Solutions

1. Cylindrical capacitor.

a) Using superposition principle plus equation 1:

i) for $0 < r < a$ inside both shells:

$$E = 0 \tag{3}$$

ii) for $a < r < b$, in between shells

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\hat{n} \tag{4}$$

iii) for $r > b$, outside both shells

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\hat{r} + \frac{-\lambda}{2\pi\varepsilon_0}\frac{1}{r}\hat{r} = 0 \tag{5}$$

b)

$$\phi_2 - \phi_1 = -\int \vec{E} \cdot d\vec{\ell} \tag{6}$$

The integrand:

$$\vec{E} \cdot d\vec{\ell} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}dr \quad \text{(field between shells)} \tag{7}$$

so

$$\phi_2 - \phi_1 = -\int_a^b \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}dr = -\frac{\lambda}{2\pi\varepsilon_0}\,\mathrm{Ln(r)}|_a^b = -\frac{\lambda}{2\pi\varepsilon_0}\mathrm{Ln}\left(\frac{b}{a}\right) \tag{8}$$

c) The capacitance per unit length

$$\frac{C}{L} = \frac{\lambda}{V} = \frac{2\pi\varepsilon_0}{\mathrm{Ln}\left(\frac{b}{a}\right)} \tag{9}$$

You can also solve this problem with a finite cylindrical capacitor (length L) but assuming the electric field created by it is like the infinite case as we showed here.

2. Capacitor filled with dielectric

1. Capacitance without dielectric using $C_0 = \frac{\epsilon_0 A}{d}$:

$$C_0 = \frac{8.854 \times 10^{-12}\,\mathrm{F/m} \times 100 \times 10^{-4}\,\mathrm{m}^2}{1.0 \times 10^{-3}\,\mathrm{m}} \tag{10}$$

$$C_0 = 8.854 \times 10^{-11}\,\mathrm{F} = 88.54\,\mathrm{pF} \tag{11}$$

2. Capacitance with dielectric using $C_d = \kappa\frac{\epsilon_0 A}{2d}$ (since half of the area is covered by the dielectric):

$$C_d = 4 \times \frac{8.854 \times 10^{-12}\,\mathrm{F/m} \times 100 \times 10^{-4}\,\mathrm{m}^2}{2 \times 1.0 \times 10^{-3}\,\mathrm{m}} \tag{12}$$

$$C_d = 1.77 \times 10^{-10}\,\mathrm{F} = 177\,\mathrm{pF} \tag{13}$$

3. The total effective capacitance C_{eff} of the capacitor when considering half filled with dielectric and half without dielectric is the sum of the two:

$$C_{eff} = C_0/2 + C_d \tag{14}$$

$$C_{eff} = 88.54\,\mathrm{pF} + 177\,\mathrm{pF} = 266\,\mathrm{pF} \tag{15}$$

   This is because these are effectively two capacitors in parallel.

4. The total stored energy U is given by:

$$U = \frac{1}{2}C_{eff}V^2 \tag{16}$$

$$U = \frac{1}{2} \times 266 \times 10^{-12}\,\mathrm{F} \times (5.0\,\mathrm{V})^2 \tag{17}$$

$$U = 3320 \times 10^{-12}\,\mathrm{J} = 3.32\,\mathrm{nJ} \tag{18}$$

3.Potential of the Earth The radius of the earth is $r = 6.5 \cdot 10^6 m$, so we have

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = \left(9 \cdot 10^9 \frac{kg\, m^3}{s^2\, C^2}\right)\frac{1C}{(6.4 \cdot 10^6 m)^2} = 2.2 \cdot 10^{-4}\frac{V}{m} \tag{19}$$

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \left(9 \cdot 10^9 \frac{kg\, m^3}{s^2\, C^2}\right)\frac{1C}{(6.4 \cdot 10^6 m)} = 1400V \tag{20}$$

This value of 1400V is larger than you might think it should be, given the large radius of the earth. The point is that a Coulomb is a large quantity of charge, on an everyday scale. Or equivalently, ε_0 has a small value in the system of unites we use.

4. Spherical capacitor

1. Inside the conductors the electric field must be zero. Which means that all the charge will be on the surface. So the charge from the sphere will be on the outer surface with a surface charge density $\sigma_{sphere} = q/4\pi a^2$. The charge on the shell will be entirely on the inside surface, so that the field inside the conductor will be zero. This comes from Gauss's Law, a Gaussian surface within the conducting shell will need to have total charge enclosed of zero to make the field zero, so having the $-q$ charge on the inner surface satisfies this condition. This means the surface charge density is $\sigma_{shell} = -q/4\pi b^2$.

   This means that everywhere except between a and b the field is zero. Between a and b the field will be identical to that of a point charge of magnitude q.

   $$\vec{E} = \frac{kq}{r^2}\hat{r}$$

2. Here we can use:

   $$V_2 - V_1 = -\int_a^b \vec{E} \cdot d\vec{s}$$

   In our case:

   $$V_2 - V_1 = -\int_a^b \frac{kq}{r^2}\hat{r} \cdot \hat{r} dr$$

   Note: a unit vector dotted into itself is 1, so:

   $$V_2 - V_1 = -kq \int_a^b \frac{1}{r^2} dr$$

   $$V_2 - V_1 = kq\frac{1}{r}\Big|_a^b$$

   $$V_2 - V_1 = kq\left(\frac{1}{b} - \frac{1}{a}\right)$$

3. Here, the potential difference is $V_1 - V_2$ (this will ensure our answer is positive). Since:

$$C = \frac{q}{V}$$

and $V = V_1 - V_2$ we can use the previous result to find:

$$C = \frac{q}{kq\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$C = \frac{1}{k\left(\frac{1}{a} - \frac{1}{b}\right)}$$

5. Slabs

1. The electric field inside a perfect conductor is always 0. So the electric field inside these conducting slabs must be zero.

2. If we construct a rectangular prism Gaussian surface, so the top is within the upper conductor, and the bottom is within the lower conductor, we know that the field must be zero inside these conductors. For that to be true, the flux through the Gaussian surface must be zero and so the total charge enclosed must be zero. Because of the symmetry, we know the charge on the lower surface of the upper slab and on the upper surface of the lower slab must be equal, so they will each have $-\frac{\sigma}{2}$ each.

3. For each conductor to have zero net charge, these must be equal and opposite to the values in part b. Alternatively, you can construct a Gaussian surface to have a top above the upper conductor and bottom below the lower conductor and constrain the total charge enclosed to only σ with the answers from part b. Each has $\frac{\sigma}{2}$.

6. Dielectric Cylinder with Axial Free Charge

1. $\vec{P} = \epsilon_0 \chi \vec{E} = \epsilon_0 \chi E \hat{r}$

2. $Q_b = \int \vec{P} \cdot d\vec{a} = (2\pi r l)(\epsilon_0 \chi E)$

3. $Q_f = \lambda_f l$

4. Flux of $\vec{E} = \int \vec{E} \cdot d\vec{a} = 2\pi r l E$ and Gauss' Law says that $\int \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0}(Q_b + Q_f) \Rightarrow 2\pi r l E = \frac{1}{\epsilon_0}(2\pi r l \chi E + \lambda_f l) \Rightarrow E = \frac{\lambda_f}{2\pi\epsilon_0}\frac{1}{1+\chi}\frac{1}{r} \Rightarrow \vec{E} = \frac{\lambda_f}{2\pi\epsilon_0}\frac{1}{1+\chi}\frac{1}{r}\hat{r}$

5. $\lambda_f l$

6. Zero

7. $2\pi r l E = \frac{\lambda_f l}{\epsilon_0} \Rightarrow E = \frac{\lambda_f}{2\pi\epsilon_0}\frac{1}{r} \Rightarrow \vec{E} = \frac{\lambda_f}{2\pi\epsilon_0}\frac{1}{r}\hat{r}$

7. Dielectric Slab

1. Zero because the flux through the top is Pa^2 since P and $\hat{n}$ are parallel, flux through the bottom is $-Pa^2$ since P and $\hat{n}$ are anti-parallel, and there is no flux through other faces since P and $\hat{n}$ are orthogonal.

2. $\vec{P} = \epsilon_0 \chi \vec{E} \Rightarrow \vec{E} = \frac{\vec{P}}{\epsilon_0 \chi} \hat{k}$

3. Zero because of Gauss' Law and the fact that flux of $\vec{E}$ is 0 by reasoning similar to part a.

4. $\sigma_b = \vec{P} \cdot \hat{n} = P\hat{k} \cdot \hat{k} \Rightarrow \sigma_b = P$

5. $\sigma_{\text{total}} = \epsilon_0 \hat{n} \cdot (\vec{E}_{\text{out}} - \vec{E}_{\text{in}}) = \epsilon_0 \hat{n} \cdot (0 - \frac{P}{\epsilon_0 \chi} \hat{k}) = -\frac{P}{\chi}$ and we know $\sigma_{\text{total}} = \sigma_b + \sigma_{\text{free}} \Rightarrow -\frac{P}{\chi} = P + \sigma_{\text{free}} \Rightarrow \sigma_{\text{free}} = -P\left(\frac{1}{\chi} + 1\right)$