Physics 122 Fall 2024

Maxwell's equations; Electromagnetic Waves

Homework 10

Due: 11:59 PM Sunday, April 27, 2025

Figure 1: Comet McNaught photographed over Australia in 2007. The tail of the comet always points away from the sun regardless of the direction in which the comet is moving. This is in part a result of radiation pressure from the sun acting on the dust and ice particles that comprise the tail of the comet.

Guidelines

1. Please write legibly. If necessary solve the problem first on scratch paper. Then write a cogent and legible solution based on your scratch work. Learning to make a clear and logical presentation of your work is an invaluable asset, well worth the trouble of having to write it twice.

2. Please begin each problem on a new page just as we have done with the questions.

3. Parts of problems marked with an asterisk (if any) are only to be read. No response is needed.

How to submit homework: Submit your homework in PDF format via Canvas. You will find a link to submit under Assignments. Please submit before the deadline. Deadlines are enforced and late homework may not be graded. If you have compelling reasons to miss the deadline please contact the instructor preferably before the deadline. In order to generate a PDF of your homework you should scan it. If you do not have access to a scanner there are a number of effective and free scanner apps available for phones. AdobeScan is one that many students have found worked well for them in previous courses. The other option is to take pictures of your homework and convert them to PDFs using your computer. Finally if you work with a tablet you can of course easily save your work as a PDF.

OPTIONAL: THIS PROBLEM IS ZERO CREDIT

1. Atom-Radiation interaction. In the Lorenz model an atom may be pictured as an immovable positively charged nucleus that sits at the origin and a negatively charged electron that is harmonically bound to the nucleus by a spring (natural frequency ω_0 and damping constant γ).

(a) An oscillating electric field

$$\mathbf{E} = E\cos\omega t\hat{\mathbf{i}} \tag{1}$$

is applied to the atom. The motion of the electron is governed by Newton's law

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = \frac{eE}{m}\cos\omega t. \tag{2}$$

Here e is the electron charge and m is the electron mass. The solution to this equation is

$$x = A\cos(\omega t - \delta). \tag{3}$$

Here the amplitude of oscillations, A, is given by

$$A = \frac{eE}{m}\frac{1}{\left[(\omega_0^2-\omega^2)^2 + 4\omega^2\gamma^2\right]^{1/2}}. \tag{4}$$

The phase of the oscillations δ is determined by

$$\sin\delta = \frac{2\omega\gamma}{\left[(\omega_0^2-\omega^2)^2 + 4\omega^2\gamma^2\right]^{1/2}}, \quad \cos\delta = \frac{(\omega_0^2-\omega^2)}{\left[(\omega_0^2-\omega^2)^2 + 4\omega^2\gamma^2\right]^{1/2}}. \tag{5}$$

These results can be obtained by exactly the same analysis you did on the LCR circuit last week. You are not required to derive them here. However you must answer this question: What is the total electric force on the atom? Assume the nucleus and the electron have equal but opposite charges.

(b) The power needed to drive the oscillations is $P = eE\cos(\omega t)dx/dt$. Since we are interested in very high frequencies ($\omega \approx 10^{15}$ Hz) the power oscillates very rapidly in time and it is more useful to consider the power averaged over one cycle of oscillation. Calculate the average power. Give your answer in terms of $e, E, m, \omega, \omega_0$ and γ. Sketch the average power absorbed by the atom as a function of frequency ω.

[*Hint:* The average value of $\cos^2\omega t$, for example, is denoted $\langle\cos^2\omega t\rangle$ and is given by

$$\langle\cos^2\omega t\rangle = \frac{1}{T}\int_0^T dt\ \cos^2\omega t = \frac{1}{2}. \tag{6}$$

Here $T = 2\pi/\omega$. Similarly $\langle\sin^2\omega t\rangle = 1/2$ and $\langle\cos\omega t\sin\omega t\rangle = 0$. Finally a useful trigonometric identity is $\sin(\omega t - \delta) = \sin\omega t\cos\delta - \cos\omega t\sin\delta$.]

(c) Now suppose that an oscillating magnetic field

$$\mathbf{B} = B \cos \omega t \hat{\mathbf{j}} \tag{7}$$

is also applied with $B = E/c$. Assume that the magnetic force on the electron is sufficiently weak that its path is still essentially as given in part (a). The magnetic force on the electron is then given by

$$\mathbf{F} = eB \cos \omega t \frac{dx}{dt} \hat{\mathbf{k}}. \tag{8}$$

Since the magnetic force on the atom also oscillates very rapidly it may be approximated by its time average. Calculate the time averaged force. *Hints:* Use the hint to the previous part. You should find $\langle \mathbf{F} \rangle = \hat{\mathbf{k}} \langle P \rangle / c$.

(d) Explain why there is no magnetic force on the nucleus. Hence conclude that the expression derived in part (c) is the total force on the atom.

*(e) The results of this problem show that an atom placed in an electromagnetic wave will absorb energy from it and will also experience a force due to the radiation. It is this force that is used in order to cool and trap atoms using light, a remarkable new technique that was developed starting in the 1980s. For more information on the subject see *Cooling and trapping atoms* by Phillips and Metcalf, Scientific American, March 1987 and *Laser Trapping of Neutral Particles* by Steven Chu, Scientific American, 1992 and the Nobel Prize website for 1997 and 2001. Although the Lorenz model might seem artificial it is extremely useful since it produces results identical to more rigorous quantum mechanical calculations.

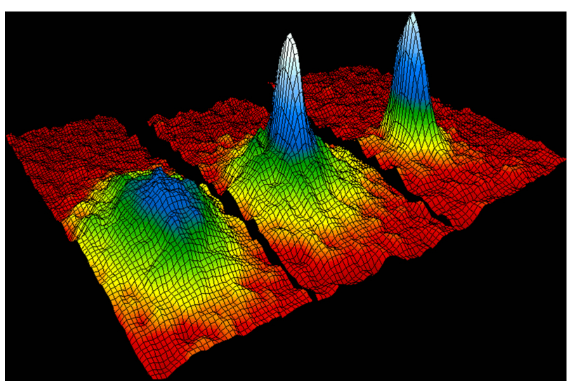


Figure 2: An iconic image from the cover of the journal *Science* (July 1995) showing a dilute vapor of rubidium atoms that have been laser cooled by means of the force analyzed in this problem to record-setting temperatures as low as tens of nano-Kelvin above absolute zero. As a result the vapor passes from a gaseous state to a new state of matter called a Bose-Einstein condensate.

2. Electromagnetic waves. An electromagnetic plane wave propagates along the positive x-axis. The electric and magnetic fields are given by

$$\mathbf{E} = \mathcal{E}\cos(kx - \omega t)\hat{\mathbf{j}}, \quad \mathbf{B} = \frac{\mathcal{E}}{c}\cos(kx - \omega t)\hat{\mathbf{k}}. \tag{9}$$

Here $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light in vacuum. Note that the electric and magnetic field amplitudes are related by the ratio c, as required by Maxwell's equations in a vacuum. The frequency ω and the wave-vector k are related via $\omega = kc$.

(a) Calculate u, the energy density of the field. Also calculate $\langle u \rangle$, the time averaged energy density defined as

$$\langle u \rangle = \frac{1}{T}\int_0^T dt\; u. \tag{10}$$

Useful info:

$$\langle \cos^2(kx - \omega t) \rangle = \frac{1}{2}. \tag{11}$$

(b) Calculate the Poynting vector $\mathbf{S}$ and the time averaged Poynting vector $\langle S \rangle$. Recall that the Poynting vector corresponds to the energy per unit area delivered by the wave to a surface perpendicular to the direction of wave propagation, $\hat{\mathbf{i}}$. The magnitude S is synonymous with the intensity of the wave sometimes denoted I.

(c) Calculate the momentum density $\mathbf{\Pi}$ and the time averaged momentum density $\langle \mathbf{\Pi} \rangle$.

(d) Calculate the time averaged radiation pressure force exerted by the wave on a surface perpendicular to $\hat{\mathbf{i}}$ and with area A assuming that the surface is (i) perfectly absorbing and (ii) perfectly reflecting.

For parts (a) - (d) give your answer in terms of the given quantities $\mathcal{E}, k$ and ω and the fundamental constants ϵ_0 and c.

(e) *Dimensional analysis:* (i) A professor points out that the time averaged intensity $I = \langle S \rangle$ is related to the energy density by $= \langle u \rangle c$. However somewhat later he tells the class that $I = \langle u \rangle / c$. Use dimensional analysis to figure out which one he really meant. (ii) The professor similarly manages to confuse the issue of whether $\langle \Pi \rangle = \langle u \rangle c$ or $\langle \Pi \rangle = \langle u \rangle / c$. Use dimensional analysis to determine which one is right.

(f) Sunlight has a time averaged intensity $I = 1.36\ \mathrm{kW/m^2}$ in the neighborhood of the Earth (it's a lot weaker near Pluto).

(i) Calculate the electric field amplitude $\mathcal{E}$ and the magnetic field amplitude $\mathcal{E}/c$ for a plane wave with the same intensity as sunlight. Give your answer in terms of V/m and T respectively.

(ii) Calculate the time averaged energy density of sunlight. Give your answer in $\mathrm{J/m^3}$.

(iii) Calculate the time averaged momentum density of sunlight. Give your answer in $\mathrm{kg/m^2 s}$.

(iv) Calculate the radiation pressure force on a perfectly reflecting solar sail of area equal to one square mile. Give your answer in N.

[`Optional - zero credit`] (v) Assuming the sail is made of aluminum foil calculate its acceleration in $\mathrm{m/s^2}$.

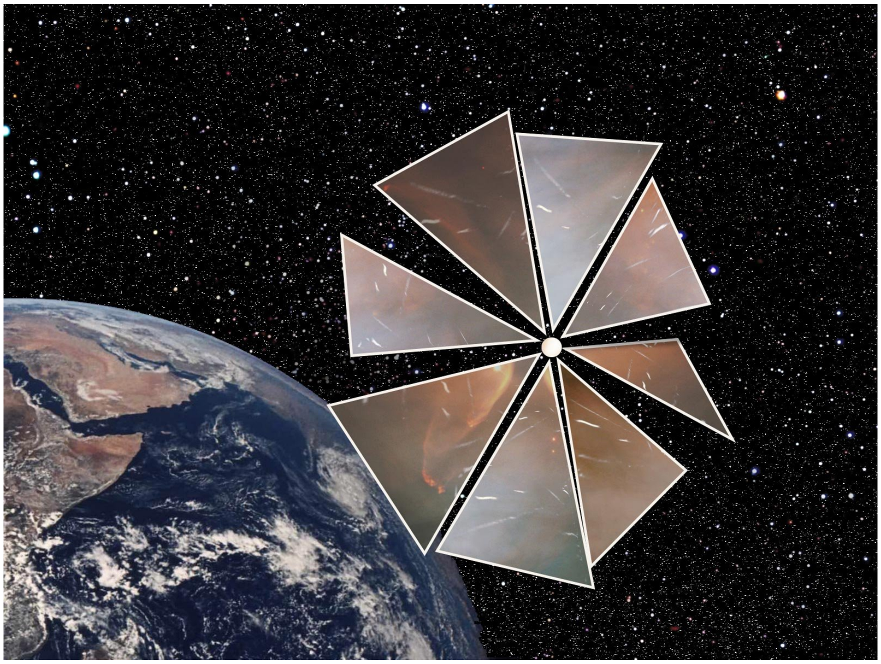
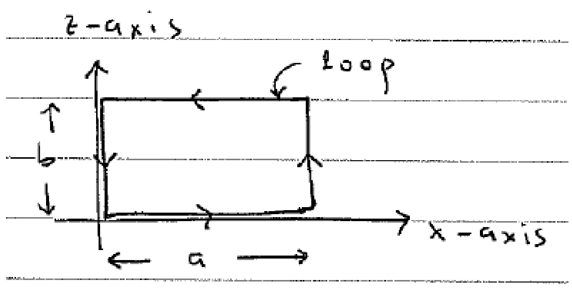


Figure 3: Artist's conception of solar sailing.

OPTIONAL - THIS PROBLEM IS ZERO CREDIT

3. Maxwell and the speed of light. Consider the electromagnetic plane wave

$$\begin{aligned} \mathbf{E} &= \alpha \cos(kx - \omega t)\, \hat{\mathbf{j}} \\ \mathbf{B} &= \beta \cos(kx - \omega t)\, \hat{\mathbf{k}}. \end{aligned} \tag{12}$$



(a) Evaluate $\oint \mathbf{B} \cdot d\ell$ around the rectangular loop in the x-z plane shown in the figure.

(b) Evaluate the flux of the electric field though the surface of the rectangle

$$\phi_E = \int da\, \hat{\mathbf{n}} \cdot \mathbf{E}. \tag{13}$$

Also calculate $d\phi_E/dt$.

(c) Apply the Ampere-Maxwell law to show that the amplitudes of the electric and magnetic waves are in the ratio

$$\frac{\alpha}{\beta} = \frac{k}{\omega} \frac{1}{\mu_0 \epsilon_0}. \tag{14}$$

(d) By applying Faraday's law to a similar loop in the x-y plane one can show

$$\frac{\alpha}{\beta} = \frac{\omega}{k} \tag{15}$$

Use eq (14) and eq (15) to show that the speed of light is

$$c = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{16}$$

and the ratio of the electric and magnetic amplitudes is

$$\frac{\alpha}{\beta} = c. \tag{17}$$

Figure 4: Maxwell Monument in Edinburgh, Scotland.

OPTIONAL - THIS PROBLEM IS ZERO CREDIT

4. Dynamics of a Comet's Tail: Cometary nuclei are mountain sized globs of dust and ice. As the nucleus approaches the sun, the surface ices vaporize and tiny particles of dust and ice are blown off to form the brilliant cometary tail (fig 5).

Figure 5: The Bayeux tapestry depicting the appearance of Halley's comet in 1066 before the Battle of Hastings.

(a) Let L denote the total power radiated by the sun. Calculate I, the intensity of sunlight, at a distance R from the sun. [*Hint:* L is the total energy per second radiated by the sun. The intensity is the energy per second per unit area. To calculate the intensity imagine that the sun is surrounded by a sphere of radius R. How much energy passes this sphere per second? How much energy passes the sphere per second per unit area?]

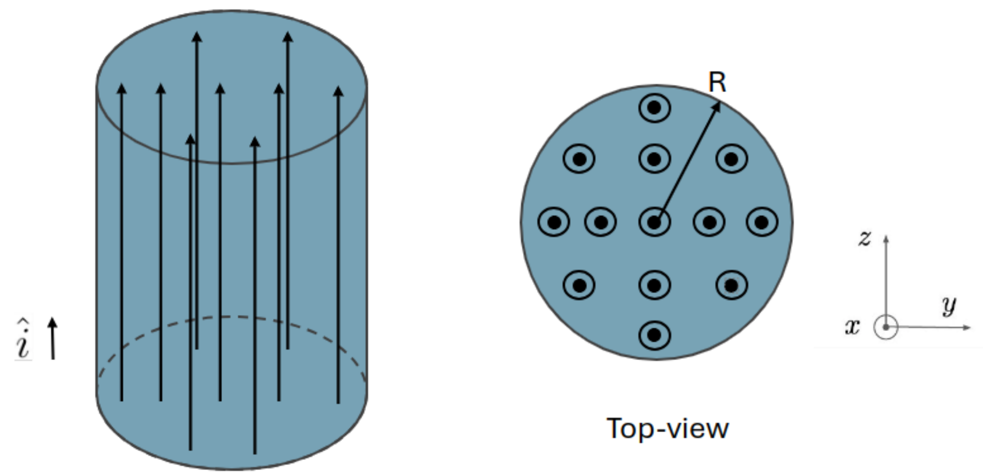
(b) Model the tail particles as dark spheres of radius r which completely absorb all incident solar radiation (together with the field momentum it carries). Calculate the magnitude of the force exerted by solar radiation on the tail particles. Give your answer in terms of L, R, r and $c =$ speed of light. [*Hint:* The force is the radiation pressure of sunlight multiplied by the cross section area of the tail particles.]

(c) Calculate the radius of a dust particle for which radiation pressure forces precisely balance the sun's gravitational attraction. Give your answer in terms of $G =$ Newton's constant of gravitation, $M =$ mass of the sun, $\rho =$ density of silicate dust, c and L.

(d) Particles of radius smaller than the critical value calculated in part (c) will be blown outward by radiation pressure. Calculate the critical radius in μm. Take $\rho = 2.5 \times 10^3$ kg/m^3 and $L = 4 \times 10^{26}$ W. Also $M = 2 \times 10^{30}$ kg, $c = 3.0 \times 10^8$ m/s, and $G = 6.67 \times 10^{-11}$

S.I. units.

Problem 5



Inside an infinitely long cylinder of radius R, there is a uniform electric field that is changing overtime as

$$\vec{E}(t) = E_0 \sin(\omega t)\hat{i}$$

a) Use Ampere-Maxwell's Law to calculate the magnetic field at the surface of the cylinder (distance R form the axis). Give magnitude and direction (seen from the top).

b) Repeat for a point that is at a distance 2R from the axis of the cylinder.