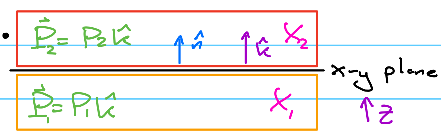
1.




a) $\sigma_{bound} = \hat{n} \cdot \vec{P}_1 - \hat{n} \cdot \vec{P}_2$ $\quad \hat{n} = \hat{k}$

$= \hat{k} \cdot (P_1 \hat{k}) - \hat{k} \cdot (P_2 \hat{k})$

$$\sigma_{bound} = P_1 - P_2$$

b) $\vec{P}_1 = \chi_1 \varepsilon_0 \vec{E}_1$

$\vec{E}_1 = \dfrac{\vec{P}_1}{\chi_1 \varepsilon_0} = \dfrac{P_1}{\chi_1 \varepsilon_0} \hat{k}$ $\quad$ Same for $P_2$ $\Rightarrow$

$$\vec{E}_1 = \frac{P_1}{\chi_1 \varepsilon_0} \hat{k}$$
$$\vec{E}_2 = \frac{P_2}{\chi_2 \varepsilon_0} \hat{k}$$

c) $\sigma_{tot} = -\varepsilon_0 \hat{n} \cdot (\vec{E}_1 - \vec{E}_2)$ $\quad \hat{n} = \hat{k}$

$= -\cancel{\varepsilon_0} \hat{n} \cdot \left( \dfrac{P_1}{\chi_1 \cancel{\varepsilon_0}} \hat{k} - \dfrac{P_2}{\chi_2 \cancel{\varepsilon_0}} \hat{k} \right) = -\left( \dfrac{P_1}{\chi_1} - \dfrac{P_2}{\chi_2} \right)$

$$\sigma_{tot} = \frac{P_2}{\chi_2} - \frac{P_1}{\chi_1}$$

d) $\sigma_{free} = \sigma_{total} - \sigma_{bound} = 0$

$\sigma_{total} = \sigma_{bound}$

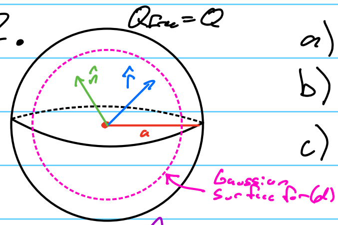
$P_1 - P_2 = \dfrac{P_2}{\chi_2} - \dfrac{P_1}{\chi_1}$

$\dfrac{P_2}{\chi_2} + P_2 = P_1 + \dfrac{P_1}{\chi_1} \rightarrow P_2 \left( \dfrac{1}{\chi_2} + 1 \right) = P_1 \left( \dfrac{1}{\chi_1} + 1 \right)$

$P_2 = P_1 \dfrac{\frac{1}{\chi_1} + 1}{\frac{1}{\chi_2} + 1}$

$= P_1 \dfrac{\frac{\chi_1 + 1}{\chi_1}}{\frac{\chi_2 + 1}{\chi_2}} = P_1 \dfrac{\chi_2 (\chi_1 + 1)}{\chi_1 (\chi_2 + 1)}$

$$P_2 = P_1 \frac{\chi_2 (\chi_1 + 1)}{\chi_1 (\chi_2 + 1)}$$

2. 

a) $Q$ uniformly distributed: $Q_{total} = Q$

b) Radially outward

c) $\frac{Q_{enc}}{\varepsilon_0} = \oint \vec{E}\, d\vec{a}$

$\frac{Q}{\varepsilon_0} = \int E\, da \cos\theta$ $\qquad E = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$

$E = \frac{Q}{\varepsilon_0 \cdot A} = \frac{Q}{\varepsilon_0} \cdot \frac{1}{4\pi r^2}$

$\hat{r} = \hat{n}$

d) Interior of Sphere: $\vec{E} = E\hat{r}$ where $E$ can be a function of radius

(i) $\vec{P} = \chi\varepsilon_0\vec{E}$ $\qquad \vec{P} = \chi\varepsilon_0 E\hat{r}$

(ii) $\vec{P} = \chi\varepsilon_0 E\hat{r}$, $\hat{n} = \hat{r}$

uniform P

$Q_{bound} = \cancel{Q_B(v)}^{\,0} - \oint_V \hat{n}\cdot\vec{P}\, da = -\oint \hat{r}\cdot\chi\varepsilon_0 E\hat{r}\, da$

$= -\chi\varepsilon_0 E a$

$\Rightarrow Q_{bound} = -\chi\varepsilon_0 E(4\pi r^2)$

(iii) $Q_{free} = Q \cdot$ (volume ratio)

$= Q \cdot \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi a^3}$ $\quad r \geq a$ $\qquad Q_{free} = Q\frac{r^3}{a^3}$

$\hat{n} = \hat{r} \rightarrow 0^\circ \rightarrow \cos\theta = 1$

(iv) $Q_{total} = Q_{free} + Q_{bound}$

$= Q\frac{r^3}{a^3} - \chi\varepsilon_0 E(4\pi r^2)$

$\frac{Q_{enc}}{\varepsilon_0} = \oint \vec{E}\, d\vec{a}$ $\quad \hat{n} = \hat{r}$

$\frac{Q_{total}}{\varepsilon_0} = \oint E \cdot da$

$\frac{Q_{total}}{\varepsilon_0} = Ea$

$E = \frac{Q_{total}}{\varepsilon_0\, a} = \frac{Q\frac{r^3}{a^3} - \chi\varepsilon_0 E(4\pi r^2)}{\varepsilon_0(4\pi r^2)}$

$= \frac{Q\frac{r^{\cancel{3}}}{a^3}}{4\pi\varepsilon_0 \cancel{r^2}} - \frac{\chi\cancel{\varepsilon_0}E(\cancel{4\pi r^2})}{\cancel{4\pi}\cancel{\varepsilon_0}\cancel{r^2}}$

$E + \chi E = \frac{Qr}{4\pi\varepsilon_0 a^3}$ $\qquad \vec{E} = \frac{Qr}{4\pi\varepsilon_0 a^3(1+\chi)}\hat{r}$

(i) $\vec{P} = \chi \varepsilon_0 \vec{E}$ $\qquad \vec{E} = \dfrac{Qr}{4\pi\varepsilon_0 a^3(1+\chi)}\hat{r}$

$$\vec{P} = \chi \cancel{\varepsilon_0}\left(\frac{Qr}{4\pi\cancel{\varepsilon_0}a^3(1+\chi)}\hat{r}\right)$$

$$\vec{P} = \frac{Qr\chi}{4\pi a^3(1+\chi)}\hat{r}$$

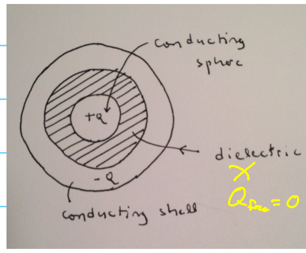
(ii) $\vec{P} = \chi \varepsilon_0 \vec{E}$ $\qquad \vec{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$

$$\vec{P} = \chi\left(\frac{Q}{4\pi r^2}\hat{r}\right)$$

(iii) $\sigma_b = \hat{n} \cdot \vec{P}$ $\qquad \hat{n} = \hat{r}$

$$\sigma_{bound} = \frac{Qr\chi}{4\pi a^3(1+\chi)}$$

3.




a) Sphere: $Q_{sphere} = Q$

Shell: $Q_{shell} = -Q$

Dielectric: $Q_{free} = 0$, only bound charge

↳ bound charge doesn't contribute to the system charge

$Q_{total} = Q + (-Q) + 0 = 0$ $\qquad Q_{total} = 0$

Gauss law: $\frac{Q_{enc}}{\varepsilon_0} = \oint \vec{E} d\vec{a} \Rightarrow Q_{enc} = 0$ for $r > c$, so $E = 0$

b) $\vec{E} = 0$ inside conducting sphere as it is a conductor. All of the charge resides on the edges of the sphere, which lines up to cancel out E.

c) Here, we are inside the conducting shell, so the electric field must be 0 to fulfill the properties of a conductor.

d) To neutralize the effect of the sphere's charge Q, the inner surface must have a charge of $-Q$. The total charge of the shell is $-Q$, so the entire charge must reside on the inner surface of the outer shell. This cancels out the electric field inside the shell, which is needed for a conductor.

e) Inside dielectric: $\vec{E} = E\hat{r}$

(i) $\vec{P} = \chi \varepsilon_0 \vec{E} \implies \vec{P} = \chi \varepsilon_0 E \hat{r}$

(ii) $Q_{bound} = -\oint \hat{n} \cdot \vec{P} da \qquad \hat{n} = \hat{r}$

$= -\int (\hat{r} \cdot \hat{r})(\chi \varepsilon_0 E) da \qquad Q_{bound} = -4\pi r^2 \chi \varepsilon_0 E$

$= -\chi \varepsilon_0 E a$

(iii) $Q_{free} = Q_{total} - Q_{bound} \qquad Q_{free} = Q$

$Q_{total} = Q - 4\pi r^2 \chi \varepsilon_0 E$

(iv) $\frac{Q_{enc}}{\varepsilon_0} = \oint \vec{E} d\vec{a} \qquad \hat{n} = \hat{r}$

$$E = \frac{Q_{enc}}{\varepsilon_0 a} = \frac{Q - 4\pi r^2 \chi \varepsilon_0 E}{\varepsilon_0 (4\pi r^2)} = \frac{Q}{\varepsilon_0 (4\pi r^2)} - \frac{\cancel{4\pi r^2} \chi \cancel{\varepsilon_0} E}{\cancel{\varepsilon_0} (\cancel{4\pi r^2})} = \frac{Q}{4\pi\varepsilon_0 r^2} - \chi E$$

$E = \frac{Q}{4\pi\varepsilon_0 r^2} + \chi E \qquad E($

$E + \chi E = \frac{Q}{4\pi\varepsilon_0 r^2} \Rightarrow \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2 (1+\chi)} \hat{r}$

(c) $V_2 - V_1 = \Delta\Phi = -\int_b^a \vec{E} \cdot d\vec{l}$

$$= -\int_b^a \frac{Q}{4\pi\varepsilon_0 r^2(1+\chi)}\,dr = \frac{-Q}{4\pi\varepsilon_0(1+\chi)}\int_b^a \frac{1}{r^2}\,dr = \frac{Q}{4\pi\varepsilon_0(1+\chi)}\left(\frac{1}{r}\right)_b^a$$

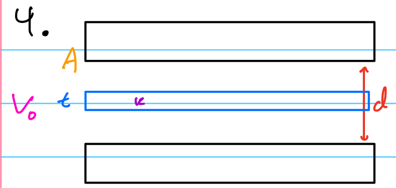
$$V_2 - V_1 = \frac{Q}{4\pi\varepsilon_0(1+\chi)}\left(\frac{1}{a} - \frac{1}{b}\right)$$

f.

$$C = \frac{Q}{\Delta V} = \frac{\cancel{Q}}{\frac{\cancel{Q}}{4\pi\varepsilon_0(1+\chi)}\left(\frac{1}{a} - \frac{1}{b}\right)} = \frac{4\pi\varepsilon_0(1+\chi)}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$C = \frac{4\pi\varepsilon_0(1+\chi)}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

4.




a) $C = \frac{Q}{V} = \frac{\sigma A}{|V|} = \frac{\sigma A}{E\int_0^d dd} = \frac{\cancel{\sigma} A}{d\left(\frac{\cancel{\sigma}}{\varepsilon_0}\right)} = \varepsilon_0\left(\frac{A}{d}\right)$

$$\boxed{C = \varepsilon_0 \frac{A}{d}}$$

b) $C = \left(\frac{1}{C_d} + \frac{1}{C_t}\right)^{-1}$ (parallel)

$C_d = \kappa\left(\varepsilon_0 \frac{A}{d}\right)$ $\quad C_{tot} = \left(\frac{1}{\kappa\varepsilon_0 \frac{A}{t}} + \frac{1}{\varepsilon_0 \frac{A}{d}}\right)^{-1}$ $\quad C_{tot} = \left(\frac{1}{\varepsilon_0 A}\left(\frac{t}{\kappa} + d\right)\right)^{-1}$

$C_t = \varepsilon_0 \frac{A}{d}$

c) $U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{(CV)^2}{C} = \frac{1}{2}CV^2$

Initially: $U_0 = \frac{1}{2}C_0V_0^2 = \frac{1}{2}\left(\varepsilon_0\frac{A}{d}\right)V_0^2$

Final: $U_f = \frac{1}{2}C_fV_0^2 = \frac{1}{2}\left[\frac{1}{\varepsilon_0 A}\left(\frac{t}{\kappa} + d\right)\right]^{-1}V_0^2$

Energy Factor

$t=d$: $U_f = \frac{1}{2}\left[\frac{1}{\varepsilon_0 A}\left(\frac{d}{\kappa} + d\right)\right]^{-1}V_0^2 = \frac{1}{2}\left[\left(\frac{d}{\varepsilon_0 A}\right)\left(\frac{1}{\kappa} + 1\right)\right]^{-1}V_0^2$

$$\frac{U_0}{U_f} = \frac{\cancel{\frac{1}{2}}\left[\left(\frac{d}{\varepsilon_0 A}\right)\left(\frac{1}{\kappa}+1\right)\right]^{-1}\cancel{V_0^2}}{\cancel{\frac{1}{2}}\left(\varepsilon_0\frac{A}{d}\right)\cancel{V_0^2}} = \frac{1}{\cancel{\varepsilon_0}\left(\frac{\cancel{A}}{\cancel{d}}\right)\left(\frac{\cancel{d}}{\cancel{\varepsilon_0 A}}\right)\left(\frac{1}{\kappa}+1\right)} = \frac{1}{\frac{1}{\kappa}+1} = \frac{1}{\frac{1}{\kappa}+\frac{\kappa}{\kappa}}$$

$$\frac{U_0}{U_f} = \frac{1}{\frac{1+\kappa}{\kappa}} = \frac{\kappa}{\kappa+1} \qquad \boxed{\frac{U_0}{U_f} = \frac{\kappa}{\kappa+1}}$$ w/ dielectric

$t=0$: $U_f = \frac{1}{2}\left[\frac{1}{\varepsilon_0 A}\left(\frac{0}{\kappa} + d\right)\right]^{-1}V_0^2 = \frac{1}{2}\left[\frac{d}{\varepsilon_0 A}\right]^{-1}V_0^2$

$$\frac{U_0}{U_f} = \frac{\cancel{\frac{1}{2}}\left[\frac{d}{\varepsilon_0 A}\right]^{-1}\cancel{V_0^2}}{\cancel{\frac{1}{2}}\left(\varepsilon_0\frac{A}{d}\right)\cancel{V_0^2}} = \frac{1}{\left(\frac{\cancel{d}}{\cancel{\varepsilon_0 A}}\right)\left(\cancel{\varepsilon_0}\frac{\cancel{A}}{\cancel{d}}\right)} = 1 \qquad \boxed{\frac{U_0}{U_f} = 1}$$ w/o dielectric