



Electric Charges & Fields, Coulomb's Law, Superposition

Electric Charge

- charge is a fundamental prop of matter
 - two types: \textcircled{P} & \textcircled{G} matter(atoms) are normally neutral
- Triboelectricity:** Friction transfers charge

Metal Rods in Demo?

- Plastic & crystal are insulators
 - charge "stays where put"
- Metals are conductors
 - charge "flow freely"

Charge is conserved

- Rubbing transfers charges, but does not create nor destroy it

Charge is quantized

- charge is always a multiple of $e = 1.6 \times 10^{-19}$ Coulombs (C) \Rightarrow electron = $-e$, proton = $+e$

Why the rods are moving!

- Attraction or Repulsion \Rightarrow There is a force between the charges

\hookrightarrow Force is known as coulomb's force

$$F = \frac{1}{4\pi\epsilon_0} \cdot \frac{qQ}{r^2} \quad \text{where } \frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}$$

\square When two charges q and Q are same signed $\Rightarrow F$ is positive = Repulsive

\square When they have opposite signs $\Rightarrow F$ is negative = Attractive

Demo - Rods

$$\textcircled{P} \textcircled{P} \textcircled{G} \leftarrow \rightarrow$$

$$\textcircled{G} \textcircled{G} \textcircled{P} \leftarrow \rightarrow$$

$$\textcircled{P} \textcircled{G} \textcircled{P} \rightarrow \leftarrow$$

Note: the metal is still neutral

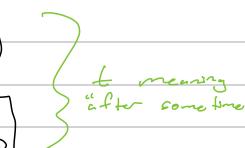
Demoval / Metal

$$\textcircled{G} \quad \textcircled{G}$$

$$\textcircled{P} \quad \textcircled{G}$$

$$\textcircled{P} \quad \textcircled{G}$$

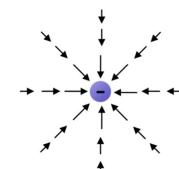
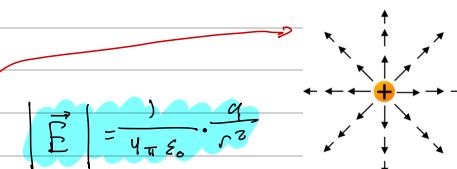
$$\textcircled{G} \quad \textcircled{G}$$



Electric Field and Coulomb's Law

Laws of Electrostatics

- Electrostatics = charges @ rest
- charges produce electric fields
- electric fields push charges



$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

$$|\vec{r}| = r$$

$$\text{Note: } r = |\vec{r}|$$

\vec{r} is a unit vector that points away from charge q

$$\text{or alternatively so } \vec{r} = \frac{\vec{r}}{|\vec{r}|} \text{ and } \vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \frac{\vec{r}}{|\vec{r}|}$$

Equivalent!

Superposition Principle

In Words:

Electric Field produced = Sum of the electric field produced by many point charges by each point charge independently

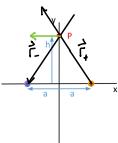
$$\text{In Maths: } \vec{E}(q_1, q_2) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_1^2} \hat{r}_1 + \frac{1}{4\pi\epsilon_0} \frac{q_2}{r_2^2} \hat{r}_2$$

Second Law of Electrostatics

$$\text{Fields push/pull charges: } \vec{F} = Q \vec{E} \Rightarrow \vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{r}$$

$\Sigma N \quad \Sigma C \quad \Sigma N \quad \Sigma C$

Example: Point Charges



Calculate the Electric field at point P.

Pos vector \vec{r}_1 from O to $p_1: -a\hat{i} + b\hat{j}$

Pos vector \vec{r}_2 from O to $p_2: a\hat{i} + b\hat{j}$ (not mass, position not dipole charge!)

Calculate the Electric Field at P:

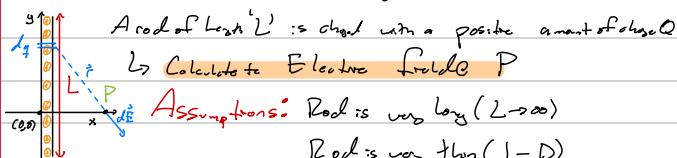
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} \vec{r} \rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0} \left(\frac{1}{(\sqrt{a^2+b^2})^3} [q(-a\hat{i}+b\hat{j}) + -q(a\hat{i}+b\hat{j})] \right) = \frac{2qa}{4\pi\epsilon_0 \sqrt{a^2+b^2}^3} \cdot (-\hat{i})$$

$$r_1 = a\hat{i} + b\hat{j} \rightarrow r_1 = \sqrt{a^2+b^2} \quad \rightarrow \vec{E} = \vec{E}_1 + \vec{E}_2$$

$$r_2 = -a\hat{i} + b\hat{j} \rightarrow r_2 = \sqrt{a^2+b^2}$$

Here we are assuming point-like charges but this is not realistic

Continuous Distribution of Charge



Calculate the Electric field at P

Assumptions: Rod is very long ($L \rightarrow \infty$)

Rod is very thin ($L \ll D$)

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} \vec{r} \rightarrow d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \vec{r}$$

$$\vec{r} = x\hat{i} - y\hat{j} \quad r = \sqrt{x^2+y^2} \quad \rightarrow d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \cdot (x\hat{i} - y\hat{j})$$

Definition: Charge per unit length or linear charge density

$$\lambda = \frac{\text{Charge}}{\text{Length}} = \frac{Q}{L}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \sum_{i=1}^N \vec{E}_i = \int_a^b d\vec{E}$$

$$\text{so, } \vec{E} = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\lambda}{\sqrt{x^2+y^2}} (x\hat{i} - y\hat{j}) dy \quad \text{Solve } x \text{ and } y \text{ separately}$$

Example

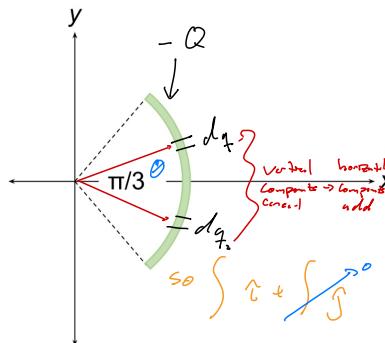
A plastic rod with a uniformly distributed charge $-Q$ is bent into a circular arc of radius r that subtends an angle of $\pi/3$ radians. We place coordinate axes such that the axis of symmetry of the rod lies along the x-axis and the origin is the center of curvature for the rod. What is the electric field (magnitude and direction) at the origin?

Hint: Imagine that the arc is made up of many infinitesimally small point charges dq . Each dq creates a differential electric field of magnitude $dE = k dq/r^2$.

Sum up all the dEs from all the dqs to get the magnitude of the overall electric field.

Integration is a way to add many infinitesimally small elements.

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \vec{r}$$



To integrate, let's use polar! $(x, y) \rightarrow (r, \theta)$

$$\int dE_x = \frac{1}{4\pi\epsilon_0} \int_{-\pi/6}^{\pi/6} \frac{r d\cos\theta}{r^2} d\theta$$

$$= \frac{1}{4\pi\epsilon_0 r} \left[\sin\theta \right]_{-\pi/6}^{\pi/6}$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{3Q}{\pi r^2}$$

$$\text{but what is lambda? } \lambda = \frac{Q}{L} = \frac{Q}{r(\pi/3)} \quad \text{Just think length = angle * radius (like in } C = 2\pi r)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{3Q}{\pi r^2} \hat{i}$$

$$|d\vec{E}| = \left| \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \vec{r} \right|$$

$$\begin{cases} \cos\theta = \frac{a}{r} \\ \sin\theta = \frac{b}{r} \end{cases} \rightarrow \vec{r} = -r\cos\theta\hat{i} - r\sin\theta\hat{j}$$

$$x\text{-component of } |dE|: dE_x = \frac{1}{4\pi\epsilon_0} \frac{da}{r^2} \cos\theta$$

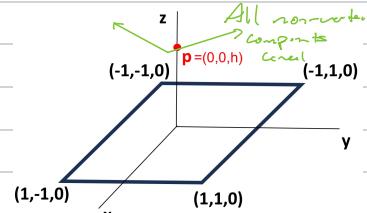
We need to integrate over all θ , b , dE_x terms of dq , not $d\theta$!

$$\Rightarrow dq = \int ds \quad \text{where } ds \text{ is segment of arc}$$

$$\Rightarrow \frac{2\pi r}{\theta} \text{ so } ds = r d\theta \rightarrow dq = r \lambda d\theta$$

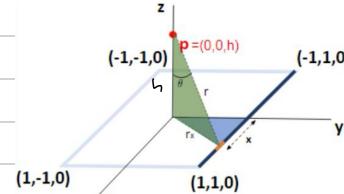
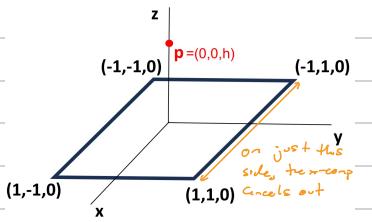
Example

A plastic rod with a uniformly distributed charge Q is shaped forming a square whose side is 1m. We place coordinate axes such that the square is on the x-y plane and the origin is the center of the square. What is the electric field (magnitude and direction) at the point $p=(0,0,h)$?



\Rightarrow The problem solution will be $\vec{E} = E \hat{k}$

So we just need to calculate the vertical (z) component of one side and add the rest!



Hint: What is the square formed by? Is there any symmetry?

Imagine that the square is made up of many infinitesimally small point charges dq . Each dq creates a differential electric field of magnitude $dE = k dq/r^2$.

Sum up all the dEs from all the dqs to get the magnitude of the overall electric field.

Integration is a way to add many infinitely small elements.

$$\begin{aligned} \vec{r}_h &= h \hat{k} \\ \vec{r}_{xy} &= -x \hat{i} - \hat{j} \Rightarrow r_{xy}^2 = x^2 + 1^2 \\ \text{so, } r^2 &= r_{xy}^2 + h^2 \\ \Rightarrow r &= \sqrt{x^2 + h^2} \end{aligned}$$

$$\begin{aligned} d\vec{E} &= \frac{1}{4\pi\epsilon_0} \frac{dq}{(x^2 + h^2)^{3/2}} (-x \hat{i} - \hat{j} + h \hat{k}) \\ \Rightarrow \vec{E} &= \int \frac{1}{4\pi\epsilon_0} \frac{dq}{(x^2 + h^2)^{3/2}} (-x \hat{i} - \hat{j} + h \hat{k}) \end{aligned}$$

only components \hat{k}

In one side \hat{j} component part, but disappears when using Left cancellation

$$\vec{E}_{\text{Top}} = \int_{-1}^1 \frac{1}{4\pi\epsilon_0} \frac{dq}{(\sqrt{x^2 + h^2})^2} (-x \hat{i} - \hat{j} + h \hat{k})$$

$$\vec{E}_{\text{Bottom}} = \int_{-1}^1 \frac{1}{4\pi\epsilon_0} \frac{dq}{(\sqrt{x^2 + h^2})^2} (+\hat{i} - \hat{j} + h \hat{k})$$

$$\text{so } \boxed{\vec{E} = \vec{E}_{\text{Top}} + \vec{E}_{\text{Bottom}}}$$

Gauss Law

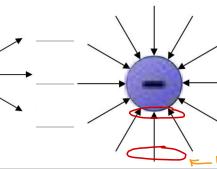
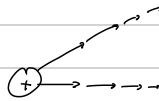
Electric Field Lines

- (i) Lines are tangent to \vec{E} at every point in space
- (ii) The density of lines is proportional to $|\vec{E}|$
- (iii) Lines never cross
- (iv) No closed loops
- (v) Lines start or stop on charges

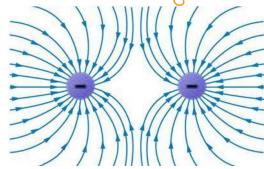
↳ Start should be a positive charge, and
should be on negative charge

↳ This rule inherently enforces part (iv)

Single Charge



Two charges

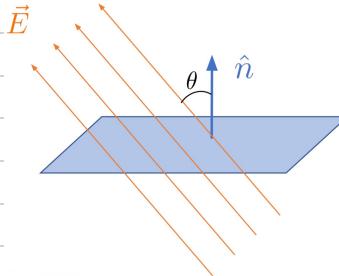


Recall Vector Multiplication

$$\left\{ \begin{array}{l} \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \\ = |\vec{A}| |\vec{B}| \cos \theta \end{array} \right. \text{ for } \theta \text{ the angle between } \vec{A} \text{ and } \vec{B}$$

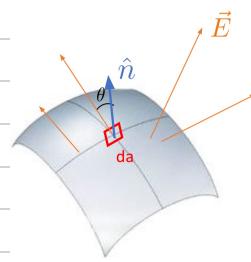
Electric Flux

- Plot surface and uniform \vec{E}
- Defined: $\Phi_E = A \hat{n} \cdot \vec{E} = A |\hat{n}| |\vec{E}| \cos \theta = A |\vec{E}| \cos \theta$
 - o \hat{n} is a unit vector perpendicular to the surface (normal vector)
 - o θ is the angle between \hat{n} and \vec{E}
 - o A is the area of the surface
- Non-Plot surface and/or Non-uniform \vec{E}



$$\Phi_E = \int \hat{n} \cdot \vec{E} dA = \int |\hat{n}| |\vec{E}| \cos \theta dA = \int |\vec{E}| \cos \theta dA$$

- o \hat{n} is a unit vector perpendicular to the surface da
- o θ is the angle between \hat{n} and \vec{E}



Gauss Law

- Rest of Maxwell's Equations

- In words: The electric flux

$$\text{through a closed surface} = \frac{\text{Amount of charge inside the surface}}{\epsilon_0}$$

- Equation:

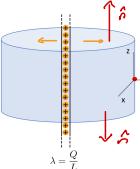
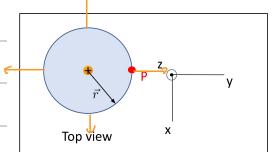
$$\Phi_E = \int_{\text{closed}} \hat{n} \cdot \vec{E} dA = \frac{Q}{\epsilon_0}$$

- \vec{E} points outside the surface by convention

$$\cdot \hat{n} \cdot d\vec{a} = \frac{d\vec{a}}{\epsilon_0} \Rightarrow \text{locally, where } d\vec{a} = da \cdot \hat{n}$$

• Example / Review: Infinite Line

Assumption: rod is very long ($L \rightarrow \infty$)
rod is very thin ($l \rightarrow 0$)



Length l of rod make cylinder

$$\Rightarrow Q_{ENC} = \lambda l$$

$$\phi_E = \frac{Q_{ENC}}{\epsilon_0} = \frac{\lambda l}{\epsilon_0}$$

Extra credit
Question

What is the direction of the total electric field @ P?

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}, \quad E = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$2 \int_{Cyl}^{ } |\vec{E}| \cos \theta da + \int_{Side}^{ } \vec{E} da$$

$\theta = 90^\circ, \text{ so } 0$

$$= 0 + \int_{Side}^{ } |\vec{E}| \cos \theta da \\ = \int_{Side}^{ } E da = E \int_{Side}^{ } da = EA$$

$$\Phi_E = E \cdot 2\pi r l$$

$$E \cdot 2\pi r l = \frac{\lambda l}{\epsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\epsilon_0 r}$$

CAPS

$$\vec{E} = \vec{E}_f \quad \left. \begin{array}{l} \vec{n} = \pm \hat{r} \\ \theta = 90^\circ \end{array} \right\} \theta = 90^\circ$$

SIDE

$$\vec{E} = E \hat{r} \quad \left. \begin{array}{l} \vec{n} = \hat{r} \\ \theta = 0 \end{array} \right\} \theta = 0$$

The magnitude is constant
for all parts on SIDE

Infinite Plane

Direction of Electric Field?

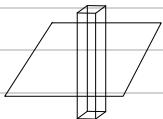
$$A = \pi r^2$$

$$\sigma = \frac{Q}{L^2} \quad \int_{Cyl}^{ } \vec{E} da = \frac{Q_{ENC}}{\epsilon_0}$$

$$Q_{ENC} = \sigma \pi r^2$$

$$\int_{Cyl}^{ } \vec{E} da = 2 \int_{Side}^{ } E da = 2EA = 2E\pi r^2$$

$$2B\pi r^2 = \frac{\sigma \pi r^2}{\epsilon_0} \Rightarrow E = \frac{\sigma}{2\epsilon_0}$$



Also works for
two conductors

