: If the student correctly identifies that V is the potential at the location of a charge due to the other charges, award partial credit for understanding the concept; Give Correctly calculating V (5'): If the student calculates the potential V at one charge due to the other three charges, award partial credit for correctly setting up the calculation. Correctly summing contributions(2').Final expression(2').

(c) [18pts] Calculate the total electrostatic potential energy (U) for System 2 (tetrahedron).

The total electrostatic potential energy for System 2 is given by:

$$U_{\text{tetrahedron}} = \frac{q^2}{4\pi\epsilon_0 a}(-4 + 2)$$

This is calculated by summing the potential energies between each pair of charges in the tetrahedron.

Rubric: we use the formula for the electrostatic potential energy between two point charges

$$U = \frac{q_1 q_2}{4\pi\epsilon_0 r}$$

(this equation or anything equivalent is 5')

For the given system, the charges are arranged as follows:

$$q_1 = +q; q_2 = -q; q_3 = +q; q_4 = -q$$

No need to take points off as long as they can build a reference system that is consistent
$(2'points)$Between q$_1 = +q; q_2 = -q : U_1 = \frac{q_1q_2}{4\pi\epsilon_0 r} = -\frac{q^2}{4\pi\epsilon_0 a}$
$(2'points)$ Between q$_1 = +q; q_3 = +q :\ U_2 = \frac{q_1q_3}{4\pi\epsilon_0 r} = \frac{q^2}{4\pi\epsilon_0 a}$
(2'points)Between q$_1 = +q; q_4 = -q : U_3 = \frac{q_1q_4}{4\pi\epsilon_0 r} = -\frac{q^2}{4\pi\epsilon_0 a}$
$(2'points)$ Between q$_2 = -q; q_3 = +q : U_4 = \frac{q_2q_3}{4\pi\epsilon_0 r} = -\frac{q^2}{4\pi\epsilon_0 a}$
$(2'points)$ Between q$_2 = -q; q_4 = -q : U_5 = \frac{q_2q_4}{4\pi\epsilon_0 r} = \frac{q^2}{4\pi\epsilon_0 a}$
$(2'points)$ Between q$_3 = +q; q_4 = -q : U_6 = \frac{q_3q_4}{4\pi\epsilon_0 r} = -\frac{q^2}{4\pi\epsilon_0 a}$
summing up all the potential energies(1 point)

$$U_{total} = U_1 + U_2 + U_3 + U_4 + U_5 + U_6 = -\frac{q^2}{2\pi\epsilon_0 a}$$

Don't punish twice for math errors; Give enough credits if they use equivalent methods
Common error: Using U=q*V gives partial credits that is no more than 14' in total. Give 5' for Understanding of U=qV: If the student correctly identifies that V is the potential at the location of a charge due to the other charges, award partial credit for understanding the concept; Give Correctly calculating V (5'): If the student calculates the potential V at one charge due to the other three charges, award partial credit for correctly setting up the calculation. Correctly summing contributions(2').Final expression(2').

[4pts] Compare the electrostatic potential energies of System 1 and System 2. Which system has a lower potential energy (that will be the more stable arrangement)?

The comparison of the electrostatic potential energies is given by comparing $(-4+2/\sqrt{2})$ with $(-4+2)$, and given the fact that the first is more negative, System 1 has a lower potential energy and is more stable.
Rubric: 4 for getting the correct answer, -1 if it's the math error. Don't punish twice if they get the wrong answer from part b and c

Formula Sheet

Helpful integral:

$$\int dr\ \frac{1}{r} = \ln r. \tag{1}$$

Electric Field by a point charge:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \quad \left(\text{alternative } \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}\right) \tag{2}$$