



Midterm Exam 1



Electric Charges & Fields, Coulomb's Law, Superposition

Electric Charge

- charge is a fundamental prop of matter
 - two types: \textcircled{P} & \textcircled{G} matter (atoms) are normally neutral
- Triboelectricity:** Friction transfers charge

Metal Rods in Demo?

- Plastic & crystal are insulators
 - charge "stays where put"
- Metals are conductors
 - charge "flow freely"

Charge is conserved

- Rubbing transfers charges, but does not create nor destroy it

Charge is quantized

- charge is always a multiple of $e = 1.6 \times 10^{-19}$ Coulombs (C) \Rightarrow electron = $-e$, proton = $+e$

Why the rods are moving!

- Attraction or Repulsion \Rightarrow There is a force between the charges

\hookrightarrow Force is known as coulomb's force

$$F = \frac{1}{4\pi\epsilon_0} \cdot \frac{qQ}{r^2} \quad \text{where } \frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}$$

\square When two charges q and Q are same signed $\Rightarrow F$ is positive = Repulsive

\square When they have opposite signs $\Rightarrow F$ is negative = Attractive

Demo - Rods

$$\textcircled{P} \textcircled{P} \textcircled{G} \leftarrow \rightarrow$$

$$\textcircled{G} \textcircled{G} \textcircled{P} \leftarrow \rightarrow$$

$$\textcircled{P} \textcircled{G} \textcircled{P} \rightarrow \leftarrow$$

Note: the metal is still neutral

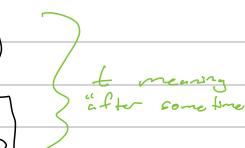
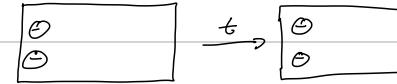
Demoval / Metal

$$\textcircled{G} \quad \textcircled{G}$$

$$\textcircled{P} \quad \textcircled{G}$$

$$\textcircled{P} \quad \textcircled{G}$$

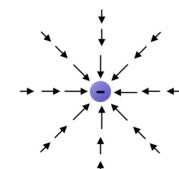
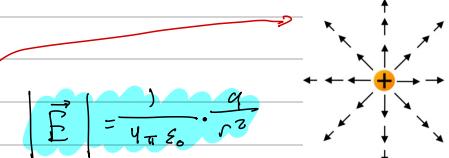
$$\textcircled{G} \quad \textcircled{G}$$



Electric Field and Coulomb's Law

Laws of Electrostatics

- Electrostatics = charges @ rest
- charges produce electric fields
- electric fields push charges



$$q \quad \vec{r} \quad r = |\vec{r}|$$

$$|\vec{E}| = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2}$$

\vec{r} is a unit vector that points away from charge q

or alternatively so $\vec{r} = \frac{\vec{r}}{|\vec{r}|}$ and $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \vec{r}$

Equivalent!

Superposition Principle

In Words:

Electric Field produced = Sum of the electric field produced by many point charges by each point charge independently

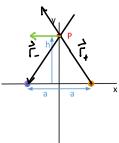
$$\text{In Maths: } \vec{E}(q_1, q_2) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_1^2} \hat{r}_1 + \frac{1}{4\pi\epsilon_0} \frac{q_2}{r_2^2} \hat{r}_2$$

Second Law of Electrostatics

$$\text{Fields push/pull charges: } \vec{F} = Q \vec{E} \Rightarrow \vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \hat{r}$$

EN? EC? EN?

Example: Point Charges



Calculate the Electric field at point P.

Pos vector \vec{r} from Q to $P: -a\hat{i} + b\hat{j}$

Pos vector \vec{r} from Q to $P: a\hat{i} + b\hat{j}$ (not mass, position not dependent charge!)

Calculate the Electric Field at P:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} \vec{r} \rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0} \left(\frac{1}{(\sqrt{a^2+b^2})^3} [q(-a\hat{i}+b\hat{j}) + -q(a\hat{i}+b\hat{j})] \right) = \frac{2qa}{4\pi\epsilon_0 \sqrt{a^2+b^2}^3} \cdot (-\hat{i})$$

$$r_- = a\hat{i} + b\hat{j} \rightarrow r_- = \sqrt{a^2+b^2} \quad \rightarrow \vec{E}_- = \vec{E}_f + \vec{E}_-$$

Continuous Distribution of Charge

$$dq = \lambda dy$$

A rod of length L is charged with a positive amount of charge Q
 \rightarrow Calculate the Electric field at P

Assumptions: Rod is very long ($L \rightarrow \infty$)

Rod is very thin ($1-D$)

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} \vec{r} \rightarrow d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \vec{r}$$

$$\vec{r} = x\hat{i} - y\hat{j} \quad \rightarrow d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \cdot (x\hat{i} - y\hat{j})$$

Example

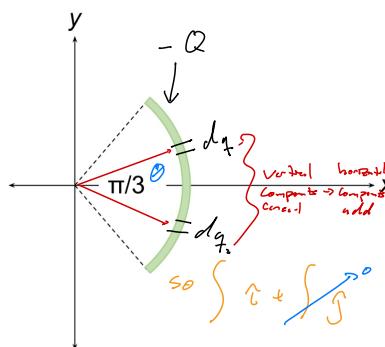
A plastic rod with a uniformly distributed charge $-Q$ is bent into a circular arc of radius r that subtends an angle of $\pi/3$ radians. We place coordinate axes such that the axis of symmetry of the rod lies along the x-axis and the origin is the center of curvature for the rod. What is the electric field (magnitude and direction) at the origin?

Hint: Imagine that the arc is made up of many infinitesimally small point charges dq . Each dq creates a differential electric field of magnitude $dE = k dq/r^2$.

Sum up all the dEs from all the dqs to get the magnitude of the overall electric field.

Integration is a way to add many infinitesimally small elements.

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \vec{r}$$



To integrate, let's use polar! $(x, y) \rightarrow (r, \theta)$

$$\int dE_x = \frac{1}{4\pi\epsilon_0} \int_{-\pi/6}^{\pi/6} \frac{r^2 \cos \theta}{r^3} d\theta$$

$$= \frac{1}{4\pi\epsilon_0 r} \left[\sin \theta \right]_{-\pi/6}^{\pi/6}$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{3Q}{\pi r^2} \rightarrow$$

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{3Q}{\pi r^2} \hat{i}}$$

Definition: Charge per unit length or linear charge density

$$\lambda = \frac{\text{Charge}}{\text{Length}} = \frac{Q}{L}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \sum_{i=1}^N \vec{E}_i = \int d\vec{E}$$

$$\text{so, } \vec{E} = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\lambda}{\sqrt{x^2+y^2}} (x\hat{i} - y\hat{j}) dy \quad \text{Solve } x \text{ and } y \text{ separately}$$

$$|d\vec{E}| = \left| \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \vec{r} \right|$$

$$\begin{cases} \cos \theta = \frac{a}{r} \\ \sin \theta = \frac{b}{r} \end{cases} \rightarrow \vec{r} = -r\cos \theta \hat{i} - r\sin \theta \hat{j}$$

$$x\text{-component of } |dE|: \quad dE_x = \frac{1}{4\pi\epsilon_0} \frac{da}{r^2} \cos \theta$$

We need to integrate over all θ , b , dE_x terms of dq , not $d\theta$!

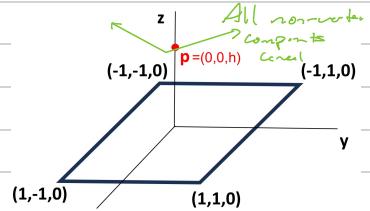
$$\Rightarrow dq = \lambda ds \quad \text{where } ds \text{ is segment of arc}$$

$$\Rightarrow \frac{2\pi r}{\theta} \text{ so } ds = r d\theta \rightarrow dq = r \lambda d\theta$$

$$\text{Just think length = angle * radius}$$

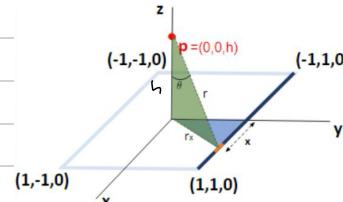
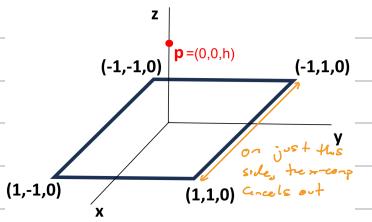
Example

A plastic rod with a uniformly distributed charge Q is shaped forming a square whose side is 1m. We place coordinate axes such that the square is on the x-y plane and the origin is the center of the square. What is the electric field (magnitude and direction) at the point $p=(0,0,h)$?



\Rightarrow The problem solution will be $\vec{E} = E \hat{k}$

So we just need to calculate the vertical (z) component of one side and add the rest!



Hint: What is the square formed by? Is there any symmetry?

Imagine that the square is made up of many infinitesimally small point charges dq . Each dq creates a differential electric field of magnitude $dE = k dq/r^2$.

Sum up all the dEs from all the dqs to get the magnitude of the overall electric field.

Integration is a way to add many infinitely small elements.

$$\text{Blue triangle: } r_x^2 = x^2 + 1^2$$

$$\text{Green triangle: } r^2 = r_x^2 + h^2$$

$$\vec{r}_h = h \hat{k}$$

$$\vec{r}_{xy} = -x \hat{i} - \hat{j} \Rightarrow r_x^2 = x^2 + 1^2$$

$$\therefore r^2 = r_x^2 + h^2$$

$$\Rightarrow r = \sqrt{x^2 + h^2}$$

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{(x^2 + h^2)^{3/2}} (-x \hat{i} - \hat{j} + h \hat{k})$$

$$\Rightarrow \vec{E} = \int \frac{1}{4\pi\epsilon_0} \frac{dq}{(x^2 + h^2)^{3/2}} (-x \hat{i} - \hat{j} + h \hat{k})$$

$$\vec{E}_{\text{Top}} = \int d\vec{E}_{\text{Top}} = \int_{-1}^1 \frac{1}{4\pi\epsilon_0} \frac{dq}{(\sqrt{x^2 + h^2})^3} (-x \hat{i} - \hat{j} + h \hat{k}) = +c \hat{j} + d \hat{k}$$

$$\Rightarrow \vec{E}_{\text{Left}} = \int d\vec{E}_{\text{Left}} = \dots = -c \hat{i} + d \hat{k}$$

In one side \hat{j} component part but disappears when using Left cancell

$$\vec{E}_{\text{Top}} = \int_{-1}^1 \frac{1}{4\pi\epsilon_0} \frac{dq}{(\sqrt{x^2 + h^2})^3} (-x \hat{i} - \hat{j} + h \hat{k})$$

$$\vec{E}_{\text{Bottom}} = \int_{-1}^1 \frac{1}{4\pi\epsilon_0} \frac{dq}{(\sqrt{x^2 + h^2})^3} (+\hat{i} - \hat{j} + h \hat{k})$$

$$\text{so } \vec{E} = \vec{E}_{\text{Top}} + 4 \hat{k}$$

Gauss Law

Electric Field Lines

- (i) Lines are tangent to \vec{E} at every point in space
- (ii) The density of lines is proportional to $|\vec{E}|$
- (iii) Lines never cross

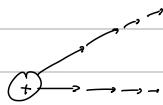
(iv) No closed loops

(v) Lines start or stop on charges

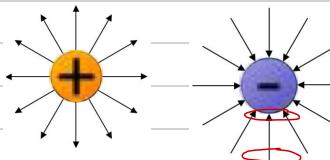
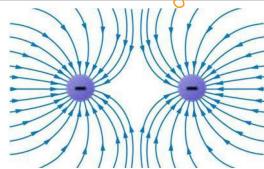
↳ Start should be a positive charge, and
should be on negative charge

↳ This rule inherently enforces part (iv)

Single Charge



Two charges



Recall Vector Multiplication

$$\begin{cases} \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \\ = |\vec{A}| |\vec{B}| \cos \theta \end{cases}$$

for θ the angle between \vec{A} and \vec{B}

Electric Flux

• Plot surface and uniform \vec{E}

• Defined: $\Phi_E = A \hat{n} \cdot \vec{E} = A |\hat{n}| |\vec{E}| \cos \theta = A |\vec{E}| \cos \theta$

o \hat{n} is a unit vector perpendicular to the surface (normal vector)

o θ is the angle between \hat{n} and \vec{E}

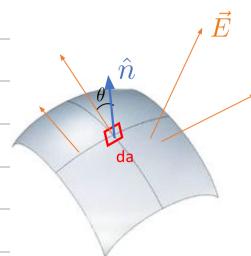
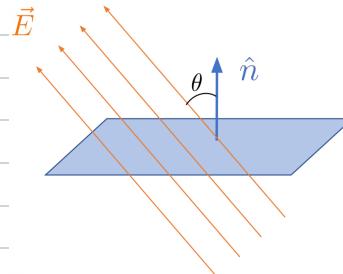
o A is the area of the surface

• Non-Plot surface and/or Non-uniform \vec{E}

$$\Phi_E = \int \hat{n} \cdot \vec{E} dA = \int |\hat{n}| |\vec{E}| \cos \theta dA = \int |\vec{E}| \cos \theta dA$$

o \hat{n} is a unit vector perpendicular to the surface dA

o θ is the angle between \hat{n} and \vec{E}



Gauss Law

• Rest of Maxwell's Equations

• In words: The electric flux

$$\text{through a closed surface} = \frac{\text{Amount of charge inside the surface}}{\epsilon_0}$$

• Equation:

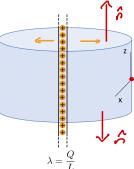
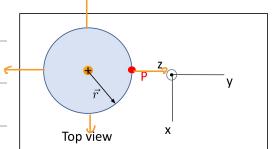
$$\Phi_E = \int_{\text{closed}} \hat{n} \cdot \vec{E} dA = \frac{Q}{\epsilon_0}$$

• \vec{E} points outside the surface by convention

$$\cdot \hat{n} \cdot d\vec{A} = \frac{dA}{\epsilon_0} \Rightarrow \text{Locally, where } d\vec{A} = dA \cdot \hat{n}$$

• Example / Review: Infinite Line

Assumption: rod is very long ($L \rightarrow \infty$)
rod is very thin ($l \rightarrow 0$)



Length l of rod make cylinder

$$\Rightarrow Q_{ENC} = \lambda l$$

$$\phi_E = \frac{Q_{ENC}}{\epsilon_0} = \frac{\lambda l}{\epsilon_0}$$

Ext records
Question

What is the direction of the total electric field at P?

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}, \quad E = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$2 \int_{\text{Cups}} |\vec{E}| \cos \theta d\alpha + \int_{\text{Side}} \vec{E} d\alpha$$

$\theta = 90^\circ, \hat{n} = \hat{z}$

$$= 0 + \int_{\text{Side}} |\vec{E}| \cos \theta d\alpha$$

$$= \int E d\alpha = E d\alpha = EA$$

$$\Phi_E = E \cdot 2\pi r l$$

$$E \cdot 2\pi r l = \frac{\lambda l}{\epsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\epsilon_0 r}$$

CAPS radial

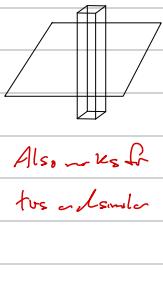
$$\vec{E} = E \hat{r} \quad \left. \begin{array}{l} \vec{E} = E \hat{r} \\ \hat{n} = \pm \hat{z} \end{array} \right\} \theta = 90^\circ$$

SIDE radial

$$\vec{E} = E \hat{r} \quad \left. \begin{array}{l} \vec{E} = E \hat{r} \\ \hat{n} = \hat{r} \end{array} \right\} \theta = 0$$

The magnitude is constant
for all parts on SIDE

Infinite Plane



Direction of Electric Field?

$$A = \pi r^2$$

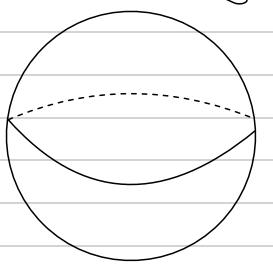
$$\sigma = \frac{Q}{L^2} \quad \int \vec{n} \cdot \vec{E} d\alpha = \frac{Q_{ENC}}{\epsilon_0}$$

$$Q_{ENC} = \sigma \pi r^2$$

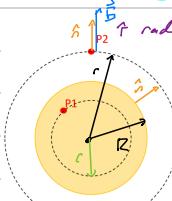
$$\int \vec{n} \cdot \vec{E} d\alpha = 2 \int E d\alpha = 2EA = 2E\pi r^2$$

$$2B\pi r^2 = \frac{\sigma \pi r^2}{\epsilon_0} \Rightarrow E = \frac{\sigma}{2\epsilon_0}$$

Spherical Symmetry



$$\text{Charge Density } \rho = \frac{Q}{V} \quad \left\{ \frac{C}{m^3} \right\}$$



Calculate the Electric field at points P1 and P2 using Gauss Law of the sphere of Radius R and charged with a density ρ

$$\hat{n} = \hat{r}$$

$$\vec{E} = E \hat{r}$$

$$\int \hat{n} \cdot \vec{E} d\alpha = \frac{Q_{enc}}{\epsilon_0}$$

$$\frac{Q_{enc}}{\epsilon_0} = \frac{\rho \frac{4}{3}\pi r^3}{\epsilon_0}$$

Flux through gaussian Surface which is outer radius of sphere $\rightarrow \int E \cos(\theta) d\alpha = E A = E(4\pi r^2)$

$$R > R: \frac{\rho \frac{4}{3}\pi R^3}{\epsilon_0} = E(4\pi r^2) \Rightarrow E = \frac{\rho R^3}{3\epsilon_0 r^2}$$

$$R < R: \hat{n} = \hat{r} \quad \frac{Q_{enc}}{\epsilon_0} = \frac{\rho V}{\epsilon_0} = \frac{\rho \frac{4}{3}\pi r^3}{\epsilon_0}$$

enclosed radius

$$\int E d\alpha = EA = E(4\pi r^2)$$

Gaussian surface is r

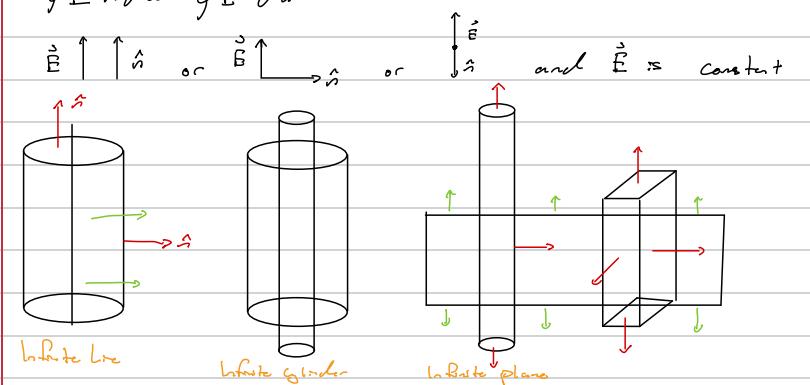
$$\left. \begin{array}{l} \int \hat{n} \cdot \vec{E} d\alpha = \frac{Q_{enc}}{\epsilon_0} \\ \frac{Q_{enc}}{\epsilon_0} = E(4\pi r^2) \end{array} \right\} E = \frac{\rho r}{3\epsilon_0}$$

Any charge outside the gaussian surface does not affect the Flux

Exam 1 Review Lecture

Gauss Law

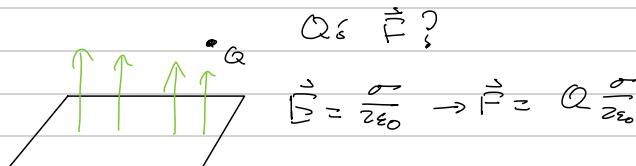
$$\oint \vec{E} \cdot d\vec{a} = \oint \vec{E} \cdot d\vec{a}$$



Coulomb's Law

- $\vec{F} = Q \vec{E}$
 - E by point q : $E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$
- $\left. \begin{array}{l} \text{if you have only} \\ \text{two charges} \end{array} \right\} \Rightarrow \vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$

But if not 2 point-line:



\vec{r} vs. \hat{r}

Direct Integration

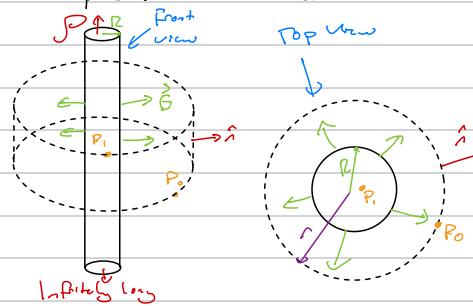
$$\begin{aligned} \vec{r} &= x\hat{i} + y\hat{j} \\ \vec{r} &= |\vec{r}| \cdot \hat{r} \\ \rightarrow |\vec{r}| &= r = \sqrt{x^2 + y^2} \\ \rightarrow \hat{r} &= \frac{x}{\sqrt{x^2 + y^2}} \hat{i} + \frac{y}{\sqrt{x^2 + y^2}} \hat{j} \end{aligned}$$

magnitude direction

charge densities

$$\begin{aligned} \lambda &= \frac{C}{m} & \sigma &= \frac{C}{m^2} & J &= \frac{C}{m^3} \\ \text{linear} & & \text{surface} & & \text{volume} & \end{aligned}$$

Example: Gauss Law



$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} = \frac{\pi D^2 l \cdot P}{\epsilon_0}$$

$$\begin{aligned} \vec{B} &= E \hat{r} \\ \text{Side } \hat{n} &= \hat{r} \\ \text{CAPS. } \hat{n} &= \pm \hat{r} \end{aligned}$$

$$\begin{aligned} &= 2 \int_{\text{CAPS}} E \cos(90^\circ) da + \int_{\text{S.DBS}} E \cos 0^\circ da \\ &= 0 + E a = E(2\pi r l) \end{aligned}$$

$$\Rightarrow E = \frac{\pi r^2 P}{2 r \epsilon_0}$$

outward

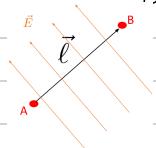
Cannot depend on parameter of Gaussian surface (in this case)

Midterm Exam 2



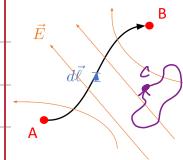
Circulation Law

Line Integrals



Straight line & uniform \vec{E}

- Line Integral = $\int \vec{E} \cdot d\vec{l}$ for $d\vec{l}$ the displacement vector
↳ For this simple application (straight line & uniform \vec{E})



General

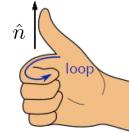
$$\bullet \text{Line integral} = \int_A^B \vec{E} \cdot d\vec{l}$$

Closed Loops

$$\bullet \text{Circulation: } \oint \vec{E} \cdot d\vec{l}$$

Right Hand Rule

- Stokes' theorem
- Use orientation of loop to determine \vec{l} direction using:



Circulation

- The circulation of the electric field equals zero

$$\rightarrow \text{Circulation} = \oint d\vec{l} \cdot \vec{E} = 0$$

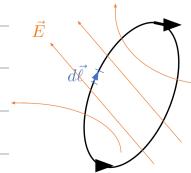
(Coulomb's law)

Alternative: The line integral between two points do not depend on path

$$\int_A^B \vec{E} \cdot d\vec{l} = - \int_B^A \vec{E} \cdot d\vec{l} \quad \left\{ \int_A^B \vec{B} \cdot d\vec{l} \right\}$$

$$\Rightarrow \oint \vec{E} \cdot d\vec{l} = \int_A^B \vec{E} \cdot d\vec{l} + \int_B^A \vec{E} \cdot d\vec{l} = 0$$

Choose the path that is most easy to integrate



No closed loops! (Proof by contradiction)

$$\begin{aligned} \text{Assume to the contrary:} \\ \oint \vec{E} \cdot d\vec{l} &= \int |E| d\ell \text{ (>> 0)} \\ \Theta &= 0 = \int |E| d\ell \\ &= (+) > 0 \end{aligned}$$

No closed loops! but $\oint \vec{E} \cdot d\vec{l} = 0$ by circ. law

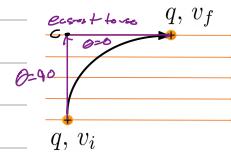
Application / PIZI

• Recall: Work-Energy Theorem

$$\int_{i \rightarrow f} d\vec{l} \cdot \vec{F}_{\text{total}} = \Delta K = \frac{1}{2} m(v_f^2 - v_i^2)$$

• But for electric force we don't need the path!

$$\rightarrow \int_{i \rightarrow f} d\vec{l} \cdot (q \vec{E}) = \Delta K = \frac{1}{2} m(v_f^2 - v_i^2) = \text{Same for all paths b/w i and f}$$

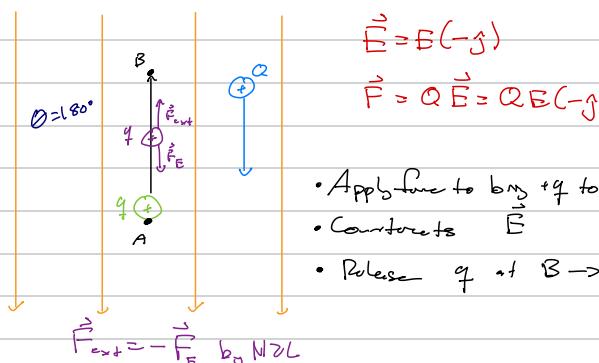


Example

$$\begin{aligned} \int_A^B \vec{E} \cdot d\vec{l} &= \int_A^C \vec{E} \cdot d\vec{l} + \int_C^B \vec{E} \cdot d\vec{l} \\ &= \int_A^C q E \cos 90^\circ d\ell + \int_C^B q E \cos 90^\circ d\ell \\ &= \int_A^B q E d\ell = E \cdot L \end{aligned}$$

Electrostatic Energy

Overview / Introduction

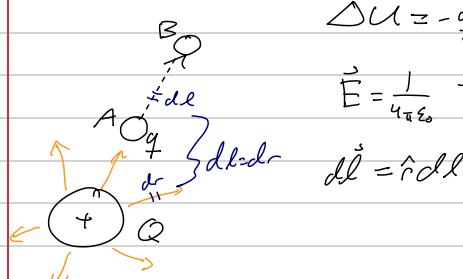


Using our assumption

- The external force matches the electric force: $U_B - U_A = \Delta U = \int_A^B d\vec{l} \cdot \vec{F}_{\text{ext}} = - \int_A^B d\vec{l} \cdot (q \vec{E})$
- In this simple case: $U_B - U_A = \Delta U = q E (z_B - z_A) \rightarrow \text{GPE } (U_0 = mg)$

This is a particular formula, not general!

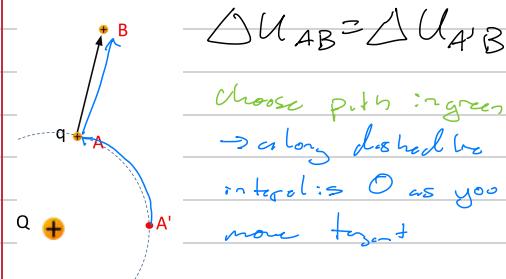
Point Charge



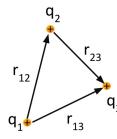
$$\Delta U = -q \int_A^B d\vec{l} \cdot \vec{E} = -q \frac{Q}{4\pi\epsilon_0} \int_A^B \frac{\hat{r} \cdot \hat{r}}{r^2} dr = -\frac{qQ}{4\pi\epsilon_0} \int_A^B \frac{dr}{r^2} = \frac{-qQ}{4\pi\epsilon_0} \int_A^B \frac{dr}{r^2}$$

$$\Rightarrow \Delta U = \frac{qQ}{4\pi\epsilon_0} \left(\frac{1}{r_B} - \frac{1}{r_A} \right)$$

$$\text{If } r_A \rightarrow \infty \text{ chooses reference: } U = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r}$$



Arrangement of Charges



- The total energy of the system equals the sum of the energy of each pair of charges

$$U = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right)$$

Electrostatic Potential

- Defined Φ , also known as V
- Do not confuse w/ electrostatic energy (U)
- Another way to describe an electric field
- Electric static potential is a scalar (not a vector)
- To know \vec{E} or Φ for any point in space provides some amount of info (on electrostatics)
- If you know one you can find out the other
- Voltage $V = \Delta \Phi$

Example: Scalar Fields

Elevation

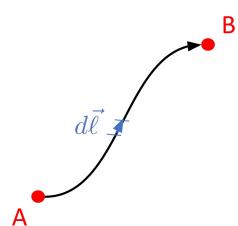
Vector Fields

Gravity

Temperature

Wind

Demonstration



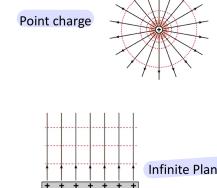
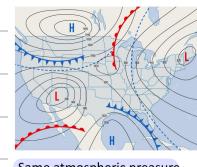
$$\Delta \Phi = \Phi(B) - \Phi(A) = - \int_A^B d\vec{l} \cdot \vec{E}$$

if we set a reference point ($\Phi = 0$)

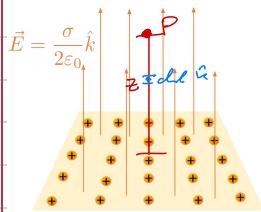
$$\Phi(r) = - \int_{\text{ref}}^r d\vec{l} \cdot \vec{E} \quad \Phi(\text{ref}) = 0$$

Scalar Quantity - Use Equipotential Lines

Equipotential lines



Example: Potential - Infinite Plane

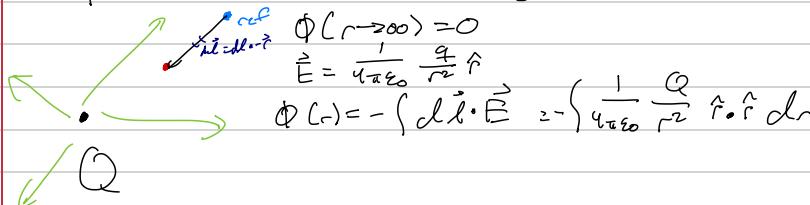


Given σ , $z=0$ so $\Phi(z=0)=0$

$$\vec{E} = \frac{\sigma}{2\epsilon_0} (\pm \hat{k}) \quad (\text{from Gauss law})$$

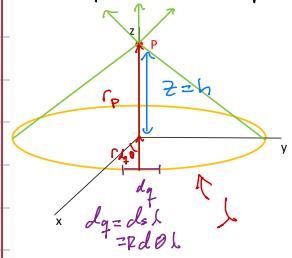
$$\Delta \Phi = \Phi_{\text{ref}} - \Phi_P = - \left(\int_{\text{ref}}^P d\vec{l} \cdot \vec{E} \right) = \int E dl \cos \theta = E \int_z^0 dz = E_z = \frac{-\sigma z}{2\epsilon_0}$$

Example: Potential - Point Line Charge



$$\Phi(r) = \frac{Q}{4\pi\epsilon_0 r}$$

Example: Superposition Principle (\vec{E} vs. Φ)



a) Develop the Field \vec{E} at P

Should point \rightarrow P

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^3} \hat{r} = \frac{1}{4\pi\epsilon_0} \frac{dq}{\sqrt{z^2 + R^2}} \{ z\hat{i} - R(\cos\theta\hat{j} + \sin\theta\hat{k}) \}$$

$$\begin{aligned} \vec{r}_P &= z\hat{i} \\ \vec{r}_{dq} &= R\cos\theta\hat{j} + R\sin\theta\hat{k} \end{aligned} \quad \left. \begin{aligned} \vec{r} + \vec{r}_{dq} &= \vec{r}_P \Rightarrow \vec{r} = z\hat{i} - R(\cos\theta\hat{j} + \sin\theta\hat{k}) \\ r &= \sqrt{z^2 + R^2} \end{aligned} \right\}$$

$$\Rightarrow \vec{E} = \frac{R\lambda}{4\pi\epsilon_0(z^2 + R^2)^{3/2}} \int_0^{2\pi} (z\hat{i} - R\cos\theta\hat{j} - R\sin\theta\hat{k}) d\theta \rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0} \frac{2\pi R\lambda z}{(z^2 + R^2)^{3/2}} \hat{i}$$

b) Develop potential without using \vec{E}

$$\Phi(P) = ?$$
$$d\Phi = \frac{1}{4\pi\epsilon_0} \frac{dq}{r}$$
$$\Phi(h) = \int_{0}^{2\pi} \frac{1}{4\pi\epsilon_0} \frac{R d\theta}{\sqrt{R^2 + h^2}} \Rightarrow$$
$$r = \sqrt{R^2 + h^2}$$

It's easier in general to find Φ

$$\Phi(h) = \frac{\Sigma R}{2\epsilon_0} \frac{1}{\sqrt{R^2 + h^2}}$$

Correlation between Φ and \vec{E} \leftarrow outside of scope of course

$$\Phi = - \oint d\vec{l} \cdot \vec{E}$$
$$\vec{E} = -\nabla \Phi = -\left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z} \right)$$

The gradient only calc B!

Conductors

Perfect Conductor Ideal doesn't exist in nature

- Electric Field inside is zero

• If $E \neq 0$, charges rearrange themselves so $B=0$ instantaneously for perfect conductors

• After some time, real conductors behave this way time is a factor for real conductors (low and)

• In the surface (outer), the Electric Field is perpendicular to the surface of the conductor

→ All net charges reside in the surface

$$Q_{enc} = \epsilon_0 \oint d\vec{a} \cdot \vec{E} = 0$$

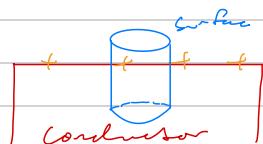
+ Enclosed charge is 0 and all charge is in surface

$$Q = Q_+ - Q_- = 0 \quad (0 \text{ charge, but particles exist inside})$$

Charge Density

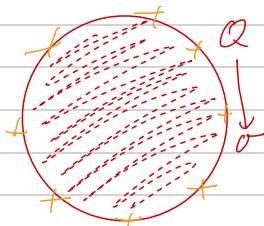
$$\oint d\vec{a} \cdot \vec{E} = \int_{\text{in}} E da + \int_{\text{out}} 0 + \int_{\text{in/out}} (0) = EA_{\text{cup}}$$

inside conductor \vec{E} pointing outwards \vec{E} is uniform $\Rightarrow \sigma = q_0 / A$



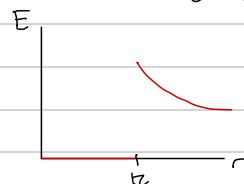
$$EA_{\text{cup}} = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\sigma \cdot A_{\text{cup}}}{\epsilon_0}$$

$$E = \frac{\sigma}{\epsilon_0}, \quad \sigma = E \epsilon_0$$

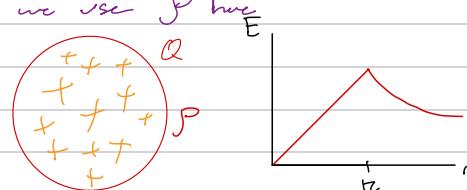


If the sphere is perfect conductor
all charges are on the surface!

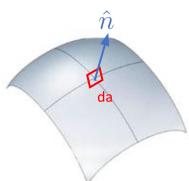
Use σ !



Differentiation: solution worked/ probably as they had charges evenly distributed throughout volume; we use P here



Force on the Conductor



$$\text{Force: } \vec{F} = q \vec{E} \quad (\text{known})$$

$$\rightarrow d\vec{F} = \frac{1}{2} \sigma da \vec{E} \hat{n}$$

$$\vec{F} = \frac{1}{2} \int \sigma \vec{E} \hat{n} da \quad (\text{use } \sigma = E \epsilon_0 \text{ or } E = \frac{\sigma}{\epsilon_0})$$

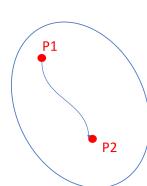
$$= \frac{1}{2} \epsilon_0 \int \vec{E}^2 \hat{n} da = \frac{1}{2 \epsilon_0} \int \sigma^2 \hat{n} da \quad \text{always perpendicular}$$

$$\vec{E} = E \hat{n} \quad \leftarrow \text{outside conductor}$$

$$\frac{1}{2} E \leftarrow \text{on surface? average}$$

$$E = 0 \quad \leftarrow \text{inside conductor}$$

Potential inside conductor



$$\Delta \phi = \int d\vec{l} \cdot \vec{E} = - \int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$$

→ but $E=0$ inside conductor $\therefore \Delta \phi = 0$

so, conductors are equipotential surfaces

Watch out!

$$\Delta \phi = \phi_B - \phi_A = 0$$

$$\rightarrow \phi_A = \phi_B \neq 0$$

↑ just their difference

Had to use fusion first last year

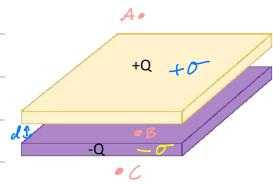
Capacitance

- Charge Q is proportional to potential difference (often called voltage) between two electrodes by a quantity called Capacitance

$$Q = CV \quad \text{where} \quad Q \equiv \text{Charge (coulombs, C)} \\ C \equiv \text{Capacitance (Farads, F)} \\ V \equiv \text{Voltage (Volts, V)}$$

Conventions:
Use Volts instead of (equiv to)
Potential difference ($\Delta\Phi$)

Parallel plates



Assume $A \gg d^2$

$$C = \frac{Q}{V} = \sigma A$$

$$V = \Delta\Phi = \int d\vec{l} \cdot \vec{E}$$

Q: What's \vec{E} at all 3 points?

above (+) $\vec{A} \uparrow \vec{E} \downarrow \rightarrow E = 0$
between $\vec{B} \uparrow \downarrow \vec{E} \downarrow \rightarrow E = \frac{\sigma}{\epsilon_0}$

below (-) $\vec{C} \downarrow \vec{E} \downarrow \rightarrow E = 0$

Derive from Gauss Law (Previous lecture)

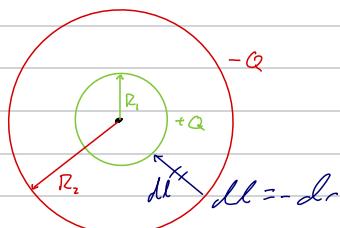
Use $\vec{E} @$ all 3 points to get Cap of parallel plate capacitors?

$$C = \frac{Q}{V} = \frac{Q}{\Delta\Phi}$$

$$\Delta\Phi = - \int_{P_1 \rightarrow P_2} d\vec{l} \cdot \vec{E} = - \int_{P_1 \rightarrow P_2} dl E_{\text{ext}} \epsilon_0 = E \int_{P_1}^{P_2} dl = d \left(\frac{Q}{\epsilon_0} \right), \quad Q = \sigma A$$

$$\therefore C = \frac{\sigma A}{d} = \frac{\epsilon_0 A}{d}$$

Example



$$C = \frac{Q}{V}, \quad V = \Delta\Phi = - \int_{-\infty}^{\infty} \vec{E} \cdot d\vec{l}$$

① Determine \vec{E} in 3 points (only really need two b/w Θ and ② just along the far periphery)

$$\vec{E}_{\text{inside}} = 0 \quad Q_{\text{enc}} = 0$$

$$\vec{E}_{\text{b/w}}: \quad Q_{\text{enc}} = Q$$

$$\frac{Q_{\text{enc}}}{\epsilon_0} = E_a \rightarrow \vec{E} = \frac{Q}{4\pi r^2 \epsilon_0}$$

$$\vec{E}_{\text{outside}} = 0 \quad Q_{\text{enc}} = Q - Q = 0$$

- Spherical & hollow shells

- Conductive materials

- Determine Capacitance (C) ② Determine $\Delta\Phi$

$$\Delta\Phi = - \int_{-\infty}^{\infty} E dl \cos 180 = \int_{R_2}^{R_1} \frac{Q}{4\pi r^2 \epsilon_0} dl = - \int_{R_2}^{R_1} \frac{Q}{4\pi r^2 \epsilon_0} dr = \frac{Q}{4\pi \epsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$\Rightarrow C = \frac{Q}{V} = \frac{Q}{4\pi \epsilon_0 \left(\frac{1}{R_1} - \frac{1}{R_2} \right)}$$

Exam 2 Review Lecture

- Dielectrics will not be on the homework

3 Key Topics

- Potential Energy
 - Electric Potential
 - Capacitance
- } holding all from Exam 1

Capacitors

$$C = \frac{Q}{V}$$

Parallel plate

Left to right $E = \sigma$

Conductor slab

$$\int \vec{E} \cdot d\vec{a} = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{A(\sigma + \sigma_u)}{\epsilon_0} = \frac{A(\sigma + 2\sigma_u)}{\epsilon_0}$$

$$2 \int_{\text{caps}} \vec{E} \cdot d\vec{a} + \int_{\text{side}} \vec{E} \cdot d\vec{a} = 0 = \frac{A(\sigma + 2\sigma_u)}{\epsilon_0}$$

$$\sigma + 2\sigma_u = 0$$

$$\sigma_u = -\sigma = -\frac{\sigma}{2}$$

Question 2
on HW 5
I got it right :-)

Outside edges of slabs should be $+\frac{\sigma}{2}$ to
keep conductors neutral

Potential by a Disk

HW 5 Q1

$$\Phi_{\text{ring}} = \frac{1}{4\pi\epsilon_0} \frac{\lambda(2\pi r)}{\sqrt{r^2 + z^2}}$$

$$2\pi r \lambda = 2\pi r d\sigma$$

$$\Rightarrow \int_0^R \frac{1}{4\pi\epsilon_0} \frac{2\pi r d\sigma}{\sqrt{r^2 + z^2}}$$

Don't expect something like this on the exam

Potential & Energy

$$\Delta\phi = - \int \vec{E} \cdot d\vec{r}$$

$$\Delta U = -Q \int \vec{E} \cdot d\vec{l}$$

$\left. \right\} \Delta U = Q \Delta \phi$

Example Regarding Capacitance

$$\int \vec{E} \cdot d\vec{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$E(2\pi rL) = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$E = \frac{Q}{2\pi rL \epsilon_0}$$

$$V = \Delta\phi = \int_{R-d}^R \frac{Q}{2\pi\epsilon_0 r L} dr \propto \log\left(\frac{R}{R-d}\right)$$

Midterm Exam 3



Dielectrics

- Dielectrics = Insulators

examples: Glass

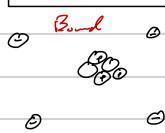
Plastic

Water

Diamond

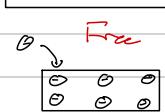
• charges stay put

Internal Charge
or. Bound Charge



vs.

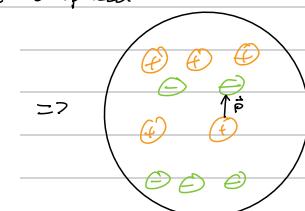
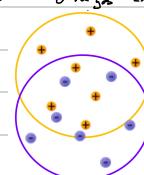
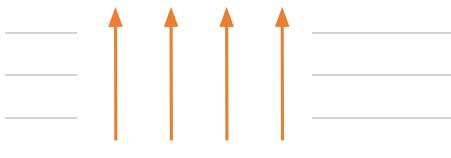
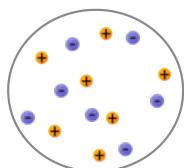
External Charge
or. Free Charge



or Poles

Response of Internal Charges to Fields

Atom of Dielectric \rightarrow Apply electric Field \rightarrow Bound charges are displaced



Displaced dipoles
in the atom

Dipole

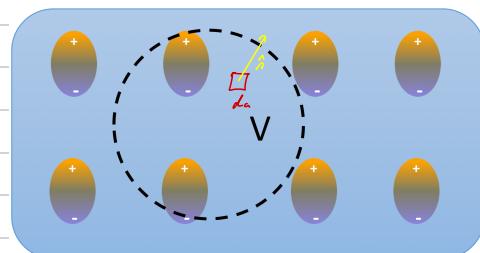


$$\vec{P} = q \vec{d}$$

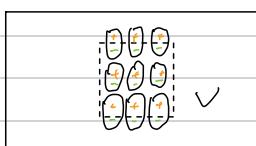
Polarized medium

\rightarrow Polarization vector \vec{P}

o Dipole Moment Density: $\vec{P} = \frac{\sum \vec{p}}{volume}$



How much bound charge makes?



$$\vec{P}$$

• \vec{P} is uniform, $Q_B(V) = 0$

• Answer depends on \vec{P} at the surface

• If \vec{P} is parallel to the surface no net charge

$$Q_B \propto \vec{P}_{surface} \checkmark$$

$$Q_{bound} = Q_B(V) - \oint_{\text{closed}} \vec{P} = -\oint |\vec{P}| |d\vec{l}| \cos 0$$

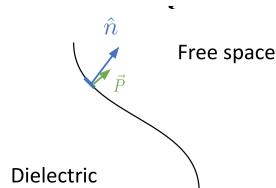
• Only for Dielectrics

• Not Gauss' Law, thus: $Q_{bound} Q_{free} = \epsilon_0 \oint d\vec{a} \cdot \vec{E}$

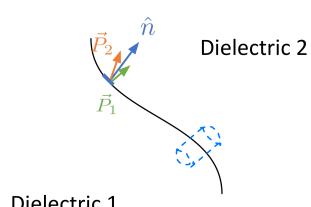
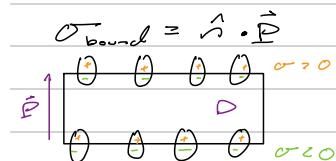
Free
Charges

$$Q_{free} = \oint d\vec{a} \cdot (\epsilon_0 \vec{E} + \vec{P})$$

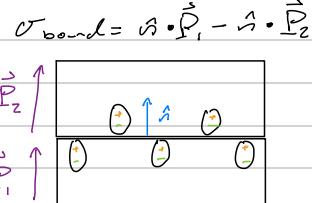
Surface Charge Density



One dielectric



Two dielectrics



Gauss Law to get σ_{total}

$$\vec{E}_1 = \frac{\sigma_{total}}{2\epsilon_0} \hat{n}$$

$$\sigma_{total} = \sigma_{free} + \sigma_{bound}$$

$$\sigma_{free} = \hat{n} \cdot (\epsilon_0 \vec{E}_2 + \vec{P}_2)$$

$$\sigma_{free} = \hat{n} \cdot (\epsilon_0 \vec{E}_1 + \vec{P}_1)$$

$$\sigma_{free} = \hat{n} \cdot (\epsilon_0 (\vec{E}_2 + \vec{P}_2) - \hat{n} (\epsilon_0 \vec{E}_1 + \vec{P}_1))$$

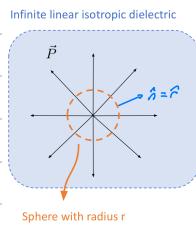
Linear Isotropic Dielectrics

$$\vec{P} \propto \vec{E}^x \quad \text{proportionality depends on the material's temperature and pressure} \Rightarrow \vec{P} = \chi \epsilon_0 \vec{E}$$

Simpler case
Gauss's Law $\vec{P} \cdot \vec{E}^x$ (to a given point)

Bliss Susceptibility

Example



Assume $\vec{P} = \frac{A}{r^2} \hat{r}$ constant

- What is the bound charge inside a sphere with radius r ?

$$Q_{\text{bound}} = \oint_C \vec{P} \cdot d\vec{l} = \int_{\text{volume}} \vec{P} \cdot \vec{dl}$$

$$= 4\pi r^2 P \cos 0$$

$$Q_{\text{bound}} = -4\pi r^2 \left(\frac{A}{r^2} \right) = -4\pi A$$

$$Q_{\text{bound}} = -4\pi A$$

Note: Q_{bound} is independent of r ; it has a constant value

↳ Why? All the charges are at the center

- What is the electric field?

$$\vec{E} = \frac{\vec{P}}{\chi \epsilon_0} = \frac{\frac{A}{r^2} \hat{r}}{\chi \epsilon_0} = \frac{1}{\chi \epsilon_0} \frac{A}{r^2} \hat{r}$$

$$\vec{E} = \frac{1}{\chi \epsilon_0} \frac{A}{r^2} \hat{r}$$

- What is the total charge inside a sphere with radius r ?

$$Q_{\text{tot}} = \oint \vec{E} \cdot d\vec{l} \Rightarrow Q_{\text{tot}} = \epsilon_0 E \oint dl = \epsilon_0 E 4\pi r^2 = \epsilon_0 \frac{1}{\chi \epsilon_0} \frac{A}{r^2} 4\pi r^2 = \frac{4\pi A}{\chi}$$

$$Q_{\text{tot}} = \frac{4\pi A}{\chi}$$

- What is the external (free) charge inside a sphere with radius r ?

$$Q_{\text{free}} = Q_{\text{tot}} - Q_{\text{bound}} = 4\pi A \frac{1+\chi}{\chi}$$

$$Q_{\text{free}} = 4\pi A \frac{1+\chi}{\chi}$$

You must see $1+\chi$ as ϵ_r (relative permittivity) or the dielectric constant K (Kappa)

- What is the electric field as a function of free charge?

$$\vec{E} (Q_f = 4\pi A \frac{1+\chi}{\chi}) ? \quad A = \frac{\chi Q_f}{4\pi(1+\chi)} \Rightarrow \vec{E} = \frac{1}{\chi \epsilon_0} \left(\frac{\chi Q_f}{4\pi(1+\chi)} \right) \hat{r}$$

$$\vec{E} = \frac{1}{4\pi \epsilon_0 (1+\chi)} \frac{Q_f}{r^2} \hat{r}$$

Basically coulomb's law, but with $(1+\chi)$ factor.

We will define $K = 1+\chi$ (Kappa)

Parallel Plate Capacitor

- Capacitance is affected: $C = K C_{\text{air}} = (K \epsilon_0) \frac{A}{d}$

Material	Dielectric constant, K
Air	1.085-1.21 at 20°C*
Water	80 at 20°C (79.50-80.24)*
Ice	3 at -5°C*
Basalt	12
Granite	7.9
Sandstone	9-11
Dry loam	3.5
Dry sand	2.5 (2.5-3.5 at 20°C)*
Ethanol liquid	15.21 at 20°C*
Ethanol Mixing Liquid	39.82 at 20°C*
Acetone	21.20 at 20°C*

*measured dielectric constant in this study

What is the new Capacitance with two dielectrics?

In Parallel

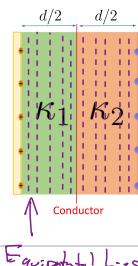
$C_1 = \frac{Q_1}{V} \quad C_2 = \frac{Q_2}{V} \quad \text{for constant } V \quad (V = Ed)$

$E \frac{A}{2} = \frac{Q_i}{\epsilon_0 \epsilon_i} \Rightarrow Q_i = (\epsilon_i \epsilon_0) E \frac{A}{2}$

$\therefore C_i = \frac{\epsilon_i \epsilon_0}{2} \frac{A}{d}$

$\Rightarrow C = \frac{Q + Q_2}{V} = \frac{K_1 + K_2}{2} \epsilon_0 \frac{A}{d} = C_1 + C_2$

In Series

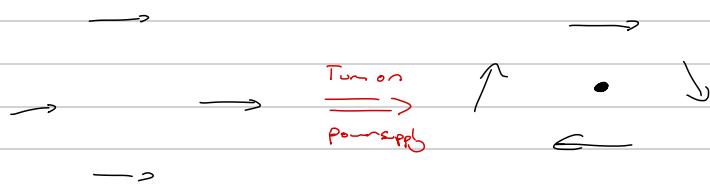


$$\Delta \phi = \int \frac{\sigma}{K \epsilon_0} dl = \frac{\sigma}{K \epsilon_0} \int dl = \frac{\sigma}{K \epsilon_0} y$$

$$C = \frac{Q}{V_1 + V_2} = \left(\frac{V_1 + V_2}{Q} \right)^{-1} = \left(\frac{1}{C_1} + \frac{1}{C_2} \right)^{-1}$$

Magnetic Fields

Demo w/ Compasses



Electric Current produces a magnetic field
is circular as seen by the direction of the compass arrows

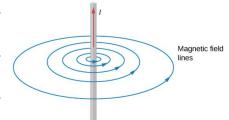
Magnetic Fields

- Moving electric charges produce magnetic fields
↳ there are no "magnetic charges"
- Magnetic fields exert forces on moving electric charges

Note: Moving charges still produce electric fields!

Law of Faraday's Law of Induction

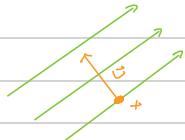
- Electric currents (or just currents) produce magnetic fields \vec{B}
- \vec{B} exert forces on moving charges



Definition of Magnetostatics: Assume that there is no accumulation of charges, but the charges can move

Lorentz Force

- The force must depend on the charge, velocity, and field



$$\vec{F} = \vec{F}(q, \vec{B}, \vec{v})$$

- The force is the vector product of \vec{v} and \vec{B}

$$\vec{F} = q \vec{v} \times \vec{B}$$

Recall - Cross Product

- The cross product of two vectors is a vector

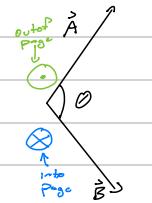
$$\vec{A} \times \vec{B} = \text{vector}$$

- Its magnitude:

$$|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin(\theta)$$

- Direction:

$\vec{A} \times \vec{B}$ is perpendicular to \vec{A} and \vec{B}



Currents

- Which charges move?

→ Metals \Rightarrow Negative charges move

→ Semiconductors \Rightarrow Positive charges move

We will assume that positive charges move!

$I \equiv$ Current flowing through surface in conductors

$$I \equiv \frac{\Delta Q}{\Delta t} \xrightarrow{\substack{\text{charge} \\ \text{crosses} \\ \text{during a certain} \\ \text{time}}} \frac{\text{charge}}{\text{time}}$$

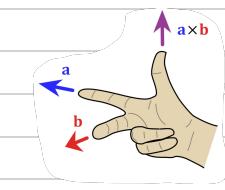
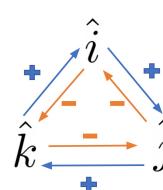
$$\text{Units: } \frac{\text{Coulombs}}{\text{Second}} = \text{Amperes}$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$\vec{A} = 3\hat{i} + 4\hat{j}$$

$$\vec{B} = 6\hat{i} + 7\hat{k}$$

$$\vec{A} \times \vec{B} = 28\hat{i} - 21\hat{j} - 24\hat{k}$$



Currents: Line Current

- Assume an ∞ wire with infinitesimal thickness

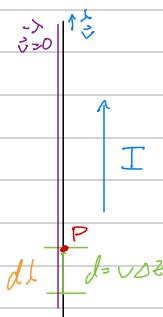
\hookrightarrow Current I means

$\rightarrow I$ coulombs going up

$\rightarrow -I$ coulombs going down

- All charges ($+$) in a small Δt will cross P in a time Δt

$$\text{The current: } I = \frac{\Delta Q}{\Delta t} = \frac{I \cdot \Delta t}{\Delta t} = I$$



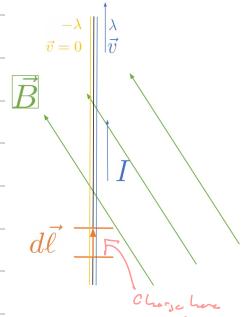
Coulomb vs. Lorentz

$$\rightarrow F_E = q \vec{E}$$

$$\rightarrow F_B = q \vec{v} \times \vec{B}$$

\hookrightarrow Do cross product, \hookrightarrow deal with q 's sign

Lorentz Force on a Current



- Force on a piece of wire $d\vec{l}$

$$d\vec{F} = (I d\vec{l}) \vec{v} \times \vec{B} \quad \text{Note: } d\vec{l} v = d\vec{l} \vec{v}$$

$\hookrightarrow q = I \cdot d\vec{l}$

- In terms of current

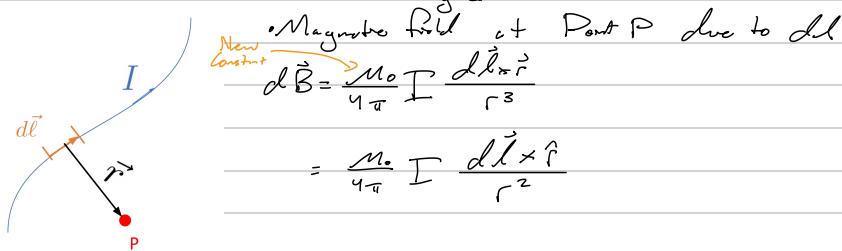
$$d\vec{F} = I \int d\vec{l} \times \vec{B}$$

Note: $I = s \cdot l$ (see previous!)

- Integrate to solve for \vec{F}

$$\vec{F} = I \int d\vec{l} \times \vec{B} = I \int |d\vec{l}| |\vec{B}| \sin \theta \quad \text{for } \theta \text{ b/w } d\vec{l} \text{ and } \vec{B}$$

Biot-Savart: Field Created by a current



Example: Infinite wire

$$\vec{r} = y\hat{i} + z\hat{k} \rightarrow r = \sqrt{y^2 + z^2}$$

$$d\vec{l} = d\ell \hat{v}$$

$$d\vec{l} \times \vec{r} = -y d\ell \hat{i}$$

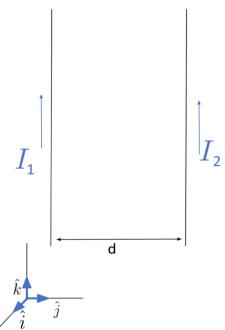
$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{-y d\ell \hat{i}}{(z^2 + y^2)^{3/2}}$$

$$\vec{B} = \int_{-\infty}^{\infty} \frac{\mu_0}{4\pi} I \frac{-y d\ell \hat{i}}{(z^2 + y^2)^{3/2}}$$

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{I}{y} (-\hat{i})$$

Force b/w 2 infinite wires

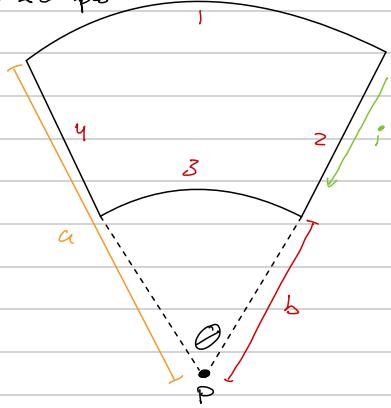
- What is the force that I_1 exerts on I_2 ?



$$d\vec{F}_2 = \frac{\mu_0 I_1 I_2}{2\pi d} d\ell (-\hat{j}) = -d\vec{F}_1$$

- If the intensities are parallel the force is attractive,
- If the intensities are anti-parallel the force is repulsive

Example



Find Magnetic Field at P

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^3}$$

(i) Determine direction of $d\vec{s}$ for col segment

a) Angles b/w $d\vec{s}$ and \vec{r} for col segment

Segment 1

Angle b/w $d\vec{s}$ and $d\vec{l}_1$?

$$|d\vec{s} \times \vec{r}| = |d\vec{s}| |\vec{r}| \sin \theta$$

$$\theta = 90^\circ$$

by symmetry, forces to same for seg 2

Segment 2

Angle b/w $d\vec{s}$ and \vec{r} in Segment 2

b) Direction of Segments and PBR

$$\textcircled{1} \quad \vec{B} = B \cdot \vec{\otimes}$$

$$\textcircled{2} \quad \vec{B} = B \cdot \vec{\odot}$$

$$\begin{cases} \vec{r} \\ d\vec{s} \\ 90^\circ \text{ & same seg.} \end{cases} \quad \left. \begin{array}{l} d\vec{B}_1 = d\vec{B}_2 = 0 \\ \vec{B} = 0 \end{array} \right\}$$

b) Integration - Bad, don't follow

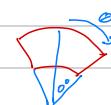
→ How is $d\vec{s}$ related to $d\theta$

$$d\vec{s} = r d\theta$$



→ What is the position vector \vec{r}

$$\vec{r} = -r \sin \theta \hat{i} - r \cos \theta \hat{j}$$



$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{s} \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi} I r \frac{d\theta \vec{r}}{r^3}$$

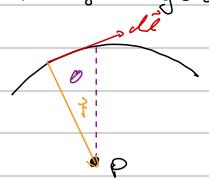
$$\int_{-\theta_2}^{\theta_2} d\vec{B}_1 = \frac{\mu_0}{4\pi} I \frac{d\theta}{a} (-\sin \theta \hat{i} - \cos \theta \hat{j}) = \frac{\mu_0}{4\pi} \cdot \frac{I}{a} \cdot \int_{-\theta_2}^{\theta_2} -\sin \theta \hat{i} - \cos \theta \hat{j} d\theta$$

This can be integrated

$$\int_{-\theta_2}^{\theta_2} d\vec{B}_2 = \frac{\mu_0}{4\pi} I \frac{d\theta}{b} (-\sin \theta \hat{i} - \cos \theta \hat{j}) = \frac{\mu_0}{4\pi} \cdot \frac{I}{b} \cdot \int_{-\theta_2}^{\theta_2} -\sin \theta \hat{i} - \cos \theta \hat{j} d\theta$$

using $\int \sin \theta d\theta = -\cos \theta$

b) Integrating good



$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^3}$$

$$\cdot |d\vec{l} \times \vec{r}| = |d\vec{l}| |\vec{r}| \sin 90^\circ = d\vec{l} \cdot r = (rd\theta) r$$

Direction $\textcircled{\times}$ $\rightarrow -\hat{k}$

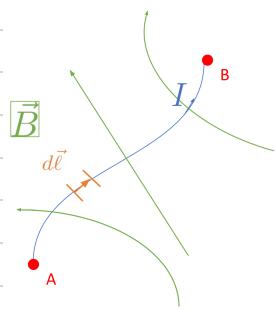
$$\begin{cases} \cdot \vec{r} = \hat{\theta} + \hat{j} \\ d\vec{l} = \hat{r} + \hat{k} \end{cases}$$

$$\Rightarrow d\vec{B}_1 = \frac{\mu_0}{4\pi} I \frac{r^2 d\theta}{r^3} \textcircled{\times} = \frac{\mu_0}{4\pi} I \frac{1}{r} \theta (-\hat{k}) \quad \vec{B}_1 = -\frac{\mu_0}{4\pi} I \frac{1}{a} \theta \hat{k}$$

$$\Rightarrow d\vec{B}_2 = \frac{\mu_0}{4\pi} I \frac{r^2 d\theta}{r^3} \textcircled{\odot} = \frac{\mu_0}{4\pi} I \frac{1}{r} \theta \hat{k} \quad \vec{B}_2 = \frac{\mu_0}{4\pi} I \frac{1}{b} \theta \hat{k}$$

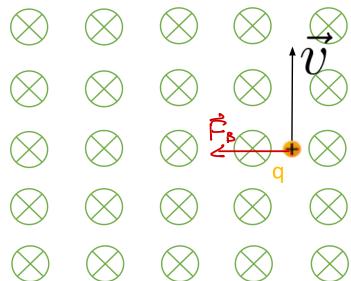
$$\boxed{\vec{B}_{\text{tot}} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0}{4\pi} I \theta \left(\frac{1}{b} - \frac{1}{a} \right) \hat{k}}$$

Acceleration and Ampere's Law



Q: Charge with a const. velocity at pos. A, \vec{v}_A , that creates a magnetic field \vec{B} , will suffer a force (Lorentz). At point B, it will have an velocity \vec{v}_B . Will: (i) $|\vec{v}_B| > |\vec{v}_A|$
 (ii) $|\vec{v}_B| = |\vec{v}_A|$ const. $\vec{F}_B = q\vec{v} \times \vec{B} \Rightarrow \vec{F}_B \perp \vec{v}$
 (iii) $|\vec{v}_B| < |\vec{v}_A|$
 90° apply \rightarrow centripital so $|\vec{v}|$ doesn't change
 $\Delta K = \int_A^B \vec{F} \cdot d\vec{\ell} = \int_A^B$ follows $q_0 = 0 \rightarrow |\vec{v}_B| = |\vec{v}_A|$ (work cons. thm)

Trajectory on a Uniform Field



• Directions of Lorentz force $(\vec{F}_B = q\vec{v} \times \vec{B})$

RHR \rightarrow must point Left

• How will q move? Upward Left

• Trajectory? Fix a const. θ

$$N2L: qvB \sin 90^\circ = m \frac{v^2}{r} \Rightarrow r = \frac{mv}{qB}$$

$$T = \frac{2\pi r}{v} = \frac{2\pi m}{qB}, \quad \omega = \frac{2\pi}{T} = \frac{qB}{m}$$

Cyclotron Frequency

Parallels

Charges produce Electric fields

- Version I
 • Coulomb's Law
 • Superposition Principle

- Version II
 • Gauss Law
 • Circulation Law

Moving charges produce magnetic fields

- Version I
 • Biot-Savart's Law
 • Superposition Principle

- Version II
 • Gauss Law of Magnetism
 • Ampere's Law

Gauss Law of Magnetism

$$\oint \vec{B} \cdot d\vec{a} = \oint \vec{n} \cdot \vec{B} da = 0$$

where \vec{n} points outside the surface by convention

Ampere's Law

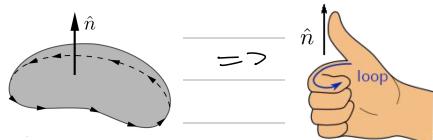
- Magnetic field across a closed loop (circulation) is proportional to the current through any surface whose boundary is the loop
- In Magnetic States: $\oint d\vec{l} \cdot \vec{B} = \mu_0 I_{enc}$ I enc is any current that goes through the loop's boundary



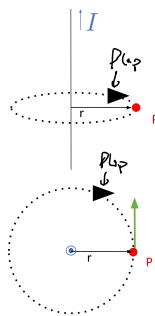
4 geometries
 will apply to (mag
 beam eng)

Stokes' Corollary

Direction of loop \Rightarrow direction of normal vector is fixed



Example: Magnetic Field by Infinite Line



- Choose an amperian loop passing by P
- \rightarrow Similar to Gaussian surface, result should not depend on it
- Since field depends on r (Field lines are circles)

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

$$\int \vec{B} \cdot d\vec{s} = \mu_0 I \Rightarrow B(2\pi r) = \frac{\mu_0 I}{2\pi r}$$

Summary of Ampère's Law (nicely written)

To find the magnetic field due to a current, Ampère's law is sometimes more efficient than the Biot-Savart law.

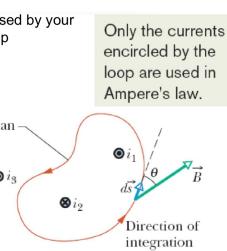
Ampère's law:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc}$$

current enclosed by your Ampérian loop

infinitesimally small length vector along an imaginary loop surrounding one or more currents

Ampère's law is always true, but it is only useful in a limited number of cases with a large amount of symmetry.



Right Hand Rule for direction of Field

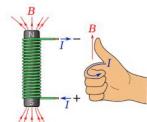
Right hand rule for the direction of the magnetic field due to a current:

- Point your thumb in the direction of the current.
- Your fingers curl in the direction of the magnetic field



OR

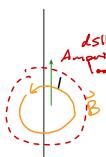
- Curl your fingers in the direction of the current.
- Your thumb points in the direction of the magnetic field.



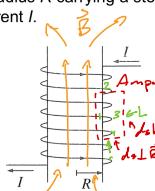
Example

Use Ampère's Law to determine the magnetic field in each of the following 3 cases:

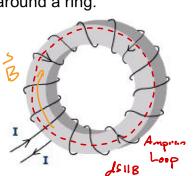
Case 1: A long straight wire carries a steady current I in the direction shown.



Case 2: A very long solenoid has n closely wound turns per unit length on a cylinder of radius R carrying a steady current I .



Case 3: A toroid consists of a long wire, carrying a steady current I , that is tightly wound N times around a ring.



$$N = \text{number of loops}$$

$$n = \frac{N}{L} = \text{density of loops}$$

Caution: It's okay for $\oint \vec{B} \cdot d\vec{s} = 0$ for some points on the Ampérian loop, but if $\oint \vec{B} \cdot d\vec{s} = 0$ for every point on the loop, then your loop will not allow you to determine the magnitude of the magnetic field.

(i) Direction of magnetic field

(ii) Draw an Ampérian loop s.t. $\oint \vec{B} \cdot d\vec{s} = B \oint ds$ or 0 at every point on the loop ($0^{\circ}, 90^{\circ}$)

(iii) Choose direction to integrate around the loop and calculate $\oint \vec{B} \cdot d\vec{s}$ (circulation)

Case 1: $\oint \vec{B} \cdot d\vec{s}$ = circulation

$$\theta = 0^\circ \quad B \parallel rd\theta$$

$$B(2\pi r)$$

Case 2: $\oint \vec{B} \cdot d\vec{s} = (\oint \vec{B} \cdot d\vec{s})_1 + (\oint \vec{B} \cdot d\vec{s})_2 + \dots$

$$(\oint \vec{B} \cdot d\vec{s})_1 + (\oint \vec{B} \cdot d\vec{s})_2 + \dots$$

Side 1: $B(L)$, 2: 0, 3: 0, 4: 0

$\theta = 90^\circ$

Assume Faraday's law! This is expected by

Same result as case 1

inside, outside sides cancel! & this approximation is fine

(iv) Determine i_{enc}

$$\text{Case 1: } i_{enc} = I$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

$$\text{Case 2: } i_{enc} = I \cdot n \cdot L$$

$$\Rightarrow B = \frac{\mu_0 I \cdot n \cdot L}{L}$$

$$\text{Case 3: } i_{enc} = N \cdot I$$

$$B = \frac{\mu_0 N \cdot I}{2\pi r}$$

(v) Summary of Example!



Case 1: A long straight wire carries a steady current I in the direction shown.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

$$B = \frac{\mu_0 I}{2\pi r}$$



Case 2: A very long solenoid has n closely wound turns per unit length on a cylinder of radius R carrying a steady current I .

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 n I_{enc}$$

$$B = \mu_0 n I$$



Case 3: A toroid consists of a long wire, carrying a steady current I , that is tightly wound N times around a ring.

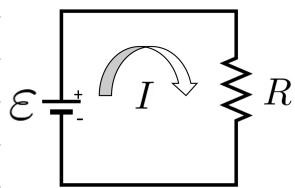
$$\oint \vec{B} \cdot d\vec{s} = \mu_0 N I_{enc}$$

$$B = \frac{\mu_0 N I}{2\pi r}$$

Circulation Law	Gauss Law	Ampere's Law	Gauss Law of B
The circulation of E equals zero	The flux of E through a closed surface equals charge enclosed	The circulation of B through a closed loop equals current enclosed	The flux of B through a closed surface equals zero
$\oint d\vec{l} \cdot \vec{E} = 0$	$\oint \hat{n} \cdot \vec{E} da = \frac{Q}{\epsilon_0}$	$\oint d\vec{l} \cdot \vec{B} = \mu_0 I$	$\oint \vec{n} \cdot \vec{B} da = 0$

but note that this will result in t_{nr}

Faraday's Law



DC Battery

- Two terminal device
- Maintains a Voltage (potential drop) across terminals (ε)
- Positive terminal (+) higher voltage
- Current I flows in both directions

Electromotive Force

\Rightarrow Ohm's Law

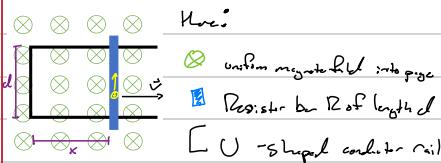
$$V = IR$$

Current flows from higher potential to lower potential

Possible

- Not always good conductor
- Potential drop across (V)
- Current can flow in both directions

Faraday's Flux rule



Here:

Bar is moved to height z uniformly

→ in bar will feel upward force (RH_{FB})

$$|\vec{F}| = |+q \vec{v} \times \vec{B}| \approx qvB \Rightarrow \vec{F} = qvB\hat{j}$$

→ Force per unit of charge

$$\vec{f} = \frac{\vec{F}}{q} = vB\hat{j} \quad \text{Looks like electric field } (\vec{B} = \frac{\vec{E}}{q}), \text{ not true!}$$

→ Effect to voltage: $V = \vec{f}d = vBd = \varepsilon$

⇒ Known as the electromotive force

$$\Rightarrow I = \frac{\varepsilon}{R} = \frac{dvB}{R}$$

→ B flux of \vec{B} , given $\varepsilon = vBd$

$$\Phi_B = \int \vec{B} \cdot d\vec{a} = \int B da \cos 0 = Bda \approx B(x)dx$$

→ Derivative over time of B flux

$$\frac{d}{dt}(\Phi_B) = \frac{d}{dt}(Bdx) \Rightarrow \frac{d\Phi_B}{dt} = Bdx \frac{dx}{dt} = BdV \Rightarrow \frac{d\Phi_B}{dt} \propto \varepsilon$$

→ If we are more careful w/ signs, then generally: $\varepsilon = -\frac{d\Phi_B}{dt}$ ↪ should always hold

We are missing something

• If we move magnet and leave wire fixed, $\vec{J} = 0$ by the flux laws, why?

↳ Change reference frames, but we really need to refine Faraday's Law

Faraday's Law of Induction

$$\varepsilon = -\frac{d\Phi_B}{dt} \Rightarrow \oint dl \cdot \vec{B} = -\frac{d}{dt} \int \vec{n} \cdot \vec{B} da$$

Faraday's Law can be used to be the circulation law

Gauss Law

Ampere's Law

Gauss Law of B

The circulation of E equals the variation in time of the flux of B

The flux of E through a closed surface equals charge enclosed

The circulation of B through a closed loop equals current enclosed

The flux of B through a closed surface equals zero

$$\oint dl \cdot \vec{E} = -\frac{d}{dt} \int \vec{n} \cdot \vec{B} da$$

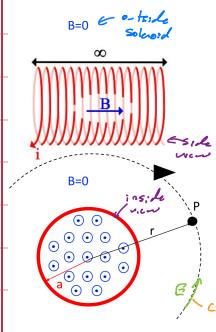
↑ not 0 in electrostatics

$$\oint \hat{n} \cdot \vec{E} da = \frac{Q}{\varepsilon_0}$$

$$\oint d\vec{l} \cdot \vec{B} = \mu_0 I$$

$$\oint \vec{n} \cdot \vec{B} da = 0$$

Electric Field Lines



If circulation is not 0 can field lines form closed loops?

• At point P: $\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_A \vec{B} da$

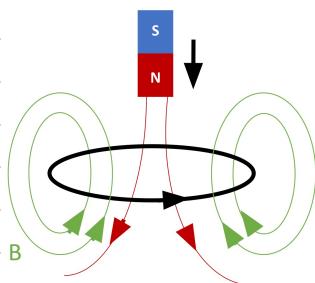
\rightarrow closed contour $\oint \vec{E}$ so that $\vec{E} \cdot d\vec{l}$ are parallel

$$\int E da = E \int dl = E 2\pi r \quad \text{Flux only through the physical area}$$

$$-\frac{d}{dt} \int A \vec{B} da = -\frac{d}{dt} (B \cdot \pi r^2) = -\pi r^2 \frac{dB}{dt}$$

$$E 2\pi r = -\pi r^2 \frac{dB}{dt} \Rightarrow E = -\frac{r^2}{2r} \frac{dB}{dt} \quad \text{also rza}$$

Lenz's Law (Rule)

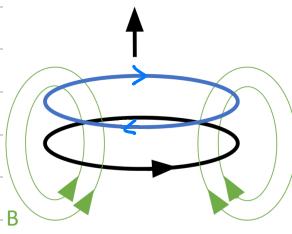


• Derived from Faraday's Law

• Direction of the induced current (or electric field) is such that the magnetic field created by it (the current), opposes the changing flux that created it.

	M _e	Faraday	B _f ? B ₀
Flux	↑	↓	Anti parallel
	↓	↑	Parallel

Example:



- From I=0 we start increasing the current in the black loop.
- What is the direction of the induced current on the blue loop?
 - Opposite direction
- What is the force on the upper ring?
 - Repulsive

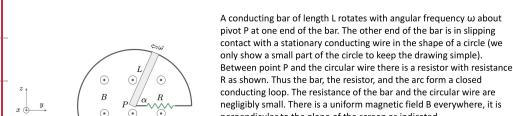
Three ways to induce current

→ You need to induce flux

- (i) $B(t)$
- (ii) $A(t)$
- (iii) $\phi(t)$

$$\phi_B = BA \cos \theta$$

Example - / Poll E



What is the magnitude and direction of the force exerted on the conducting bar?

$$\vec{F}_{Bn} \leftarrow I \leftarrow \mathcal{E} \leftarrow \frac{d\phi_B}{dt} \text{ steps!}$$

$$\Phi_B = \oint \vec{B} \cdot d\vec{a} \quad \text{Full circle}$$

$$= B \cdot A = B \cdot \pi L^2 \Rightarrow B \cdot \frac{\pi}{2} L^2$$

$$\Phi_B = \frac{1}{2} BL^2 \omega$$

What is the direction of the current induced in the conductive sector when the angle θ is increased? Use Lenz's law

Need to produce field \vec{B} , so **clockwise**

What is the role of the current?

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{1}{2} BL^2 \omega \quad \text{so } \frac{d\Phi}{dt} = \omega$$

$$I = \frac{\mathcal{E}}{R} = -\frac{1}{2R} BL^2 \omega \quad \text{you can drop minus sign because deal with it using Lenz's law}$$

Finally, to find:

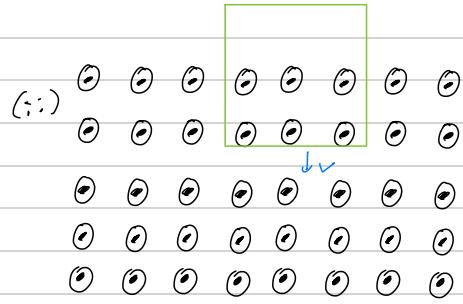
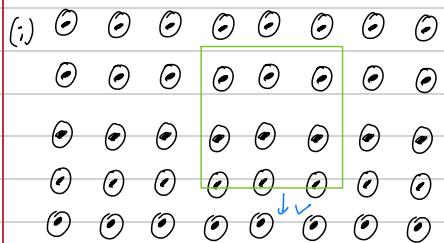
$$d\vec{F} = I \cdot d\vec{l} \times \vec{B}$$

$$d\vec{F} = I \cdot d\vec{l} \cdot B$$

$$\vec{F} = IB \int_0^L d\vec{l}$$

Using BHR, Force points to the right, can parametrize if needed

Example



Direction of Induced Current?

Need to produce Θ Clockwise

\rightarrow Increase Flux over time
Me F
↑ ↓

Direction of Induced Current?

No induced current!

$\rightarrow A, B, \Theta$ overall constant

Review

Capacitors & Dielectrics

A

d

K

$$\text{without dielectric: } C_0 = \epsilon_0 \frac{A}{d}$$

$$\hookrightarrow \text{add dielectric: } C_K = K \epsilon_0 \frac{A}{d}$$

A

d

K_1

K_2

$$\text{Half of fringing region: } C_0^{1/2} = \epsilon_0 \frac{A}{2d} \quad \left\{ \frac{A}{2} \right\}$$

$$\hookrightarrow \text{add } K_i \rightarrow C_{K_i}^{1/2} = K_i \epsilon_0 \frac{A}{2d}$$

$$\text{Total Capacitance: } C_{(K_1, K_2)} = C_{K_1}^{1/2} + C_{K_2}^{1/2} \quad \begin{array}{l} \text{Capacitors add} \\ \text{Resistors have opposite} \\ \text{eq-eq's} \end{array}$$

(parallel)

A

d

K_1

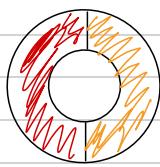
K_2

$$\text{Half of fringing region: } C_0^{1/2} = \epsilon_0 \frac{2A}{d} \quad \left\{ \frac{A}{d/2} \right\}$$

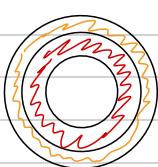
$$\hookrightarrow \text{add: } C_{K_i}^{1/2} = K_i \epsilon_0 \frac{2A}{d}$$

$$\text{Total Capacitance: } C_{(series)} = \left(\frac{1}{C_{K_1}^{1/2}} + \frac{1}{C_{K_2}^{1/2}} \right)^{-1}$$

Parallel

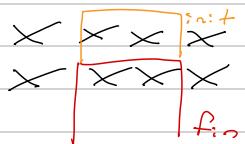


Series



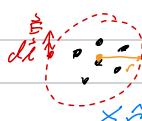
Lenz's Law

Question 4 on HW



Flux decreasing. $\Phi \downarrow \Rightarrow \frac{d\Phi_B}{dt} < 0$

Faraday $\Phi \uparrow$ created



$$\oint \vec{E} \cdot d\vec{l} = - \frac{d}{dt} \left(\iint \vec{B} \cdot d\vec{a} \right) = - \frac{d}{dt} \left(\iint \vec{B} \cdot \hat{n} da \right)$$

$$E(2\pi r) = - \frac{d(\text{area})}{dt} (\pi r^2) (\cos 180^\circ)$$

$$= \frac{dB}{dt} (\pi r^2)$$

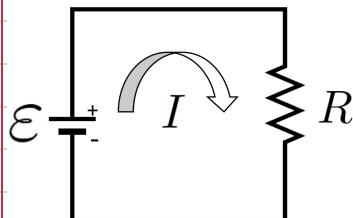
Stokes' contribution important

Final Exam



DC Circuits

Recall: Basic DC Circuit



- DC Battery
 - Two terminal device
 - Maintains a Voltage (potential drop) across terminals (ϵ)
 - Positive terminal at higher voltage
 - Current can flow in both directions
- Resistor
 - Not very good conductor
 - Potential drop across (V)
 - Current can flow in both directions

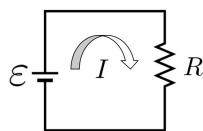
Electromotive Force

Kirchoff Rule Steps

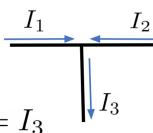
1. Assume Dir I
2. Walk Loop
3. 1st K rule (to n loops)
4. 2nd K rule (to n loops)
5. Solve sys of eq.

Kirchoff's Rules

- Potential drop around a loop add up to zero



Example 1



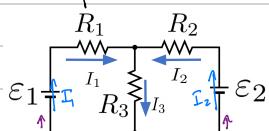
- In a junction the amount of current flowing in equals the flowing out

$$\epsilon - IR = 0$$

$$\Rightarrow I = \frac{\epsilon}{R}$$

If you assume opposite flow, you just get $I = 0$

Example 2



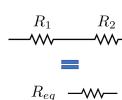
$$\begin{aligned} \epsilon_1 - I_1 R_1 - I_3 R_3 &= 0 \\ \epsilon_2 - I_2 R_2 - I_3 R_3 &= 0 \\ I_1 + I_2 &= I_3 \end{aligned}$$

$$\begin{aligned} I_1 &= \frac{\epsilon_1 R_2 + \epsilon_2 R_3 - \epsilon_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} \\ I_2 &= \frac{\epsilon_2 R_1 + \epsilon_2 R_3 - \epsilon_1 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} \\ I_3 &= \frac{\epsilon_1 R_2 + \epsilon_2 R_1}{R_1 R_2 + R_1 R_3 + R_2 R_3} \end{aligned}$$

Just solve
the system

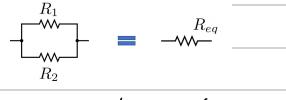
Resistance is source of problem

Series:
The current through the resistor is the same



$$R_{eq} = R_1 + R_2$$

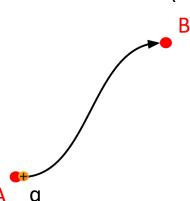
Parallel:
The voltage drop through the resistors is the same



$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

Recall: Energy - Potential

$$\Delta U = q (\phi_{CB} - \phi_{CA})$$



Energy Flow: Battery

- Energy delivered to system by battery

$$dU = dq \epsilon$$

- If an amount of charge dq goes through the battery in a time dt , we can write the power: $P = \frac{dU}{dt} = \frac{dq}{dt} \epsilon = I \epsilon$

Energy Flow: Resistor

- Amount of charge dq that goes through resistor loses energy

$$dU = dq (C - IR) \leftarrow \epsilon = IR, just reg'd from above$$

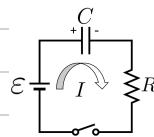
- Resistor gains an energy $dU = dq IR$

- The power of the resistor is

$$P = \frac{dU}{dt} = \frac{dq}{dt} IR = I^2 R \leftarrow This is called Joule Heating$$

RC Circuits

Resistor Capacitor Circuits



- Capacitor stores charge

*The charge is proportional to the Potential difference (Voltage) between the two electrodes by a quantity called Capacitance (C)

$$\rightarrow Q = CV$$

• $Q \equiv$ Charge (Coulombs, C)

• $C \equiv$ Capacitance (Farads, F)

• $V \equiv$ Voltage (Volts, V)

- In terms of current: $dQ = I dt$

- Current can flow in both directions (from positive to negative plates)

- The charge changes smoothly!

Solving the RC Circuit

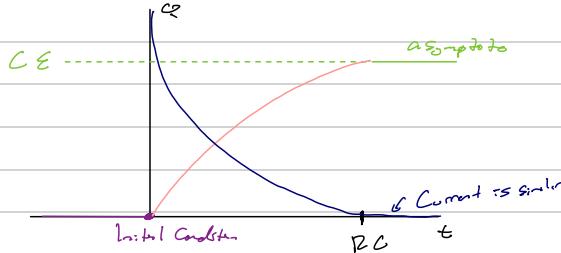
$$E - RI - \frac{Q}{C} = 0 \quad \leftarrow \text{From Kirchhoff's Loop rule}$$

$$\rightarrow \text{One equation, two unknowns: } I = \frac{dQ}{dt} \quad \text{Cannot change abruptly}$$

$$\rightarrow \text{Initial Condition of LDE: } @ t=0^+, Q=0$$

$$\rightarrow \text{At } t=\infty, \text{ system reaches steady state} \rightarrow \frac{dQ}{dt}=0$$

$$\therefore E - RI - \frac{Q}{C} = 0 \quad Q = EC \quad \text{& this is the asymptote}$$



$$\Rightarrow Q(t) = CE(1 - e^{-t/RC})$$

$$\Rightarrow I = \frac{dQ}{dt} = \frac{CE}{RC} e^{-t/RC}$$

- At $t < 0$: Switch is open; $I=0$
Capacitor discharged; $Q=0$

- At $t=0$: Close switch; $I \neq 0$
Capacitor starts charging

- At $t=\infty$: Charge plateaus

Solving the DE units of time

$$CE - RI - \frac{dQ}{dt} - Q = 0 \quad (\text{multipy DE by } C)$$

$\rightarrow RC$ is the time it takes to reach steady state

\rightarrow Guess a solution: $Q = CE(1 - e^{-t/RC})$

• Using Solution/guess

$$\frac{dQ}{dt} = \frac{d}{dt}(CE - CEe^{-t/RC}) = \frac{1}{RC} CE e^{-t/RC}$$

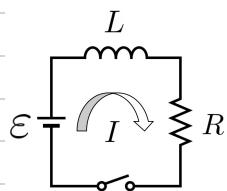
$$CE - \frac{RC}{t} CE e^{-t/RC} - CE + CE e^{-t/RC} = 0$$

$$e^{-t/RC} - \frac{RC}{t} e^{-t/RC} = 0 \Rightarrow \frac{RC}{t} = 1$$

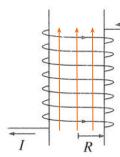
$t = RC$ verifies our solution!

RL Circuits

Resistor Inductor Circuits



• Inductor is typically a wire coil (solenoid)



$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{core}}$$

$$B = \mu_0 I n$$

$$\Phi_B = B N A = \mu_0 I n^2 l$$

$$\frac{d\Phi_B}{dt} = \mu_0 \frac{dI}{dt} n^2 l A = -E$$

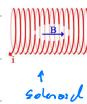
$$E = -L \frac{dI}{dt}$$

inductance Δ

$$\rightarrow \text{Inductance } (L) : L = \mu_0 n^2 l A$$

$$\rightarrow \text{In general, the inductance: } L = \frac{\mu_0 B}{I}$$

Inductance (L)

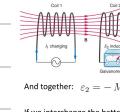


- The change in the current will produce a change in the magnetic field together with a change in flux of B through the solenoid. This will induce a electromotive force in the solenoid (self inductance)

$$V = -L \frac{dI}{dt}$$

- $I \equiv$ Current (Amperes, A)
- $L \equiv$ Inductance (Henrys, H)
- $V \equiv$ Voltage (Volts, V)

Mutual Inductance



The induced electromotive force in coil 2

$$\varepsilon_2 = -N_2 \frac{d\Phi_{B,2}}{dt}$$

and the flux has to be proportional to the current in coil 1:

$$N_2 \Phi_{B,2} = [M_{21}] I_1$$

proportionality constant

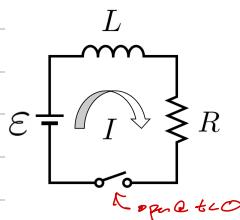
$$\text{If we interchange the battery and galvanometer } \varepsilon_1 = -M_{12} \frac{dI_2}{dt}$$

It can be proved that $M_{21} = M_{12} \equiv M$

$$\varepsilon_1 = -M \frac{dI_2}{dt}$$

$$\varepsilon_2 = -M \frac{dI_1}{dt}$$

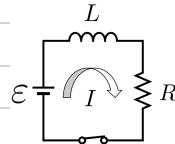
RL Circuit



- Inductor \rightarrow wire coil (solenoid)
- Voltage drop $\rightarrow V = -L \frac{dI}{dt}$
- Current doesn't change abruptly
- Current can flow in both directions
 \hookrightarrow from positive to negative potential

Solving the RL Circuit

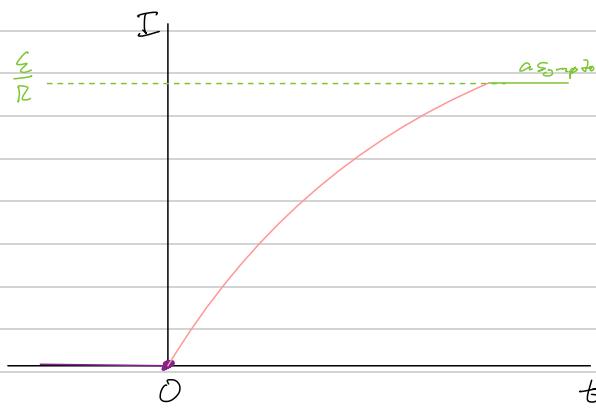
- At $t < 0$
 \hookrightarrow Switch is open, $I=0$
- At $t=0$
 \hookrightarrow close switch, $I \neq 0$



Kirchhoff's Loop rule

$$E - L \frac{dI}{dt} - IR = 0$$

Solving the DE



• At $t=0^+$ $\rightarrow I=0$ (current doesn't change abruptly)

• At $t=\infty$ $\rightarrow \frac{dI}{dt}=0$ (steady-state)

$$I = \frac{E}{R}$$

units of Amperes

$$\frac{E}{R} - \frac{L}{R} \frac{dI}{dt} - I = 0$$

\rightarrow Guess a solution: $I = \frac{E}{R} \left(1 - e^{-t/\tau} \right)$

• Plug in to equation to find τ and write:

$$I(t) = \frac{E}{R} \left[1 - e^{-\frac{L}{R}t} \right]$$

AC Circuits

Complex numbers: Review

- A complex number $z = x + iy$ is formed by

- Real part: $\operatorname{Re}[z] = x$

- Imaginary part: $\operatorname{Im}[z] = y$

- The imaginary unit: $i \leftarrow$ sometimes just notation
 $\hookrightarrow i^2 = -1$

- In Polar Coordinates: $z = \rho \cos \theta + i \rho \sin \theta$

$$\begin{cases} \rho = \sqrt{x^2 + y^2} = |z| \\ \theta = \arctan(\frac{y}{x}) \end{cases} \quad (\text{xy}) \leftrightarrow (\rho, \theta)$$

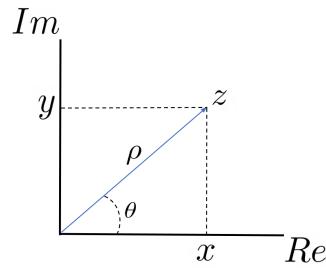
- Euler's formula: $e^{i\theta} = \cos \theta + i \sin \theta$

$$\Rightarrow z = \rho e^{i\theta} = \rho \cos \theta + i \rho \sin \theta$$

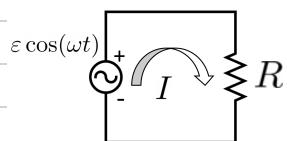
$$e^{i\pi} + 1 = 0$$

\rightarrow exponential form of complex number: $e^z = e^{wt+iy} = e^{wt} e^{iy}$

\rightarrow important function: $f(t) = e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$



AC Circuits: Resistor



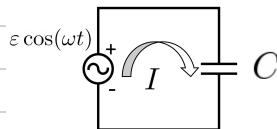
- Voltage of AC Source

$$\sqrt{V} = \Phi_+ - \Phi_- = E \cos(\omega t)$$

- Kirchhoff Rule

$$E \cos(\omega t) - IR = 0 \Rightarrow I = \frac{E}{R} \cos(\omega t)$$

AC Circuits: Capacitor



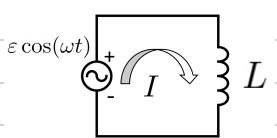
- Voltage of AC Source is sine

- Kirchhoff Rule

$$E \cos(\omega t) - \frac{Q}{C} = 0 \Rightarrow Q = EC \cos(\omega t)$$

$$I = \frac{dQ}{dt} = -EC \omega \sin(\omega t)$$

AC Circuits: Inductor



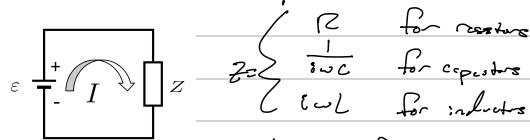
- Voltage of AC Source is sine

- Kirchhoff Rule

$$E \cos(\omega t) - L \frac{dI}{dt} = 0 \Rightarrow \frac{dI}{dt} = \frac{E}{L} \cos(\omega t)$$

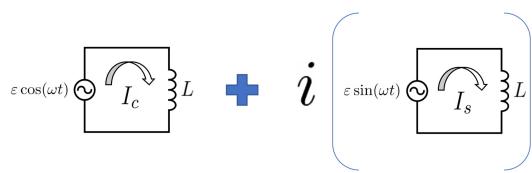
AC Circuits: Impedance

- We can study the equivalent DC circuit since impedance reacts



$$\rightarrow \text{Kirchhoff Rule: } \tilde{I} = \frac{\tilde{E}}{Z} \Rightarrow I(t) = I_0 [\tilde{I} e^{i\omega t}]$$

Impedance Complex Ohm's Law Proof



Recall: $(\rho, \theta) \Leftrightarrow (\rho e^{i\theta}, \theta)$

$$\rho e^{i\theta} = \rho [\cos \theta + i \sin \theta]$$

$$\epsilon \cos(\omega t) - L \frac{dI_c}{dt} = 0 + i (\epsilon \sin(\omega t) - L \frac{dI_s}{dt} = 0)$$

$$\epsilon \cos(\omega t) + i \epsilon \sin(\omega t) - L \left(\frac{dI_c}{dt} + i \frac{dI_s}{dt} \right) = 0$$

$$\epsilon e^{i\omega t} - L \frac{dI_e}{dt} = 0 \quad \text{where } I_e = I_c + iI_s$$

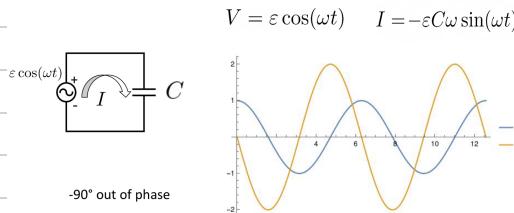
$$I_e = \tilde{I}_e e^{i\omega t} \quad \epsilon e^{i\omega t} - L (i \tilde{I}_e e^{i\omega t}) = 0$$

$$\frac{dI_e}{dt} = i\omega \tilde{I}_e e^{i\omega t} \quad \epsilon = L i \omega \tilde{I}_e \Rightarrow \epsilon = Z \tilde{I}_e \quad \text{Make a guess as to what the diff eq's sol. is!}$$

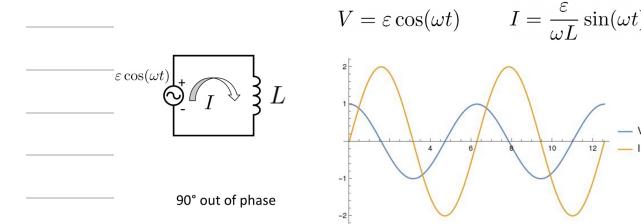
$$I_e = I_c + I_s = \tilde{I}_e e^{i\omega t} \Rightarrow I_c = R_c [\tilde{I}_e] = R_c [\tilde{I}_e e^{i\omega t}] \quad (\text{we want } I_c)$$

Phase

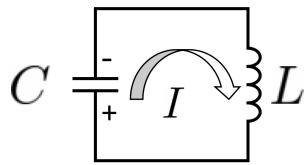
Current-Voltage Phase Capacitor



Current-Voltage Phase Inductor



LC Circuit



• Kirchhoff's Rule

$$-L \frac{dI}{dt} - \frac{Q}{C} = 0 \quad \left\{ \begin{array}{l} L \frac{d^2Q}{dt^2} + \frac{Q}{C} = 0 \\ I = \frac{dQ}{dt} \end{array} \right.$$

→ Analogous to Harmonic oscillations

$$m \frac{d^2x}{dt^2} + kx = 0 \rightarrow x = x_0 \cos(\sqrt{\frac{k}{m}}t) + v_0 \sqrt{\frac{m}{k}} \sin(\sqrt{\frac{k}{m}}t)$$

Our solution: $Q(t) = Q_0 \cos(\sqrt{\frac{1}{LC}}t) + I_0 \sqrt{LC} \sin(\sqrt{\frac{1}{LC}}t)$

$$I(t) = \frac{dQ}{dt} = \frac{Q_0}{\sqrt{LC}} \sin(\sqrt{\frac{1}{LC}}t)$$

Energy stored in Capacitors & Inductors

• Capacitor

$$dU = V dq = \frac{q}{C} dq, \quad U = \int_0^Q \frac{q}{C} dq = \frac{1}{2} \frac{Q^2}{C}$$

$U = \frac{1}{2} CV^2$

• Inductor

$$P = \frac{dU}{dt} = \frac{dq}{dt} V = IV = IL \frac{dI}{dt} = \frac{1}{2} L \frac{d(I^2)}{dt}$$

$$U = \int_0^{t_0} P dt = \frac{1}{2} L \int_0^{t_0} \frac{d(I^2)}{dt} dt = \frac{1}{2} LI^2$$

Ampere-Maxwell Equation

Maxwell Equations

Faraday's Law	Gauss Law	Ampere's Law	Gauss Law of B
The circulation of E equals the variation in time of the flux of B	The flux of E through a closed surface equals charge enclosed	The circulation of B through a closed loop equals current enclosed by any surface whose boundary is the loop	The flux of B through a closed loop equals zero

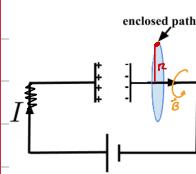
$\oint d\ell \cdot \vec{E} = -\frac{d}{dt} \int \hat{n} \cdot \vec{B} da$ $\int \hat{n} \cdot \vec{E} da = \frac{Q}{\epsilon_0}$ $\oint d\ell \cdot \vec{B} = \mu_0 I$ $\oint \hat{n} \cdot \vec{B} da = 0$

Recall: Ampere Law Scenario

Ampere's Law: The magnetic field across a closed loop (circulation) is proportional to the current through the any surface whose boundary is the loop.

But something is missing...

→ Assume infinite wire



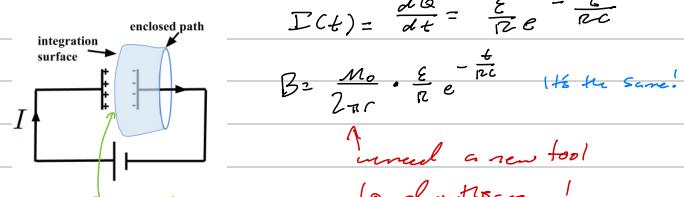
$$I(t) = \frac{dQ}{dt} = \frac{\epsilon}{\epsilon_0} e - \frac{t}{RC}$$

$$\oint d\ell \cdot \vec{B} = \mu_0 I$$

$$B \cdot 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0}{2\pi r} \cdot \frac{\epsilon}{\epsilon_0} e - \frac{t}{RC}$$

What if we choose a different surface



$$I(t) = \frac{dQ}{dt} = \frac{\epsilon}{\epsilon_0} e - \frac{t}{RC}$$

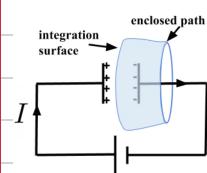
$$B = \frac{\mu_0}{2\pi r} \cdot \frac{\epsilon}{\epsilon_0} e - \frac{t}{RC}$$

It's the same!

↑ created a new tool
to do this easier!

We're missing a term in Ampere's Law

Maxwell-Ampere's Law: $\oint d\ell \cdot \vec{B} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$ → Flux := to electric field over time



Since $Q(t) \rightarrow E(t)$

then $\Phi_E(t) \propto S(t)$

$$\Rightarrow B = \frac{\mu_0}{2\pi r} \cdot \frac{\epsilon}{\epsilon_0} e - \frac{t}{RC}$$

displacement current

Example: Mag field inside capacitor -

$$\text{Ans: } \oint \vec{B} \cdot d\ell = B \cdot 2\pi r = B \cdot 2\pi r$$

$$\text{RHS: } \text{Mag Field} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

$$\Phi_E = \int \vec{E} \cdot d\vec{a} = EA$$

$$= E(t) \frac{\pi r^2}{\epsilon_0}$$

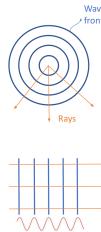
$$\text{inside } \parallel \text{ plate cap} \Rightarrow E(t) = \frac{Q(t)}{\epsilon_0} = \frac{A \epsilon_0}{\pi r^2} = \frac{Q(t)}{\pi r^2 \epsilon_0}$$

$$B(2\pi r) = \frac{1}{2\pi r} \left(\frac{Q(t)}{\pi r^2 \epsilon_0} \right) \cdot \mu_0 \epsilon_0$$

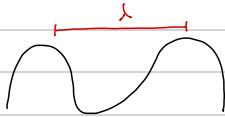
$$B = \frac{dQ}{dt} \frac{\mu_0 r^2}{\epsilon_0} \cdot \frac{1}{2\pi r}$$

Electromagnetic Waves

Recall: Waves



- Circular (spherical) waves
 - Phase fronts: contours of maximum amplitude
 - Displacement in a water wave
 - The distance between two is called wave length
 - Similar to equipotential surfaces
- Rays: Lines perpendicular to phase fronts pointing in the displacement direction
 - Similar to field lines
- Plane waves
 - Far away from the source, spherical waves look like plane waves

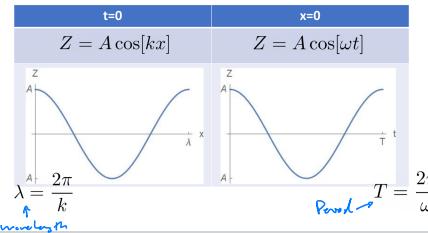


• Assume circular waves \Rightarrow From source are planar
 \hookrightarrow Calculations done should use reference at far (∞) distances

Equations → Amplitude vs speed

$$Z = A \cos[k(x - ct)]$$

note: $\omega = kc$
 Wave vector (just $k = \frac{\omega}{c}$, not a vector, it's a scalar)



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$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$	$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$	$\nabla \cdot \vec{B} = 0$

Different forms are easier for F1, but a Physics teacher

on the ball!
 Integral Form

Diff. Form

Physical terms
 bound to scope
 to class

Waves as Solutions of Maxwell Eqs

• Longitudinal? \rightarrow

◦ Oscillation direction same as propagation

$$\vec{E} = \hat{i} \epsilon \cos(kx - \omega t)$$

Not a solution of Maxwell Equations!

• Transversal? \rightarrow

◦ Direction \perp Propagation

$$\vec{E} = \hat{j} \epsilon \cos(kx - \omega t)$$

also, $\vec{B} = \hat{k} \beta \cos(kx - \omega t)$

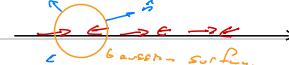
These are a solution

to an Equation

$\int \vec{E} \cdot d\vec{a}$
 \vec{B} not only
 if my not hold

Proof of Longitudinal wave not being a Solution

$$\vec{E} = \hat{i} \epsilon \cos(kx - \omega t)$$



$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} = 0 \quad (\text{no enclosed charge})$$

$$\Rightarrow 0 = \text{Flux} = \int_{\text{left}} \vec{E} \cdot d\vec{a} + \int_{\text{right}} \vec{E} \cdot d\vec{a} \quad 0 = 0 \text{ R-L}$$

$$0 = \int \vec{E} da + \int \vec{E} da$$

So E must be 0, but $\vec{E} = \hat{i} \epsilon \cos(kx - \omega t)$

Plugging into Maxwell Equations: $B = \frac{\epsilon}{c} \cdot j$, $C = \frac{\omega}{k} = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$

Sources of EM Waves

• Single charges accelerating

• Multiple particles moving slow: acceleration of dipole moment

◦ Dipole antenna: systems small compared with the wave length produced
 \rightarrow Atoms: Visible light (microns) whole size $10,000 \times$ smaller

◦ Molecules: Gamma and x-rays

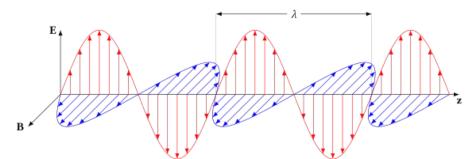
Poynting Vector

• The energy current or Poynting vector: $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$

• This fixes oscillation direction of \vec{B}

$$\vec{E} = \hat{i} \vec{j}, \vec{B} = \hat{k} \vec{k} \rightarrow \vec{S} = S \hat{i}$$

Energy flows in the \hat{i} direction



Polarization

Polarization: Linear

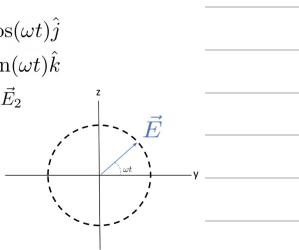
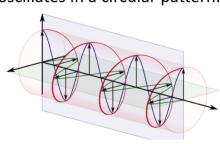
- The electric field parametrized as: $\vec{E} = \vec{\varepsilon} \cos(kx - \omega t)$
 \hookrightarrow oscillates in one direction
- Linear polarization covers the waves



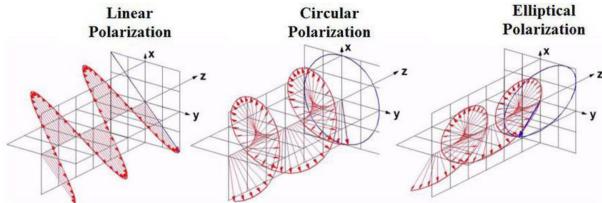
Polarization: Circular

- The Electric field parametrized as

$$\begin{aligned}\vec{E}_1 &= \varepsilon \cos(kx - \omega t) \hat{j} \\ \vec{E}_2 &= \varepsilon \cos(kx - \omega t + \frac{\pi}{2}) \hat{k}\end{aligned}$$
 $\xrightarrow{x=0} \vec{E}_1 = \varepsilon \cos(\omega t) \hat{j}$
 $\vec{E}_2 = \varepsilon \sin(\omega t) \hat{k}$
 $\vec{E} = \vec{E}_1 + \vec{E}_2$



Polarization: Summary



Energy, Momentum and energy current

- Time average intensity (Energy per second per square meter)

$$I = \langle \mathcal{E} \rangle$$

- Momentum flux (Momentum per second per square meter)

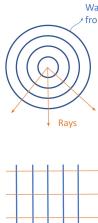
$$P = \frac{I}{c} \quad \text{Known as Radiation Pressure}$$

- How do transparently absorbing, or reflecting surfaces behave?

$$\hookrightarrow P = 2 \frac{I}{c}$$

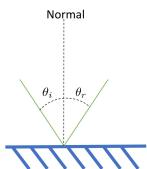
Geometric Optics

Geometric optics or Ray optics



- Describes light propagation in terms of rays
- Good approach if the wavelength is smaller than objects
- Effects
 - (i) Reflections
 - (ii) Refraction
- Applications
 - (i) Lenses
 - (ii) Cameras

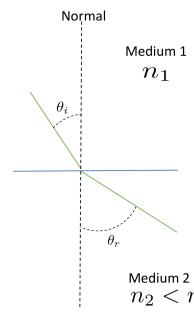
Reflection



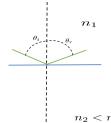
- An incident ray in a reflector with an angle θ_i with respect the normal direction of the reflector
- The reflected ray exits with an angle θ_r
- $\theta_i = \theta_r$
- Applications: Convex reflector - moon mirrors



Refraction



- A ray going from medium 1 to 2
 - Gets refracted (deviated)
 - The deviation depends on the refractive index (n)
 - Refractive index related with the dielectric susceptibility
- Snell's Law: $n_1 \sin(\theta_i) = n_2 \sin(\theta_r)$
- Total internal reflection
 - critical angle θ_c
- Speed of light in a medium: $v_n = \frac{c}{n}$



This can happen if
 $\theta_r > 90^\circ \sin \theta_c$
 "wraps" around

Interference

- Two waves out of phase

$$\vec{E}_1 = \varepsilon \cos(kx - \omega t) \hat{j} \quad \vec{E}_1 + \vec{E}_2 = 2\varepsilon \cos\left(\frac{\phi}{2}\right) \cos(kx - \omega t + \frac{\phi}{2}) \\ \vec{E}_2 = \varepsilon \cos(kx - \omega t + \phi) \hat{j}$$

- Two waves in phase travelling different distances

$$\vec{E}_1 = \varepsilon \cos(kx_1 - \omega t) \hat{j} \quad \vec{E}_1 + \vec{E}_2 = 2\varepsilon \cos\left(\frac{k(x_2 - x_1)}{2}\right) \cos\left(\frac{k(x_2 + x_1)}{2} - \omega t\right) \\ \vec{E}_2 = \varepsilon \cos(kx_2 - \omega t) \hat{j}$$

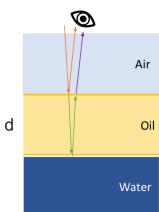
Recall:

$$\cos(a) + \cos(b) = 2 \cos\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right)$$

Applications

Noise cancelling uses play wave 90° to outside noise to try and cancel the sound

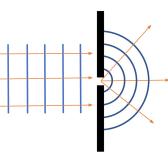
Light on an Oil Blob



- When light reaches the interface to a higher refractive medium it shifts a phase π upon reflection.
- The transmitted wave travels extra distance ($2d$)
 - Assume perpendicular rays to the surface
 - Note as well that the wavelength in air and oil is not the same
- The amplitude of the interference is proportional to

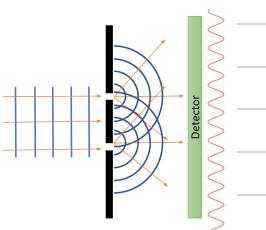
$$\sin\left(\frac{2dn_{oil}}{\lambda} \pi\right) \begin{cases} \frac{2dn_{oil}}{\lambda} \pi = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots & \text{Max (bright)} \\ \frac{2dn_{oil}}{\lambda} \pi = 0, \pm 1, \pm 2, \dots & \text{Min (dark)} \end{cases}$$

Diffracton



- When a wave encounters an obstacle of a similar size to its wavelength, it gets diffracted
- The diffracting object or aperture effectively becomes a second source of the propagating wave
- Think about sound, wavelength 2cm-10cm

Diffraction & Young Experiment



- The waves diffracted by two slits (d is the distance between them) interfere producing an interference pattern in the screen.
- Assuming that the screen is far away from the slits:
 - Maximum (bright) if $\sin \theta d \propto \lambda$
 - Constructive Interference
 - Minimum (dark) if $\sin \theta d \propto \frac{\lambda}{2}$
 - Destructive Interference



wave like behavior

Thats what happens if an observer is present...