On Perturbative Chern–Simons Theory

Chi Zhang

E-mail: zhangchi2018@itp.ac.cn

ABSTRACT: Reading notes for [AS].

Contents

Some notations.

$M-$ a compact closed manifold

$G-$ a semisimple Lie group with $\mathfrak{g}$

Met– the moduli space of Riemannian metrics on M

$V-$ the number of internal vertices of a Feynman diagram

$I-$ the number of internal edges of a Feynman diagram

$M(V)-$ the Vth configuration space of M

$B = \mathrm{Bl}(M \times M, \Delta)-$ the differential geometric blow up of $M(2)$ along the diagonal $\Delta \subset M \times M$

$M[V]-$ the Axelrod–Singer compactification of $M(V)$

1 Settings and Definitions

Recall that the classical Chern-Simons actions is

$$CS(A) = \frac{1}{4\pi}\int_M \mathrm{tr} A \wedge dA + \frac{2}{3} A \wedge A \wedge A$$

M is a closed and oriented 3-manifold. The partition function is

$$Z_k(M, G) = \int \mathcal{D}A e^{\mathrm{i}kCS(A)}.$$

To define the Lorentz gauge condition, we choose a metric g on M, and the Hodge star $* : \Omega^k(M, \mathfrak{g}) \to \Omega^{3-k}(M, \mathfrak{g})$, with respect to g and the Killing form on $\mathfrak{g}$ The Hodge dual of d is denoted as δ.

Using the standard BRST argument, the gauge fixed action can be obtained by

$$S_{\mathrm{gf}} = S + Q\psi$$

with the *gauge fixing fermion*

$$\psi = \langle \bar{c}, \phi \rangle := -\mathrm{tr} {*\bar{c}}, \phi.$$

Here ϕ is the gauge fixing condition and $\bar{c}$ the Faddeev-Popov anti-ghost, Q the BRST operator.

In our case, we choose Lorentz gauge condition

$$\phi = \delta A,$$

and our BRST operator reads

$$\begin{cases} QA & = d_A c \\ Qc & = \frac{1}{2}[c, c] \\ Q\bar{c} & = \lambda \\ Q\lambda & = 0 \end{cases} \tag{1.1}$$

with

$$d_A c = dc + [A, c].$$

thus the gauge fixed Chern-Simons action is of the form

$$S_{\mathrm{gf}} = kCS(A) + \frac{k}{2\pi}\int_M \langle \lambda, \delta A \rangle + \frac{k}{2\pi}\int_M \langle \bar{c}, \delta d_A c \rangle$$

Now consider the field

$$\mathcal{A} = c + \bar{c} + A + \lambda$$

, then we can write S_{gf} in a compact form, by straight calculations

$$S_{\mathrm{gf}} = \frac{k}{2\pi}\int_M (\frac{1}{2}\langle \mathcal{A}, d\mathcal{A}\rangle + \frac{1}{6}\langle \mathcal{A}, [\mathcal{A}, \mathcal{A}]\rangle) = kCS(\mathcal{A}).$$

For the following, we fix a flat connection a, and work in terms $\hat{\mathcal{A}} \coloneqq \mathcal{A} - a$, we get

$$S_{\mathrm{gf}} = kCS(a + \hat{\mathcal{A}}) = kCS(a) + \frac{k}{2\pi}\int_M (\frac{1}{2}\left\langle \hat{\mathcal{A}}, d_a\hat{\mathcal{A}}\right\rangle + \frac{1}{6}\left\langle \hat{\mathcal{A}}, [\hat{\mathcal{A}}, \hat{\mathcal{A}}]\right\rangle)$$

and our path integral becomes

$$Z_k(M, a, g) = e^{\mathrm{i}kCS(a)}\int D\hat{\mathcal{A}} e^{-\mathrm{i}S(\hat{\mathcal{A}})}$$

with

$$S(\hat{\mathcal{A}}) = \frac{k}{2\pi}\int_M (\frac{1}{2}\left\langle \hat{\mathcal{A}}, d_a\hat{\mathcal{A}}\right\rangle + \frac{1}{6}\left\langle \hat{\mathcal{A}}, [\hat{\mathcal{A}}, \hat{\mathcal{A}}]\right\rangle).$$

From now on, we shall assume that a is isolated up to gauge transformations and that the stabilizer of a is discerete. Or equivalently, we assume that

Assumption 1.1. The cohomology $H^*(M, \mathfrak{g})$ of d_a vanishes.

Let

$$W[\mathcal{J}] \coloneqq \int D\mathcal{A} e^{-\int dx \frac{\mathrm{i}k}{2\pi}\mathcal{A}(x)\wedge d_a\mathcal{A}(x) - \mathcal{J}(x)\wedge\mathcal{A}(x)}$$

be the generating functional coupled to a source $\mathcal{J}$. By completing the square

$$\begin{aligned} &\int dx \frac{\mathrm{i}k}{2\pi}\mathcal{A}(x)\wedge d_a\mathcal{A}(x) - \mathcal{J}(x)\wedge\mathcal{A}(x) \\ =&\int dx \frac{\mathrm{i}k}{2\pi}(\mathcal{A}(x) - \frac{2\pi}{\mathrm{i}k}d_a^{-1}\mathcal{J}(x))\wedge d_a(\mathcal{A}(x) - \frac{2\pi}{\mathrm{i}k}d_a^{-1}\mathcal{J}(x)) - \frac{\pi}{\mathrm{i}k}d_a^{-1}\mathcal{J}(x)\wedge\mathcal{J}(x)\end{aligned}$$

and substituting $D\mathcal{A}$ as $D(\mathcal{A} - 2\pi(\mathrm{i}k)^{-1}d_a^{-1}\mathcal{J})$, we get a standard form of the generating functional

$$W[\mathcal{J}] = W[0]e^{\frac{\lambda^{-1}}{2}\int dx (d_a^{-1}\mathcal{J}(x))\wedge\mathcal{J}(x)}$$

with $\lambda = \frac{ik}{2\pi}$, and d_a^{-1} the inverse of d_a.

If we denote the integral kernel of d_a^{-1}, i.e. the propagator, as $L(x, y)$

$$d_a^{-1}\mathcal{J}(x) = \int L(x, y)\wedge\mathcal{J}(y)$$

and thus the generating functional reads

$$W[\mathcal{J}] = W[0]\exp\Big(-\frac{\lambda^{-1}}{2}\int dydz\mathcal{J}(y)\wedge L(y, z)\wedge\mathcal{J}(z)\Big)$$

We can use $W[\mathcal{J}]$ to compute $Z_k(M, a, g)$, by the standard result in quantum field theory

$$Z_k(M, a, g) = e^{\mathrm{i}kCS(a)}\int D\mathcal{A} e^{-\mathrm{i}S(\mathcal{A})} = e^{\mathrm{i}kCS(a)}\exp\Big(-\frac{\lambda}{6}\int dx f_{abc}\frac{\delta}{\delta\mathcal{J}^a(x)}\frac{\delta}{\delta\mathcal{J}^b(x)}\frac{\delta}{\delta\mathcal{J}^c(x)}\Big)W[J]\Big|_{\mathcal{J}=0}.$$

Expending exponentials as series

$$Z_k = Z_k^{\mathrm{sc}} \sum_{V=0}^{\infty} \frac{1}{V!} \sum_{I=0}^{\infty} \frac{1}{I!} \Big[-\frac{\lambda}{3!} \int_M dx f_{abc} \frac{\delta}{\delta \mathcal{J}^a(x)} \frac{\delta}{\delta \mathcal{J}^b(x)} \frac{\delta}{\delta \mathcal{J}^c(x)}\Big]^V \Big[\frac{\lambda^{-1}}{2!} \int_M dydz \mathcal{J}^a(y) L^{ab}(y,z) \mathcal{J}^b(z)\Big]^I \Bigg|_{\mathcal{J}=0}. \tag{1.2}$$

Note that terms in (1.2) will vanish unless $3V = 2I$ for any V, I. Thus the Feynman diagrams to our concern are only vacuum bubbles with no external vertices and edges.

Since propagators play a crucial role in perturbative theory, let us take a closer look into the propagator $L(x,y)$ and its dependence on the Riemannian metric g.

2 Properties of the Propagator

Actually we don't know yet what d_a^{-1} is, so now we are going to show its existence and specify its expression. Let $*$ be the Hodge operator with respect to a Riemannian metric g on M and the Killing form on $\mathfrak{g}$, and $\delta_a := \pm * d_a *$ be the Hodge dual of d_a.
By the Hodge theorem, we have the following decomposition of the differential forms

$$\Omega^*(M,\mathfrak{g}) = \mathrm{Harm}^*(M,\mathfrak{g}) \oplus \Omega^*_{\mathrm{ex}}(M,\mathfrak{g}) \oplus \Omega^*_{\mathrm{coex}}(M,\mathfrak{g}),$$

where

$$\begin{aligned} \Omega^*_{\mathrm{ex}} &:= \mathrm{im}\, d_a \\ \Omega^*_{\mathrm{coex}} &:= \mathrm{im}\, \delta_a \end{aligned}$$

are exact and coexact forms respectively.

However, by our assumption and easy observations, we have

Lemma 2.1. The arrows in the diagram

$$\begin{tikzcd} \Omega^{k+1}_{\mathrm{ex}}(M,\mathfrak{g}) \arrow[dr, bend left, "\delta_a^{k+1}"] & \Omega^{k+1}_{\mathrm{coex}}(M,\mathfrak{g}) \\ \Omega^{k}_{\mathrm{ex}}(M,\mathfrak{g}) & \Omega^{k}_{\mathrm{coex}}(M,\mathfrak{g}) \arrow[ul, bend left, "d_a^k"] \arrow[loop right, "\Delta_a"] \end{tikzcd}$$

are all isomorphisms between vector spaces for all k's, with $\Delta_a := d_a\delta_a + \delta_a + d_a$ the Hodge Laplacian.

Now we consider the composition of the operators

$$\delta_a \circ \Delta_a^{-1} : \Omega^*(M,\mathfrak{g}) \to \Omega^{*-1}(M,\mathfrak{g}),$$

which makes sense, since here the Laplacian is invertible. For any $\omega \in \Omega^*_{\mathrm{coex}(M,\mathfrak{g})}$, we have

$$d_a(\delta_a \circ \Delta_a^{-1})\omega = (\delta_a \circ \Delta_a^{-1}) d_\alpha \omega = \omega$$

thus it is an iverse of d_a on $\Omega^*_{\mathrm{coex}}(M,\mathfrak{g})$ and is denoted as

$$d_a^{-1} = \delta_a \Delta_a^{-1}.$$

Easy computation follows that:

$$\{ d, d^{-1} \} = \{ d, \delta \circ \Delta_M \} = \{ d, \delta \} \circ \Delta_M - \delta \circ \{ d, \Delta_M^{-1} \} = \Delta_M \circ \Delta_M^{-1} = \mathrm{id}.$$

Or equivalently,

$$d \circ d^{-1} + d^{-1} \circ d = \mathrm{id}. \tag{2.1}$$

Since the propagator $L(x, y)$ is the integral kernel of d_a^{-1}, Equation (2.1) tells us that:

Lemma 2.2. $L(x, y)$ is closed on M^2/Δ, here Δ is the diagonal in M^2.

Proof. For any $\psi \in \Omega^*(M, \mathfrak{g})$, using the integral version of the equation (2.1), we have

$$\begin{aligned}
\psi(x) &= d_x \int_y L(x, y) \wedge \psi(y) + \int_y L(x, y) \wedge (d\psi(y)) \\
&= \int_y (d_x L(x, y)) \wedge \psi(y) - \int_y d_y (L(x, y) \wedge \psi(y)) + \int_y (d_y L(x, y)) \wedge \psi(y) \\
&= \int_y [(d_x + d_y) L(x, y)] \wedge \psi(y) \\
&= \int_y (d_{M^2} L(x, y)) \wedge \psi(y),
\end{aligned}$$

which implies that $d_{M^2} L(x, y)$ is the integral kernel of identity, thus is supported on $\Delta \subset M^2$, which says that $d_{M^2} L(x, y) = 0$ on M^2/Δ. $\square$

Our goal is to study the gravitational anomaly of Chern-Simons theory, i.e. the dependence on the Riemannian structure of the perturbative expansion terms in Equation (1.2). As we have seen, the propagator $L(x, y)$ in fact depends on a Riemannian metric g on M, since the Hodge operator $*$ does. For later convenience, we'd better view $L(x, y)$ as a 2-form in $\Omega^*(M \times M \times \mathbf{Met}, \mathfrak{g} \otimes \mathfrak{g})$ rather than in $\Omega^*(M \times M, \mathfrak{g} \otimes \mathfrak{g})$, where $\mathbf{Met}$ is the moduli space of all Riemannian metric on M and $\Omega^*(M \times M \times \mathbf{Met}, \mathfrak{g})$ is the pullback of $\Omega^*(M \times M, \mathfrak{g})$ along the projection $M \times \mathbf{Met} \to M$. From now on we denote $L(x, y)$ as $L(x, y; g)$ to emphasize its dependence on the metric.

Our next step is to find a parallel of Equation (2.1) on $M \times \mathbf{Met}$. Let D be the covariant exterior derivative operator on $\Omega^*(M \times \mathbf{Met}, \mathfrak{g})$, that is,

$$D = d_a \otimes \mathrm{id} \pm \mathrm{id} \otimes d_{\mathbf{Met}},$$

where the sign before the second term on the right hand side depends on the δ_a degree of the form acted on by D. Since the volume form on $\mathbf{Met}$ may be ill-defined, we can only define a "partial Hodge dual" of D on $M \times \mathbf{Met}$:

$$D^* := \delta_a \otimes \mathrm{id}$$

. The corresponding "Hodge Laplacian" is simply

$$O := \{ D, D^\dagger \} = DD^\dagger + D^\dagger D$$

since we endow $d_{\mathbf{Met}}$ of cohomological degree $+1$.

For any $\omega \otimes \Theta \in \Omega^*(M \times \mathbf{Met}, \mathfrak{g})$ with ω of degree p, we have

$$\begin{aligned} O(\omega \otimes \Theta) &= (DD^* + D^*D)(\omega \otimes \Theta) \\ &= (d_a \otimes \mathrm{id} + (-1)^{p-1}\mathrm{id} \otimes d_{\mathbf{Met}})(\delta_a \omega \otimes \Theta) \\ &+ (\delta_a \otimes \mathrm{id})(d_a \omega \otimes \Theta + (-1)^p \omega \otimes d_{\mathbf{Met}} \Theta) \\ &= (\Delta_a \omega) \otimes \Theta + (-1)^{p-1} \delta_a \omega \otimes d_{\mathbf{Met}} \Theta + (-1)^p \delta_a \omega \times d_{\mathbf{Met}} \Theta \\ &= (\Delta_a \omega) \otimes \Theta \end{aligned}$$

thus, we have

$$O = \Delta_a \otimes \mathrm{id}.$$

O is invertible as Δ_a and we can define the inverse of D as

$$D^{-1} := D^* \circ O^{-1} = d^{-1} \otimes \mathrm{id}.$$

Without suprise, we have the commutation relation for D and D^{-1}

$$\{ D, D^{-1} \} = DD^{-1} + D^{-1}D = \mathrm{id} \tag{2.2}$$

Using Equation (2.2) and repeating a parallel argument, we have

Lemma 2.3. *$L(x, y; g)$ is D-closed away from the diagonal $\Delta \in M \times M$.*

To determine the singularities of $L(x, y; g)$, let $G \in \Omega^3(M \times M \times \mathbf{Met}, \mathfrak{g} \otimes \mathfrak{g})$ be the *Green's function* for O^{-1}

$$O^{-1}\psi(x) = \int_y G(x, y; g) \wedge \psi(y),$$

then

$$L(x, y; g) = D_x^* G(x, y; g).$$

That means if we know the singularities of $G(x, y; g)$, we know the same of $L(x, y; g)$.

Proposition 2.4. For a generalized Laplacian Δ on a compact manifold M, let G be its Green function, we have

$$G(x, y) = \int_0^\infty K(t, x, y) dt,$$

in the distributional sense, where $K(t, x, y)$ the heat kernel for Δ.

Proof. The proof is formal. For any differential form $\psi in \Omega^*(M)$

$$\begin{aligned} &\int_{y \in M} (\int_0^\infty K(t, x, y) dt) \wedge \psi(y) \\ =&\int_0^\infty e^{-t\Delta} \psi(x) dt \\ =&(P - \Delta^{-1}) \int_0^\infty d(e^{-t\Delta}) \psi(x) \\ =&(P - \Delta^{-1})(P - 1)\psi(x) \\ =&\Delta^{-1}\psi(x) \end{aligned}$$

where $P : \Omega^*(M) \to \mathrm{Harm}^*(M)$ is the projection to harmonics. The third line holds because by the definition of Δ^{-1}

$$\Delta^{-1}\Delta = 1 - P,$$

thus

$$\Delta^{-1}\frac{d}{dt}(e^{-t\Delta}) = \Delta^{-1}\Delta e^{-t\Delta} = (1-P)e^{-t\Delta}.$$

□

In our case, if we had known the heat kernel for the generalized Laplacian O, we know the Green function and thus the propagator. We expect to use the asymptotic expansion of the heat kernel to obtain the singular part of the Green function. To do so, we need to make sure that the curvature forms on manifold $M \times \mathbf{Met}$ is non singular.

Denote $\nabla^{M\times\mathbf{Met}}$ by the Levi-Civita connection on the infinite dimensional manifold $M \times \mathbf{Met}$ with respect to the Riemannian metric of the form

$$\langle (v_1, m_1), (v_2, m_2) \rangle_{(x,g)} = g_x(v_1, v_2) + \Gamma_g(m_1, m_2)$$

with $v_1, v_2 \in T_xM, m_1, m_2 \in T_g\mathbf{Met}$, and Γ is any Riemannian metric on $\mathbf{Met}$, since we actually don't need the information of Γ.
We have a natural projection $M \times \mathbf{Met} \to M$, and denote $\widetilde{TM}$ to be the pull-back bundle of the diagram below:

$$\begin{array}{ccc} \widetilde{TM} & \longrightarrow & TM \\ \downarrow & & \downarrow \\ M \times \mathbf{Met} & \longrightarrow & M. \end{array}$$

Then $\widetilde{TM} \hookrightarrow T(M \times \mathbf{Met})$ is a subbundle of $T(M \times \mathbf{Met})$ over $M \times \mathbf{Met}$. Assume that $M \times \mathbf{Met}$ is paracompact, then we have the vector bundle decomposition $T(M \times \mathbf{Met}) = (\widetilde{TM})^\perp \oplus \widetilde{TM}$. Then we have a projection of bundles $\pi_{\widetilde{TM}} : T(M \times \mathbf{Met}) \to \widetilde{TM}$, and let $\tilde{\nabla}$ to be the connection defined as

$$\tilde{\nabla} = \pi_{\widetilde{TM}} \circ \nabla^{M\times\mathbf{Met}}$$

.

At a point $(x, g) \in M \times \mathbf{Met}$, we have the local expression:

$$(\tilde{\nabla}_{\partial/\partial x^i}\frac{\partial}{\partial x^k})(x,g) = \Gamma^j_{ik}(x)\frac{\partial}{\partial x^j}$$
$$(\tilde{\nabla}_m\frac{\partial}{\partial x^k})(x,g) = \frac{1}{2}g^{jl}(x)(\delta g)_{lk}(x)\frac{\partial}{\partial x^j} = \frac{1}{2}(g^{-1}(x)\delta g)^j_k\frac{\partial}{\partial x^j}$$

If u is a section of $\widetilde{TM}$, and its coordinates in the trivialization $\{\partial/\partial_{x^j}\}$ is $\{u^j\}$, then we have

$$\tilde{\nabla}u^j = du^j + \Gamma^j_{ik}u^k dx^i + \frac{1}{2}(g^{-1}m)^j_k u^k.$$

Now we see that the curvature tensors for the connection $\tilde{\nabla}$ is non singular on $M \times \mathbf{Met}$. If we denote the curvature 2-form as $Om\tilde{e}ga$, then it can be decomposed as

$$\tilde{\Omega} = \tilde{\Omega}^{(2.0)} + \tilde{\Omega}^{(1,1)} + \tilde{\Omega}^{(0,2)}.$$

Its degree in $\mathbf{Met}$ direction is non-zero.

2.1 Extending L to the Blow-ups

For simplicity, in this subsection we assume the Lie group G to be trivial. Knowing that $\tilde{\Omega}$ is non-singular, we can use 2.4 boldly to estimate the order of singularities of the Green's function $G(x, y; g)$ of the generalized Laplacian O.

Reacall that the asymptotic expension of heat kernel $K(t,\ x,\ y)$ for a generalized Laplacian on an n–dimensional compact Riemannian manifold

$$K(t,\ x,\ y) \sim \sum_{k=0}^{\infty} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{d^2(x,\ y)}{4t}} u_k(x, y) t^{k-\frac{n}{2}} \tag{2.3}$$

where $u_0(x,\ y) = \frac{1}{\sqrt{\det\left(d\exp_x^{-1}(y)\right)}}$ and other $u_k(x,\ y)$'s are determined recursively.

Proposition 2.5. The leading singularity of $G(x, y; g)$ is $\frac{1}{d}$, where $d = d(x, y)$ is the distant between x, y for x, y close enough.

Proof. In our case, M is a 3-dimensional manifold. The asymptotic expansion of heat kernel is

$$K(t,\ x,\ y) \sim \sum_{k=0}^{\infty} \frac{1}{(4\pi)^{\frac{3}{2}}} e^{-\frac{d^2(x,\ y)}{4t}} u_k(x, y) t^{k-\frac{3}{2}}$$

For $k = 0$, we need to compute the intagral

$$u_0(x, y) \int_0^{\infty} \frac{1}{(4\pi t)^{3/2}} e^{-\frac{d^2}{4t}} dt = \frac{2u_0(x, y)}{(4\pi)^{3/2}} \int_0^{\infty} e^{-\frac{d^2 s^2}{4}} ds = \frac{1}{2\pi} \frac{u_0(x, y)}{d}$$

where in the second line we substitute $t = \frac{1}{s^2}$.
For $k = 1$, we the integral becomes

$$\begin{aligned}
&\int_0^{\infty} e^{-d^2/4t} t^{-1/2} dt \\
=&2 \int_0^{\infty} e^{-d^2 s^2/4} s^2 ds \\
=&2 \int_0^{\infty} e^{-d^2 s^2/4} d(-\frac{1}{s}) \\
=&+\infty - d^2 \int_0^{\infty} e^{-d^2 s^2/4} ds \\
=&+\infty - 2\sqrt{\pi} d
\end{aligned}$$

The infinity appears for all close enough x, y, not only on the diagonal, thus it contains no information of heat transfer on M and should be discarded.
For $k \geq 2$, we can evaluate the integral recursively and find that the results consists of $\mathcal{O}(d)$-terms and infinity terms of the same type as in the case $k = 1$, thus they have no contribution to singularities of $G(x, y; g)$. □

For a point $x \in M$, we choose a Riemannian normal coordinate $\{\, u_i \,\}$ around x. With some elaboration, we can work out the concrete form of the leading singularities of $G(x, y; g)$ in normal coordinate

Proposition 2.6.

$$G(x,y) = \frac{1}{24\pi\,\|u\|}\epsilon_{ijk}(\tilde{\nabla}u^i \wedge \tilde{\nabla}u^j \wedge \tilde{\nabla}u^k - 3\,\|u\|^2\,\tilde{\Omega}^i_j \wedge \tilde{\nabla}u^k) + \mathcal{O}(\|u\|^2) \tag{2.4}$$

with $y = \exp_x(u)$.

Remark. In [AS], this result is obtained by solving the equation inductively

$$O_y G(x,y;g) = 0$$

via the Hadamard parametrix construction, with technical calculations.

Using the expression of D^* in normal coordinate

$$D^* = -\iota(\partial_{u^i})\tilde{\nabla}_{\partial_{u^i}}$$

we finally have derived the singular term of $L(x,y;g)$

$$L(x,y;g) = -\frac{1}{8\pi}\epsilon_{ijk}\hat{u}^i(\tilde{\nabla}\hat{u}^j \wedge \tilde{\nabla}\hat{u}^k + \tilde{\Omega}^j_k) + \mathcal{O}(\|u\|).$$

Denote $B \coloneqq \mathrm{Bl}_2(M \times M, \Delta)$ as the blow up of configuration space $M(2)$ along the diagonal in $M \times M$, and let $b : B \to M(2)$ be the blow–down map. b is identity when restricted to $B^\circ = M(2)$, and can be identified with the map $S(TM) \to M$ when restricted to $\partial B = S(TM)$, where $S(TM)$ denotes the unit sphere bundle of the tangent bundle $T\Delta = TM$ of the diagonal Δ in $M \times M$.

Since $L(x,y;g)$ is non-singular viewed as a form in $\Omega^2(S(TM) \times \mathbf{Met}, \mathfrak{g} \otimes \mathfrak{g})$, we can extend $L(x,y;g)$ from $M(2)$ to B smoothly. We denote this smooth extension as L_B. We may characterize L_B as

$$L_B(x,y;g)|_{\partial B} = \lambda(x,u;g) \otimes \mathrm{id}_{\mathfrak{g}} + b^*\rho(x) \tag{2.5}$$

where ρ is a smooth form defined on the diagonal Δ, and $\lambda(x,u;g) \in \Omega^2(\partial B \times \mathbf{Met})$

$$\lambda(x,u;g) \coloneqq -\frac{1}{8\pi}\epsilon_{ijk}\hat{u}^i(\tilde{\nabla}\hat{u}^j \wedge \tilde{\nabla}\hat{u}^k + \tilde{\Omega}^j_k(x;g)). \tag{2.6}$$

And we have the following important statement

Proposition 2.7. L_B is D-closed on $B \times \mathbf{Met}$.

Proof. Since L is D-closed on $M(2)$ and L_B is a smooth extension of L. □

2.2 Compactifaction of Configuration Spaces

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3 Main Theorem

The quantities of our concern are the Feynman diagrams

$$I_l = \int_{M[V]} \mathrm{Tr}^{(V)}((L_{C,tot})^I)$$

the integral is performed on $M[V]$ to avoid divergents.

The main theorem of the paper is that

Theorem 3.1. There is a constant β_l depending only on l and the Killing form $\langle -, - \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ so that the quantity

$$\hat{I}_l^{conn}(M, a, s) := I_l^{conn}(M, a, g) - \beta_l CS_{grav}(g, s)$$

is independent of g. $\hat{I}_l^{conn}$ is therefore a topological invariant depending on the choice of the manifold M, homotopy framing s, and flat connection a.

Proof. Using Stokes' Theorem and degree-counting techniques. □

4 Miscellaneous

Lemma 4.1 ([FQ]). If G is simply connected, then any principal G-bundle over a manifold of dimension ≤ 3 is trivial.

Proof. Since $\pi_0(G) = \pi_1(G) = \pi_2(G) = 0$, the Lemma follows by obstruction theory. □

Proposition 4.2. Let M be a compact oriented 3-manifold with $\partial M \neq \varnothing$. Then we have

$$CS(g^*A) = CS(A) + \frac{1}{8\pi^2}\int_{\partial M} \operatorname{Tr} A \wedge dgg^{-1} - \int_M g^*\sigma$$

where σ is the volume form of $SU(2)$.

References

[AS] Scott Axelrod and I. M. Singer. Chern-Simons perturbation theory. II. 39(1):173–213.

[FQ] Daniel S. Freed and Frank Quinn. Chern-Simons Theory with Finite Gauge Group. 156(3):435–472.