Physics 122 Spring 2025

Magnetostatics: Biot-Savart law and Lorentz Force law

Short Homework 7

Due: Sunday March 30 before 11:59 PM

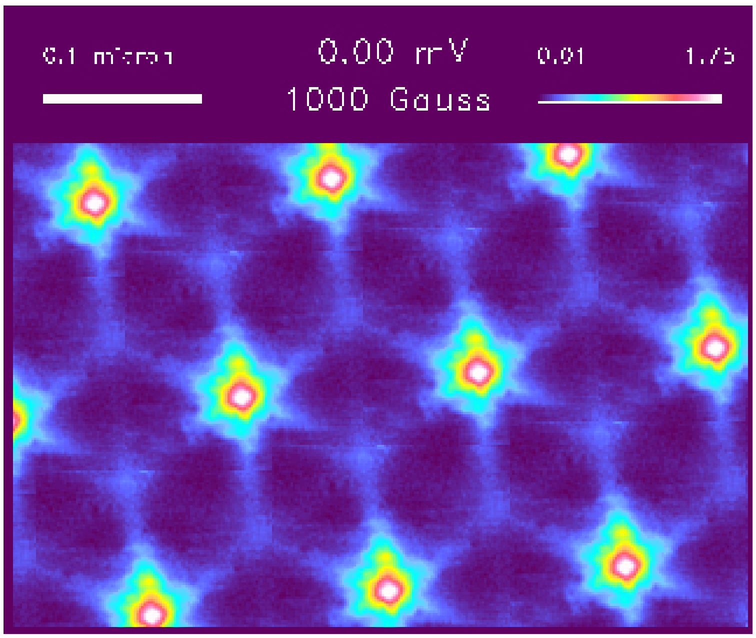

Figure 1: Vortex lattice in a type II superconductor. The bright yellow circles are the locations of vortices: whirlpools of circulating electrical current seen from above. Ampere's law dictates that there is a strong magnetic field perpendicular to the page in the core of the vortices and that there is no magnetic field outside the cores. Hence superconducting vortices are also called magnetic flux tubes. This image was made using scanning probe microscopy by Harald Hess at Bell Labs.

Guidelines

1. Please write legibly. If necessary solve the problem first on scratch paper. Then write a cogent and legible solution based on your scratch work. Learning to make a clear and logical presentation of your work is an invaluable asset, well worth the trouble of having to write it twice.

2. Please begin each problem on a new page just as we have done with the questions.

3. Parts of problems marked with an asterisk (if any) are only to be read. No response is needed.

How to submit homework: Submit your homework in PDF format via Canvas. You will find a link to submit under Assignments. Please submit before the deadline. Deadlines are enforced and late homework may not be graded. If you have compelling reasons to miss the deadline please contact the instructor preferably before the deadline. In order to generate a PDF of your homework you should scan it. If you do not have access to a scanner there are a number of effective and free scanner apps available for phones. AdobeScan is one that many students have found worked well for them in previous courses. The other option is to take pictures of your homework and convert them to PDFs using your computer. Finally if you work with a tablet you can of course easily save your work as a PDF.

1. Units and dimensional analysis. (Recap) Dimensional analysis is concerned with checking the consistency of units in an equation. Dimensional considerations put surprisingly tight constraints on the equations we can sensibly write down. This makes dimensional analysis a remarkably powerful tool as illustrated below.

(a) In the SI system the units of all quantities can be expressed as a combination of kg, m, s and C. Express the units of the following quantities in these terms.

(i) Electric field, $\mathbf{E}$.

(ii) Permittivity of free space, ϵ_0.

(iii) Electrostatic potential, ϕ.

(iv) Magnetic field, $\mathbf{B}$.

(v) Electric current, i.

(vi) Permeability of free space, μ_0.

(vii) The electric dipole moment, $\mathbf{p}$.

(viii) Energy.

Hint: For each quantity think of some defining equation in which it appears and in which you know the units of all but the quantity of interest. For example use the force law to get the units of the electric field, Coulomb's force law to get units of the permittivity etc.

(b) *Name that unit:* The units of some of the quantities listed above have names. Here is a list of unit names. Match the named unit to the corresponding physical quantity. (i) Volt. (ii) Ampere. (iii) Tesla. (iv) Joule. (v) eV.

(c) *Comparing units:* (i) Explain why 1 N/C = 1 V/m. [*Hint:* Reduce both to a combination of kg, m, s and C.] (ii) Which quantity in part (a) has units of N/C or equivalently V/m?

(ii) Let $[\mathbf{E}]$ denote the units of electric field and $[\mathbf{B}]$ denote the units of magnetic field. By what factor do the two sets of units differ? In other words, determine the ratio $[\mathbf{E}]/[\mathbf{B}]$.

(d) *Dimensional analysis: Magnetic energy density.* A professor tells his class that just as the energy density in a static electric field is given by $(1/2)\epsilon_0 E^2$, so also the energy density in a static magnetic field is given by

$$\frac{1}{2}\mu_0 B^2. \tag{1}$$

However in his written lecture notes the professor writes that the magnetic energy density is given by

$$\frac{1}{2\mu_0} B^2. \tag{2}$$

Either the professor made a careless mistake in class or there is a typo in his lecture notes. Use dimensional analysis to determine which of these expressions is correct.

(e) *Dimensional analysis and the speed of light:* (i) Form a combination of the quantities ϵ_0 and μ_0 that has units of speed. [*Hint:* Assume that $\mu_0^n \epsilon_0^m$ has units of speed where the exponents n and m are numbers to be determined.]

(ii) Calculate the numerical value of this speed in m/s. *Useful data:* $\epsilon_0 = 8.85 \times 10^{-12}$ S.I. units; $\mu_0 = 4\pi \times 10^{-7}$ S.I. units.

(iii)* Maxwell discovered that light is an electromagnetic phenomenon. The speed calculated in part (ii) is the speed of light!

Figure 2: (*Left*) CWRU undergraduate Polykarp Kusch, (BS Physics 1931), winner of the 1955 Nobel Prize in physics for the first high precision measurement of the magnetic moment of the electron. (*Right*) CWRU undergraduate David Hanneke (BS Physics 2001) holds the current world record for the most high precision measurement of the magnetic moment of the electron, having measured it to thirteen decimal places for his doctoral thesis work at Harvard.

(f) [Optional. Zero credit]*Dimensional analysis: radiation by an accelerating point charge.* A point charge q has an acceleration a. We will learn later that accelerating charges give off electromagnetic radiation. The purpose of this problem is to use dimensional analysis to estimate the power P (energy per unit time) given off by the accelerating charge. To this end we assume

$$P = \kappa q^\alpha \epsilon_0^\beta a^\gamma c^\delta. \tag{3}$$

Here κ is a numerical constant with no units. ϵ_0 is the permittivity of free space and c is the speed of light. α, β, γ and δ are unknown numerical exponents. By matching units on both sides of eq (3) determine α, β, γ and δ.

The formula for the power radiated by an accelerating charge is called Larmor's formula. It is necessary to go beyond dimensional analysis to determine that $\kappa = 1/6\pi$; however it is remarkable how far one can go based just on dimensional analysis.

(g) [Optional. Zero credit] *Dimensional analysis: Quantum Electrodynamics.* Quantum electrodynamics (QED) is the theory that describes the interaction of electrons with electric and magnetic fields. Applications of QED range from lasers and light emitting diodes to more fundamental problems of physics. The key natural constants that appear in QED are the fundamental unit of charge $e = 1.6 \times 10^{-19}$ C; Planck's constant, $\hbar = 1.055 \times 10^{-34}$ kg m^2/s, a constant that appears in all formulae of quantum physics; the speed of light, $c = 3.0 \times 10^8$ m/s; and $1/4\pi\epsilon_0 = 9 \times 10^9$ kg m^3/s^2 C^2. Out of these constants form a quantity α that is dimensionless (i.e. has no units). α is called the fine-structure constant. Calculate the numerical value of α. The fact that α is dimensionless is key to the consistency of quantum electrodynamics. That it has a small numerical value makes QED a theory of great predictive power.

In addition to its electric charge an electron has a magnetic moment (see problem 2 for a definition of magnetic moment). The magnetic moment of the electron has been measured to thirteen decimal places; QED predicts the value of the moment correctly to the same number of places. For this reason QED is sometimes called the most accurate theory in all of science.

2. Torque on a loop: A square loop of side ℓ lies in the horizontal x-y plane as shown in the figure. A current i flows counterclockwise around the loop. The z-axis points vertically out of the page.

(a) Suppose that a vertical magnetic field $\mathbf{B} = B_z\hat{\mathbf{k}}$ is applied.

(i) Determine the magnetic force on each segment of the loop.

(ii) Assuming the force on each segment acts at the center of the segment, compute the magnetic torque on each segment about the center of the square loop. (Recall that the torque $\boldsymbol{\tau}$ due to a force $\mathbf{F}$ applied at position $\mathbf{r}$ is given by $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$).

(iii) What is the net torque on the loop?

(b) Suppose a horizontal magnetic field $\mathbf{B} = B_x\hat{\mathbf{i}}$ is applied instead. Repeat parts (i), (ii) and (iii) above for this circumstance.

(c) Define the magnetic moment of the loop as a vector $\mathbf{m}$ that has magnitude ia where i is the current and a is the area of the loop. The direction of $\mathbf{m}$ is perpendicular to the loop and out of the page; it is determined by a right hand rule (point fingers along current, then thumb points along $\mathbf{m}$). Verify that the results of parts (a) and (b) are consistent with the general formula

$$\boldsymbol{\tau}_{\text{net}} = \mathbf{m} \times \mathbf{B} \tag{4}$$

where $\boldsymbol{\tau}_{\text{net}}$ is the net torque on the loop and $\mathbf{B}$ is the applied magnetic field.

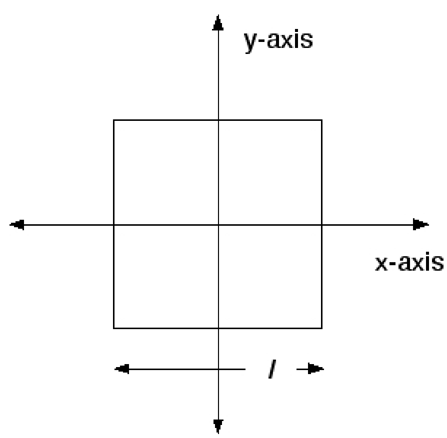

Figure 3: Square loop lies in a horizontal plane. The current flows counterclockwise.

The definition of magnetic moment $\mathbf{m}$ and the formula for the torque given here are generally valid for loops of arbitrary shape, not only square.

3. Circular loop. A circular loop of wire of radius a lies in the x-y plane centered about the origin. A current i flows through the wire counter-clockwise as seen from above.

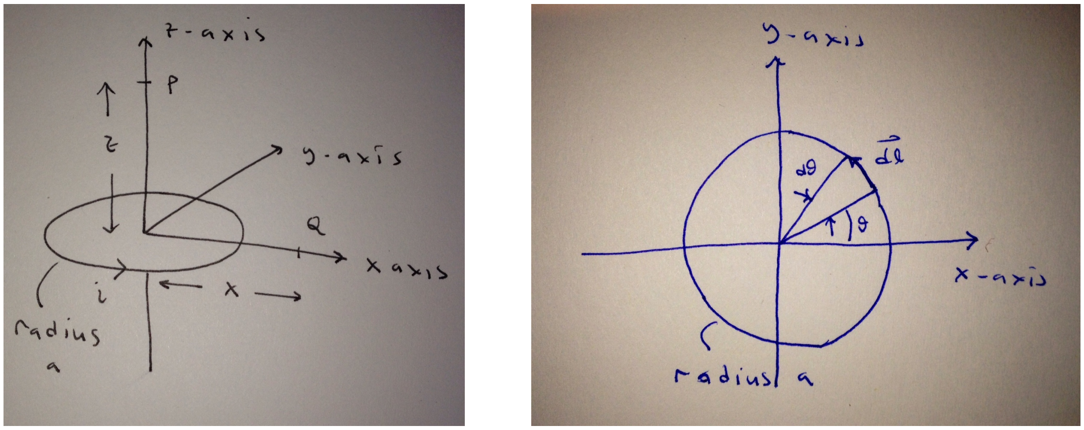

Figure 4: A circular loop.

(a) *Magnetic dipole moment:* What is $\mathbf{m}$, the magnetic dipole moment of the loop? Give your answer in terms of $i, a, \hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

The magnetic dipole moment of a flat current loop is a vector. The magnitude of the dipole moment is equal to iA where i is the current and A is the area of the loop. The direction of the dipole moment is perpendicular to the loop in the direction that the right thumb points when the right hand fingers are curled around the loop in the direction of the current flow.

(b) *Axial field:* Determine the magnetic field at a point P that lies at an elevation z above the center of the circle using the Biot-Savart law. To this end consider the infinitesimal segment $d\boldsymbol{\ell}$ of the ring shown in the figure.

(i) Write down the vector $d\boldsymbol{\ell}$ in terms of $a, d\theta, \theta, \hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

(ii) Determine $\mathbf{r}$, the displacement vector from the segment $d\boldsymbol{\ell}$ to P in terms of $a, \theta, z, \hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. Also for later convenience determine r.

(iii) Evaluate $d\boldsymbol{\ell} \times \mathbf{r}$. Give your answer in terms of $a, z, \theta, d\theta, \hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

(iv) Recall that according to the Biot-Savart law the magnetic field due to an infinitesimal segment is given by

$$d\mathbf{B} = \frac{\mu_0 i}{4\pi} \frac{d\boldsymbol{\ell} \times \mathbf{r}}{r^3}. \tag{5}$$

Here $\mathbf{r}$ represents the vector from the position of the segment $d\boldsymbol{\ell}$ to the point P and r is the magnitude of $\mathbf{r}$. Use eq (5) and your results of parts (ii) and (iii) to determine $d\mathbf{B}$, the field produced by the infinitesimal segment $d\boldsymbol{\ell}$.

(v) Integrate the result of part (iv) to determine the total magnetic field at the point P.

(vi) Simplify your result in the limit that $z \gg a$. Verify that the magnetic field is given approximately by

$$\mathbf{B} \approx \frac{\mu_0}{2\pi} \frac{\mathbf{m}}{z^3}. \tag{6}$$

(c) Now let us determine the magnetic field at the point Q on the x-axis. To this end consider the infinitesimal segment $d\boldsymbol{\ell}$ of the ring shown in the figure.

(i)* You have already determined $d\boldsymbol{\ell}$ in part (b-i).

(ii) Determine $\mathbf{r}$, the displacement from the segment $d\boldsymbol{\ell}$ to Q. Give your answer in terms of $a, \theta, d\theta, x, \hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

(iii) Evaluate $d\boldsymbol{\ell} \times \mathbf{r}$. Give your answer in terms of $a, \theta, d\theta, x, \hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

(iv) Recall that according to Biot-Savart the magnetic field due to an infinitesimal segment is given by eq (5) with $\mathbf{r}$ the displacement from $d\boldsymbol{\ell}$ to Q and r the magnitude of $\mathbf{r}$. Use eq (5) and your results from parts (ii) and (iii) to determine $d\mathbf{B}$, the field produced by the infinitesimal segment $d\boldsymbol{\ell}$.

(v) It follows from part (iv) that the total field is given by the integral

$$\mathbf{B} = -\frac{\mu_0 i a}{4\pi x^2}\hat{\mathbf{k}} \int_0^{2\pi} d\theta \, \frac{\cos\theta - (a/x)}{[1 - 2(a/x)\cos\theta + (a/x)^2]^{3/2}}. \tag{7}$$

This integral cannot be evaluated exactly. Further calculation has to be carried out numerically. However at large distances, $x \gg a$, the integrand can be approximated as

$$\frac{\cos\theta - (a/x)}{[1 - 2(a/x)\cos\theta + (a/x)^2]^{3/2}} \approx -\cos\theta + (3\cos^2\theta - 1)\frac{a}{x} + \dots \tag{8}$$

With this approximation it is now possible to evaluate the integral. Verify that the magnetic field is approximately given by

$$\mathbf{B} \approx -\frac{\mu_0 \mathbf{m}}{4\pi x^3}. \tag{9}$$