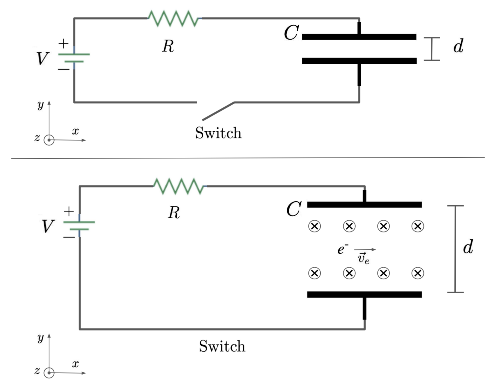
Final Review PHYS 122 Spring 2025

05/03/2025

Problem 1: RC Circuits and Magnetism



1. Consider the following circuit consisting of a battery of voltage V, switch, resistor of resistance R, and capacitor of capacitance C with distance between the plates d. The capacitor begins uncharged with the switch open (top figure). At time $t = 0$, the switch is closed.

   (a) Write the expression for the voltage drop across the capacitor at a given time t as a function of V, R, and C. Use that expression to find the voltage across the capacitor at $t = 0$ and the limit as $t \to \infty$ (at very long time).

   (b) Suppose an electron (charge $-e$) is placed at rest in between the plates of the capacitor. Assume the electric field is uniform inside the plates. Show that the force on the electron at a given time t is:

   $$\vec{F}(t) = \frac{Ve}{d}\left(1 - e^{-t/RC}\right)\hat{\jmath}.$$

   (c) Now suppose the capacitor is fully charged (we are in the limit of $t \to \infty$) and there is now a uniform magnetic field between the plates of the capacitor $\vec{B} = -B\hat{k}$ (bottom figure). The electron, initially at rest, is given a kick in the $\hat{\imath}$ direction so it has velocity $\vec{v}_e = v_e\hat{\imath}$ and travels in a straight path. What is the strength of the magnetic field in terms of v_e, V, and d?

Solution

(a) The voltage across the capacitor is given by:

$$V_C = V\left(1 - e^{-t/\tau}\right)$$

Plugging in $\tau = RC$:

$$V_C = V\left(1 - e^{-t/RC}\right)$$

At $t = 0$, there is no drop across the capacitor as it acts as a wire. The entire voltage drop is across the resistor. At long times, when the capacitor is charged, the voltage across the capacitor is V, the same as the battery used to charge it.

(b) The force on the electron due to the electric field in between the plates is given by:

$$\vec{F} = q\vec{E}$$

The electric field between the plates is:

$$|\vec{E}| = \frac{V}{d}$$

Substituting:

$$\vec{F} = \frac{eV}{d}\left(1 - e^{-t/RC}\right)\hat{\jmath}$$

(c) For it to travel in a straight line we need the electric force to be balanced by the magnetic force. So we want:

$$\vec{F} = q\vec{v} \times \vec{B} + q\vec{E} = 0$$

Rearranged, we want:

$$q\vec{v} \times \vec{B} = -q\vec{E}$$

We already have this electric force above:

$$(-e)\vec{v}_e \times \vec{B} = -\frac{e}{d}V\left(1 - e^{-t/RC}\right)\hat{\jmath}$$

We just have to take the long time limit:

$$(-e)\vec{v}_e \times \vec{B} = -\frac{e}{d}V\hat{\jmath}$$

Since the velocity is orthogonal to the magnetic field, the magnitude of the magnetic field is:

$$\vec{v}_e \times \vec{B} = \frac{V}{d}\hat{\jmath}$$

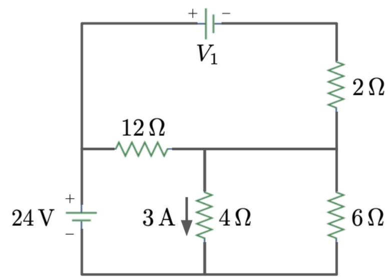
$$|\vec{v}_e||\vec{B}| = \frac{V}{d}$$

$$|\vec{B}| = \frac{V}{d|\vec{v}_e|}$$

Then we can get the direction from knowing that the cross product for velocity and the magnetic field must point in the positive y-direction, so the magnetic field must point into the page ($-\hat{k}$ direction):

$$\vec{B} = -\frac{V}{d|\vec{v}_e|}\hat{k}$$

Problem 2: Graphite and Circuit Analysis



1. Consider a cylindrical piece of graphite with radius $r = 1\,\text{mm}$ and resistivity $\rho = 8 \times 10^{-6}\,\Omega \cdot \text{m}$.

   (a) What length of graphite would be needed to have a total resistance $R = 1\,\Omega$, assuming the current flows along the length? Round your answer to 2 decimals in SI notation in meters.

   (b) Several of these pieces of graphite (with integer multiples of the length from part (a)) are used to build the circuit in Figure 3. The current through the $4\,\Omega$ resistor is measured with an ammeter as $I = 3\,\text{A}$. What is the current through each of the other 3 resistors?

   (c) What is the voltage of the battery at the top?

   (d) What is the power delivered to the circuit by the $24\,\text{V}$ battery?

Solution

(a) **Length of Graphite Needed:**

Using the formula for resistance:

$$R = \rho \frac{L}{A},$$

where $A = \pi r^2$ is the cross-sectional area of the graphite. Solving for L:

$$L = \frac{RA}{\rho}.$$

Substituting the given values:

$$L = \frac{(1\,\Omega) \cdot \pi (0.001\,\text{m})^2}{8 \times 10^{-6}\,\Omega \cdot \text{m}}.$$

Simplifying:

$$L = 0.3927\,\text{m}.$$

Writing in scientific notation:

$$L = 3.93 \times 10^{-1}\,\text{m}.$$

(b) **Currents Through the Resistors:**

Consider the loop in the lower left corner. Using Kirchhoff's Loop Law:

$$24\,\text{V} = (12\,\Omega)(I_{12}) + (4\,\Omega)(3\,\text{A}),$$

where I_{12} is the current through the $12\,\Omega$ resistor. Solving for I_{12}:

$$I_{12} = 1\,\text{A}.$$

For the lower right corner loop:

$$(4\,\Omega)(3\,\text{A}) = (6\,\Omega)(I_6).$$

Solving for I_6:

$$I_6 = 2\,\text{A}.$$

Using Kirchhoff's Node Law at the middle node:

$$1\,\text{A} + I_x = 3\,\text{A},$$

where $I_x = 2\,\text{A}$. For the right node:

$$I_2 = 2\,\text{A} + 2\,\text{A}.$$

Thus:

$$I_2 = 4\,\text{A}.$$

The gathered solutions are:

$$I_{12} = 1\,\text{A}, \quad I_6 = 2\,\text{A}, \quad I_2 = 4\,\text{A}, \quad I_4 = 3\,\text{A}.$$

(c) **Voltage of the Top Battery:**

Using the top loop:

$$V_1 = (12\,\Omega)(1\,\mathrm{A}) - (4\,\Omega)(2\,\mathrm{A}).$$

Simplifying:

$$V_1 = 4\,\mathrm{V}.$$

Alternatively, using the outer loop:

$$V_1 = 24\,\mathrm{V} - (6\,\Omega)(2\,\mathrm{A}) - (2\,\Omega)(4\,\mathrm{A}).$$

Simplifying:

$$V_1 = 4\,\mathrm{V}.$$

(d) **Power Delivered by the 24 V Battery:**

The total current through the 24 V battery is:

$$I_{24\,\mathrm{V}} = 2\,\mathrm{A} + 3\,\mathrm{A}.$$

Thus:

$$I_{24\,\mathrm{V}} = 5\,\mathrm{A}.$$

The power is:

$$P = IV.$$

Substituting:

$$P = (5\,\mathrm{A})(24\,\mathrm{V}).$$

Simplifying:

$$P = 120\,\mathrm{W}.$$

Problem 3: Conservation of Energy, Electric and Magnetic Fields

Problem Solving Technique for Ampère's Law

The calculation of the magnetic field of a given current distribution by means of Ampère's Law is possible only if the current distribution and the field are highly symmetric. If so, the calculation involves the following steps, which are analogous to those we used in the calculation of the electric field by means of Gauss' Law:

1. Determine the direction of the magnetic field from considerations of symmetry. For example, for current flow along a straight line, the field lines will be loops around the line; for current flow around a cylinder, the field lines will resemble those of an infinite solenoid.

2. Select a closed path such that for some portion of the path the magnetic field is parallel to the path and of constant magnitude B, and for the remaining portion (if any) the magnetic field is perpendicular to the path or is zero. Keep in mind that the path for Ampère's Law is a mathematical path; it does not have to coincide with a wire or with the surface of some physical body. Instead, the path is chosen to coincide with points where the magnetic field B is to be evaluated.

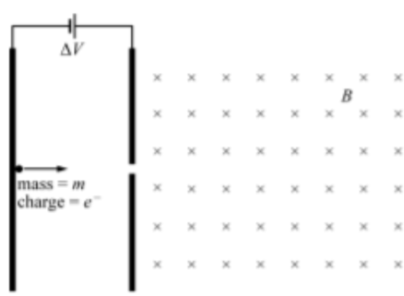
3. Express the integral $\oint \vec{B} \cdot d\vec{s}$ in terms of the magnitude B of the magnetic field, $\oint \vec{B} \cdot d\vec{s} = B\,l$, where l is the length of the portion of the path where the magnetic field is tangent to the path.

4. Use Ampère's Law to relate Bl to the current enclosed by the path: When evaluating I, make sure to include only the portion of the total current that is enclosed by the path.

5. Solve the resulting equation for the magnitude of the magnetic field.

Problem

Two plates are set up with a potential difference V between them. A negatively charged particle of mass m and charge e is placed at the left-hand plate, which has a negative charge, and is allowed to accelerate across the space between the plates and pass through a small opening. After passing through the small opening, the particle enters a region in which there is a uniform magnetic field of magnitude B, directed into the page.



(a) Which way is the magnetic field pointing?

(b) Will the particle feel a force in this magnetic field? If yes, which direction will the force point?

(c) What kind of path will the particle follow?

(d) Derive an expression for the speed of the particle as it passes through the small opening in terms of V, m, and e.

(e) Calculate the height H at which the particle will collide with the screen.

Solution

(a) The magnetic field points in the negative z-direction, which is **into the page**.

(b) Yes, the particle will experience a Lorentz force. Using the right-hand rule for the cross product:

$$\vec{F} \propto -\hat{\imath} \times -\hat{k} = -\hat{\jmath}$$

So the force is in the **negative y-direction**.

(c) The particle will follow a **circular path** because the Lorentz force is always perpendicular to the velocity of the particle, providing the centripetal force.

(d) We relate kinetic energy to the electrostatic potential energy:

$$\frac{1}{2}mv^2 = eV \Rightarrow v = \sqrt{\frac{2eV}{m}}$$

(e) The centripetal force is provided by the magnetic (Lorentz) force:

$$\frac{mv^2}{R} = evB \Rightarrow R = \frac{mv}{eB}$$

Since the particle enters at the center and completes a half-circle before hitting the screen:

$$H = 2R = 2 \cdot \frac{mv}{eB}$$

Useful Formula For Dielectric Potential Problem

Displacement field

$$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

Polarization vector

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

Relative permittivity (for linear material)

$$\varepsilon_r = 1 + \chi_e$$

Total surface charge

$$\hat{\mathbf{n}} \cdot (\mathbf{E_2} - \mathbf{E_1}) = \frac{\sigma_{tot}}{\varepsilon_0}$$

Total bound surface charge

$$\hat{\mathbf{n}} \cdot (\mathbf{P_1} - \mathbf{P_2}) = \sigma_{bound}$$

Total free surface charge

$$\hat{\mathbf{n}} \cdot (\mathbf{D_2} - \mathbf{D_1}) = \sigma_{free}$$

Total free charge

$$\oiint_A \mathbf{D} \cdot d\mathbf{A} = Q_{free}$$

Total bound charge

$$-\oiint_A \mathbf{P} \cdot d\mathbf{A} = Q_{bound}$$

Total charge

$$\varepsilon_0 \oiint_A \mathbf{E} \cdot d\mathbf{A} = Q_{tot}$$

Potential

$$V_a - V_b = -\int_b^a \mathbf{E} \cdot d\mathbf{l}$$

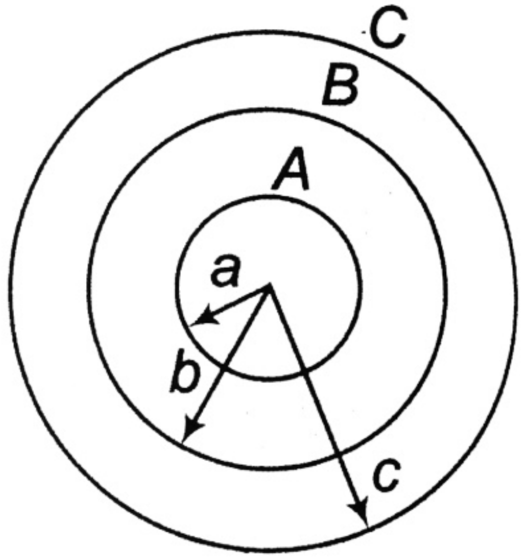
Problem 1: Potential

We have a capacitor with three concentric shells A, B, C of radius a, b, and c. Let a = 2R, b = 3R, c = 4R. The shell A and C are connected by wires. The charge on the shell B is -Q. The total charge on A and C is ~~0.~~ Q.

(a) Find the charge on A and C?

(b) Find the electric field at each region. Plot the electric field as a function of r.

(c) Find the capacitance. Here, you have two capacitor (inner, middle and middle, outer). Let Q for each region be $Q = |Q_1 - Q_2|$.



(a) We have

$$Q_A + Q_C = Q$$

To have equipotential, we have

$$\int_a^b \frac{Q_A}{4\pi\varepsilon_0 r^2} dr + \int_b^c \frac{Q_A - Q}{4\pi\varepsilon_0 r^2} dr = 0$$

$$\frac{Q_A}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{Q_A - Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{c}\right) = 0$$

$$Q_A\left(\frac{1}{2R} - \frac{1}{3R}\right) + (Q_A - Q)\left(\frac{1}{3R} - \frac{1}{4R}\right) = 0$$

$$\frac{Q_A}{6R} + (Q_A - Q)\frac{1}{12R} = 0$$

$$2Q_1 + Q_1 - Q = 0$$

$$\boxed{Q_A = \frac{Q}{3}} \rightarrow \boxed{Q_C = \frac{2Q}{3}}$$

(b) $r < a \rightarrow \boxed{E = 0}$

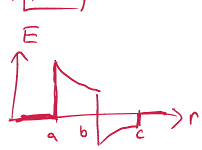
$a < r < b \rightarrow E 4\pi r^2 = \frac{Q/3}{\varepsilon_0}$

$$\boxed{\vec{E} = \frac{Q}{12\pi\varepsilon_0 r^2}\hat{r}}$$

$b < r < c \rightarrow E 4\pi r^2 = \frac{Q/3 - Q}{\varepsilon_0}$

$$\boxed{\vec{E} = \frac{-2Q}{12\pi\varepsilon_0 r^2}\hat{r}}$$

$r > c \rightarrow \boxed{E = 0}$



c) Capacitance in series

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$C_1 = \frac{|Q_A - Q|}{\left|\frac{Q_A}{6R}\right|} = \frac{2Q/3}{\frac{Q/3}{6R}} = 12R$$

$$C_2 = \frac{|-Q + Q_C|}{\left|\frac{Q_A}{6R}\right|} = \frac{Q/3}{\frac{Q/3}{6R}} = 6R$$

$$\frac{1}{C_{eq}} = \frac{1}{12R} + \frac{1}{6R} = \frac{1}{4R}$$

$$\rightarrow \boxed{C_{eq} = 4R}$$

Problem 2: Dielectric Slab

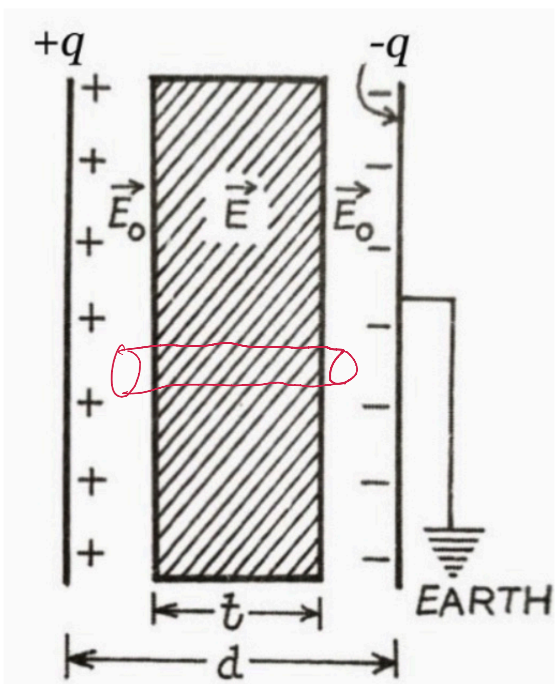
Consider the following configuration where we have a parallel plate capacitor with surface charge density σ and $-\sigma$ with spacing d. We set the voltage of the negative plate as 0. We insert a neutral block at the center of the capacitor with thickness t.

(a) Suppose the block is a conductor. What is the electric field inside the block? What is the charge at the left edge and right edge of the block?

(b) Now, instead consider the block as a dielectric. The dielectric has a relative permittivity of ε_r. What is the electric field inside the block (consider both external and induced electric field)? What is the charge at the left edge and right edge of the block?

(c) What is the total free charge in the dielectric slab? What is the bound charge in the slab?

(d) What is the capacitance of the capacitor without the block? Now, consider the block as dielectric inserted. What is the capacitance now?



(a) $E = 0$ inside conductor

$\boxed{\sigma_{left} = -\sigma}$

$\boxed{\sigma_{right} = +\sigma}$

(b) $\vec{E}_{ext} = \frac{\sigma}{\varepsilon_0}\hat{x}$

$\vec{P} = \varepsilon_0 \chi \vec{E}_{ext}$

$\vec{P} = \varepsilon_0 (\varepsilon_r - 1)\frac{\sigma}{\varepsilon_0}\hat{x}$

$\boxed{\sigma_{right} = \vec{P}\cdot\hat{n} = \sigma(\varepsilon_r - 1)}$

$\boxed{\sigma_{left} = -\sigma(\varepsilon_r - 1)}$

$\vec{E}_{int} = \frac{\sigma(\varepsilon_r - 1)}{\varepsilon_0}(-\hat{x})$

$\vec{E}_{tot} = \vec{E}_{int} + \vec{E}_{ext}$

$\vec{E}_{tot} = \frac{\sigma}{\varepsilon_0}(1 - (\varepsilon_r - 1))$

$\boxed{\vec{E}_{tot} = \frac{\sigma}{\varepsilon_0}(2 - \varepsilon_r)}$

c) Neutral slab $\rightarrow$ $\boxed{Q_{free} = 0}$

$Q_{bound} = -\oint \vec{P}\cdot\hat{n}\,da$

$= -(\sigma(\varepsilon_r - 1)A - \sigma(\varepsilon_r - 1)A)$

$\boxed{Q_{bound} = 0}$

d) $\boxed{C_{empty} = \frac{\varepsilon_0 A}{d}}$

$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$

$C_1 = \frac{\varepsilon_0 A}{\frac{(d-t)}{2}} = C_3 \rightarrow \frac{1}{C_1} + \frac{1}{C_3} = \frac{2}{C_1} = \frac{d-t}{\varepsilon_0 A}$

$C_2 = \frac{\varepsilon_0 \varepsilon_r A}{t}$

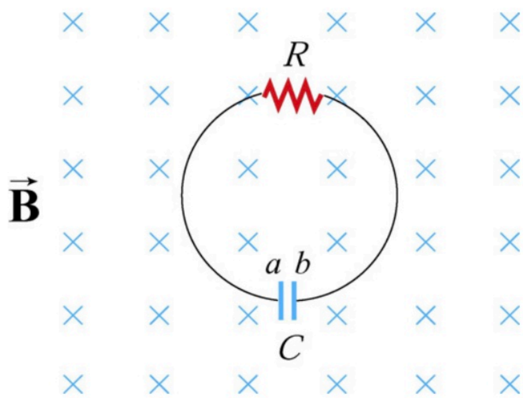
$\frac{1}{C_{eq}} = \frac{d-t}{\varepsilon_0 A} + \frac{t}{\varepsilon_0 \varepsilon_r A} \rightarrow \boxed{C_{eq} = \frac{\varepsilon_0 \varepsilon_r A}{(d-t)\varepsilon_r + t}}$

Problem 3 (Bonus): Faraday + RC Circuit

Consider a circular loop of wire of radius r lying in the xy plane, as shown below. The loop contains a resistor R and a capacitor C, and is placed in a uniform magnetic field which points into the page and decreases at a rate dB/dt = -α , with $\alpha > 0$.

(a) Find the current in the circuit.

(b) What is the maximum charge on the capacitor.



$$\mathcal{E}mf = -\frac{d\Phi_B}{dt} = -\pi r^2 \frac{dB}{dt} = \pi r^2 \alpha$$

→ RC circuit with battery = emf

$$\rightarrow I(t) = \frac{\mathcal{E}mf}{R} e^{-t/RC}$$

$$\boxed{I(t) = \frac{\pi r^2 \alpha}{R} e^{-t/RC}}$$

$$q(t) = \int I(t)\,dt = \frac{\pi r^2 \alpha}{R}(-RC)e^{-t/RC} + C_0$$

$$q(t) = -\pi r^2 \alpha C e^{-t/RC} + C_0$$

$$q(0) = -\pi r^2 \alpha C + C_0 = 0$$

$$\rightarrow C_0 = \pi r^2 \alpha C$$

$$\rightarrow \boxed{q(t) = \pi r^2 \alpha C\,[1 - e^{-t/RC}]}$$

$$\boxed{q_{max}(t \rightarrow \infty) = \pi r^2 \alpha C}$$

Dielectric (IF YOU SEE MISTAKES, PLEASE EMAIL US RIGHT AWAY)

We have conductor so E field is zero. Using Gauss's law, we have

$$\sigma_l = -\sigma, \sigma_r = \sigma$$

Electric field inside the dielectric

$$E_{ins} = \frac{E_{ext}}{\varepsilon_r} = \frac{\sigma}{\varepsilon_0 \varepsilon_r}$$

Polarization vector inside the slab

$$\mathbf{P} = \varepsilon_0(\varepsilon_r - 1)\mathbf{E}_{ins} = \frac{\varepsilon_r - 1}{\varepsilon_r}\sigma$$

Induced electric field

$$\mathbf{E}_{ind} = \mathbf{E}_{ext} - \mathbf{E}_{ins}$$

$$\mathbf{E}_{ind} = \frac{\sigma}{\varepsilon_0} - \frac{\sigma}{\varepsilon_0 \varepsilon_r} = \frac{\sigma}{\varepsilon_0}(1 - \frac{1}{\varepsilon_r})$$

Bound charge to the left and right

$$\sigma_{bl} = -\frac{\varepsilon_r - 1}{\varepsilon_r}\sigma$$

$$\sigma_{br} = \frac{\varepsilon_r - 1}{\varepsilon_r}\sigma$$

We have

$$Q_{free} = Q_{bound} = 0$$

Free because the slab is netraul. Bound is 0 because the polarization is uniform.
The capacitance without dielectric

$$C = \frac{\varepsilon_0 A}{d}$$

With dielectric, we have

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

where

$$C_1 = C_3 = \frac{\varepsilon_0 A}{(d-t)/2}$$

$$C_2 = \frac{\varepsilon_0 \varepsilon_r A}{t}$$

$$C_{eq} = \frac{\varepsilon_0 \varepsilon_r A}{(d-t)\varepsilon_r + t}$$