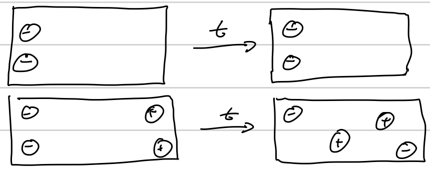
Midterm Exam 1

Need to write the Maxwell equations on the final exam!

Electric Charges & Fields, Coulomb's Law, Superposition

Electric Charge

- charge is a fundamental prop. of matter
  - ↳ two types: ⊕ & ⊖   matter (atoms) are normally neutral
- Triboelectricity: Friction transfers charge
- Metal Rod in demo?
  - Plastic & crystal are insulators
    - → charge "stays where put"
  - Metals are conductors
    - → charge "flow freely"



t meaning "after some time"

Demo - Rods

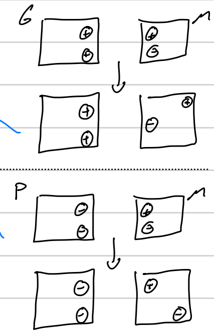
⊖ P P ⊖ ← →

⊕ G G ⊕ ← →

⊖ P G ⊕ → ←

Demo w/ Metal

Note: the metal is still neutral



- Charge is conserved
  - Rubbing transfers charges, but does not create nor destroy it
- Charge is quantized
  - Charge is always a multiple of $e = 1.6 \times 10^{-19}$ Coulombs (C) → electron $= -e$, proton $= e$
- why the rods are moving!
  - Attraction or Repulsion ⇒ There is a force between the charges
    - ↳ Force is known as coulomb's force

$$F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{qQ}{r^2} \qquad \text{where } \frac{1}{4\pi\varepsilon_0} = 9\times10^9 \ \frac{N \cdot m^2}{C^2}$$

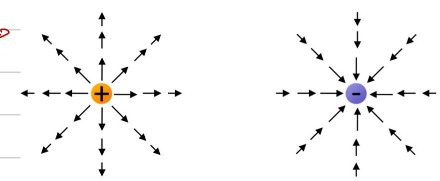
- When two charges q and Q are same signed → F is positive = Repulsive
- when they have opposite sign → F is negative = Attractive

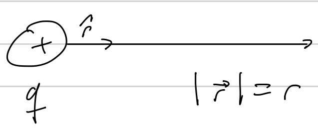
Electric Field and Coulomb's Law

Laws of Electrostatics

- Electrostatics ≡ charges @ rest
  - charges produce electric fields
  - electric fields push charges

$$|\vec{E}| = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$





$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

$\hat{r}$ is a unit vector that points away from charge q

Note: $r = |\vec{r}|$

or alternatively (use $\vec{v} = |\vec{v}|\,(\hat{v})$) so $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ and

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^3} \vec{r}$$

Equivalent!

Superposition Principle

In Words:

Electric Field produced by many point charges = Sum of the electric field produced by each point charge independently

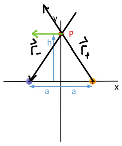
In Math:

$$\vec{E}(q_1, q_2) = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{r_1^3} \vec{r}_1 + \frac{1}{4\pi\varepsilon_0} \frac{q_2}{r_2^3} \vec{r}_2$$

Second Law of Electrostatics

Fields push/pull charges: $\underset{\{N\}}{\vec{F}} = \underset{\{C\}}{Q}\,\underset{\{\frac{N}{C}\}}{\vec{E}} \Rightarrow \vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{r^2} \hat{r}$

Example: PollEv



Calculate the Electric field at point P.

Pos vector $\vec{r}$ from ⊕ to P: $-a\hat{\imath} + h\hat{\jmath}$

Pos vector $\vec{r}$ from ⊖ to P: $a\hat{\imath} + h\hat{\jmath}$ (not moving, position vector not dependent on charge!)

Here, we are assuming point-like charges but this is not realistic

Calculate the Electric Field @ P:

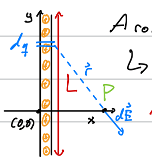
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r} \Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{(\sqrt{a^2+h^2})^3}\left[q(-a\hat{\imath}+h\hat{\jmath}) + -q(a\hat{\imath}+h\hat{\jmath})\right]\right) = \frac{2qa}{4\pi\varepsilon_0\sqrt{a^2+h^2}^3}\cdot(-\hat{\imath})$$

$r_- = a\hat{\imath} + h\hat{\jmath} \rightarrow r_- = \sqrt{a^2+h^2}$

$r_+ = -a\hat{\imath} + h\hat{\jmath} \rightarrow r_+ = \sqrt{a^2+h^2}$

$\Big\rangle\ \vec{E} = \vec{E}_+ + \vec{E}_-$

Continuous Distribution of Charge



$dq = \lambda dy$

A rod of Length 'L' is charged with a positive amount of charge Q

↳ Calculate the Electric field @ P

Assumptions: Rod is very long ($L \to \infty$)

Rod is very thin (1-D)

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r} \rightarrow d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}$$

$\vec{r} = x\hat{\imath} - y\hat{\jmath}$

$r = \sqrt{x^2+y^2}$

$$\Big\rangle\ d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{x^2+y^2}^3}\cdot(x\hat{\imath} - y\hat{\jmath})$$

Definition: Charge per unit length, or linear charge density

$$\lambda = \frac{\text{Charge}}{\text{Length}} = \frac{Q}{L}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \sum_{i=1}^{N}\vec{E}_i = \int_a^b d\vec{E}$$

So, $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\lambda}{\sqrt{x^2+y^2}^3}(x\hat{\imath} - y\hat{\jmath})\,dy$$

Solve $\hat{\imath}$ and $\hat{\jmath}$ components separately

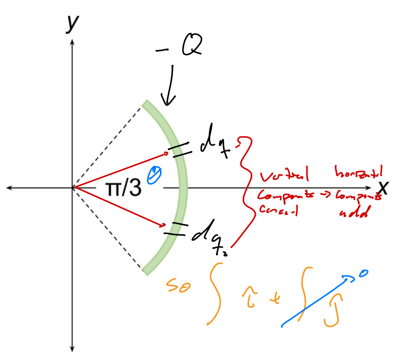
Example

A plastic rod with a uniformly distributed charge –Q is bent into a circular arc of radius *r* that subtends an angle of π/3 radians. We place coordinate axes such that the axis of symmetry of the rod lies along the *x*-axis and the origin is the center of curvature for the rod. What is the electric field (magnitude and direction) at the origin?

Hint: Imagine that the arc is made up of many infinitesimally small point charges *dq*. Each *dq* creates a differential electric field of magnitude *dE* = *k dq*/*r*².

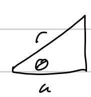
Sum up all the *dE*s from all the *dq*s to get the magnitude of the overall electric field.

Integration is a way to add many infinitely small elements.



$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}$$

$$|d\vec{E}| = \left|\frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}\right|$$



$\cos\theta = \frac{a}{r}$

$\sin\theta = \frac{b}{r}$

$\Big\}\ \vec{r} = -r\cos\theta\,\hat{\imath} - r\sin\theta\,\hat{\jmath}$

x-component of $|d\vec{E}|$: $$dE_x = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\cos\theta$$

we need to integrate over all θ, but dE_x is in terms of dq, not $d\theta$!

To integrate let's use polar! $(x, y) \to (r, \theta)$

$\Rightarrow dq = \lambda\, ds$ where ds is segment of arc

(λ: linear charge density)

$\Rightarrow \underbrace{2\pi r}_{\theta}$ so $ds = r\,d\theta \rightarrow dq = r\lambda\, d\theta$

$$\int d\vec{E}_x = \frac{1}{4\pi\varepsilon_0}\int_{-\pi/6}^{\pi/6}\frac{r\lambda\cos\theta\, d\theta}{r^2}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 r}\left[\sin\theta\right]_{-\pi/6}^{\pi/6}$$

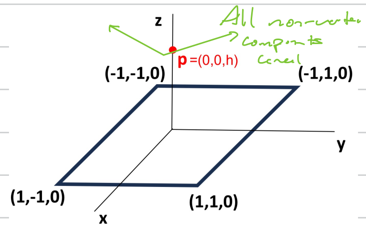
but what is lambda $= \frac{Q}{L} = \frac{Q}{r(\pi/3)}$

Just think length = angle • radius (like in $C = 2\pi r$)

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{3Q}{\pi r^2} \rightarrow \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{3Q}{\pi r^2}\hat{\imath}}$$

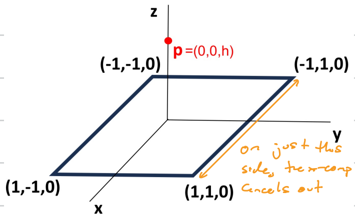
Example

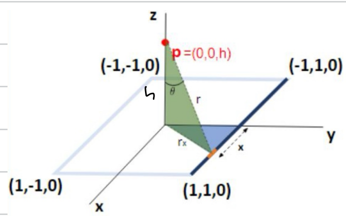
A plastic rod with a uniformly distributed charge Q is shaped forming a square whose side is 1m. We place coordinate axes such that the square is on the x-y plane and the origin is the center of the square. What is the electric field (magnitude and direction) at the point p=(0,0,h)?



$\Rightarrow$ This problem's solution will be $\vec{E} = E\hat{k}$

So we just need to calculate the vertical (z) component of one side and add the rest!





Blue triangle: $r_x^2 = x^2 + 1^2$

Green triangle: $r^2 = r_x^2 + h^2$

$\vec{r}_h = h\hat{k}$

$\vec{r}_{xy} = -x\hat{i} - \hat{j} \Rightarrow r_x^2 = x^2 + 1^2$ $\Big\}\ \vec{r} = -x\hat{i} - \hat{j} + h\hat{k}$

So, $r^2 = r_x^2 + h^2$

$\rightarrow r = \sqrt{x^2+h^2+1}$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(x^2+h^2+1)^{3/2}}\left(-x\hat{i}-\hat{j}+h\hat{k}\right)$$

x and y components cancel

$$\Rightarrow \vec{E} = \int \frac{1}{4\pi\varepsilon_0}\frac{dq}{(x^2+h^2+1)^{3/2}}\left(\cancel{-x\hat{i}}-\cancel{\hat{j}}+h\hat{k}\right)$$

Hint: What is the square formed by? Is there any symmetry?

Imagine that the square is made up of many infinitesimally small point charges *dq*. Each *dq* creates a differential electric field of magnitude $dE = k\,dq/r^2$.

Sum up all the *dE*s from all the *dq*s to get the magnitude of the overall electric field.

Integration is a way to add many infinitely small elements.

$$\vec{E}_{Right} = \int d\vec{E}_{Right} = \int_{-1}^{1} \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx}{\left(\sqrt{x^2+h^2+1}\right)^3}\left(-x\hat{i}-\hat{j}+h\hat{k}\right) = +c\hat{j} + d\hat{k}$$

In one side $\hat{j}$ component persists, but disappears when using Left as well

$$\Rightarrow \vec{E}_{Left} = \int d\vec{E}_{Left} = \ldots = -c\hat{j} + d\hat{k}$$

$$\vec{E}_{Top} = \int_{-1}^{1}\frac{1}{4\pi\varepsilon_0}\frac{dy}{\left(\sqrt{x^2+h^2+1}\right)^2}\left(-\hat{i} - y\hat{j} + h\hat{k}\right)$$

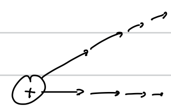
$$\vec{E}_{Bottom} = \int_{-1}^{1}\frac{1}{4\pi\varepsilon_0}\frac{dy}{\left(\sqrt{x^2+h^2+1}\right)^2}\left(+\hat{i} - y\hat{j} + hk\right)$$

So $\boxed{\vec{E} = \vec{E}_{Right}\cdot 4\hat{k}}$

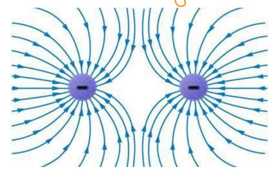
Gauss Law

Electric Field Lines

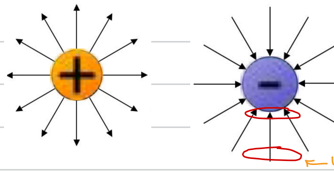
Single Charge



Two charges



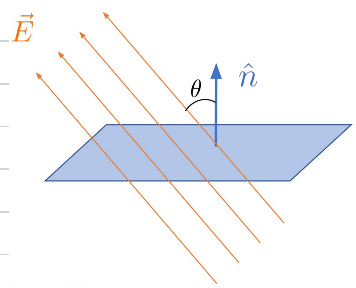
(i) Lines are tangent to $\vec{E}$ at every point in space
(ii) The density of lines is proportional to $|\vec{E}|$
(iii) Lines never cross
(iv) No closed loops
(v) Lines start or stop on charges
↳ start should be a positive charge, end should be a negative charge
↳ This rule inherently enforces part (iv)



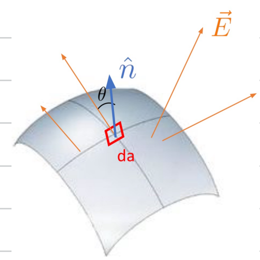
Recall Vector Multiplication
$$\begin{cases} \vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z \\ \quad = |\vec{A}||\vec{B}|\cos\theta \end{cases}$$
for θ the angle between $\vec{A}$ and $\vec{B}$

Electric Flux

- Flat surface and uniform $\vec{E}$
- Defined:
$$\Phi_E = A\hat{n}\cdot\vec{E} = A\,\underbrace{|\hat{n}|}_{1}\,|\vec{E}|\cos\theta = A|\vec{E}|\cos\theta$$
    - $\hat{n}$ is a unit vector perpendicular to the surface (normal vector)
    - θ is the angle between $\hat{n}$ and $\vec{E}$
    - A is the area of the surface



- Non-Flat surface and/or Non-uniform $\vec{E}$
$$\Phi_E = \int \hat{n}\cdot\vec{E}\,da = \int \underbrace{|\hat{n}|}_{1}|\vec{E}|\cos\theta\,da = \int |\vec{E}|\cos\theta\,da$$
    - $\hat{n}$ is a unit vector perpendicular to the surface da
    - θ is the angle between $\hat{n}$ and $\vec{E}$



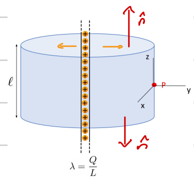
Gauss Law

- First of Maxwell's Equations
- In words: The electric flux through a closed surface = $\dfrac{\text{Amount of charge inside the surface}}{\varepsilon_0}$
- Equation:
$$\Phi_{E_{\text{closed}}} = \oint \hat{n}\cdot\vec{E}\,da = \frac{Q}{\varepsilon_0}$$
    - $\hat{n}$ points outside the surface by convention
    - $\oint \vec{E}\cdot d\vec{a} = \frac{Q}{\varepsilon_0}$ is also valid, where $d\vec{a} = da\cdot\hat{n}$

• **Example / Review:** Infinite Line

Assumptions: rod is very long ($L \to \infty$)

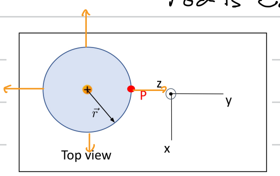
rod is very thin (1-D)



Length ℓ of rod inside cylinder

$\Rightarrow Q_{ENC} = \lambda \ell$

$$\phi_E = \frac{Q_{ENC}}{\varepsilon_0} = \frac{\lambda \ell}{\varepsilon_0}$$



$$= \int_{Cap\uparrow} \hat{n}\,\vec{E}\,da + \int_{Cap\downarrow} \hat{n}\,\vec{E}\,da + \int_{side} \hat{n}\,\vec{E}\,da$$

CAPS

$\vec{E} = E\hat{r}$ — radial

$\hat{n} = \pm\hat{k}$ $\Big\}\ \theta = 90°$

Extra credit Question

What is the direction of the total electric field @ P? $\hat{S}$

$$\boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\hat{n}\ ,\quad E = \frac{\lambda}{2\pi\varepsilon_0 r}}$$

$$\underbrace{2\int_{Caps} |\hat{n}||\vec{E}|\cos\theta\, da}_{\theta = 90°,\ so\ 0} + \int_{side} \hat{n}\,\vec{E}\,da$$

SIDE

$\vec{E} = E\hat{r}$ — radial

$\hat{n} = \hat{r}$ $\Big\}\ \theta = 0$

$$= 0 + \int |\hat{n}||E|\cos\theta\, da$$

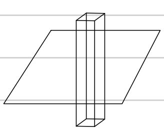
$$= \int E\,da = E\int da = EA$$

The magnitude is constant for all points on SIDE

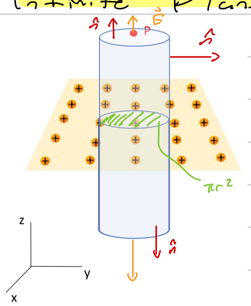
$$\Phi_E = E \cdot 2\pi r \ell$$

$$E \cdot 2\pi r \cancel{\ell} = \frac{\lambda \cancel{\ell}}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Infinite Plane



Also works for this and smaller



Direction of Electric field?

$A = \pi r^2$

$$\sigma = \frac{Q}{L^2} \qquad \oint \hat{n}\,\vec{E}\,da = \frac{Q_{ENC}}{\varepsilon_0} \qquad Q_{ENC} = \sigma\pi r^2$$

↑ Surface Charge density

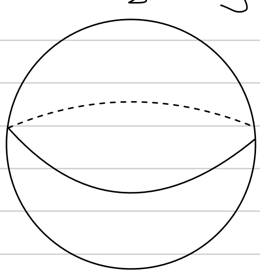
$$= \int_{Caps} \hat{n}\,\vec{E}\,da = 2\int E\,da = 2EA = 2E\pi r^2$$

SIDE $\hat{n} = \hat{r}$ $\theta = 90°$

CAPS $\hat{n} = \pm\hat{k}$ $\theta = 0$

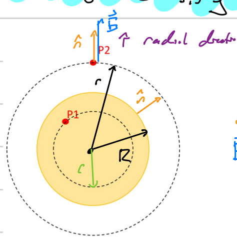
$$2E\cancel{\pi r^2} = \frac{\sigma\cancel{\pi r^2}}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{2\varepsilon_0}$$

Spherical Symmetry



Charge Density: $\rho = \frac{Q}{V}$ $\left\{\frac{C}{m^3}\right\}$

$$\oint \hat{n}\cdot\vec{E}\,da = \frac{Q_{enc}}{\varepsilon_0}$$



Calculate the Electric field at points P1 and P2 **using Gauss Law** of the sphere of Radius R and charged with a density ρ

$\hat{n} = \hat{r}$

$\vec{E} = E\hat{r}$

$$\frac{Q_{enc}}{\varepsilon_0} = \frac{\rho\frac{4}{3}\pi R^3}{\varepsilon_0}$$

Flux through gaussian surface which is outer radius sphere → $\oint E\cos(0)\,da = Ea = E(4\pi r^2)$

$r > R$: $\frac{\rho\frac{4}{3}\pi R^3}{\varepsilon_0} = E(4\pi r^2) \Rightarrow \boxed{E = \frac{\rho R^3}{3\varepsilon_0 r^2}}$

outside the sphere

enclosed radius

$r < R$: $\hat{n} = \hat{r}$

$\vec{E} = E\hat{r}$

$$\frac{Q_{enc}}{\varepsilon_0} = \frac{V\rho}{\varepsilon_0} = \frac{\frac{4}{3}\pi r^3\rho}{\varepsilon_0}$$

Gaussian surface is r

$$\oint E\,da = EA = E(4\pi r^2)$$

$$\Bigg\}\ \frac{\rho\frac{4}{3}\pi r^3}{\varepsilon_0} = E(4\pi r^2) \Rightarrow \boxed{E = \frac{\rho r}{3\varepsilon_0}}$$

inside the sphere

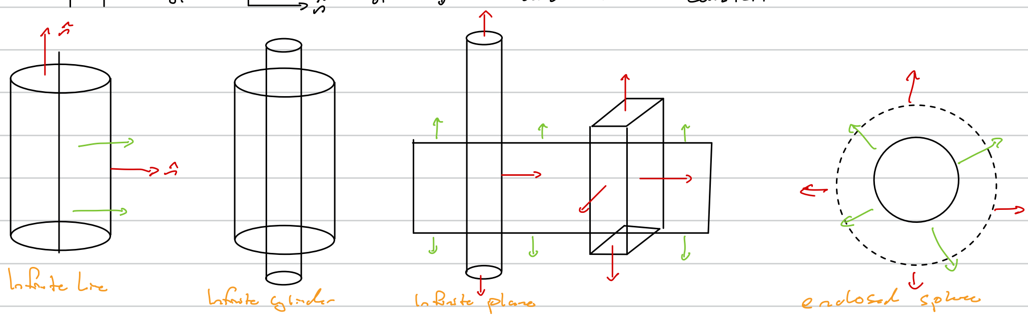
Any charge outside the gaussian surface does not affect the flux

Exam 1 Review Lecture

Gauss Law

$$\oint \vec{E}\,\hat{n}\,da = \oint \vec{E}\cdot d\vec{a}$$

$\vec{E} \uparrow \uparrow \hat{n}$ or $\vec{E} \perp \hat{n}$ or $\vec{E} \updownarrow \hat{n}$ and $\vec{E}$ is constant



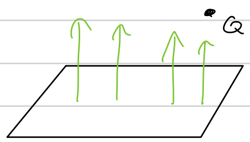
Coulombs Law

- $\vec{F} = Q\vec{E}$
- E by point q: $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

If you have only two charges: $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$

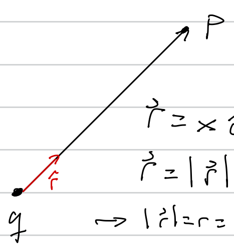
But if not 2 point-like:

Q's $\vec{F}$?



$$\vec{E} = \frac{\sigma}{2\varepsilon_0} \rightarrow \vec{F} = Q\frac{\sigma}{2\varepsilon_0}$$

$\hat{r}$ vs. $\vec{r}$



$\vec{r} = x\hat{i} + y\hat{j}$

$\vec{r} = |\vec{r}|\cdot\hat{r}$

Direct Integration

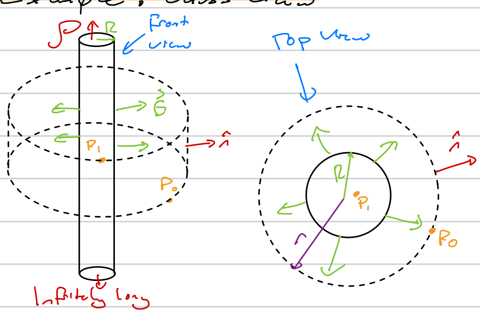
$\rightarrow |\vec{r}| = r = \sqrt{x^2+y^2}$ magnitude

$\rightarrow \hat{r} = \frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j}$ direction

charge densities

$\lambda = \frac{C}{m}$ linear $\quad \sigma = \frac{C}{m^2}$ surface $\quad \rho = \frac{C}{m^3}$ volume

Example: Gauss Law



$$\oint \vec{E}\cdot\hat{n}\,da = \frac{Q_{enc}}{\varepsilon_0} = \frac{\pi R^2 l\cdot\rho}{\varepsilon_0}$$

$\vec{E} = E\hat{r}$

Sides: $\hat{n} = \hat{r}$

CAPS: $\hat{n} = \pm\hat{k}$

$$= 2\int_{CAPS} E\cos(90)\,da + \int_{SIDES} E\cos 0\,da$$

$$= 0 + Ea = E(2\pi r l)$$

area of sides

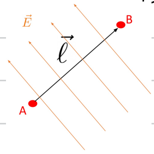
$$\Rightarrow \boxed{E = \frac{R^2\rho}{2r\varepsilon_0}}$$ outside the cylinder

← Cannot depend on parameter of Gauss surface (l in this case)

Midterm Exam 2

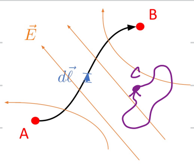
Circulation Law

Line Integrals



Straight line & uniform $\vec{E}$

- Line Integral $= \vec{\ell} \cdot \vec{E}$ for $\vec{\ell}$ the displacement vector
  - ↳ For this simple application (straight line & uniform $\vec{E}$)



General

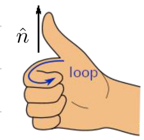
- Line integral $= \int_A^B \vec{E} \cdot d\vec{\ell}$

Closed Loops

- Circulation: $\oint \vec{E} \cdot d\vec{\ell}$

Right Hand Rule

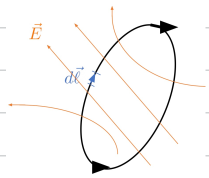
- Stokes convention
- Use direction of loop to determine $\hat{n}$ direction using:



Circulation

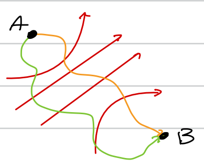
- The circulation of the electric field equals zero

→ Circulation $= \oint d\vec{\ell} \cdot \vec{E} = 0$
(electrostatics)



- **Alternative:** The line integral between two points do not depend on path

$$\int_A^B \vec{E} \cdot d\vec{\ell} = -\int_B^A \vec{E} \cdot d\vec{\ell} \left\{ \int_A^B \vec{E} \cdot d\vec{\ell} \right\}$$



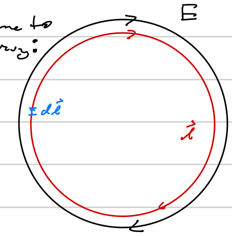
$$\Rightarrow \oint \vec{E} \cdot d\vec{\ell} = \int_A^B \vec{E} \cdot d\vec{\ell} + \int_B^A \vec{E} \cdot d\vec{\ell} = 0$$

Choose the path that is most easy to integrate

No closed Loops!

(Proof by Contradiction)

Assume the contrary:



So, $\oint \vec{E} \cdot d\vec{\ell} = \oint |\vec{E}||d\vec{\ell}|\cos\theta$

>0 strictly

$\theta = 0$ $\quad = \oint |\vec{E}||d\vec{\ell}|$

$= (+) > 0$

$\boxed{\text{No closed loops!}}$ but $\oint \vec{E} \cdot d\vec{\ell} = 0$ by circ. law

Application

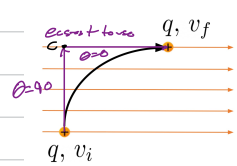
P121

- **Recall:** Work-Energy Theorem

$$\int_{i \to f} d\vec{\ell} \cdot \vec{F}_{Total} = \Delta K = \frac{1}{2} m (v_f^2 - v_i^2)$$

- But for electric force we don't need the path!

$$\to \int_{i \to f} d\vec{\ell} \cdot (q\vec{E}) = \Delta K = \frac{1}{2} m (v_f^2 - v_i^2) = \text{Same for all paths btwn } i \text{ and } f$$



Example

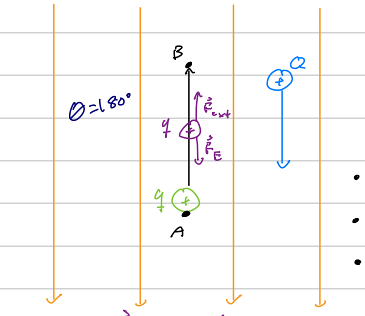
$$\int_A^B \vec{E} d\vec{\ell} = \int_A^C \vec{E} d\vec{\ell} + \int_C^B \vec{E} d\vec{\ell}$$

$$= \int_A^C E\,d\ell \cos 90 + \int_C^B E\,d\ell \cos 0$$

$$= E \int_C^B d\ell = E \cdot L$$

Electrostatic Energy

Overview / Introduction



$\vec{E} = E(-\hat{y})$

$\vec{F} = Q\vec{E} = QE(-\hat{y})$

- Apply force to bring $+q$ to B from A
- Counteracts $\vec{E}$
- Release q at B → moves down

$\vec{F}_{ext} = -\vec{F}_E$ by N2L

Assumption: We move the charge very slowly, so the initial, final, and intermediate velocity is zero and hence there is No Kinetic Energy

↳ Spend an infinite amount of time from A to B ($v = \frac{\Delta x}{\Delta t}$)

→ Energy Input = Work done by external agent $= \int_A^B \vec{F}_{ext} \cdot d\vec{l}$

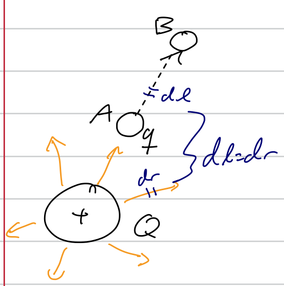
Using our assumption

- The external force matches the electric force: $U_B - U_A \equiv \Delta U = \int_A^B d\vec{l} \cdot \vec{F}_{ext} = -\int_A^B d\vec{l} \cdot (\underbrace{q\vec{E}}_{\vec{F}_E})$, with $\vec{F}_{ext} = -\vec{F}_E$

  ↳ In this simple (uniform $\vec{E}$ ✓) case: $U_B - U_A \equiv \Delta U = qE(z_B - z_A)$ wow! GPE ($U_G = mgy$)

This is a particular formula, not general!

Point Charge



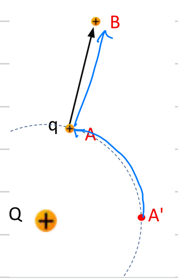
$\Delta U = -q\int_A^B d\vec{l} \cdot \vec{E} = -q\frac{Q}{4\pi\varepsilon_0}\int_A^B \frac{\hat{r} \cdot \hat{r}}{r^2}\, dl = -\frac{qQ}{4\pi\varepsilon_0}\int_A^B \frac{dl}{r^2} = \frac{-qQ}{4\pi\varepsilon_0}\int_A^B \frac{dr}{r^2}$

$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$

$d\vec{l} = \hat{r}\, dl$

$\Rightarrow \Delta U = \frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$

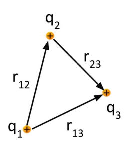
If $r_A \to \infty$ chosen as reference: $U = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r}$



$\Delta U_{AB} = \Delta U_{A'B}$

choose path in green
→ along dashed line
integral is 0 as you
move tangent

Arrangement of Charges



- The total energy of the system equals the sum of the energy of each pair of charges

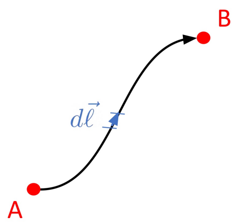
$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right)$$

Electrostatic Potential

- Defined Φ, also seen as V
- Do not confuse w/ electrostatic energy (U)
- Another way to describe an electric field
- Electrostatic potential is a scalar (not a vector)
- To know $\vec{E}$ or Φ for any position space provides same amount of info (only electrostatics)
- If you know one you can figure out the other $\quad \vec{E}(r) \rightleftarrows \Phi(r)$ ← not discussed here, not easily but derivable
- Voltage $\quad V = \Delta\Phi$

Examples of... Scalar Fields: Elevation, Temperature. Vector Fields: Gravity, Wind.

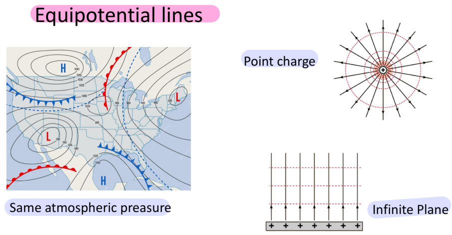
Definition



- $\Delta\Phi = \Phi(B) - \Phi(A) \equiv -\int_A^B d\vec{\ell}\cdot\vec{E}$
- If we set a reference point ($\Phi = 0$)

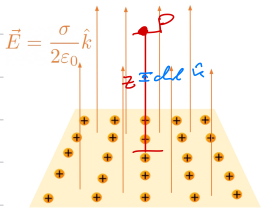
$$\Phi(\vec{r}) \equiv -\int_{ref\to\vec{r}} d\vec{\ell}\cdot\vec{E} \qquad \Phi(ref) = 0$$

Scalar Quantity - Use Equipotential Lines



- Always perpendicular to electric lines

Example: Potential - Infinite Plane

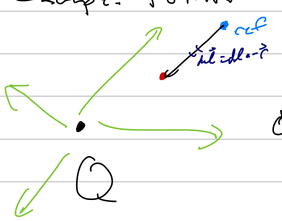


- Choose $z = 0$ so $\Phi(z=0) = 0$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}(\pm\hat{k}) \quad \text{(from gauss law)}$$

$$\Delta\Phi = \cancel{\Phi_{ref}} - \Phi_P = -\left(-\int_{ref\to P} d\vec{\ell}\cdot\vec{E}\right) = \int E\, d\ell\cos\theta = E\int_{z_P}^{0_{ref}} dz = Ez = \boxed{-\frac{\sigma z}{2\varepsilon_0}}$$

Example: Potential Point Like Charge



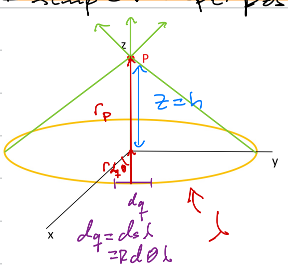
$$\Phi(r\to\infty) = 0$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

$$\Phi(r) = -\int d\vec{\ell}\cdot\vec{E} = -\int \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}\cdot\hat{r}\, dr$$

$$\boxed{\Phi(r) = \frac{Q}{4\pi\varepsilon_0 r}}$$

Example: Superposition Principle ($\vec{E}$ vs. Φ)



a) Develop the field @ P

Should point up

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{z^2+R^2}^{\,3}}\left(z\hat{k} - R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\right)$$

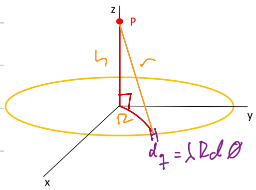
$$\left.\begin{aligned}\vec{r}_P &= z\hat{k}\\ \vec{r}_{dq} &= R\cos\theta\,\hat{\imath} + R\sin\theta\,\hat{\jmath}\end{aligned}\right\}\ \vec{r} + \vec{r}_{dq} = \vec{r}_P \rightarrow \vec{r} = z\hat{k} - R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$

$$\hookrightarrow\ r = \sqrt{z^2 + R^2(1)}$$

$$\Rightarrow \vec{E} = \frac{R\lambda}{4\pi\varepsilon_0(z^2+R^2)^{3/2}}\int_0^{2\pi}\left(z\hat{k} - \cancel{R\cos\theta\,\hat{\imath}} - \cancel{R\cos\theta\,\hat{\jmath}}\right)d\theta \rightarrow \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\pi R\lambda z}{(z^2+R^2)^{3/2}}\hat{k}}$$

$\cos\theta$ and $\sin\theta$ over 0 to 2π = 0

b) Develop potential without using $\vec{E}$



$\Phi(P) = ?$

$$d\Phi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$$

Its easier, in general, to find Φ

$$\Phi(h) = \int_0^{2\pi} \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\, d\theta}{\sqrt{R^2+z^2}} \Rightarrow \boxed{\Phi(h) = \frac{\lambda R}{2\varepsilon_0}\frac{1}{\sqrt{R^2+h^2}}}$$

$r = \sqrt{R^2+h^2}$

Converting between Φ and $\vec{E}$ ← outside of scope of course

$$\begin{tikzcd} \vec{E} \arrow[r, bend left, "\Phi = -\int d\vec{l}\cdot\vec{E}"] & \Phi \arrow[l, bend left, "\vec{E} = -\nabla\Phi"] \end{tikzcd}$$

The gradient
Omg calc 3!

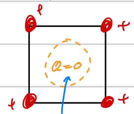
$$\vec{E} = -\nabla\Phi = -\left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right)$$

Conductors

Perfect Conductor — Ideal, doesn't exist in nature

- Electric Field inside is zero
  - If $E \neq 0$, charges rearrange themselves so $E = 0$ — instantaneous for perfect conductors
  - After some time, real conductors behave this way — time is a factor for real conductors (low amt)
- In the surface (outside), the Electric field is perpendicular to the surface of the conductor

→ All net charges reside in the surface

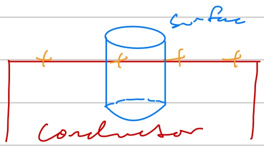


$Q_{enc} = \varepsilon_0 \oint d\vec{a} \cdot \vec{E} = 0$

Enclosed charge is 0, and all charge is in surface

$Q = Q_+ - Q_- = 0$ (0 charge, but particles exist inside)

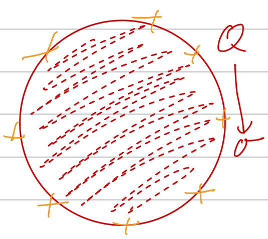
Charge Density



$$\oint d\vec{a} \cdot \vec{E} = \int_{\uparrow} E\, da + \int_{\downarrow} 0 + \int_{side} (0) = E A_{cap}$$

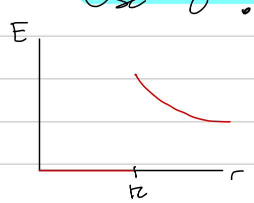
inside conductor; $\hat{n}$ beam; $\vec{E}$ is vertical; $\theta = 90°$

$$\longrightarrow E \cancel{A_{cap}} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma \cdot \cancel{A_{cap}}}{\varepsilon_0} \quad \text{so} \quad \boxed{E = \frac{\sigma}{\varepsilon_0}, \quad \sigma = E\varepsilon_0}$$

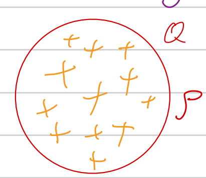


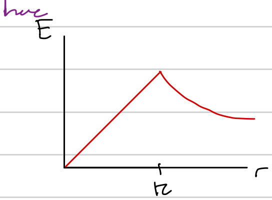
If the sphere is a perfect conductor, all charges are on the surface!

Use σ!

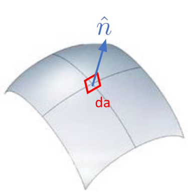


Different from solutions worked w/ previously as they had charges evenly distributed throughout volume; we use ρ here





Force on the Conductor



Force: $\vec{F} = qE$ (known)

$\rightarrow d\vec{F} = \frac{1}{2}\sigma\, da\, E\hat{n}$

$\vec{E} = E\hat{n}$ ← outside conductor

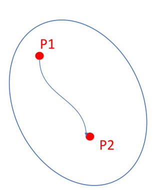
$\frac{1}{2}E$ ← on surface? average

$E = 0$ ← inside conductor

$$\vec{F} = \frac{1}{2}\int \sigma\, E\hat{n}\, da \quad \left(\text{use } \sigma = E\varepsilon_0 \text{ or } E = \frac{\sigma}{\varepsilon_0}\right)$$

$$= \frac{1}{2}\varepsilon_0 \int E^2 \hat{n}\, da = \frac{1}{2\varepsilon_0}\int \sigma^2 \hat{n}\, da$$ ← always positive (outwards)

Potential inside conductor



$$\Delta\Phi = \int d\vec{l} \cdot \vec{E} = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$$

→ but $E = 0$ inside conductor ∴ $\Delta\Phi = 0$

So, conductors are equipotential surfaces

Watch out!

$\Delta\Phi = \Phi_B - \Phi_A = 0$

$\longrightarrow \Phi_A = \Phi_B \neq 0$

↑ just their difference

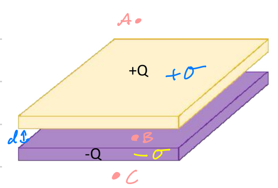
Had to use this on final last year

Capacitance

- Charge Q is proportional to potential difference (often called voltage) between two electrodes by a quantity called Capacitance

$Q = CV$ where $Q \equiv$ Charge (coolombs, C)
$C \equiv$ Capacitance (Farads, F)
$V \equiv$ Voltage (Volts, V) ⟵ Convention: Use volts instead of potential difference (equiv to $\Delta\Phi$)

Parallel plates



Assume Area $A \gg d^2$

$C = \frac{Q}{V} = \frac{\sigma A}{}$

$V = \Delta\Phi = -\int_1^2 d\vec{l}\cdot\vec{E}$

Q: What's $\vec{E}$ at all 3 Points?

above (+)	A	$\uparrow\frac{\sigma}{2\varepsilon_0}$ $\cancel{\downarrow -\frac{\sigma}{2\varepsilon_0}}$	$+\sigma$	$E = 0$
btwn	B	$d\updownarrow$ $\downarrow 2\times\frac{\sigma}{2\varepsilon_0}\downarrow$		$E = \frac{\sigma}{\varepsilon_0}$
below (-)	C	$\cancel{\downarrow -\frac{\sigma}{2\varepsilon_0}}$ $\cancel{\uparrow\frac{\sigma}{2\varepsilon_0}}$	$-\sigma$	$E = 0$

Derive from Gauss Law (previous lectures)

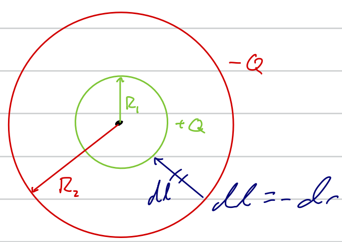
Use $\vec{E}$ @ all 3 points to get C of parallel plate capacitors?

$C = \frac{Q}{|V|} > 0$, $|V| = \Delta\phi$

$$\Delta\phi = -\int_{P_1\to P_2} d\vec{l}\cdot\vec{E} = -\int_{P_1\to P_2} dl\,E\cos 180 = E\int_0^d dl \quad d\left(\frac{\sigma}{\varepsilon_0}\right), \quad Q = \sigma A$$

So $\boxed{C = \frac{\sigma A}{\frac{d\sigma}{\varepsilon_0}} = \frac{\varepsilon_0 A}{d}}$

Example



$C = \frac{Q}{V}$, $V = \Delta\phi = -\int_{-\to+} \vec{E}\cdot d\vec{l}$

① Determine $\vec{E}$ in 3 points (only really need term btwn ⊖ and ⊕, just doing this for practice)

$\vec{E}_{inside} = 0$ $\quad Q_{enc} = 0$

$\vec{E}_{btwn}$: $Q_{enc} = Q$

$\frac{Q_{enc}}{\varepsilon_0} = E_a \rightarrow E = \frac{Q}{4\pi r^2\varepsilon_0}$ (Area of Gaussian surface)

- spherical & hollow shells
- Conductive materials
- Determine Capacitance (C)

$\vec{E}_{outside} = 0$ $\quad Q_{enc} = Q - Q = 0$

② Determine $\Delta\phi$

$$\Delta\Phi = -\int_{-\to+} E\,dl\cos 180 = \int \frac{Q}{4\pi r^2\varepsilon_0}\,dl = -\int_{R_2}^{R_1} \frac{Q}{4\pi r^2\varepsilon_0}\,dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\Rightarrow \boxed{C = \frac{Q}{V} = 4\pi\varepsilon_0\left(\frac{1}{R_1} - \frac{1}{R_2}\right)^{-1}}$$