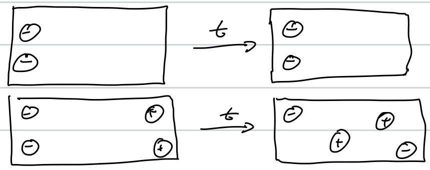
Midterm Exam 1

Need to write the Max well equations on the final exam!

Electric Charges & Fields, Coulomb's Law, Superposition

Electric Charge

- charge is a fundamental prop. of matter
  - ↳ two types: ⊕ & ⊖ matter (atoms) are normally neutral
- Triboelectricity: Friction transfers charge
- Metal Rod in demo?
  - Plastic & crystal are insulators
    - → charge "stays where put"
  - Metals are conductors
    - → charge "flow freely"

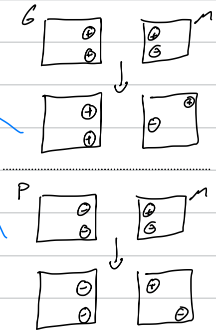


t meaning "after some time"

Demo - Rods

⊖ P P ⊖ ← →
⊕ G G ⊕ ← →
⊖ P G ⊕ → ←

Demo w/ Metal



Note: the metal is still neutral

- Charge is conserved
  - Rubbing transfers charges, but does not create nor destroy it
- Charge is quantized
  - Charge is always a multiple of $e = 1.6 \times 10^{-19}$ Coulombs (C) → electron = $-e$, proton = e
- why the rods are moving!
  - Attraction or Repulsion ⇒ There is a force between the charges
    - ↳ Force is known as coulomb's force

$$F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{qQ}{r^2}$$

where $\frac{1}{4\pi\varepsilon_0} = 9\times10^9 \ \frac{N \cdot m^2}{C^2}$

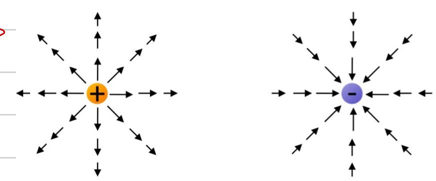
- When two charges q and Q are same signed → F is positive = Repulsive
- when they have opposite sign → F is negative = Attractive

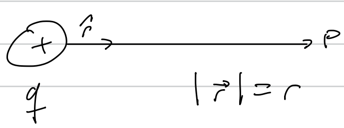
Electric Field and Coulomb's Law

Laws of Electrostatics

- Electrostatics ≡ charges @ rest
  - charges produce electric fields
  - electric fields push charges

$$|\vec{E}| = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$





$\hat{r}$ is a unit vector that points away from charge q

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

Note: $r = |\vec{r}|$

or alternatively (use $\vec{v} = |\vec{v}|\,(\hat{v})$) so $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ and

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^3} \vec{r}$$

Equivalent!

Superposition Principle

In Words:

Electric Field produced by many point charges = Sum of the electric field produced by each point charge independently

In Math:

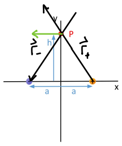
$$\vec{E}(q_1, q_2) = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{r_1^3} \vec{r}_1 + \frac{1}{4\pi\varepsilon_0} \frac{q_2}{r_2^3} \vec{r}_2$$

Second Law of Electrostatics

Fields push/pull charges: $\vec{F} = Q\vec{E}$ ⇒ $\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{r^2} \hat{r}$

$\{N\}$ $\{C\}$ $\{\frac{N}{C}\}$

Example: PollEv



Calculate the Electric field at point P.

Pos vector $\vec{r}$ from $\ominus$ to p: $-a\hat{\imath}+h\hat{\jmath}$

Pos vector $\vec{r}$ from $\oplus$ to p: $a\hat{\imath}+h\hat{\jmath}$ (not moving, position vector not dependent on charge!)

Here, we are assuming point-like charges but this is not realistic

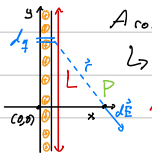
Calculate the Electric Field @ P:

$$\vec{E}=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r} \Rightarrow \vec{E}=\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{(\sqrt{a^2+h^2})^3}\left[q(-a\hat{\imath}+h\hat{\jmath})+-q(a\hat{\imath}+h\hat{\jmath})\right]\right)=\frac{2qa}{4\pi\varepsilon_0\sqrt{a^2+h^2}^{\,3}}\cdot(-\hat{\imath})$$

$$\left.\begin{array}{l} r_- = a\hat{\imath}+h\hat{\jmath} \rightarrow r_-=\sqrt{a^2+h^2} \\ r_+ = -a\hat{\imath}+h\hat{\jmath} \rightarrow r_+=\sqrt{a^2+h^2}\end{array}\right\rangle \vec{E}=\vec{E}_+ + \vec{E}_-$$

Continuous Distribution of Charge

$dq=\lambda\, dy$



A rod of Length 'L' is charged with a positive amount of charge Q

↳ Calculate the Electric field @ P

Assumptions: Rod is very long ($L\to\infty$)

Rod is very thin (1-D)

Definition: Charge per unit length or linear charge density

$$\lambda=\frac{\text{Charge}}{\text{Length}}=\frac{Q}{L}$$

$$\vec{E}=\vec{E}_1+\vec{E}_2=\sum_{i=1}^{N}\vec{E}_i=\int_a^b d\vec{E}$$

$$\vec{E}=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}\rightarrow d\vec{E}=\frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}$$

$$\left.\begin{array}{l}\vec{r}=x\hat{\imath}-y\hat{\jmath}\\ r=\sqrt{x^2+y^2}\end{array}\right\rangle d\vec{E}=\frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{x^2+y^2}^{\,3}}\cdot(x\hat{\imath}-y\hat{\jmath})$$

$$\text{So, } \vec{E}=\frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\lambda}{\sqrt{x^2+y^2}^{\,3}}(x\hat{\imath}-y\hat{\jmath})\,dy$$

Solve $\hat{\imath}$ and $\hat{\jmath}$ components separately

Example

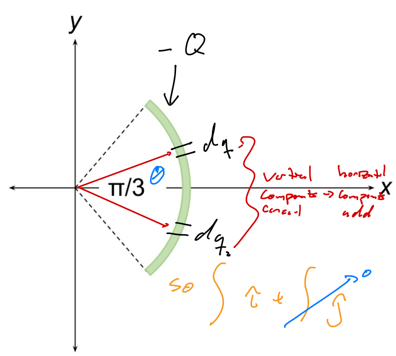
A plastic rod with a uniformly distributed charge $-Q$ is bent into a circular arc of radius r that subtends an angle of $\pi/3$ radians. We place coordinate axes such that the axis of symmetry of the rod lies along the x-axis and the origin is the center of curvature for the rod. What is the electric field (magnitude and direction) at the origin?

Hint: Imagine that the arc is made up of many infinitesimally small point charges dq. Each dq creates a differential electric field of magnitude $dE = k\, dq/r^2$.

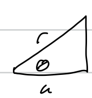
Sum up all the dEs from all the dqs to get the magnitude of the overall electric field.

Integration is a way to add many infinitely small elements.

$$d\vec{E}=\frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}$$



$$|d\vec{E}|=\left|\frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}\right|$$



$$\left.\begin{array}{l}\cos\theta=\frac{a}{r}\\ \sin\theta=\frac{b}{r}\end{array}\right\}\ \vec{r}=-r\cos\theta\,\hat{\imath}-r\sin\theta\,\hat{\jmath}$$

x-component of $|d\vec{E}|$: $dE_x=\frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\cos\theta$

we need to integrate over all θ, but dE_x is in terms of dq, not $d\theta$!

$\Rightarrow dq=\lambda\, ds$ where ds is segment of arc

↖ linear charge density

$\Rightarrow \underbrace{2\pi r}_{\theta}$ so $ds=r\,d\theta \rightarrow dq=r\lambda\, d\theta$

To integrate let's use polar! $(x,y)\to(r,\theta)$

$$\int d\vec{E}_x=\frac{1}{4\pi\varepsilon_0}\int_{-\pi/6}^{\pi/6}\frac{r\lambda\cos\theta\, d\theta}{r^2}$$

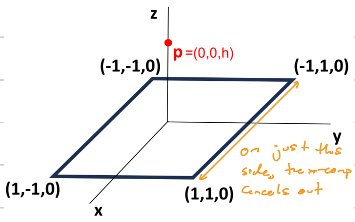
$$=\frac{\lambda}{4\pi\varepsilon_0 r}\Big[\sin\theta\Big]_{-\pi/6}^{\pi/6}$$

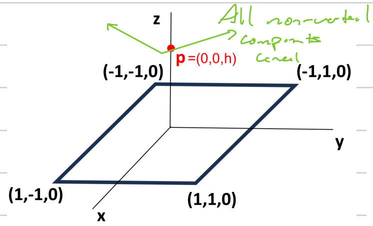
$$=\frac{1}{4\pi\varepsilon_0}\cdot\frac{3Q}{\pi r^2}\rightarrow \boxed{\vec{E}=\frac{1}{4\pi\varepsilon_0}\frac{3Q}{\pi r^2}\hat{\imath}}$$

but what is lambda $=\frac{Q}{L}=\frac{Q}{r(\pi/3)}$ Just think length = angle · radius (like in $C=2\pi r$)

Example

A plastic rod with a uniformly distributed charge Q is shaped forming a square whose side is 1m. We place coordinate axes such that the square is on the x-y plane and the origin is the center of the square. What is the electric field (magnitude and direction) at the point p=(0,0,h)?





$\Rightarrow$ This problem's solution will be $\vec{E} = E\hat{k}$

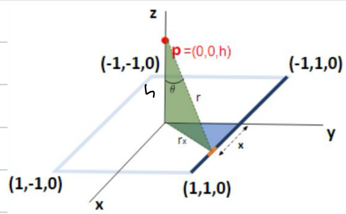
So we just need to calculate the vertical (z) component of one side and add the rest!

Hint: What is the square formed by? Is there any symmetry?

Imagine that the square is made up of many infinitesimally small point charges *dq*. Each *dq* creates a differential electric field of magnitude $dE = k\,dq/r^2$.

Sum up all the *dE*s from all the *dq*s to get the magnitude of the overall electric field.

Integration is a way to add many infinitely small elements.



Blue triangle: $r_x^2 = x^2 + 1^2$

Green triangle: $r^2 = r_x^2 + h^2$

$$\left.\begin{aligned} \vec{r}_h &= h\hat{k} \\ \vec{r}_{xy} &= -x\hat{i} - \hat{j} \;\rightarrow\; r_x^2 = x^2 + 1^2 \end{aligned}\right\} \vec{r} = -x\hat{i} - \hat{j} + h\hat{k}$$

so, $r^2 = r_x^2 + h^2$

$$\rightarrow r = \sqrt{x^2 + h^2 + 1}$$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{dq}{(x^2+h^2+1)^{3/2}} \left(-x\hat{i} - \hat{j} + h\hat{k}\right)$$

x and y components cancel

$$\Rightarrow \vec{E} = \int \frac{1}{4\pi\varepsilon_0} \frac{dq}{(x^2+h^2+1)^{3/2}} \left(\cancel{-x\hat{i}}^{\,0} \cancel{-\hat{j}}^{\,0} + h\hat{k}\right)$$

$$\vec{E}_{Right} = \int d\vec{E}_{Right} = \int_{-1}^{1} \frac{1}{4\pi\varepsilon_0} \frac{\lambda\, dx}{\left(\sqrt{x^2+h^2+1}\right)^3} \left(-x\hat{i} - \hat{j} + h\hat{k}\right) = +c\,\hat{j} + d\,\hat{k}$$

↑ In one side $\hat{j}$ component present, but disappears when using Left as well

$$\Rightarrow \vec{E}_{Left} = \int d\vec{E}_{Left} = \ldots = -c\hat{j} + d\hat{k}$$

$$\vec{E}_{Top} = \int_{-1}^{1} \frac{1}{4\pi\varepsilon_0} \frac{dy}{\left(\sqrt{x^2+h^2+1}\right)^2} \left(-\hat{i} - y\hat{j} + h\hat{k}\right)$$

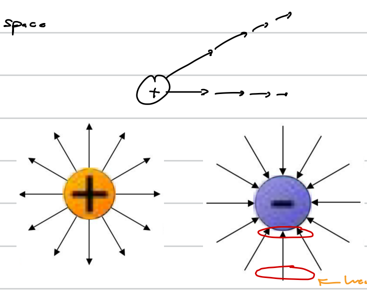
$$\vec{E}_{Bottom} = \int_{-1}^{1} \frac{1}{4\pi\varepsilon_0} \frac{dy}{\left(\sqrt{x^2+h^2+1}\right)^2} \left(+\hat{i} - y\hat{j} + h\hat{k}\right)$$

so $\boxed{\vec{E} = \vec{E}_{Right} \cdot 4\hat{k}}$

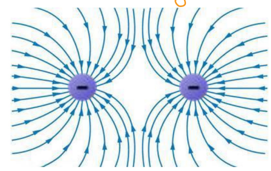
Gauss Law

Electric Field Lines

Single Charge



Two charges



(i) Lines are tangent to $\vec{E}$ at every point in space

(ii) The density of lines is proportional to $|\vec{E}|$

(iii) Lines never cross

(iv) No closed loops

(v) Lines start or stop on charges

↳ start should be a positive charge, and should be a negative charge

↳ This rule inherently enforces part (iv)

← weaker charge → fewer lines

Recall Vector Multiplication

$$\begin{cases} \vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z \\ \quad = |\vec{A}||\vec{B}|\cos\theta \end{cases}$$

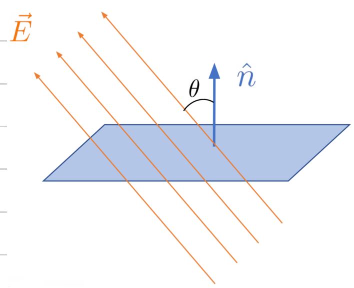
for θ the angle between $\vec{A}$ and $\vec{B}$

Electric Flux

- Flat surface and uniform $\vec{E}$
- Defined:

$$\Phi_E = A\hat{n}\cdot\vec{E} = A\underbrace{|\hat{n}|}_{1}|\vec{E}|\cos\theta = A|\vec{E}|\cos\theta$$

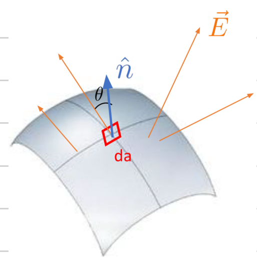
  - $\hat{n}$ is a unit vector perpendicular to the surface (normal vector)
  - θ is the angle between $\hat{n}$ and $\vec{E}$
  - A is the area of the surface



- Non-flat surface and/or Non-uniform $\vec{E}$

$$\Phi_E = \int \hat{n}\cdot\vec{E}\,da = \int \underbrace{|\hat{n}|}_{1}|\vec{E}|\cos\theta\,da = \int|\vec{E}|\cos\theta\,da$$

  - $\hat{n}$ is a unit vector perpendicular to the surface da
  - θ is the angle between $\hat{n}$ and $\vec{E}$



Gauss Law

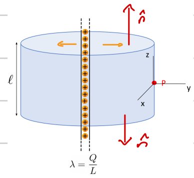
- First of Maxwell's Equations
- In words: The electric flux through a closed surface = $\dfrac{\text{Amount of charge inside the surface}}{\varepsilon_0}$
- Equation:

$$\Phi_{E_{\text{closed}}} = \oint \hat{n}\cdot\vec{E}\,da = \frac{Q}{\varepsilon_0}$$

- $\hat{n}$ points outside the surface by convention
- $\oint \vec{E}\cdot d\vec{a} = \frac{Q}{\varepsilon_0}$ is also valid, where $d\vec{a} = da\cdot\hat{n}$

• Example / Review: Infinite Line

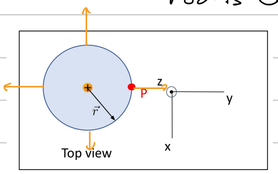
Assumptions: rod is very long ($L \to \infty$)
rod is very thin (1-D)



Length ℓ of rod inside cylinder

$\Rightarrow Q_{ENC} = \lambda \ell$

$$\phi_E = \frac{Q_{ENC}}{\varepsilon_0} = \frac{\lambda \ell}{\varepsilon_0}$$



$$= \int_{Cap\uparrow} \hat{n}\vec{E}\,da + \int_{Cap\downarrow} \hat{n}\vec{E}\,da + \int_{Side} \hat{n}\vec{E}\,da$$

Extra credit Question

What is the direction of the total electric field @ P? $\hat{S}$

CAPS — radial

$\vec{E} = E\hat{r}$, $\hat{n} = \pm\hat{k}$ } $\theta = 90°$

SIDE — radial

$\vec{E} = E\hat{r}$, $\hat{n} = \hat{r}$ } $\theta = 0$

$$\boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\hat{r}, \quad E = \frac{\lambda}{2\pi\varepsilon_0 r}}$$

$$\underbrace{2\int_{Caps} |\hat{n}||\vec{E}|\cos\theta\,da}_{\theta = 90°,\text{ so } 0} + \int_{Side} \hat{n}\vec{E}\,da$$

$$= 0 + \int |\hat{n}||E|\cos\theta\,da$$

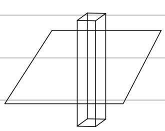
$$= \int E\,da = E\int da = EA$$

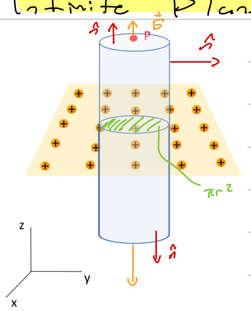
The magnitude is constant for all points on SIDE

$$\Phi_E = E \cdot 2\pi r \ell$$

$$E \cdot 2\pi r \cancel{\ell} = \frac{\lambda \cancel{\ell}}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Infinite Plane





Also works for this and smaller

Direction of Electric field?

$A = \pi r^2$

$$\sigma = \frac{Q}{L^2} \qquad \oint \hat{n}\vec{E}\,da = \frac{Q_{ENC}}{\varepsilon_0} \qquad Q_{ENC} = \sigma\pi r^2$$

Surface charge density

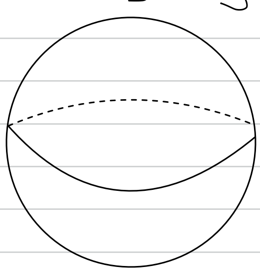
$$= \int_{Caps} \hat{n}\vec{E}\,da = 2\int E\,da = 2EA = 2E\pi r^2$$

SIDE $\hat{n} = \hat{r}$ $\theta = 90°$

CAPS $\hat{n} = \pm\hat{k}$ $\theta = 0$

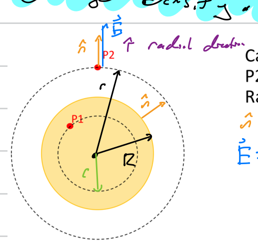
$$2E\cancel{\pi r^2} = \frac{\sigma\cancel{\pi r^2}}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{2\varepsilon_0}$$

Spherical Symmetry



Charge Density: $\rho = \frac{Q}{V}$ $\left\{\frac{C}{m^3}\right\}$

$$\oint \hat{n}\cdot\vec{E}\,da = \frac{Q_{enc}}{\varepsilon_0}$$



Calculate the Electric field at points P1 and P2 **using Gauss Law** of the sphere of Radius R and charged with a density ρ

$\hat{n} = \hat{r}$

$\vec{E} = E\hat{r}$

$$\frac{Q_{enc}}{\varepsilon_0} = \frac{\rho\frac{4}{3}\pi R^3}{\varepsilon_0}$$

Flux through gaussian surface which is outer radius sphere $\rightarrow \oint E\cos(0)\,da = Ea = E(4\pi r^2)$

$r > R$:

$$\frac{\rho\frac{4}{3}\pi R^3}{\varepsilon_0} = E(4\pi r^2) \Rightarrow \boxed{E = \frac{\rho R^3}{3\varepsilon_0 r^2}}$$

outside the sphere

enclosed radius

$r < R$: $\hat{n} = \hat{r}$, $\vec{E} = E\hat{r}$

$$\frac{Q_{enc}}{\varepsilon_0} = \frac{V\rho}{\varepsilon_0} = \frac{\frac{4}{3}\pi r^3\rho}{\varepsilon_0}$$

Gaussian surface is r

$$\oint E\,da = EA = E(4\pi r^2)$$

$$\frac{\rho\frac{4}{3}\pi r^3}{\varepsilon_0} = E(4\pi r^2) \Rightarrow \boxed{E = \frac{\rho r}{3\varepsilon_0}}$$

inside the sphere

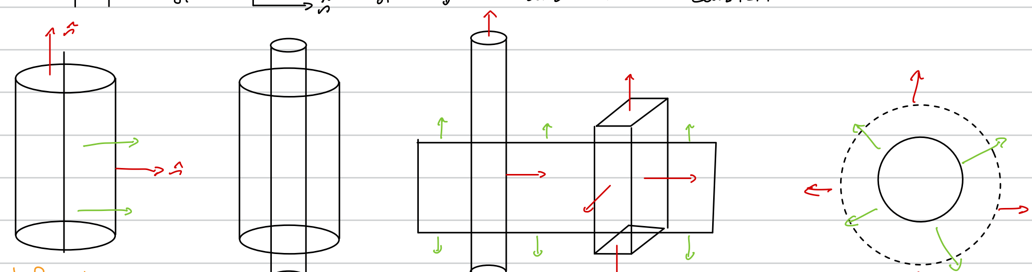
Any charge outside the gaussian surface does not affect the flux

Exam 1 Review Lecture

Gauss Law

$$\oint \vec{E}\,\hat{n}\,da = \oint \vec{E}\cdot d\vec{a}$$

$\vec{E} \uparrow \uparrow \hat{n}$ or $\vec{E} \perp \hat{n}$ or $\vec{E} \uparrow\downarrow \hat{n}$ and $\vec{E}$ is constant



Infinite Line   Infinite Cylinder   Infinite plane   enclosed sphere

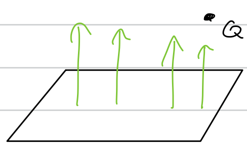
Coulombs Law

- $\vec{F} = Q\vec{E}$
- E by point q: $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

} If you have only two charges: $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$

But if not 2 point-like:

Q's $\vec{F}$?

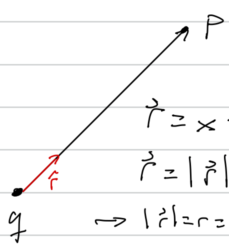


$$\vec{E} = \frac{\sigma}{2\varepsilon_0} \rightarrow \vec{F} = Q\frac{\sigma}{2\varepsilon_0}$$

$\hat{r}$ vs. $\vec{r}$

charge densities

$\lambda = \frac{C}{m}$ linear $\sigma = \frac{C}{m^2}$ surface $\rho = \frac{C}{m^3}$ volume



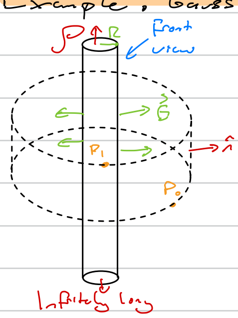
$\vec{r} = x\hat{\imath} + y\hat{\jmath}$

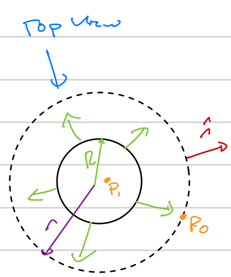
$\vec{r} = |\vec{r}|\cdot\hat{r}$

$\rightarrow |\vec{r}| = r = \sqrt{x^2+y^2}$ magnitude

Direct Integration $\rightarrow \hat{r} = \frac{x}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{y}{\sqrt{x^2+y^2}}\hat{\jmath}$ direction

Example: Gauss Law





$$\oint \vec{E}\cdot\hat{n}\,da = \frac{Q_{enc}}{\varepsilon_0} = \frac{\pi R^2 l \cdot \rho}{\varepsilon_0}$$

$\vec{E} = E\hat{r}$

Sides: $\hat{n} = \hat{r}$

CAPS: $\hat{n} = \pm\hat{k}$

$$= 2\int_{CAPS} E\cos(90)\,da + \int_{S.DES} E\cos 0\,da$$

area of sides

$$= 0 + Ea = E(2\pi r l)$$

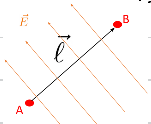
$$\Rightarrow \boxed{E = \frac{R^2\rho}{2r\varepsilon_0}}$$ outside the cylinder

← Cannot depend on parameter of Gauss surface (l in this case)

Midterm Exam 2

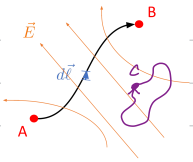
Circulation Law

Line Integrals



Straight line & uniform $\vec{E}$

- Line Integral $= \vec{\ell} \cdot \vec{E}$ for $\vec{\ell}$ the displacement vector
  - ↳ For this simple application (straight line & uniform $\vec{E}$)



General

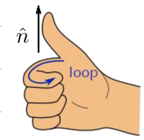
- Line integral $= \int_A^B \vec{E} \cdot d\vec{\ell}$

Closed Loops

- Circulation: $\oint \vec{E} \cdot d\vec{\ell}$

Right Hand Rule

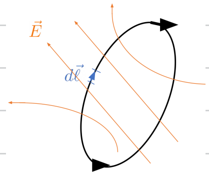
- Stokes convention
- Use direction of loop to determine $\hat{n}$ direction using:



Circulation

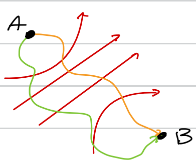
- The circulation of the electric field equals zero

→ Circulation $= \oint d\vec{\ell} \cdot \vec{E} = 0$
(electrostatics)



- Alternative: The line integral between two points do not depend on path

$$\int_A^B \vec{E} \cdot d\vec{\ell} = -\int_B^A \vec{E} \cdot d\vec{\ell} \left\{ \int_A^B \vec{E} \cdot d\vec{\ell} \right\}$$

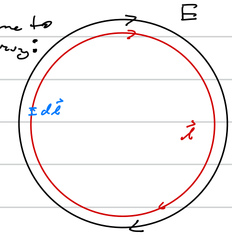


$$\Rightarrow \oint \vec{E} \cdot d\vec{\ell} = \int_A^B \vec{E} \cdot d\vec{\ell} + \int_B^A \vec{E} \cdot d\vec{\ell} = 0$$

Choose the path that is most easy to integrate

No closed Loops! (Proof by Contradiction)

Assume to the contrary:



So, $\oint \vec{E} \cdot d\vec{\ell} = \oint |\vec{E}||d\vec{\ell}|\cos$

>0 strictly

$\theta = 0$ $\quad = \oint |\vec{E}||d\vec{\ell}|$

$= (+) > 0$

No closed loops! but $\oint \vec{E} \cdot d\vec{\ell} = 0$ by circ. law

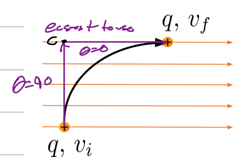
Application — P121

- Recall: Work-Energy Theorem

$$\int_{i \to f} d\vec{\ell} \cdot \vec{F}_{Total} = \Delta K = \frac{1}{2} m (v_f^2 - v_i^2)$$

- But for electric force we don't need the path!

$$\rightarrow \int_{i \to f} d\vec{\ell} \cdot (q\vec{E}) = \Delta K = \frac{1}{2} m (v_f^2 - v_i^2) = \text{Same for all paths btwn } i \text{ and } f$$

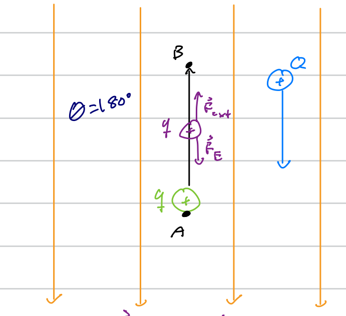


Example

$$\int_A^B \vec{E} d\vec{\ell} = \int_A^C \vec{E} d\vec{\ell} + \int_C^B \vec{E} d\vec{\ell}$$
$$= \int_A^C E d\ell \cos 90 + \int_C^B E d\ell \cos 0$$
$$= E \int_C^B d\ell = E \cdot L$$

Electrostatic Energy

Overview / Introduction



$\vec{E} = E(-\hat{y})$

$\vec{F} = Q\vec{E} = QE(-\hat{y})$

- Apply force to bring $+q$ to B from A
- Counteracts $\vec{E}$
- Release q at B → moves down

$\vec{F}_{ext} = -\vec{F}_E$ by N2L

Assumption: We move the charge very slowly, so the initial, final, and intermediate velocity is zero and hence there is No Kinetic Energy

↳ Spend an infinite amount of time from A to B $(v = \frac{\Delta x}{\Delta t})$

→ Energy Input = Work done by external agent $= \int_A^B \vec{F}_{ext} \cdot d\vec{l}$

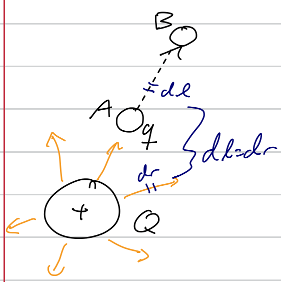
Using our assumption

- The external force matches the electric force: $U_B - U_A = \Delta U = \int_A^B d\vec{l} \cdot \vec{F}_{ext} = -\int_A^B d\vec{l} \cdot (q\vec{E})$ ($\vec{F}_{ext} = -\vec{F}_E$; $\vec{F}_E$)

  ↳ In this simple (uniform $\vec{E}$ ✓) case: $U_B - U_A = \Delta U = qE(z_B - z_A)$ wow! GPE $(U_G = mgy)$

This is a particular formula, not general!

Point Charge



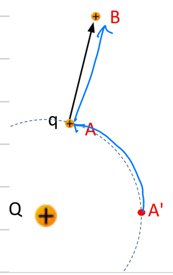
$$\Delta U = -q\int_A^B d\vec{l} \cdot \vec{E} = -q\frac{Q}{4\pi\varepsilon_0}\int_A^B \frac{\hat{r}\cdot\hat{r}}{r^2}\, d\ell = -\frac{qQ}{4\pi\varepsilon_0}\int_A^B \frac{d\ell}{r^2} = \frac{-qQ}{4\pi\varepsilon_0}\int_A^B \frac{dr}{r^2}$$

$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$

$d\vec{\ell} = \hat{r}\, d\ell$

$$\Rightarrow \Delta U = \frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

If $r_A \to \infty$ chosen as reference: $U = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r}$



$\Delta U_{AB} = \Delta U_{A'B}$

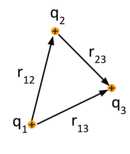
choose path in green

→ along dashed line

integral is 0 as you

move tangent

Arrangement of Charges



- The total energy of the system equals the sum of the energy of each pair of charges

$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right)$$

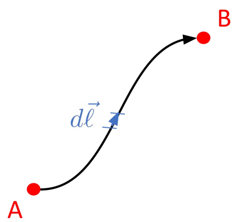
Electrostatic Potential

- Defined Φ, also seen as V
- Do not confuse w/ electrostatic energy (U)
- Another way to describe an electric field
- Electrostatic potential is a scalar (not a vector)
- To know $\vec{E}$ or Φ for any position space provides same amount of info (only electrostatics)
- If you know one you can figure out the other
- Voltage $V = \Delta\Phi$

Examples of... Scalar Fields: • Elevation • Temperature
Vector Fields: • Gravity • Wind

$\vec{E}(r) \rightleftarrows \Phi(r)$ ← not discussed here; not easily but derivable

Definition

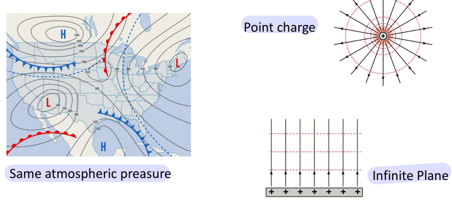


- $\Delta\Phi = \Phi(B) - \Phi(A) \equiv -\int_A^B d\vec{\ell}\cdot\vec{E}$
- If we set a reference point ($\Phi = 0$)

$$\Phi(\vec{r}) \equiv -\int_{ref\to\vec{r}} d\vec{\ell}\cdot\vec{E} \qquad \Phi(ref) = 0$$

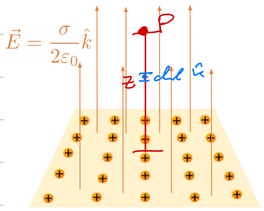
Scalar Quantity - Use Equipotential Lines

Equipotential lines



- Always perpendicular to electric lines

Example: Potential - Infinite Plane

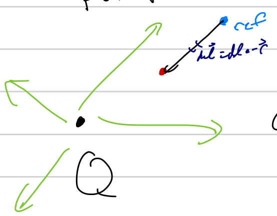


- Choose $z = 0$ so $\Phi(z=0) = 0$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}(\pm\hat{k}) \quad \text{(from gauss law)}$$

$$\Delta\Phi = \cancel{\Phi_{ref}} - \Phi_P = -\left(-\int_{ref\to P} d\vec{\ell}\cdot\vec{E}\right) = \int E\,d\ell\cos\theta = E\int_{z_P}^{ref=0} dz = Ez = \boxed{-\frac{\sigma z}{2\varepsilon_0}}$$

Example: Potential Point Like Charge



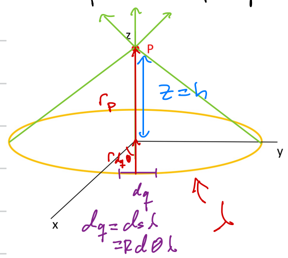
$$\Phi(r\to\infty) = 0$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

$$\Phi(r) = -\int d\vec{\ell}\cdot\vec{E} = -\int \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}\cdot\hat{r}\,dr$$

$$\boxed{\Phi(r) = \frac{Q}{4\pi\varepsilon_0 r}}$$

Example: Superposition Principle ($\vec{E}$ vs. Φ)



a) Develop the field @ P

Should point up

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{z^2+R^2}^3}\left(z\hat{k} - R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\right)$$

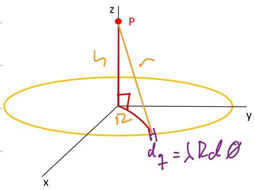
$$\left.\begin{aligned}\vec{r}_P &= z\hat{k}\\ \vec{r}_{dq} &= R\cos\theta\,\hat{\imath} + R\sin\theta\,\hat{\jmath}\end{aligned}\right\}\ \vec{r} + \vec{r}_{dq} = \vec{r}_P \to \vec{r} = z\hat{k} - R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$

$$\hookrightarrow r = \sqrt{z^2 + R^2(1)}$$

$$\Rightarrow \vec{E} = \frac{R\lambda}{4\pi\varepsilon_0(z^2+R^2)^{3/2}}\int_0^{2\pi}\left(z\hat{k} - \cancel{R\cos\theta\,\hat{\imath}} - \cancel{R\sin\theta\,\hat{\jmath}}\right)d\theta \to \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\pi R\lambda z}{(z^2+R^2)^{3/2}}\hat{k}}$$

$\cos\theta$ and $\sin\theta$ over 0 to 2π = 0

b) Develop potential without using $\vec{E}$



$\Phi(P) = ?$

$d\Phi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$

Its easier, in general, to find Φ

$$\Phi(h) = \int_0^{2\pi} \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\, d\theta}{\sqrt{R^2+z^2}} \Rightarrow \boxed{\Phi(h) = \frac{\lambda R}{2\varepsilon_0}\frac{1}{\sqrt{R^2+h^2}}}$$

$r = \sqrt{R^2+h^2}$

Converting between Φ and $\vec{E}$ ← outside of scope of course

$$\vec{E} \xrightarrow{\Phi = -\int d\vec{l}\cdot\vec{E}} \Phi \qquad \Phi \xrightarrow{\vec{E} = -\nabla\Phi} \vec{E}$$

The gradient
Omg calc 3!

$$\vec{E} = -\nabla\Phi = -\left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right)$$

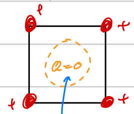
Conductors

Perfect Conductor

Ideal, doesn't exist in nature

- Electric Field inside is zero
  - If $E \neq 0$, charges rearrange themselves so $E = 0$ — instantaneous for perfect conductors
  - After some time, real conductors behave this way — time is a factor for real conductors (low amt)
- In the surface (outside), the Electric field is perpendicular to the surface of the conductor

→ All net charges reside in the surface



$Q_{enc} = \varepsilon_0 \oint d\vec{a} \cdot \vec{E} = 0$

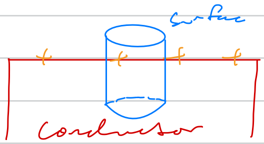
Enclosed charge is 0, and all charge is in surface

$Q = Q_+ - Q_- = 0$ (0 charge, but particles exist inside)

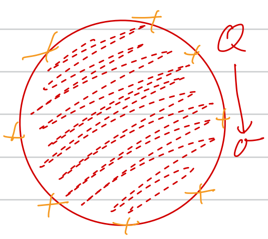
Charge Density

$$\oint d\vec{a} \cdot \vec{E} = \int_{\uparrow} E\, da + \int_{\downarrow} 0 + \int_{side} (0) = E A_{cap}$$

inside conductor; $\hat{n}$ horiz, $\vec{E}$ is vertical $\Big\}\ \theta = 90°$

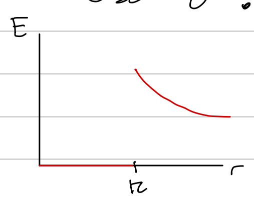


$$\longrightarrow E \cancel{A_{cap}} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma \cdot \cancel{A_{cap}}}{\varepsilon_0} \quad \text{so} \quad \boxed{E = \frac{\sigma}{\varepsilon_0}, \quad \sigma = E\varepsilon_0}$$

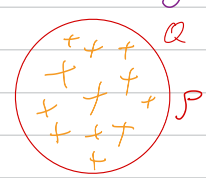


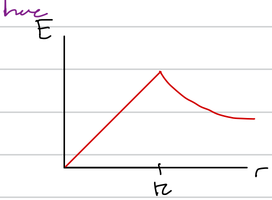
If the sphere is a perfect conductor, all charges are on the surface!

Use σ!

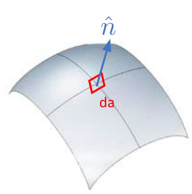


Different from insulators worked w/ previously as they had charges evenly distributed throughout volume; we use ρ here





Force on the Conductor



Force: $\vec{F} = qE$ (known)

$\rightarrow d\vec{F} = \frac{1}{2}\sigma\, da\, E\, \hat{n}$

$\vec{E} = E\hat{n}$ ← outside conductor

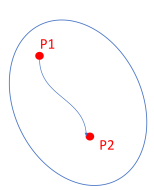
$\frac{1}{2}E$ ← on surface? average

$E = 0$ ← inside conductor

$$\vec{F} = \frac{1}{2}\int \sigma\, E\, \hat{n}\, da \quad \left(\text{use } \sigma = E\varepsilon_0, \text{ or } E = \frac{\sigma}{\varepsilon_0}\right)$$

$$= \frac{1}{2}\varepsilon_0 \int E^2 \hat{n}\, da = \frac{1}{2\varepsilon_0}\int \sigma^2 \hat{n}\, da$$ ← always positive (outwards)

Potential inside conductor



$$\Delta\Phi = \int d\vec{l} \cdot \vec{E} = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$$

→ but $E = 0$ inside conductor $\therefore \Delta\Phi = 0$

So, conductors are equipotential surfaces

Watch out!

$\Delta\Phi = \Phi_B - \Phi_A = 0$

$\rightarrow \Phi_A = \Phi_B \neq 0$ ← just their difference

Had to use this on final last year

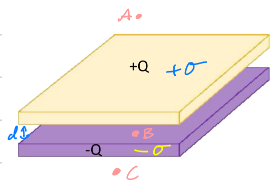
Capacitance

• Charge Q is proportional to potential difference (often called voltage) between two electrodes by a quantity called Capacitance

$Q = CV$ where $Q \equiv$ Charge (coulombs, C)
$C \equiv$ Capacitance (Farads, F)
$V \equiv$ Voltage (Volts, V)

Convention: Use volts instead of potential difference (equiv to $\Delta\Phi$)

Parallel plates



Assume Area $A \gg d^2$

$C = \frac{Q}{V} = \frac{\sigma A}{}$

$V = \Delta\Phi = -\int_1^2 d\vec{l}\cdot\vec{E}$

Q: What's $\vec{E}$ at all 3 Points?

above (+) • A: $\uparrow \frac{\sigma}{2\varepsilon_0}$, $\downarrow -\frac{\sigma}{2\varepsilon_0}$ $\longrightarrow E = 0$

$+\sigma$

btwn • B: $d\updownarrow$ $\downarrow 2\times\frac{\sigma}{2\varepsilon_0}\downarrow$ $\longrightarrow E = \frac{\sigma}{\varepsilon_0}$

$-\sigma$

below (−) • C: $\downarrow -\frac{\sigma}{2\varepsilon_0}$, $\uparrow \frac{\sigma}{2\varepsilon_0}$ $\longrightarrow E = 0$

Derive from Gauss Law (previous lecture)

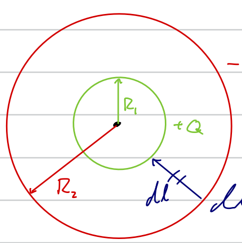
Use $\vec{E}$ @ all 3 points to get C of parallel plate capacitors?

$C = \frac{Q}{|V|} > 0$, $|V| = \Delta\phi$

$$\Delta\phi = -\int_{P_1\to P_2} d\vec{l}\cdot\vec{E} = -\int_{P_1\to P_2} dl\, E\cos 180 = E\int_0^d dl = d\left(\frac{\sigma}{\varepsilon_0}\right), \quad Q = \sigma A$$

So, $\boxed{C = \frac{\sigma A}{\frac{d\sigma}{\varepsilon_0}} = \frac{\varepsilon_0 A}{d}}$

Example



$dl = -dr$

$C = \frac{Q}{V}$, $V = \Delta\phi = -\int_{-\to+} \vec{E}\cdot d\vec{l}$

① Determine $\vec{E}$ in 3 points (only really need to be btwn ⊖ and ⊕, just doing this for practice)

$\vec{E}_{inside} = 0$ $Q_{enc} = 0$

$\vec{E}_{btwn}$: $Q_{enc} = Q$

$\frac{Q_{enc}}{\varepsilon_0} = E a \longrightarrow E = \frac{Q}{4\pi r^2 \varepsilon_0}$ Area of Gaussian surface

• spherical & hollow shells
• Conductive materials
• Determine Capacitance (C)

$\vec{E}_{outside} = 0$ $Q_{enc} = Q - Q = 0$

② Determine $\Delta\phi$

$$\Delta\phi = -\int_{-\to+} E\, dl \cos 180 = \int \frac{Q}{4\pi r^2\varepsilon_0}\, dl = -\int_{R_2}^{R_1} \frac{Q}{4\pi r^2 \varepsilon_0}\, dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$\Rightarrow \boxed{C = \frac{Q}{V} = 4\pi\varepsilon_0\left(\frac{1}{R_1} - \frac{1}{R_2}\right)^{-1}}$

Exam 2 Review Lecture

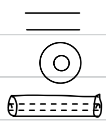
- Dielectrics will not be on the homework

3 Key Topics
- Potential Energy
- Electric Potential
- Capacitance

including all from Exam 1

Capacitance

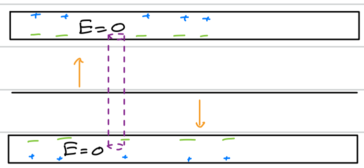
$$C = \frac{Q}{V} \begin{cases} \varepsilon_0 \frac{A}{d} \\ \\ \\ \end{cases}$$



Parallel plate

$\sigma_u = \sigma_l$ by symmetry

Infinite length σ



Conductor Slab

Conductor Slab

$E = \frac{\sigma}{2\varepsilon_0}$

$$2\int_{caps} \vec{E} \cdot d\vec{a} + \int_{side} 0 = 0 = \frac{A(\sigma + 2\sigma_u)}{\varepsilon_0}$$

$\hat{n} \perp \vec{E}$

$$\sigma + 2\sigma_u = 0$$

$$\boxed{\sigma_u = \sigma_l = -\frac{\sigma}{2}}$$

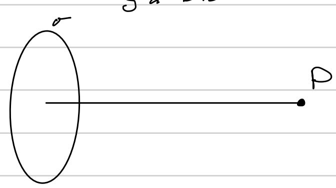
Question 2 on hw 5, I got it right!

Outside edges of slabs should be $+\frac{\sigma}{2}$ to keep conductors neutral

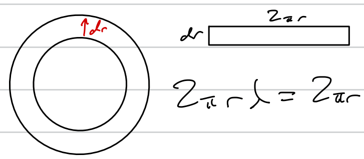
$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} = \frac{A(\sigma + \sigma_u + \sigma_l)}{\varepsilon_0} = \frac{A(\sigma + 2\sigma_u)}{\varepsilon_0}$$

Potential by a Disk HW5 Q1

σ



$$\Phi_{ring} = \frac{1}{4\pi\varepsilon_0} \frac{\overbrace{\lambda(2\pi r)}^{q}}{\sqrt{r^2 + z^2}} \Rightarrow \int_0^R \frac{1}{4\pi\varepsilon_0} \frac{2\pi r \sigma \, dr}{\sqrt{r^2 + z^2}}$$



$2\pi r \lambda = 2\pi r \, dr \, \sigma$

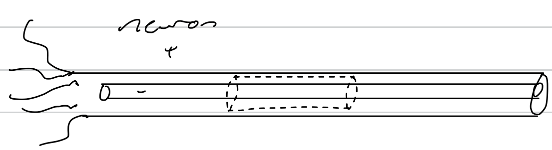
Don't expect something like this on the exam

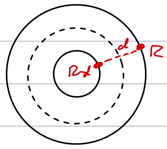
Potential & Energy

$$\left.\begin{aligned} \Delta\phi &= -\int \vec{E} \cdot d\vec{l} \\ \Delta U &= -Q\int \vec{E} \cdot d\vec{l} \end{aligned}\right\} \Delta U = Q\Delta\phi$$

Example Regarding Capacitance

neuron





$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

$$E(2\pi r L) = \frac{Q_{enc}}{\varepsilon_0}$$

$$E = \frac{Q}{2\pi r L \varepsilon_0}$$

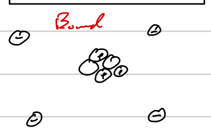
$$V = \Delta\phi = \int_{R-d}^{R} \frac{Q}{2\pi\varepsilon_0 r L} dr \propto \log\left(\frac{R}{R-d}\right)$$

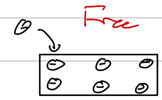
Midterm Exam 3

Dielectrics

- Dielectrics = Insulators
  - examples: Glass, Plastic, Water, Diamond
- charges stay put

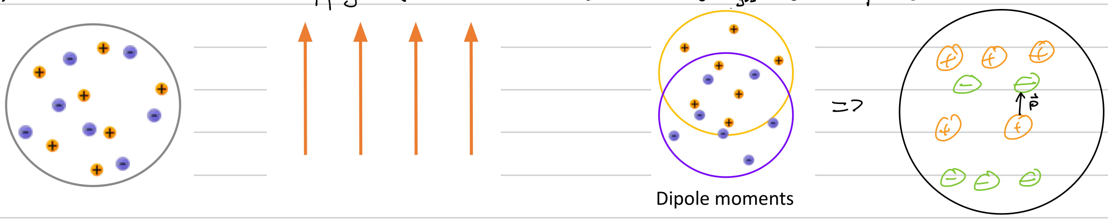
Internal charge a.k.a. Bound Charge vs. External charge a.k.a. Free Charge or Fields





Response of Internal Charges to Fields

Atom of Dielectric → Apply electric Field → Bound charges are displaced

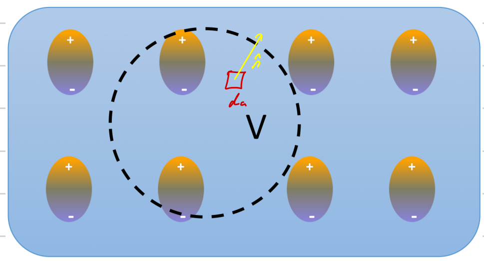


Displaced densities in the atom

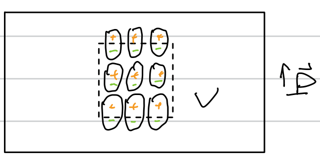
Dipole

$\ominus \xrightarrow{\vec{d}} \oplus$

- $\vec{p} = q\,\vec{d}$
- Polarized medium
  → Polarization vector $\vec{P}$
  - Dipole Moment Density: $\vec{P} = \frac{\sum \vec{p}}{V}$ ← add all, ← volume



How much bound charge inside V?



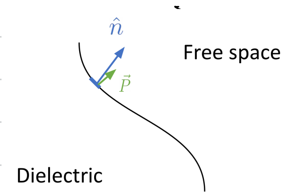
- $\vec{P} \equiv$ uniform, $Q_B(V) = 0$
- Answer depends on $\vec{P}$ at the surface $\quad Q_B \propto \vec{P}_{surface}\ V$
- If $\vec{P}$ is parallel to the surface, no net charge

$$Q_{bound} = Q_B(V) = -\oint_V da\,\hat{n}\cdot\vec{P} = -\oint |\vec{P}|\,|d\vec{a}|\cos\theta$$

- Only for Dielectrics
- Not Gauss' Law, this is: $Q_{bound} + Q_{free} = \varepsilon_0 \oint da\,\hat{n}\cdot\vec{E}$ — Lone Charge

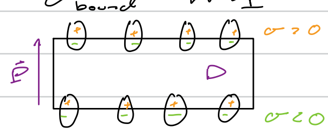
$$Q_{free} = \oint da\,\hat{n}\cdot\underbrace{(\varepsilon_0\vec{E} + \vec{P})}_{\vec{D}}$$

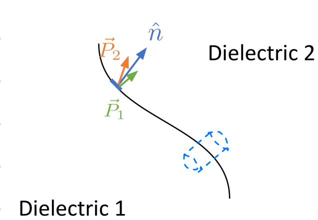
Surface Charge Density



One dielectric

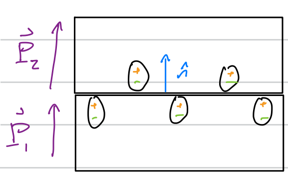
$$\sigma_{bound} = \hat{n}\cdot\vec{P}$$



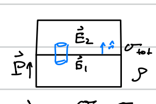


Two dielectrics

$$\sigma_{bound} = \hat{n}\cdot\vec{P}_1 - \hat{n}\cdot\vec{P}_2$$



Gauss Law to get σ_{total}



$\vec{E}_i = 2\frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$

$\sigma = \varepsilon_0 E_i$

$$\sigma_{total} = -\varepsilon_0\,\hat{n}\cdot(\vec{E}_1 - \vec{E}_2)$$

$$\sigma_{free} = \sigma_{total} - \sigma_{bound}$$

$$\sigma_{free} = \hat{n}\cdot\underbrace{(\varepsilon_0\vec{E}_2 + \vec{P}_2)}_{\vec{D}_2} - \hat{n}\underbrace{(\varepsilon_0\vec{E}_1 + \vec{P}_1)}_{\vec{D}_1}$$

Linear Isotropic Dielectrics

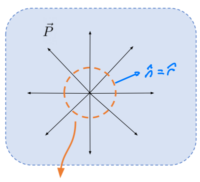
$\vec{P} \propto \vec{E}^{1}$ ← Simplest case

proportionality depends on the material's temperature and pressure $\Rightarrow$ $\vec{P} = \chi \varepsilon_0 \vec{E}$

Generally $\vec{P} \propto \vec{E}^{x}$ (to a given power)

χ: Electric Susceptibility

Example



• Assume $\vec{P} = \frac{A}{r^2}\hat{r}$ (A: constant)

• What is the bound charge inside a sphere with radius r?

$$Q_{bound} = \underbrace{\cancel{Q_B(V)}}_{\text{uniform } \vec{P}} - \oint da\, \hat{n}\cdot\vec{P} = 4\pi r^2 P\cos(0)$$

$$Q_{bound} = -4\pi r^2\left(\frac{A}{r^2}\right) = -4\pi A$$

$$\boxed{Q_{bound} = -4\pi A}$$

Note: Q_{bound} independent of r, in fact it's a constant value

↳ Why? All the charges at the center

• What is the electric field?

$$\vec{E} = \frac{\vec{P}}{\chi\varepsilon_0} = \frac{\frac{A}{r^2}\hat{r}}{\chi\varepsilon_0} = \frac{1}{\chi\varepsilon_0}\frac{A}{r^2}\hat{r}$$

$$\boxed{\vec{E} = \frac{1}{\chi\varepsilon_0}\frac{A}{r^2}\hat{r}}$$

• What is the total charge inside a sphere with radius r

$$\frac{Q_{enc}}{\varepsilon_0} = \oint \vec{E}\cdot d\vec{a} \Rightarrow Q_{enc} = \varepsilon_0 E \oint da = \varepsilon_0 E\, 4\pi r^2 = \cancel{\varepsilon_0}\frac{1}{\chi\cancel{\varepsilon_0}}\frac{A}{\cancel{r^2}}4\pi\cancel{r^2} = \frac{4\pi A}{\chi}$$

$$\boxed{Q_{total} = \frac{4\pi A}{\chi}}$$

• What is the external (free) charge inside a sphere with radius r?

$$Q_{free} = Q_{total} - Q_{bound} = 4\pi A\,\frac{1+\chi}{\chi}$$

$$\boxed{Q_{free} = 4\pi A\,\frac{1+\chi}{\chi}}$$

You might see $1+\chi$ as ε_r (relative permittivity) or the dielectric constant Kappa (K)

• What is the electric field as a function of free charge?

$$\vec{E}\left(Q_F = 4\pi A\frac{1+\chi}{\chi}\right)?\quad A = \frac{\chi Q_F}{4\pi(1+\chi)} \Rightarrow \vec{E} = \frac{1}{\cancel{\chi}\varepsilon_0 r^2}\left(\frac{\cancel{\chi} Q_F}{4\pi(1+\chi)}\right)\hat{r}$$

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0(1+\chi)}\frac{Q_F}{r^2}\hat{r}}$$

Basically coulombs law, but with $(1+\chi)$ factor.

We will define $K = 1+\chi$ (Kappa)

Parallel Plate Capacitor

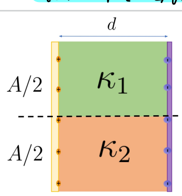
• Capacitance is affected: $C = K C_{air} = (K\varepsilon_0)\frac{A}{d}$

Material	Dielectric constant, K
Air	1(0.85-1.21 at 20°C)*
Water	80 at 20°C (79.50-80.24)*
Ice	3 at -5°C*
Basalt	12
Granite	7-9
Sandstone	9-11
Dry loam	3.5
Dry sand	2.5 (2.5-3.5 at 20°C)*
Ethanol liquid	15.21 at 20°C*
Ethanol Mixing Liquid	39.82 at 20°C*
Acetone	21.20 at 20°C*

*measured dielectric constant in this study

What is the new Capacitance with two dielectrics?

In Parallel



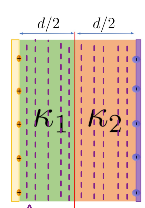
$C_1 = \frac{Q_1}{V}$, $C_2 = \frac{Q_2}{V}$ for constant V $(V = Ed)$

Gauss Law: $E\frac{A}{2} = \frac{Q_i}{K_i\varepsilon_0} \Rightarrow Q_i = (K_i\varepsilon_0)E\frac{A}{2}$

$$\therefore\ C_i = \frac{K_i\varepsilon_0}{2}\frac{A}{d}$$

$$\Rightarrow C = \frac{Q_1 + Q_2}{V} = \frac{K_1 + K_2}{2}\varepsilon_0\frac{A}{d} = C_1 + C_2$$

In Series



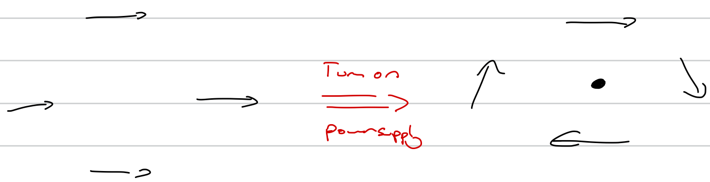
Equipotential Lines

$$\Delta\phi = \int \frac{\sigma}{K\varepsilon_0}dl = \frac{\sigma}{K\varepsilon_0}\int dl = \frac{\sigma}{K\varepsilon_0}y$$

$$\boxed{C = \frac{Q}{V_1+V_2} = \left(\frac{V_1+V_2}{Q}\right)^{-1} = \left(\frac{1}{C_1}+\frac{1}{C_2}\right)^{-1}}$$

Magnetic Fields

Demo w/ Compasses



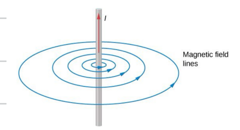
Electric Current produces a magnetic field which is circular as seen by the direction of the compass arrows

Magnetic Fields

- Moving electric charges produce magnetic fields
  - ↳ there are no "magnetic charges"
- Magnetic fields exert forces on moving electric charges

Note: Moving charges still produce electric fields!

Law of Magnetostatics

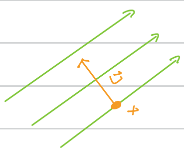


- Electric currents (or just currents) produce magnetic fields $\vec{B}$
- $\vec{B}$ exert forces on moving charges

Definition of Magnetostatics: Assume that there is no accumulation of charges, but the charges can move

Lorentz Force

- The force must depend on the charge, velocity, and field

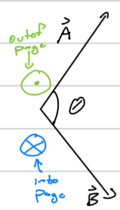


$$\vec{F} = \vec{F}(q, \vec{B}, \vec{v})$$

- The force is the vector product of $\vec{v}$ and $\vec{B}$

$$\vec{F} = q\,\vec{v} \times \vec{B}$$

Recall - Cross Product



- The cross product of two vectors is a vector
  $\vec{A} \times \vec{B} = \text{vector}$
- Its magnitude:
  $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|\sin(\theta)$
- Direction:
  $\vec{A} \times \vec{B}$ is perpendicular to $\vec{A}$ and $\vec{B}$

Currents

- Which charges move?
  - → Metals ⟹ Negative charges move
  - → Semiconductors ⟹ Positive charges move

We will assume that positive charges move!

$I \equiv$ Current flowing through surface in direction

$$I \equiv \frac{\Delta Q}{\Delta t}$$

ΔQ → charge that cross surface; Δt → during a certain time

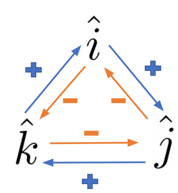
Units $\frac{\text{Coulombs}}{\text{Second}} = \text{Ampere}$

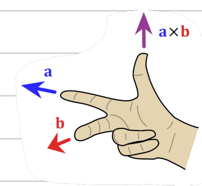
$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$\vec{A} = 3\hat{i} + 4\hat{j}$

$\vec{B} = 6\hat{i} + 7\hat{k}$

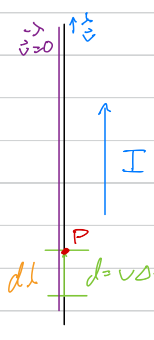
$\vec{A} \times \vec{B} = 28\hat{i} - 21\hat{j} - 24\hat{k}$





Currents: Line Wire

- Assume an ∞ wire with infinitesimal thickness
  - ↳ current I means
    - → I coulombs going up
    - → $-I$ coulombs going down
- All charges (+) in the interval $v\Delta t$ will cross P in a time Δt
- The current: $I = \frac{\Delta Q}{\Delta t} = \frac{\lambda v \Delta t}{\Delta t} = \lambda v$



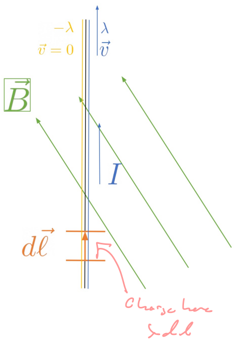
Coulomb vs. Lorentz

→ $F_E = q\vec{E}$

→ $F_B = q\,\vec{v} \times \vec{B}$

↳ Do cross product, then deal with q's sign

Lorentz Force on a Current



• Force on a piece of wire $d\ell$

$$d\vec{F} = (\lambda\, d\ell)\,\vec{v}\times\vec{B} \qquad d\vec{F} = (\lambda\, d\vec{\ell}\, v)\times\vec{B}$$

$\leftarrow q = \lambda \cdot d\ell$

Note: $d\vec{\ell}\, v = d\ell\, \vec{v}$

• In terms of current

$$\boxed{d\vec{F} = I\, d\vec{\ell}\times\vec{B}}$$

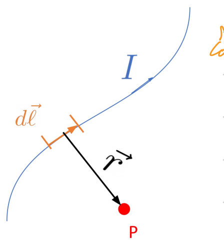
Note: $I = \lambda v$ (see previous!)

• Integrate to solve for $\vec{F}$

$$\boxed{\vec{F} = I\int d\vec{\ell}\times\vec{B} = I\int |d\vec{\ell}|\,|\vec{B}|\sin\theta}$$

for θ btwn $d\vec{\ell}$ and $\vec{B}$

Biot-Savart: Field Created by a current



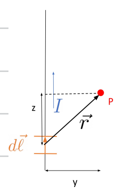
• Magnetic field at Point P due to $d\ell$

New Constant

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{\ell}\times\vec{r}}{r^3}$$

$$= \frac{\mu_0}{4\pi} I \frac{d\vec{\ell}\times\hat{r}}{r^2}$$

Example: Infinite wire



$\vec{r} = y\hat{j} + z\hat{k} \rightarrow r = \sqrt{y^2+z^2}$

$d\vec{\ell} = d\ell\,\hat{k}$

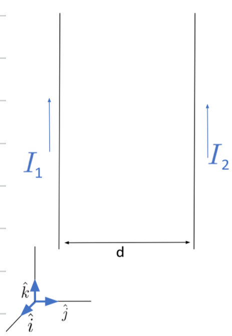
$d\vec{\ell}\times\vec{r} = -y\,d\ell\,\hat{i}$

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{-y\,d\ell\,\hat{i}}{(z^2+y^2)^{3/2}}$$

$$\vec{B} = \int_{-\infty}^{\infty} \frac{\mu_0}{4\pi} I \frac{-y\,d\ell\,\hat{i}}{(z^2+y^2)^{3/2}}$$

$$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{y}(-\hat{i})$$

Force btwn 2 infinite wires



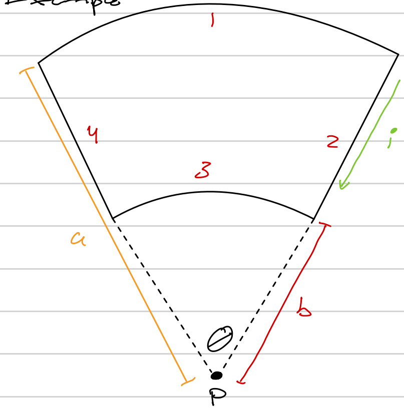
• What is the force that I_1 exert on I_2 ?

$$d\vec{F}_2 = \frac{\mu_0 I_1 I_2}{2\pi d} d\ell(-\hat{j}) = -d\vec{F}_1$$

• If the intensities are parallel the force is attractive,

• If the intensities are anti-parallel the force is repulsive

Example



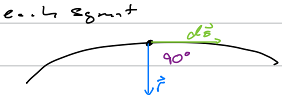
Find Magnetic Field @ P

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^3}$$

(i) Determine direction of $d\vec{s}$ for each segment

a) Angles btwn $d\vec{s}$ and $\vec{r}$ for each segment

→ Segment 1

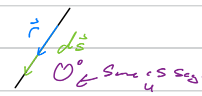


Angle btwn $d\vec{s}$ and $d\vec{r}$?

$|d\vec{s} \times \vec{r}| = |d\vec{s}||\vec{r}|\sin\theta$ $\theta = 90°$ by symmetry, true is the same for seg 3

→ Segment 2

Angle btwn $d\vec{s}$ and $\vec{r}$ in segment 2



$dB_2 = dB_4 = 0$

b) Direction of Segments w/ RHR

① $\vec{B} = B \cdot \otimes$

② $\vec{B} = B \cdot \odot$

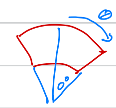
b) Integration – Bad, don't follow

→ How is ds related to $d\theta$

$ds = r\,d\theta$

→ What is the position vector $\vec{r}$

$\vec{r} = -r\sin\theta\hat{i} - r\cos\theta\hat{j}$



not recommended to parametrize

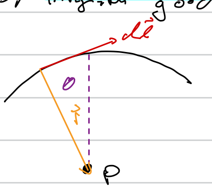
$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{s} \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi} I\, r \frac{d\theta\, \vec{r}}{r^3}$$

$$\int_{-\theta/2}^{\theta/2} d\vec{B}_1 = \int \frac{\mu_0}{4\pi} I \frac{d\theta}{a}\left(-\sin\theta\hat{i} - \cos\theta\hat{j}\right) = \frac{\mu_0}{4\pi} \cdot \frac{I}{a} \cdot \int_{-\theta/2}^{\theta/2} -\sin\theta\hat{i} - \cos\theta\hat{j}\, d\theta$$

$$\int_{-\theta/2}^{\theta/2} d\vec{B}_3 = \int \frac{\mu_0}{4\pi} I \frac{d\theta}{b}\left(-\sin\theta\hat{i} - \cos\theta\hat{j}\right) = \frac{\mu_0}{4\pi} \cdot \frac{I}{b} \cdot \int_{-\theta/2}^{\theta/2} -\sin\theta\hat{i} - \cos\theta\hat{j}\, d\theta$$

This can be tricky to work out many pitfalls

b) Integration good



$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^3}$$

- $|d\vec{l} \times \vec{r}| = |d\vec{l}||\vec{r}|\sin 90 = dl\,r = (r\,d\theta)r$

  Direction $\otimes$ → $-\hat{k}$

- $\vec{r} = \quad \hat{i} + \quad \hat{j}$

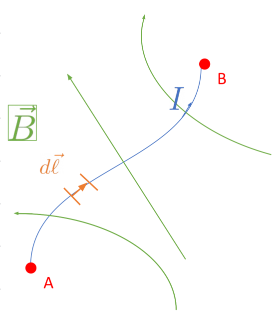
  $d\vec{l} = \quad \hat{i} + \quad \hat{j}$

$$\Rightarrow d\vec{B}_1 = \frac{\mu_0}{4\pi} I \frac{r^2 d\theta}{r^3} \otimes = \frac{\mu_0}{4\pi} I \frac{1}{r} \theta(-\hat{k}) \qquad \vec{B}_1 = -\frac{\mu_0}{4\pi} I \frac{1}{a} \theta \hat{k}$$

$$\Rightarrow d\vec{B}_2 = \frac{\mu_0}{4\pi} I \frac{r^2 d\theta}{r^3} \odot = \frac{\mu_0}{4\pi} I \frac{1}{r} \theta \hat{k} \qquad \vec{B}_2 = \frac{\mu_0}{4\pi} I \frac{1}{b} \theta \hat{k}$$

$$\boxed{\vec{B}_{tot} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0}{4\pi} I \theta \left(\frac{1}{b} - \frac{1}{a}\right)\hat{k}}$$

Acceleration and Ampere's Law



Q: Charge with a certain velocity at point A, $\vec{v}_A$, that enters a magnetic field $\vec{B}$, will suffer a force (Lorentz). At point B, it will have another velocity $\vec{v}_B$. Will: (i) $|\vec{v}_B| > |\vec{v}_A|$

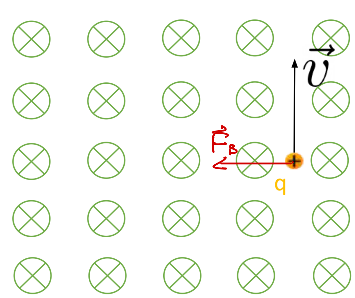
constant angle: $\vec{F}_B = q\vec{v} \times \vec{B} \Rightarrow \vec{F}_B \perp \vec{v}$

(ii) $|\vec{v}_B| = |\vec{v}_A|$ $90°$ applied → centripetal so $|\vec{v}|$ doesn't change

(iii) $|\vec{v}_B| < |\vec{v}_A|$

$\Delta K = \int_A^B \vec{F} \cdot d\vec{\ell} = \int_A^B F\, d\ell \cos 90 = 0 \rightarrow |\vec{v}_B| = |\vec{v}_A|$ (work energy theorem)

Trajectory on a Uniform Field



- Direction of Lorentz force $(\vec{F}_B = q\vec{v} \times \vec{B})$
  RHR → must point Left
- How will q move? Up and left
- Trajectory? F is a centripetal force
  N2L: $qvB \sin 90 = m\frac{v^2}{r} = \boxed{r = \frac{mv}{qB}}$

$$T = \frac{2\pi r}{v} = \frac{2\pi m}{qB}, \quad \omega = \frac{2\pi}{T} = \frac{qB}{m}$$

Time to complete a rotation — frequency — Cyclotron Frequency

Parallels

Charges produce Electric fields			
Version I		Version II	
• Coulomb´s Law	⟺	• Gauss Law	→ Flux
• Superposition Principle		• Circulation Law	→ Circulation

Moving charges produce magnetic fields			
Version I		Version II	
• Biot-Savart´s Law	⟺	• Gauss Law of Magnetism	→ Flux
• Superposition Principle		• Ampere's Law	→ Circulation

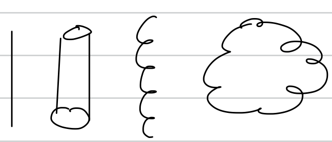
Gauss Law of Magnetism

$$\Phi_B = \oint \vec{B} \cdot d\vec{a} = \oint \hat{n} \cdot \vec{B}\, da = 0$$

where $\hat{n}$ points outside the surface by convention

Ampere's Law

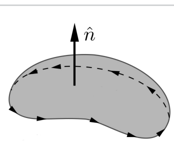
- Magnetic field across a closed loop (circulation) is proportional to the current through any surface whose boundary is the loop
- In Magnetostatics: $\oint d\vec{\ell} \cdot \vec{B} = \mu_0 I_{enc}$ — I_{enc} is any current that goes through the loop's boundary



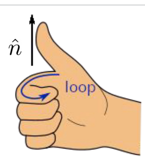
4 geometries we will apply to (may be on exam)

Stokes Convention

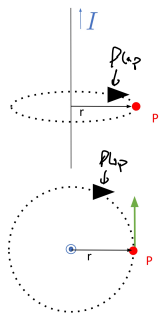
Direction of loop ⇒ direction of normal vector is fixed



⇒



Example: Magnetic Field by Infinite Line



• Choose an amperian loop passing by P

↳ Similar to Gaussian surface, result should not depend on it

• we know field depends on r (field lines are circles)

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$\int B\,dl\cos\theta = \mu_0 I$$

$$B(2\pi r) = \mu_0 I \Rightarrow \boxed{B = \frac{\mu_0 I}{2\pi r}}$$

Summary of Ampère's law (nicely written)

To find the magnetic field due to a current, Ampère's law is sometimes more efficient than the Biot-Savart law.

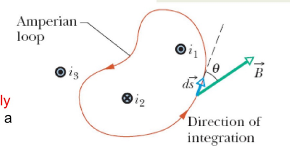
Ampère's law:

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 i_{enc}$$

current enclosed by your Ampèrian loop

infinitesimally small length vector along an imaginary loop surrounding one or more currents

Only the currents encircled by the loop are used in Ampere's law.

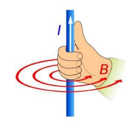


Ampère's law is always true, but it is only useful in a limited number of cases with a large amount of symmetry.

Right Hand Rule for direction of mag field

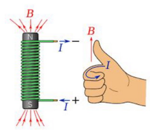
Right hand rule for the direction of the magnetic field due to a current:

1) Point your thumb in the direction of the current.
2) Your fingers curl in the direction of the magnetic field



OR

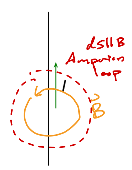
1) Curl your fingers in the direction of the current.
2) Your thumb points in the direction of the magnetic field.



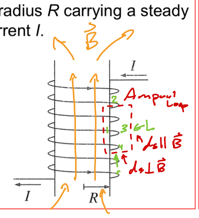
Example

Use Ampère's Law to determine the magnetic field in each of the following 3 cases:

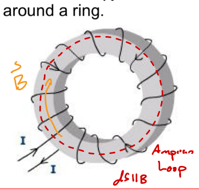
Case 1: A long straight wire carries a steady current I in the direction shown.



Case 2: A very long solenoid has n closely wound turns per unit length on a cylinder of radius R carrying a steady current I.



Case 3: A toroid consists of a long wire, carrying a steady current I, that is tightly wound N times around a ring.



N = number of loops

$n = \frac{N}{L}$ = density of loops

Caution: It's okay for $\oint \vec{B}\cdot d\vec{s} = 0$ for some points on the Ampèrian loop, but if $\oint \vec{B}\cdot d\vec{s} = 0$ for every point on the loop, then your loop will not allow you to determine the magnitude of the magnetic field.

(i) Direction of magnetic field

(ii) Draw an Amperian loop s.t. $\oint \vec{B}\cdot d\vec{s} = B\int ds$ or 0 at every point on the loop ($\theta = 0°$ or $90°$)

(iii) Choose direction to integrate around the loop and calculate $\oint \vec{B}\cdot d\vec{s}$ (circulation)

Case 1: $\oint \vec{B}\cdot d\vec{s}$ = circulation

$\theta = 0°$ $B\int r\,d\theta$

$B(2\pi r)$

Case 2: $\oint \vec{B}\cdot d\vec{s} = \left(\int \vec{B}\cdot d\vec{s}\right)_1 + \left(\int \vec{B}\cdot d\vec{s}\right)_2 + \left(\int \vec{B}\cdot d\vec{s}\right)_3 + \left(\int \vec{B}\cdot d\vec{s}\right)_4$

side 1: $B(L)$, 2: 0, 3: 0, 4: 0

$\theta = 90°$

Assume field is trapped inside, and the side is outside! ← this approximation is fine

Case 3: $B(2\pi r)$ = circulation

↑ Same result as Case 1, this is expected

(iv) Determine i_{enc}

Case 1: $i_{enc} = I$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

Case 2: $i_{enc} = I\cdot n\cdot L$

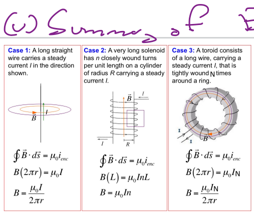
accounts for current crossing loop n times

$$\Rightarrow B = \frac{\mu_0 I n L}{L}$$

Case 3: $i_{enc} = I\cdot N$

$$B = \frac{\mu_0 I N}{2\pi r}$$

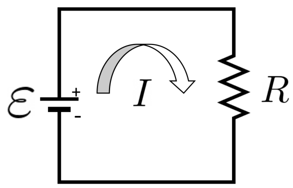
(v) Summary of Example!



Circulation Law	Gauss Law	Ampere's Law	Gauss Law of B
The circulation of E equals zero	The flux of E through a closed surface equals charge enclosed	The circulation of B through a closed loop equals current enclosed	The flux of B through a closed surface equals zero
$\oint d\vec{\ell}\cdot\vec{E} = 0$	$\oint \hat{n}\cdot\vec{E}\,da = \frac{Q}{\varepsilon_0}$	$\oint d\vec{\ell}\cdot\vec{B} = \mu_0 I$	$\oint \vec{n}\cdot\vec{B}da = 0$

but not in electrostatics, will revisit later

Faraday's Law



DC Battery

- Two terminal device
- Maintains a Voltage (potential drop) across terminals ($\mathcal{E}$) — electromotive force
- Positive terminal @ higher voltage
- Current can flow in both directions

Resistor

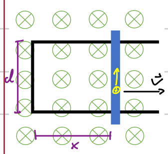
- Not a very good conductor
- Potential drop across (V)
- Current can flow in both directions

=> Ohm's Law

$V = IR$

current flows from high to lower potential

Faraday's flux rule



Here:

- ⊗ uniform magnetic field into page
- Resistor bar R of length d
- [U-shaped conductor rail

Bar is moved to the right @ velocity v

(+) in bar will feel upward force (RHR)

$|\vec{F}| = |+q\,\vec{v}\times\vec{B}| = qvB \Rightarrow \vec{F} = qvB\hat{y}$

-> Force per unit of charge

$\vec{f} = \frac{\vec{F}}{q} = vB\hat{y}$ Looks like electric field ($\vec{E} = \frac{\vec{F}}{q}$), not the same!

-> Effective voltage: $V = fd = vBd = \mathcal{E}$

↳ Known as the electromotive force

=> $I \overset{\text{ohm's law}}{=} \frac{\mathcal{E}}{R} = \frac{dvB}{R}$

-> Flux of $\vec{B}$, given $\mathcal{E} = vBd$

$$\Phi_B = \oint \vec{B}\cdot d\vec{a} = \int B\,da\cos 0 = Bda = B(xd)$$

-> Derivative over time of flux

$$\frac{d}{dt}(\Phi_B) = \frac{d}{dt}(Bdx) \Rightarrow \frac{d\Phi_B}{dt} = \underbrace{Bd}_{\text{constant}}\frac{dx}{dt} = Bdv \Rightarrow \frac{d\Phi_B}{dt} \propto \mathcal{E}$$

-> If we're more careful w/ signs, then generally: $\mathcal{E} = -\frac{d\Phi_B}{dt}$ ← should always hold

We are missing something

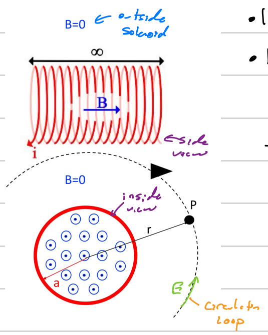
- If we move magnet and leave wire fixed, $\vec{v}=0$ but the flux changes, why?
  ↳ Can argue reference frames, but we really need to refine Faraday's Law

Faraday's Law of Induction

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \Rightarrow \oint d\vec{l}\cdot\vec{E} = -\frac{d}{dt}\int \hat{n}\cdot\vec{B}\,da$$

Faraday's Law (what used to be the circulation law)	Gauss Law	Ampere's Law	Gauss Law of B
The circulation of E equals the variation in time of the flux of B	The flux of E through a closed surface equals charge enclosed	The circulation of B through a closed loop equals current enclosed	The flux of B through a c losed surface equals zero
$\oint d\ell\cdot\vec{E} = -\frac{d}{dt}\int \hat{n}\cdot\vec{B}\,da$ (not 0 in electrostatics)	$\oint \hat{n}\cdot\vec{E}\,da = \frac{Q}{\varepsilon_0}$	$\oint d\vec{\ell}\cdot\vec{B} = \mu_0 I$	$\oint \vec{n}\cdot\vec{B}da = 0$

Electric Field Lines



- If circulation is not 0, can field lines form closed loops?
- E @ point P: $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \hat{n}\cdot\vec{B}\,da$

→ choose direction of $\vec{E}$ so that $\vec{E}$ & $d\vec{l}$ are parallel

$$\int E\,dl\cos 0 = E\int dl = E\,2\pi r$$

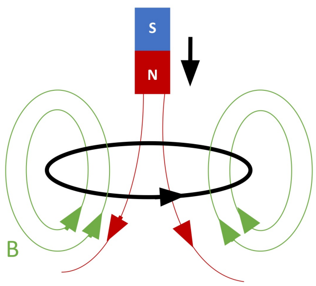
Flux only through the physical area

$$-\frac{d}{dt}\int \vec{B}\cdot d\vec{A} = -\frac{d}{dt}\left(B\cdot\pi a^2\right) = -\pi a^2\frac{dB}{dt}$$

$$E2\cancel{\pi}r = -\cancel{\pi}a^2\frac{dB}{dt} \Rightarrow \boxed{E = -\frac{a^2}{2r}\frac{dB}{dt}}$$

when $r \geq a$

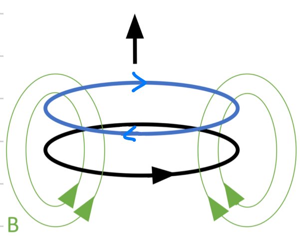
Lenz's Law (Rule)



- Derived from Faraday's Law
- Direction of the induced current (or electric field) is such that the magnetic field created by it (the current), opposes the changing flux that created it.

	Me	Faraday	B_F ? B_0
Flux	↑	↓	Anti parallel
	↓	↑	Parallel

Example:



- From I=0 we start increasing the current in the black loop.
- What is the direction of the induced current on the blue loop?
  - Opposite direction
- What is the force on the upper ring?
  - Repulsive

Three ways to induce current

→ You need to induce flux

(i) $B(t)$

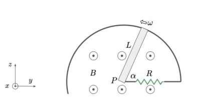
(ii) $A(t)$

(iii) $\theta(t)$

all impact Φ_B

$$\Phi_B = BA\cos\theta$$

Example w/ Poll Ev



A conducting bar of length L rotates with angular frequency ω about pivot P at one end of the bar. The other end of the bar is in slipping contact with a stationary conducting wire in the shape of a circle (we only show a small part of the circle to keep the drawing simple). Between point P and the circular wire there is a resistor with resistance R as shown. Thus the bar, the resistor, and the arc form a closed conducting loop. The resistance of the bar and the circular wire are negligibly small. There is a uniform magnetic field B everywhere, it is perpendicular to the plane of the screen as indicated.

What is the magnitude and direction of the force exerted on the conducting bar?

$\vec{F}_{Bar} \leftarrow I \leftarrow \varepsilon \leftarrow \frac{d\Phi_B}{dt}$ Steps!

$\Phi_B = \oint \vec{B}\cdot d\vec{a}$ Full circle

$= B\cdot A = B\cdot\pi L^2 \Rightarrow B\cdot\frac{\alpha}{2}L^2$

$\Phi_B = \frac{1}{2}BL^2\alpha$

What is the direction of the current induced in the conducting sector when the angle α is increased? Use Lenz law

Need to produce field ⊗, so clockwise

What is the value of the current?

$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{1}{2}BL^2\omega$ ← α is a function of t so $\frac{d\alpha}{dt} = \omega$

$I = \frac{\varepsilon}{R} = -\frac{1}{2R}BL^2\omega$ ← you can drop minus sign here as we deal with that using lenz's law

Finally, the force:

$d\vec{F} = I\,d\vec{l}\times\vec{B}$

$dF = I\,dl\,B$

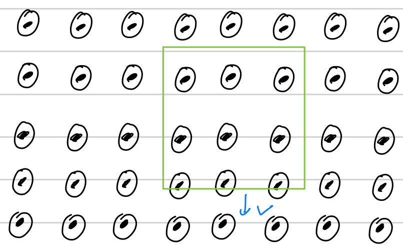
$\vec{F} = IB\int_0^l dl$

$$F = \frac{1}{2R}B^2L^3\omega$$

Using RHR, Force points to the right, can parametrize if needed

Example

(i)

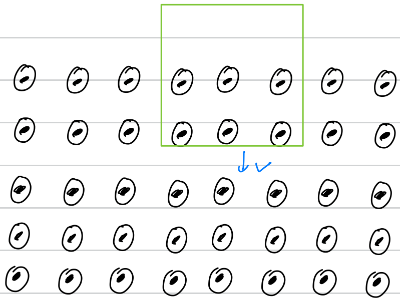


Direction of Induced Current?

No induced current!

→ A, B, θ are all constant!

(ii)



Direction of Induced Current?

Need to produce ⊗

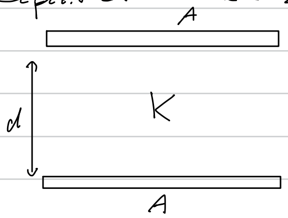
Clockwise

→ Increasing flux over time

Me ↑ F ↓

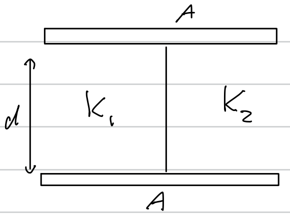
Review

Capacitor & Dielectrics



without dielectric: $C_0 = \varepsilon_0 \frac{A}{d}$

↳ add dielectric: $C_K = K \varepsilon_0 \frac{A}{d}$



area wise

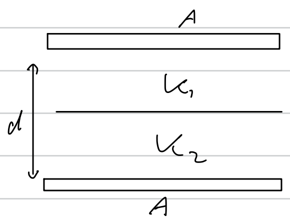
Half of the empty region: $C_0^{1/2} = \varepsilon_0 \frac{A}{2d}$ $\left\{ \frac{A/2}{d} \right\}$

↳ add K_i → $C_{K_i}^{1/2} = K_i \varepsilon_0 \frac{A}{2d}$

Total Capacitance: $C_{K_1 | K_2} = C_{K_1}^{1/2} + C_{K_2}^{1/2}$

(parallel)

Capacitors and resistors have opposite equations



distance wise

Half of region: $C_0^{1/2} = \varepsilon_0 \frac{2A}{d}$ $\left\{ \frac{A}{d/2} \right\}$

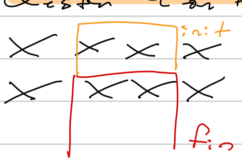
↳ add: $C_{K_i}^{1/2} = K_i \varepsilon_0 \frac{2A}{d}$

Total Capacitance: $C_{\frac{K_1}{K_2}} = \left(\frac{1}{C_{K_1}^{1/2}} + \frac{1}{C_{K_2}^{1/2}} \right)^{-1}$

(series)

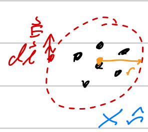
Lenz's Law

Question 4 on HW



Flux decreasing. $\Phi \downarrow$ $\therefore \frac{d\Phi_B}{dt} < 0$

Faraday $\Phi \uparrow$ created



$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\left(\overbrace{\int \vec{B} \cdot d\hat{a}}^{\phi_B} \right) = -\frac{d}{dt}\left(\int \vec{B} \cdot \hat{n}\, da \right)$$

$$E(2\pi r) = -\frac{d(B(t))}{dt}(\pi r^2)(\cos 180)$$

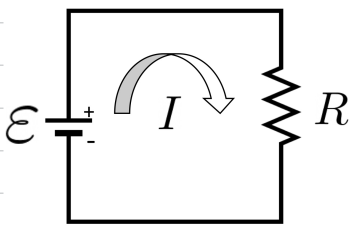
$$= \frac{dB}{dt}(\pi r^2)$$

Stokes convention is important

Final Exam

DC Circuits

Recall: Basic DC Circuit



Electromotive Force

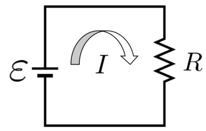
- DC Battery
  - Two terminal device
  - Maintains a Voltage (potential drop) across terminals ($\mathcal{E}$)
  - Positive terminal at higher voltage
  - Current can flow in both directions
- Resitor
  - Not very good conductor
  - Potential drop across (V)
  - Current can flow in both directions

Kirchoff Rule Steps

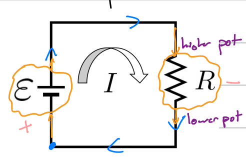
1. Assume Dir I
2. Walk Loop
3. 1st K rule (to n loops)
4. 2nd K rule (to n-1 loops)
5. Solve sys of eq.

Kirchoff's Rules

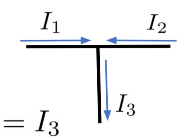
- Potential drop around a loop add up to zero



Example 1



- In a junction the amount of current flowing in equals the flowing out



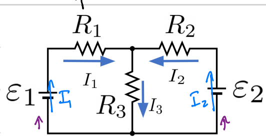
$$I_1 + I_2 = I_3$$

$\mathcal{E} - IR = 0$

$\Rightarrow I = \frac{\mathcal{E}}{R}$

If you assume opposite flow, you just get $I < 0$

Example 2



$\mathcal{E}_1 - I_1R_1 - I_3R_3 = 0$

$\mathcal{E}_2 - I_2R_2 - I_3R_3 = 0$

$I_1 + I_2 = I_3$

$$I_1 = \frac{\varepsilon_1 R_2 + \varepsilon_1 R_3 - \varepsilon_2 R_3}{R_1R_2 + R_1R_3 + R_2R_3}$$

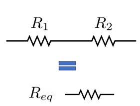
$$I_2 = \frac{\varepsilon_2 R_1 + \varepsilon_2 R_3 - \varepsilon_1 R_3}{R_1R_2 + R_1R_3 + R_2R_3}$$

$$I_3 = \frac{\varepsilon_1 R_2 + \varepsilon_2 R_1}{R_1R_2 + R_1R_3 + R_2R_3}$$

Just solve the system

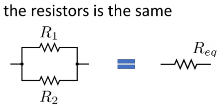
Resistance in Series & parallel

Series:
The current through the resistor is the same



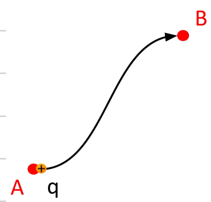
$R_{eq} = R_1 + R_2$

Parallel:
The voltage drop through the resistors is the same



$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$

Recall: Energy - Potential



$\Delta U = q(\phi(B) - \phi(A))$

Energy Flow: Battery

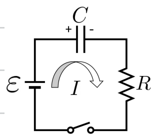
- Energy delivered to system by battery
  $dU = dq\,\mathcal{E}$
- If an amount of charge dq goes through the battery in a time dt, we can write the power:
  $P = \frac{dU}{dt} = \frac{dq}{dt}\mathcal{E} = I\mathcal{E}$

Energy Flow: Resistor

- Amount of charge dq that goes through a resistor looses an energy
  $dU = dq(-IR)$ ← $\mathcal{E} = IR$, just negated from above
- Resistor gains an energy $dU = dq\,IR$
- The power of the resistor is
  $P = \frac{dU}{dt} = \frac{dq}{dt}IR = I^2R$ ← This is called Joule heating

RC Circuits

Resistor Capacitor Circuits



- Capacitor stores charge
  - The **charge (Q)** is proportional to the Potential difference (often called **voltage, V**) between the two electrodes by a quantity called **Capacitance (C)** → $Q = CV$

- Q ≡ Charge (Coulombs, C)
- C ≡ Capacitance (Farads, F)
- V ≡ Voltage (Volts, V)

- In terms of current: $dQ = I\,dt$
- Current can flow in both directions (from positive to negative plates)

- The charge changes smoothly!

Solving the RC Circuit

$\mathcal{E} - RI - \frac{Q}{C} = 0$ ← from Kirchoff's Loop rule

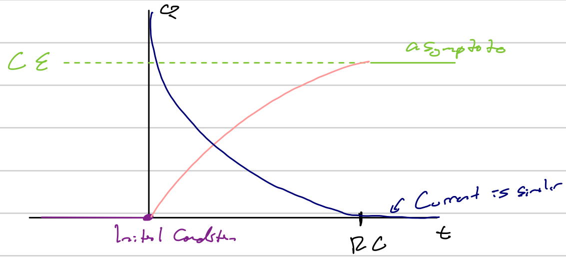
→ One equation, two unknowns: $I = \frac{dQ}{dt}$

→ Initial Condition of LDE: @ $t=0^+$, $Q = 0$ ← Cannot change abruptly

→ At $t = \infty$, system @ steady state → $\frac{dQ}{dt} = 0$

$\therefore \mathcal{E} - R\cancel{\frac{dQ}{dt}} - \frac{Q}{C} = 0 \qquad Q = \mathcal{E}C$ ← this is the asymptote

- At $t < 0$: Switch is open; $I = 0$
  Capacitor discharged; $Q = 0$
- At $t = 0$: Close switch; $I \neq 0$
  Capacitor starts charging
- At $t = \infty$: Charge plateaus



$$\Rightarrow \boxed{Q(t) = C\mathcal{E}\left(1 - e^{-t/\tau}\right)}$$

$$\Rightarrow \boxed{I = \frac{dQ}{dt} = \frac{\mathcal{E}}{R} e^{-\frac{t}{RC}}}$$

- Solving the DE

$C\mathcal{E} - RC\frac{dQ}{dt} - Q = 0$ (multiply DE by C)

units of time

↳ RC is the time it takes to reach steady state

→ Guess a solution: $Q = C\mathcal{E}\left(1 - e^{-t/\tau}\right)$

- Verify Solution/guess

$$\frac{dQ}{dt} = \frac{d}{dt}\left(C\mathcal{E} - C\mathcal{E}e^{-t/\tau}\right) = \frac{1}{\tau}C\mathcal{E}e^{-t/\tau}$$

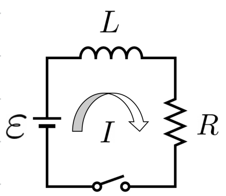
$$\cancel{C\mathcal{E}} - \frac{RC}{\tau}\cancel{C\mathcal{E}}e^{-t/\tau} - \cancel{C\mathcal{E}} + \cancel{C\mathcal{E}}e^{-t/\tau} = 0$$

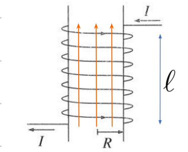
$$e^{-t/\tau} - \frac{RC}{\tau}e^{-t/\tau} = 0 \Rightarrow \frac{RC}{\tau} = 1$$

$\tau = RC$ verifies our solution!

RL Circuits

Resistor Inductor Circuits





• Inductor is typically a wire coil (solenoid)

$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$

$B = \mu_0 I n$

$\Phi_B = BNA = \mu_0 I n^2 \ell$ → $\frac{d\Phi_B}{dt} = \mu_0 \frac{dI}{dt} n^2 \ell A = -\varepsilon$

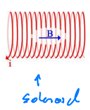
$\varepsilon = -L \frac{dI}{dt}$

inductance

→ Inductance (L): $L = \mu_0 n^2 \ell A$

→ In general, the inductance: $L = \frac{N\Phi_B}{I}$

Inductance (L)

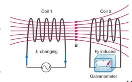


• The change in the current will produce a change in the magnetic field together with a change in flux of B through the solenoid. This will induce a electromotive force in the solenoid (self inductance)

$$V = -L\frac{dI}{dt}$$

• I ≡ Current (Amperes, A)
• L ≡ Inductance (Henrys, H)
• V ≡ Voltage (Volts, V)

Mutual Induction



The induced electromotive force in coil 2

$$\varepsilon_2 = -N_2 \frac{d\Phi_{B,2}}{dt}$$

and the flux has to be proportional to the current in coil 1:

$$N_2 \Phi_{B,2} = M_{21} I_1$$

Proportionality constant

And together: $\varepsilon_2 = -M_{21}\frac{dI_1}{dt}$

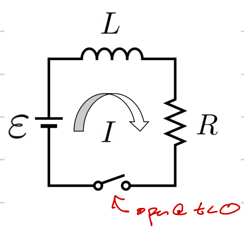
If we interchange the battery and galvanometer $\varepsilon_1 = -M_{12}\frac{dI_2}{dt}$

It can be proved that $M_{21} = M_{12} \equiv M$

$$\varepsilon_1 = -M\frac{dI_2}{dt}$$

$$\varepsilon_2 = -M\frac{dI_1}{dt}$$

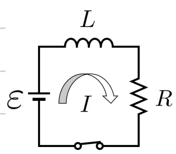
RL Circuit



• Inductor - wire coil (solenoid)
• Voltage drop: $V = -L\frac{dI}{dt}$
• Current doesn't change abruptly
• Current can flow in both directions
↳ from positive to negative potential

Solving the RL Circuit

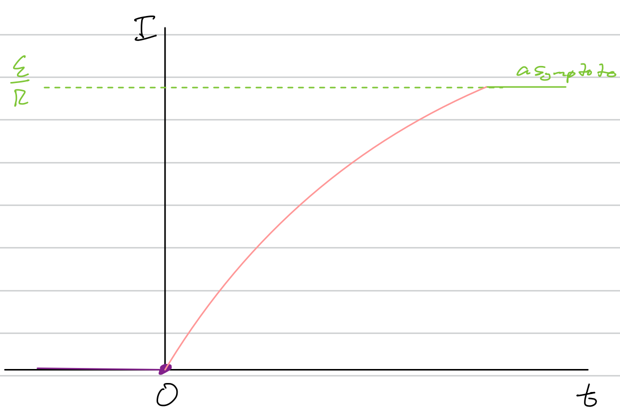
• At $t<0$
↳ Switch is open, $I=0$
• At $t=0$
↳ close switch, $I \neq 0$



• Kirchoff's Loop rule

$$\varepsilon - L\frac{dI}{dt} - IR = 0$$

Solving the DE



• At $t = 0^+$ → $I = 0$ (Current doesn't change abruptly)

• At $t = \infty$ → $\frac{dI}{dt} = 0$ (steady-state)

$I = \frac{\varepsilon}{R}$

units of time

$\frac{\varepsilon}{R} - \frac{L}{R}\frac{dI}{dt} - I = 0$ (divide DE by R)

→ Guess a solution: $I = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau}\right)$

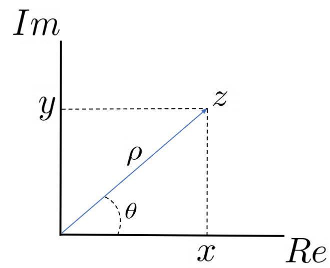
• Plug into equation to find τ and verify:

$$\boxed{I(t) = \frac{\varepsilon}{R}\left[1 - e^{-\frac{L}{R}t}\right]}$$

AC Circuits

Complex numbers: Review

- A complex number $z = x + iy$ is formed by
  - Real part: $Re[z] = x$
  - Imaginary part: $Im[z] = y$
  - The imaginary unit: i ← sometimes j, just notation
    - ↳ $i^2 = -1$



- In Polar Coordinates: $z = \rho\cos\theta + i\rho\sin\theta$

$$\begin{cases} \rho = \sqrt{x^2+y^2} = |z| \\ \theta = \arctan(y/x) \end{cases} \qquad (x,y) \longleftrightarrow (\rho,\theta)$$

- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

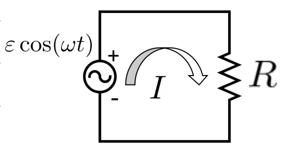
$\Rightarrow z = \rho e^{i\theta} = \rho\cos\theta + i\rho\sin\theta$

e.g. $e^{i\pi} + 1 = 0$

→ exponentiate complex number: $e^{z} = e^{x+iy} = e^{x}e^{iy}$

→ Important function: $f(t) = e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

AC Circuits: Resistor



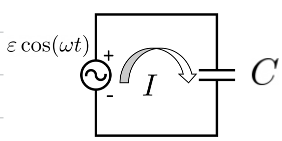
- Voltage of AC source

$V = \Phi_+ - \Phi_- = \varepsilon\cos(\omega t)$

- Kirchhoff Rule

$$\varepsilon\cos\omega t - IR = 0 \Rightarrow \boxed{I = \frac{\varepsilon}{R}\cos(\omega t)}$$

AC Circuits: Capacitor

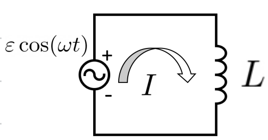


- Voltage of AC Source is same
- Kirchhoff Rule

$$\varepsilon\cos(\omega t) - \frac{Q}{C} = 0 \Rightarrow Q = \varepsilon C\cos(\omega t)$$

$$I = \frac{dQ}{dt} = -\varepsilon\omega C\sin(\omega t)$$

AC Circuits: Inductor

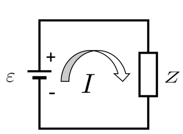


- Voltage of AC Source is same
- Kirchhoff Rule

$$\varepsilon\cos\omega t - L\frac{dI}{dt} = 0 \Rightarrow \frac{dI}{dt} = \frac{\varepsilon}{L}\cos(\omega t)$$

AC Circuits: Impedance

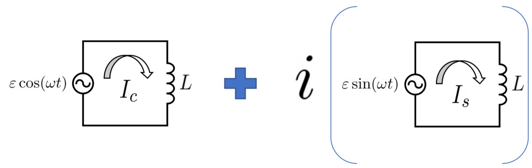
- We can study the equivalent DC circuit where impedance reads



$$Z = \begin{cases} R & \text{for resistors} \\ \frac{1}{i\omega C} & \text{for capacitors} \\ i\omega L & \text{for inductors} \end{cases}$$

→ Kirchhoff Rule: $\tilde{I} = \frac{\varepsilon}{Z} \Rightarrow I(t) = Re[\tilde{I}e^{i\omega t}]$

Impedance: Complex Ohm's Law Proof



Recall: $(x, y) \Leftrightarrow (\rho, \theta)$

$\rho e^{i\theta} = \rho[\cos\theta + i\sin\theta]$

$$\varepsilon\cos(\omega t) - L\frac{dI_c}{dt} = 0 \quad + \quad i\left(\varepsilon\sin(\omega t) - L\frac{dI_s}{dt} = 0\right)$$

$$\varepsilon\cos(\omega t) + i\varepsilon\sin(\omega t) - L\left(\frac{dI_c}{dt} + i\frac{dI_s}{dt}\right) = 0$$

$$\varepsilon e^{i\omega t} - L\frac{dI_e}{dt} = 0 \quad \text{where } I_e = I_c + iI_s \leftarrow$$ Make a guess as to what the diff eq's sol. is!

$$\left.\begin{aligned} I_e &= \tilde{I}e^{i\omega t} \\ \frac{dI_e}{dt} &= i\omega\tilde{I}e^{i\omega t} \end{aligned}\right\} \quad \cancel{\varepsilon e^{i\omega t}} - L(i\omega\tilde{I}\cancel{e^{i\omega t}}) = 0$$

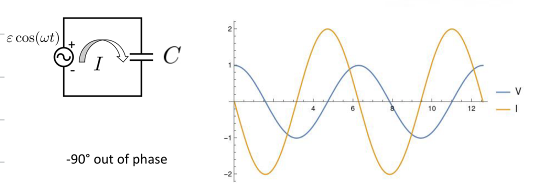
$$\varepsilon = \underbrace{Li\omega}_{=Z}\tilde{I} \Rightarrow \varepsilon = Z\tilde{I}, \quad \tilde{I} = \frac{\varepsilon}{Z}$$ (we want I_c)

$$I_e = I_c + I_s = \tilde{I}e^{i\omega t} \Rightarrow I_c = \underbrace{Re}_{\text{real part}}[I_e] = Re[\tilde{I}e^{i\omega t}]$$

Phase

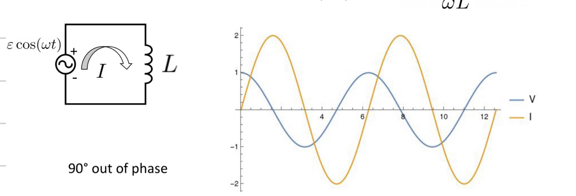
Current-Voltage Phase Capacitor

$V = \varepsilon\cos(\omega t)$ $\quad I = -\varepsilon C\omega\sin(\omega t)$

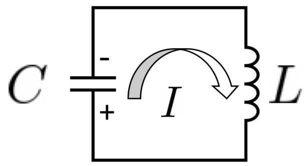


Current-Voltage Phase Inductor

$V = \varepsilon\cos(\omega t)$ $\quad I = \frac{\varepsilon}{\omega L}\sin(\omega t)$



LC Circuit



- Kirchhoff Rule

$$\left.\begin{aligned} -L\frac{dI}{dt} - \frac{Q}{C} &= 0 \\ I &= \frac{dQ}{dt} \end{aligned}\right\} L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

→ Analogous to Harmonic oscillators

$$m\frac{d^2x}{dt^2} + kx = 0 \rightarrow x = x_0\cos\left(\sqrt{\frac{k}{m}}t\right) + v_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right)$$

Our solution: $Q(t) = Q_0\cos\left(\sqrt{\frac{1}{LC}}t\right) + \cancel{\underbrace{I_0}_{\text{no initial current}}\sqrt{LC}\sin\left(\sqrt{\frac{1}{LC}}t\right)}$

$$I(t) = \frac{dQ}{dt} = \frac{Q_0}{\sqrt{LC}}\sin\left(\sqrt{\frac{1}{LC}}t\right)$$

Energy Stored in Capacitors & Inductors

- **Capacitor**

$$dU = V\,dq = \frac{q}{C}dq, \qquad U = \int_0^Q \frac{q}{C}dq = \boxed{\frac{1}{2}\frac{Q^2}{C}}$$

$$\boxed{U = \frac{1}{2}CV^2}$$

- **Inductor**

$$P = \frac{dU}{dt} = \frac{dq}{dt}V = IV = IL\frac{dI}{dt} = \frac{1}{2}L\frac{d(I^2)}{dt}$$

$$U = \int_0^{t_0} P\,dt = \frac{1}{2}L\int_0^{t_0}\frac{d(I^2)}{dt}dt = \boxed{\frac{1}{2}LI^2}$$