



Midterm Exam 1



Electric Charges & Fields, Coulomb's Law, Superposition

Electric Charge

- charge is a fundamental prop of matter
 - two types: \textcircled{P} & \textcircled{G} matter(atoms) are normally neutral
- Triboelectricity:** Friction transfers charge

Metal Rods in Demo?

- Plastic & crystal are insulators
 - charge "stays where put"
- Metals are conductors
 - charge "flow freely"

Charge is conserved

- Rubbing transfers charges, but does not create nor destroy it

Charge is quantized

- charge is always a multiple of $e = 1.6 \times 10^{-19}$ Coulombs (C) \Rightarrow electron = $-e$, proton = $+e$

Why the rods are moving!

- Attraction or Repulsion \Rightarrow There is a force between the charges

\hookrightarrow Force is known as coulomb's force

$$F = \frac{1}{4\pi\epsilon_0} \cdot \frac{qQ}{r^2} \quad \text{where } \frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}$$

\square When two charges q and Q are same signed $\Rightarrow F$ is positive = Repulsive

\square When they have opposite signs $\Rightarrow F$ is negative = Attractive

Demo - Rods

$$\textcircled{P} \textcircled{P} \textcircled{G} \leftarrow \rightarrow$$

$$\textcircled{G} \textcircled{G} \textcircled{P} \leftarrow \rightarrow$$

$$\textcircled{P} \textcircled{G} \textcircled{P} \rightarrow \leftarrow$$

Note: the metal is still neutral

Demoval / Metal

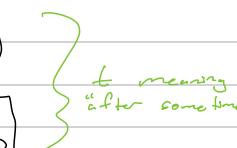
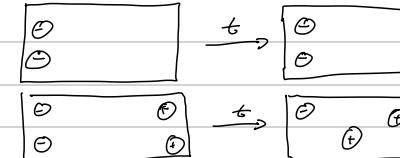
$$\textcircled{G} \quad \textcircled{G}$$

$$\textcircled{P} \quad \textcircled{G}$$

$$\textcircled{G} \quad \textcircled{G}$$

$$\textcircled{P} \quad \textcircled{G}$$

$$\textcircled{G} \quad \textcircled{G}$$



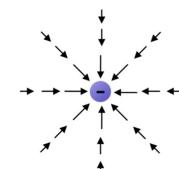
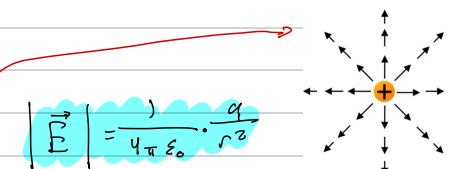
$$\textcircled{P} \quad \textcircled{G}$$

$$\textcircled{G} \quad \textcircled{G}$$

Electric Field and Coulomb's Law

Laws of Electrostatics

- Electrostatics = charges @ rest
- charges produce electric fields
- electric fields push charges



$$q \quad \vec{r} \quad r = |\vec{r}|$$

$$|\vec{E}| = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \quad \text{Note: } r = |\vec{r}|$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

\vec{r} is a unit vector that points away from charge q

or alternatively so $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ and $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \vec{r}$

(so $\vec{v} = \vec{r}/t$)

Equivalent!

Superposition Principle

In Words:

Electric Field produced = Sum of the electric field produced by many point charges by each point charge independently

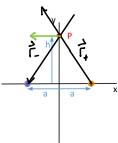
$$\text{In Maths: } \vec{E}(q_1, q_2) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_1^2} \hat{r}_1 + \frac{1}{4\pi\epsilon_0} \frac{q_2}{r_2^2} \hat{r}_2$$

Second Law of Electrostatics

$$\text{Fields push/pull charges: } \vec{F} = Q \vec{E} \Rightarrow \vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{r}$$

EN? EC? EN?

Example: Point Charges



Calculate the Electric field at point P.

Pos vector \vec{r} from Q to $P: -a\hat{i} + b\hat{j}$

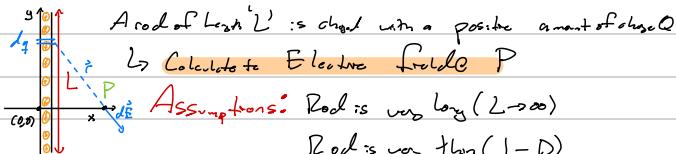
Pos vector \vec{r} from Q to $P: a\hat{i} + b\hat{j}$ (not mass, position not dependent charge!)

Calculate the Electric Field at P:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} \vec{r} \rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0} \left(\frac{1}{(\sqrt{a^2+b^2})^3} [q(-a\hat{i}+b\hat{j}) + -q(a\hat{i}+b\hat{j})] \right) = \frac{2qa}{4\pi\epsilon_0 \sqrt{a^2+b^2}^3} \cdot (-\hat{i})$$

$$r_- = a\hat{i} + b\hat{j} \rightarrow r_- = \sqrt{a^2+b^2} \quad \rightarrow \vec{E}_- = \vec{E}_f + \vec{E}_-$$

Continuous Distribution of Charge



Calculate the Electric field at P

Assumptions: Rod is very long ($L \rightarrow \infty$)

Rod is very thin ($L \ll D$)

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} \vec{r} \rightarrow d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \vec{r}$$

$$\vec{r} = x\hat{i} - y\hat{j} \quad r = \sqrt{x^2+y^2} \quad \rightarrow d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \cdot (x\hat{i} - y\hat{j})$$

Definition: Charge per unit length or linear charge density

$$\lambda = \frac{\text{Charge}}{\text{Length}} = \frac{Q}{L}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \sum_{i=1}^N \vec{E}_i = \int_0^L d\vec{E}$$

$$\text{so, } \vec{E} = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\lambda}{\sqrt{x^2+y^2}} (x\hat{i} - y\hat{j}) dy \quad \text{Solve } x \text{ and } y \text{ separately}$$

Example

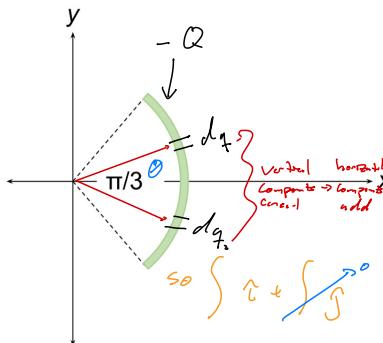
A plastic rod with a uniformly distributed charge $-Q$ is bent into a circular arc of radius r that subtends an angle of $\pi/3$ radians. We place coordinate axes such that the axis of symmetry of the rod lies along the x-axis and the origin is the center of curvature for the rod. What is the electric field (magnitude and direction) at the origin?

Hint: Imagine that the arc is made up of many infinitesimally small point charges dq . Each dq creates a differential electric field of magnitude $dE = k dq/r^2$.

Sum up all the dEs from all the dqs to get the magnitude of the overall electric field.

Integration is a way to add many infinitesimally small elements.

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \vec{r}$$



To integrate, let's use polar! $(x, y) \rightarrow (r, \theta)$

$$\int dE_x = \frac{1}{4\pi\epsilon_0} \int_{-\pi/6}^{\pi/6} \frac{r \cos \theta}{r^2} d\theta$$

$$= \frac{1}{4\pi\epsilon_0 r} \left[\sin \theta \right]_{-\pi/6}^{\pi/6}$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{3Q}{\pi r^2} \rightarrow$$

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{3Q}{\pi r^2} \hat{i}}$$

$$|d\vec{E}| = \left| \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \vec{r} \right|$$

$$\begin{cases} \cos \theta = \frac{a}{r} \\ \sin \theta = \frac{b}{r} \end{cases} \rightarrow \vec{r} = -r \cos \theta \hat{i} - r \sin \theta \hat{j}$$

$$x\text{-component of } |dE|: dE_x = \frac{1}{4\pi\epsilon_0} \frac{da}{r^2} \cos \theta$$

We need to integrate over all θ , b , dE_x , ds terms of dq , not $d\theta$!

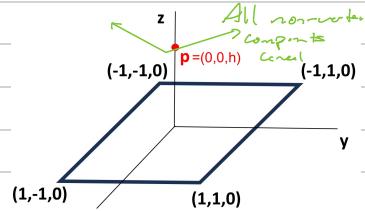
$$\Rightarrow dq = \int ds \quad \text{where } ds \text{ is segment of arc}$$

$$\Rightarrow \frac{2\pi r}{\theta} \text{ so } ds = r d\theta \rightarrow dq = r \lambda d\theta$$

$$\text{Just think length = angle * radius}$$

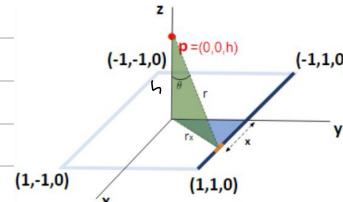
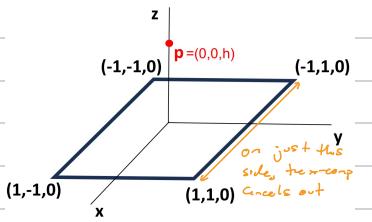
Example

A plastic rod with a uniformly distributed charge Q is shaped forming a square whose side is 1m. We place coordinate axes such that the square is on the x-y plane and the origin is the center of the square. What is the electric field (magnitude and direction) at the point $p=(0,0,h)$?



\Rightarrow The problem solution will be $\vec{E} = E \hat{k}$

So we just need to calculate the vertical (z) component of one side and add the rest!



Hint: What is the square formed by? Is there any symmetry?

Imagine that the square is made up of many infinitesimally small point charges dq . Each dq creates a differential electric field of magnitude $dE = k dq/r^2$.

Sum up all the dEs from all the dqs to get the magnitude of the overall electric field.

Integration is a way to add many infinitely small elements.

$$\begin{aligned} \vec{r}_h &= h \hat{k} \\ \vec{r}_{xy} &= -x \hat{i} - \hat{j} \Rightarrow r_{xy}^2 = x^2 + 1^2 \\ \text{so, } r^2 &= r_{xy}^2 + h^2 \\ \Rightarrow r &= \sqrt{x^2 + h^2} \end{aligned}$$

$$\begin{aligned} d\vec{E} &= \frac{1}{4\pi\epsilon_0} \frac{dq}{(x^2 + h^2)^{3/2}} (-x \hat{i} - \hat{j} + h \hat{k}) \\ \Rightarrow \vec{E} &= \int \frac{1}{4\pi\epsilon_0} \frac{dq}{(x^2 + h^2)^{3/2}} (-x \hat{i} - \hat{j} + h \hat{k}) \end{aligned}$$

only components \hat{k}

In one side \hat{j} component part, but disappears when using Left cancellation

$$\vec{E}_{\text{left}} = \int d\vec{E}_{\text{left}} = \int \frac{1}{4\pi\epsilon_0} \frac{dq}{(x^2 + h^2)^{3/2}} (-x \hat{i} - \hat{j} + h \hat{k}) = -c \hat{j} + d \hat{k}$$

$$\vec{E}_{\text{left}} = \int d\vec{E}_{\text{left}} = \int \frac{1}{4\pi\epsilon_0} \frac{dq}{(x^2 + h^2)^{3/2}} (-x \hat{i} - \hat{j} + h \hat{k}) = -c \hat{j} + d \hat{k}$$

$$\text{so, } \vec{E} = \vec{E}_{\text{left}} \cdot 4 \hat{k}$$

$$\vec{E}_{\text{Top}} = \int_{-1}^1 \frac{1}{4\pi\epsilon_0} \frac{dq}{(\sqrt{x^2 + h^2})^2} (-\hat{i} - \hat{j} + h \hat{k})$$

$$\vec{E}_{\text{Bottom}} = \int_{-1}^1 \frac{1}{4\pi\epsilon_0} \frac{dq}{(\sqrt{x^2 + h^2})^2} (+\hat{i} - \hat{j} + h \hat{k})$$

Gauss Law

Electric Field Lines

- (i) Lines are tangent to \vec{E} at every point in space
- (ii) The density of lines is proportional to $|\vec{E}|$
- (iii) Lines never cross

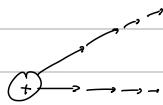
(iv) No closed loops

(v) Lines start or stop on charges

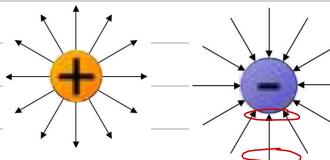
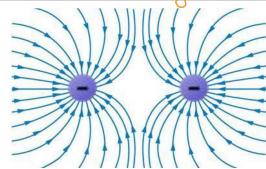
↳ Start should be a positive charge, and
should be on negative charge

↳ This rule inherently enforces part (iv)

Single Charge



Two charges



For weaker charge \rightarrow fewer lines

Recall Vector Multiplication

$$\left\{ \begin{array}{l} \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \\ = |\vec{A}| |\vec{B}| \cos \theta \end{array} \right. \text{ for } \theta \text{ the angle between } \vec{A} \text{ and } \vec{B}$$

Electric Flux

• Plot surface and uniform \vec{E}

• Defined: $\Phi_E = A \hat{n} \cdot \vec{E} = A |\hat{n}| |\vec{E}| \cos \theta = A |\vec{E}| \cos \theta$

o \hat{n} is a unit vector perpendicular to the surface (normal vector)

o θ is the angle between \hat{n} and \vec{E}

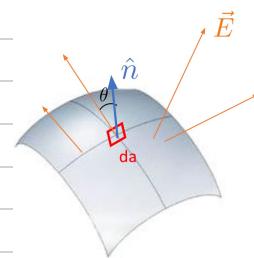
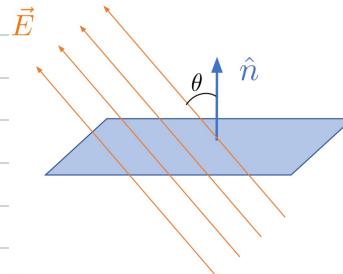
o A is the area of the surface

• Non-Plot surface and/or Non-uniform \vec{E}

$\Phi_E = \int \hat{n} \cdot \vec{E} da = \int |\hat{n}| |\vec{E}| \cos \theta da = \int |\vec{E}| \cos \theta da$

o \hat{n} is a unit vector perpendicular to the surface da

o θ is the angle between \hat{n} and \vec{E}



Gauss Law

• Rest of Maxwell's Equations

• In words: The electric flux

$$\text{through a closed surface} = \frac{\text{Amount of charge inside the surface}}{\epsilon_0}$$

• Equation:

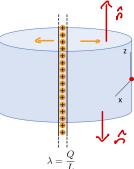
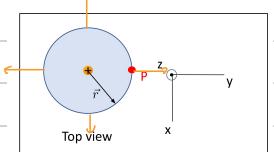
$$\Phi_E = \int_{\text{closed}} \hat{n} \cdot \vec{E} da = \frac{Q}{\epsilon_0}$$

• \hat{n} points outside the surface by convention

$$\cdot \hat{n} \cdot da = \frac{d\vec{a}}{\epsilon_0} \Rightarrow \text{locally, where } d\vec{a} = da \cdot \hat{n}$$

• Example / Review: Infinite Line

Assumption: rod is very long ($L \rightarrow \infty$)
rod is very thin ($l \rightarrow 0$)



Length l of rod make cylinder

$$\Rightarrow Q_{ENC} = \lambda l$$

$$\phi_E = \frac{Q_{ENC}}{\epsilon_0} = \frac{\lambda l}{\epsilon_0}$$

Ext records
Question

What is the direction of the total electric field @ P?

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{n}, \quad E = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$2 \int_{\text{Cups}} |\vec{E}| \cos \theta da + \int_{\text{Side}} \vec{n} \cdot \vec{E} da$$

$\theta = 90^\circ, \text{ so } 0$

$$= 0 + \int_{\text{Side}} |\vec{E}| \cos \theta da$$

$$= \int E da = E dA = EA$$

$$\Phi_E = E \cdot 2\pi r l$$

$$E \cdot 2\pi r l = \frac{\lambda l}{\epsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\epsilon_0 r}$$

CAPS radial

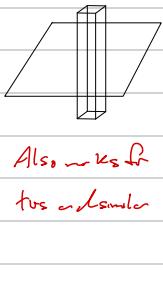
$$\vec{E} = E \hat{r} \quad \left. \begin{array}{l} \vec{n} = \pm \hat{r} \\ \theta = 90^\circ \end{array} \right\}$$

SIDE radial

$$\vec{E} = E \hat{r} \quad \left. \begin{array}{l} \vec{n} = \hat{r} \\ \theta = 0 \end{array} \right\}$$

The magnitude is constant
for all parts on SIDE

Infinite Plane



Direction of Electric Field?

$$A = \pi r^2$$

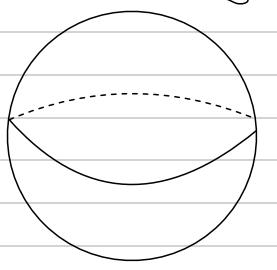
$$\sigma = \frac{Q}{l^2} \quad \int \vec{n} \cdot \vec{E} da = \frac{Q_{ENC}}{\epsilon_0}$$

SIDE $\vec{n} = \hat{z} \quad \theta = 90^\circ$
CAPS $\vec{n} = \pm \hat{x} \quad \theta = 0$

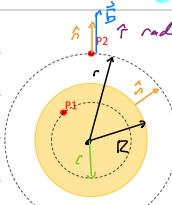
$$\int_{\text{Cups}} \vec{n} \cdot \vec{E} da = 2 \int E da = 2EA = 2E\pi r^2$$

$$2B\pi r^2 = \frac{\sigma \pi r^2}{\epsilon_0} \Rightarrow E = \frac{\sigma}{2\epsilon_0}$$

Spherical Symmetry



Charge Density $\rho = \frac{Q}{V} \quad \left\{ \frac{4}{3}\pi r^3 \right\}$



Calculate the Electric field at points P1 and P2 using Gauss Law of the sphere of Radius R and charged with a density ρ

$$\vec{E} = E \hat{r}$$

$\vec{n} = \hat{r}$

Flux through Gaussian Surface $\rightarrow \int E \cos(\theta) da = E A = E(4\pi r^2)$

$$\frac{Q_{enc}}{\epsilon_0} = \rho \frac{4}{3}\pi R^3$$

$$R > R: \frac{\rho \frac{4}{3}\pi R^3}{\epsilon_0} = E(4\pi r^2) \Rightarrow E = \frac{\rho R^3}{3\epsilon_0 r^2}$$

$R < R: \vec{n} = \hat{r}$

$$\frac{Q_{enc}}{\epsilon_0} = \frac{\rho V}{\epsilon_0} = \frac{\rho \frac{4}{3}\pi r^3}{\epsilon_0}$$

$$\int E da = EA = E(4\pi r^2)$$

Gaussian surface is r

$$\left. \begin{array}{l} \int \vec{n} \cdot \vec{E} da = \frac{Q_{enc}}{\epsilon_0} \\ \frac{\rho \frac{4}{3}\pi r^3}{\epsilon_0} = E(4\pi r^2) \end{array} \right\} E = \frac{\rho r}{3\epsilon_0}$$

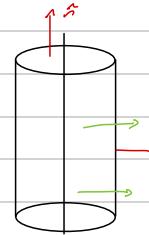
Any charge outside the gaussian surface does not affect the Flux

Exam 1 Review Lecture

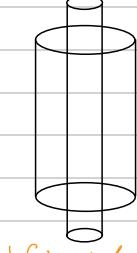
Gauss Law

$$\oint \vec{E} \cdot \hat{n} da = \oint \vec{E} \cdot d\vec{a}$$

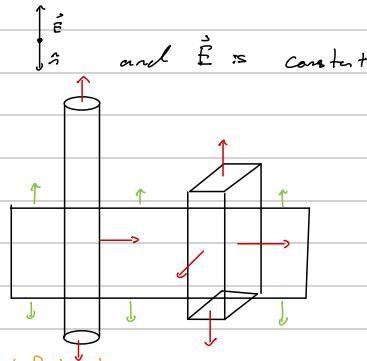
and E is constant



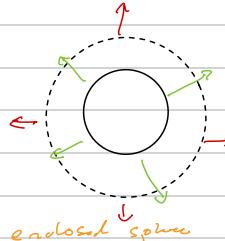
Infinite Loop



Lafayette



↓
↓
↓



a - short [↓] space

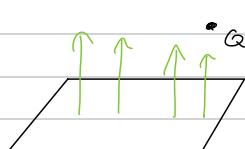
Colombia L...^{as}

- $\vec{F} = Q \vec{E}$
 - E by point q : $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$

} If you have only
 two charges

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$$

But if not 2 point-line



Q₆ T₁₁ C₃

$$[\vec{J}]^z = \frac{\sigma}{2\varepsilon_0} \rightarrow [\vec{F}]^z = Q \frac{\sigma}{2\varepsilon_0}$$

↑ vs. ↑

Direct Integrals

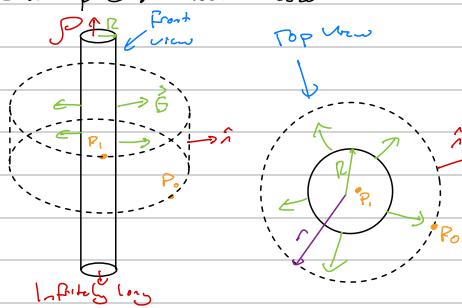
A diagram illustrating vector components. A vector \vec{r} is shown originating from the origin, pointing into the first quadrant. It is decomposed into two components: $x\hat{i} + y\hat{j}$, where \hat{i} and \hat{j} are unit vectors along the x and y axes respectively. The magnitude of the vector is given as $r = |\vec{r}| = \sqrt{x^2 + y^2}$. The direction of the vector is indicated by an angle θ measured counter-clockwise from the positive x-axis.

Charge densities

$$I = \frac{C}{m} \quad O = \frac{C}{m^2} \quad S = \frac{C}{m^3}$$

free surface volume

Example: Gauss Law



$$\oint \vec{E} \cdot \hat{n} d\alpha = \frac{Q_{enc}}{\epsilon_0} = \frac{\pi D^2 l \cdot P}{\epsilon_0}$$

$$\vec{B} = E \hat{i}$$

Solenoids $\hat{z} = \hat{i}$

CAPS: $\hat{z} = \pm \hat{k}$

$$= 2 \int_{CAPS} E \cos(90^\circ) da + \int_{SPBS} E \cos 0^\circ da$$

area of plates

$$= 0 + E a = E(2\pi r l)$$

$$\Rightarrow E = \frac{12^2 p}{2 \times \epsilon}$$

outside the cylinder

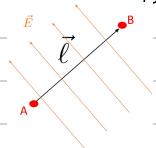
Conot depasos parametru
Gussi surface (l:ntro cco)

Midterm Exam 2



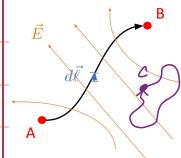
Circulation Law

Line Integrals



Straight line & uniform \vec{E}

- Line Integral = $\int \vec{E} \cdot d\vec{l}$ for $d\vec{l}$ the displacement vector
↳ For this simple application (straight line & uniform \vec{E})



General

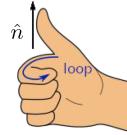
$$\bullet \text{Line integral} = \int_A^B \vec{E} \cdot d\vec{l}$$

Closed Loops

$$\bullet \text{Circulation: } \oint \vec{E} \cdot d\vec{l}$$

Right Hand Rule

- Stokes' theorem
- Use orientation of loop to determine \vec{l} direction using:



Circulation

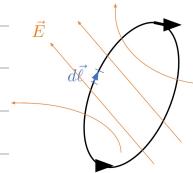
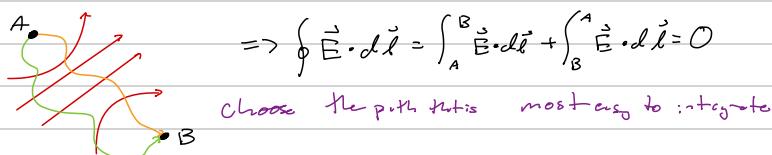
- The circulation of the electric field equals zero

$$\rightarrow \text{Circulation} = \oint d\vec{l} \cdot \vec{E} = 0$$

(Coulomb's law)

Alternative: The line integral between two points do not depend on path

$$\int_A^B \vec{E} \cdot d\vec{l} = - \int_B^A \vec{E} \cdot d\vec{l} \quad \left\{ \int_A^B \vec{B} \cdot d\vec{l} \right\}$$



No closed loops! (Proof by contradiction)

$$\begin{aligned} \text{Assume to the contrary:} \\ \text{So, } \oint \vec{E} \cdot d\vec{l} &= \int |E| d\ell \text{ (only } \vec{E} \text{)} \\ \Theta &= 0 = \int |E| d\ell \\ &= (+) > 0 \end{aligned}$$

No closed loops! but $\oint \vec{E} \cdot d\vec{l} = 0$ by circ. law

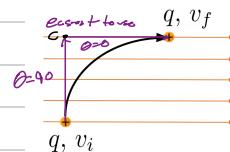
Application / PIZI

• Recall: Work-Energy Theorem

$$\int_{i \rightarrow f} d\vec{l} \cdot \vec{F}_{\text{total}} = \Delta K = \frac{1}{2} m(v_f^2 - v_i^2)$$

• But for electric force we don't need the path!

$$\rightarrow \int_{i \rightarrow f} d\vec{l} \cdot (q \vec{E}) = \Delta K = \frac{1}{2} m(v_f^2 - v_i^2) = \text{Same for all paths b/w i and f}$$

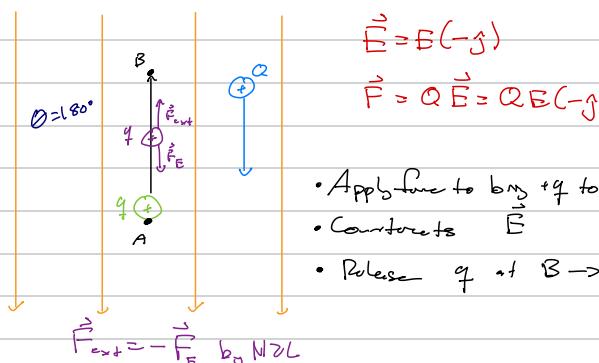


Example

$$\begin{aligned} \int_A^B \vec{E} \cdot d\vec{l} &= \int_A^C \vec{E} \cdot d\vec{l} + \int_C^B \vec{E} \cdot d\vec{l} \\ &= \int_A^C q E \cos 90^\circ + \int_C^B q E \cos 90^\circ \\ &= E \int_C^B d\ell = E \cdot L \end{aligned}$$

Electrostatic Energy

Overview / Introduction

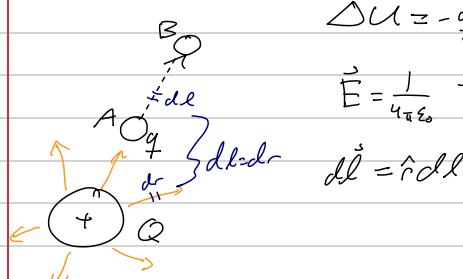


Using our assumption

- The external force matches the electric force: $U_B - U_A = \Delta U = \int_A^B d\vec{l} \cdot \vec{F}_{\text{ext}} = - \int_A^B d\vec{l} \cdot (q \vec{E})$
- In this simple case: $U_B - U_A = \Delta U = q E (z_B - z_A) \rightarrow \text{GPE } (U_0 = mgz)$

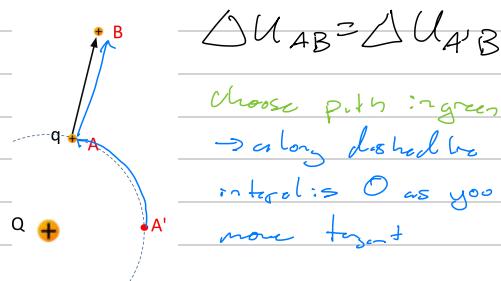
This is a particular formula, not general!

Point Charge

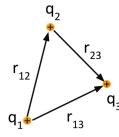


$$\Rightarrow \Delta U = \frac{qQ}{4\pi\epsilon_0} \left(\frac{1}{r_B} - \frac{1}{r_A} \right)$$

$$\text{If } r_A \rightarrow \infty \text{ chooses reference: } U = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r}$$



Arrangement of Charges



- The total energy of the system equals the sum of the energy of each pair of charges

$$U = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right)$$

Electrostatic Potential

- Defined Φ , also known as V
- Do not confuse w/ electrostatic energy (U)
- Another way to describe an electric field
- Electric static potential is a scalar (not a vector)
- To know \vec{E} or Φ for any point in space provides some amount of info (on electrostatics)
- If you know one you can find out the other
- Voltage $V = \Delta \Phi$

Example: Scalar Fields

Elevation

Vector Fields

Gravity

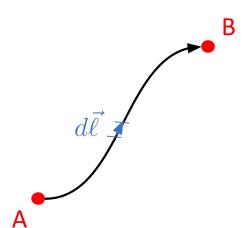
Temperature

Wind

$$\vec{E}(r) \leftrightarrow \Phi(r)$$

$\vec{E}(r)$ not discussed here
not easily measurable

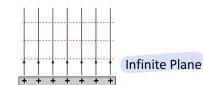
Demonstration



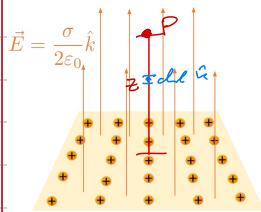
$$\begin{aligned}\Delta \Phi &= \Phi(B) - \Phi(A) = - \int_A^B d\vec{l} \cdot \vec{E} \\ \text{if we set a reference point } (\Phi=0) \\ \Phi(r) &= - \int_{\text{ref}}^r d\vec{l} \cdot \vec{E} \quad \Phi(\text{ref}) = 0\end{aligned}$$

Scalar Quantities - Use Equipotential Lines

Equipotential lines



Example: Potential - Infinite Plane



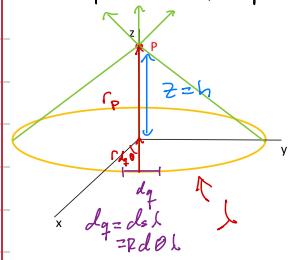
$$\begin{aligned}\text{Charging } z=0 \text{ so } \Phi(z=0)=0 \\ \vec{E} &= \frac{\sigma}{2\epsilon_0} (\pm \hat{k}) \quad (\text{from Gauss law}) \\ \Delta \Phi &= \Phi_{\text{ref}} - \Phi_P = - \left(\int_{\text{ref}}^P d\vec{l} \cdot \vec{E} \right) = \int E dl \cos \theta = E \int_z^0 dz = E_z = \frac{-\sigma z}{2\epsilon_0}\end{aligned}$$

Example: Potential - Point Line Charge

$$\begin{aligned}\vec{E} &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \\ \Phi(r) &= - \int d\vec{l} \cdot \vec{E} = - \int \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \cdot \hat{r} dr\end{aligned}$$

$$\Phi(r) = \frac{Q}{4\pi\epsilon_0 r}$$

Example: Superposition Principle (\vec{E} vs. Φ)



a) Develop the Field \vec{E} at P

Choose point P

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \hat{r} = \frac{1}{4\pi\epsilon_0} \frac{dq}{\sqrt{z^2 + R^2}} \{ z\hat{i} - R(\cos\theta\hat{j} + \sin\theta\hat{k}) \}$$

$$\begin{aligned}\vec{r}_P &= z\hat{i} \\ \vec{r}_{dq} &= R\cos\theta\hat{j} + R\sin\theta\hat{k}\end{aligned} \quad \left. \begin{aligned}\vec{r} + \vec{r}_{dq} &= \vec{r}_P \\ \vec{r} &= z\hat{i} - R(\cos\theta\hat{j} + \sin\theta\hat{k})\end{aligned} \right\} \quad \begin{aligned}r &= \sqrt{z^2 + R^2}\end{aligned}$$

$$\Rightarrow \vec{E} = \frac{R}{4\pi\epsilon_0(z^2 + R^2)^{3/2}} \int_0^{2\pi} (z\hat{i} - R\cos\theta\hat{j} - R\sin\theta\hat{k}) d\theta \rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0} \frac{2\pi R z}{(z^2 + R^2)^{3/2}} \hat{i}$$

b) Develop potential without using \vec{E}

$$\Phi(P) = ?$$
$$d\Phi = \frac{1}{4\pi\epsilon_0} \frac{dq}{r}$$
$$d\Phi = \frac{1}{4\pi\epsilon_0} \frac{R^2 d\theta}{\sqrt{R^2 + z^2}}$$
$$d\Phi = \frac{1}{4\pi\epsilon_0} \frac{R^2 d\theta}{r}$$

It's easier in general to find Φ

$$\Phi(h) = \frac{\Sigma R}{2\epsilon_0} \frac{1}{\sqrt{R^2 + h^2}}$$

Correlation between Φ and \vec{E} \leftarrow outside of scope of course

$$\Phi = - \int d\vec{l} \cdot \vec{E}$$
$$\vec{E} = -\nabla \Phi = - \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z} \right)$$
$$\vec{E} = -\nabla \Phi$$

The gradient only calc B!

Conductors

Perfect Conductor Ideal doesn't exist in nature

- Electric Field inside is zero

• If $E \neq 0$, charges rearrange themselves so $B=0$ instantaneously for perfect conductors

• After some time, real conductors behave this way time is a factor for real conductors (low and)

• In the surface (outer), the Electric Field is perpendicular to the surface of the conductor

→ All net charges reside in the surface

$$Q_{\text{enc}} = \epsilon_0 \oint d\vec{a} \cdot \vec{E} = 0$$

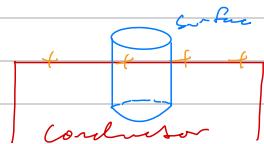
+ Enclosed charge is 0, and all charge is in surface

$$Q = Q_+ - Q_- = 0 \quad (\text{0 charge, but particles exist inside})$$

Charge Density

$$\oint d\vec{a} \cdot \vec{E} = \int_{\text{P}} E da + \int_{\text{O}} 0 + \int_{\text{S.E.O.}} (0) = EA_{\text{cup}}$$

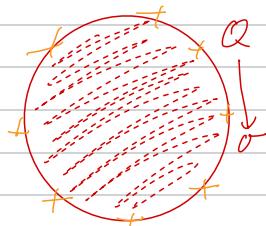
inside conductor \vec{E} is zero \vec{E} is outside $\vec{E} = 0$



$$\rightarrow E A_{\text{cup}} = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\sigma \cdot A_{\text{cup}}}{\epsilon_0}$$

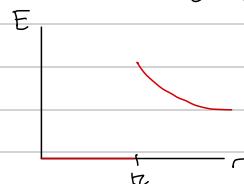
so

$$E = \frac{\sigma}{\epsilon_0}, \quad \sigma = E \epsilon_0$$

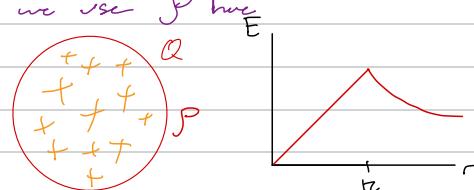


If the sphere is perfect conductor
all charges are on the surface!

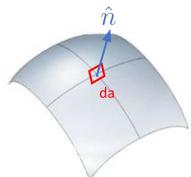
Use σ !



Differentiation: isolates variables/ probably as they had charges evenly distributed throughout volume; we use P here



Force on the Conductor



$$\text{Force: } \vec{F} = q \vec{E} \quad (\text{known})$$

$$\rightarrow d\vec{F} = \frac{1}{2} \sigma da \vec{E} \hat{n}$$

$$\vec{F} = \frac{1}{2} \int \sigma \vec{E} \hat{n} da \quad (\text{use } \sigma = E \epsilon_0 \text{ or } E = \frac{\sigma}{\epsilon_0})$$

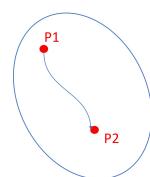
$$= \frac{1}{2} \epsilon_0 \int \vec{E}^2 \hat{n} da = \frac{1}{2 \epsilon_0} \int \sigma^2 \hat{n} da \quad \text{always perpendicular}$$

$$\vec{E} = E \hat{n} \quad \leftarrow \text{outside conductor}$$

$$\frac{1}{2} E \leftarrow \text{on surface? average}$$

$$E = 0 \quad \leftarrow \text{inside conductor}$$

Potential inside conductor



$$\Delta \phi = \int d\vec{l} \cdot \vec{E} = - \int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$$

→ but $E=0$ inside conductor $\therefore \Delta \phi = 0$

so, conductors are equipotential surfaces

Watch out!

$$\Delta \phi = \phi_B - \phi_A = 0$$

$$\rightarrow \phi_A = \phi_B \neq 0$$

↑ just their difference

Had to use fusion first last year

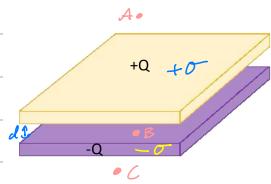
Capacitance

- Charge Q is proportional to potential difference (often called voltage) between two electrodes by a quantity called Capacitance

$$Q = CV \quad \text{where} \quad Q \equiv \text{Charge (coulombs, C)} \\ C \equiv \text{Capacitance (Farads, F)} \\ V \equiv \text{Voltage (Volts, V)}$$

Conversion:
Use Volts instead of (equiv to)
Potential difference ($\Delta\Phi$)

Parallel plates



Assume $A \gg d^2$

$$C = \frac{Q}{V} = \sigma A$$

$$V = \Delta\Phi = \int d\vec{l} \cdot \vec{E}$$

above (↑) Q : What's \vec{E} at all 3 points?

$$\rightarrow A \quad \vec{E}_{\text{in}} = \frac{\sigma}{\epsilon_0} \vec{i}_z \quad \vec{E} = 0$$

$$\text{below } \rightarrow B \quad \vec{E} = \frac{\sigma}{\epsilon_0} \vec{i}_z$$

$$\rightarrow C \quad \vec{E}_{\text{out}} = \frac{\sigma}{\epsilon_0} \vec{i}_z \quad \vec{E} = 0$$

Derive from Gauss Law (Previous lecture)

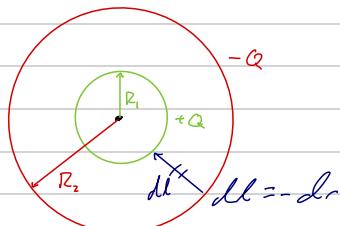
Use $\vec{E} \propto \vec{Q}$ and 3 points to get Cap of parallel plate capacitors?

$$C = \frac{Q}{V} = \frac{Q}{\Delta\Phi}$$

$$\Delta\Phi = - \int_{P_1 \rightarrow P_2} d\vec{l} \cdot \vec{E} = - \int_{P_1 \rightarrow P_2} dl E_{\text{in}} \epsilon_0 = E \int_0^d dl \cdot d\left(\frac{\sigma}{\epsilon_0}\right), \quad Q = \sigma A$$

$$\therefore C = \frac{\sigma A}{\frac{d\sigma}{\epsilon_0}} = \frac{\epsilon_0 A}{d}$$

Example



$$C = \frac{Q}{V}, \quad V = \Delta\Phi = - \int_{-\infty}^{\infty} \vec{E} \cdot d\vec{l}$$

① Determine \vec{E} in 3 points (only really need two b/w θ and ② just along the far perimeter)

$$\vec{E}_{\text{inside}} = 0 \quad Q_{\text{enc}} = 0$$

$$\vec{E}_{\text{outer}}: \quad Q_{\text{enc}} = Q$$

$$\frac{Q_{\text{enc}}}{\epsilon_0} = E_a \rightarrow E = \frac{Q}{4\pi r^2 \epsilon_0}$$

$$\vec{E}_{\text{outside}} = 0 \quad Q_{\text{enc}} = Q - Q = 0$$

- Spherical & hollow shells

- Conductive materials

- Determine Capacitance (C) ② Determine $\Delta\Phi$

$$\Delta\Phi = - \int_{-\infty}^{\infty} E dl \cos 180 = \int_{R_2}^{R_1} \frac{Q}{4\pi r^2 \epsilon_0} dl = - \int_{R_2}^{R_1} \frac{Q}{4\pi r^2 \epsilon_0} dr = \frac{Q}{4\pi \epsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$\Rightarrow C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi \epsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)} = 4\pi \epsilon_0 \left(\frac{1}{R_1} - \frac{1}{R_2} \right)^{-1}$$

Midterm Exam 3



Dielectrics

- Dielectrics = Insulators

examples: Glass

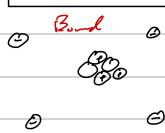
Plastic

Water

Diamond

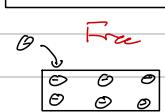
• charges stay put

Internal Charge
or. Bound Charge



vs.

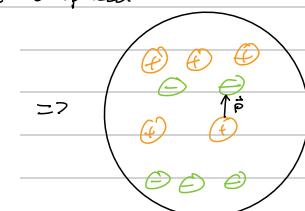
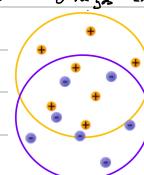
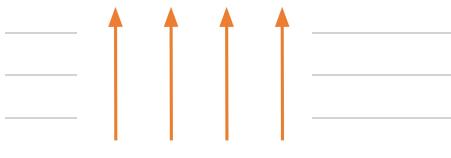
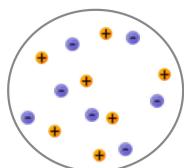
External Charge
or. Free Charge



or Poles

Response of Internal Charges to Fields

Atom of Dielectric \rightarrow Apply electric Field \rightarrow Bound charges are displaced



Displaced dipoles
in the atom

Dipole

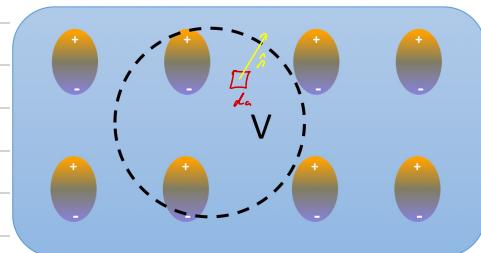


$$\vec{P} = q \vec{d}$$

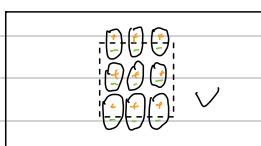
Polarized medium

\rightarrow Polarization vector \vec{P}

o Dipole Moment Density: $\vec{P} = \frac{\sum \vec{p}}{volume}$



How much bound charge makes?



$$\vec{P}$$

• \vec{P} is uniform, $Q_B(V) = 0$

• Answer depends on \vec{P} at the surface

• If \vec{P} is parallel to the surface no net charge

$$Q_B \propto \vec{P}_{surface} \checkmark$$

$$Q_{bound} = Q_B(V) - \oint_{\text{closed}} \vec{P} = - \oint |\vec{P}| |d\vec{l}| \cos 0$$

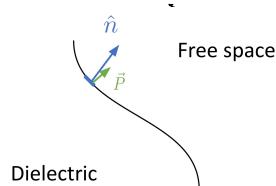
• Only for Dielectrics

• Not Gauss' Law, thus: $Q_{bound} Q_{free} = \epsilon_0 \oint d\vec{a} \cdot \vec{E}$

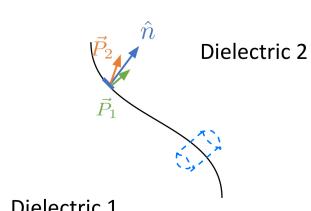
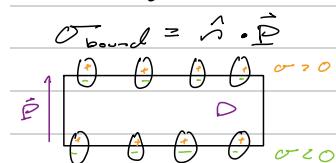
Free
Charges

$$Q_{free} = \oint d\vec{a} \cdot (\epsilon_0 \vec{E} + \vec{P})$$

Surface Charge Density

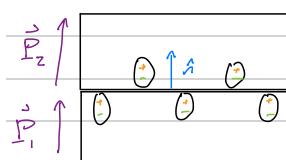


One dielectric



Two dielectrics

$$\sigma_{bound} = \hat{n} \cdot \vec{P}_1 - \hat{n} \cdot \vec{P}_2$$



Gauss Law to get σ_{total}

$$\vec{E}_1 = \frac{\sigma_{total}}{2\epsilon_0} \hat{n}$$

$$\sigma_{total} = \sigma_{free} + \sigma_{bound}$$

$$\vec{E}_1 = \frac{\sigma_{free}}{2\epsilon_0} \hat{n}$$

$$\sigma_{free} = \hat{n} \cdot (\epsilon_0 \vec{E}_2 + \vec{P}_2) - \hat{n} \cdot (\epsilon_0 \vec{E}_1 + \vec{P}_1)$$

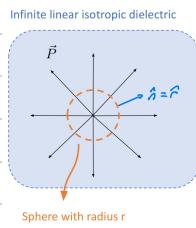
Linear Isotropic Dielectrics

$$\vec{P} \propto \vec{E}^2 \quad \text{proportional dep on the material temperature and pressure} \Rightarrow \vec{P} = \chi \epsilon_0 \vec{E}$$

Simpler case
Gauss's Law $\vec{P} \cdot \vec{E}^x$ (to g.p.m)

Bliss Susceptibility

Example



Assume $\vec{P} = \frac{A}{r^2} \hat{r}$ const

- What is the bound charge inside a sphere with radius r ?

$$Q_{\text{bound}} = \oint_C \vec{P} \cdot d\vec{l}$$

up from \vec{P}

$$= 4\pi r^2 P \cos 0$$

$$Q_{\text{bound}} = -4\pi r^2 \left(\frac{A}{r^2}\right) = -4\pi A$$

$$Q_{\text{bound}} = -4\pi A$$

Note: Q_{bound} is independent of r ; it has a constant value

↳ Why? All the charges are at the center

- What is the electric field?

$$\vec{E} = \frac{\vec{P}}{\chi \epsilon_0} = \frac{\frac{A}{r^2} \hat{r}}{\chi \epsilon_0} = \frac{1}{\chi \epsilon_0} \frac{A}{r^2} \hat{r}$$

$$\vec{E} = \frac{1}{\chi \epsilon_0} \frac{A}{r^2} \hat{r}$$

- What is the total charge inside a sphere with radius r ?

$$Q_{\text{tot}} = \oint \vec{E} d\vec{l} \Rightarrow Q_{\text{tot}} = \epsilon_0 E \oint d\vec{a} = \epsilon_0 E 4\pi r^2 = \epsilon_0 \frac{1}{\chi \epsilon_0} \frac{A}{r^2} 4\pi r^2 = \frac{4\pi A}{\chi}$$

$$Q_{\text{tot}} = \frac{4\pi A}{\chi}$$

- What is the external (free) charge inside a sphere with radius r ?

$$Q_{\text{free}} = Q_{\text{tot}} - Q_{\text{bound}} = 4\pi A \frac{1+\chi}{\chi}$$

$$Q_{\text{free}} = 4\pi A \frac{1+\chi}{\chi}$$

You must see $1+\chi$ as ϵ_r (relative permittivity) or the dielectric constant Kappa (K)

$$\vec{E} (Q_F = 4\pi A \frac{1+\chi}{\chi}) ? A = \frac{\chi Q_F}{4\pi(1+\chi)} \Rightarrow \vec{E} = \frac{1}{\chi \epsilon_0} \left(\frac{\chi Q_F}{4\pi(1+\chi)} \right) \hat{r}$$

Basically coulomb's law, but with $(1+\chi)$ factor.

We will define $K = 1+\chi$ (Kappa)

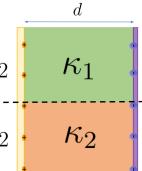
Parallel Plate Capacitor

- Capacitance is affected: $C = K C_{\text{air}} = (K \epsilon_0) \frac{A}{d}$

Material	Dielectric constant, K
Air	1.085-1.21 at 20°C*
Water	80 at 20°C (79.50-80.24)*
Ice	3 at -5°C*
Basalt	12
Granite	7.9
Sandstone	9-11
Dry loam	3.5
Dry sand	2.5 (2.5-3.5 at 20°C)*
Ethanol liquid	15.21 at 20°C*
Ethanol Mixing Liquid	39.82 at 20°C*
Acetone	21.20 at 20°C*

*measured dielectric constant in this study

What is the new Capacitor with two dielectrics?



$$C_i = \frac{Q_i}{V}, C_2 = \frac{Q_2}{V} \quad \text{for constant } V \quad (V = Ed)$$

$$E \frac{A}{2} = \frac{Q_2}{\epsilon_0 K_2} \Rightarrow Q_2 = (K_2 \epsilon_0) E \frac{A}{2}$$

$$\therefore C_i = \frac{K_i \epsilon_0}{z} \frac{A}{d}$$