Primer on Topology for AlgTop

This is a collection of notes intended to provide the minimal necessary topology background to ramp someone up to algebraic topology.

Trey Plante

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1 Topological Spaces

Definition 1.1 (Lee)**.** A *toplogy* on a set X is a collection $\mathcal{T}$ of subsets of X, called *open sets*, satisfying the following:

1. $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$
2. If $U_1, \cdots, U_n \in \mathcal{T}$, then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$
3. If $\{U_\alpha\}_{\alpha \in A}$ is a collection of elements of $\mathcal{T}$, then $\cup_{\alpha \in A} U_\alpha$ is in $\mathcal{T}$.

Definition 1.2 (Lee)**.** A *topological space* is a pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X.

Remark 1.3 (Lee). A neighborhood of $q \in X$ is an open set containing q.

Lemma 1.4 (Exercise 2.1 (Munkres))**.** *Let $(X, \mathcal{T})$ be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Then, A is open in X.*

Proof. Suppose that $A \subset X$ and that for all $x \in A$ there is an open set U containing x such that $U \subset A$. By definition, $\cup_{x \in A} U_x \in \mathcal{T}$ because each $U_x \in \mathcal{T}$. Then, $\cup_{x \in A} U_x = A$ so $A \in \mathcal{T}$. $\square$

Lemma 1.5 (Exercise 2.3 (Munkres))**.** *Is the collection*

$$\mathcal{T}_{\text{inf}} = \{U | X - U \textit{ is infinite or empty or all of } X\}$$

a topology on X?

Proof.

1. If the condition is all of X, then $U = \emptyset$, so $\emptyset \in \mathcal{T}$. If the condition is empty, then $U = X$, so $X \in \mathcal{T}$.
2. Let $U_1, \cdots, U_n \in \mathcal{T}$. For $U = \cap_i U_i$, we have

   $$X - U = X - \cap_i U_i = \cup_i (X - U_i)$$

   .

   If any of the $X - U_i$ is infinite, all of X, or empty, the union meets the corresponding condition.
3. Let $U_1, U_2, \cdots \in \mathcal{T}$. For $U = \cup_i U_i$, we have

   $$X - U = X - \cup_i U_i = \cap_i (X - U_i)$$

   .

   If all of the $X - U_i$ are infinite, the intersection does not necessarily meet any of the conditions. Thus, $\mathcal{T}_{\text{inf}}$ is not a topology on X.

□

Definition 1.6 (Munkres)**.** A topology $\mathcal{T}$ is finer than $\mathcal{T}'$ if $\mathcal{T}' \subseteq \mathcal{T}$ and, analogously, $\mathcal{T}'$ is coarser.

1.1 Bases

Definition 1.7 (Munkres)**.** If X is a set, a *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X such that

1. For each $x \in X$, there is at least one basis element B containing x.
2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing such that $B_3 \subset B_1 \cap B_2$.

Remark 1.8. If $\mathcal{B}$ satisfies both of the above conditions, then we can define the topology $\mathcal{T}$ generated by $\mathcal{B}$ as follows: A subset U of X is said to be open in X if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Each basis element itself is an element of $\mathcal{T}$.

exercises + lemmas

Lemma 1.9 (Munkres)**.** *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U and $x \in U$, there is $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.*

Proof. It suffices to show (1) that $\mathcal{C}$ is a basis and (2) the topology generated by $\mathcal{C}$ is the same as the topology of X.

Part 1: $\mathcal{C}$ is a basis

For the first condition, we must check that for each $x \in X$, there is a $C \in \mathcal{C}$ containing x. Let $x \in X$. Because X is an open set of X, we have, by supposition, that there is $C \in \mathcal{C}$ such that $x \in C \subset X$.

For the second condition, we must check that if $x \in C_1 \cap C_2$, then there is $C_3 \subset C_1 \cap C_2$. Let $x \in C_1 \cap C_2$. Because C_1 and C_2 are open, so is $C_1 \cap C_2$. By supposition, there is $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset U$.

Part 2: $\mathcal{C}$ generates the topology of X

Let $\mathcal{T}$ be the collection of open sets of X and let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$. □

1.2 Product Topology

Definition 1.10 (Lee)**.** Suppose $X_1, \cdots, X_n$ are topological spaces. Then let $\mathcal{B} = \{U_1 \times \cdots \times U_n : U_i \text{ is open in } X_i, i = 1, \cdots, n\}$. The topology generated by $\mathcal{B}$ is the product topology.

Remark 1.11. We can say that the product topology $X \times Y$ is the topology that has as its basis $\mathcal{B}$ the collection of all sets of the form $U \times V$ where U is open in X and V is open in Y.

Then, $\mathcal{B}$ meets the first condition of a basis because $X \times Y$ is itself a basis element. For the second condition, consider $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$. Because the binary intersections are in X and Y (open), we have that the intersection is a basis element. We can generalize this to n-ary products.

Definition 1.12. Subspace topology ...
ch3

Definition 1.13. Closed set and limit point ...

Definition 1.14. Continuous function ...

Definition 1.15. Metric space ....

Definition 1.16. Quotient topology ...

Definition 1.17. Connected space

Definition 1.18. Component and path component ...

Definition 1.19. Compact space ...

Definition 1.20. hausdorff space ...

Definition 1.21. The separation axioms, Urysohn's lemma and the Tietze extension theorem

Definition 1.22. homotopy and the fundamental group ... ch7

Definition 1.23. group theory ...
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