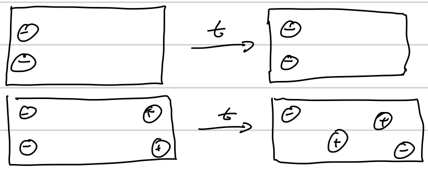
Midterm Exam 1

Need to write the Maxwell equations on the final exam!

Electric Charges & Fields, Coulomb's Law, Superposition

Electric Charge

- charge is a fundamental prop. of matter
  - ↳ two types: ⊕ & ⊖ — matter (atoms) are normally neutral
- **Triboelectricity:** Friction transfers charge
- Metal Rod in demo?
  - Plastic & crystal are **insulators**
    - → charge "**stays where put**"
  - Metals are **conductors**
    - → charge "**flow freely**"



t meaning "after some time"

Demo - Rods

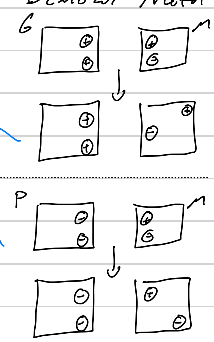
⊖ P P ⊖ ← →

⊕ G G ⊕ ← →

⊖ P G ⊕ → ←

Demo w/ Metal

Note: the metal is still neutral



- Charge is conserved
  - Rubbing transfers charges, but does not create nor destroy it
- Charge is quantized
  - Charge is always a multiple of $e = 1.6\times10^{-19}$ Coulombs (C) → electron $= -e$, proton $= e$
- why the rods are moving!
  - Attraction or Repulsion ⇒ There is a force between the charges
    - ↳ Force is known as **coulomb's force**

$$F = \frac{1}{4\pi\varepsilon_0}\cdot\frac{qQ}{r^2} \quad \text{where } \frac{1}{4\pi\varepsilon_0} = 9\times10^9\ \frac{N\cdot m^2}{C^2}$$

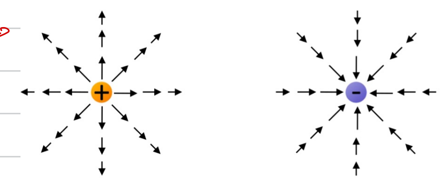
- When two charges q and Q are same signed → F is positive = Repulsive
- when they have opposite sign → F is negative = Attractive

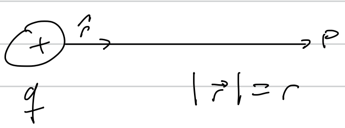
Electric Field and Coulomb's Law

Laws of Electrostatics

- **Electrostatics** ≡ charges @ rest
  - charges produce electric fields
  - electric fields push charges

$$|\vec{E}| = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}$$





q $\quad |\vec{r}| = r$

$\hat{r}$ is a unit vector that points away from charge q

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

Note: $r = |\vec{r}|$

or alternatively (use $\vec{v} = |\vec{v}|\,(\hat{v})$) so $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ and

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}$$

Equivalent!

Superposition Principle

In Words:

Electric Field produced by many point charges = Sum of the electric field produced by each point charge independently

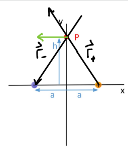
In Math:

$$\vec{E}(q_1, q_2) = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1^3}\vec{r}_1 + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2^3}\vec{r}_2$$

Second Law of Electrostatics

Fields push/pull charges: $\underset{\{N\}}{\vec{F}} = \underset{\{C\}}{Q}\,\underset{\{\frac{N}{C}\}}{\vec{E}} \Rightarrow \vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$

Example: PollEv



Calculate the Electric field at point P.

Pos vector $\vec{r}$ from ⊕ to p: $-a\hat{\imath} + h\hat{\jmath}$

Pos vector $\vec{r}$ from ⊖ to p: $a\hat{\imath} + h\hat{\jmath}$ (not moving, position vector not depending on charge!)

Here, we are assuming point-like charges but this is not realistic

Calculate the Electric Field @ P:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r} \Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{(\sqrt{a^2+h^2})^3}\left[q(-a\hat{\imath}+h\hat{\jmath}) + -q(a\hat{\imath}+h\hat{\jmath})\right]\right) = \frac{2qa}{4\pi\varepsilon_0\sqrt{a^2+h^2}^{\,3}}\cdot(-\hat{\imath})$$

$r_- = a\hat{\imath} + h\hat{\jmath} \rightarrow r_- = \sqrt{a^2+h^2}$

$r_+ = -a\hat{\imath} + h\hat{\jmath} \rightarrow r_+ = \sqrt{a^2+h^2}$

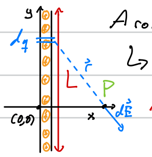
$\vec{E} = \vec{E}_+ + \vec{E}_-$

Continuous Distribution of Charge

Definition: Charge per unit length, or linear charge density

$$\lambda = \frac{\text{Charge}}{\text{Length}} = \frac{Q}{L}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \sum_{i=1}^{N}\vec{E}_i = \int_a^b d\vec{E}$$



$dq = \lambda\, dy$

A rod of Length 'L' is charged with a positive amount of charge Q

↳ Calculate the Electric field @ P

Assumptions: Rod is very long ($L \to \infty$)

Rod is very thin (1-D)

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r} \rightarrow d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}$$

$\vec{r} = x\hat{\imath} - y\hat{\jmath}$

$r = \sqrt{x^2+y^2}$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{x^2+y^2}^{\,3}}\cdot(x\hat{\imath} - y\hat{\jmath})$$

$$\text{so, } \vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\lambda}{\sqrt{x^2+y^2}^{\,3}}(x\hat{\imath} - y\hat{\jmath})\,dy$$

Solve $\hat{\imath}$ and $\hat{\jmath}$ components separately

Example

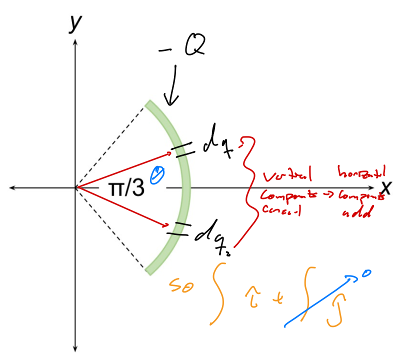
A plastic rod with a uniformly distributed charge –Q is bent into a circular arc of radius *r* that subtends an angle of π/3 radians. We place coordinate axes such that the axis of symmetry of the rod lies along the *x*-axis and the origin is the center of curvature for the rod. What is the electric field (magnitude and direction) at the origin?

Hint: Imagine that the arc is made up of many infinitesimally small point charges *dq*. Each *dq* creates a differential electric field of magnitude $dE = k\,dq/r^2$.

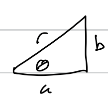
Sum up all the *dE*s from all the *dq*s to get the magnitude of the overall electric field.

Integration is a way to add many infinitely small elements.

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}$$



$$|d\vec{E}| = \left|\frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}\right|$$



$\cos\theta = \frac{a}{r}$, $\sin\theta = \frac{b}{r}$ ⟹ $\vec{r} = -r\cos\theta\,\hat{\imath} - r\sin\theta\,\hat{\jmath}$

x-component of $|d\vec{E}|$: $dE_x = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\cos\theta$

we need to integrate over all θ, but dE_x is in terms of dq, not $d\theta$!

⟹ $dq = \lambda\, ds$ where ds is segment of arc

↖ linear charge density

⟹ $\frac{2\pi r}{\theta}$ so $ds = r\,d\theta \rightarrow dq = r\lambda\, d\theta$

To integrate let's use polar! $(x, y) \rightarrow (r, \theta)$

$$\int d\vec{E}_x = \frac{1}{4\pi\varepsilon_0}\int_{-\pi/6}^{\pi/6}\frac{r\lambda\cos\theta\, d\theta}{r^2}$$

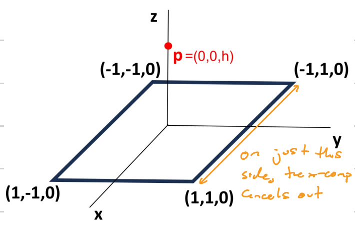
$$= \frac{\lambda}{4\pi\varepsilon_0 r}\left[\sin\theta\right]_{-\pi/6}^{\pi/6}$$

but what is lambda = $\frac{Q}{L} = \frac{Q}{r(\pi/3)}$ Just think length = angle · radius (like in $C = 2\pi r$)

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{3Q}{\pi r^2} \rightarrow \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{3Q}{\pi r^2}\hat{\imath}}$$

Example

A plastic rod with a uniformly distributed charge Q is shaped forming a square whose side is 1m. We place coordinate axes such that the square is on the x-y plane and the origin is the center of the square. What is the electric field (magnitude and direction) at the point p=(0,0,h)?

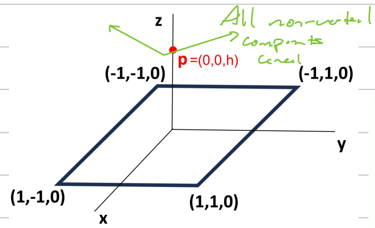


Hint: What is the square formed by? Is there any symmetry?

Imagine that the square is made up of many infinitesimally small point charges *dq*. Each *dq* creates a differential electric field of magnitude $dE = k\, dq/r^2$.

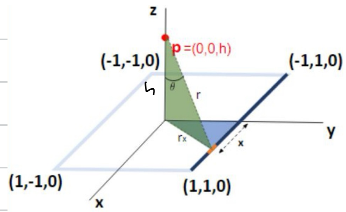
Sum up all the *dE*s from all the *dq*s to get the magnitude of the overall electric field.

Integration is a way to add many infinitely small elements.



$\Rightarrow$ This problem's solution will be $\vec{E} = E\hat{k}$

So we just need to calculate the vertical (z) component of one side and add the rest!



Blue triangle: $r_x^2 = x^2 + 1^2$

Green triangle: $r^2 = r_x^2 + h^2$

$\vec{r}_h = h\hat{k}$

$\vec{r}_{xy} = -x\hat{i} - \hat{j} \Rightarrow r_x^2 = x^2 + 1^2 \quad \Big\} \; \vec{r} = -x\hat{i} - \hat{j} + h\hat{k}$

So, $r^2 = r_x^2 + h^2$

$\to r = \sqrt{x^2 + h^2 + 1}$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(x^2+h^2+1)^{3/2}}\left(-x\hat{i} - \hat{j} + h\hat{k}\right)$$

x and y components cancel

$$\Rightarrow \vec{E} = \int \frac{1}{4\pi\varepsilon_0}\frac{dq}{(x^2+h^2+1)^{3/2}}\left(\cancel{-x\hat{i}}^{\,0}\ \cancel{-\hat{j}}^{\,0} + h\hat{k}\right)$$

$$\vec{E}_{Right} = \int d\vec{E}_{Right} = \int_{-1}^{1} \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx}{\left(\sqrt{x^2+h^2+1}\right)^3}\left(-x\hat{i} - \hat{j} + h\hat{k}\right) = +c\,\hat{j} + d\,\hat{k}$$

↑ In one side $\hat{j}$ component persists, but disappears when using Left as well

$$\Rightarrow \vec{E}_{left} = \int d\vec{E}_{left} = \ldots = -c\hat{j} + d\hat{k}$$

$$\vec{E}_{Top} = \int_{-1}^{1} \frac{1}{4\pi\varepsilon_0}\frac{dy}{\left(\sqrt{x^2+h^2+1}\right)^2}\left(-\hat{i} - y\hat{j} + h\hat{k}\right)$$

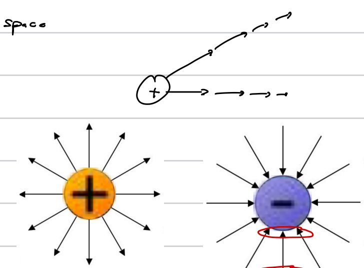
$$\vec{E}_{Bottom} = \int_{-1}^{1} \frac{1}{4\pi\varepsilon_0}\frac{dy}{\left(\sqrt{x^2+h^2+1}\right)^2}\left(+\hat{i} - y\hat{j} + hk\right)$$

So $\boxed{\vec{E} = \vec{E}_{right} \cdot 4\hat{k}}$

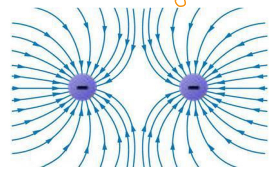
Gauss Law

Electric Field Lines

Single Charge



Two charges



(i) Lines are tangent to $\vec{E}$ at every point in space

(ii) The density of lines is proportional to $|\vec{E}|$

(iii) Lines never cross

(iv) No closed loops

(v) Lines start or stop on charges

↳ start should be a positive charge, end should be a negative charge

↳ This rule inherently enforces part (iv)

← weaker charge → fewer lines

Recall Vector Multiplication

$$\begin{cases} \vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z \\ \quad = |\vec{A}||\vec{B}|\cos\theta \end{cases}$$

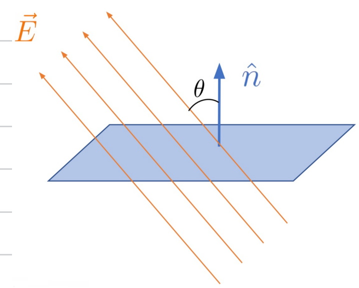
for θ the angle between $\vec{A}$ and $\vec{B}$

Electric Flux

- Flat surface and uniform $\vec{E}$
- Defined:

$$\Phi_E = A\hat{n}\cdot\vec{E} = A\,\overbrace{|\hat{n}|}^{1}\,|\vec{E}|\cos\theta = A|\vec{E}|\cos\theta$$

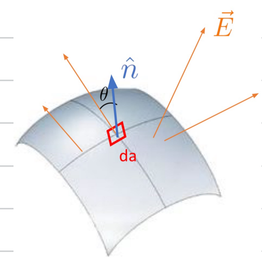
  - $\hat{n}$ is a unit vector perpendicular to the surface (normal vector)
  - θ is the angle between $\hat{n}$ and $\vec{E}$
  - A is the area of the surface



- Non-flat surface and/or Non-uniform $\vec{E}$

$$\Phi_E = \int \hat{n}\cdot\vec{E}\,da = \int \overbrace{|\hat{n}|}^{1}\,|\vec{E}|\cos\theta\,da = \int |\vec{E}|\cos\theta\,da$$

  - $\hat{n}$ is a unit vector perpendicular to the surface da
  - θ is the angle between $\hat{n}$ and $\vec{E}$



Gauss Law

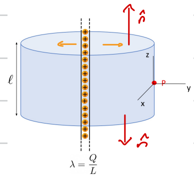
- First of Maxwell's Equations
- In words: The electric flux through a closed surface $=$ $\dfrac{\text{Amount of charge inside the surface}}{\varepsilon_0}$
- Equation:

$$\Phi_{E_{\text{closed}}} = \oint \hat{n}\cdot\vec{E}\,da = \frac{Q}{\varepsilon_0}$$

- $\hat{n}$ points outside the surface by convention
- $\oint \vec{E}\cdot d\vec{a} = \frac{Q}{\varepsilon_0}$ is also valid, where $d\vec{a} = da\cdot\hat{n}$

• Example / Review: Infinite Line

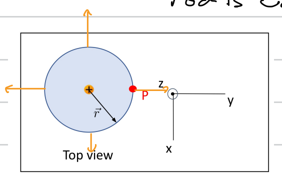
Assumption: rod is very long ($L \to \infty$)
rod is very thin (1-D)



Length ℓ of rod inside cylinder

$\Rightarrow Q_{ENC} = \lambda \ell$

$$\phi_E = \frac{Q_{ENC}}{\varepsilon_0} = \frac{\lambda \ell}{\varepsilon_0}$$



$$= \int_{Cap\uparrow} \hat{n}\vec{E}\,da + \int_{Cap\downarrow} \hat{n}\vec{E}\,da + \int_{side} \hat{n}\vec{E}\,da$$

Extra credit Question

What is the direction of the total electric field @ P? $\hat{s}$

CAPS — radial
$\vec{E} = E\hat{r}$
$\hat{n} = \pm\hat{k}$
$\Big\}\ \theta = 90°$

SIDE — radial
$\vec{E} = E\hat{r}$
$\hat{n} = \hat{r}$
$\Big\}\ \theta = 0$

$$\boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\hat{n}, \quad E = \frac{\lambda}{2\pi\varepsilon_0 r}}$$

$$\underbrace{2\int_{Caps} |\hat{n}||\vec{E}|\cos\theta\,da}_{\theta = 90°,\ so\ 0} + \int_{side} \hat{n}\vec{E}\,da$$

$$= 0 + \int |\hat{n}||E|\cos\theta\,da$$

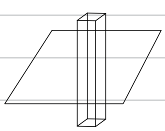
$$= \int E\,da = E\int da = EA$$

The magnitude is constant for all points on SIDE

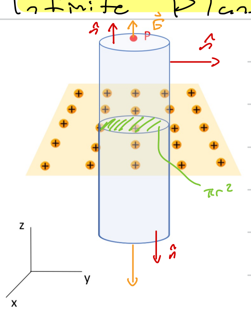
$$\Phi_E = E \cdot 2\pi r \ell$$

$$E \cdot 2\pi r \cancel{\ell} = \frac{\lambda \cancel{\ell}}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Infinite Plane



Also works for this and smaller



Direction of Electric Field?

$A = \pi r^2$

$$\sigma = \frac{Q}{L^2} \qquad \oint \hat{n}\vec{E}\,da = \frac{Q_{ENC}}{\varepsilon_0} \qquad Q_{ENC} = \sigma\pi r^2$$

Surface Charge density

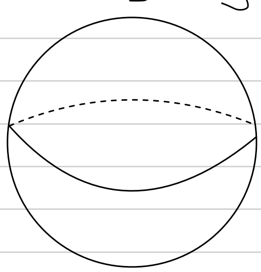
SIDE $\hat{n} = \hat{r}$ $\theta = 90°$

CAPS $\hat{n} = \pm\hat{k}$ $\theta = 0$

$$= \int_{Caps} \hat{n}\vec{E}\,da = 2\int E\,da = 2EA = 2E\pi r^2$$

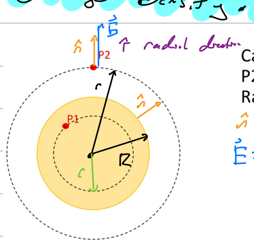
$$2E\cancel{\pi r^2} = \frac{\sigma\cancel{\pi r^2}}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{2\varepsilon_0}$$

Spherical Symmetry



Charge Density: $\rho = \frac{Q}{V}$ $\left\{\frac{C}{m^3}\right\}$

$$\oint \hat{n}\cdot\vec{E}\,da = \frac{Q_{enc}}{\varepsilon_0}$$



Calculate the Electric field at points P1 and P2 **using Gauss Law** of the sphere of Radius R and charged with a density ρ

$\hat{n} = \hat{r}$

$\vec{E} = E\hat{r}$

$$\frac{Q_{enc}}{\varepsilon_0} = \frac{\rho\frac{4}{3}\pi R^3}{\varepsilon_0}$$

Flux through gaussian surface which's outer radius sphere $\rightarrow \oint E\cos(0)\,da = Ea = E(4\pi r^2)$

$r > R$: $\frac{\rho\frac{4}{3}\pi R^3}{\varepsilon_0} = E(4\pi r^2) \Rightarrow \boxed{E = \frac{\rho R^3}{3\varepsilon_0 r^2}}$

outside the sphere

enclosed radius

$r < R$: $\hat{n} = \hat{r}$
$\vec{E} = E\hat{r}$

$$\frac{Q_{enc}}{\varepsilon_0} = \frac{V\rho}{\varepsilon_0} = \frac{\frac{4}{3}\pi r^3\rho}{\varepsilon_0}$$

Gaussian surface is r

$$\oint E\,da = EA = E(4\pi r^2)$$

$$\Bigg\}\ \frac{\rho\frac{4}{3}\pi r^3}{\varepsilon_0} = E(4\pi r^2) \Rightarrow \boxed{E = \frac{\rho r}{3\varepsilon_0}}$$

inside the sphere

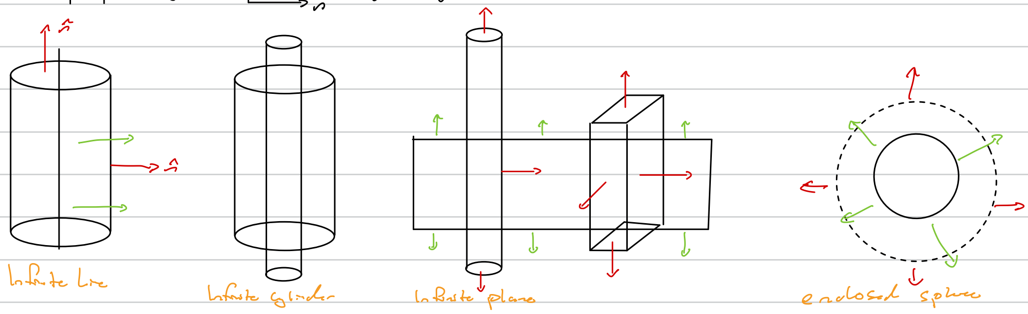
Any charge outside the gaussian surface does not affect the flux

Exam 1 Review Lecture

Gauss Law

$\oint \vec{E}\,\hat{n}\,da = \oint \vec{E}\cdot d\vec{a}$

$\vec{E} \uparrow \uparrow \hat{n}$ or $\vec{E} \perp \hat{n}$ or $\updownarrow \vec{E}, \hat{n}$ and $\vec{E}$ is constant



Coulombs Law

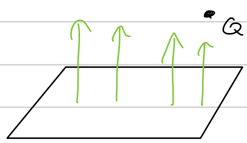
- $\vec{F} = Q\vec{E}$
- E by point q: $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

} If you have only two charges

$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$

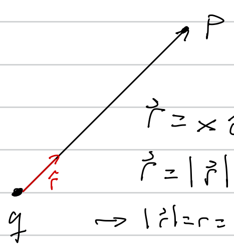
But if not 2 point-like:

Q's $\vec{F}$?



$\vec{E} = \frac{\sigma}{2\varepsilon_0} \rightarrow \vec{F} = Q\frac{\sigma}{2\varepsilon_0}$

$\hat{r}$ vs. $\vec{r}$



$\vec{r} = x\hat{i} + y\hat{j}$

$\vec{r} = |\vec{r}|\cdot\hat{r}$

$\rightarrow |\vec{r}| = r = \sqrt{x^2+y^2}$ magnitude

$\rightarrow \hat{r} = \frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j}$ direction

Direct Integration

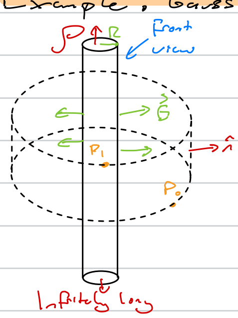
charge densities

$\lambda = \frac{C}{m}$ linear

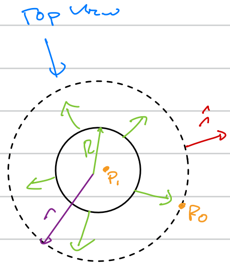
$\sigma = \frac{C}{m^2}$ surface

$\rho = \frac{C}{m^3}$ volume

Example: Gauss Law



Top view



$\oint \vec{E}\cdot\hat{n}\,da = \frac{Q_{enc}}{\varepsilon_0} = \frac{\pi R^2 l\cdot\rho}{\varepsilon_0}$

$\vec{E} = E\hat{r}$

Sides: $\hat{n} = \hat{r}$

CAPS: $\hat{n} = \pm\hat{k}$

$= 2\int_{CAPS} E\cos(90)\,da + \int_{SIDES} E\cos 0\,da$

$= 0 + Ea = E(2\pi r l)$ area of sides

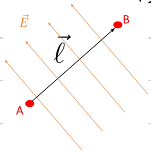
$\Rightarrow \boxed{E = \frac{R^2\rho}{2r\varepsilon_0}}$ outside the glass

← Cannot depend on parameter of Gaussian surface (l in this case)

Midterm Exam 2

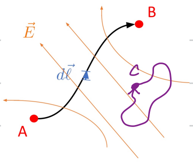
Circulation Law

Line Integrals



Straight line & uniform $\vec{E}$

- Line Integral $= \vec{\ell} \cdot \vec{E}$ for $\vec{\ell}$ the displacement vector
  - ↳ For this simple application (straight line & uniform $\vec{E}$)



General

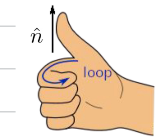
- Line integral $= \int_A^B \vec{E} \cdot d\vec{\ell}$

Closed Loops

- Circulation: $\oint \vec{E} \cdot d\vec{\ell}$

Right Hand Rule

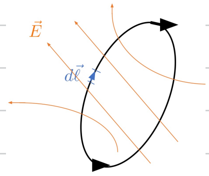
- Stokes convention
- Use direction of loop to determine $\hat{n}$ direction using:



Circulation

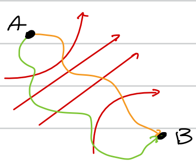
- The circulation of the electric field equals zero

→ Circulation $= \oint d\vec{\ell} \cdot \vec{E} = 0$
(electrostatics)



- **Alternative:** The line integral between two points do not depend on path

$$\int_A^B \vec{E} \cdot d\vec{\ell} = -\int_B^A \vec{E} \cdot d\vec{\ell} \left\{ \int_A^B \vec{E} \cdot d\vec{\ell} \right\}$$



$$\Rightarrow \oint \vec{E} \cdot d\vec{\ell} = \int_A^B \vec{E} \cdot d\vec{\ell} + \int_B^A \vec{E} \cdot d\vec{\ell} = 0$$

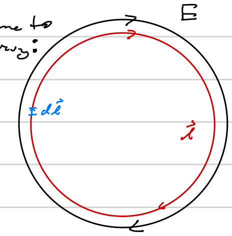
Choose the path that is most easy to integrate

No closed Loops!

(Proof by Contradiction)

Assume to the contrary:

$\vec{E}$



so, $\oint \vec{E} \cdot d\vec{\ell} = \oint |\vec{E}||d\vec{\ell}| \cos\theta$

>0 strictly

$\theta = 0$ $\quad = \oint |\vec{E}||d\vec{\ell}|$

$= (+) > 0$

No closed loops! but $\oint \vec{E} \cdot d\vec{\ell} = 0$ by circ. law

Application

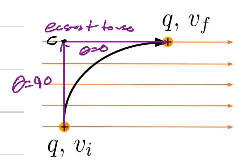
P121

- **Recall:** Work-Energy Theorem

$$\int_{i \to f} d\vec{\ell} \cdot \vec{F}_{Total} = \Delta K = \frac{1}{2} m (v_f^2 - v_i^2)$$

- But for electric force we don't need the path!

$$\rightarrow \int_{i \to f} d\vec{\ell} \cdot (q\vec{E}) = \Delta K = \frac{1}{2} m (v_f^2 - v_i^2) = \text{Same for all paths btwn } i \text{ and } f$$



Example

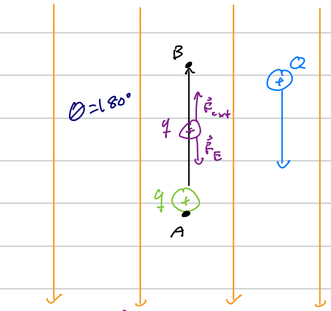
$$\int_A^B \vec{E} d\vec{\ell} = \int_A^C \vec{E} d\vec{\ell} + \int_C^B \vec{E} d\vec{\ell}$$

$$= \int_A^C E d\ell \cos 90 + \int_C^B E d\ell \cos 0$$

$$= E \int_C^B d\ell = E \cdot L$$

Electrostatic Energy

Overview / Introduction



$\vec{E} = E(-\hat{j})$

$\vec{F} = Q\vec{E} = QE(-\hat{j})$

- Apply force to bring +q to B from A
- Counteracts $\vec{E}$
- Release q at B → moves down

$\vec{F}_{ext} = -\vec{F}_E$ by N2L

Assumption: We move the charge very slowly, so the initial, final, and intermediate velocity is zero and hence there is No Kinetic Energy

↳ Spend an infinite amount of time from A to B $(v = \frac{\Delta x}{\Delta t})$

→ Energy Input = Work done by external agent $= \int_A^B \vec{F}_{ext} \cdot d\vec{l}$

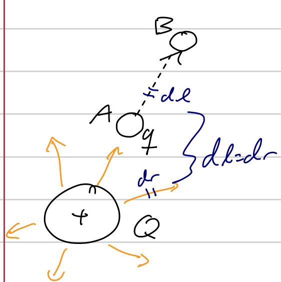
Using our assumption

- The external force matches the electric force: $U_B - U_A = \Delta U = \int_A^B d\vec{l} \cdot \vec{F}_{ext} = -\int_A^B d\vec{l} \cdot (q\vec{E})$

  ↳ In this simple (uniform $\vec{E}$ ✓) case: $U_B - U_A = \Delta U = qE(z_B - z_A)$ wow! GPE $(U_G = mgy)$

This is a particular formula, not general!

Point Charge



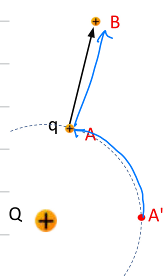
$$\Delta U = -q\int_A^B d\vec{l} \cdot \vec{E} = -q\frac{Q}{4\pi\varepsilon_0}\int_A^B \frac{\hat{r}\cdot\hat{r}}{r^2}\,dl = -\frac{qQ}{4\pi\varepsilon_0}\int_A^B \frac{dl}{r^2} = \frac{-qQ}{4\pi\varepsilon_0}\int_A^B \frac{dr}{r^2}$$

$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$

$d\vec{l} = \hat{r}\,dl$

$$\Rightarrow \Delta U = \frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

If $r_A \to \infty$ chosen as reference: $U = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r}$



$\Delta U_{AB} = \Delta U_{A'B}$

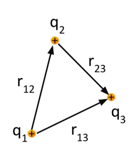
choose path in green

→ along dashed line

integral is 0 as you

move tangent

Arrangement of Charges



- The total energy of the system equals the sum of the energy of each pair of charges

$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right)$$

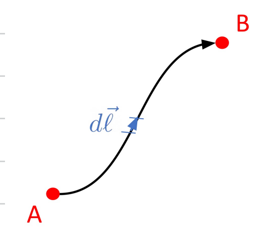
Electrostatic Potential

- Defined Φ, also seen as V
- Do not confuse w/ electrostatic energy (U)
- Another way to describe an electric field
- Electrostatic potential is a scalar (not a vector)
- To know $\vec{E}$ or Φ for any position space provides same amount of info (only electrostatics)
- If you know one you can figure out the other $\vec{E}(r) \rightleftarrows \Phi(r)$ ← not discussed here, not easily but doable
- Voltage $V = \Delta\Phi$

Examples of... Scalar Fields
- Elevation
- Temperature

Vector Fields
- Gravity
- Wind

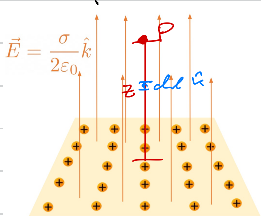
Definition



- $\Delta\Phi = \Phi(B) - \Phi(A) \equiv -\int_A^B d\vec{\ell}\cdot\vec{E}$
- If we set a reference point ($\Phi = 0$)

$$\Phi(\vec{r}) \equiv -\int_{ref\to\vec{r}} d\vec{\ell}\cdot\vec{E} \qquad \Phi(ref) = 0$$

Example: Potential - Infinite Plane

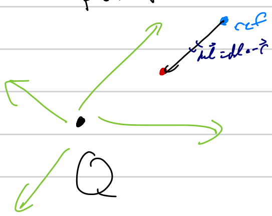


- Choose $z=0$ so $\Phi(z=0)=0$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}(\pm\hat{k}) \quad \text{(from gauss law)}$$

$$\Delta\Phi = \cancel{\Phi_{ref}} - \Phi_P = -\left(-\int_{ref\to P} d\vec{\ell}\cdot\vec{E}\right) = \int E\,d\ell\cos\theta = E\int_{z}^{0} dz = Ez = \boxed{-\frac{\sigma z}{2\varepsilon_0}}$$

Example: Potential Point Like Charge



$$\Phi(r\to\infty) = 0$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

$$\Phi(r) = -\int d\vec{\ell}\cdot\vec{E} = -\int \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}\cdot\hat{r}\,dr$$

$$\boxed{\Phi(r) = \frac{Q}{4\pi\varepsilon_0 r}}$$