1. Dimensional Analysis

a) SI Units

(i) E

$\vec{F} = Q\vec{E}$ $[=]$ $N = C \cdot \vec{E}$

$\vec{E}\, [=]\, \frac{N}{C} = \frac{kg\, m}{s^2 \cdot C}$

(ii) ε_0

$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}$

$[=]$ $N = \frac{1}{\varepsilon_0}\frac{C^2}{m^2}$

$\varepsilon_0\, [=]\, \frac{C^2}{N \cdot m^2} = \frac{C^2}{\frac{kg \cdot m}{s^2} \cdot m^2}$

$\varepsilon_0\, [=]\, \frac{C^2 \cdot s^2}{kg \cdot m^3}$

(iii) ϕ

$\phi = -\int \vec{E} \cdot d\vec{l}$

$[=]$ $\frac{kg \cdot m}{s^2 \cdot C} \cdot m$

$\phi\, [=]\, \frac{kg \cdot m^2}{s^2 \cdot C}$

(iv) B

$\vec{F}_B = q\,\vec{v} \times \vec{B}$

$[=]$ $N = C\left(\frac{m}{s}\right)(B)$

$\frac{kg \cdot m}{s^2} = C\left(\frac{m}{s}\right)B$

$B\, [=]\, \frac{kg}{C \cdot s}$

(v) I

$\vec{F} = I\int d\vec{l} \times \vec{B}$

$[=]$ $\frac{kg \cdot \cancel{m}}{s^{\cancel{2}}} = I\,(\cancel{m})\left(\frac{kg}{C \cdot \cancel{s}}\right)$

$I\, [=]\, \frac{C}{s}$

(vi) μ_0

$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{y}(-\hat{z})$

$[=]$ $\frac{kg}{C\,\cancel{s}} = \mu_0 \frac{C}{\cancel{s}\, m}$

$\mu_0\, [=]\, \frac{kg \cdot m}{C^2}$

(vii) p

$\vec{p} = q\,\vec{d}$

$[=]$ $p = C \cdot m$

$p\, [=]\, C\, m$

(viii) Energy

$W = \vec{F} \cdot \vec{d}$

$[=]$ $W = N \cdot m$

$= \frac{kg \cdot m}{s^2} \cdot m$

Energy $[=]$ $\frac{kg \cdot m^2}{s^2}$

b) Naming other units

(i) Volt $\left\{\frac{kg \cdot m^2}{s^2 \cdot C}\right\}$

Electrostatic Potential ϕ

(ii) Ampere $\left\{\frac{C}{s}\right\}$

Electric current I

(iii) Tesla $\left\{\frac{kg}{C \cdot s}\right\}$

Magnetic Field

(iv) Joule $\left\{\frac{kg \cdot m^2}{s^2}\right\}$

Energy

(v) eV $\left\{\frac{kg \cdot m^2}{s^2}\right\}$

Energy

c) Comparing units

(i) $\frac{N}{C} = \frac{V}{m}$?

Breaking down into fundamental units shows $1\frac{N}{C} = 1\frac{V}{m}$ as both are $\frac{kg \cdot m}{C \cdot s^2}$.

$\frac{N}{C} = \frac{\frac{kg \cdot m}{s^2}}{C} = \frac{kg \cdot m}{C \cdot s^2}$ $\qquad$ $\frac{V}{m} = \frac{\frac{kg \cdot m^2}{s^2 \cdot C}}{m} = \frac{kg \cdot m}{C \cdot s^2}$

(ii) Electric Field

d) Energy & Magnetic Energy Density

Should be $\frac{\text{Energy}}{\text{Volume}} = \frac{\frac{kg \cdot m^2}{s^2}}{m^3} = \frac{kg}{m \cdot s^2}$

$\frac{1}{2}\varepsilon_0 E^2 [=] \frac{\cancel{C^2} \cdot s^2}{\cancel{kg} \cdot m^3} \cdot \frac{kg^{\cancel{2}} \cdot m^{\cancel{2}}}{\cancel{C^2} \cdot s^{\cancel{4}2}} = \frac{kg}{m \cdot s^2}$

$\frac{1}{2}\mu_0 B^2 [=] \frac{kg \cdot m}{C^2}\left(\frac{kg}{C \cdot s}\right)^2 = \frac{kg^3 \cdot m}{C^4 \cdot s^2}$

$\frac{1}{2\mu_0} B^2 [=] \frac{\cancel{C^2}}{\cancel{kg} \cdot m} \cdot \frac{kg^{\cancel{2}}}{\cancel{C^2} \cdot s^2} = \frac{kg}{m \cdot s^2}$

Therefore $\frac{1}{2\mu_0} B^2$ should be correct and the professor didn't make a typo

e) $\mu_0^n \varepsilon_0^m = \frac{m}{s}$

(i) $\left(\frac{kg \cdot m}{C^2}\right)^n \left(\frac{C^2 \cdot s^2}{kg \cdot m^3}\right)^m = \frac{m}{s}$

$\left(\frac{kg^n m^n}{C^{2n}}\right)\left(\frac{C^{2m} s^{2m}}{kg^m \cdot m^{3m}}\right) = \frac{kg^{n-m} \cdot m^{n-3m} \cdot C^{2m-2n}}{s^{-2m}}$

$n - m = 0 \qquad n = -\frac{1}{2}$

$n - 3m = 1 \qquad -\frac{1}{2} - 3\left(-\frac{1}{2}\right) = -\frac{1}{2} + \frac{3}{2} = 1$

$2m - 2n = 0 \qquad 2\left(-\frac{1}{2}\right) - 2\left(-\frac{1}{2}\right) = 0$

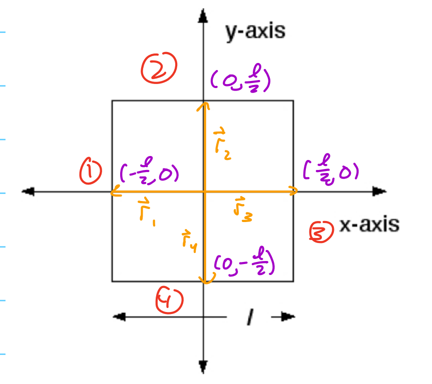
$-2m = 1 \qquad m = -\frac{1}{2}$

$\mu_0^{-\frac{1}{2}} \varepsilon_0^{-\frac{1}{2}} = \frac{m}{s}$

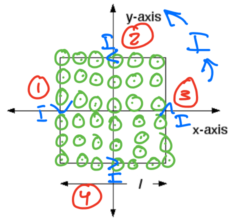
(ii) $\varepsilon_0 = 8.85 \times 10^{-12}$; $\mu_0 = 4\pi \times 10^{-7}$

$\mu_0^{-\frac{1}{2}} \varepsilon_0^{-\frac{1}{2}} = \left(4\pi \times 10^{-7}\right)^{-\frac{1}{2}} \left(8.85 \times 10^{-12}\right)^{-\frac{1}{2}} = 299{,}863{,}380.5 \ m/s$

2. Torque on a loop



a) Suppose $\vec B = B_z \hat k$



(i) ① $\vec F = I \int d\vec l \times \vec B$ RHR: $(-\hat i)$

$= I B_z (-\hat i) \int dl$

$\vec F_{B_1} = I B_z l (-\hat i)$

② RHR: $\hat j$ $\vec F_{B_2} = I B_z l (\hat j)$

③ RHR: $(\hat i)$ $\vec F_{B_3} = I B_z l (\hat i)$

④ RHR: $(-\hat j)$ $\vec F_{B_4} = I B_z l (-\hat j)$

(ii) $\vec \tau = \vec r \times \vec F$

① $\left.\begin{array}{l} \vec r_1 \ \& \ -\hat i \\ \vec F_1 \ \& \ -\hat i \end{array}\right\}$ $\theta = 0$ $\sin\theta = 0 \rightarrow \vec\tau_1 = 0$

① $\left.\begin{array}{l} \vec r_2 \ \& \ \hat j \\ \vec F_2 \ \& \ \hat j \end{array}\right\}$ $\theta = 0$ $\sin\theta = 0 \rightarrow \vec\tau_2 = 0$

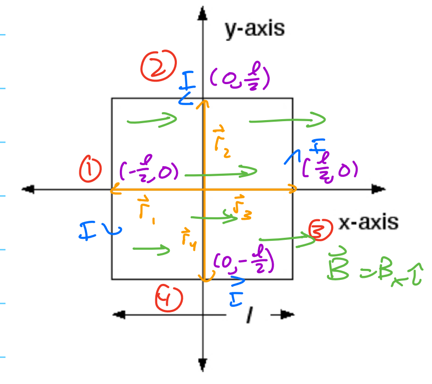
① $\left.\begin{array}{l} \vec r_3 \ \& \ \hat i \\ \vec F_3 \ \& \ \hat i \end{array}\right\}$ $\theta = 0$ $\sin\theta = 0 \rightarrow \vec\tau_3 = 0$

① $\left.\begin{array}{l} \vec r_4 \ \& \ -\hat j \\ \vec F_4 \ \& \ -\hat j \end{array}\right\}$ $\theta = 0$ $\sin\theta = 0 \rightarrow \vec\tau_4 = 0$

(iii) $\vec\tau_{net} = \vec\tau_1 + \vec\tau_2 + \vec\tau_3 + \vec\tau_4$
$= 0$

$\vec\tau_{net} = 0$

b) Suppose $\vec{B} = B_x\hat{i}$



(i) ① $\vec{F} = I\int d\vec{l}\times\vec{B} \Rightarrow \vec{F}_1 = I\,l\,B_x\,\hat{k}$

RHR: $-\hat{j}\times\hat{i} = \hat{k}$

② RHR: $\hat{i}\times\hat{i} = 0 \Rightarrow \vec{F}_2 = 0$

③ RHR: $\hat{j}\times\hat{i} = -\hat{k} \Rightarrow \vec{F}_3 = I l B_x(-\hat{k})$

④ RHR: $-\hat{i}\times\hat{i} = 0$ $\quad \vec{F}_4 = 0$

(ii) $\vec{\tau} = \vec{r}\times\vec{F}$

① $\vec{r}_1 \propto -\hat{i}$, $\vec{F}_1 \propto \hat{k}$ $\quad$ RHR: $-\hat{i}\times\hat{k} = \hat{j}$, $\theta = 90$ $\quad \vec{\tau}_1 = (\frac{l}{2}) I l B_x\hat{j} = \frac{1}{2} I l^2 B_x\hat{j}$

② $\vec{r}_2 \propto \hat{j}$, $\vec{F}_2 = 0 \Rightarrow \tau_2 = 0$

③ $\vec{r}_3 \propto \hat{i}$, $\vec{F}_3 \propto -\hat{k}$ $\quad$ RHR: $\hat{i}\times -\hat{k} = \hat{j}$, $\theta = 90$ $\quad \vec{\tau}_3 = (\frac{l}{2}) I l B_x\hat{j} = \frac{1}{2} I l^2 B_x\hat{j}$

④ $\vec{r}_4 \propto -\hat{j}$, $\vec{F}_4 = 0 \Rightarrow \tau_4 = 0$

(iii) $\tau_{net} = \vec{\tau}_1 + \cancel{\vec{\tau}_2}^{\,0} + \vec{\tau}_3 + \cancel{\vec{\tau}_4}^{\,0}$

$= \frac{1}{2} I l^2 B_x\hat{j} + \frac{1}{2} I l^2 B_x\hat{j} \Rightarrow \vec{\tau}_{net} = I l^2 B_x\hat{j}$

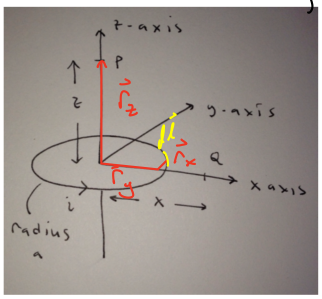
c) $\tau_{net} = \vec{m}\times\vec{B}$

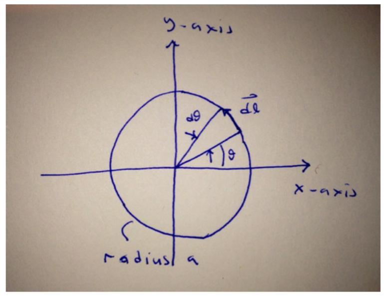
(i) $\vec{m}$ points $\hat{k}$
(A) $\vec{B}$ points $\hat{k}$
$\Big\}$ $\vec{m}\times\vec{B} = |mB|\sin\theta \to \theta = 0$ $\quad \vec{\tau}_{net} = 0$

(ii) $\vec{m}$ points $\hat{k}$
$\vec{B}$ points $\hat{i}$
$\Big\}$ $\vec{m}\times\vec{B} = |Ia||B_x|\sin\theta$ $\quad \vec{\tau}_{net} = I l^2 B_x\hat{j}$
RHR: $\hat{k}\times\hat{i} = \hat{j}$ $\quad \theta = 90°$

Both results are consistent!

3) Circular Loop





a) Magnetic Dipole Moment

RHR: Current CC, so $\vec{m} = m\hat{k}$

$m = iA \rightarrow A = \pi a^2$ (a = radius)

$\vec{m} = i\pi a^2\hat{k}$

b) Axial Field

(i) Point P at elevation z

$d\vec{l}$ = length(position vector) $= a\,d\theta\,(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$ ✓

$\theta = 0°: d\vec{l} \Rightarrow \uparrow$

$\theta = 90°: d\vec{l} \Rightarrow \leftarrow$

(ii) Displacement Vector

$\vec{r} = \vec{r}_x + \vec{r}_y + \vec{r}_z = a(-\cos\theta\,\hat{i} - \sin\theta\,\hat{j}) + z\hat{k}$

$r = \sqrt{[a(-\cos\theta\,\hat{i} - \sin\theta\,\hat{j})]^2 + z^2} = \sqrt{a^2+z^2}$ $\quad r = \sqrt{a^2+z^2}$

(iii) $d\vec{l} \times \vec{r} = \langle -a\,d\theta\sin\theta,\ a\,d\theta\cos\theta,\ 0\rangle \times \langle -a\cos\theta,\ -a\sin\theta,\ z\rangle$

$= \langle a\,d\theta\cos\theta\, z - 0,\ 0 - (-a\,d\theta\sin\theta)(z),\ (-a\,d\theta\sin\theta)(-a\sin\theta) - (a\,d\theta\cos\theta)(-a\cos\theta)\rangle$

$= \langle az\,d\theta\cos\theta,\ az\,d\theta\sin\theta,\ a^2 d\theta\sin^2\theta + a^2 d\theta\cos^2\theta\rangle$

$= az\,d\theta\cos\theta\,\hat{i} + az\,d\theta\sin\theta\,\hat{j} + a^2 d\theta\,\hat{k}$

$d\vec{l} \times \vec{r} = a\,d\theta\,(z\cos\theta\,\hat{i} + z\sin\theta\,\hat{j} + a\hat{k})$

(iv) $d\vec{B} = \frac{\mu_0 i}{4\pi} \cdot \frac{1}{\sqrt{a^2+z^2}^3}\left[a\,d\theta\,(z\cos\theta\,\hat{i} + z\sin\theta\,\hat{j} + a\hat{k})\right]$

(v) $\vec{B} = \int_0^{2\pi} \frac{\mu_0 i}{4\pi} \cdot \frac{1}{\sqrt{a^2+z^2}^3}\left[a\,d\theta\,(z\cos\theta\,\hat{i} + z\sin\theta\,\hat{j} + a\hat{k})\right]$

$= \frac{\mu_0 i}{4\pi} \cdot \frac{a}{\sqrt{a^2+z^2}^3}\left[z\cancelto{0}{\int_0^{2\pi}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\,d\theta} + a\int_0^{2\pi}\hat{k}\,d\theta\right]$

$= \frac{\mu_0 i}{\cancel{4\pi}\,2} \cdot \frac{a}{\sqrt{a^2+z^2}^3}\left[a\,2\cancel{\pi}\,\hat{k}\right]$

$\vec{B} = \frac{\mu_0 i}{2} \cdot \frac{a^2}{\sqrt{a^2+z^2}^3}\,\hat{k}$

(vi) $\vec{B} = \frac{\mu_0 i}{2} \cdot \frac{a^2}{\sqrt{a^2+z^2}^3} \hat{k}$

$\Rightarrow z \gg a$

$$\vec{B} = \frac{\mu_0 i}{2} \cdot \frac{a^2}{z^3} \hat{k} = \frac{\mu_0 (\overbrace{i a^2 \pi \hat{k}}^{\vec{m}})}{2\pi z^3} = \frac{\mu_0 \vec{m}}{2\pi z^3} \qquad \vec{B} = \frac{\mu_0 \vec{m}}{2\pi z^3}$$

c) Point Q on the x-axis

(i) From b-i) $d\vec{l} = a\,d\theta\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$

(ii) Add x to $\hat{i}$

$\vec{r} = (x - a\cos\theta)\hat{i} - a\sin\theta\,\hat{j}$

$r = \sqrt{(x - a\cos\theta)^2 + (-a\sin\theta)^2}$
$= \sqrt{x^2 - 2ax\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta}$
$r = \sqrt{x^2 - 2ax\cos\theta + a^2}$

(iii) $d\vec{l} \times \vec{r} = \langle -a\,d\theta\sin\theta, a\,d\theta\cos\theta, 0\rangle \times \langle x - a\cos\theta, -a\sin\theta, 0\rangle$
$= \langle 0-0, 0-0, (-a\,d\theta\sin\theta)(-a\sin\theta) - (a\,d\theta\cos\theta)(x - a\cos\theta)\rangle$
$= \langle 0, 0, a^2\sin^2\theta\,d\theta - ax\,d\theta\cos\theta + a^2\cos^2\theta\,d\theta\rangle$
$= \langle 0, 0, a^2\,d\theta - ax\,d\theta\cos\theta\rangle$

$d\vec{l} \times \vec{r} = a\,d\theta\,(a - x\cos\theta)\,\hat{k}$

(iv) $d\vec{B} = \frac{\mu_0 i}{4\pi} \frac{a}{\sqrt{x^2 - 2ax\cos\theta + a^2}^3} (a - x\cos\theta)\,\hat{k}\,d\theta$

(v) $$\mathbf{B} = -\frac{\mu_0 i a}{4\pi x^2}\hat{\mathbf{k}} \int_0^{2\pi} d\theta \, \frac{\cos\theta - (a/x)}{[1 - 2(a/x)\cos\theta + (a/x)^2]^{3/2}}.$$

Use: $$\frac{\cos\theta - (a/x)}{[1 - 2(a/x)\cos\theta + (a/x)^2]^{3/2}} \approx -\cos\theta + (3\cos^2\theta - 1)\frac{a}{x} + \ldots$$

$$\vec{B} = -\frac{\mu_0 i a}{4\pi x^2}\hat{k}\int_0^{2\pi}\left(\cancel{-\cos\theta}^{\,0} + (3\cos^2\theta - 1)\frac{a}{x} + \ldots\right)d\theta$$

$$= -\frac{\mu_0 i a}{4\pi x^2}\hat{k}\left(\pi\frac{a}{x}\right) = -\frac{\mu_0 (\overbrace{i\pi a^2\hat{k}}^{\vec{m}})}{4\pi x^3} \Rightarrow \vec{B} = -\frac{\mu_0 \vec{m}}{4\pi x^3}$$