Physics 122 Spring 2025

Electrostatics

Homework 3

Due: 2/16/2025

Figure 1: Niels Bohr and Wolfgang Pauli study a tippe top. Their attempts to understand the energy levels of atoms, for which they won Nobel Prizes in 1922 and 1945 respectively, are analyzed in problems 1 and 4.

Guidelines

1. Please write legibly. If necessary solve the problem first on scratch paper. Then write a cogent and legible solution based on your scratch work. Learning to make a clear and logical presentation of your work is an invaluable asset, well worth the trouble of having to write it twice.

2. Please begin each problem on a new page just as we have done with the questions.

3. Parts of problems marked with an asterisk (if any) are only to be read. No response is needed.

How to submit homework: Submit your homework in PDF format via Canvas. You will find a link to submit under Assignments. Please submit before the deadline. Deadlines are enforced and late homework may not be graded. If you have compelling reasons to miss the deadline please contact the instructor preferably before the deadline. In order to generate a PDF of your homework you should scan it. If you do not have access to a scanner there are a number of effective and free scanner apps available for phones. AdobeScan is one that many students have found worked well for them in previous courses. The other option is to take pictures of your homework and convert them to PDFs using your computer. Finally if you work with a tablet you can of course easily save your work as a PDF.

OPTIONAL. ZERO CREDIT.

1. Electric field of a hydrogen atom.

In our study of electrostatics we have considered both point charges and continuous distributions of charge. But so far, for simplicity, we have assumed that the continuous distributions have uniform density. In this problem we relax that assumption.

In the quantum mechanical description of a hydrogen atom the proton is a point charge with charge $+q$ located at the origin. The electron is a cloud; it may be modeled as a continuous charge distribution. The electron cloud is spherically symmetrical and the charge density depends only on distance from the proton. The charge density is given by

$$\rho(r) = -\frac{q}{8\pi}\frac{1}{a^3}\exp\left(-\frac{r}{a}\right). \tag{1}$$

Here $a = 0.53$ Å $= 5.3 \times 10^{-11}$ m is the "Bohr radius" of the hydrogen atom.

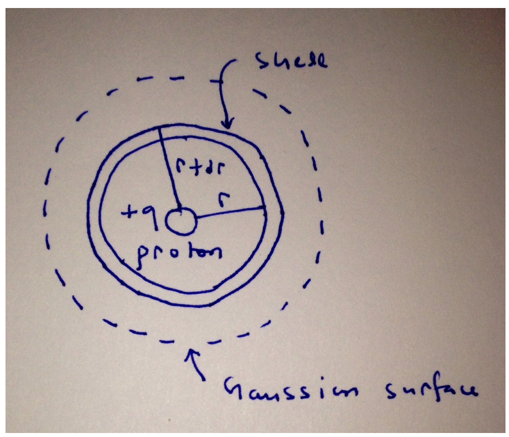


Figure 2: A proton surrounded by an invisible electron cloud. Shown in the figure are a shell and a Gaussian surface.

(a) Consider a spherical shell of inner radius r and outer radius $r + dr$ centered about the proton. What is the charge within this shell? [*Answer:* $-qr^2\exp(-r/a)/2a^3$].

(b) What is the charge Q within a Gaussian sphere of radius R centered about the proton? Give your answer in terms of q, r, dr and a. [*Hint:* Integrate the result of part (a). Don't forget the charge of the proton].

Helpful integral:

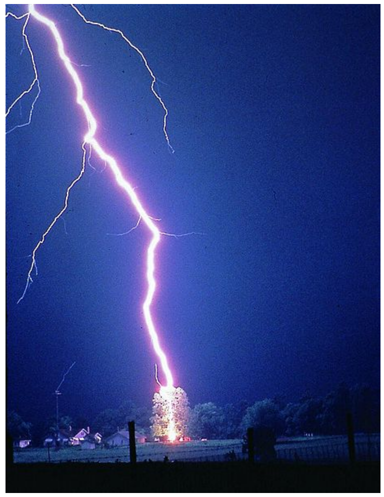
$$\int dr\, r^2 \exp\left(-\frac{r}{a}\right) = -(2a^3 + 2a^2 r + ar^2)\exp\left(-\frac{r}{a}\right). \tag{2}$$

(c) What is the total charge of the atom? [*Hint:* Set $R \to \infty$ in the result of part (b)].

(d) By spherical symmetry the electric field magnitude depends only on distance from the proton. Thus the magnitude of the electric field is constant on the Gaussian surface of part (b). In what direction does this electric field point?

(e) Let E denote the magnitude of the electric field at a distance R from the proton. Determine the electric flux through the Gaussian surface of part (b). Give your answer in terms of E and R.

(f) Use Gauss's law to determine E. Give your answer in terms of Q, R and ϵ_0. (Even more optional: Make a graph of E as a function of distance R from the proton).



2. Dielectric breakdown of a gas.

(a) An electron starts from rest and moves a distance ℓ under the influence of a uniform electric field of magnitude E. Assume that the displacement is in the direction of the electric field. What is the final kinetic energy of the electron? Give your answer in terms of q = charge of the electron, m = mass of the electron, E and ℓ.

(b) Suppose the final kinetic energy of the electron is 10 eV and $\ell = 1.0 \times 10^{-6}$ m. Determine E in V/m. *Possibly useful data:* $q = -1.6 \times 10^{-19}$ C, $m = 9.1 \times 10^{-31}$ kg.

(c)* Air is primarily composed of neutral N_2 and O_2 molecules and is an insulator. There are always some free electrons in air (created if nothing else by radioactivity and cosmic rays) but their concentration is too low to make air significantly conducting. Free electrons move a typical distance ℓ between collisions with air molecules; ℓ is called the mean free path of the electrons. If an intense electric field is present (e.g. due to a thundercloud; see problem 8 of homework 2) the electron will be accelerated between collisions. If electrons are accelerated sufficiently, their collision will knock additional electrons out of the molecules as well as excite the molecules, causing them to subsequently relax by giving off light. If each collision yields several electrons, a chain reaction is established whereby each newly liberated electron undergoes further collisions liberating additional electrons. Under these conditions the air undergoes "dielectric breakdown": it gives off light and becomes conducting. Lightning bolts are conducting channels created by dielectric breakdown of air by the intense electric fields of thunderclouds. In fluorescent lights a low density gas (with a correspondingly long mean free path ℓ) undergoes dielectric breakdown under comparatively weaker electric fields.

3. Electrostatic energy of complex ions. A complex ion consists of a central cation, M^{4+}, with a charge $4e$ and four "ligands" , N^{-}, each with charge $-e$.

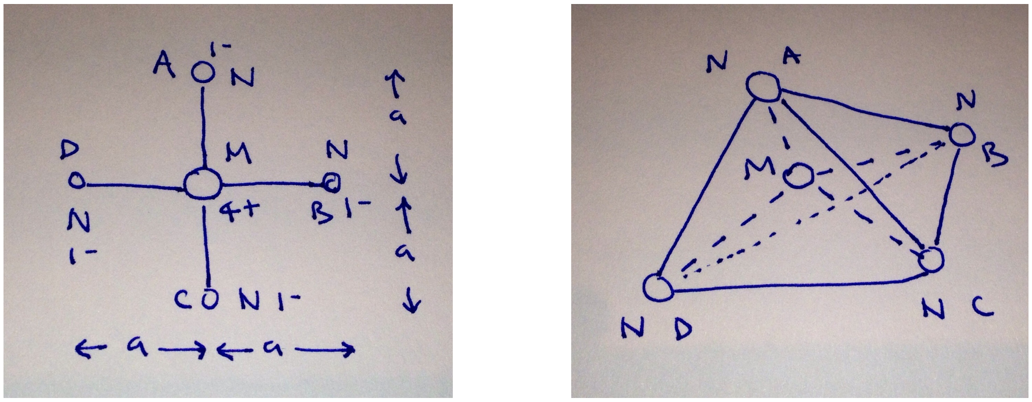


Figure 3: A planar complex (left) and a tetrahedral complex (right).

(a) *Planar complex:* In a planar complex the four N anions sit at the corners of a square. and the M cation sits at the center. a denotes the distance between the central M cation and the four N anions. Calculate the electrostatic energy of the complex treating the ions as point charges. Give your answer in terms of $e^2, 4\pi\epsilon_0$ and a. [*Hint:* The electrostatic energy is a sum of the energy of each pair of ions. What is the energy of the pair M-N_A? Here, N_A denotes the N anion marked A in the figure. What is the energy of the pair N_A-N_B? Of N_A-N_C? Notice that all other pairs are equivalent to one of these three].

(b) *Tetrahedral complex:* In a tetrahedral complex the four N anions site at the vertices of a tetrahedron and the M cation sits at the center of the tetrahedron. Again a denotes the distance between the M cation and each of the N anions. We place the origin of our co-ordinates at M and align the z-axis with N_A where N_A denotes the N anion marked A in the figure. The position vectors of the four N anions are then

$$A \to a\hat{\mathbf{k}}, \quad B \to -\frac{1}{3}a\hat{\mathbf{k}}+\frac{\sqrt{8}}{3}a\hat{\mathbf{i}}, \quad C \to -\frac{1}{3}a\hat{\mathbf{k}}-\frac{\sqrt{2}}{3}a\hat{\mathbf{i}}+\sqrt{\frac{2}{3}}a\hat{\mathbf{j}}, \quad D \to -\frac{1}{3}a\hat{\mathbf{k}}-\frac{\sqrt{2}}{3}a\hat{\mathbf{i}}-\sqrt{\frac{2}{3}}a\hat{\mathbf{j}}.$$

What is the electrostatic energy of the tetrahedral complex? You may treat the ions as point charges. Give your answer in terms of $e^2, 4\pi\epsilon_0$ and a. [*Hint:* Again the electrostatic energy is the sum of the energy of all pairs. What is the energy of the pair M-N_A? Here N_A denotes the N anion marked A in the figure. What is the energy of the pair N_A-N_B? Note that by symmetry all M-N pairs have the same energy and all N-N pairs have the same energy].

(c) Which configuration has lower energy, the planar geometry or the tetrahedral? If electrostatics was the only consideration, the complex ion would take on the configuration that has lowest electrostatic energy.

4. Electrostatic energy of a salt crystal. The figure shows an arrangement of Na^+ ions (charge e) and Cl^- ions (charge $-e$) along a straight line. The separation between consecutive ions is a. This "one-dimensional crystal" is a cartoon of common salt which is a three dimensional alternating pattern of the same ions.

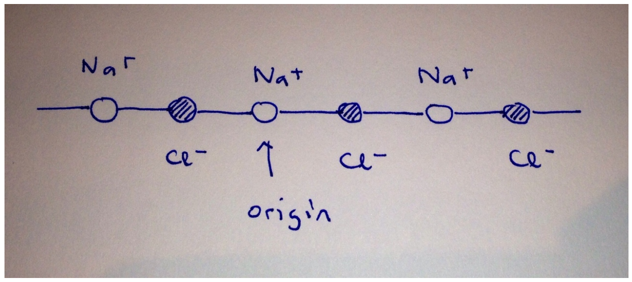


Figure 4: A one dimensional crystal of sodium chloride.

The total electrostatic energy of the one-dimensional crystal is ∞. However the energy per ion is finite and the purpose of this problem is to calculate it. The total electrostatic energy of the crystal is the sum of the energy of every pair. To calculate the energy per Na ion focus on the pairs that involve the Na ion located at the origin.

(a) Write an expression for the electrostatic energy of all pairs that involve the Na ion at the origin. [*Hint:* This expression is an infinite series].

(b) Sum the infinite series making use of the identity

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots \tag{3}$$

You should find that the total electrostatic energy per Na ion is

$$-\frac{e^2}{2\pi\epsilon_0}\frac{\ln 2}{a}. \tag{4}$$