Primer on Topology for AlgTop

This is a collection of notes intended to provide the minimal necessary topology background to ramp someone up to algebraic topology.

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Contents

1 Topological Spaces

Definition 1.1 (Lee)**.** A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X, called *open sets*, satisfying the following:

1. $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$
2. If $U_1, \cdots, U_n \in \mathcal{T}$, then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$
3. If $\{U_\alpha\}_{\alpha \in A}$ is a collection of elements of $\mathcal{T}$, then $\cup_{\alpha \in A} U_\alpha$ is in $\mathcal{T}$.

Definition 1.2 (Lee)**.** A *topological space* is a pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X.

Remark 1.3 (Lee). A neighborhood of $q \in X$ is an open set containing q.

Lemma 1.4 (Exercise 2.1 (Munkres))**.** *Let $(X, \mathcal{T})$ be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Then, A is open in X.*

Proof. Suppose that $A \subset X$ and that for all $x \in A$ there is an open set U containing x such that $U \subset A$. By definition, $\cup_{x \in A} U_x \in \mathcal{T}$ because each $U_x \in \mathcal{T}$. Then, $\cup_{x \in A} U_x = A$ so $A \in \mathcal{T}$. □

Lemma 1.5 (Exercise 2.3 (Munkres))**.** *Is the collection*

$$\mathcal{T}_{\text{inf}} = \{U | X - U \textit{ is infinite or empty or all of } X\}$$

a topology on X?

Proof.

1. If the condition is all of X, then $U = \emptyset$, so $\emptyset \in \mathcal{T}$. If the condition is empty, then $U = X$, so $X \in \mathcal{T}$.

2. Let $U_1, \cdots, U_n \in \mathcal{T}$. For $U = \cap_i U_i$, we have

$$X - U = X - \cap_i U_i = \cup_i (X - U_i)$$

.

If any of the $X - U_i$ is infinite, all of X, or empty, the union meets the corresponding condition.

3. Let $U_1, U_2, \cdots \in \mathcal{T}$. For $U = \cup_i U_i$, we have

$$X - U = X - \cup_i U_i = \cap_i (X - U_i)$$

.

If all of the $X - U_i$ are infinite, the intersection does not necessarily meet any of the conditions. Thus, $\mathcal{T}_{\text{inf}}$ is not a topology on X.

□

Definition 1.6 (Munkres). A topology $\mathcal{T}$ is finer than $\mathcal{T}'$ if $\mathcal{T}' \subseteq \mathcal{T}$ and, analogously, $\mathcal{T}'$ is coarser.

1.1 Bases

Definition 1.7 (Munkres). If X is a set, a *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X such that

1. For each $x \in X$, there is at least one basis element B containing x.

2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Remark 1.8. If $\mathcal{B}$ satisfies both of the above conditions, then we can define the topology $\mathcal{T}$ generated by $\mathcal{B}$ as follows: A subset U of X is said to be open in X if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Each basis element itself is an element of $\mathcal{T}$.

Lemma 1.9. *Let $\mathcal{T}$ be the collection generated by the basis $\mathcal{B}$ on X. $\mathcal{T}$ is a topology.*

Proof.

- $X \in \mathcal{T}$ because for each $x \in X$ there is a basis element B with $x \in B$ and $B \subset X$. $\emptyset \in \mathcal{T}$ vacuously because for each $x \in \emptyset$ is a vacuous statement.

- Suppose $U_1, \cdots, U_n \in \mathcal{T}$. We want to show $U_1 \cap \cdots \cap U_n \in \mathcal{T}$. For each $x \in U_1 \cap \cdots \cap U_n$, we have that there are basis elements $x \in B_1, \cdots, x \in B_n$ because each $U_i \in \mathcal{T}$. We also know $U_i \subset B_i$. By the definition of basis, there is some basis element $B \subset B_1 \cap \cdots \cap B_n$ such that $x \in B$. Then, $x \in B$ and $B \subset U_1 \cap \cdots \cap U_n$ (by $B \subset U_i$). Thus, $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

- Suppose $U_i \in \mathcal{T}$. We want to show that $\cup_i U_i \in \mathcal{T}$. For each $x \in \cup_i U_i$, we have that $x \in U_i$ and there is a basis element $x \in B$ with $B \subset U_i$. Then, $B \subset \cup_i U_i$. Thus, $\cup_i U_i \in \mathcal{T}$.

□

exercises + lemmas

Lemma 1.10 (Munkres)**.** *Let X be a set and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.*

Remark 1.11. This lemma means that we can express every open set U of X as a union of basis elements.

Lemma 1.12 (Munkres)**.** *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U and $x \in U$, there is $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.*

Proof. It suffices to show (1) that $\mathcal{C}$ is a basis and (2) the topology generated by $\mathcal{C}$ is the same as the topology of X.

Part 1: $\mathcal{C}$ is a basis

For the first condition, we must check that for each $x \in X$, there is a $C \in \mathcal{C}$ containing x. Let $x \in X$. Because X is an open set of X, we have, by supposition, that there is $C \in \mathcal{C}$ such that $x \in C \subset X$.

For the second condition, we must check that if $x \in C_1 \cap C_2$, then there is $C_3 \subset C_1 \cap C_2$. Let $x \in C_1 \cap C_2$. Because C_1 and C_2 are open, so is $C_1 \cap C_2$. By supposition, there is $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset U$.

Part 2: $\mathcal{C}$ generates the topology of X

Let $\mathcal{T}$ be the collection of open sets of X and let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$.

Suppose that U belongs to $\mathcal{T}$. Then for each $x \in U$, there is $C \in \mathcal{C}$ such that $x \in C \subset U$. Then, by the definition of basis, U is in $\mathcal{T}'$.

Suppose that U belongs to $\mathcal{T}'$. Then, U is a union of elements of $\mathcal{C}$ by the preceding lemma. Each element of $\mathcal{C}$ belongs to $\mathcal{T}$ and a topology is closed under arbitrary unions so U belongs to $\mathcal{T}$. □

1.2 Product Topology

Definition 1.13 (Lee)**.** Suppose $X_1, \cdots, X_n$ are topological spaces. Then let $\mathcal{B} = \{U_1 \times \cdots \times U_n : U_i \text{ is open in } X_i, i = 1, \cdots, n\}$. The topology generated by $\mathcal{B}$ is the product topology.

Remark 1.14. We can say that the product topology $X \times Y$ is the topology that has as its basis $\mathcal{B}$ the collection of all sets of the form $U \times V$ where U is open in X and V is open in Y.

Then, $\mathcal{B}$ meets the first condition of a basis because $X \times Y$ is itself a basis element. For the second condition, consider $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$. Because the binary intersections are in X and Y (open), we have that the intersection is a basis element. We can generalize this to n-ary products.

Theorem 1.15. *If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then*

$$\mathcal{D} = \{B \times C | B \in \mathcal{B} \ \textit{and}\ C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Proof. Suppose that $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y.

By Lemma 1.12, it suffices to show that $\mathcal{D}$ is a collection of open sets of $X \times Y$ such that for each open set Z and $z \in Z$, there is a $D \in \mathcal{D}$ such that $x \in D \subset Z$.

Let Z be an arbitrary open set of $X \times Y$ and $z \in Z$. Notice that for every $x \in X$ and $y \in Y$ we have that $x \in B$ and $y \in C$ for basis elements $B \subset U$ and $C \subset V$ for open sets U and V.

Then, observe that $z = x \times y$ such that $x \times y \in U \times V \subset Z$, where $U \in \mathcal{B}$ and $V \in \mathcal{C}$.

!!! □

Definition 1.16. A *subbasis* $\mathcal{S}$ for a topology on X is a collection of subsets D_i of X such that $\cup_i D_i = X$. The topology generated by the subbasis $\mathcal{S}$ is the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

Definition 1.17. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be defined by the equations

$$\pi_1(x, y) = x$$
$$\pi_2(x, y) = y$$

These maps are called the first and second projections of $X \times Y$ onto their first and second factors.

Remark 1.18. The projection maps are surjective, unless one of the spaces X or Y happens to be empty, in which case $X \times Y$ is empty.

Definition 1.19. If U is an open subset of X and V is an open subset of Y, then

$$\pi_1^{-1}(U) = U \times Y$$
$$\pi_2^{-1}(V) = X \times V$$

which are both open in $X \times Y$.

Theorem 1.20. *The collection*

$$\mathcal{S} = \{\pi_1^{-1}(U) | U \ \textit{open in}\ X\} \cup \{\pi_2^{-1}(V) | V \ \textit{open in}\ Y\}$$

is a subbasis for the product topology on $X \times Y$.

Definition 1.21. Subspace topology ...
ch3

Definition 1.22. Closed set and limit point ...

Definition 1.23. Continuous function ...

Definition 1.24. Metric space ....

Definition 1.25. Quotient topology ...

Definition 1.26. Connected space

Definition 1.27. Component and path component ...

Definition 1.28. Compact space ...

Definition 1.29. hausdorff space ...

Definition 1.30. The separation axioms, Urysohn's lemma and the Tietze extension theorem

Definition 1.31. homotopy and the fundamental group ... ch7

Definition 1.32. group theory ...
ch9