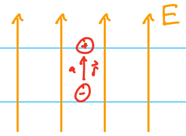
1. Energy of a Dipole

*a) Dipole moment of a $+q/-q$ pair: $p = qr$ r is displacement: $\ominus \xrightarrow[\vec{r}]{a} \oplus$

↳ Magnitude of dipole moment: $p = qa$

b) Force on a dipole

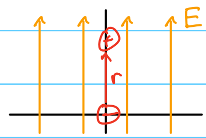


$F_- = -qE$

$F_+ = qE$

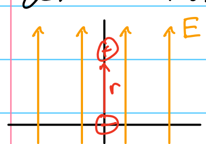
$F_{total} = F_- + F_+ = 0$

c) Torque on a dipole



$$\tau = r \times F$$
$$= \vec{r} \times F_+ + 0 \times F_-$$
$$= r \times (qE)$$
$$\tau = p \times E$$

d) Electrostatic Energy

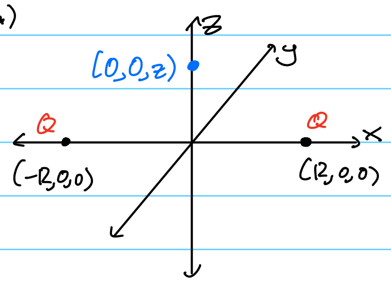


$$\Delta U = -\int_-^+ d\vec{l} \cdot (q\vec{E}) = -\int_0^r q\vec{E}\, d\vec{l} = -(\vec{r}\, d\, \vec{E})$$

$$\Rightarrow U = -p \cdot E$$

2. Equipotential Lines: Equipotentials for point charges

a)



Choose some point $(0, 0, z)$ on the z-axis

$$r = \sqrt{R^2 + z^2}$$

$$R: \ \vec{r}_R = -R\hat{i} + z\hat{k}$$

$$\vec{E}_R = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{\sqrt{R^2+z^2}^3}\right)\left[-R\hat{i} + z\hat{k}\right]$$

$$-R: \ \vec{r}_{-R} = R\hat{i} + z\hat{k}$$

$$\vec{E}_{-R} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{\sqrt{R^2+z^2}^3}\right)\left[R\hat{i} + z\hat{k}\right]$$

$$E_z = \vec{E}_R + \vec{E}_{-R} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{\sqrt{R^2+z^2}^3}\right)\left[\cancel{-R\hat{i}}^{\,0} + z\hat{k}\right] + \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{\sqrt{R^2+z^2}^3}\right)\left[\cancel{R\hat{i}} + z\hat{k}\right]$$

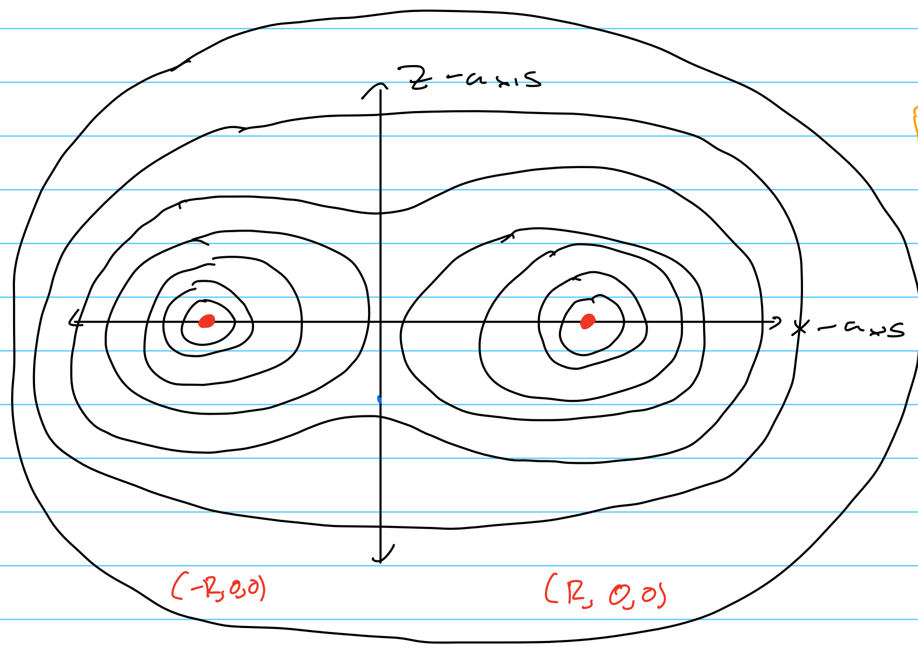
$$\Rightarrow \quad \vec{E}_z = \frac{z}{4\pi\varepsilon_0}\frac{2Q}{\sqrt{R^2+z^2}^3}\hat{k}, \quad E_z = \frac{z}{4\pi\varepsilon_0}\frac{2Q}{\sqrt{R^2+z^2}^3}$$

Maximum when $\frac{dE_z}{dz} = 0$

$$\frac{dE_z}{dz} = \frac{d}{dz}\left(\frac{Q}{2\pi\varepsilon_0}\frac{z}{\sqrt{R^2+z^2}^3}\right) = \frac{Q}{2\pi\varepsilon_0}\left[\frac{R^2-2z^2}{(R^2+z^2)^{5/2}}\right] = 0$$

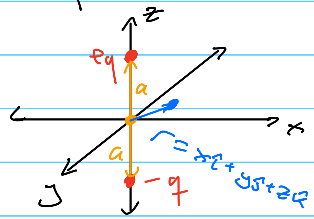
$$\Rightarrow R^2 - 2z^2 = 0 \quad \text{so} \quad z = \pm\frac{R}{\sqrt{2}}$$

b)



Become more circular (like a single charge) as we get far away

3. Dipole Potential



a) $\vec{R} = 0\hat{i} + 0\hat{j} + 2a\hat{k}$

$\Rightarrow \quad p = q\vec{R} = q(2a\hat{k})$

$$p = 2qa\hat{k}$$

b) $r_+ = \sqrt{x^2+y^2+(z-a)^2}$

$r_- = \sqrt{x^2+y^2+(z+a)^2}$

Point like charges

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_+} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{r_-} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} + \frac{-q}{r_-}\right)$$

$$\Rightarrow \quad U = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+a)^2}}\right)$$

c) $$\frac{1}{\sqrt{x^2+y^2+(z-a)^2}} \approx \frac{1}{r} + \frac{az}{r^3}$$

$$\frac{1}{\sqrt{x^2+y^2+(z+a)^2}} \approx \frac{1}{r} - \frac{az}{r^3}$$

$$\Rightarrow \quad U = \frac{q}{4\pi\varepsilon_0}\left(\left(\cancel{\frac{1}{r}} + \frac{az}{r^3}\right) - \left(\cancel{\frac{1}{r}} - \frac{az}{r^3}\right)\right) = \frac{q}{4\pi\varepsilon_0}\frac{2az}{r^3}$$

d) $p = qb$, where a = distance

so, here $b = 2a$

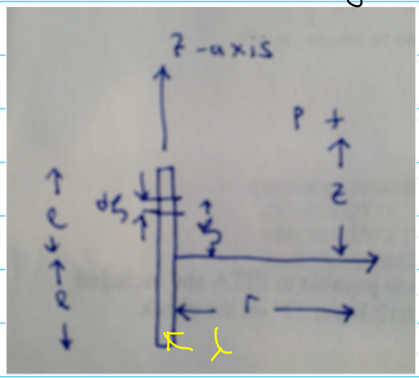
Define $z \equiv \vec{r}$, displacement vector $r = z\hat{k}$

$$\Rightarrow \quad U = \frac{2qaz}{4\pi\varepsilon_0 r^3} = \frac{bqz}{4\pi\varepsilon_0 r^3} = \frac{p \cdot r}{4\pi\varepsilon_0 r^3}$$

$$\Rightarrow \quad U = \frac{1}{4\pi\varepsilon_0} \cdot \frac{p \cdot r}{r^3}$$

matches given ideal potential

4. Potential of a charged rod



a) $\lambda = \frac{\text{charge}}{\text{length}} = \frac{Q}{L}$

$Q = L\lambda = 2\ell\lambda$ $\qquad Q_{tot} = 2\lambda\ell$

b) Dist from $d\xi$ to P:

$$r = \sqrt{r^2 + (\xi - z)^2}$$

Infinitesimal Segment $d\xi \to dQ$

$$dQ = \lambda\, d\xi$$

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} \Rightarrow dV = d\Phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{\lambda\, d\xi}{\sqrt{r^2+(\xi-z)^2}}\right)$$

c) $$\Phi_P = \int_{\text{Bottom}}^{\text{Top}} d\Phi = \int_{\xi=-\ell}^{\xi=\ell} \frac{1}{4\pi\varepsilon_0}\left(\frac{\lambda\, d\xi}{\sqrt{r^2+(\xi-z)^2}}\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\int_{\xi=-\ell}^{\xi=\ell}\left(\frac{d\xi}{\sqrt{r^2+(\xi-z)^2}}\right)$$

Using an integral

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{\xi - z + \sqrt{r^2+(\xi-z)^2}}{r}\right]_{\xi=-\ell}^{\xi=\ell}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left(\ln\left[\frac{\ell - z + \sqrt{r^2+(\ell-z)^2}}{r}\right] - \ln\left[\frac{-\ell - z + \sqrt{r^2+(-\ell-z)^2}}{r}\right]\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\cdot\ln\left[\frac{\frac{\ell - z + \sqrt{r^2+(\ell-z)^2}}{\cancel{r}}}{\frac{-\ell - z + \sqrt{r^2+(-\ell-z)^2}}{\cancel{r}}}\right] = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{\ell - z + \sqrt{r^2+(\ell-z)^2}}{-\ell - z + \sqrt{r^2+(-\ell-z)^2}}\right]$$

$$\Rightarrow \Phi_P = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{\ell - z + \sqrt{r^2+(\ell-z)^2}}{-\ell - z + \sqrt{r^2+(-\ell-z)^2}}\right]$$