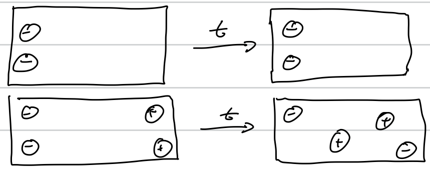
Midterm Exam 1

Need to write the Maxwell equations on the final exam!

Electric Charges & Fields, Coulomb's Law, Superposition

Electric Charge

- charge is a fundamental prop. of matter
  - ↳ two types: ⊕ & ⊖ — matter (atoms) are normally neutral
- **Triboelectricity**: Friction transfers charge
- Metal Rod in demo?
  - Plastic & crystal are insulators
    - → charge "stays where put"
  - Metals are conductors
    - → charge "flow freely"



t meaning "after some time"

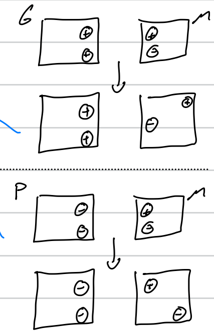
Demo - Rods

⊖P P⊖ ← →

⊕G G⊕ ← →

⊖P G⊕ → ←

Demo w/ Metal



Note: the metal is still neutral

- Charge is conserved
  - Rubbing transfers charges, but does not create nor destroy it
- Charge is quantized
  - Charge is always a multiple of $e = 1.6 \times 10^{-19}$ Coulombs (C) → electron $= -e$, proton $= e$
- why the rods are moving!
  - Attraction or Repulsion => There is a force between the charges
    - ↳ Force is known as coulomb's force

$$F = \frac{1}{4\pi\varepsilon_0}\cdot\frac{qQ}{r^2}$$ where $\frac{1}{4\pi\varepsilon_0} = 9\times10^9 \ \frac{N\cdot m^2}{C^2}$

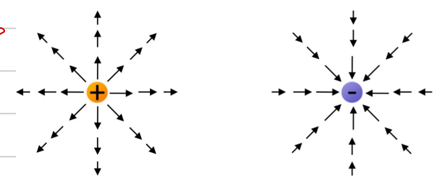
- When two charges q and Q are same signed → F is positive = Repulsive
- when they have opposite sign → F is negative = Attractive

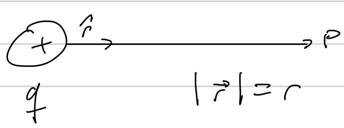
Electric Field and Coulomb's Law

Laws of Electrostatics

- Electrostatics ≡ charges @ rest
  - charges produce electric fields
  - electric fields push charges

$$\left|\vec{E}\right| = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}$$





$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

$\hat{r}$ is a unit vector that points away from charge q

Note: $r = |\vec{r}|$

or alternatively (use $\vec{v} = |\vec{v}|\,\hat{v}$) so $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ and $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}$

Equivalent!

Superposition Principle

In Words:

Electric Field produced by many point charges = Sum of the electric field produced by each point charge independently

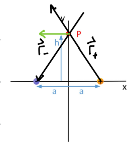
In Math: $$\vec{E}(q_1, q_2) = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1^3}\vec{r}_1 + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2^3}\vec{r}_2$$

Second Law of Electrostatics

Fields push/pull charges: $\vec{F} = Q\vec{E}$ => $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$

$\{N\}$ $\{C\}$ $\{\frac{N}{C}\}$

Example: PollEv



Calculate the Electric field at point P.

Pos vector $\vec{r}$ from ⊕ to P: $-a\hat{\imath} + h\hat{\jmath}$

Pos vector $\vec{r}$ from ⊖ to P: $a\hat{\imath} + h\hat{\jmath}$ (not moving, position vector not dependent on charge!)

Here, we are assuming point-like charges but this is not realistic

Calculate the Electric Field @ P:

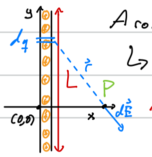
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r} \rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{(\sqrt{a^2+h^2})^3}\left[q(-a\hat{\imath}+h\hat{\jmath}) + -q(a\hat{\imath}+h\hat{\jmath})\right]\right) = \frac{2qa}{4\pi\varepsilon_0\sqrt{a^2+h^2}^3}\cdot(-\hat{\imath})$$

$r_- = a\hat{\imath} + h\hat{\jmath} \rightarrow r_- = \sqrt{a^2+h^2}$

$r_+ = -a\hat{\imath} + h\hat{\jmath} \rightarrow r_+ = \sqrt{a^2+h^2}$

$\vec{E} = \vec{E}_+ + \vec{E}_-$

Continuous Distribution of Charge



$dq = \lambda dy$

A rod of length 'L' is charged with a positive amount of charge Q

↳ Calculate the Electric field @ P

Assumptions: Rod is very long ($L \to \infty$)

Rod is very thin (1-D)

Definition: Charge per unit length, or linear charge density

$$\lambda = \frac{\text{Charge}}{\text{Length}} = \frac{Q}{L}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \sum_{i=1}^{N}\vec{E}_i = \int_a^b d\vec{E}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r} \rightarrow d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}$$

$\vec{r} = x\hat{\imath} - y\hat{\jmath}$

$r = \sqrt{x^2+y^2}$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{x^2+y^2}^3}\cdot(x\hat{\imath} - y\hat{\jmath})$$

$$\text{so, } \vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\lambda}{\sqrt{x^2+y^2}^3}(x\hat{\imath} - y\hat{\jmath})\,dy$$

Solve $\hat{\imath}$ and $\hat{\jmath}$ components separately

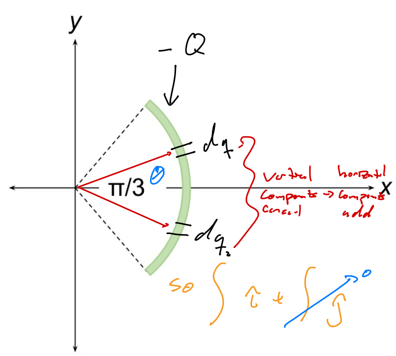
Example

A plastic rod with a uniformly distributed charge –Q is bent into a circular arc of radius *r* that subtends an angle of π/3 radians. We place coordinate axes such that the axis of symmetry of the rod lies along the *x*-axis and the origin is the center of curvature for the rod. What is the electric field (magnitude and direction) at the origin?

Hint: Imagine that the arc is made up of many infinitesimally small point charges *dq*. Each *dq* creates a differential electric field of magnitude $dE = k\,dq/r^2$.

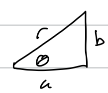
Sum up all the *dE*s from all the *dq*s to get the magnitude of the overall electric field.

Integration is a way to add many infinitely small elements.



$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}$$

$$|d\vec{E}| = \left|\frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}\right|$$



$\cos\theta = \frac{a}{r}$

$\sin\theta = \frac{b}{r}$

$\vec{r} = -r\cos\theta\,\hat{\imath} - r\sin\theta\,\hat{\jmath}$

x-component of $|d\vec{E}|$: $dE_x = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\cos\theta$

we need to integrate over all θ, but dE_x is in terms of dq, not $d\theta$!

$\Rightarrow dq = \lambda\, ds$ where ds is segment of arc

↖ linear charge density

$\Rightarrow 2\pi r$ so $ds = r\,d\theta \rightarrow dq = r\lambda\, d\theta$

To integrate let's use polar! $(x, y) \rightarrow (r, \theta)$

$$\int d\vec{E}_x = \frac{1}{4\pi\varepsilon_0}\int_{-\pi/6}^{\pi/6}\frac{r\lambda\cos\theta\, d\theta}{r^2}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 r}\left[\sin\theta\right]_{-\pi/6}^{\pi/6}$$

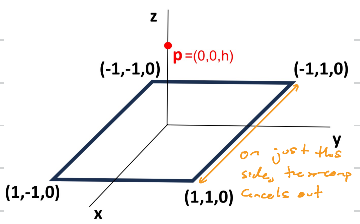
but what is lambda $= \frac{Q}{L} = \frac{Q}{r(\pi/3)}$

Just think length = angle · radius (like in $C = 2\pi r$)

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{3Q}{\pi r^2} \rightarrow \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{3Q}{\pi r^2}\hat{\imath}}$$

Example

A plastic rod with a uniformly distributed charge Q is shaped forming a square whose side is 1m. We place coordinate axes such that the square is on the x-y plane and the origin is the center of the square. What is the electric field (magnitude and direction) at the point p=(0,0,h)?

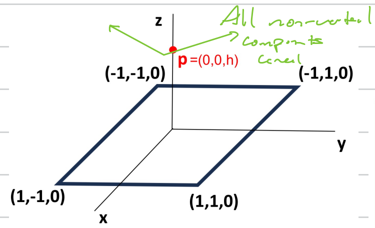


Hint: What is the square formed by? Is there any symmetry?

Imagine that the square is made up of many infinitesimally small point charges dq. Each dq creates a differential electric field of magnitude $dE = k\, dq/r^2$.

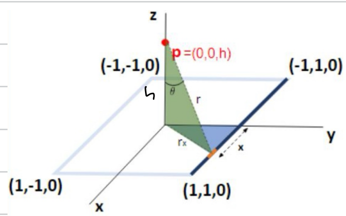
Sum up all the dEs from all the dqs to get the magnitude of the overall electric field.

Integration is a way to add many infinitely small elements.



=> This problem's solution will be $\vec{E} = E\hat{k}$

So we just need to calculate the vertical (z) component of one side and add the rest!



Blue triangle: $r_x^2 = x^2 + 1^2$

Green triangle: $r^2 = r_x^2 + h^2$

$\vec{r}_h = h\hat{k}$

$\vec{r}_{xy} = -x\hat{i} - \hat{j} \Rightarrow r_x^2 = x^2 + 1^2$ $\Big\}\ \vec{r} = -x\hat{i} - \hat{j} + h\hat{k}$

So, $r^2 = r_x^2 + h^2$

$\rightarrow r = \sqrt{x^2 + h^2 + 1}$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(x^2+h^2+1)^{3/2}}\left(-x\hat{i} - \hat{j} + h\hat{k}\right)$$

x and y components cancel

$$\Rightarrow \vec{E} = \int \frac{1}{4\pi\varepsilon_0}\frac{dq}{(x^2+h^2+1)^{3/2}}\left(\cancel{-x\hat{i}} \cancel{-\hat{j}} + h\hat{k}\right)$$

$$\vec{E}_{Right} = \int d\vec{E}_{Right} = \int_{-1}^{1} \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx}{\left(\sqrt{x^2+h^2+1}\right)^3}\left(-x\hat{i} - \hat{j} + h\hat{k}\right) = +c\hat{j} + d\hat{k}$$

↑ In one side $\hat{j}$ component present, but disappears when using Left as well

$$\Rightarrow \vec{E}_{Left} = \int d\vec{E}_{Left} = \ldots = -c\hat{j} + d\hat{k}$$

$$\vec{E}_{Top} = \int_{-1}^{1} \frac{1}{4\pi\varepsilon_0}\frac{dy}{\left(\sqrt{x^2+h^2+1}\right)^2}\left(-\hat{i} - y\hat{j} + h\hat{k}\right)$$

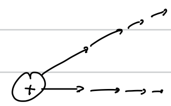
$$\vec{E}_{Bottom} = \int_{-1}^{1} \frac{1}{4\pi\varepsilon_0}\frac{dy}{\left(\sqrt{x^2+h^2+1}\right)^2}\left(+\hat{i} - y\hat{j} + hk\right)$$

So $\boxed{\vec{E} = \vec{E}_{Right} \cdot 4\hat{k}}$

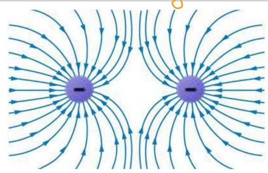
Gauss Law

Electric Field Lines

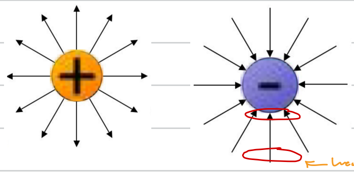
Single Charge



Two charges



(i) Lines are tangent to $\vec{E}$ at every point in space
(ii) The density of lines is proportional to $|\vec{E}|$
(iii) Lines never cross
(iv) No closed loops
(v) Lines start or stop on charges
↳ start should be a positive charge, end should be a negative charge
↳ This rule inherently enforces part (iv)



← weaker charge → fewer lines

Recall Vector Multiplication

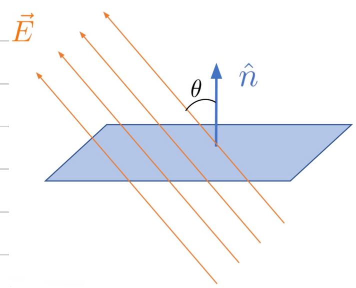
$$\left\{ \begin{aligned} \vec{A}\cdot\vec{B} &= A_x B_x + A_y B_y + A_z B_z \\ &= |\vec{A}||\vec{B}|\cos\theta \end{aligned} \right.$$

for θ the angle between $\vec{A}$ and $\vec{B}$

Electric Flux

- Flat surface and uniform $\vec{E}$
- Defined:

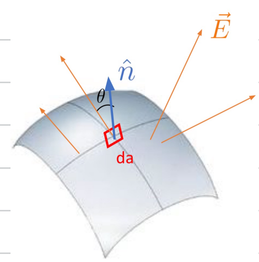
$$\Phi_E = A\hat{n}\cdot\vec{E} = A\underbrace{|\hat{n}|}_{1}|\vec{E}|\cos\theta = A|\vec{E}|\cos\theta$$



- $\hat{n}$ is a unit vector perpendicular to the surface (normal vector)
- θ is the angle between $\hat{n}$ and $\vec{E}$
- A is the area of the surface

- Non-Flat surface and/or Non-uniform $\vec{E}$

$$\Phi_E = \int \hat{n}\cdot\vec{E}\,da = \int \underbrace{|\hat{n}|}_{1}|\vec{E}|\cos\theta\,da = \int |\vec{E}|\cos\theta\,da$$



- $\hat{n}$ is a unit vector perpendicular to the surface da
- θ is the angle between $\hat{n}$ and $\vec{E}$

Gauss Law

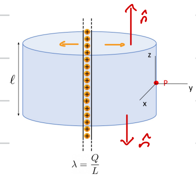
- First of Maxwell's Equations
- In words: The electric flux through a closed surface $=$ $\dfrac{\text{Amount of charge inside the surface}}{\varepsilon_0}$
- Equation:

$$\Phi_{E\,\text{closed}} = \oint \hat{n}\cdot\vec{E}\,da = \frac{Q}{\varepsilon_0}$$

- $\hat{n}$ points outside the surface by convention
- $\oint \vec{E}\cdot d\vec{a} = \frac{Q}{\varepsilon_0}$ is also valid, where $d\vec{a} = da\cdot\hat{n}$

• **Example / Review:** Infinite Line

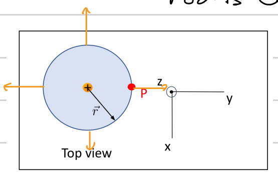
Assumptions: rod is very long ($L \to \infty$)
rod is very thin (1-D)



Length ℓ of rod inside cylinder

$\Rightarrow Q_{ENC} = \lambda \ell$

$\phi_E = \frac{Q_{ENC}}{\varepsilon_0} = \frac{\lambda \ell}{\varepsilon_0}$



$= \int_{Cap \uparrow} \hat{n}\cdot\vec{E}\,da + \int_{Cap \downarrow} \hat{n}\cdot\vec{E}\,da + \int_{side} \hat{n}\cdot\vec{E}\,da$

Extra credit Question: What is the direction of the total electric field @ P? $\hat{s}$

CAPS — radial
$\vec{E} = E\hat{r}$
$\hat{n} = \pm\hat{k}$ $\Big\}\ \theta = 90°$

SIDE — radial
$\vec{E} = E\hat{r}$
$\hat{n} = \hat{r}$ $\Big\}\ \theta = 0$

$$\boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\hat{r}\ ,\quad E = \frac{\lambda}{2\pi\varepsilon_0 r}}$$

$$\underbrace{2\int_{Caps} |\hat{n}||\vec{E}|\cos\theta\,da}_{\theta = 90°,\ so\ 0} + \int_{side} \hat{n}\cdot\vec{E}\,da$$

$= 0 + \int |\hat{n}||E|\cos\theta\,da$

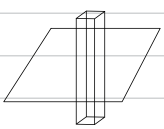
$= \int E\,da = E\int da = EA$

The magnitude is constant for all points on SIDE

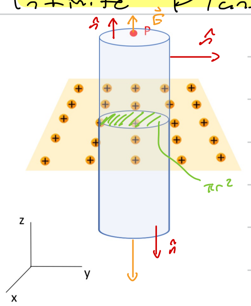
$\Phi_E = E \cdot 2\pi r \ell$

$$E \cdot 2\pi r \cancel{\ell} = \frac{\lambda \cancel{\ell}}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Infinite Plane



Also works for this and smaller



Direction of Electric field?

$A = \pi r^2$

$\sigma = \frac{Q}{L^2}$ — Surface Charge density

$\oint \hat{n}\cdot\vec{E}\,da = \frac{Q_{ENC}}{\varepsilon_0}$

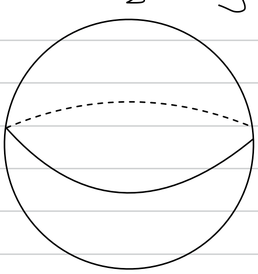
$Q_{ENC} = \sigma\pi r^2$

SIDE $\hat{n} = \hat{r}$ $\theta = 90°$
CAPS $\hat{n} = \pm\hat{k}$ $\theta = 0$

$= \int_{Caps} \hat{n}\cdot\vec{E}\,da = 2\int E\,da = 2EA = 2E\pi r^2$

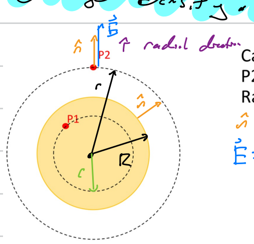
$$2E\cancel{\pi r^2} = \frac{\sigma\cancel{\pi r^2}}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{2\varepsilon_0}$$

Spherical Symmetry



Charge Density: $\rho = \frac{Q}{V}$ $\left\{\frac{C}{m^3}\right\}$

$\oint \hat{n}\cdot\vec{E}\,da = \frac{Q_{enc}}{\varepsilon_0}$



Calculate the Electric field at points P1 and P2 **using Gauss Law** of the sphere of Radius R and charged with a density ρ

$\hat{n} = \hat{r}$
$\vec{E} = E\hat{r}$

$\frac{Q_{enc}}{\varepsilon_0} = \frac{\rho\frac{4}{3}\pi R^3}{\varepsilon_0}$

Flux through gaussian surface which's outer radius sphere $\rightarrow \oint E\cos(0)\,da = Ea = E(4\pi r^2)$

$r > R$: $\frac{\rho\frac{4}{3}\pi R^3}{\varepsilon_0} = E(4\pi r^2) \Rightarrow \boxed{E = \frac{\rho R^3}{3\varepsilon_0 r^2}}$ outside the sphere

$r < R$: $\hat{n} = \hat{r}$
$\vec{E} = E\hat{r}$

$\frac{Q_{enc}}{\varepsilon_0} = \frac{V\rho}{\varepsilon_0} = \frac{\frac{4}{3}\pi r^3\rho}{\varepsilon_0}$ — enclosed radius

Gaussian surface is r

$\oint E\,da = EA = E(4\pi r^2)$

$\frac{\rho\frac{4}{3}\pi r^3}{\varepsilon_0} = E(4\pi r^2) \Rightarrow \boxed{E = \frac{\rho r}{3\varepsilon_0}}$ inside the sphere

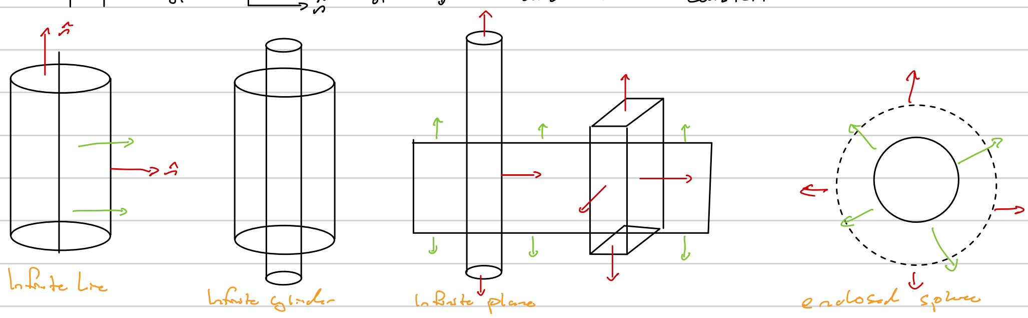
Any charge outside the gaussian surface does not affect the flux

Exam 1 Review Lecture

Gauss Law

$$\oint \vec{E}\,\hat{n}\,da = \oint \vec{E}\cdot d\vec{a}$$

$\vec{E} \uparrow \uparrow \hat{n}$ or $\vec{E} \perp \hat{n}$ or $\vec{E} \updownarrow \hat{n}$ and $\vec{E}$ is constant



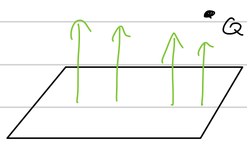
Coulombs Law

- $\vec{F} = Q\vec{E}$
- E by point q: $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

} If you have only two charges $\quad \vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$

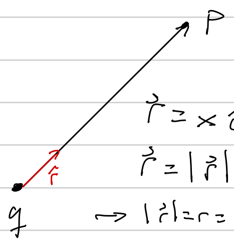
But if not 2 point-like:

Q's $\vec{F}$?



$$\vec{E} = \frac{\sigma}{2\varepsilon_0} \rightarrow \vec{F} = Q\frac{\sigma}{2\varepsilon_0}$$

$\hat{r}$ vs. $\vec{r}$



$\vec{r} = x\hat{i} + y\hat{j}$

$\vec{r} = |\vec{r}|\cdot\hat{r}$

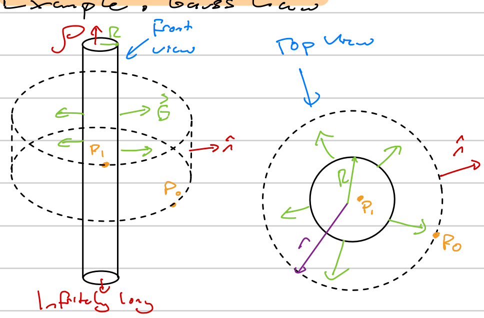
$\rightarrow |\vec{r}| = r = \sqrt{x^2+y^2}$ magnitude

Direct Integration $\rightarrow \hat{r} = \frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j}$ direction

charge densities

$\lambda = \frac{C}{m}$ linear $\quad \sigma = \frac{C}{m^2}$ surface $\quad \rho = \frac{C}{m^3}$ volume

Example: Gauss Law



$$\oint \vec{E}\cdot\hat{n}\,da = \frac{Q_{enc}}{\varepsilon_0} = \frac{\pi R^2 l\cdot\rho}{\varepsilon_0}$$

$\vec{E} = E\hat{r}$

Sides: $\hat{n} = \hat{r}$

CAPS: $\hat{n} = \pm\hat{k}$

$$= 2\int_{CAPS} E\cos(90)\,da + \int_{S.DES} E\cos 0\,da$$

area of sides

$$= 0 + Ea = E(2\pi r l)$$

$$\Rightarrow \boxed{E = \frac{R^2\rho}{2r\varepsilon_0}}$$ outside the cylinder

$\leftarrow$ Constant dependent parameter of Gauss surface (l in this case)