Notes

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Contents

1 The type theory

The type theory is a two-level type theory, meaning it contains two notions of equality.

1.1 UIP Layer

1.2 Cubical Layer

The cubical type theory is parameterized by a rich cofibration logic with the interval structure being that of a distributive lattice.

cofibration specification ...

Definition 1.1. Partial types are defined by the following rules:

$$\frac{\Gamma \vdash \mathrm{A} : \mathcal{U}_l}{\Gamma \vdash [\alpha] \to A}\ \text{form} \qquad \frac{\Gamma, \alpha \vdash a : A}{\Gamma \vdash \mathrm{in}_\psi(f) : [\alpha] \to \mathrm{A}}\ \text{intro}$$

$$\frac{\Gamma \vdash f : [\alpha] \to A}{\Gamma \vdash \mathrm{out}_\psi(f) : A}\ \text{elim} \qquad \frac{\Gamma \vdash f : [\alpha] \to \mathrm{A}}{\Gamma \vdash \mathrm{out}_\psi(f) \equiv a}\ \text{comp}$$

Definition 1.2. Extension types are defined by the following rules:

$$\frac{\Gamma \vdash A : \mathcal{U}_l \qquad \Gamma \vdash \psi\ \mathbf{cof} \qquad \Gamma \vdash t : [\psi] \to \mathrm{A}}{\Gamma \vdash \mathrm{A}[\psi \to t]}\ \text{form}$$

$$\frac{\Gamma \vdash a : \mathrm{A} \quad \Gamma \vdash \psi\,\mathbf{cof} \quad \Gamma, \psi \vdash t : \mathrm{A} \quad \Gamma, \psi \vdash t \equiv a}{\Gamma \vdash \mathrm{in}_{\psi}(a) : \mathrm{A}[\psi \to t]}\ \text{intro}$$

$$\frac{\Gamma \vdash a : \mathrm{A}[\psi \to t]}{\Gamma \vdash \mathrm{out}_{\psi}(a) : \mathrm{A}}\ \text{elim} \qquad \frac{\Gamma \vdash a : A[\psi \to t] \quad \Gamma \vdash \psi}{\Gamma \vdash \mathrm{out}_{\psi}(a) \equiv t}\ \text{comp1} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash \mathrm{out}_{\psi}(\mathrm{in}_{\psi}(a)) \equiv a}\ \text{comp2}$$

Definition 1.3. The elimination form of disjunctions is introduced as follows:

$$\frac{\Gamma \vdash A : \mathcal{U}_l \quad \Gamma \vdash_{\mathbf{\Psi}} \alpha \vee \beta \quad \Gamma, \alpha \vdash e_1 : A \quad \Gamma, \beta \vdash e_2 : A \quad \Gamma, \alpha, \beta \vdash e \equiv f : A}{\Gamma \vdash [e_1, e_2]_{\alpha,\beta} : A}\ \text{intro}$$

2 Equational Theory

Definition 2.1. The equational theory of disjunctions is as follows:

$$\frac{\Gamma \vdash \alpha}{\Gamma \vdash [e_1, e_2]_{\alpha,\beta} \equiv e_1}\ \text{comp1} \qquad \frac{\Gamma \vdash \beta}{\Gamma \vdash [e_1, e_2]_{\alpha,\beta} \equiv e_2}\ \text{comp2}$$

$$\frac{\Gamma_{\mathbf{\Psi}} \vdash \alpha \vee \beta \quad \Gamma, \alpha \vdash e \equiv f : A \quad \Gamma, \beta \vdash e \equiv f : A}{\Gamma \vdash e \equiv f : A}\ \text{ext} \qquad \frac{\Gamma_{\mathbf{\Psi}} \vdash \alpha \vee \beta \quad \Gamma \vdash e : A}{e \equiv [e, e]_{\alpha,\beta} : A}\ \text{eta}$$

$$\frac{\Gamma \vdash \alpha \Leftrightarrow \gamma \quad \Gamma \vdash \beta \Leftrightarrow \delta \quad \Gamma, \alpha \vdash e_1 \equiv f_1 : A \quad \Gamma, \beta \vdash e_2 \equiv f_2 : A}{\Gamma \vdash [e_1, e_2]_{\alpha,\beta} \equiv [f_1, f_2]_{\gamma,\delta} : A}\ \text{cong}$$

3 Algorithmic Definitions

We present an algorithmic procedure for deciding the judgments of the type theory.

Definition 3.1. Head reduction is defined as follows:

$$\begin{array}{l}
\Gamma \rhd (\lambda x.e)f \to e[f/x] \\
\Gamma \rhd ef \to e'f \qquad \text{if } \Gamma \rhd e \to e' \\
\Gamma \rhd [\alpha \to e, \beta \to f] \to e \qquad \text{if } \Gamma \vdash \alpha \\
\Gamma \rhd [\alpha \to e, \beta \to e] \to f \qquad \text{if } \Gamma \vdash \beta \\
\Gamma \rhd \mathrm{out}_{\psi}(a) \to t \qquad \text{if } \Gamma \rhd a \uparrow A[\psi \to t] \text{ and } \Gamma \vdash \psi \\
\Gamma \rhd \mathrm{out}_{\psi}(in(a)) \to a \\
\Gamma \rhd \mathrm{out}_{\psi}(a) \to a' \qquad \text{if } \Gamma \rhd a \to a' \\
\Gamma \rhd \mathrm{out}_{[\alpha]}(in(a)) \to a \\
\Gamma \rhd \mathrm{out}_{[\alpha]}(a) \to a' \qquad \text{if } \Gamma \rhd a \to a'
\end{array}$$

Definition 3.2. Term equivalence is defined as follows:

$$\begin{array}{l}
\Gamma \rhd M \Leftrightarrow N : A[\psi \to t] \qquad \text{if } \Gamma \rhd \mathrm{out}_{\psi}(M) \Leftrightarrow \mathrm{out}_{\psi}(N) : A \\
\Gamma \rhd M \Leftrightarrow N : [\alpha] \to A \qquad \text{if } \Gamma, \alpha \rhd \mathrm{out}_{[\alpha]}(M) \Leftrightarrow \mathrm{out}_{[\alpha]}(N) : A \\
\Gamma \rhd M \Leftrightarrow N : (x : \mathbb{I}) \to A \qquad \text{if } \Gamma, x : \mathbb{I} \rhd Mx \Leftrightarrow Nx : A(x) \\
\Gamma \rhd M \Leftrightarrow N : (x : \mathbf{cof}) \to A \qquad \text{if } \Gamma, x : \mathbf{cof} \rhd Mx \Leftrightarrow Nx : A(x)
\end{array}$$

Definition 3.3. Neutral equivalence is defined as follows:

$$\begin{array}{ll}
\Gamma \triangleright [\alpha \to p_1, \beta \to p_2] \leftrightarrow q \uparrow A & \text{if } \Gamma, \alpha \triangleright p_1 \Leftrightarrow q : A \text{ and } \Gamma, \beta \triangleright p_2 \Leftrightarrow q : A \\
\Gamma \triangleright p \leftrightarrow [\alpha \to q_1, \beta \to q_2] \uparrow A & \text{if } \Gamma, \alpha \triangleright p \Leftrightarrow q_1 : A \text{ and } \Gamma, \beta \triangleright p \Leftrightarrow q_2 : A \\
\Gamma \triangleright \mathrm{out}_{\psi}(p) \Leftrightarrow \mathrm{out}_{\psi}(q) \uparrow A & \text{if } \Gamma \triangleright p \leftrightarrow q \uparrow A[\psi \to t] \\
\Gamma \triangleright \mathrm{out}_{[\alpha]}(p) \Leftrightarrow \mathrm{out}_{\psi}(q) \uparrow A & \text{if } \Gamma \triangleright p \leftrightarrow q \uparrow [\alpha] \to A
\end{array}$$