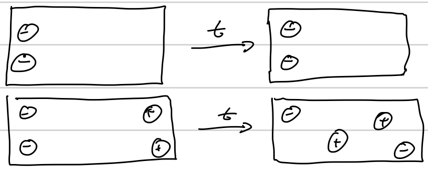
Midterm Exam 1

Need to write the Maxwell equations on the final exam!

Electric Charges & Fields, Coulomb's Law, Superposition

Electric Charge

- charge is a fundamental prop. of matter
  - ↳ two types: ⊕ & ⊖ matter (atoms) are normally neutral
- Triboelectricity: Friction transfers charge
- Metal Rod in demo?
  - Plastic & crystal are insulators
    - → charge "stays where put"
  - Metals are conductors
    - → charge "flow freely"



t meaning "after some time"

Demo - Rods

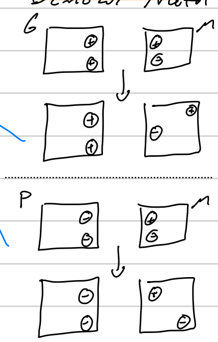
⊖ P P ⊖ ← →

⊕ G G ⊕ ← →

⊖ P G ⊕ → ←

Demo w/ Metal

Note: the metal is still neutral



- Charge is conserved
  - Rubbing transfers charges, but does not create nor destroy it
- Charge is quantized
  - Charge is always a multiple of $e = 1.6\times10^{-19}$ Coulombs (C) → electron $= -e$, proton $= e$
- why the rods are moving!
  - Attraction or Repulsion => There is a force between the charges
    - ↳ Force is known as coulomb's force

$$F = \frac{1}{4\pi\varepsilon_0}\cdot\frac{qQ}{r^2} \quad \text{where } \frac{1}{4\pi\varepsilon_0} = 9\times10^9\ \frac{N\cdot m^2}{C^2}$$

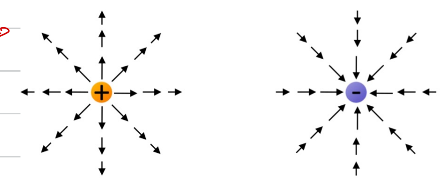
- when two charges q and Q are same signed → F is positive = Repulsive
- when they have opposite sign → F is negative = Attractive

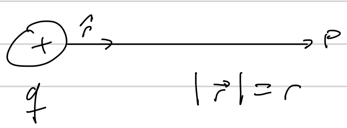
Electric Field and Coulomb's Law

Laws of Electrostatics

- Electrostatics ≡ charges @ rest
  - charges produce electric fields
  - electric fields push charges

$$|\vec{E}| = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}$$





$\hat{r}$ is a unit vector that points away from charge q

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

Note: $r = |\vec{r}|$

or alternatively (use $\vec{v} = |\vec{v}|(\hat{v})$) so $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ and

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}$$

Equivalent!

Superposition Principle

In Words:

Electric Field produced by many point charges = Sum of the electric field produced by each point charge independently

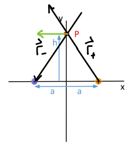
In Math:

$$\vec{E}(q_1, q_2) = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1^3}\vec{r}_1 + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2^3}\vec{r}_2$$

Second Law of Electrostatics

Fields push/pull charges: $\underset{\{N\}}{\vec{F}} = \underset{\{C\}}{Q}\,\underset{\{\frac{N}{C}\}}{\vec{E}} \Rightarrow \vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$

Example: PollEv



Calculate the Electric field at point P.

Pos vector $\vec{r}$ from ⊖ to p: $-a\hat{\imath} + h\hat{\jmath}$

Pos vector $\vec{r}$ from ⊕ to p: $a\hat{\imath} + h\hat{\jmath}$ (not moving, position vector not dependent on charge!)

Here, we are assuming point-like charges but this is not realistic

Calculate the Electric Field @ P:

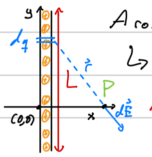
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r} \rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{(\sqrt{a^2+h^2})^3}\left[q(-a\hat{\imath}+h\hat{\jmath}) + -q(a\hat{\imath}+h\hat{\jmath})\right]\right) = \frac{2q_a}{4\pi\varepsilon_0\sqrt{a^2+h^2}^3}\cdot(-\hat{\imath})$$

$r_- = a\hat{\imath} + h\hat{\jmath} \rightarrow r_- = \sqrt{a^2+h^2}$

$r_+ = -a\hat{\imath} + h\hat{\jmath} \rightarrow r_+ = \sqrt{a^2+h^2}$

$\vec{E} = \vec{E}_+ + \vec{E}_-$

Continuous Distribution of Charge



$dq = \lambda dy$

A rod of Length 'L' is charged with a positive amount of charge Q

↳ Calculate the Electric field @ P

Assumptions: Rod is very long ($L \rightarrow \infty$)

Rod is very thin (1-D)

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r} \rightarrow d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}$$

$\vec{r} = x\hat{\imath} - y\hat{\jmath}$

$r = \sqrt{x^2+y^2}$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{x^2+y^2}^3}\cdot(x\hat{\imath} - y\hat{\jmath})$$

Definition: Charge per unit length, or linear charge density

$$\lambda = \frac{\text{Charge}}{\text{Length}} = \frac{Q}{L}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \sum_{i=1}^{N}\vec{E}_i = \int_a^b d\vec{E}$$

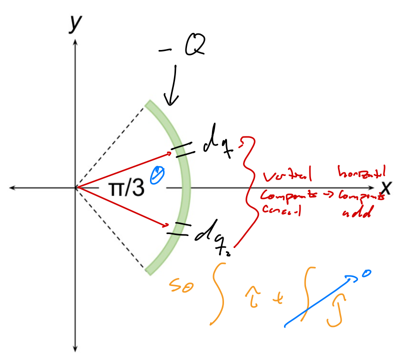
$$\text{so, } \vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\lambda}{\sqrt{x^2+y^2}^3}(x\hat{\imath} - y\hat{\jmath})\,dy$$

Solve $\hat{\imath}$ and $\hat{\jmath}$ components separately

Example

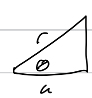
A plastic rod with a uniformly distributed charge –Q is bent into a circular arc of radius *r* that subtends an angle of π/3 radians. We place coordinate axes such that the axis of symmetry of the rod lies along the *x*-axis and the origin is the center of curvature for the rod. What is the electric field (magnitude and direction) at the origin?

> Hint: Imagine that the arc is made up of many infinitesimally small point charges *dq*. Each *dq* creates a differential electric field of magnitude $dE = k\,dq/r^2$.
>
> Sum up all the *dE*s from all the *dq*s to get the magnitude of the overall electric field.
>
> Integration is a way to add many infinitely small elements.



$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}$$

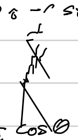
$$|d\vec{E}| = \left|\frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r}\right|$$



$\cos\theta = \frac{a}{r}$

$\sin\theta = \frac{b}{r}$

$\vec{r} = -r\cos\theta\,\hat{\imath} - r\sin\theta\,\hat{\jmath}$



x-component of $|d\vec{E}|$: $dE_x = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\cos\theta$

we need to integrate over all θ, but dE_x is in terms of dq, not dθ!

$\Rightarrow dq = \lambda\,ds$ where ds is segment of arc

linear charge density

$\Rightarrow \underbrace{2\pi r}_{\theta}$ so $ds = r\,d\theta \rightarrow dq = r\lambda\,d\theta$

To integrate let's use polar! $(x,y) \rightarrow (r,\theta)$

$$\int d\vec{E}_x = \frac{1}{4\pi\varepsilon_0}\int_{-\pi/6}^{\pi/6}\frac{r\lambda\cos\theta\,d\theta}{r^2}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 r}\left[\sin\theta\right]_{-\pi/6}^{\pi/6}$$

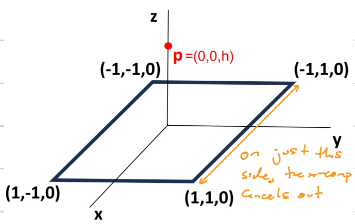
but what is lambda $= \frac{Q}{L} = \frac{Q}{r(\pi/3)}$

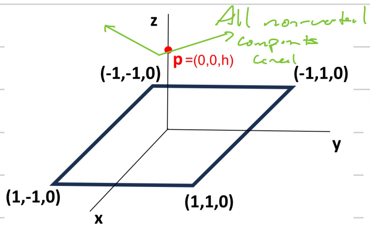
Just think length = angle · radius (like in $C = 2\pi r$)

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{3Q}{\pi r^2} \rightarrow \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{3Q}{\pi r^2}\hat{\imath}}$$

Example

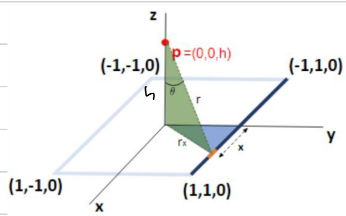
A plastic rod with a uniformly distributed charge Q is shaped forming a square whose side is 1m. We place coordinate axes such that the square is on the x-y plane and the origin is the center of the square. What is the electric field (magnitude and direction) at the point p=(0,0,h)?





⇒ This problem's solution will be $\vec{E} = E\hat{k}$

So we just need to calculate the vertical (z) component of one side and add the rest!



Blue triangle: $r_x^2 = x^2 + 1^2$

Green triangle: $r^2 = r_x^2 + h^2$

Hint: What is the square formed by? Is there any symmetry?

Imagine that the square is made up of many infinitesimally small point charges *dq*. Each *dq* creates a differential electric field of magnitude *dE = k dq/r*^{2}.

Sum up all the *dE*s from all the *dq*s to get the magnitude of the overall electric field.

Integration is a way to add many infinitely small elements.

$\vec{r}_h = h\hat{k}$

$\vec{r}_{xy} = -x\hat{i} - \hat{j} \Rightarrow r_x^2 = x^2 + 1^2$ $\Big\}$ $\vec{r} = -x\hat{i} - \hat{j} + h\hat{k}$

So, $r^2 = r_x^2 + h^2$

$\rightarrow r = \sqrt{x^2+h^2+1}$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(x^2+h^2+1)^{3/2}}(-x\hat{i}-\hat{j}+h\hat{k})$$

x and y components cancel

$$\Rightarrow \vec{E} = \int \frac{1}{4\pi\varepsilon_0}\frac{dq}{(x^2+h^2+1)^{3/2}}(\cancel{-x\hat{i}}\cancel{-\hat{j}}+h\hat{k})$$

$$\vec{E}_{Right} = \int d\vec{E}_{Right} = \int_{-1}^{1}\frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx}{(\sqrt{x^2+h^2+1})^3}(-x\hat{i}-\hat{j}+h\hat{k}) = +c\hat{j} + d\hat{k}$$

↑ In one side $\hat{j}$ component present, but disappears when using Left as well

$$\Rightarrow \vec{E}_{left} = \int d\vec{E}_{left} = \dots = -c\hat{j} + d\hat{k}$$

$$\vec{E}_{Top} = \int_{-1}^{1}\frac{1}{4\pi\varepsilon_0}\frac{dy}{(\sqrt{x^2+h^2+1})^2}(-\hat{i}-y\hat{j}+h\hat{k})$$

$$\vec{E}_{Bottom} = \int_{-1}^{1}\frac{1}{4\pi\varepsilon_0}\frac{dy}{(\sqrt{x^2+h^2+1})^2}(+\hat{i}-y\hat{j}+hk)$$

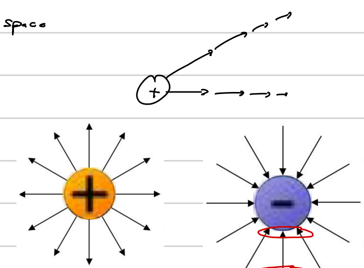
So $\boxed{\vec{E} = \vec{E}_{Right}\cdot 4\hat{k}}$

Gauss Law

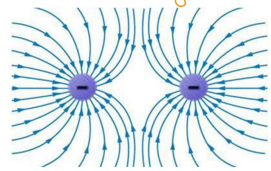
Electric Field Lines

(i) Lines are tangent to $\vec{E}$ at every point in space
(ii) The density of lines is proportional to $|\vec{E}|$
(iii) Lines never cross
(iv) No closed loops
(v) Lines start or stop on charges
↳ start should be a positive charge, end should be on negative charge
↳ This rule inherently enforces part (iv)

Single Charge



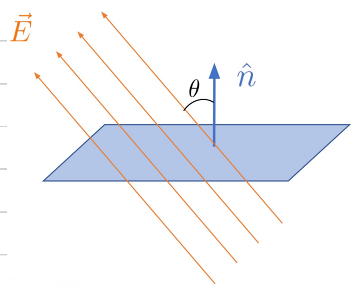
Two charges



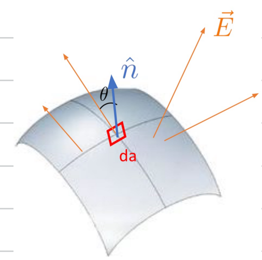
Recall Vector Multiplication
$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$
$$= |\vec{A}||\vec{B}|\cos\theta$$
for θ the angle between $\vec{A}$ and $\vec{B}$

Electric Flux

- Flat surface and uniform $\vec{E}$
- Defined:
$$\Phi_E = A\hat{n}\cdot\vec{E} = A\,\overbrace{|\hat{n}|}^{1}\,|\vec{E}|\cos\theta = A|\vec{E}|\cos\theta$$
  - $\hat{n}$ is a unit vector perpendicular to the surface (normal vector)
  - θ is the angle between $\hat{n}$ and $\vec{E}$
  - A is the area of the surface



- Non-Flat surface and/or Non-uniform $\vec{E}$
$$\Phi_E = \int \hat{n}\cdot\vec{E}\,da = \int \overbrace{|\hat{n}|}^{1}|\vec{E}|\cos\theta\,da = \int|\vec{E}|\cos\theta\,da$$
  - $\hat{n}$ is a unit vector perpendicular to the surface da
  - θ is the angle between $\hat{n}$ and $\vec{E}$



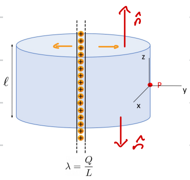
Gauss Law

- First of Maxwell's Equations
- In words: The electric flux through a closed surface $=$ $\dfrac{\text{Amount of charge inside the surface}}{\varepsilon_0}$
- Equation:
$$\Phi_{E_{\text{closed}}} = \oint \hat{n}\cdot\vec{E}\,da = \frac{Q}{\varepsilon_0}$$

- $\hat{n}$ points outside the surface by convention
- $\oint \vec{E}\cdot d\vec{a} = \frac{Q}{\varepsilon_0}$ is also valid, where $d\vec{a} = da\cdot\hat{n}$

• **Example / Review:** Infinite Line

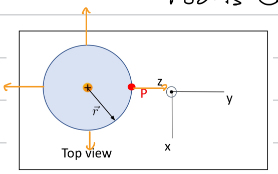
Assumptions: rod is very long ($L \to \infty$)
rod is very thin (1-D)



Length ℓ of rod inside cylinder

$\Rightarrow Q_{ENC} = \lambda \ell$

$$\Phi_E = \frac{Q_{ENC}}{\varepsilon_0} = \frac{\lambda \ell}{\varepsilon_0}$$



$$= \int_{Cap\uparrow} \hat{n}\vec{E}\, da + \int_{Cap\downarrow} \hat{n}\vec{E}\, da + \int_{side} \hat{n}\vec{E}\, da$$

Extra credit Question: What is the direction of the total electric field @ P? $\hat{s}$

CAPS
$\vec{E} = E\hat{r}$ (radial)
$\hat{n} = \pm\hat{k}$
$\theta = 90°$

$$\boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\hat{n}, \quad E = \frac{\lambda}{2\pi\varepsilon_0 r}}$$

$$\underbrace{2\int_{Caps} |\hat{n}||\vec{E}|\cos\theta\, da}_{\theta = 90°,\ so\ 0} + \int_{side} \hat{n}\vec{E}\, da$$

SIDE
$\vec{E} = E\hat{r}$ (radial)
$\hat{n} = \hat{r}$
$\theta = 0$

$= 0 + \int |\hat{n}||\vec{E}|\cos\theta\, da$

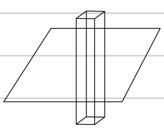
$= \int E\, da = E\int da = EA$

The magnitude is constant for all points on SIDE

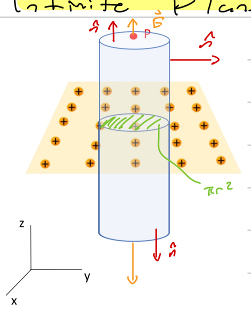
$\Phi_E = E \cdot 2\pi r \ell$

$$E \cdot 2\pi r \cancel{\ell} = \frac{\lambda \cancel{\ell}}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Infinite Plane



Also works for this and smaller



Direction of Electric Field?

$A = \pi r^2$

$\sigma = \frac{Q}{L^2}$ (Surface Charge density)

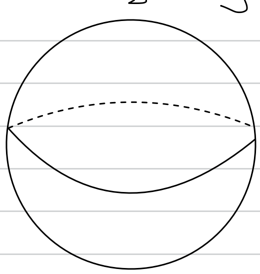
$$\oint \hat{n}\vec{E}\, da = \frac{Q_{ENC}}{\varepsilon_0} \qquad Q_{ENC} = \sigma\pi r^2$$

$$= \int_{Caps} \hat{n}\vec{E}\, da = 2\int E\, da = 2EA = 2E\pi r^2$$

SIDE $\hat{n} = \hat{r}$ $\theta = 90°$
CAPS $\hat{n} = \pm\hat{k}$ $\theta = 0$

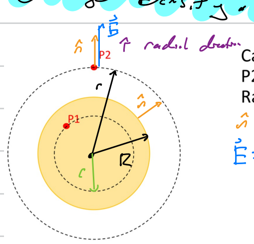
$$2E\cancel{\pi r^2} = \frac{\sigma\cancel{\pi r^2}}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{2\varepsilon_0}$$

Spherical Symmetry



Charge Density: $\rho = \frac{Q}{V}$ $\left\{\frac{C}{m^3}\right\}$

$$\oint \hat{n}\cdot\vec{E}\, da = \frac{Q_{enc}}{\varepsilon_0}$$



Calculate the Electric field at points P1 and P2 **using Gauss Law** of the sphere of Radius R and charged with a density ρ

$\hat{n} = \hat{r}$
$\vec{E} = E\hat{r}$

$$\frac{Q_{enc}}{\varepsilon_0} = \frac{\rho\frac{4}{3}\pi R^3}{\varepsilon_0}$$

Flux through gaussian surface which's outer radius sphere $\rightarrow \oint E\cos(0)\, da = Ea = E(4\pi r^2)$

$r > R$: $\frac{\rho\frac{4}{3}\pi R^3}{\varepsilon_0} = E(4\pi r^2) \Rightarrow \boxed{E = \frac{\rho R^3}{3\varepsilon_0 r^2}}$ outside the sphere

$r < R$: $\hat{n} = \hat{r}$, $\vec{E} = E\hat{r}$

$$\frac{Q_{enc}}{\varepsilon_0} = \frac{V\rho}{\varepsilon_0} = \frac{\frac{4}{3}\pi r^3\rho}{\varepsilon_0}$$ (enclosed radius)

Gaussian surface is r

$$\oint E\, da = EA = E(4\pi r^2)$$

$$\frac{\rho\frac{4}{3}\pi r^3}{\varepsilon_0} = E(4\pi r^2) \Rightarrow \boxed{E = \frac{\rho r}{3\varepsilon_0}}$$ inside the sphere

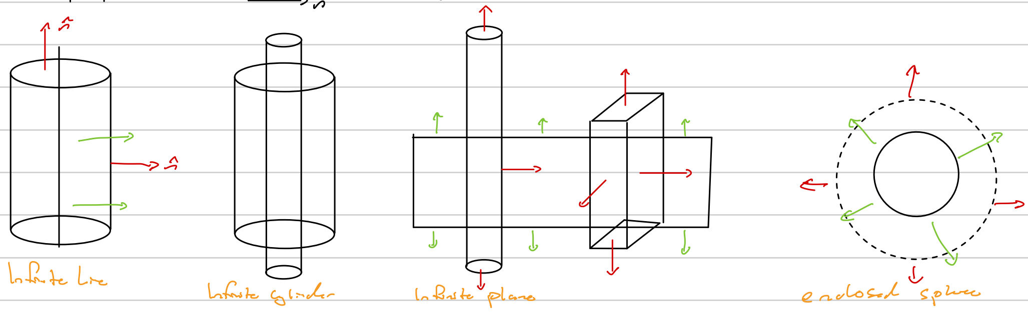
Any charge outside the gaussian surface does not affect the flux

Exam 1 Review Lecture

Gauss Law

$$\oint \vec{E}\,\hat{n}\,da = \oint \vec{E}\cdot d\vec{a}$$

$\vec{E} \uparrow \uparrow \hat{n}$ or $\vec{E} \perp \hat{n}$ or $\vec{E} \uparrow\downarrow \hat{n}$ and $\vec{E}$ is constant



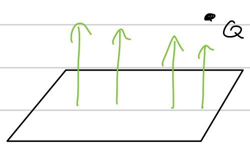
Coulombs Law

- $\vec{F} = Q\vec{E}$
- E by point q: $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

} If you have only two charges $\quad \vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$

But if not 2 point-like:

Q's $\vec{F}$?

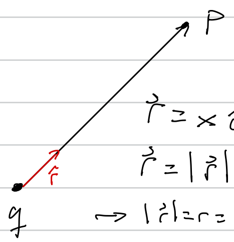


$$\vec{E} = \frac{\sigma}{2\varepsilon_0} \rightarrow \vec{F} = Q\frac{\sigma}{2\varepsilon_0}$$

$\hat{r}$ vs. $\vec{r}$

charge densities

$\lambda = \frac{C}{m}$ linear $\quad \sigma = \frac{C}{m^2}$ surface $\quad \rho = \frac{C}{m^3}$ volume



$\vec{r} = x\hat{i} + y\hat{j}$

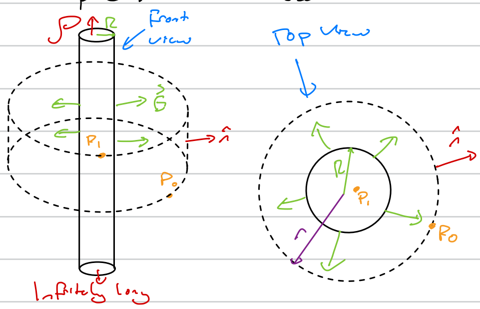
$\vec{r} = |\vec{r}|\cdot\hat{r}$

Direct Integration

$\rightarrow |\vec{r}| = r = \sqrt{x^2+y^2}$ magnitude

$\rightarrow \hat{r} = \frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j}$ direction

Example: Gauss Law



$$\oint \vec{E}\cdot\hat{n}\,da = \frac{Q_{enc}}{\varepsilon_0} = \frac{\pi R^2 l\cdot\rho}{\varepsilon_0}$$

$\vec{E} = E\hat{r}$

Sides: $\hat{n} = \hat{r}$

CAPS: $\hat{n} = \pm\hat{k}$

$$= 2\int_{CAPS} E\cos(90)\,da + \int_{SIDES} E\cos 0\,da$$

area of sides

$$= 0 + Ea = E(2\pi r l)$$

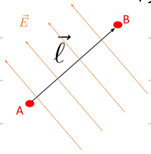
$$\Rightarrow \boxed{E = \frac{R^2\rho}{2r\varepsilon_0}}$$ outside the cylinder

← Cannot depend on parameter of Gauss surface (l in this case)

Midterm Exam 2

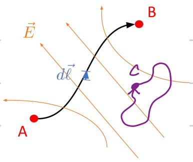
Circulation Law

Line Integrals



Straight line & uniform $\vec{E}$

- Line Integral $= \vec{\ell} \cdot \vec{E}$ for $\vec{\ell}$ the displacement vector
  - ↳ For this simple application (straight line & uniform $\vec{E}$)



General

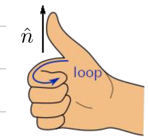
- Line integral $= \int_A^B \vec{E} \cdot d\vec{\ell}$

Closed Loops

- Circulation: $\oint \vec{E} \cdot d\vec{\ell}$

Right Hand Rule

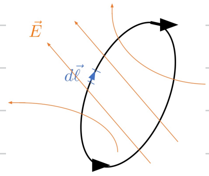
- Stokes convention
- Use direction of loop to determine $\hat{n}$ direction using:



Circulation

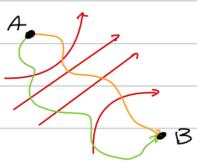
- The circulation of the electric field equals zero

→ Circulation $= \oint d\vec{\ell} \cdot \vec{E} = 0$
(electrostatics)



- **Alternative:** The line integral between two points do not depend on path

$$\int_A^B \vec{E} \cdot d\vec{\ell} = -\int_B^A \vec{E} \cdot d\vec{\ell} \quad \left\{ \int_A^B \vec{E} \cdot d\vec{\ell} \right\}$$



$$\Rightarrow \oint \vec{E} \cdot d\vec{\ell} = \int_A^B \vec{E} \cdot d\vec{\ell} + \int_B^A \vec{E} \cdot d\vec{\ell} = 0$$

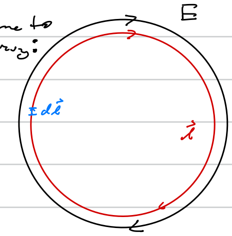
Choose the path that is most easy to integrate

No closed Loops!

(Proof by Contradiction)

$\vec{E}$

Assume to the contrary:



So, $\oint \vec{E} \cdot d\vec{\ell} = \oint |\vec{E}||d\vec{\ell}| \cos\theta$

>0 strictly

$\theta = 0$ $= \oint |\vec{E}||d\vec{\ell}|$

$= (+) > 0$

No closed loops! but $\oint \vec{E} \cdot d\vec{\ell} = 0$ by circ. law

Application

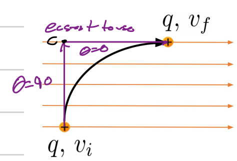
P121

- **Recall:** Work-Energy Theorem

$$\int_{i \to f} d\vec{\ell} \cdot \vec{F}_{Total} = \Delta K = \frac{1}{2} m (v_f^2 - v_i^2)$$

- But for electric force we don't need the path!

$$\rightarrow \int_{i \to f} d\vec{\ell} \cdot (q\vec{E}) = \Delta K = \tfrac{1}{2} m (v_f^2 - v_i^2) = \text{Same for all paths btwn } i \text{ and } f$$

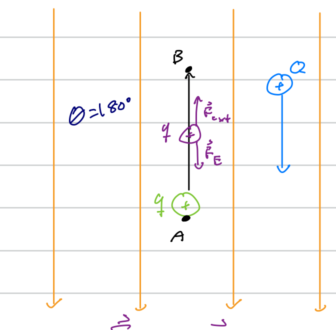


Example

$$\int_A^B \vec{E} d\vec{\ell} = \int_A^C \vec{E} d\vec{\ell} + \int_C^B \vec{E} d\vec{\ell}$$
$$= \int_A^C E d\ell \cos 90 + \int_C^B E d\ell \cos 0$$
$$= E \int_C^B d\ell = E \cdot L$$

Electrostatic Energy

Overview / Introduction



$\vec{E} = E(-\hat{y})$

$\vec{F} = Q\vec{E} = QE(-\hat{y})$

- Apply force to bring $+q$ to B from A
- Counteracts $\vec{E}$
- Release q at B → moves down

$\vec{F}_{ext} = -\vec{F}_E$ by N2L

Assumption: We move the charge very slowly, so the initial, final, and intermediate velocity is zero and hence there is No Kinetic Energy

↳ Spend an infinite amount of time from A to B $(v = \frac{\Delta x}{\Delta t})$

→ Energy Input = Work done by external agent $= \int_A^B \vec{F}_{ext} \cdot d\vec{l}$

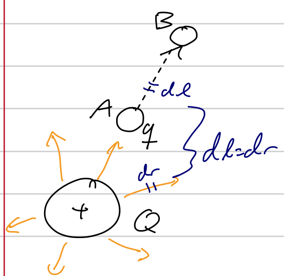
Using our assumption

- The external force matches the electric force: $U_B - U_A \equiv \Delta U = \int_A^B d\vec{l} \cdot \vec{F}_{ext} = -\int_A^B d\vec{l} \cdot \underbrace{(q\vec{E})}_{\vec{F}_E}$ ($\vec{F}_{ext} = -\vec{F}_E$)

  ↳ In this simple (uniform $\vec{E}$ ✓) case: $U_B - U_A \equiv \Delta U = qE(z_B - z_A)$ wow! GPE $(U_G = mgy)$

This is a particular formula, not general!

Point Charge



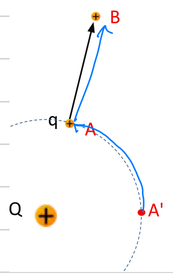
$$\Delta U = -q\int_A^B d\vec{l} \cdot \vec{E} = -q\frac{Q}{4\pi\varepsilon_0}\int_A^B \frac{\hat{r} \cdot \hat{r}}{r^2}\, dl = -\frac{qQ}{4\pi\varepsilon_0}\int_A^B \frac{dl}{r^2} = \frac{-qQ}{4\pi\varepsilon_0}\int_A^B \frac{dr}{r^2}$$

$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$

$d\vec{l} = \hat{r}\, dl$

$$\Rightarrow \Delta U = \frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

If $r_A \to \infty$ chosen as reference: $U = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r}$



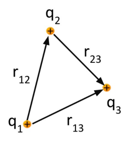
$\Delta U_{AB} = \Delta U_{A'B}$

choose path in green

→ along dashed line

integral is 0 as you move tangent

Arrangement of Charges



- The total energy of the system equals the sum of the energy of each pair of charges

$$U = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

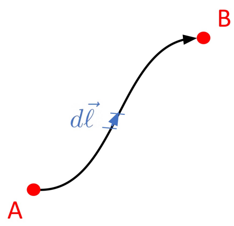
Electrostatic Potential

- Defined Φ, also seen as V
- Do not confuse w/ electrostatic energy (U)
- Another way to describe an electric field
- Electrostatic potential is a scalar (not a vector)
- To know $\vec{E}$ or Φ for any position space provides same amount of information (only electrostatics)
- If you know one you can figure out the other $\quad \vec{E}(r) \rightleftarrows \Phi(r) \leftarrow$ not discussed here, not easily but derivable
- Voltage $V = \Delta\Phi$

Examples of... Scalar Fields: • Elevation • Temperature

Vector Fields: • Gravity • Wind

Definition

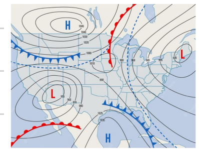


- $\Delta\Phi = \Phi(B) - \Phi(A) \equiv -\int_A^B d\vec{\ell}\cdot\vec{E}$
- If we set a reference point ($\Phi = 0$)

$$\Phi(\vec{r}) \equiv -\int_{ref \to \vec{r}} d\vec{\ell}\cdot\vec{E} \qquad \Phi(ref) = 0$$

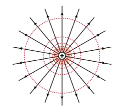
Scalar Quantity - Use Equipotential Lines

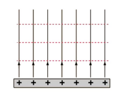
Equipotential lines



Same atmospheric preasure

Point charge

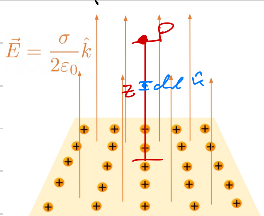




Infinite Plane

- Always perpendicular to electric lines

Example: Potential - Infinite Plane

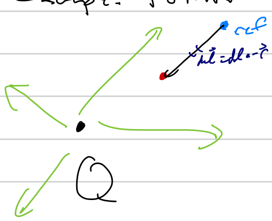


- Choose $z = 0$ so $\Phi(z=0) = 0$

$$\vec{E} = \frac{\sigma}{2\epsilon_0}(\pm\hat{k}) \quad \text{(from gauss law)}$$

$$\Delta\Phi = \cancel{\Phi_{ref}} - \Phi_P = -\left(-\int_{ref \to P} d\vec{\ell}\cdot\vec{E}\right) = \int E\, d\ell \cos\theta = E\int_{z_P}^{0_{ref}} dz = Ez = \boxed{-\frac{\sigma z}{2\epsilon_0}}$$

Example: Potential Point Like Charge



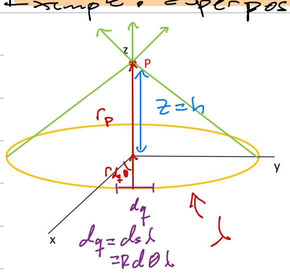
$$\Phi(r \to \infty) = 0$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

$$\Phi(r) = -\int d\vec{\ell}\cdot\vec{E} = -\int \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}\cdot\hat{r}\, dr$$

$$\boxed{\Phi(r) = \frac{Q}{4\pi\epsilon_0 r}}$$

Example: Superposition Principle ($\vec{E}$ vs. Φ)



a) Develop the field @ P

Should point up

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^3}\vec{r} = \frac{1}{4\pi\epsilon_0}\frac{dq}{\sqrt{z^2+R^2}^{\,3}}\left(z\hat{k} - R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\right)$$

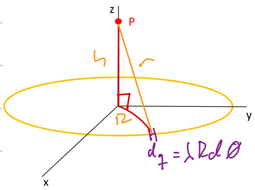
$$\left.\begin{array}{l}\vec{r}_P = z\hat{k} \\ \vec{r}_{dq} = R\cos\theta\,\hat{\imath} + R\sin\theta\,\hat{\jmath}\end{array}\right\} \vec{r} + \vec{r}_{dq} = \vec{r}_P \rightarrow \vec{r} = z\hat{k} - R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$

$$\hookrightarrow r = \sqrt{z^2 + R^2(1)}$$

$$\Rightarrow \vec{E} = \frac{R\lambda}{4\pi\epsilon_0(z^2+R^2)^{3/2}}\int_0^{2\pi}\left(z\hat{k} - \cancel{R\cos\theta\,\hat{\imath}} - \cancel{R\cos\theta\,\hat{\jmath}}\right)d\theta \rightarrow \boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2\pi R\lambda z}{(z^2+R^2)^{3/2}}\hat{k}}$$

$\cos\theta$ and $\sin\theta$ are 0 to 2π = 0

b) Develop potential without using $\vec{E}$



$\Phi(P) = ?$

$d\Phi = \frac{1}{4\pi\varepsilon_0} \frac{dq}{r}$

It's easier, in general, to find Φ

$\Phi(h) = \int_0^{2\pi} \frac{1}{4\pi\varepsilon_0} \frac{\lambda R d\theta}{\sqrt{R^2+z^2}} \Rightarrow \boxed{\Phi(h) = \frac{\lambda R}{2\varepsilon_0} \frac{1}{\sqrt{R^2+h^2}}}$

$r = \sqrt{R^2+h^2}$

Converting between Φ and $\vec{E}$ ← outside of scope of course

$$\vec{E} \xrightarrow{\Phi = -\int d\vec{l} \cdot \vec{E}} \Phi$$

$$\Phi \xrightarrow{\vec{E} = -\nabla \Phi} \vec{E}$$

← The gradient Omg calc 3!

$\vec{E} = -\nabla \Phi = -\left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}\right)$

Conductors

Perfect Conductor Ideal, doesn't exist in nature

- Electric Field inside is zero
  - If $E \neq 0$, charges rearrange themselves so $E = 0$ instantaneous for perfect conductors
  - After some time, real conductors behave this way time is a factor for real conductors (low amt)
- In the surface (outside), the Electric field is perpendicular to the surface of the conductor

→ All net charges reside in the surface

$Q_{enc} = \varepsilon_0 \oint d\vec{a} \cdot \vec{E} = 0$

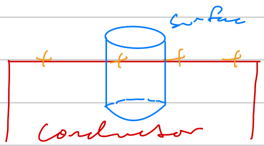
Enclosed charge is 0, and all charge is in surface

$Q = Q_+ - Q_- = 0$ (0 charge, but particles exist inside)

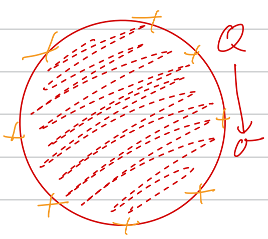
Charge Density

$$\oint d\vec{a} \cdot \vec{E} = \int_{t} E\, da + \int_{b} 0 + \int_{side} (0) = E A_{cap}$$

inside conductor → 0; $\hat{n}$ horizontal, $\vec{E}$ is vertical } $\theta = 90°$

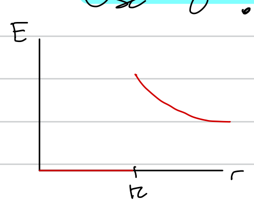


$$\longrightarrow E \cancel{A_{cap}} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma \cdot \cancel{A_{cap}}}{\varepsilon_0} \quad \text{so} \quad \boxed{E = \frac{\sigma}{\varepsilon_0}, \quad \sigma = E\varepsilon_0}$$

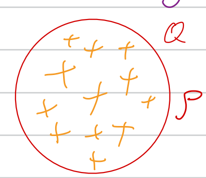


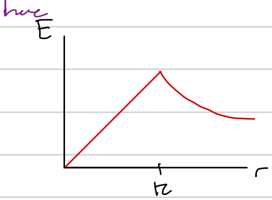
If the sphere is a perfect conductor, all charges are on the surface!

Use σ!

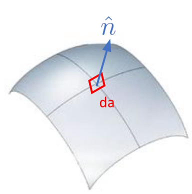


Different from insulators worked w/ previously as they had charges evenly distributed throughout volume; we use ρ here





Force on the Conductor



Force: $\vec{F} = qE$ (known)

$\vec{E} = E\hat{n}$ ← outside conductor

$\frac{1}{2}E$ ← on surface? average

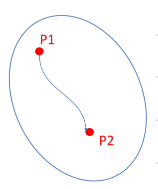
$E = 0$ ← inside conductor

$\rightarrow d\vec{F} = \frac{1}{2}\sigma\, da\, E\hat{n}$

$\vec{F} = \frac{1}{2}\int \sigma E \hat{n}\, da$ (use $\sigma = E\varepsilon_0$, or $E = \frac{\sigma}{\varepsilon_0}$)

$= \frac{1}{2}\varepsilon_0 \int E^2 \hat{n}\, da = \frac{1}{2\varepsilon_0}\int \sigma^2 \hat{n}\, da$ ← always positive (outwards)

Potential inside conductor



$\Delta \Phi = \int d\vec{l} \cdot \vec{E} = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}$

→ but $E = 0$ inside conductor $\therefore \Delta\Phi = 0$

So, conductors are equipotential surfaces

Watch out!

$\Delta\Phi = \Phi_B - \Phi_A = 0$

$\rightarrow \Phi_A = \Phi_B \neq 0$

↑ just their difference

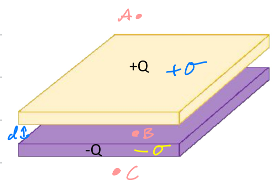
Had to use this on final last year

Capacitance

- Charge Q is proportional to potential difference (often called voltage) between two electrodes by a quantity called Capacitance

$Q = CV$ where $Q \equiv$ Charge (coulombs, C)
$C \equiv$ Capacitance (Farads, F)
$V \equiv$ Voltage (Volts, V) $\leftarrow$ Convention: Use volts instead of potential difference $\left(\begin{matrix}\text{equiv to}\\ \Delta\Phi\end{matrix}\right)$

Parallel plates



Assume Area $A >> d^2$

$C = \frac{Q}{V} = \frac{\sigma A}{}$

$V = \Delta\Phi = -\int_1^2 d\vec{l}\cdot\vec{E}$

Q: What's $\vec{E}$ at all 3 Points?

above (+) • A $\uparrow\frac{\sigma}{2\varepsilon_0}$ $\cancel{\downarrow -\frac{\sigma}{2\varepsilon_0}}$ $+\sigma$ $\longrightarrow E = 0$

btwn $\rightarrow$ • B $d\updownarrow$ $\downarrow 2 \times \frac{\sigma}{2\varepsilon_0}\downarrow$ $-\sigma$ $\longrightarrow E = \frac{\sigma}{\varepsilon_0}$

below (−) • C $\cancel{\downarrow -\frac{\sigma}{2\varepsilon_0}}$ $\cancel{\uparrow\frac{\sigma}{2\varepsilon_0}}$ $\longrightarrow E = 0$

Derive from Gauss Law (Previous lectures)

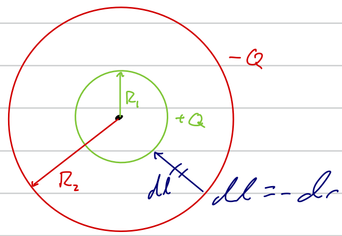
Use $\vec{E}$ @ all 3 points to get C of parallel plate capacitors?

$C = \frac{Q}{|V|} > 0$ ($|V| = \Delta\phi$)

$\Delta\phi = -\int_{P_1\to P_2} d\vec{l}\cdot\vec{E} = -\int_{P_1\to P_2} dl\, E\cos 180 = E\int_0^d dl \quad d\left(\frac{\sigma}{\varepsilon_0}\right), \quad Q = \sigma A$

So, $\boxed{C = \frac{\sigma A}{\frac{d\sigma}{\varepsilon_0}} = \frac{\varepsilon_0 A}{d}}$

Example



$C = \frac{Q}{V}, \quad V = \Delta\phi = -\int_{-\to+} \vec{E}\cdot d\vec{l}$

① Determine $\vec{E}$ in 3 points (only really need to one btwn ⊖ and ⊕, just doing this for practice)

$\vec{E}_{inside} = 0 \quad Q_{enc} = 0$

$\vec{E}_{btwn}: \quad Q_{enc} = Q$

$\frac{Q_{enc}}{\varepsilon_0} = E a$ (Area of Gaussian surface) $\longrightarrow E = \frac{Q}{4\pi r^2\varepsilon_0}$

- spherical & hollow shells
- Conductive materials

$\vec{E}_{outside} = 0 \quad Q_{enc} = Q - Q = 0$

- Determine Capacitance (C) ② Determine $\Delta\phi$

$\Delta\phi = -\int_{-\to+} E\,dl\cos 180 = \int \frac{Q}{4\pi r^2\varepsilon_0} dl = -\int_{R_2}^{R_1} \frac{Q}{4\pi r^2\varepsilon_0} dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

$\Rightarrow \boxed{C = \frac{Q}{V} = 4\pi\varepsilon_0\left(\frac{1}{R_1} - \frac{1}{R_2}\right)^{-1}}$

Exam 2 Review Lecture

- Dielectrics will <u>not</u> be on the homework

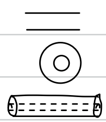
3 Key Topics

- Potential Energy
- Electric Potential
- Capacitance

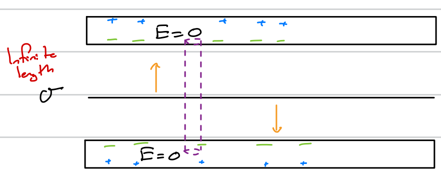
} Including all from Exam 1

Capacitance

$$C = \frac{Q}{V} \begin{cases} \varepsilon_0 \frac{A}{d} \\ \\ \\ \end{cases}$$



Parallel plate



$E = \frac{\sigma}{2\varepsilon_0}$

$\hat{n} \perp \vec{E}$

$\sigma_u = \sigma_l$ by symmetry

$$2\int_{caps} \vec{E}\cdot d\vec{a} + \int_{side} 0 = 0 = \frac{A(\sigma + 2\sigma_u)}{\varepsilon_0}$$

$$\sigma + 2\sigma_u = 0 \qquad \boxed{\sigma_u = \sigma_l = -\frac{\sigma}{2}}$$

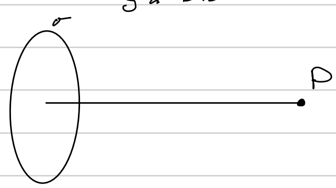
Question 2 on hw 5, I got it right!

Outside edges of slabs should be $+\frac{\sigma}{2}$ to keep conductors neutral

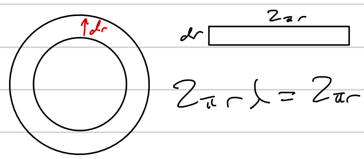
$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} = \frac{A(\sigma + \sigma_u + \sigma_l)}{\varepsilon_0} = \frac{A(\sigma + 2\sigma_u)}{\varepsilon_0}$$

Potential by a Disk

HW5 Q1



$$\Phi_{ring} = \frac{1}{4\pi\varepsilon_0} \frac{\overbrace{\lambda(2\pi r)}^{q}}{\sqrt{r^2 + z^2}} \Rightarrow \int_0^R \frac{1}{4\pi\varepsilon_0} \frac{2\pi r\sigma\, dr}{\sqrt{r^2+z^2}}$$



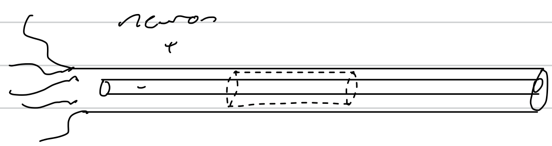
$$2\pi r \lambda = 2\pi r\, dr\, \sigma$$

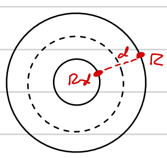
Don't expect something like this on the exam

Potential & Energy

$$\left.\begin{aligned} \Delta\phi &= -\int \vec{E}\cdot d\vec{l} \\ \Delta U &= -Q\int \vec{E}\cdot d\vec{l} \end{aligned}\right\} \Delta U = Q\Delta\phi$$

Example Regarding Capacitance





$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

$$E(2\pi r L) = \frac{Q_{enc}}{\varepsilon_0}$$

$$E = \frac{Q}{2\pi r L \varepsilon_0}$$

$$V = \Delta\phi = \int_{R-d}^{R} \frac{Q}{2\pi\varepsilon_0 r L}\, dr \propto \log\left(\frac{R}{R-d}\right)$$

Midterm Exam 3

Dielectrics

- Dielectrics = Insulators

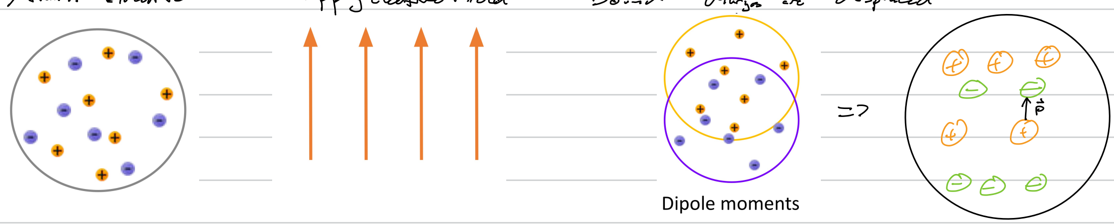
examples: Glass
Plastic
Water
Diamond

- charges stay put

Internal charge a.k.a. Bound charge vs. External charge a.k.a. Free charge or Fields

Response of Internal Charges to Fields

Atom of Dielectric → Apply electric field → Bound charges are displaced

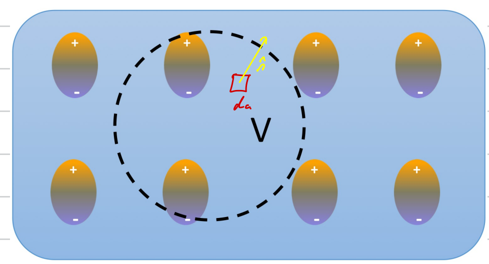


Displaced densities in the atom

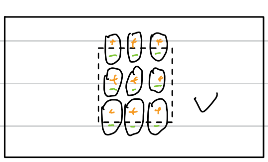
Dipole

$$\ominus \xrightarrow{\vec{d}} \oplus$$

- $\vec{p} = q\,\vec{d}$
- Polarized medium
  → Polarization vector $\vec{P}$
  - Dipole Moment Density: $\vec{P} = \frac{\sum \vec{p}}{V}$ ← add all; ← volume



How much bound charge inside V?



- $\vec{P}$ = uniform, $Q_B(V) = 0$
- Answer depends on $\vec{P}$ at the surface
- If $\vec{P}$ is parallel to the surface, no net charge

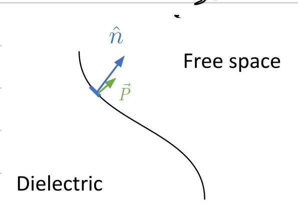
$Q_B \propto \vec{P}_{surface}\,\checkmark$

$$Q_{bound} = Q_B(V) - \oint da\,\hat{n}\cdot\vec{P} = -\oint |\vec{P}|\,|d\vec{a}|\cos\theta$$

- Only for Dielectrics
- Not Gauss' Law, this is: $Q_{bound} + Q_{free} = \varepsilon_0 \oint da\,\hat{n}\cdot\vec{E}$ — Lone Charge

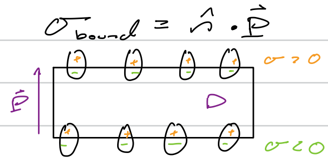
$$Q_{free} = \oint da\,\hat{n}\cdot\underbrace{(\varepsilon_0\vec{E} + \vec{P})}_{\vec{D}}$$

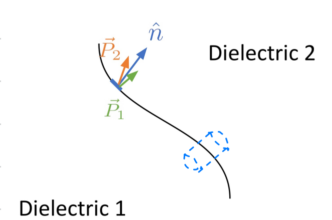
Surface Charge Density



One dielectric

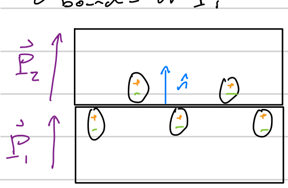
$$\sigma_{bound} = \hat{n}\cdot\vec{P}$$



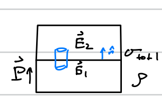


Two dielectrics

$$\sigma_{bound} = \hat{n}\cdot\vec{P}_1 - \hat{n}\cdot\vec{P}_2$$



Gauss Law to get σ_{total}



$$\vec{E}_i = 2\frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

$$\sigma = \varepsilon_0 E_i$$

$$\sigma_{total} = -\varepsilon_0\hat{n}\cdot(\vec{E}_1 - \vec{E}_2)$$

$$\sigma_{free} = \sigma_{total} - \sigma_{Bound}$$

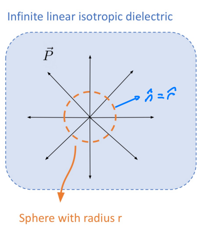
$$\sigma_{free} = \hat{n}\cdot(\underbrace{\varepsilon_0\vec{E}_2 + \vec{P}_2}_{\vec{D}_2}) - \hat{n}\cdot(\underbrace{\varepsilon_0\vec{E}_1 + \vec{P}_1}_{\vec{D}_1})$$

Linear Isotropic Dielectrics

$\vec{P} \propto \vec{E}^1$ — simplest case — proportionality depends on the material, temperature and pressure $\Rightarrow$ $\vec{P} = \chi \varepsilon_0 \vec{E}$ ↳ Electric Susceptibility

Generally $\vec{P} \propto \vec{E}^x$ (to a given power)

Example



• Assume $\vec{P} = \frac{A}{r^2}\hat{r}$ (A constant)

• What is the bound charge inside a sphere with radius r?

$$Q_{bound} = \cancel{Q_B(V)} - \oint da\, \hat{n}\cdot\vec{P}$$

uniform $\vec{P}$

$$= 4\pi r^2 P \cos 0$$

$$Q_{bound} = -4\pi r^2\left(\frac{A}{r^2}\right) = -4\pi A \qquad \boxed{Q_{bound} = -4\pi A}$$

Note: Q_{bound} independent of r, in fact it's a constant value
↳ Why? All the charge is at the center

• What is the electric field?

$$\vec{E} = \frac{\vec{P}}{\chi\varepsilon_0} = \frac{\frac{A}{r^2}\hat{r}}{\chi\varepsilon_0} = \frac{1}{\chi\varepsilon_0}\frac{A}{r^2}\hat{r} \qquad \boxed{\vec{E} = \frac{1}{\chi\varepsilon_0}\frac{A}{r^2}\hat{r}}$$

• What is the total charge inside a sphere with radius r

$$\frac{Q_{enc}}{\varepsilon_0} = \oint \vec{E}\cdot d\vec{a} \Rightarrow Q_{enc} = \varepsilon_0 E \oint da = \varepsilon_0 E\, 4\pi r^2 = \cancel{\varepsilon_0}\frac{1}{\chi\cancel{\varepsilon_0}}\frac{A}{\cancel{r^2}}4\pi\cancel{r^2} = \frac{4\pi A}{\chi} \qquad \boxed{Q_{total} = \frac{4\pi A}{\chi}}$$

• What is the external (free) charge inside a sphere with radius r?

$$Q_{free} = Q_{total} - Q_{bound} = 4\pi A\,\frac{1+\chi}{\chi} \qquad \boxed{Q_{free} = 4\pi A\,\frac{1+\chi}{\chi}}$$

You might see $1+\chi$ as ε_r (relative permittivity) or the dielectric constant Kappa (K)

• What is the electric field as a function of free charge?

$$\vec{E}\left(Q_F = 4\pi A\frac{1+\chi}{\chi}\right)? \quad A = \frac{\chi Q_F}{4\pi(1+\chi)} \Rightarrow \vec{E} = \frac{1}{\cancel{\chi}\varepsilon_0 r^2}\left(\frac{\cancel{\chi} Q_F}{4\pi(1+\chi)}\right)\hat{r} \qquad \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0(1+\chi)}\frac{Q_F}{r^2}\hat{r}}$$

Basically coulombs law, but with $(1+\chi)$ factor.

We will define $K = 1+\chi$ (Kappa)

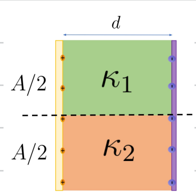
Parallel Plate Capacitor

• Capacitance is affected: $C = K C_{air} = (K\varepsilon_0)\frac{A}{d}$

Material	Dielectric constant, κ
Air	1(0.85-1.21 at 20°C)*
Water	80 at 20°C (79.50-80.24)*
Ice	3 at -5°C*
Basalt	12
Granite	7-9
Sandstone	9-11
Dry loam	3.5
Dry sand	2.5 (2.5-3.5 at 20°C)*
Ethanol liquid	15.21 at 20°C*
Ethanol Mixing Liquid	39.82 at 20°C*
Acetone	21.20 at 20°C*

*measured dielectric constant in this study

What is the new Capacitance with two dielectrics?



$C_1 = \frac{Q_1}{V}$, $C_2 = \frac{Q_2}{V}$ for constant V ($V = Ed$)

Gauss Law: $E\frac{A}{2} = \frac{Q_i}{K_i\varepsilon_0} \Rightarrow Q_i = (K_i\varepsilon_0)E\frac{A}{2}$

$\therefore\ C_i = \frac{K_i\varepsilon_0}{2}\frac{A}{d}$