Lab 6c: LQR Controller Design for Inverted Pendulum

"If you optimize everything, you will always be unhappy." – Donald Knuth

1 Objectives

The objective of this lab is to design a full-state feedback controller using the Linear Quadratic Regulator (LQR) design technique and to understand the effect of varying the penalty matrices P and Q in the cost functional on the performance of the closed-loop system.

2 Theory

Pole placement for controller design relies on specification of the desired closed-loop poles of the system. This is usually difficult to specify, especially for systems with a large number of states. Furthermore, with pole placement design there is hard to take the "amount" of actuation (called actuation or control effort) that gets used during closed-loop operation into account.

Good regulation of the system can usually be achieved by using high amount of actuation (for example in a P-controller, higher K_p, and thus greater actuation effort, gives faster rise time). But in reality, we are often limited by power and energy constraints. Ideally, we would like to achieve good system performance while at the same time minimizing the amount of actuation used in achieving the desired performance. One way of expressing this mathematically is through a cost functional of the form:

$$J = \int_0^\infty x^T Q x + u^T R u \, dt \tag{1}$$

where Q and R are weighting matrices (these are the design parameters).

The LQR design problem is to design a state-feedback controller K (i.e. for $u = -Kx$) such that the cost functional J is minimized[1]. The cost functional (2) consists of two terms, the first of which you can think of as being the cost of regulating the state x (regulatory term) and the second being the cost of actuation u (actuation term). Both of these terms depend on a weighting matrix, Q and R, respectively. These matrices are the design parameters, assumed positive semidefinite. The regulatory term will "penalize" deviations from the desired state (here $x = 0$), while the actuation term will "penalize" you for any actuation effort $u \neq 0$.

For simplicity we assume in this lab that the matrices Q and R are diagonal: $Q = \mathrm{diag}(q_1, \dots, q_n)$ and $R = \mathrm{diag}(r_1, \dots, r_m)$. Thus, the objective J reduces to

$$J = \int_0^\infty \left(\sum_{i=1}^{n} q_i x_i^2 + \sum_{j=1}^{m} r_j u_j^2 \right) dt \tag{2}$$

The scalars $q_1, \dots, q_n$ and $r_1, \dots, r_m$ can be seen as relative weights between different performance terms in the objective J. For Q and R to be positive semidefinite, we need $q_i \geq 0$ and $r_i \geq 0$ for all i. The key

[1] In fact, one can show that even when optimizing over a larger class of controllers, it turns out that the optimal controller is a linear time-invariant state-feedback controller of the form $u = -Kx$.

design problem of LQR is to translate performance specifications in terms of the rise time, overshoot, bandwidth, etc. into relative weights of the above form. There is no straightforward way of doing this and it is usually done through an iterative process either in simulations or on an experimental setup. Once the matrices Q and R are completely specified, the controller gain K is found by solving the so-called Algebraic Riccati Equation (ARE), which can be done numerically in MATLAB.

3 Pre-Lab

The model for the inverted pendulum system is the same as in Lab 6a and Lab 6b. We have a four-state model with states $x, \dot{x}, \theta, \dot{\theta}$ and one input, the motor voltage V.

- What will the dimensions of Q and R be?

The Pre-Lab of this lab mainly consists of translating the stated performance specifications into matrices Q and R. For simplicity, we assume Q and R to be diagonal.

Consider the following control objective: Given that the cart and the pendulum are $x_0 = 30$ cm and $\theta_0 = 0.05$ radians (≈ 2.5 deg) displaced from their desired positions $x_{des} = 0$ and $\theta_{des} = 0$ at time $t = 0$, the objective is to get the system to the desired state as soon as possible, but without using, say, more than 6 volts of the input at any point in time. For now, however, we will ignore the constraint on the input. For our problem, we set the scalars q_2 and q_4 to zero, as we have no inherent restriction on how $\dot{x}$ and $\dot{\theta}$ vary with time. Now, in order to use scalars q_1, q_3 and r as relative weights, we will normalize them based on their initial conditions. The modified weights are:

$$\overline{q}_1 = \frac{q_1}{0.3^2} \qquad \overline{q}_3 = \frac{q_3}{0.05^2} \qquad \overline{r} = \frac{r}{6^2}$$

The weights have been normalized with square terms because the integrand of our objective functional J is a quadratic function of x and u (so the matrix Q will use $\overline{q}_1$ and $\overline{q}_3$, and $R = \overline{r}$).

1. For nominal weights $q_1 = 1$, $q_3 = 1$, and $r = 1$ (giving equal weight to each term of the objective function), determine the gain matrix K which minimizes the objective function and its associated closed-loop pole locations. You may use the `lqr` command in MATLAB to do this. Simulate the closed-loop system including the observer from Lab 6b. That is, use the state estimate $\hat{x}$ to control the system – your input is $u = K(r - \hat{x})$. Make sure to use the same initial condition for observer and system. Report the value of your observer gain matrix L. Plot output y and control action u for initial conditions of $x_0 = 30\,\text{cm}$ and $\theta_0 = 0.05\,\text{rad}$.
2. Individually vary the weights from their nominal values and study the influence of the weights on how the system outputs and control effort varies with time. The weights are relative, so you may assume $q_1 = 1$ in all cases, and vary only the other two. Choose your weights such that you can clearly see the effect in the system behavior (you can restrict your weights to the range $0 - 100$). Consider the following five cases: (nominal, $q_3 \ll 1$, $q_3 \gg 1$, $r \ll 1$, and $r \gg 1$). For each case:
   (a) report the value of K and the closed-loop pole locations
   (b) plot the output y and the control action u
   (c) report the maximum deviations in x and θ as well as u_{max}, the maximal (absolute) value of u

(d) describe briefly the effect of changing the weights on the closed-loop system behavior

3. You will observe that the position x will first increase before converging to zero. What is the physical reason for this behavior?

4 Lab

Implement the controllers you designed in the Pre-Lab on the hardware. Don't forget to include the observer, as you did in the Pre-Lab, and use the state estimate $\hat{x}$ to control the system. Use a step input of the form $r = [0.3\ 0\ 0\ 0]^T$. Make sure to set the observer initial state to zero.

Consider the following controllers:

1. nominal weights
2. a higher relative weight q_1, other weights nominal
3. a higher relative weight q_3, other weights nominal
4. a higher relative weight of r, other weights nominal

In each case, observe the output response of the system, note the variation of the position of the cart and the pendulum with time and the control input. In addition, plot output y and control u and discuss the effect of the weights on the system behavior. Make sure that the differences are noticeable on your plots. For each case,

- discuss how the closed-loop system behavior changes w.r.t. the nominal case
- discuss if and how your results differ from the ones obtained in the simulations in the Pre-Lab

Run the sinusoidal reference from lab 6b ($r_1(t) = .05 \sin(t)$) using the weights that you think are best. How does the performance compare to Lab 6b?