Physics 122 Spring 2025

Electrostatics

Homework 4

Due: Feb 23rd, 23.59PM

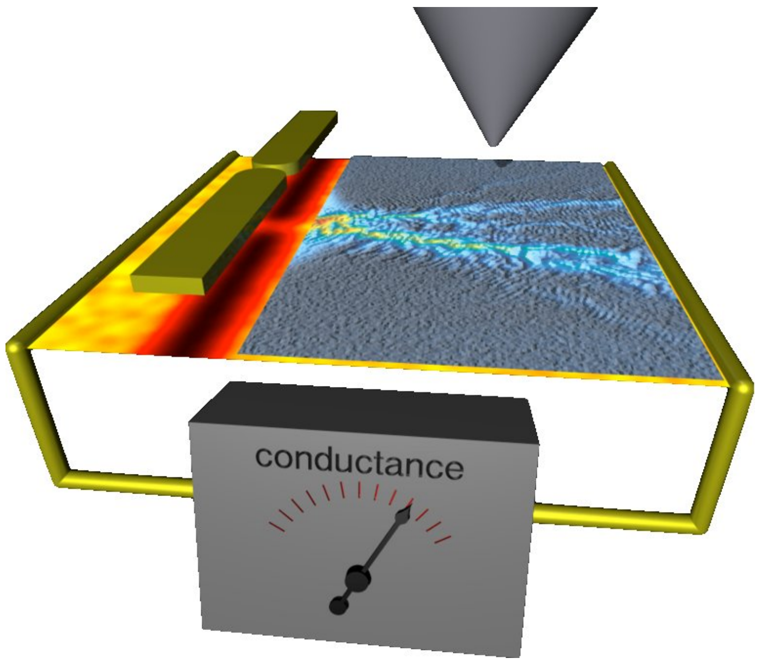

Figure 1: Schematic of a nanoscale capacitor formed by the conducting tip of a scanning probe microscope (conical object in figure) and a flat conducting sheet in a semi-conductor device (a "two-dimensional electron gas" shown grey in the figure). The tip and the semi-conductor surface are only 10 nm apart. The gold bars to the left are "gates" that create a constriction called a quantum point contact through which electrons must pass in flowing from left to right. The blue streaks show electrons squirting as they pass through the point contact. The streaks represent actual data obtained by scanning the probe tip over the sample [Topinka *et al.*, Nature (2001).] The image is courtesy Professor Jesse Berezovsky who has performed similar scanning probe measurements in graphene.

Guidelines

1. Please write legibly. If necessary solve the problem first on scratch paper. Then write a cogent and legible solution based on your scratch work. Learning to make a clear and logical presentation of your work is an invaluable asset, well worth the trouble of having to write it twice.

2. Please begin each problem on a new page just as we have done with the questions.

3. Parts of problems marked with an asterisk (if any) are only to be read. No response is needed.

How to submit homework: Submit your homework in PDF format via Canvas. You will find a link to submit under Assignments. Please submit before the deadline. Deadlines are enforced and late homework may not be graded. If you have compelling reasons to miss the deadline please contact the instructor preferably before the deadline. In order to generate a PDF of your homework you should scan it. If you do not have access to a scanner there are a number of effective and free scanner apps available for phones. AdobeScan is one that many students have found worked well for them in previous courses. The other option is to take pictures of your homework and convert them to PDFs using your computer. Finally if you work with a tablet you can of course easily save your work as a PDF.

1. **Energy of a dipole.**

*(a) *Dipole moment:* An ideal electric dipole moment is a pair of equal and opposite charges $+q$ and $-q$ separated by a distance a. The "dipole moment" of the pair is a vector $\mathbf{p}$ defined as

$$\mathbf{p} = q\mathbf{r} \tag{1}$$

where $\mathbf{r}$ is the displacement vector from the negative to the positive charge. The magnitude of the dipole moment is qa.

(b) *Force on a dipole:* Consider a dipole moment immersed in a uniform electric field $\mathbf{E}$. What is the total force on the dipole?

(c) *Torque on a dipole:* Assume the negative charge is at the origin and the position of the positive charge is $\mathbf{r}$. Assume that the dipole is immersed in a uniform electric field $\mathbf{E}$. What is the torque on the dipole? Give your answer in terms of $\mathbf{p}$ and $\mathbf{E}$.

(d) *Electrostatic Energy:* Again assume that the negative charge is at the origin and the position of the positive charge is $\mathbf{r}$. Assume that the dipole is immersed in a uniform electric field $\mathbf{E}$. What is the electrostatic potential energy of the dipole? Give your answer in terms of $\mathbf{p}$ and $\mathbf{E}$.

[*Hint:* Recall that for a charge Q at position $\mathbf{R}$ in a uniform electric field $E\hat{\mathbf{k}}$ the electrostatic energy is $-Q\mathbf{E}\cdot\mathbf{R}$].

2. **Equipotential Lines: Equipotentials for two point charges**

   (a) Two point charges Q are located at $(\pm R, 0, 0)$. Find the electric field at all points on the z-axis. For what value of z is the field maximum?

   (b) Make a rough sketch of the equipotential curves everywhere in space (or rather, everywhere in the xz plane). Be sure to indicate what the curves look like very close to and very far from the charges, and how the transition from close to far occurs.

3. **Dipole potential.** Two particles of charge $+q$ and $-q$ respectively are placed on the z-axis symmetrically about the origin, each a distance a from the origin.

(a) What is the dipole moment $\mathbf{p}$ of the two charges? Give your answer in terms of $q, a, \hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. [*Hint:* Recall that the dipole moment $\mathbf{p}$ is defined as $\mathbf{p} = q\mathbf{R}$ where $\mathbf{R}$ is the displacement vector from the negative charge to the positive.]

(b) What is the electrostatic potential at a point $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. [*Hint:* Use superposition]. Give your answer in terms of x, y, z, a, q and ϵ_0.

(c) Simplify your result in the limit that $r \gg a$.Here $r = \sqrt{x^2 + y^2 + z^2}$ is the magnitude of $\mathbf{r}$. You may use the approximation

$$\frac{1}{\sqrt{x^2 + y^2 + (z-a)^2}} \approx \frac{1}{r} + \frac{az}{r^3}. \tag{2}$$

(d) Show that your result for part (c) is equivalent to

$$\phi(\mathbf{r}) \approx \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}. \tag{3}$$

Eq (3) is the potential of an ideal dipole.

4. **Potential of a charged rod.** A uniformly charged line segment of length 2ℓ lies on the z-axis symmetrically about the origin. The charge per unit length is λ.

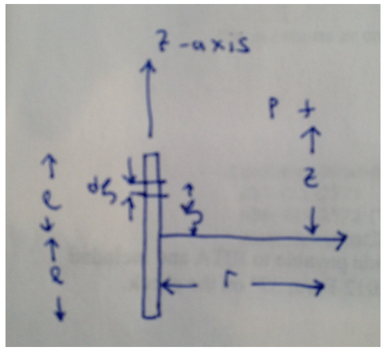

Figure 2: A uniformly charged line segment of length 2ℓ. The infinitesimal segment $d\zeta$ is at an elevation ζ above the origin. The point P is at a distance r from the z-axis and at an elevation z above the origin.

(a) What is the total charge of the line?

(b) Consider the infinitesimal segment of length $d\zeta$ a distance ζ above the origin. What is the electrostatic potential due to this segment at the point P?

(c) Integrate the result of part (a) to obtain the electrostatic potential at P due to the entire line segment.

Helpful integral:

$$\int d\zeta \, \frac{1}{\sqrt{r^2 + (\zeta - z)^2}} = \ln \left[\frac{\zeta - z + \sqrt{r^2 + (\zeta - z)^2}}{r} \right]. \tag{4}$$

(d) [*Optional. Very hard. Zero credit*] Show that the equipotential surfaces are ellipsoids with the two ends of the rod the focii of the ellipsoids.

5. **Potential of a disc** (Optional Zero Credit)

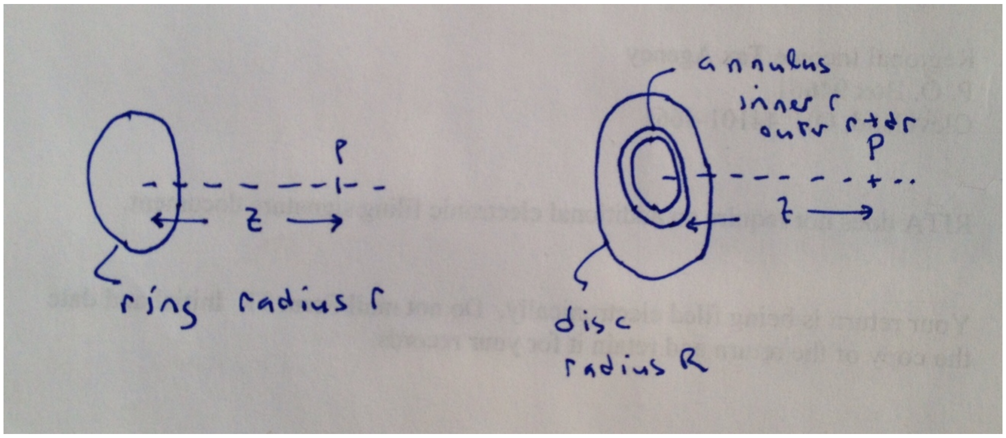

Figure 3: A charged ring (left) and disc (right). The disc may be subdivided into concentric annuli in order calculate its potential.

(a) *The Ring:* A ring of radius r has a charge q. The charge is distributed uniformly around the circumference of the ring. Determine the electrostatic potential ϕ at the point P that lies on the axis of the ring at a distance z from the center of the ring.

(b) *The Disc:* Now consider a uniformly charged disc with a radius R and a surface charge per unit area σ. We wish to determine the potential at a point that lies on the axis of the disc at a distance z from the center of the disc.

(i) To this end first consider the infinitesimal annulus of the inner radius r and outer radius $r + dr$. Determine the charge of this annulus. Determine the potential at P due to this infinitesimal annulus. [*Hint:* Use your result of part (a)].

(ii) Now integrate the result of part (i) to obtain the potential at P due to the disc.

Helpful integral:

$$\int dr \frac{2r}{\sqrt{r^2 + z^2}} = 2\sqrt{r^2 + z^2}. \tag{5}$$

(iii) What form do you expect the potential to have for $z \gg R$. Explain briefly. Does your result of part (ii) reduced to the expected result?

Helpful approximation: For $z \gg R$

$$\sqrt{z^2 + r^2} \approx z + \frac{R^2}{2z}. \tag{6}$$