



Midterm Exam 1



Electric Charges & Fields, Coulomb's Law, Superposition

Electric Charge

- charge is a fundamental prop of matter
 - two types: \textcircled{P} & \textcircled{G} matter(atoms) are normally neutral
- Triboelectricity:** Friction transfers charge

Metal Rods in Demo?

- Plastic & crystal are insulators
 - charge "stays where put"
- Metals are conductors
 - charge "flow freely"

Charge is conserved

- Rubbing transfers charges, but does not create nor destroy it

Charge is quantized

- charge is always a multiple of $e = 1.6 \times 10^{-19}$ Coulombs (C) \Rightarrow electron = $-e$, proton = $+e$

Why the rods are moving!

- Attraction or Repulsion \Rightarrow There is a force between the charges

\hookrightarrow Force is known as coulomb's force

$$F = \frac{1}{4\pi\epsilon_0} \cdot \frac{qQ}{r^2} \quad \text{where } \frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}$$

\square When two charges q and Q are same signed $\Rightarrow F$ is positive = Repulsive

\square When they have opposite signs $\Rightarrow F$ is negative = Attractive

Demo - Rods

$$\textcircled{P} \textcircled{P} \textcircled{G} \leftarrow \rightarrow$$

$$\textcircled{G} \textcircled{G} \textcircled{P} \leftarrow \rightarrow$$

$$\textcircled{P} \textcircled{G} \textcircled{P} \rightarrow \leftarrow$$

Note: the metal is still neutral

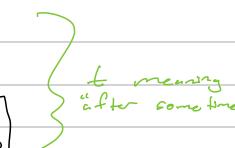
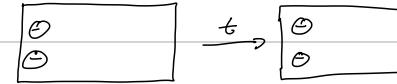
Demoval / Metal

$$\textcircled{G} \quad \textcircled{G}$$

$$\textcircled{P} \quad \textcircled{G}$$

$$\textcircled{P} \quad \textcircled{G}$$

$$\textcircled{G} \quad \textcircled{G}$$

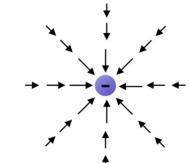
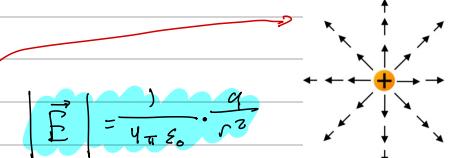


t meaning "after sometime"

Electric Field and Coulomb's Law

Laws of Electrostatics

- Electrostatics = charges @ rest
- charges produce electric fields
- electric fields push charges



$$q \quad \vec{r} \quad | \vec{r} | = r$$

$$|\vec{E}| = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2} \quad \text{Note: } r = |\vec{r}|$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

\vec{r} is a unit vector that points away from charge q

or alternatively so $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$

(so $\vec{v} = |\vec{v}| \hat{v}$)

Equivalent!

Superposition Principle

In Words:

Electric Field produced = Sum of the electric field produced by many point charges by each point charge independently

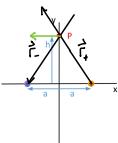
$$\text{In Maths: } \vec{E}(q_1, q_2) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_1^2} \hat{r}_1 + \frac{1}{4\pi\epsilon_0} \frac{q_2}{r_2^2} \hat{r}_2$$

Second Law of Electrostatics

$$\text{Fields push/pull charges: } \vec{F} = Q \vec{E} \Rightarrow \vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{r}$$

$\vec{E} \propto \vec{E} \propto \vec{E}$

Example: Point Charges



Calculate the Electric field at point P.

Pos vector \vec{r}_1 from O to $p_1: -a\hat{i} + b\hat{j}$

Pos vector \vec{r}_2 from O to $p_2: a\hat{i} + b\hat{j}$ (not mass, position not dipole charge!)

Calculate the Electric Field at P:

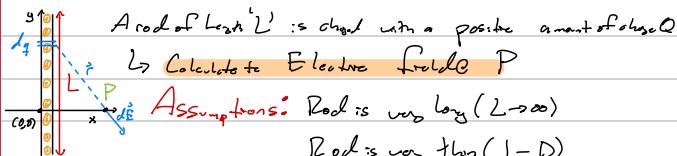
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} \vec{r} \rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0} \left(\frac{1}{(\sqrt{a^2+b^2})^3} [q(-a\hat{i}+b\hat{j}) + -q(a\hat{i}+b\hat{j})] \right) = \frac{2qa}{4\pi\epsilon_0 \sqrt{a^2+b^2}^3} \cdot (-\hat{i})$$

$$r_1 = a\hat{i} + b\hat{j} \rightarrow r_1 = \sqrt{a^2+b^2} \quad \rightarrow \vec{E} = \vec{E}_1 + \vec{E}_2$$

$$r_2 = -a\hat{i} + b\hat{j} \rightarrow r_2 = \sqrt{a^2+b^2}$$

Here we are assuming point-like charges but this is not realistic

Continuous Distribution of Charge



Calculate the Electric field at P

Assumptions: Rod is very long ($L \rightarrow \infty$)

Rod is very thin ($L \ll D$)

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} \vec{r} \rightarrow d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \vec{r}$$

$$\vec{r} = x\hat{i} - y\hat{j} \quad r = \sqrt{x^2+y^2} \quad \rightarrow d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \cdot (x\hat{i} - y\hat{j})$$

Definition: Charge per unit length or linear charge density

$$\lambda = \frac{\text{Charge}}{\text{Length}} = \frac{Q}{L}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \sum_{i=1}^N \vec{E}_i = \int_a^b d\vec{E}$$

$$\text{so, } \vec{E} = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\lambda}{\sqrt{x^2+y^2}} (x\hat{i} - y\hat{j}) dy \quad \text{Solve } x \text{ and } y \text{ separately}$$

Example

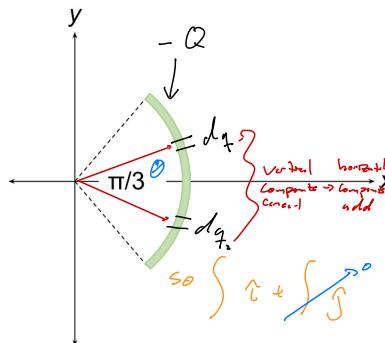
A plastic rod with a uniformly distributed charge $-Q$ is bent into a circular arc of radius r that subtends an angle of $\pi/3$ radians. We place coordinate axes such that the axis of symmetry of the rod lies along the x-axis and the origin is the center of curvature for the rod. What is the electric field (magnitude and direction) at the origin?

Hint: Imagine that the arc is made up of many infinitesimally small point charges dq . Each dq creates a differential electric field of magnitude $dE = k dq/r^2$.

Sum up all the dEs from all the dqs to get the magnitude of the overall electric field.

Integration is a way to add many infinitesimally small elements.

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \vec{r}$$



To integrate, let's use polar! $(x, y) \rightarrow (r, \theta)$

$$\int dE_x = \frac{1}{4\pi\epsilon_0} \int_{-\pi/6}^{\pi/6} \frac{r d\cos\theta}{r^2} d\theta$$

$$= \frac{1}{4\pi\epsilon_0 r} \left[\sin\theta \right]_{-\pi/6}^{\pi/6}$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{3Q}{\pi r^2}$$

$$\text{but what is lambda? } \lambda = \frac{Q}{L} = \frac{Q}{r(\pi/3)} \quad \text{Just think length = angle * radius (like in } C = 2\pi r)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{3Q}{\pi r^2} \hat{i}$$

$$|d\vec{E}| = \left| \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \vec{r} \right|$$

$$\begin{cases} \cos\theta = \frac{a}{r} \\ \sin\theta = \frac{b}{r} \end{cases} \rightarrow \vec{r} = -r\cos\theta\hat{i} - r\sin\theta\hat{j}$$

$$x\text{-component of } |dE|: dE_x = \frac{1}{4\pi\epsilon_0} \frac{da}{r^2} \cos\theta$$

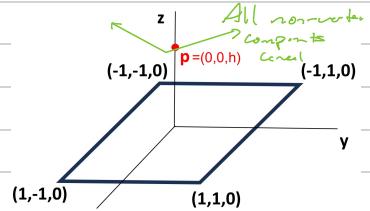
We need to integrate over all θ , b , dE_x terms of dq , not $d\theta$!

$$\Rightarrow dq = \int ds \quad \text{where } ds \text{ is segment of arc}$$

$$\Rightarrow \frac{2\pi r}{\theta} \text{ so } ds = r d\theta \rightarrow dq = r \lambda d\theta$$

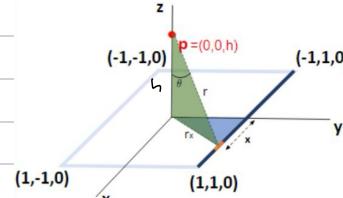
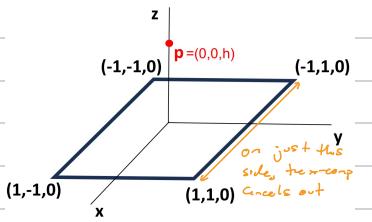
Example

A plastic rod with a uniformly distributed charge Q is shaped forming a square whose side is 1m. We place coordinate axes such that the square is on the x-y plane and the origin is the center of the square. What is the electric field (magnitude and direction) at the point $p=(0,0,h)$?



\Rightarrow The problem solution will be $\vec{E} = E \hat{k}$

So we just need to calculate the vertical (z) component of one side and add the rest!



Hint: What is the square formed by? Is there any symmetry?

Imagine that the square is made up of many infinitesimally small point charges dq . Each dq creates a differential electric field of magnitude $dE = k dq/r^2$.

Sum up all the dEs from all the dqs to get the magnitude of the overall electric field.

Integration is a way to add many infinitely small elements.

$$\begin{aligned} \vec{r}_h &= h \hat{k} \\ \vec{r}_{xy} &= -x \hat{i} - y \hat{j} \Rightarrow r_{xy}^2 = x^2 + y^2 \\ \text{so, } r^2 &= r_{xy}^2 + h^2 \\ \Rightarrow r &= \sqrt{x^2 + y^2 + h^2} \end{aligned}$$

$$\begin{aligned} d\vec{E} &= \frac{1}{4\pi\epsilon_0} \frac{dq}{(x^2 + y^2 + h^2)^{3/2}} (-x \hat{i} - y \hat{j} + h \hat{k}) \\ \Rightarrow \vec{E} &= \int \frac{1}{4\pi\epsilon_0} \frac{dq}{(x^2 + y^2 + h^2)^{3/2}} (-x \hat{i} - y \hat{j} + h \hat{k}) \end{aligned}$$

only components \hat{k}

In one side \hat{j} component part, but disappears when using Left cancellation

$$\vec{E}_{\text{Top}} = \int_{-1}^1 \frac{1}{4\pi\epsilon_0} \frac{dy}{(\sqrt{x^2 + y^2 + h^2})^3} (-x \hat{i} - y \hat{j} + h \hat{k})$$

$$\vec{E}_{\text{Bottom}} = \int_{-1}^1 \frac{1}{4\pi\epsilon_0} \frac{dy}{(\sqrt{x^2 + y^2 + h^2})^3} (+\hat{i} - y \hat{j} + h \hat{k})$$

$$\Rightarrow \vec{E}_{\text{Left}} = \int d\vec{E}_{\text{Left}} = \dots = -c \hat{j} + d \hat{k}$$

$$\text{so } \boxed{\vec{E} = \vec{E}_{\text{Top}} + 4 \hat{k}}$$

Gauss Law

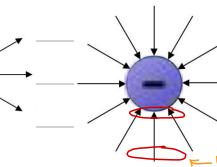
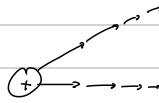
Electric Field Lines

- (i) Lines are tangent to \vec{E} at every point in space
- (ii) The density of lines is proportional to $|\vec{E}|$
- (iii) Lines never cross
- (iv) No closed loops
- (v) Lines start or stop on charges

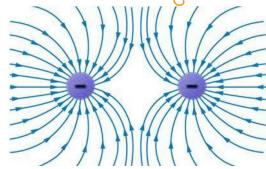
↳ Start should be a positive charge, and
should be on negative charge

↳ This rule inherently enforces part (iv)

Single Charge



Two charges

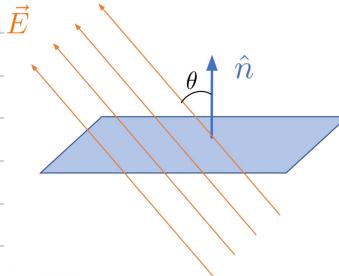


Recall Vector Multiplication

$$\begin{cases} \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \\ = |\vec{A}| |\vec{B}| \cos \theta \end{cases} \quad \text{for } \theta \text{ the angle between } \vec{A} \text{ and } \vec{B}$$

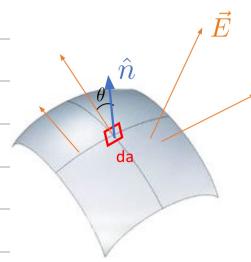
Electric Flux

- Plot surface and uniform \vec{E}
- Defined: $\Phi_E = A \hat{n} \cdot \vec{E} = A |\hat{n}| |\vec{E}| \cos \theta = A |\vec{E}| \cos \theta$
 - o \hat{n} is a unit vector perpendicular to the surface (normal vector)
 - o θ is the angle between \hat{n} and \vec{E}
 - o A is the area of the surface
- Non-Plot surface and/or Non-uniform \vec{E}



$$\Phi_E = \int \hat{n} \cdot \vec{E} dA = \int |\hat{n}| |\vec{E}| \cos \theta dA = \int |\vec{E}| \cos \theta dA$$

- o \hat{n} is a unit vector perpendicular to the surface da
- o θ is the angle between \hat{n} and \vec{E}



Gauss Law

- Rest of Maxwell's Equations

- In words: The electric flux

$$\text{through a closed surface} = \frac{\text{Amount of charge inside the surface}}{\epsilon_0}$$

- Equation:

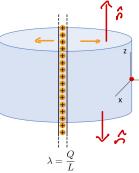
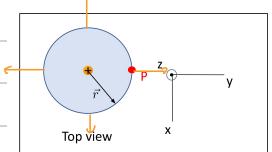
$$\Phi_E = \int_{\text{closed}} \hat{n} \cdot \vec{E} dA = \frac{Q}{\epsilon_0}$$

- \vec{E} points outside the surface by convention

$$\cdot \hat{n} \cdot d\vec{A} = \frac{d\vec{A}}{\epsilon_0} \Rightarrow \text{locally, where } d\vec{A} = da \cdot \hat{n}$$

• Example / Review: Infinite Line

Assumption: rod is very long ($L \rightarrow \infty$)
rod is very thin ($l \rightarrow 0$)



Length l of rod make cylinder

$$\Rightarrow Q_{ENC} = \lambda l$$

$$\phi_E = \frac{Q_{ENC}}{\epsilon_0} = \frac{\lambda l}{\epsilon_0}$$

Ext records
Question

What is the direction of the total electric field @ P?

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{n}, \quad E = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$2 \int_{\text{Cups}} |\vec{E}| \cos \theta d\alpha + \int_{\text{Side}} \vec{n} \cdot \vec{E} d\alpha$$

$\theta = 90^\circ, \hat{n} = \vec{0}$

$$= 0 + \int_{\text{Side}} |\vec{E}| \cos \theta d\alpha$$

$$= \int E d\alpha = E d\alpha = EA$$

$$\Phi_E = E \cdot 2\pi r l$$

$$E \cdot 2\pi r l = \frac{\lambda l}{\epsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\epsilon_0 r}$$

CAPS radial

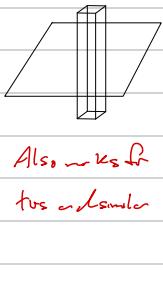
$$\vec{E} = E \hat{r} \quad \left. \begin{array}{l} \vec{E} = E \hat{r} \\ \hat{n} = \pm \hat{r} \end{array} \right\} \theta = 90^\circ$$

SIDE radial

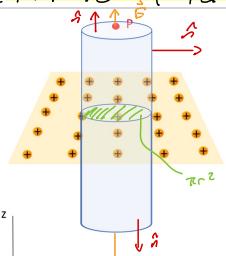
$$\vec{E} = E \hat{r} \quad \left. \begin{array}{l} \vec{E} = E \hat{r} \\ \hat{n} = \hat{r} \end{array} \right\} \theta = 0$$

The magnitude is constant
for all parts on SIDE

Infinite Plane



Also works for
two conductors



Direction of Electric Field?

$$A = \pi r^2$$

$$\sigma = \frac{Q}{L^2} \quad \int \vec{n} \cdot \vec{E} d\alpha = \frac{Q_{ENC}}{\epsilon_0}$$

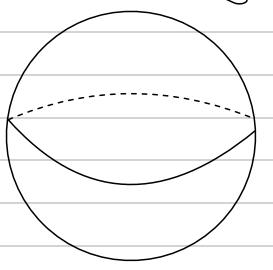
$$Q_{ENC} = \sigma \pi r^2$$

SIDE $\hat{n} = \vec{z}$ $\theta = 90^\circ$
CAPS $\hat{n} = \pm \hat{z}$ $\theta = 0$

$$\int_{\text{Cups}} \vec{n} \cdot \vec{E} d\alpha = 2 \int E d\alpha = 2EA = 2E\pi r^2$$

$$2B\pi r^2 = \frac{\sigma \pi r^2}{\epsilon_0} \Rightarrow E = \frac{\sigma}{2\epsilon_0}$$

Spherical Symmetry



Charge Density $\rho = \frac{Q}{V}$

$$\left\{ \frac{C}{m^3} \right\}$$

$$\int \vec{n} \cdot \vec{E} d\alpha = \frac{Q_{enc}}{\epsilon_0}$$

Calculate the Electric field at points P1 and P2 using Gauss Law of the sphere of Radius R and charged with a density ρ

$$\hat{n} = \vec{r}$$

$$\vec{E} = E \vec{r}$$

$$\frac{Q_{enc}}{\epsilon_0} = \rho \frac{4}{3}\pi R^3$$

Flux through gaussian Surface which is outer radius of sphere $\rightarrow \int E \cos(\theta) d\alpha = E A = E(4\pi r^2)$

$$R > R: \frac{\rho \frac{4}{3}\pi R^3}{\epsilon_0} = E(4\pi r^2) \Rightarrow E = \frac{\rho R^3}{3\epsilon_0 r^2}$$

$R < R: \hat{n} = \vec{r}$

$$\vec{E} = E \vec{r}$$

$$\frac{Q_{enc}}{\epsilon_0} = \frac{\rho V}{\epsilon_0} = \frac{\rho \frac{4}{3}\pi r^3}{\epsilon_0}$$

$$\int E d\alpha = EA = E(4\pi r^2)$$

Gaussian surface is r

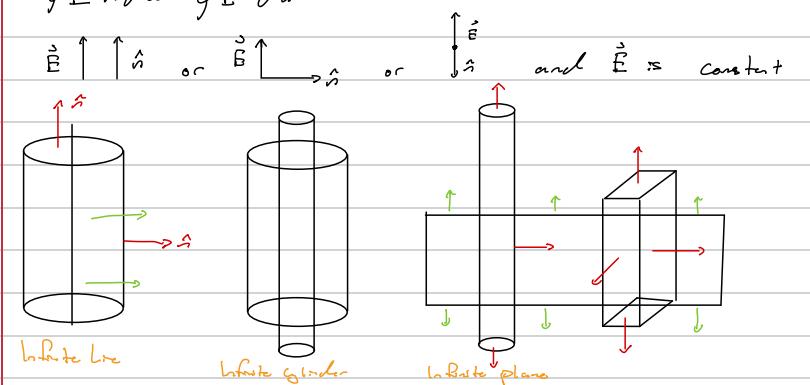
$$\left\{ \frac{\rho \frac{4}{3}\pi r^3}{\epsilon_0} = E(4\pi r^2) \Rightarrow E = \frac{\rho r}{3\epsilon_0} \right\}$$

Any charge outside the gaussian surface does not affect the Flux

Exam 1 Review Lecture

Gauss Law

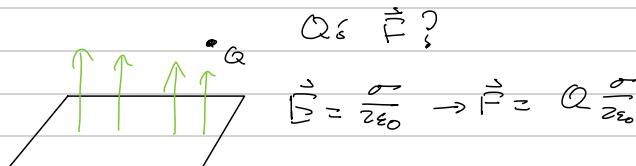
$$\oint \vec{E} \cdot d\vec{a} = \oint \vec{E} \cdot d\vec{a}$$



Coulomb's Law

- $\vec{F} = Q \vec{E}$
 - E by point q : $E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$
- $\left. \begin{array}{l} \text{If you have only} \\ \text{two charges} \end{array} \right\} \Rightarrow \vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$

But if not 2 point-line:



\vec{r} vs. \hat{r}

$$\begin{aligned}
 \vec{r} &= x\hat{i} + y\hat{j} \\
 \hat{r} &= |\vec{r}| \cdot \hat{r} \\
 \rightarrow |\vec{r}| &= r = \sqrt{x^2 + y^2} \\
 \rightarrow \hat{r} &= \frac{x}{\sqrt{x^2 + y^2}} \hat{i} + \frac{y}{\sqrt{x^2 + y^2}} \hat{j}
 \end{aligned}$$

magnitude

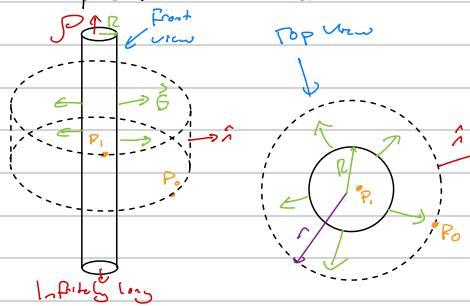
direction

Direct Integration

charge density

$$\begin{aligned}
 \lambda &= \frac{C}{m} & \sigma &= \frac{C}{m^2} & J &= \frac{C}{m^3} \\
 \text{linear} & & \text{surface} & & \text{volume} &
 \end{aligned}$$

Example: Gauss Law



$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} = \frac{\pi D^2 l \cdot P}{\epsilon_0}$$

$$\begin{aligned}
 \vec{E} &= E\hat{r} \\
 \text{Since } \hat{r} &= \hat{z} \\
 \text{CAPS. } z &= \pm r\hat{z}
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \int_{CAPS} E \cos(90^\circ) da + \int_{S,DBS} E \cos 0 da \\
 &= 0 + E_a = E(2\pi r l)
 \end{aligned}$$

$$\Rightarrow E = \frac{\lambda^2 l}{2\pi r \epsilon_0}$$

outward

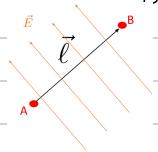
Cannot depend on parameter of Gaussian surface (in this case)

Midterm Exam 2



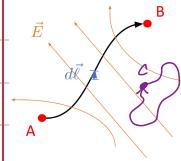
Circulation Law

Line Integrals



Straight line & uniform \vec{E}

- Line Integral = $\int \vec{E} \cdot d\vec{l}$ for $d\vec{l}$ the displacement vector
↳ For this simple application (straight line & uniform \vec{E})



General

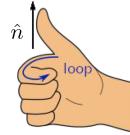
$$\bullet \text{Line integral} = \int_A^B \vec{E} \cdot d\vec{l}$$

Closed Loops

$$\bullet \text{Circulation: } \oint \vec{E} \cdot d\vec{l}$$

Right Hand Rule

- Stokes' theorem
- Use orientation of loop to determine \vec{l} direction using:



Circulation

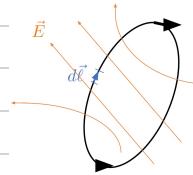
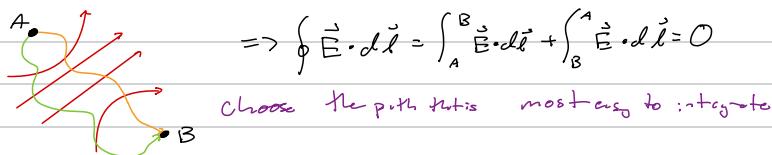
- The circulation of the electric field equals zero

$$\rightarrow \text{Circulation} = \oint d\vec{l} \cdot \vec{E} = 0$$

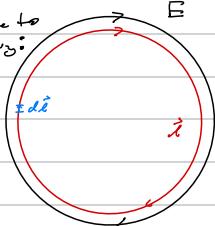
(Coulomb's law)

Alternative: The line integral between two points do not depend on path

$$\int_A^B \vec{E} \cdot d\vec{l} = - \int_B^A \vec{E} \cdot d\vec{l} \quad \left\{ \int_A^B \vec{B} \cdot d\vec{l} \right\}$$



Assume to recenter:



No closed loops! (Proof by contradiction)

$$\begin{aligned} \text{So, } \oint \vec{E} \cdot d\vec{l} &= \int |E| d\vec{l} \text{ (area)} \\ &> \text{smallly} \\ \Theta &= 0 = \int |E| d\vec{l} \\ &= (+) > 0 \end{aligned}$$

No closed loops!

$$\text{but } \oint \vec{E} \cdot d\vec{l} = 0 \text{ by circ. law}$$

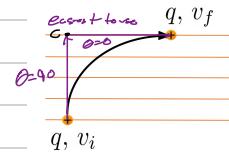
Application / PIZI

• Recall: Work-Energy Theorem

$$\int_{i \rightarrow f} d\vec{l} \cdot \vec{F}_{\text{total}} = \Delta K = \frac{1}{2} m(v_f^2 - v_i^2)$$

• But for electric force we don't need the path!

$$\rightarrow \int_{i \rightarrow f} d\vec{l} \cdot (q \vec{E}) = \Delta K = \frac{1}{2} m(v_f^2 - v_i^2) = \text{Same for all paths b/w i and f}$$

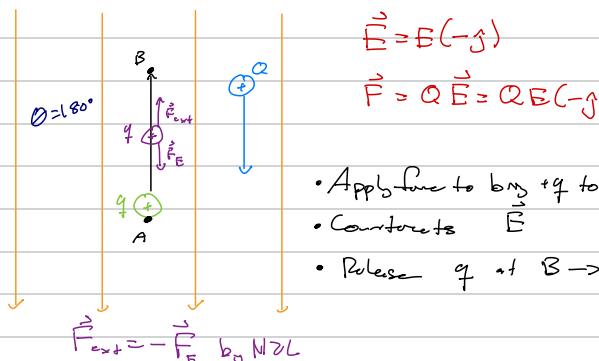


Example

$$\begin{aligned} \int_A^B \vec{E} \cdot d\vec{l} &= \int_A^C \vec{E} \cdot d\vec{l} + \int_C^B \vec{E} \cdot d\vec{l} \\ &= \int_A^C q E \cos 90^\circ d\vec{l} + \int_C^B q E \cos 90^\circ d\vec{l} \\ &= q E \int_C^B d\vec{l} = q E L \end{aligned}$$

Electrostatic Energy

Overview / Introduction

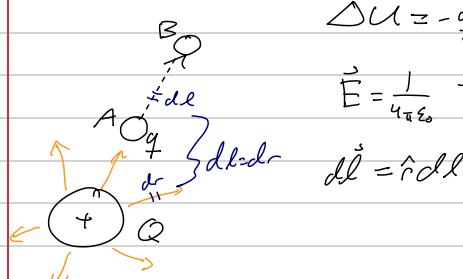


Using our assumption

- The external force matches the electric force: $U_B - U_A = \Delta U = \int_A^B d\vec{l} \cdot \vec{F}_{\text{ext}} = - \int_A^B d\vec{l} \cdot (q \vec{E})$
- In this simple case: $U_B - U_A = \Delta U = q E (z_B - z_A) \rightarrow \text{GPE } (U_0 = mg)$

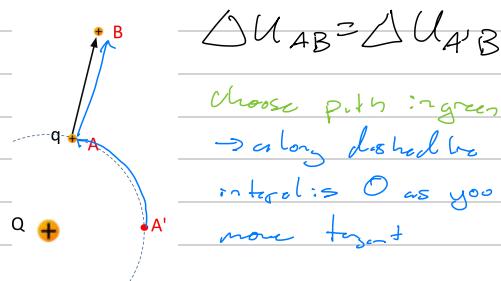
This is a particular formula, not general!

Point Charge

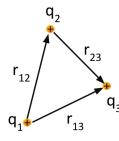


$$\Rightarrow \Delta U = \frac{qQ}{4\pi\epsilon_0} \left(\frac{1}{r_B} - \frac{1}{r_A} \right)$$

$$\text{If } r_A \rightarrow \infty \text{ chooses reference: } U = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r}$$



Arrangement of Charges



- The total energy of the system equals the sum of the energy of each pair of charges

$$U = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right)$$

Electrostatic Potential

- Defined Φ , also known as V
- Do not confuse w/ electrostatic energy (U)
- Another way to describe an electric field
- Electric static potential is a scalar (not a vector)
- To know \vec{E} or Φ for any point in space provides some amount of info (on electrostatics)
- If you know one you can find out the other
- Voltage $V = \Delta \Phi$

Example: Scalar Fields

Elevation

Vector Fields

Gravity

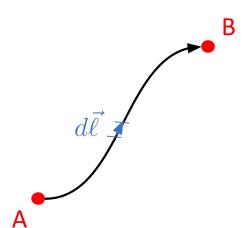
Temperature

Wind

$$\vec{E}(r) \leftrightarrow \Phi(r)$$

$\vec{E}(r)$ not discussed here
not easily measurable

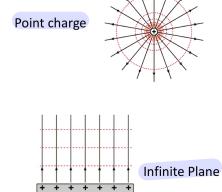
Demonstration



$$\begin{aligned}\Delta \Phi &= \Phi(B) - \Phi(A) = - \int_A^B d\vec{l} \cdot \vec{E} \\ \text{if we set a reference point } (\Phi=0) \\ \Phi(r) &= - \int_{\text{ref}}^r d\vec{l} \cdot \vec{E} \quad \Phi(\text{ref}) = 0\end{aligned}$$

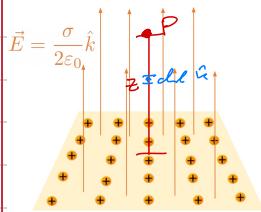
Scalar Quantity - Use Equipotential Lines

Equipotential lines



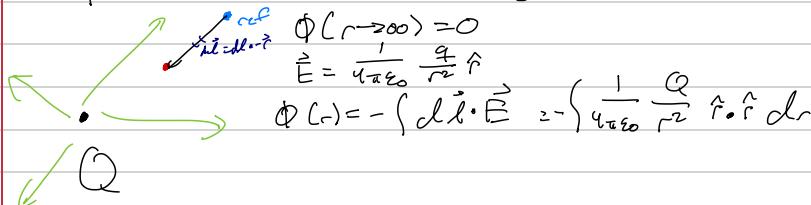
Always perpendicular to electric lines

Example: Potential - Infinite Plane



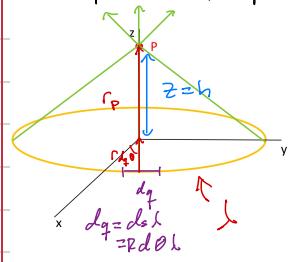
$$\begin{aligned}\text{Chg. } \sigma, z=0 \text{ so } \Phi(z=0)=0 \\ \vec{E} = \frac{\sigma}{2\epsilon_0} (\pm \hat{k}) \quad (\text{from Gauss law}) \\ \Delta \Phi = \Phi_{\text{ref}} - \Phi_P = - \left(\int_{\text{ref}}^P d\vec{l} \cdot \vec{E} \right) = \int E d\vec{l} \cos \theta = E \int_z^0 dz = E_z = \frac{-\sigma z}{2\epsilon_0}\end{aligned}$$

Example: Potential - Point Line Charge



$$\Phi(r) = \frac{Q}{4\pi\epsilon_0 r}$$

Example: Superposition Principle (\vec{E} vs. Φ)



a) Develop the Field \vec{E} at P

Should point \vec{r}_P

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \hat{r} = \frac{1}{4\pi\epsilon_0} \frac{dq}{\sqrt{z^2+R^2}} \{ z\hat{i} - R(\cos\theta\hat{j} + \sin\theta\hat{k}) \}$$

$$\begin{aligned}\vec{r}_P = z\hat{i} \\ \vec{r}_{dq} = R\cos\theta\hat{j} + R\sin\theta\hat{k}\end{aligned} \quad \left. \begin{aligned}\vec{r} + \vec{r}_{dq} &= \vec{r}_P \\ \vec{r} &= z\hat{i} - R(\cos\theta\hat{j} + \sin\theta\hat{k})\end{aligned} \right\} \quad \begin{aligned}r &= \sqrt{z^2 + R^2} \quad (1)\end{aligned}$$

$$\Rightarrow \vec{E} = \frac{R}{4\pi\epsilon_0(z^2+R^2)^{3/2}} \int_0^{2\pi} (z\hat{i} - R\cos\theta\hat{j} - R\sin\theta\hat{k}) d\theta \quad \text{cos}\theta \text{ and } \sin\theta \text{ are 0 to } 2\pi = 0$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{2\pi R z}{(z^2+R^2)^{3/2}} \hat{i}$$

b) Develop potential without using \vec{E}

$$\Phi(P) = ?$$
$$d\Phi = \frac{1}{4\pi\epsilon_0} \frac{dq}{r}$$
$$\Phi(h) = \int_{0}^{2\pi} \frac{1}{4\pi\epsilon_0} \frac{R d\theta}{\sqrt{R^2 + h^2}} \Rightarrow$$
$$r = \sqrt{R^2 + h^2}$$

It's easier in general to find Φ

$$\Phi(h) = \frac{\Sigma R}{2\epsilon_0} \frac{1}{\sqrt{R^2 + h^2}}$$

Correlation between Φ and \vec{E} \leftarrow outside of P scope of course

$$\Phi = - \int d\vec{l} \cdot \vec{E}$$
$$\vec{E} = -\nabla \Phi = - \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z} \right)$$

The gradient only calc B!

Conductors

Perfect Conductor Ideal doesn't exist in nature

- Electric Field inside is zero

• If $E \neq 0$, charges rearrange themselves so $B=0$ instantaneously for perfect conductors

• After some time, real conductors behave this way (time is a factor for real conductors (low σ))

• In the surface (outer), the Electric Field is perpendicular to the surface of the conductor

→ All net charges reside in the surface

$$Q_{\text{enc}} = \epsilon_0 \oint d\vec{a} \cdot \vec{E} = 0$$

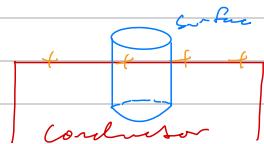
+ Enclosed charge is 0, and all charge is in surface

$$Q = Q_+ - Q_- = 0 \quad (\text{0 charge, but particles exist inside})$$

Charge Density

$$\oint d\vec{a} \cdot \vec{E} = \int_{\text{p}} E da + \int_{\text{O}} 0 + \int_{\text{S.E.O.}} (0) = EA_{\text{cup}}$$

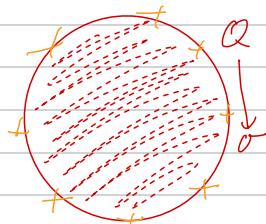
inside conductor \vec{E} is zero \vec{E} is outside $\vec{E} = 0$ $\sigma = q_0$



$$\rightarrow E A_{\text{cup}} = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\sigma \cdot A_{\text{cup}}}{\epsilon_0}$$

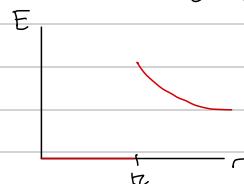
so

$$E = \frac{\sigma}{\epsilon_0}, \quad \sigma = E \epsilon_0$$

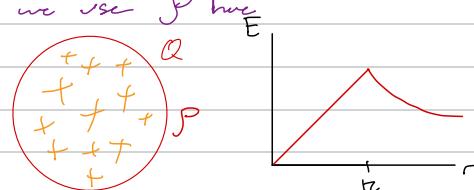


If the sphere is perfect conductor
all charges are on the surface!

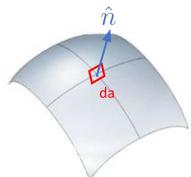
Use σ !



Differentiation: solution worked/ probably as they had charges evenly distributed throughout volume; we use ρ here



Force on the Conductor



$$\text{Force: } \vec{F} = q \vec{E} \quad (\text{known})$$

$$\rightarrow d\vec{F} = \frac{1}{2} \sigma da \vec{E} \hat{n}$$

$$\vec{F} = \frac{1}{2} \int \sigma \vec{E} \hat{n} da \quad (\text{use } \sigma = E \epsilon_0 \text{ or } E = \frac{\sigma}{\epsilon_0})$$

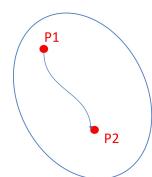
$$= \frac{1}{2} \epsilon_0 \int \vec{E}^2 \hat{n} da = \frac{1}{2 \epsilon_0} \int \sigma^2 \hat{n} da \quad \text{always perpendicular}$$

$$\vec{E} = E \hat{n} \quad \text{outside conductor}$$

$$\frac{1}{2} E \hat{n} \quad \text{on surface? average}$$

$$E = 0 \quad \text{inside conductor}$$

Potential inside conductor



$$\Delta \phi = \int d\vec{l} \cdot \vec{E} = - \int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$$

→ but $E=0$ inside conductor $\therefore \Delta \phi = 0$

so, conductors are equipotential surfaces

Watch out!

$$\Delta \phi = \phi_B - \phi_A = 0$$

$$\rightarrow \phi_A = \phi_B \neq 0$$

↑ just their difference

Had to use fusion first last year

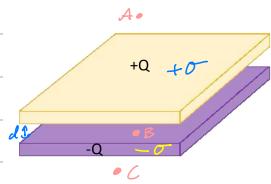
Capacitance

- Charge Q is proportional to potential difference (often called voltage) between two electrodes by a quantity called Capacitance

$$Q = CV \quad \text{where} \quad Q \equiv \text{Charge (coulombs, C)} \\ C \equiv \text{Capacitance (Farads, F)} \\ V \equiv \text{Voltage (Volts, V)}$$

Conversion:
Use Volts instead of (equiv to)
Potential difference ($\Delta\Phi$)

Parallel plates



Assume $A \gg d^2$

$$C = \frac{Q}{V} = \sigma A$$

$$V = \Delta\Phi = \int d\vec{l} \cdot \vec{E}$$

above (↑) Q : What's \vec{E} at all 3 points?

$$\rightarrow A \quad \vec{E}_{\text{in}} = \frac{\sigma}{\epsilon_0} \vec{i}_z \quad \vec{E} = 0$$

$$\text{below } \rightarrow B \quad \vec{E} = \frac{\sigma}{\epsilon_0} \vec{i}_z$$

$$\rightarrow C \quad \vec{E}_{\text{out}} = \frac{\sigma}{\epsilon_0} \vec{i}_z \quad \vec{E} = 0$$

Derive from Gauss Law (Previous lecture)

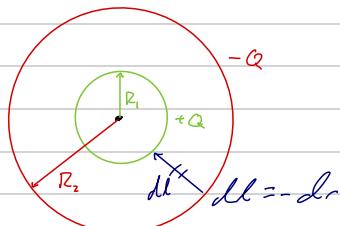
Use $\vec{E} \propto \vec{Q}$ and 3 points to get Cap of parallel plate capacitors?

$$C = \frac{Q}{V} = \frac{Q}{\Delta\Phi}$$

$$\Delta\Phi = - \int_{P_1 \rightarrow P_2} d\vec{l} \cdot \vec{E} = - \int_{P_1 \rightarrow P_2} dl E_{\text{in}} \epsilon_0 = E \int_0^d dl = d \left(\frac{\sigma}{\epsilon_0} \right), \quad Q = \sigma A$$

$$\therefore C = \frac{\sigma A}{\frac{d\sigma}{\epsilon_0}} = \frac{\epsilon_0 A}{d}$$

Example



$$C = \frac{Q}{V}, \quad V = \Delta\Phi = - \int_{-\infty}^{\infty} \vec{E} \cdot d\vec{l}$$

① Determine \vec{E} in 3 points (only really need two b/w θ and ② just along the far periphery)

$$\vec{E}_{\text{inside}} = 0 \quad Q_{\text{enc}} = 0$$

$$\vec{E}_{\text{outer}}: \quad Q_{\text{enc}} = Q$$

$$\frac{Q_{\text{enc}}}{\epsilon_0} = E_a \rightarrow E = \frac{Q}{4\pi r^2 \epsilon_0}$$

$$\vec{E}_{\text{outside}} = 0 \quad Q_{\text{enc}} = Q - Q = 0$$

- Spherical & hollow shells

- Conductive materials

- Determine Capacitance (C) ② Determine $\Delta\Phi$

$$\Delta\Phi = - \int_{-\infty}^{\infty} E dl \cos 180 = \int_{R_2}^{R_1} \frac{Q}{4\pi r^2 \epsilon_0} dl = - \int_{R_2}^{R_1} \frac{Q}{4\pi r^2 \epsilon_0} dr = \frac{Q}{4\pi \epsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$\Rightarrow C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi \epsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)} = 4\pi \epsilon_0 \left(\frac{1}{R_1} - \frac{1}{R_2} \right)^{-1}$$

Exam 2 Review Lecture

- Dielectrics will not be on the homework

3 Key Topics

- Potential Energy
 - Electric Potential
 - Capacitance
- } holding all from Exam 1

Capacitors

$$C = \frac{Q}{V}$$

Parallel plate

Left to right

$$\int \vec{E} \cdot d\vec{r} = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{A(\sigma + \sigma_u)}{\epsilon_0} = \frac{A(\sigma + 2\sigma_u)}{\epsilon_0}$$

$$2 \int_{\text{caps}} \vec{E} \cdot d\vec{a} + \int_{\text{side}} \hat{n} \perp \vec{E} = 0 = \frac{A(\sigma + 2\sigma_u)}{\epsilon_0}$$

$$\sigma_u + 2\sigma_u = 0$$

$\sigma_u = \sigma_l$ by symmetry

$$\sigma_u = \sigma_l = -\frac{\sigma}{2}$$

Question 2
on LWS
I got it right

Outside edges of slabs should go $\pm \frac{\sigma}{2}$ to
keep conductors neutral

Potential by a Disk

HWS Q1

$$\Phi_{\text{ring}} = \frac{1}{4\pi\epsilon_0} \frac{\lambda(2\pi r)}{\sqrt{r^2 + z^2}}$$

$$2\pi r \lambda = 2\pi r d\sigma$$

$$\Rightarrow \int_0^R \frac{1}{4\pi\epsilon_0} \frac{2\pi r d\sigma}{\sqrt{r^2 + z^2}}$$

Don't expect something like this on the exam

Potential & Energy

$$\Delta\phi = - \int \vec{E} \cdot d\vec{r}$$

$$\Delta U = -Q \int \vec{E} \cdot d\vec{l}$$

Example Regarding Capacitor



$$\int \vec{E} \cdot d\vec{r} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$E(2\pi r L) = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$E = \frac{Q}{2\pi r L \epsilon_0}$$

$$V = \Delta\phi = \int_{R-d}^R \frac{Q}{2\pi\epsilon_0 r L} dr \propto \log\left(\frac{R}{R-d}\right)$$

Midterm Exam 3



Dielectrics

- Dielectrics = Insulators

examples: Glass

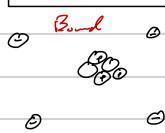
Plastic

Water

Diamond

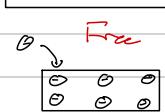
• charges stay put

Internal Charge
or. Bound Charge



vs.

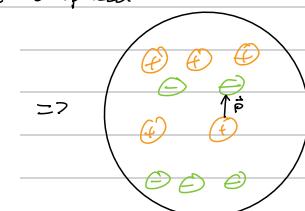
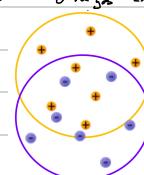
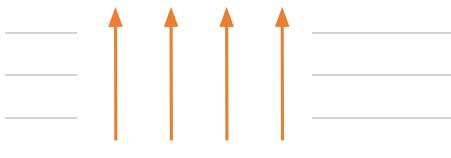
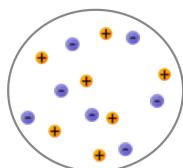
External Charge
or. Free Charge



or Poles

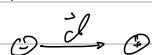
Response of Internal Charges to Fields

Atom of Dielectric \rightarrow Apply electric Field \rightarrow Bound charges are displaced



Displaced dipoles
in the atom

Dipole

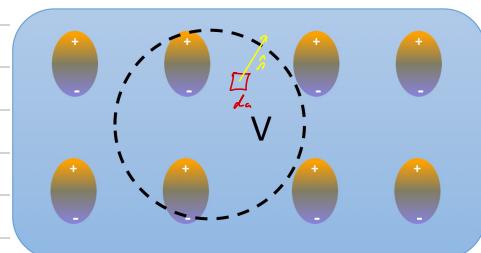


$$\vec{P} = q \vec{d}$$

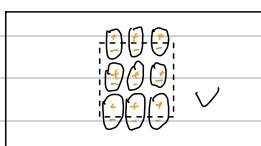
Polarized medium

\rightarrow Polarization vector \vec{P}

$$\circ \text{ Dipole Moment Density: } \vec{P} = \frac{\sum \vec{p}}{\text{volume}}$$



How much bound charge makes?



$$\int \vec{P}$$

$$\bullet \vec{P} \equiv \text{constant}, Q_B(V) = 0$$

• Answer depends on \vec{P} at the surface

• If \vec{P} is parallel to the surface no net charge

$$Q_B \propto \vec{P}_{\text{surface}}$$

$$Q_{\text{bound}} = Q_B(V) - \oint_{\text{closed}} \vec{P} = - \oint |\vec{P}| |d\vec{l}| \cos 0$$

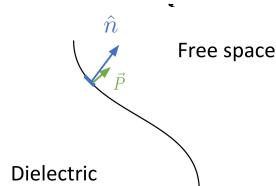
• Only for Dielectrics

$$\bullet \text{ Not Gauss' Law, thus: } Q_{\text{bound}}(Q_{\text{free}}) = \epsilon_0 \oint_{\text{closed}} \vec{E}$$

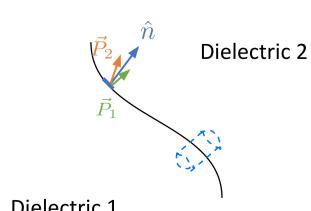
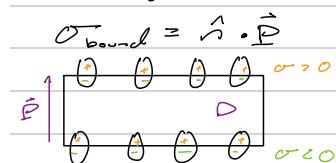
Free
Charges

$$Q_{\text{free}} = \oint_{\text{closed}} \cdot (\epsilon_0 \vec{E} + \vec{P})$$

Surface Charge Density

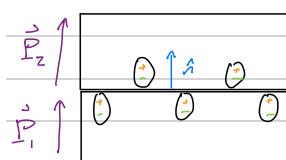


One dielectric



Two dielectrics

$$\sigma_{\text{bound}} = \hat{n} \cdot \vec{P}_1 - \hat{n} \cdot \vec{P}_2$$



Gauss Law to get σ_{total}

$$\vec{E}_1 = \frac{\sigma_{\text{total}}}{2\epsilon_0} \hat{n}$$

$$\sigma_{\text{free}} = \sigma_{\text{total}} - \sigma_{\text{bound}}$$

$$\vec{P}_1 = \epsilon_0 \sigma_{\text{free}}$$

$$\sigma_{\text{free}} = \hat{n} \cdot (\epsilon_0 \vec{E}_2 + \vec{P}_2) - \hat{n} \cdot (\epsilon_0 \vec{E}_1 + \vec{P}_1)$$

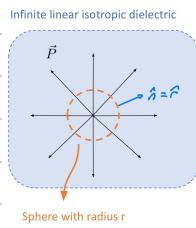
Linear Isotropic Dielectrics

$$\vec{P} \propto \vec{E}^x \quad \text{proportionality depends on the material's temperature and pressure} \Rightarrow \vec{P} = \chi \epsilon_0 \vec{E}$$

Simpler case
Gauss's Law $\vec{P} \cdot \vec{E}^x$ (to a given point)

Bliss Susceptibility

Example



Assume $\vec{P} = \frac{A}{r^2} \hat{r}$ constant

- What is the bound charge inside a sphere with radius r ?

$$Q_{\text{bound}} = \oint_C \vec{P} \cdot d\vec{l} = \int_{\text{surface}} \vec{P} \cdot \hat{n} dA$$

$$= 4\pi r^2 P \cos 0$$

$$Q_{\text{bound}} = -4\pi r^2 \left(\frac{A}{r^2} \right) = -4\pi A$$

$$Q_{\text{bound}} = -4\pi A$$

Note: Q_{bound} is independent of r ; it has a constant value

↳ Why? All the charges are at the center

- What is the electric field?

$$\vec{E} = \frac{\vec{P}}{\chi \epsilon_0} = \frac{\frac{A}{r^2} \hat{r}}{\chi \epsilon_0} = \frac{1}{\chi \epsilon_0} \frac{A}{r^2} \hat{r}$$

$$\vec{E} = \frac{1}{\chi \epsilon_0} \frac{A}{r^2} \hat{r}$$

- What is the total charge inside a sphere with radius r ?

$$Q_{\text{tot}} = \oint \vec{E} d\vec{l} \Rightarrow Q_{\text{tot}} = \epsilon_0 E \oint dA = \epsilon_0 E 4\pi r^2 = \epsilon_0 \frac{1}{\chi \epsilon_0} \frac{A}{r^2} 4\pi r^2 = \frac{4\pi A}{\chi}$$

$$Q_{\text{tot}} = \frac{4\pi A}{\chi}$$

- What is the external (free) charge inside a sphere with radius r ?

$$Q_{\text{free}} = Q_{\text{tot}} - Q_{\text{bound}} = 4\pi A \frac{1+\chi}{\chi}$$

$$Q_{\text{free}} = 4\pi A \frac{1+\chi}{\chi}$$

You must see $1+\chi$ as ϵ_r (relative permittivity) or the dielectric constant K (Kappa K)

$$\vec{E} = \frac{1}{4\pi \epsilon_0 (K) \frac{A}{r^2}} \hat{r}$$

Basically coulomb's law, but with $(1+\chi)$ factor.

We will define $K = 1+\chi$ (Kappa)

Parallel Plate Capacitor

- Capacitance is affected: $C = K C_{\text{air}} = (K \epsilon_0) \frac{A}{d}$

Material	Dielectric constant, K
Air	1.085-1.21 at 20°C*
Water	80 at 20°C (79.50-80.24)*
Ice	3 at -5°C*
Basalt	12
Granite	7.9
Sandstone	9-11
Dry loam	3.5
Dry sand	2.5 (2.5-3.5 at 20°C)*
Ethanol liquid	15.21 at 20°C*
Ethanol Mixing Liquid	39.82 at 20°C*
Acetone	21.20 at 20°C*

*measured dielectric constant in this study

What is the new capacitance with two dielectrics?

In Parallel

d

$A/2 \quad K_1$

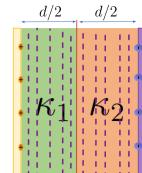
$A/2 \quad K_2$

$E \frac{A}{2} = \frac{Q_i}{\epsilon_0 \epsilon_i} \Rightarrow Q_i = (K_i \epsilon_0) E \frac{A}{2}$

$\therefore C_i = \frac{K_i \epsilon_0}{d} \frac{A}{d}$

$\Rightarrow C = \frac{Q + Q_2}{V} = \frac{K_1 + K_2}{2} \epsilon_0 \frac{A}{d} = C_1 + C_2$

In Series

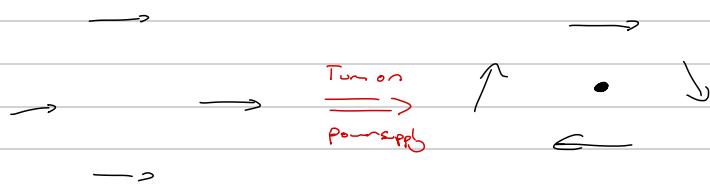


$$\Delta \phi = \int \frac{\sigma}{K \epsilon_0} dl = \frac{\sigma}{K \epsilon_0} \int dl = \frac{\sigma}{K \epsilon_0} y$$

$$C = \frac{Q}{V_1 + V_2} = \left(\frac{V_1 + V_2}{Q} \right)^{-1} = \left(\frac{1}{C_1} + \frac{1}{C_2} \right)^{-1}$$

Magnetic Fields

Demo w/ Compasses



Electric Current produces
a magnetic field
is circular as seen by
the direction of the compass
arrows

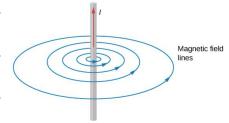
Magnetic Fields

- Moving electric charges produce magnetic fields
↳ there are no "magnetic charges"
- Magnetic fields exert forces on moving electric charges

Note: Moving charges
still produce electric fields!

Law of Faraday's Law of Induction

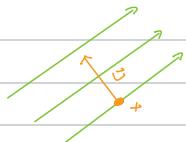
- Electric currents (or just currents) produce magnetic fields \vec{B}
- \vec{B} exert forces on moving charges



Definition of Magnetostatics: Assume that there is no accumulation of charges, but the charges can move

Lorentz Force

- The force must depend on the charge, velocity, and field



$$\vec{F} = \vec{F}(q, \vec{v}, \vec{B})$$

- The force is the vector product of \vec{v} and \vec{B}

$$\vec{F} = q \vec{v} \times \vec{B}$$

Recall - Cross Product

- The cross product of two vectors is a vector

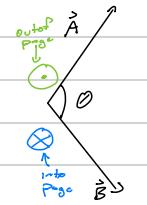
$$\vec{A} \times \vec{B} = \text{vector}$$

- Its magnitude:

$$|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin(\theta)$$

- Direction:

$\vec{A} \times \vec{B}$ is perpendicular to \vec{A} and \vec{B}



Currents

- Which charges move?

→ Metals \Rightarrow Negative charges move

→ Semiconductors \Rightarrow Positive charges move

We will assume that positive charges move!

$I \equiv$ Current flowing through surface in conductors

$$I \equiv \frac{\Delta Q}{\Delta t} \xrightarrow{\substack{\text{charge} \\ \text{crosses} \\ \text{during a certain} \\ \text{time}}} \frac{\text{charge}}{\text{time}}$$

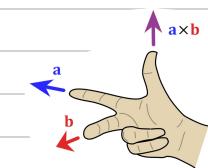
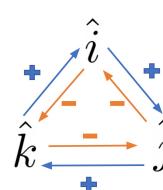
$$\text{Units: } \frac{\text{Coulombs}}{\text{Second}} = \text{Amperes}$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$\vec{A} = 3\hat{i} + 4\hat{j}$$

$$\vec{B} = 6\hat{i} + 7\hat{k}$$

$$\vec{A} \times \vec{B} = 28\hat{i} - 21\hat{j} - 24\hat{k}$$



Currents: Line Current

- Assume an ∞ wire with infinitesimal thickness

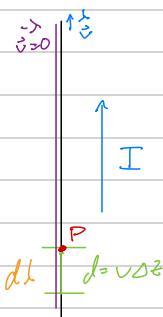
\hookrightarrow Current I means

$\rightarrow I$ coulombs going up

$\rightarrow -I$ coulombs going down

All charges ($+$) in a small Δt will cross P in a time Δt

$$\text{The current: } I = \frac{\Delta Q}{\Delta t} = \frac{I v \Delta t}{\Delta t} = I v$$



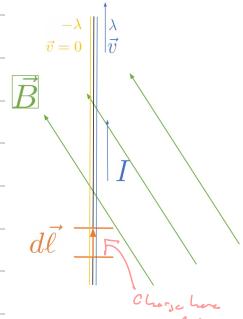
Coulomb vs. Lorentz

$$\rightarrow F_E = q \vec{E}$$

$$\rightarrow F_B = q \vec{v} \times \vec{B}$$

\hookrightarrow Do cross product, \hookrightarrow
deal with q 's sign

Lorentz Force on a Current



- Force on a piece of wire $d\vec{l}$

$$d\vec{F} = (I d\vec{l}) \vec{v} \times \vec{B} \quad \text{Note: } d\vec{l} = d\vec{l} \vec{v}$$

$\hookrightarrow q = I \cdot d\vec{l}$

- In terms of current

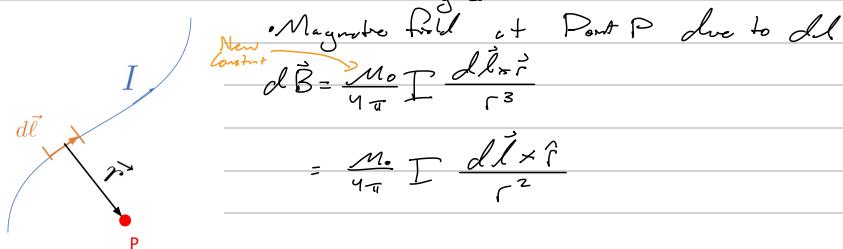
$$d\vec{F} = I \int d\vec{l} \times \vec{B}$$

Note: $I = s \cdot l$ (see previous!)

- Integrate to solve for \vec{F}

$$\vec{F} = I \int d\vec{l} \times \vec{B} = I \int |d\vec{l}| |\vec{B}| \sin \theta \quad \text{for } \theta \text{ b/w } d\vec{l} \text{ and } \vec{B}$$

Biot-Savart: Field Created by a current



Example: Infinite wire

$$\vec{r} = y\hat{j} + z\hat{k} \rightarrow r = \sqrt{y^2 + z^2}$$

$$d\vec{l} = d\vec{l} \hat{v}$$

$$d\vec{l} \times \vec{r} = -y d\vec{l} \hat{i}$$

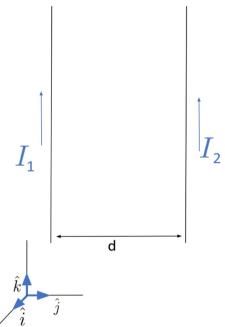
$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{-y d\vec{l} \hat{i}}{(z^2 + y^2)^{3/2}}$$

$$\vec{B} = \int_{-\infty}^{\infty} \frac{\mu_0}{4\pi} I \frac{-y d\vec{l} \hat{i}}{(z^2 + y^2)^{3/2}}$$

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{I}{y} (-\hat{i})$$

Force b/w 2 infinite wires

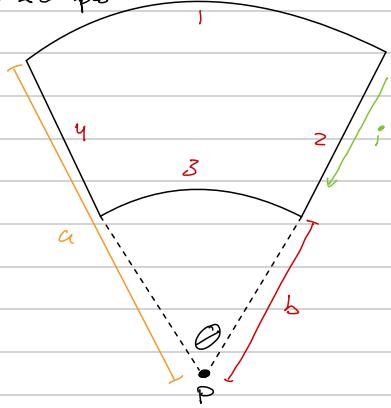
- What is the force that I_1 exerts on I_2 ?



$$d\vec{F}_2 = \frac{\mu_0 I_1 I_2}{2\pi d} d\vec{l} (-\hat{j}) = -d\vec{F}_1$$

- If the intensities are parallel the force is attractive,
- If the intensities are anti-parallel the force is repulsive

Example



Find Magnetic Field at P

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^3}$$

(i) Determine direction of $d\vec{s}$ for col segment

a) Angles b/w $d\vec{s}$ and \vec{r} for col segment

Segment 1

Angle b/w $d\vec{s}$ and $d\vec{l}_1$?

$$|d\vec{s} \times \vec{r}| = |d\vec{s}| |\vec{r}| \sin \theta$$

$$\theta = 90^\circ$$

by symmetry, forces to same for seg 2

Segment 2

Angle b/w $d\vec{s}$ and \vec{r} in Segment 2

b) Direction of Segments and PBR

$$\textcircled{1} \quad \vec{B} = B \cdot \vec{\otimes}$$

$$\textcircled{2} \quad \vec{B} = B \cdot \vec{\odot}$$

$$\begin{cases} \vec{r} \\ d\vec{s} \\ 90^\circ \text{ & same seg.} \end{cases} \quad \left. \begin{array}{l} d\vec{B}_1 = d\vec{B}_2 = 0 \\ d\vec{B}_3 = d\vec{B}_4 = 0 \end{array} \right\}$$

b) Integration - Bad, don't follow

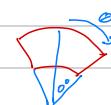
→ How is $d\vec{s}$ related to $d\theta$

$$d\vec{s} = r d\theta$$



→ What is the position vector \vec{r}

$$\vec{r} = -r \sin \theta \hat{i} - r \cos \theta \hat{j}$$



$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{s} \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi} I r \frac{d\theta \vec{r}}{r^3}$$

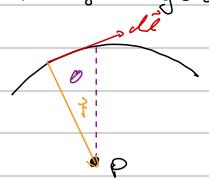
$$\int_{-\frac{\theta_2}{2}}^{\frac{\theta_2}{2}} d\vec{B}_1 = \frac{\mu_0}{4\pi} I \frac{d\theta}{a} (-\sin \theta \hat{i} - \cos \theta \hat{j}) = \frac{\mu_0}{4\pi} \cdot \frac{I}{a} \cdot \int_{-\frac{\theta_2}{2}}^{\frac{\theta_2}{2}} -\sin \theta \hat{i} - \cos \theta \hat{j} d\theta$$

This can be integrated

$$\int_{-\frac{\theta_2}{2}}^{\frac{\theta_2}{2}} d\vec{B}_2 = \frac{\mu_0}{4\pi} I \frac{d\theta}{b} (-\sin \theta \hat{i} - \cos \theta \hat{j}) = \frac{\mu_0}{4\pi} \cdot \frac{I}{b} \cdot \int_{-\frac{\theta_2}{2}}^{\frac{\theta_2}{2}} -\sin \theta \hat{i} - \cos \theta \hat{j} d\theta$$

using $\int \sin \theta d\theta = -\cos \theta$

b) Integrating good



$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^3}$$

$$|d\vec{l} \times \vec{r}| = |d\vec{l}| |\vec{r}| \sin 90^\circ = d\vec{l} r = (rd\theta) r$$

Direction $\textcircled{\times}$ $\rightarrow -\hat{k}$

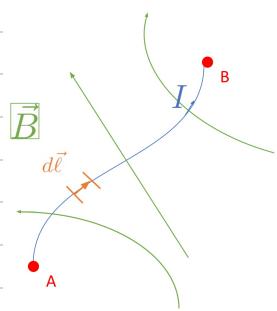
$$\begin{cases} \vec{r} = \hat{\theta} + \hat{j} \\ d\vec{l} = \hat{r} + \hat{k} \end{cases}$$

$$\Rightarrow d\vec{B}_1 = \frac{\mu_0}{4\pi} I \frac{r^2 d\theta}{r^3} \textcircled{\times} = \frac{\mu_0}{4\pi} I \frac{1}{r} \theta (-\hat{k}) \quad \vec{B}_1 = -\frac{\mu_0}{4\pi} I \frac{1}{a} \theta \hat{k}$$

$$\Rightarrow d\vec{B}_2 = \frac{\mu_0}{4\pi} I \frac{r^2 d\theta}{r^3} \textcircled{\odot} = \frac{\mu_0}{4\pi} I \frac{1}{r} \theta \hat{k} \quad \vec{B}_2 = \frac{\mu_0}{4\pi} I \frac{1}{b} \theta \hat{k}$$

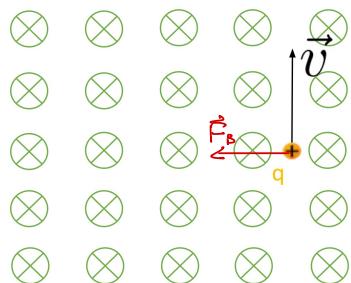
$$\boxed{\vec{B}_{\text{tot}} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0}{4\pi} I \theta \left(\frac{1}{b} - \frac{1}{a} \right) \hat{k}}$$

Acceleration and Ampere's Law



Q: Charge with a const. velocity at pos. A, \vec{v}_A , that creates a magnetic field \vec{B} , will suffer a force (Lorentz). At point B, it will have an velocity \vec{v}_B . Will: (i) $|\vec{v}_B| > |\vec{v}_A|$
 (ii) $|\vec{v}_B| = |\vec{v}_A|$ const. $\vec{F}_B = q\vec{v} \times \vec{B} \Rightarrow \vec{F}_B \perp \vec{v}$
 (iii) $|\vec{v}_B| < |\vec{v}_A|$
 90° apply \rightarrow centripital so $|\vec{v}|$ doesn't change
 $\Delta K = \int_A^B \vec{F} \cdot d\vec{\ell} = \int_A^B$ follows $q_0 = 0 \rightarrow |\vec{v}_B| = |\vec{v}_A|$ (work cons. thm)

Trajectory on a Uniform Field



- Directions of Lorentz force $C \vec{F}_B = q\vec{v} \times \vec{B}$
- B must point left
- How will q move? Upward left
- Trajectory? Fix a const. θ
- NZL: $qvB\sin 90^\circ = m\frac{v^2}{r} \Rightarrow r = \frac{mv}{qB}$
- $T = \frac{2\pi r}{v} = \frac{2\pi m}{qB}$, $\omega = \frac{2\pi}{T} = \frac{qB}{m}$
- ↑ Time to complete orbit
- ↑ Frequency

Parallels

Charges produce Electric fields

Version I	Version II
• Coulomb's Law	• Gauss Law
• Superposition Principle	• Circulation Law

Moving charges produce magnetic fields

Version I	Version II
• Biot-Savart's Law	• Gauss Law of Magnetism
• Superposition Principle	• Ampere's Law

Gauss Law of Magnetism

$$\oint_B \vec{B} \cdot d\vec{a} = \oint A \cdot \vec{B} da = 0 \quad \text{where } S \text{ points outside the surface by convention}$$

Ampere's Law

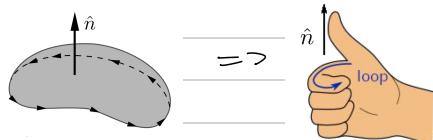
- Magnetic field across a closed loop (circulation) is proportional to the current through any surface whose boundary is the loop
- In Magnetic states: $\oint d\vec{l} \cdot \vec{B} = \mu_0 I_{enc}$ I enc is any current that goes through the loop's boundary



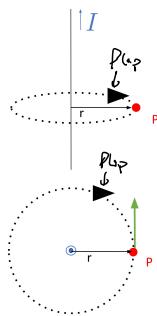
4 geometries
will apply to (mag
beam eng)

Stokes' Corollary

Direction of loop \Rightarrow direction of normal vector is fixed



Example: Magnetic Field by Infinite Line



- Choose an amperian loop passing by P
- \rightarrow Similar to Gaussian surface, result should not depend on it
- Since field depends on r (Field lines are circles)

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

$$\int \vec{B} \cdot d\vec{s} = \mu_0 I \Rightarrow B(2\pi r) = \frac{\mu_0 I}{2\pi r}$$

Summary of Ampère's Law (nicely written)

To find the magnetic field due to a current, Ampère's law is sometimes more efficient than the Biot-Savart law.

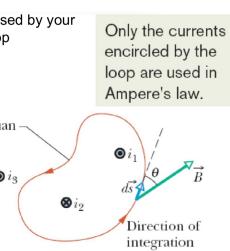
Ampère's law:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc}$$

current enclosed by your Ampérian loop

infinitesimally small length vector along an imaginary loop surrounding one or more currents

Ampère's law is always true, but it is only useful in a limited number of cases with a large amount of symmetry.



Right Hand Rule for direction of Field

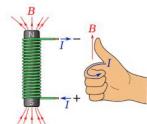
Right hand rule for the direction of the magnetic field due to a current:

- Point your thumb in the direction of the current.
- Your fingers curl in the direction of the magnetic field



OR

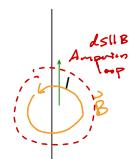
- Curl your fingers in the direction of the current.
- Your thumb points in the direction of the magnetic field.



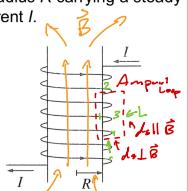
Example

Use Ampère's Law to determine the magnetic field in each of the following 3 cases:

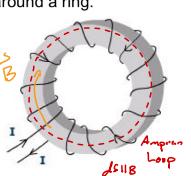
Case 1: A long straight wire carries a steady current I in the direction shown.



Case 2: A very long solenoid has n closely wound turns per unit length on a cylinder of radius R carrying a steady current I .



Case 3: A toroid consists of a long wire, carrying a steady current I , that is tightly wound N times around a ring.



$$N = \text{number of loops}$$

$$n = \frac{N}{L} = \text{density of loops}$$

Caution: It's okay for $\oint \vec{B} \cdot d\vec{s} = 0$ for some points on the Ampérian loop, but if $\oint \vec{B} \cdot d\vec{s} = 0$ for every point on the loop, then your loop will not allow you to determine the magnitude of the magnetic field.

(i) Direction of magnetic field

(ii) Draw an Ampérian loop s.t. $\oint \vec{B} \cdot d\vec{s} = B \oint d\vec{s}$ or 0 at every point on the loop ($0^{\circ}, 90^{\circ}$)

(iii) Choose direction to integrate around the loop and calculate $\oint \vec{B} \cdot d\vec{s}$ (circulation)

Case 1: $\oint \vec{B} \cdot d\vec{s}$ = circulation

$$\theta = 0^\circ \quad B \int r d\theta$$

$$B(2\pi r)$$

Case 2: $\oint \vec{B} \cdot d\vec{s} = (\int \vec{B} \cdot d\vec{s})_1 + (\int \vec{B} \cdot d\vec{s})_2 + \dots + (\int \vec{B} \cdot d\vec{s})_4$

$$\text{side 1: } B(L)$$

$$\text{side 2: } 0$$

$$\text{side 3: } 0$$

$$\text{side 4: } 0$$

$$\theta = 90^\circ$$

$$\text{Assume field is tangential}$$

$$\text{this is expected}$$

$$\text{inside solenoid is ok!}$$

$$\text{thus approximation is fine}$$

Same result as case 1

(iv) Determine i_{enc}

$$\text{Case 1: } i_{enc} = I$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

$$\text{Case 2: } i_{enc} = I \cdot n \cdot L$$

$$\Rightarrow B = \frac{\mu_0 I \cdot n \cdot L}{L}$$

$$\text{Case 3: } i_{enc} = N \cdot I$$

$$B = \frac{\mu_0 N \cdot I}{2\pi r}$$

(v) Summary of Example!



$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

$$B = \frac{\mu_0 I}{2\pi r}$$



$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

$$B = \mu_0 n I$$



$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

$$B = \frac{\mu_0 I}{2\pi r}$$

Circulation Law
The circulation of E equals zero

$$\oint d\vec{l} \cdot \vec{E} = 0$$

but not in electrodynamics with resistive term

Gauss Law
The flux of E through a closed surface equals charge enclosed

$$\oint \hat{n} \cdot \vec{E} da = \frac{Q}{\epsilon_0}$$

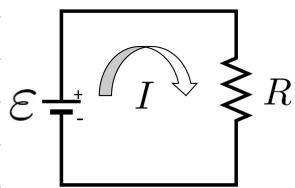
Ampere's Law
The circulation of B through a closed loop equals current enclosed

$$\oint d\vec{l} \cdot \vec{B} = \mu_0 I$$

Gauss Law of B
The flux of B through a closed surface equals zero

$$\oint \vec{n} \cdot \vec{B} da = 0$$

Faraday's Law



DC Battery

- Two terminal device
- Maintains a Voltage (potential drop) across terminals (ε)
- Positive terminal (+) higher voltage
- Current I flows in both directions

Electromotive Force

\Rightarrow Ohm's Law

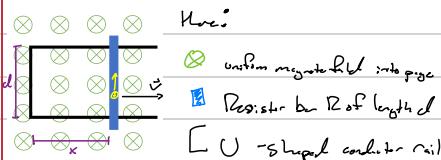
$$V = IR$$

Current flows from higher potential to lower potential

Possible

- Not always good conductor
- Potential drop across (V)
- Current can flow in both directions

Faraday's Flux rule



Here:

$\otimes \otimes \otimes \otimes$ uniform magnetic field into page

\square Resistor bar R of length d

\square U-shaped conductor rail

Bar is moved to the right at velocity v

\oplus in bar will feel upward force (RH Rule)

$$|\vec{F}| = |q\vec{v} \times \vec{B}| \approx qvB \Rightarrow \vec{F} = qvB\hat{j}$$

\rightarrow Force per unit of charge

$$\vec{f} = \frac{\vec{F}}{q} = vB\hat{j} \quad \text{Looks like electric field } (\vec{B} = \frac{\vec{E}}{q}), \text{ not true!}$$

\rightarrow Effect to voltage: $V = \vec{F}d = vBd = \varepsilon$

$$\Rightarrow I = \frac{\varepsilon}{R} = \frac{dvB}{R}$$

\rightarrow $B \propto v$ of \vec{B} , given $\varepsilon = vBd$

$$\Phi_B = \int \vec{B} \cdot d\vec{a} = \int B da \cos 0 = Bda \approx B(\pi d)$$

\rightarrow Derivative over time of Flux

$$\frac{d}{dt}(\Phi_B) = \frac{d}{dt}(Bda) \Rightarrow \frac{d\Phi_B}{dt} = B \frac{da}{dt} = BdV \Rightarrow \frac{d\Phi_B}{dt} \propto \varepsilon$$

\rightarrow If we are more careful w/ signs, then generally: $\varepsilon = -\frac{d\Phi_B}{dt}$ ↪ should always hold

We are missing something

• If we move magnet and leave wire fixed, $\vec{J} = 0$ by the flux laws, why?

↳ Change reference frames, but we really need to refine Faraday's Law

Faraday's Law of Induction

$$\varepsilon = -\frac{d\Phi_B}{dt} \Rightarrow \oint dl \cdot \vec{B} = -\frac{d}{dt} \int \vec{n} \cdot \vec{B} da$$

Faraday's Law can be used to be the circulation law

Gauss Law

Ampere's Law

Gauss Law of B

The circulation of E equals the variation in time of the flux of B

The flux of E through a closed surface equals charge enclosed

The circulation of B through a closed loop equals current enclosed

The flux of B through a closed surface equals zero

$$\oint dl \cdot \vec{E} = -\frac{d}{dt} \int \vec{n} \cdot \vec{B} da$$

↑ not 0 in electrostatics

$$\oint \hat{n} \cdot \vec{E} da = \frac{Q}{\varepsilon_0}$$

$$\oint d\ell \cdot \vec{B} = \mu_0 I$$

$$\oint \vec{n} \cdot \vec{B} da = 0$$