Assignment

Zhang Chi (201828001207022)
zhangchi2018@itp.ac.cn

June 26, 2021

Exercise 1

Let X, Y be two connected Riemann surfaces and $f : X \to Y$ be a proper map, then f is a branched covering. In particular, for any $y, y' \in Y$ we have $\#f^{-1}(y) = \#f^{-1}(y')$ counting multiplicity.

Solution. Since f is a branched covering, for any $y \in Y$, we can find charts $U_x \subseteq X$ about each point $x \in f^{-1}(y)$ and a corresponding $V \subseteq Y$ about y, with respect to which f is expressed locally as $z \mapsto z^k$, also notice that there the number of U_x is finite since f is a branched covering. Using the properness of f, we can make sure that $f^{-1}(V)$ is contained in the union of the U_x. Now we can view $V \simeq \mathbb{C}$ and $U_x \simeq \mathbb{C}$, where f is just

$$\begin{aligned} f : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z^k. \end{aligned}$$

In this case, any $y, y' \in \mathbb{C}$, we have $\#f^{-1}(y) = \#f^{-1}(y')$, which shows that $f^{-1}(y)$ is a locally constant function. Combining the fact that both X, Y are connected, we have shown that $\#f^{-1}(y)$ is a constant, completing the proof. □

Exercise 2

Show that if X is a compact surface of genus g, then $\chi(X) = 2 - 2g$.

Solution. To show this, we need some preparations. First we need to compute the Euler characteristics of sphere, torus and disk. We may view S^2 obtained identifying B and B' and edges AB, AB' as well as $BC, B'C$ along the illustrated directions. So we find a triangulation of S^2 as follows, with 3 vertices (B, B' identified), 3 edges and 2 faces.

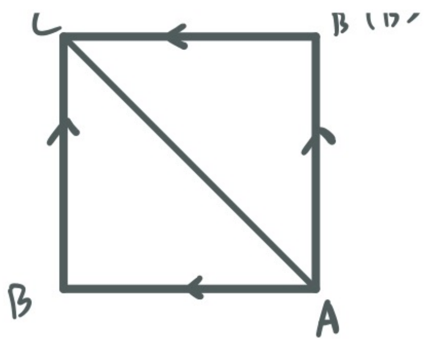

Figure 1: a triangulation of S^2.

By the characteristic formula

$$\chi = V - E + F$$

for polyhedra, we have
$$\chi(S^2) = 3 - 3 + 2 = 2.$$
Similarly, we could obtain the Euler characteristic of a torus $\mathbb{T}$
$$\chi(\mathbb{T}) = 1 - 3 + 2 = 0$$
and of a disk D
$$\chi(D) = 4 - 6 + 3 = 1,$$
via the illustrated triangulations of $\mathbb{T}$ and D. Next we claim that for the connected sum $M\#N$

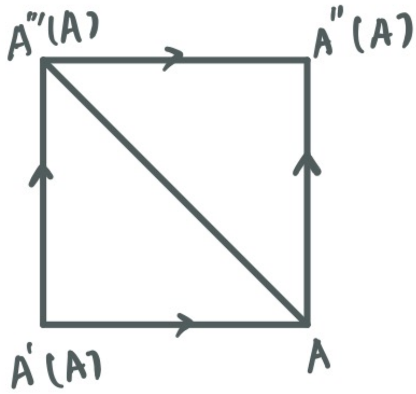

Figure 2: a triangulation of $\mathbb{T}$.

Figure 3: a triangulation of D.

of two surfaces M and N, the Euler characteristic of $M\#N$ follows as
$$\chi(M\#N) = \chi(M) + \chi(N) - 2. \tag{1}$$

This is because that we could find a triangulation T_M of M and a triangulation T_N of N, with sub-triangulations $T_{M'}, T_{D_1}$ and $T_{N'}, T_{D_2}$ when restricted to M', D_1 and N', D_2, such that their edges and vertices on ∂D_1 and ∂D_2 coincide after identifying ∂D_1 and ∂D_2, as in the following figure. But since $\partial D_1, \partial D_2$ are closed circles, so the number of edges $E(\partial D_1)$ and the number

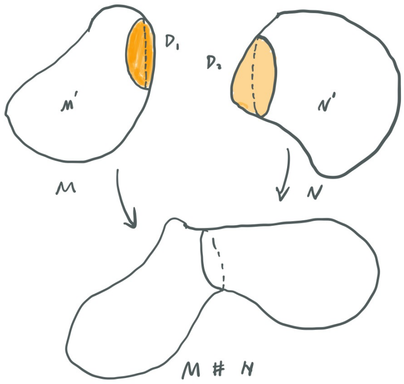

Figure 4: the connected sum $M\#N$ of M and N.

$V(\partial D_1)$ of vertices must be equal on them, and similarly for $E(\partial D_2)$ and $V(\partial D_2)$. In this spirit,

$$\begin{aligned}
\chi(M) &= E(T_M) - V(T_M) + F(T_M) \\
&= (E(T_{M'}) + E(T_{D_1}) - E(\partial D_1)) \\
&- (V(T_{M'}) + V(T_{D_1}) - V(\partial D_1)) + F(T_{M'}) + F(T_{D_1}) \\
&= (E(T_{M'}) - V(T_{M'}) + F(T_{M'})) \\
&+ E(T_{D_1}) - V(T_{D_1}) + F(T_{D_1})) \\
&= \chi(M') + \chi(D_1) \\
&= \chi(M') + 1.
\end{aligned} \tag{2}$$

In parallel,

$$\chi(N) = \chi(N') + 1 \tag{3}$$

and

$$\begin{aligned}
\chi(M\#N) &= (E(T_{M'}) - E(\partial D_1) + E(T_{N'}) - E(\partial D_2)) \\
&+ (V(T_{M'}) - V(\partial D_1) + V(T_{N'}) - V(\partial D_2)) \\
&+ (F(T_{M'} + F(T_{N'})) \\
&= (E(T_{M'}) - V(T_{M'}) + F(T_{M'})) \\
&+ (E(T_{N'}) - V(T_{N'}) + F(T_{N'})) \\
&= \chi(M') + \chi(N').
\end{aligned} \tag{4}$$

Putting (2), (3) and (4) together, we have proved (1).

Now we can show our assertion using induction. When $g(X) = 0$, we have already showed that

$$\chi(X) = \chi(S^2) = 2.$$

Now suppose X is a surface of $g \geq 1$, X can be thus presented as

$$X = X'\#\mathbb{T},$$

where X' is a surface of genus $g-1$. So we have

$$\begin{aligned}
\chi(X) &= \chi(X'\#\mathbb{T}) \\
&= \chi(X') + \chi(\mathbb{T}) - 2 \\
&= 2 - 2(g-1) - 2 \\
&= 2 - 2g,
\end{aligned}$$

which completes the proof. □

Exercise 3

Let X be a m compact Riemann surface of genus g, and $f : X \to \mathbb{P}^1$ be a holomorphic map of degree 2, compute the number of branching points of f.

Solution. By the previous exercise, we know the Euler characteristics

$$\chi(X) = 2 - 2g,$$
$$\chi(\mathbb{P}^1) = 2 - 2 \cdot 0 = 2$$

of X and $\mathbb{P}^1$. Then by plugging $\chi(X)$ and $\chi(\mathbb{P}^1$ as well as $\deg f = 2$ into the mighty Riemann-Hurwitz formula

$$\chi(X) = d\chi(\mathbb{P}^1) - b_f,$$

we finally get

$$b_f = 2 + 2g.$$

So the total number of branching of f is $2 + 2g$, counting multiplicity. □

Exercise 4

Show that $\{\frac{\partial}{\partial x}|_p, \frac{\partial}{\partial y}|_p\}$ is a basis of T_pX over $\mathbb{C}$.

Solution. Suppose that X is a Riemann surface and $p \in X$. For any $\mathbb{C}$-valued smooth function $f \in C^\infty(X)$ on X, it is always possible to choose a neighborhood $U \simeq \mathbb{C} \simeq \mathbb{R}^2$ of p and endow U with a coordinate in which $f(p) = 0$. Without loss of generality, we may also set $p = (0, 0)$.

Thus at any $q \in U$ with $q = (x, y)$, we may expand $f(q)$ by Taylor's Theorem

$$\begin{aligned} f(q) &= x\frac{\partial f}{\partial x}(p) + y\frac{\partial f}{\partial y}(p) \\ &\quad + \int_0^1 (1-t)\{x^2\frac{\partial^2 f}{\partial x^2}(tq) + 2xy\frac{\partial^2 f}{\partial x \partial y}(tq) + y^2\frac{\partial^2 f}{\partial y^2}(tq)\}dt. \end{aligned} \tag{5}$$

Taking any tangent vector $v \in T_pX$, we exert v on f and obtain that

$$(vf)(p) = \frac{\partial f}{\partial x}(p)v_x(p) + \frac{\partial f}{\partial y}(p)v_y(p)$$

with the help of (5). In the above equation we denote by

$$\begin{aligned} v_x &:= v(x), \\ v_y &:= v(y), \end{aligned} \tag{6}$$

and the terms quadratic in x, y in (5) vanishes after exerted by v and evaluated at $p = (0, 0)$ since both x and y vanish at p. Thus (6) shows that $\{\frac{\partial}{\partial x}|_p, \frac{\partial}{\partial y}|_p\}$ spans any tangent vector $v \in T_pX$ over $\mathbb{C}$.

Now we need show that $\frac{\partial}{\partial x}|_p$ and $\frac{\partial}{\partial y}|_p$ are linear independent over $\mathbb{C}$. If they were not, we have some non-zero $a, b \in \mathbb{C}$ such that

$$0 = a\frac{\partial}{\partial x}|_p + \frac{\partial}{\partial y}|_p$$

in T_pX, *i.e.*, for any $f \in C^\infty(X)$, we have

$$a\frac{\partial f}{\partial x}(p) + b\frac{\partial f}{\partial y}(p) = 0.$$

In particular we first take $f = x$ and then $f = y$, showing that $a = b = 0$, a contradiction. So they are indeed linearly independent, from which our assertion follows. □

Exercise 5

Compute the number of zeros of any holomorphic vector fields on $\mathbb{P}^1$, counting multiplicity.

Solution. By the maneuvers mentioned in class, we have already been acquainted with the space L of holomorphic vector fields on $\mathbb{P}^1$ with

$$L = \left\{ (a + bz + cz^2)\partial_z \;\middle|\; a, b, c \in \mathbb{C} \right\},$$

near a neighborhood of 0 homeomorphic to $\mathbb{C}$. So given an arbitrary holomorphic vector field v on $\mathbb{P}^1$ which can be locally expressed as $v = (a + bz + cz^2)\partial_z$, there are the following possiblities:

i $a = b = c = 0$, the vector field is just zero, so its zeros are the whole $\mathbb{P}^1$.

ii $c = 0, b \neq 0$. There is a zero $z = -a/b$ near 0. However, after the coordinate change $w = \frac{1}{z}$, we have $v = -w^2(a + b/w)\partial_w = -(aw^2 + bw)\partial_w$, showing that ∞ is also a zero of v. So in this case, v has two isolated zeros.

iii $c \neq 0$, there are two zeros of v near 0.

In summary, there are 2 isolated zeros of any holomorphic vector field on $\mathbb{P}^1$. □

Exercise 6

Give a description of all holomorphic vector fields on an elliptic curve $\mathbb{C}/\Gamma$.

Solution. Suppose that v is any holomorphic vector field on $\mathbb{C}/\Gamma$. In some neighborhood U with coordinate z, v must be presented as

$$v(z) = f(z)\partial_z,$$

with $f(z)$ some holomorphic function in z. Since v is the vector field on $\mathbb{C}/\Gamma$, it is invariant under the action of Γ, that is

$$v(z) = v(z + \Gamma).$$

But as $\partial_{z+\Gamma} = \partial$, the above condition of translation invariance of v is equivalent to that

$$f(z)\partial_z = f(z + \Gamma)\partial_z$$

for all $z \in U \simeq \mathbb{C}$, which is furthermore equivalent to

$$f(z) = f(z + \Gamma). \tag{7}$$

We expand the right hand side of (7) as power series

$$f(z + \Gamma) = f(z) + \frac{\partial f}{\partial z}(z)\Gamma + \frac{1}{2!}\frac{\partial^2 f}{\partial z^2}(z)\Gamma^2 + \cdots = f(z).$$

Since the above equation holds for all $z \in \mathbb{C}$, f must satisfy that

$$f^{(n)} = 0, n \geq 1.$$

So $f(z)$ must be a constant function. Hence the space of all holomorphic functions on $\mathbb{C}/\Gamma$ is isomorphic to $\mathbb{C}$ as a complex vector space. □

Exercise 7

Let L be the space of the holomophic vector fields on $\mathbb{P}^1$ with the operation of Lie bracket of vector fields, then L is a Lie algebra over $\mathbb{C}$. Show that L and $\mathfrak{sl}(2,\mathbb{C})$ are isomorphic Lie algebras, where

$$\mathfrak{sl}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{C}, a+d=0 \right\}.$$

Solution. Consider the map

$$\begin{gathered} \phi : L \to \mathfrak{sl}(2,\mathbb{C}), \\ (a+bz+cz^2)\partial_z \mapsto \begin{pmatrix} \frac{b}{2} & c \\ -a & -\frac{b}{2} \end{pmatrix}, \end{gathered}$$

which is clearly an isomorphism between $\mathbb{C}$-vector spaces. What need we show is that ϕ preserves the Lie brackets.

For any two holomorphic vector fields represented by $u = (a+bz+cz^2)\partial_z$ and $v = (d+ez+fz^2)\partial_z$, we have

$$\begin{aligned} [\phi(u),\phi(v)] &= \begin{pmatrix} \frac{b}{2} & c \\ -a & -\frac{b}{2} \end{pmatrix} \begin{pmatrix} \frac{e}{2} & f \\ -d & -\frac{e}{2} \end{pmatrix} - \begin{pmatrix} \frac{e}{2} & f \\ -d & -\frac{e}{2} \end{pmatrix} \begin{pmatrix} \frac{b}{2} & c \\ -a & -\frac{b}{2} \end{pmatrix} \\ &= \begin{pmatrix} af-cd & bf-ce \\ bd-ae & cd-af \end{pmatrix} \\ &= \phi(((ae-bd)+2(af-cd)z+(bf-ce)z^2)\partial_z) \end{aligned}$$

and

$$\begin{aligned} \phi([u,v]) &= \phi((a+bz+cz^2)(e+2fz)\partial_z) - \phi((d+ez+fz^2)(b+2cz)\partial_z) \\ &= \phi(((ae-bd)+2(af-cd)z+(bf-ce)z^2)\partial_z) \end{aligned},$$

which shows that

$$\phi([u,v]) = [\phi(u),\phi(v)]$$

holds for any $u,v \in L$. Hence ϕ is an isomorphism of Lie algebras L and $\mathfrak{sl}(2,\mathbb{C})$. $\square$

Exercise 8

Show that there are no non-zero holomorphic 1-forms on $\mathbb{P}^1$.

Solution. View $\mathbb{P}^1$ as $\mathbb{C} \cup \{\infty\}$, so we can cover $\mathbb{P}^1$ with two neighborhoods U,V such that U,V are all homeomorphic to $\mathbb{C}$ with $0 \in U, \infty \in V$. Then we may take the coordinate near 0 as z meanwhile the coordinate near ∞ as w. Of course, for any point in $U \cap V$ we have $w = \frac{1}{z}$.

If there were a holomorphic 1-form ω on $\mathbb{P}^1$, locally ω could be presented as

$$\omega = \begin{cases} f(z)dz, \text{on } U, \\ g(w)dw, \text{on } V, \end{cases}$$

where $f(z)$ and $g(w)$ are holomorphic functions of z and w, respectively. Since 1-forms are coordinate-free objects, for any any point $p \in U \cap V$ with coordinate z on U and coordinate w on V, the equation

$$\omega(z) = \omega(p) = \omega(w) \tag{8}$$

must hold. We may take the expansions

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots,$$
$$g(w) = b_0 + b_1 w + b_2 w^2 + \cdots$$

of f and g, under which (8) becomes

$$-(a_0 + a_1 z + a_2 z^2 + \cdots) = b_0/z^2 + b_1/z^3 + b_2/z^4 + \cdots.$$

Since the above equation holds for all $z \neq 0$, we finally conclude that

$$a_i = b_i = 0, i \geq 0,$$

which in turn tells us that $\omega = 0$, a contradiction. So there must be no non-zero 1-forms on $\mathbb{P}^1$. □

Exercise 9

Assume that $\{ e_1, \ldots, e_n \}$ is a basis of V, then

$$\{ e_i \wedge e_j \mid 1 \leq i < j \leq n \}$$

is a basis of $V \wedge V$. In particular, $\dim V \wedge V = \frac{n(n-1)}{2}$.

Solution. For any $u, v \in V$, we may expand them in basis $\{e_1, \ldots, e_n\}$:

$$u = \sum_{i=1}^{n} a_i e_i,$$
$$v = \sum_{j=1}^{n} b_j e_j.$$

Hence

$$u \wedge v = (\sum_{i=1}^{n} a_i e_i) \wedge (v = \sum_{j=1}^{n} b_j e_j) = \sum_{1 \leq i < j \leq n} (a_i b_j - b_i a_j) e_i \wedge e_j,$$

which shows that $\{ e_i \wedge e_j \mid 1 \leq i < j \leq n \}$ indeed spans $V \wedge V$.

Now we show that elements in $\{ e_i \wedge e_j \mid 1 \leq i < j \leq n \}$ are linearly independent. If they were not, there must be coefficients $c_{ij} \in k, 1 \leq i < j \leq n$ that are not all zero such that

$$\sum_{1 \leq i < j \leq n} c_{ij} e_i \wedge e_j = 0. \tag{9}$$

For expediency, we may assume that $c_{12} \neq 0$ in (9) and may view $V \wedge V$ as a subspace of $\wedge^n V$ in the natural way. By wedging $e_2 \wedge \cdots \wedge e_n$ to both sides of (9) in $\wedge^n V$, we have

$$c_1 2 e_1 \wedge e_2 \wedge \cdots \wedge e_n = 0,$$

which says that $c_1 2 = 0$, a contradiction.

So we have shown that $\{ e_i \wedge e_j \mid 1 \leq i < j \leq n \}$ is a basis of $V \wedge V$. The dimension of $V \wedge V$ is obtained by simple counting. □

Exercise 10

Define $\sigma : V^* \wedge V^* \to L$ to be

$$\sigma(f \wedge g)(u,v) = f(u)g(v) - f(v)g(u),$$

where

$$L = \{ \text{ anti-symmetric forms on } V \}.$$

Show that σ is a well-defined linear isomorphism from $V^* \wedge V^*$ to L.

Solution. First of all, for any $f, g \in V^*$ and $u, v \in V$ we have

$$\sigma(f \wedge g)(u,v) = f(u)g(v) - f(v)g(u) = -(f(v)g(u) - f(u)g(v)) = -\sigma(f \wedge g)(v,u),$$

which shows that $\sigma(f \wedge g)$ is indeed an anti-symmetric form on V. And manifestly $\sigma : V^* \wedge V^*$ is a linear map. What we need to show is that σ is an isomorphism.

We first show the injectivity. To do so we pick a basis $\{e_1, \dots, e_n\}$ for V. If $\sigma(f \wedge g) = 0$ for some $0 \neq f \wedge g \in V^* \wedge V^*$, we must have

$$0 = \sigma(f \wedge g)(e_i, e_j) = f(e_i)g(e_j) - f(e_j)g(e_i) = f_i g_j - f_j g_i, \forall i, j,$$

which means either f or g equals 0 or that f parallels to g, all causing $f \wedge g = 0$, a contradiction.

Since V is of finite dimension n, we have $V^* \simeq V$ (although not canonical), and by the previous exercise we have $\dim(V^* \wedge V^*) = \frac{n(n-1)}{2}$. Recall that any anti-symmetric form Q on V can be presented as a matrix

$$\begin{pmatrix} 0 & Q_{12} & \cdots & Q_{1n} \\ -Q_{12} & 0 & \cdots & Q_{2n} \\ \vdots & \vdots & & \vdots \\ -Q_{1n} & -Q_{2n} & \cdots & 0 \end{pmatrix},$$

where $Q_{ij} = Q(e_i, e_j)$. So the vector space L consists of all anti-symmetric metrices, thus is also of dimension $\frac{n(n-1)}{2}$. Now that $\sigma : V^* \wedge V^* \to L$ is injective, it must be isomorphic. $\square$

Exercise 11

Show that all Riemann surfaces are orientable.

Solution. For any Riemann surface X and any point $p \in X$, we may choose two neighborhood U, V of p on which the coordinates of p are $z = x + \mathrm{i}y, w = u + \mathrm{i}v$, respectively. Since a Riemann surface is by definition a 1-dimensional complex manifold, the coordinates z, w must be related by some holomorphic coordinate transformation $w = f(z)$.

To see the orientability of X, we must compute the Jacobian $J(f)(p)$ of the transformation f near p, as well as its determinant. In terms of the real variables, we have

$$J(f)(p) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix},$$

and

$$\det(J(f)(p)) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}. \tag{10}$$

Now we need to rewrite (10) in terms of complex variables, to see things clearly. In light of the relations

$$\begin{aligned} z &= x + \mathrm{i}y, \\ \bar{z} &= x - \mathrm{i}y, \end{aligned}$$

and

$$\begin{aligned} w &= u + \mathrm{i}v, \\ \bar{w} &= u - \mathrm{i}v, \end{aligned}$$

we find

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial x}\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \\ \frac{\partial}{\partial y} &= \frac{\partial z}{\partial y}\frac{\partial}{\partial z} + \frac{\partial \bar{z}}{\partial y}\frac{\partial}{\partial \bar{z}} = \mathrm{i}(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}). \end{aligned}$$

Hence

$$\frac{\partial u}{\partial x} = \frac{1}{2}(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}})(w + \bar{w}) = \frac{1}{2}(\frac{\partial w}{\partial z} + \frac{\partial \bar{w}}{\partial \bar{z}}),$$

of which the last equality follows by the holomorphicity of $w = f(z)$. Similarly,

$$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{1}{2}(\frac{\partial w}{\partial z} + \frac{\partial \bar{w}}{\partial \bar{z}}), \\ \frac{\partial u}{\partial y} &= \frac{\mathrm{i}}{2}(\frac{\partial w}{\partial z} - \frac{\partial \bar{w}}{\partial \bar{z}}), \\ \frac{\partial v}{\partial x} &= -\frac{\mathrm{i}}{2}(\frac{\partial w}{\partial z} - \frac{\partial \bar{w}}{\partial \bar{z}}). \end{aligned}$$

Finally, by plugging all these pieces into (10), we have

$$\det(J(f)(p)) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \frac{1}{4}(\frac{\partial w}{\partial z} + \frac{\partial \bar{w}}{\partial \bar{z}})^2 - \frac{1}{4}(\frac{\partial w}{\partial z} - \frac{\partial \bar{w}}{\partial \bar{z}})^2 = \frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}} = |\frac{\partial w}{\partial z}|^2 > 0.$$

Since f and p is arbitrary, the orientability of X has been proved. □

Exercise 12

Let $I = \{(x, y) \in \mathbb{R}^2 | 0 < x, y < 1\}$ and $\mu \in \Omega^2(X)$, $\mathrm{supp}\mu \subseteq cI$ for some $c \in \mathbb{C}$. Show that if $\int_I \mu = 0$, there exists some $\xi \in \Omega^1(I)$ with compact support, such that $\mu = d\xi$.

Proof. Since $\mathrm{supp}\mu \in cI$, μ can be viewed as a compactly supported 2-form on $\mathbb{R}^2$. Conversely, if we have found some ξ satisfying $\mu = d\xi$ in on I, then we can extend ξ to X by zero. Thus without loss of generality, we may take $X = \mathbb{R}^2$.

To prove the statement, suppose $\mu = R(x, y)dx \wedge dy$ such that

$$\int_X \mu = \int_{\mathbb{R}^2} R(x, y)dx \wedge dy = 0.$$

We can choose a function ψ on $\mathbb{R}$ with support in cI and with

$$\int_{-\infty}^{\infty} \psi(t)dt = 1.$$

Let

$$r(x) := \int_{-\infty}^{\infty} R(x,y)dy$$

and

$$\tilde{R}(x,y) = R(x,y) - r(x)\psi(y).$$

Notice that for all x, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \tilde{R}(x,y)dy &= \int_{-\infty}^{\infty} R(x,y)dy - \int_{-\infty}^{\infty} R(x,t)dt \int_{-\infty}^{\infty} \psi(y)dy \\ &= \int_{-\infty}^{\infty} R(x,y)dy - \int_{-\infty}^{\infty} R(x,t)dt \\ &= 0.\end{aligned}$$

Define

$$P(x,y) = \int_{-\infty}^{y} R(x,t)dt.$$

Then P is has support in cI and

$$\frac{\partial P}{\partial y} = \tilde{R}(x,y).$$

Put

$$Q(x,y) = \psi(y) \int_{-\infty}^{x} r(t)dt.$$

Then Q has support in cI and

$$\frac{\partial Q}{\partial x} = r(x)\psi(y).$$

Thus by construction of R, P, Q,

$$R = \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}.$$

Take

$$\xi = -Pdx + Qdy,$$

we have

$$d\xi = \frac{\partial P}{\partial y} dx \wedge dy + \frac{\partial Q}{\partial x} dx \wedge dy = R dx \wedge dy = \mu,$$

completing the proof. □

Exercise 13

Use the above exercise to show that

Theorem 1. If X is a connected Riemann surface (may be non-compact), then we have an isomorphism

$$\int_X : H_c^2(X, \mathbb{C}) \to \mathbb{C}$$

.

Proof. Fisrtly, we show that the map $\int_X$ is surjective. Given any $k \in \mathbb{C}$, we can any local chart (U, z) of X, such that cI is contained in U. Then take

$$\eta = k\psi(x)\psi(y)dx \wedge dy,$$

where ψ is the function on $\mathbb{R}$ with support in cI and

$$\int_{-\infty}^{\infty} \psi(x)dx = 1.$$

With extension to X by 0, we may view η as a 2-form on X with support in $cI \subseteq U$. And

$$\int_X [\eta] = \int_X \eta = \int_{\mathbb{C}} k\psi(x)\psi(y)dx \wedge dy = k$$

shows that $\int_X$ is a surjection.

Next we prove that $\int_X$ is an injection. More precisely, if there is a compactly supported 2-form μ with

$$\int_X \mu = 0,$$

we have to find a compactly supported 1-form ξ on X such that

$$\mu = d\xi.$$

Since μ is compactly, supported, we can cover its support by finitely many open sets $U_1, \ldots, U_n$. Taking refinements if necessary, we may assume that $U_1 \cap U_2 \cap \cdots \cap U_n \simeq \mathbb{C}$ so as to choose a 2-form τ with support contained in $U_1 \cap \cdots U_n$ and

$$\int_X \tau = 1$$

. Then

$$\mu = \rho_1\mu + \rho_2\mu + \cdots \rho_n\mu,$$

where $\{\rho_i\}_{i=1}^n$ is a partition of unity subordinate to $U_1, \ldots, U_n$. We may denote the integral of $\rho_1\mu$ on X by I, and note that

$$I = \int_X \rho_1\mu = -\int_X (\rho_2 + \cdots + \rho_n)\mu$$

since $\int_X \mu = 0$. Then $\rho_1\mu - I\tau$ and $(\rho_2 + \cdots \rho_n)\mu + I\tau$ are two forms with support in U_1 and $V = U_2 \cup \cdots \cup U_n$, and with integral 0.

Now we show the existence of ξ by induction on n. For U_1, using the result of **Exercise 1**, there exists a 1-form α of compact support such that $\rho_1\mu - I\tau = d\alpha$. For $V = U_2 \cup \cdots U_n$, by the inductive hypothesis, there is another 1-form β with compact support in V with $(\rho_2 + \cdots \rho_n)\mu + I\tau = d\beta$. Then take $\xi = \alpha + \beta$, we have

$$\mu = d(\alpha + \beta) = d\xi,$$

completing the proof. □

Exercise 14

Prove that

Theorem 2. Let X be a Riemann surface, and γ_1, γ_2 be two transversal smooth simply closed curves in X. Then

$$\gamma_1 \cdot \gamma_2 = \int_{\gamma_1} \eta_{\gamma_2},$$

where η_{γ_1} is the Poincaré dual of γ_1.

Proof. Given any two transversal simply closed curves γ_1, γ_2 in X, for any $p \in L \cap S$, we can find a local chart (U_p, x, y) of containing p such that

$$U_p \cap \gamma_2 = \{(x, y) | y = 0\},$$
$$U_p \cap \gamma_1 = \{(x, y) | x = 0\}$$

by transversality. Moreover, the orientation of $U_p \cap \gamma_2$ is determined by dx while the orientation of $U_p \cap \gamma_1$ is determined by dy. Also from transversality we know that $\dim \gamma_1 \cap \gamma_2 = 0$, and since X is compact, the cardinality of $\gamma_1 \cap \gamma_2$ is finite. So the intersection number $\gamma_1 \cdot \gamma_2$ is well-defined.

We may construct a tubular neighborhood N of γ_2. For each point $p \in \gamma_1 \cap \gamma_2$, let

$$N_p := U_p \cap \gamma_1 \cap N.$$

With this notation, we have

$$\gamma_1 \cap N = \cup_{p\in\gamma_1\cap\gamma_2} N_p.$$

Also note that each N_p is endowed with an orientation induced by the orientation of N and γ_2 in X.

Then take η_{γ_1} and η_{γ_2} to be the Poincaré duals of γ_1 and γ_2, respectively. Since η_{γ_2} is a 1-form with support in N and η_{γ_1} has its support in a tubular neighborhood of γ_1, the 2-form $\eta_{\gamma_1} \wedge \eta_{\gamma_2}$ is supported only in $\gamma_1 \cap N = \cup_{p\in\gamma_1\cap\gamma_2} N_p$. So we have

$$\int_X \eta_{\gamma_1} \wedge \eta_{\gamma_2} = \int_{\gamma_1} \eta_{\gamma_2} = \sum_{p\in\gamma_1\cap\gamma_2} \epsilon(p) \int_{N_p} \eta_{\gamma_2},$$

where $\epsilon(p) \in \{\pm\}$ are numbers making the orientations of $\int_{N_p} \eta_{\gamma_2}$ and $\int_X \eta_{\gamma_1} \wedge \eta_{\gamma_2}$ compatible. With some proper normalization of η_{γ_2}, the above equation becomes

$$\int_{\gamma_1} \eta_{\gamma_2} = \sum_{p\in\gamma_1\cap\gamma_2} \epsilon(p) = \gamma_1 \cdot \gamma_2,$$

completing the proof. □

Exercise 15

Show that

Theorem 3. Assume $\{\alpha_i, \beta_i\}_{i=1}^g$ is a canonical basis of $H_1(X;\mathbb{Z})$a. For any closed $\xi, \eta \in \Omega^1(X)$, we have

$$\int_X \xi \wedge \eta = \sum_{i=1}^g \left(\int_{\alpha_i} \xi \int_{\beta_i} \eta - \int_{\alpha_i} \eta \int_{\beta_i} \xi\right).$$

Proof. We have showed in **Exercise 3** that for any two transversal smooth simply closed curves γ_1, γ_2, their intersection number has the following relation with their Poincaré duals:

$$\omega(\gamma_1, \gamma_2) := \gamma_1 \cdot \gamma_2 = \int_{\gamma_2} \eta_{\gamma_1} = \int_X \eta_{\gamma_2} \wedge \eta_{\gamma_1}. \tag{11}$$

In other words, the symplectic form $\int_X () \wedge (-)$ on $H^1(X;\mathbb{C})$ induces a symplectic form ω on the symplectic vector space $H_1(X;\mathbb{C})$, via the Poincaré duality $H^1(X;\mathbb{C}) \simeq H_1(X;\mathbb{C})$. We will exploit (11) to compute the integral $\int_X \xi \wedge \eta$.

Since the integral $\int_X \xi \wedge \eta$ only depends on the cohomology classes $[\xi], [\eta]$ of the 1-forms ξ and η, and denote the Poincaré duals of $[\xi], [\eta]$ in $H_1(X;\mathbb{C})$ by $p^{-1}([\xi]), p^{-1}([\eta])$, we have

$$\int_X \xi \wedge \eta = \int_X [\xi] \wedge [\eta] = p^{-1}([\xi]) \cdot p^{-1}([\eta]).$$

Provided a canonical basis $\{\alpha_i, \beta_i\}_{i=1}^g$ of $H_1(X;\mathbb{C})$, we can expand $p^{-1}([\xi])$ and $p^{-1}([\eta])$ in terms of this basis

$$p^{-1}([\xi]) = \sum_{i=1}^g f^i \alpha_i + \sum_{i=1}^g g^i \beta_i,$$
$$p^{-1}([\eta]) = \sum_{i=1}^g u^j \alpha_j + \sum_{i=1}^g v^j \beta_j.$$

Moreover, the coefficients f^i, g^i, u^j, v^j can be computed by means of the symplectic form under the canonical basis,

$$f^i = \omega(p^{-1}([\xi]), \beta_i) = p^{-1}([\xi]) \cdot \beta_i = \int_{\beta_i} [\xi] = \int_{\beta_i} \xi,$$
$$g^i = \omega(\alpha_i, p^{-1}([\xi])) = -\omega(p^{-1}([\xi]), \alpha_i) = -\int_{\alpha_i} [\xi] = -\int_{\alpha_i} \xi,$$

and similarly

$$u^j = \omega(p^{-1}([\eta]), \beta_j) = \int_{\beta_j} \eta,$$
$$v^j = \omega(\alpha_j, p^{-1}([\eta])) = -\int_{\alpha_i} \eta.$$

So we have

$$\begin{aligned}
\int_X \xi \wedge \eta &= p^{-1}([\xi]) \cdot p^{-1}([\eta]) \\
&= \omega(\sum_{i=1}^g f^i \alpha_i + \sum_{i=1}^g g^i \beta_i, \sum_{i=1}^g u^j \alpha_j + \sum_{i=1}^g v^j \beta_j) \\
&= \sum_{i=1}^g \sum_{j=1}^g f^i v^j \delta_{ij} + \sum_{i=1}^g \sum_{j=1}^g g^i u^j (-\delta_{ij}) \\
&= \sum_{i=1}^g f^i v^i - \sum_{i=1}^g g^i u^i \\
&= \sum_{i=1}^g (\int_{\alpha_i} \xi \int_{\beta_i} \eta - \int_{\beta_i} \xi \int_{\alpha_i} \eta)
\end{aligned}$$

□

Exercise 16

Assume that $f \in L^1_{\text{loc}}$ satisfies $\frac{\partial f}{\partial \bar{z}} = 0$, prove that $f \in C^\infty(\mathbb{C})$, hence $f \in \mathcal{O}(D)$.

Proof. Since $\partial f / \partial \bar{z} = 0$, we have

$$-\int_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}} \frac{\partial g}{\partial z} = 0 \tag{12}$$

for any smooth function $g \in C^\infty(\mathbb{C})$. However, by definition, (12) says that

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{1}{2\mathrm{i}} \Delta f = 0.$$

By Weyl's Lemma, f is smooth, and since $\partial f / \partial \bar{z} = 0$ it is holomorphic. □

Exercise 17

Assume $f \in L^1_{\text{loc}}$ satisfies $\frac{df}{dx} = 0$, prove that $f = c$ almost everywhere on $\mathbb{R}$, where c is a constant.

Proof. This exercise is optional for those who have little acquaintance with real analysis. Unfortunately, I had never taken any course covering stuffs like measure theory or distribution theory. So I omit this exercise. It's really a shame. □

Exercise 18

Show that $\xi \in \Omega^1(X)$ iff locally ξ is the differential of a harmonic function.

Proof. $\Leftarrow$ Suppose $\xi = df$ with f harmonic. We want to show that ξ is holomorphic, that is,

$$\bar{\partial}\xi = 0. \tag{13}$$

Applying $\bar{\partial}$ on ξ we have

$$\bar{\partial}\xi = \bar{\partial}df = \bar{\partial}(\partial + \bar{\partial})f = \bar{\partial}\partial f,$$

but $\bar{\partial}\partial f = 0$ holds manifestly as f is harmonic. So we have shown that if ξ is the differential of a harmonic function f it is then holomorphic.

$\Rightarrow$ Suppose ξ is holomorphic, we need to find a harmonic function f such that $\xi = df$. In any local coordinate $(U, z = x + \mathrm{i}y)$, we have

$$\xi|_U = a(z)dz$$

with $a(z)$ a holomorphic function on $U \simeq \mathbb{C}$. We may take

$$f(z) := \int_\gamma a(z)dz,$$

where γ is any smooth curve in $U \simeq \mathbb{C}$ connecting 0 and z. By the monodromy theorem f is independent of the choice of γ, and the construction of f assures that $df = \xi|_U$. Now left to us is to show that f is harmonic on U. Indeed,

$$2\mathrm{i}\bar{\partial}\partial f = 2\mathrm{i}\bar{\partial}a = 0$$

since a is holomorphic on U, which completes the proof. □

Exercise 19

Assume that $f : D \to D'$ is a harmonic map between two domains in $\mathbb{C}$, and $u : D' \to \mathbb{C}$ is a harmonic function, then $u \circ f : D \to \mathbb{C}$ is a harmonic function.

Proof. Take the coordinates on D', D to be $z' = x' + \mathrm{i}y'$ and $z = x + \mathrm{i}y$, respectively. Since the coordinate transformation relating z' and z is holomorphic, the Cauchy-Riemann relation

$$\begin{cases} \frac{\partial x'}{\partial x} = \frac{\partial y'}{\partial y}, \\ \frac{\partial x'}{\partial y} = -\frac{\partial y'}{\partial x}, \end{cases} \tag{14}$$

holds. By assumption, u is harmonic on D'

$$\frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2} = 0, \tag{15}$$

we what to show that $u^* = u \circ f$ is harmonic on D, *id est*,

$$\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2} = 0$$

holds.

The proof is rather direct. First note that

$$\begin{aligned}\frac{\partial u^*}{\partial x} &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x},\\ \frac{\partial u^*}{\partial y} &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y}.\end{aligned}$$

Then by derivation again the applying the chain rules, we have

$$\frac{\partial^2 u^*}{\partial x^2} = \frac{\partial^2 u}{\partial x'^2}(\frac{\partial x'}{\partial x})^2 + \frac{\partial^2 u}{\partial y'^2}(\frac{\partial y'}{\partial x})^2 + 2\frac{\partial^2 u}{\partial x'\partial y'}\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial x} + \frac{\partial u}{\partial x'}\frac{\partial^2 x'}{\partial x^2} + \frac{\partial u}{\partial y'}\frac{\partial^2 y'}{\partial x^2}, \tag{16}$$

and

$$\frac{\partial^2 u^*}{\partial y^2} = \frac{\partial^2 u}{\partial x'^2}(\frac{\partial x'}{\partial y})^2 + \frac{\partial^2 u}{\partial y'^2}(\frac{\partial y'}{\partial y})^2 + 2\frac{\partial^2 u}{\partial x'\partial y'}\frac{\partial x'}{\partial y}\frac{\partial y'}{\partial y} + \frac{\partial u}{\partial x'}\frac{\partial^2 x'}{\partial y^2} + \frac{\partial u}{\partial y'}\frac{\partial^2 y'}{\partial y^2}. \tag{17}$$

Plugging the Cauchy-Riemann relation (14) into (16) and (17), the equations become

$$\frac{\partial^2 u^*}{\partial x^2} = \frac{\partial^2 u}{\partial x'^2}(\frac{\partial x'}{\partial x})^2 + \frac{\partial^2 u}{\partial y'^2}(\frac{\partial x'}{\partial y})^2 - 2\frac{\partial^2 u}{\partial x'\partial y'}\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial x'}\frac{\partial^2 y'}{\partial x\partial y} - \frac{\partial u}{\partial y'}\frac{\partial^2 x'}{\partial x\partial y},$$

and

$$\frac{\partial^2 u^*}{\partial y^2} = \frac{\partial^2 u}{\partial x'^2}(\frac{\partial x'}{\partial y})^2 + \frac{\partial^2 u}{\partial y'^2}(\frac{\partial x'}{\partial x})^2 + 2\frac{\partial^2 u}{\partial x'\partial y'}\frac{\partial x'}{\partial y}\frac{\partial x'}{\partial x} - \frac{\partial u}{\partial x'}\frac{\partial^2 y'}{\partial x\partial y} + \frac{\partial u}{\partial y'}\frac{\partial^2 x'}{\partial x\partial y}.$$

Summing up both sides of the above equations, we have

$$\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2} = [(\frac{\partial x'}{\partial x})^2 + (\frac{\partial x'}{\partial y})^2](\frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2}) = 0,$$

completing the proof.

□

Exercise 20

Show that $(\eta, \xi) = \overline{(\xi, \eta)}$ and $(\star\xi, \star\eta) = (\xi, \eta)$.

Proof. We first show the second identity $(\star\xi, \star\eta) = (\xi, \eta)$, then use the second identity to prove the first identity.

Since (ξ, η) is a coordinate-free object, we prove $(\star\xi, \star\eta) = (\xi, \eta)$ with the help of coordinates. For any two measurable 1-forms ξ, η on X, we may find a local coordinate $(U, z = x + \mathrm{i}y)$, in which we can express

$$\begin{aligned}\xi|_U &= adx + bdy,\\ \eta_U &= fdx + gdy,\end{aligned}$$

with a, b, f, g distribution-valued function on $U \simeq \mathbb{C}$. So locally we have

$$\begin{aligned}(\star\xi, \star\eta) &= \int_X (\star\xi) \wedge \overline{\star^2\eta} = -\int_X (\star\xi) \wedge \overline{\eta} = \int_X \overline{\eta} \wedge (\star\xi)\\ &= \int_X (fdx - gdy) \wedge (ady - bdx)\\ &= \int_X (af - bg)dx \wedge dy.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\xi,\eta) &= \int_X \xi \wedge \overline{\star\eta} \\ &= \int_X (adx + bdy) \wedge (fdy + gdx) \\ &= \int_X (af - bg) dx \wedge dy.\end{aligned}$$

So the second identity

$$(\star\xi, \star\eta) = (\xi, \eta) \tag{18}$$

has been proved.

In order to prove the first identity, we notice that

$$\star\overline{\eta} = -\overline{\star\omega} \tag{19}$$

for any $\eta \in L_1^2(X)$. Indeed, locally we have

$$\star\overline{\eta} = \star(fdx - gdy) = -(fdy + gdx)$$

and

$$\overline{\star\eta} = \overline{fdy - gdx} = fdy + gdx,$$

thus (19) holds.

Finally, for any $\xi, \eta \in L_1^2(X)$, we have

$$\begin{aligned}\overline{(\xi,\eta)} &= \overline{\int_X \xi \wedge \overline{\star\eta}} \\ &= \int_X \overline{\xi} \wedge (\star\eta) \\ &= -\int_X (\star\eta) \wedge \overline{\xi} \\ &\overset{\star^2=-1}{=} \int_X (\star\eta) \wedge \star^2\overline{\xi} \\ &\overset{(19)}{=} -\int_X (\star\eta) \wedge \star(\overline{\star\xi}) \\ &\overset{(19)}{=} \int (\star\eta) \wedge \overline{\star(\star\xi)} \\ &= (\star\eta, \star\xi) \\ &\overset{(18)}{=} (\eta, \xi),\end{aligned}$$

proving the first identity.

□

Exercise 21

Show that p given in the proof of Theorem 2.4.1 is smooth.

Proof. Since smoothness is a local property, it is enough to show that p is smooth near $\forall z \in X$. Let $(U, z = x + iy$ be an arbitrary coordinate chart containing z, such that $\xi_U = pdx + qdy$ in this local coordinate. By the assumption that $d\xi = d \star \xi = 0$, for any distribution $\phi \in \mathcal{D}(U)$, we have

$$\begin{aligned}0 = \int_X \frac{\partial\phi}{\partial y} \wedge d\xi = -\int_X d(\frac{\partial\phi}{\partial y}) \wedge \xi, \\ 0 = \int_X \frac{\partial\phi}{\partial x} \wedge d\star\xi = -\int_X d(\frac{\partial\phi}{\partial x}) \wedge \star\xi.\end{aligned} \tag{20}$$

On the other hand, we can express the first equality in (20) in coordinates x, y as

$$
\begin{aligned}
0 &= \int_X d(\frac{\partial \phi}{\partial y}) \wedge \xi \\
&= \int_X (\frac{\partial^2 \phi}{\partial x \partial y} dx + \frac{\partial^2 \phi}{\partial y^2} dy) \wedge (pdx + qdy) \\
&= \int_X (q\frac{\partial^2 \phi}{\partial x \partial y} - p\frac{\partial^2 \phi}{\partial y^2}) dx \wedge dy,
\end{aligned} \tag{21}
$$

as well as the second equality

$$
\begin{aligned}
0 &= \int_X d(\frac{\partial \phi}{\partial x}) \wedge \star\xi \\
&= \int_X (\frac{\partial^2 \phi}{\partial x^2} dx + \frac{\partial^2 \phi}{\partial x \partial y} dy) \wedge (pdy - qdx) \\
&= \int_X (p\frac{\partial^2 \phi}{\partial x^2} + q\frac{\partial^2 \phi}{\partial x \partial y}) dx \wedge dy.
\end{aligned} \tag{22}
$$

Then we subtract (22) by (21) to get

$$0 = \int_X p(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}) dx \wedge dy = p\Delta\phi dx \wedge dy,$$

which implies that

$$0 = \int_X \Delta p \phi dx \wedge dy$$

holds for any $\phi \in \mathcal{D}(U)$. Hence $\Delta p = 0$, by Weyl's Lemma, p is smooth. So the assertion follows. $\square$

Exercise 22

Let X be a compact Riemann surface, show that there is a natural linear isomorphism between $H^{1,1}(X)$ and $H^2(X; \mathbb{C})$

Proof. By definition, we have

$$H^{1,1}(X) := \operatorname{coker}(\bar{\partial} : \mathcal{E}^{1,0} \to \mathcal{E}^2)$$

and

$$H^2(X; \mathbb{C}) := \operatorname{coker}(d : \mathcal{E}^1 \to \mathcal{E}^{\in}).$$

Now we define a map

$$
\begin{aligned}
\phi : H^{1,1} &\to H^2(X; \mathbb{C}), \\
[\xi]_D &\mapsto [\xi]_{dR}
\end{aligned}
$$

where $[\xi]_D$ is the equivalent class of $\xi \in \mathcal{E}^2$ in $H^{1,1}(X)$, and $[\xi]_{dR}$ is the equivalent class of ξ in $H^1(X; \mathbb{C})$. Now we show that this map is well defined. Indeed, taking a representative $\xi + \bar{\partial}\eta$ of $[\xi]_D$, we note that $\bar{\partial}\eta = d\eta$ as $\eta \in \mathcal{E}^{0,1}$, so we have $[\xi + \bar{\partial}\eta]_{dR} = [\xi + d\eta]_{dR} = [\xi]_{dR}$, showing that ϕ is independent of η.

Now we show that ϕ is surjective and injective. For surjectivity, choose any $[\sigma]_{dR}$, we want to show that $\phi([\sigma]_D) = [\sigma]_{dR}$. Indeed, we choose any boudary $d\alpha$. Observe that

$$\int_X d\alpha = 0,$$

so by the $\partial\bar{\partial}$-Lemma, we have $d\alpha = \bar{\partial}\partial f$ for some smooth function f. Thus

$$[\sigma + d\alpha]_D = [\sigma + \bar{\partial}\partial f]_D = [\sigma]_D,$$

showing that $\phi([\sigma]_D) = [\sigma]_{dR}$. For injectivity, if there is some $[\beta]_D$ with

$$\phi([\beta]_D) = 0,$$

we have to show that $[\beta]_D = 0$, that is, $\beta = \bar{\partial}\gamma$ for some $\gamma \in \mathcal{E}^{1,0}$. Note that

$$\int_X \beta = \int_X [\beta] = 0,$$

as exact forms don't contribute to the integral, so we can use the powerful $\partial\bar{\partial}$-Lemma again to conclude that $\beta = \bar{\partial}\partial g$ for some smooth function g. Taking $\gamma = \partial g \in \mathcal{E}^{0,1}$, the proof is complete. □

Exercise 23

Let X be a Riemann surface, $p_1, \ldots, p_r \in X, n_1, \ldots, n_r \geq 1$. If $\dim H^{0,1}(X) < n_1 + \cdots + n_r$, then there exists non-constant $f \in \mathcal{O}(X \backslash \{p_1, \ldots, p_r\}) \cap (X)$, such that $\mathrm{ord}_f(p_i) \geq -n_i, i = 1, \ldots, r$.

Proof. The proof is essentially the same as that of Theorem 2.5.1. First we take (U_i, z_i) to be the local coordinates of p_i, with $z_i(p_i) = 0$ for $i = 1, \ldots, r$. For each i, we can take $\rho^{(i)} \in C^\infty(X)$ with $\mathrm{supp}\rho^{(i)} \subseteq U_i$ and $\rho^{(i)} \equiv 1$ near p_i, further we define

$$\rho_j^{(i)} := \rho^{(i)} \frac{1}{(z_i)^j}, 1 \leq j \leq n_i.$$

and

$$\xi_j^{(i)} := \bar{\partial}\phi_j^{(i)}.$$

Since $\dim H^{0,1}(X) < n_1 + \cdots + n_r$, the classes $[\xi_1^{(1)}], \ldots, [\xi_{n_1}^{(1)}], [\xi_1^{(2)}], \ldots, [\xi_{n_2}^{(2)}], \ldots, [\xi_1^{(r)}], \ldots, [\xi_{n_r}^{(r)}]$ are linearly independent in $H^{0,1}(X)$. So we can find constants $a_1^{(1)}, \ldots, a_{n_1}^{(1)}, a_1^{(2)}, \ldots, a_{n_2}^{(1)}, \ldots, a_1^{(r)}, \ldots, a_{n_r}^{(r)}$, such that

$$\sum_{i=1}^{r} \sum_{j=1}^{n_i} a_j^{(i)} [\xi_j^{(i)}] = [\sum_{i=1}^{r} \sum_{j=1}^{n_i} a_j^{(i)} \xi_j^{(i)}] = 0 \in H^{0,1}(X).$$

Thus there is some holomorphic function g satisfying

$$\sum_{i=1}^{r} \sum_{j=1}^{n_i} a_j^{(i)} \bar{\partial}\phi_j^{(i)} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} a_j^{(i)} \xi_j^{(i)} = \bar{\partial}g.$$

Finally we take

$$f := \sum_{i=1}^{r} \sum_{j=1}^{n_i} a_j^{(i)} \phi_j^{(i)} - g,$$

which satisfies $\bar{\partial}f = 0$ on X except $p_1, \ldots, p_n$. Thus $f \in \mathcal{O}(X \backslash \{p_1, \ldots, p_r\})$, and the principal part of f at each p_i is by construction

$$\frac{a_{n_i}^{(i)}}{(z_i)^{n_i}} + \cdots \frac{a_1^{(i)}}{z_i},$$

completing the proof. □

Exercise 24

Prove Theorem 2.5.5.

Proof. We first let

$$\eta := \xi_1 + \cdots + \xi_r.$$

By assumption, we can see that ξ is a meromorphic 1-form so we have

$$d\eta = \bar{\partial}\eta.$$

Then consider the integral of $\bar{\partial}\xi$ over X, we have

$$\int_X \bar{\partial}\eta = \int_X d\eta = \sum_{i=1}^{r} \int_X d\xi_i = \sum_{i=1}^{r} \int_{X \setminus D_i} d\xi_i = -\sum_{i=1}^{r} \int_{D_i} \xi_i = -\sum_{i=1}^{r} \mathrm{Res}_{\xi_i}(p_i) = 0,$$

by assumption and the definition of residue. Then by the $\partial\bar{\partial}$-Lemma we have $\bar{\partial}\eta = \bar{\partial}\partial f$ for some $f \in C^\infty(X)$, namely,

$$\eta - \partial f = \xi$$

with ξ a meromorphic 1-form on X with poles at $p_1, \ldots, p_r$. On each D_i, $\xi - \xi_i$ is smooth by construction, completing the proof. □

Exercise 25

Show that $\ker f$ is a sheaf on X, and $(\ker f)_x = \ker f_x$.

Proof. To show that $\ker f$ is a sheaf, we begin with the observation that there are restriction maps

$$\mathrm{res}_V^U : (\ker f)(U) \to (\ker f)(V)$$

for all open subsets $V \subseteq U$. Indeed, for any $s \in (\ker f)(U)$, we have

$$f|_V(\mathrm{res}_V^U(s)) = f|_V(s|_V) = f(s)|_V = 0|_V = 0,$$

showing that $U|_V(s)$ is an element of $(\ker f)(V)$. So $\mathrm{res}_V^U : (\ker f)(U) \to (\ker f)(V)$ are well-defined. Since the restriction maps are induced from the restriction maps of the sheaf $\mathcal{F}$, the relations

$$\mathrm{res}_W^V \circ \mathrm{res}_V^U = \mathrm{res}_W^U$$

hold automatically for all open sets $W \subseteq V \subseteq U$ of X.

Next we have to show that the presheaf $\ker f$ satisfies the gluing property, say, for any open set $U \subseteq X$ and any open cover $\{U_i\}$ of U, if there are $s_i \in (\ker f)(U_i)$ satisfying $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there is a unique $s \in (\ker f)(U)$ such that $s|_{U_i} = s_i$. By the gluing property of $\mathcal{F}$, there is a unique $s \in \mathcal{F}(U)$ satisfying $s|_{U_i} = s_i$, as $(\ker f)(U_i) := \ker f|_{\mathcal{F}(U_i)} \subseteq \mathcal{F}(U_i)$ by definition. What left to us is to show that $s \in (\ker f)(U)$, that is,

$$f(s) = 0.$$

Since $f(s)$ is a section of $\mathcal{G}(U)$, we have

$$f(s)|_{U_i} = f|_{U_i}(s|_{U_i}) = f|_{U_i}(s_i) = 0.$$

By the gluing property of $\mathcal{G}$, $f(s) = 0$ in $\mathcal{G}(U)$, as desired.

Finally, we are asked to compute the stalk of the sheaf $\ker f$ at an arbitrary point $x \in X$. By definition, we have

$$(\ker f)_x = \operatorname*{colim}_{U \ni x} (\ker f)(U) = \operatorname*{colim}_{U \ni x} \ker f|_{\mathcal{F}(U)}.$$

Since $\{\mathcal{F}(U)\}_{x\in U}$ is a filtered direct system of abelian groups, by the fact that filtered colimits are exact, we have

$$\operatorname*{colim}_{U\ni x} \ker f|_{\mathcal{F}(U)} = \ker \operatorname*{colim}_{U\ni x} f|_{\mathcal{F}(U)} = \ker f_x,$$

which completes the proof. □

Exercise 26

(a) Show that for any $f \in C_c^\infty(\Delta)$, there exists $u \in C^\infty(\Delta)$ such that $\bar\partial u = f(z)d\bar z$, *id est*, $\frac{\partial u}{\partial \bar z} = f$.

(b) Prove the exactness ofthe sequence

$$0 \to \mathcal{O} \to \mathcal{E}^{0,0} \xrightarrow{\bar\partial} \mathcal{E}^{0,1} \to 0, \tag{23}$$

and

$$0 \to \Omega \to \mathcal{E}^{0,1} \xrightarrow{\bar\partial} \mathcal{E}^{1,1} \to 0. \tag{24}$$

Proof. **(a)** Let

$$u(z) := \frac{1}{2\pi \mathrm{i}} \int_\Delta \frac{f(w)}{w-z} dw \wedge d\bar w.$$

We need to show that $u(z)$ is smooth for all $z \in \Delta$, and $\partial u/\partial \bar z = f$. We show these by introducing the polar coordinates as follows

$$w = z + re^{\mathrm{i}\theta}.$$

Under the polar coordinates, we have

$$dw \wedge d\bar w = -2\mathrm{i} r dr \wedge d\theta.$$

Thus

$$\begin{aligned} u(z) &= -\frac{1}{\pi} \int_\Delta \frac{f(z+re^{\mathrm{i}\theta})}{re^{\mathrm{i}\theta}} r dr \wedge d\theta \\ &= -\frac{1}{\pi} \int_\Delta f(z+re^{\mathrm{i}\theta}) e^{-\mathrm{i}\theta} dr \wedge d\theta. \end{aligned}$$

Since f has compact support in Δ, the above integral converges and depends smoothly on $z \in \Delta$. Hence u is a smooth function in x, y via $z = x + \mathrm{i}y$, so is the partial derivative $\partial u/\partial \bar z$. Now back to the $w, \bar w$ coordinates, we have

$$\begin{aligned} (\frac{\partial u}{\partial \bar z})(z) &= \frac{1}{2\pi \mathrm{i}} \int_\Delta \frac{\partial}{\partial \bar z} (\frac{f(w)}{w-z}) dw \wedge d\bar w \\ &= \frac{1}{2\pi \mathrm{i}} \int_\Delta \frac{\partial}{\partial \bar w} (\frac{f(w)}{w-z}) dw \wedge d\bar w \\ &= \int_\Delta d\mu, \end{aligned}$$

where

$$\mu(w) = -\frac{1}{2\pi \mathrm{i}} \frac{f(w)}{w-z} dw.$$

By the previous argument, we already know that $\partial u/\partial \bar z(z)$ is a smooth function of z, so the limit

$$\lim_{\epsilon \to 0} \int_{\Delta - B_\epsilon(z)} d\mu$$

converges to it, with $B_\epsilon(z)$ the small open ball centered at z with radius ϵ. By the Stokes' Theorem,

$$\begin{aligned}
(\frac{\partial u}{\partial \bar{z}})(z) &= \lim_{\epsilon\to 0}\int_{\Delta - B_\epsilon(z)} d\mu \\
&= -\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(z)} \mu \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi\mathrm{i}}\int_{|w-z|=\epsilon}\frac{f(w)}{w-z}dw \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi\mathrm{i}}\int_0^{2\pi}\frac{f(z+\epsilon e^{\mathrm{i}\theta})}{\epsilon e^{\mathrm{i}\theta}}\epsilon\mathrm{i}e^{\mathrm{i}\theta}d\theta \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^{2\pi} f(z+\epsilon e^{\mathrm{i}\theta})d\theta \\
&= f(z).
\end{aligned}$$

The above analysis shows that the Cauchy-Riemann equation

$$\frac{\partial u}{\partial \bar{z}} = f$$

always has a local solution u on Δ.

(b) The sequences (23) and (24) are exact iff they are exact stalk-wise. It suffices to show that

$$0 \to \mathcal{O}_x \to \mathcal{E}_x^{0,0} \xrightarrow{\bar{\partial}} \mathcal{E}_x^{0,1} \to 0$$

and

$$0 \to \Omega_x \to \mathcal{E}_x^{0,1} \xrightarrow{\bar{\partial}} \mathcal{E}_x^{1,1} \to 0$$

are exact for all x. Equivalently, if we pick a small enough neighborhood U_x of x, we need to show that

$$0 \to \mathcal{O}(U_x) \to \mathcal{E}^{0,0}(U_x) \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1}(U_x) \to 0 \tag{25}$$

and

$$0 \to \Omega(U_x) \to \mathcal{E}^{0,1}(U_x) \xrightarrow{\bar{\partial}} \mathcal{E}^{1,1}(U_x) \to 0 \tag{26}$$

are exact.

For (25), exactness at $\mathcal{E}^{0,0}(U_x)$ is obvious, since a function f is holomorphic at x iff $\bar{\partial}f(x) = 0$, by definition. **(a)** tells us that $\bar{\partial} : \mathcal{E}^{0,0}(U_x) \to \mathcal{E}^{0,1}(U_x)$ is surjective, since U_x is small enough, any element in $\mathcal{E}^{0,1}(U_x)$ is of the form $g(z)d\bar{z}$ with $g(z)$ viewed as a compactly supported function on Δ. The exactness of (26) holds for the same reason. □

Exercise 27

Let X be a Riemann surface and $\mathbb{C}_p$ be the skyscraper sheaf on X based at $p \in X$, then $H^1(X;\mathbb{C}_p) = 0$.

Proof. Take any open cover $\mathfrak{U}$ of X, we claim that it has a refinement $\mathfrak{U}'$ such that there is exactly one open subset containing p in $\mathfrak{U}'$. Indeed, $p \in U_0$ for some U_0 in $\mathfrak{U}$. For any other $U_i \in \mathfrak{U}$, we let $U_i' = U_i - \{p\}$ and let $\mathfrak{U}' = \{U_0, U_i'\}$, as desired. So we may assume that any open cover $\mathfrak{U}$ has only one open subset U_0 containing p. Thus

$$\check{C}^0(\mathfrak{U}, \mathbb{C}_p) = \prod_i \mathbb{C}_p(U_i) = \mathbb{C}_p(U_0) = \mathbb{C},$$

and

$$\check{C}^1(\mathfrak{U}, \mathbb{C}_p) = \prod_{i,j} \mathbb{C}_p(U_i \cap U_j) = \mathbb{C}_p(U_0 \cap U_0) = \mathbb{C},$$
$$\check{C}^2(\mathfrak{U}, \mathbb{C}_p) = \prod_{i,j,k} \mathbb{C}_p(U_i \cap U_j \cap U_k) = \mathbb{C}_p(U_0 \cap U_0 \cap U_0) = \mathbb{C}.$$

And the differentials $\check{C}^0(\mathfrak{U}, \mathbb{C}_p) \xrightarrow{d^0} \check{C}^1(\mathfrak{U}, \mathbb{C}_p) \xrightarrow{d^1} \check{C}^2(\mathfrak{U}, \mathbb{C}_p)$ are essentially induced by $\mathrm{id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$. So we have

$$H^1(\mathfrak{U}; \mathbb{C}_p) = \ker d^1 / \operatorname{im} d^0 = \ker \mathrm{id}_{\mathbb{C}} / \operatorname{im} \mathrm{id}_{\mathbb{C}} = 0.$$

Thus

$$H^1(X; \mathbb{C}_p) = \operatorname*{colim}_{\mathfrak{U}} H^1(\mathfrak{U}; \mathbb{C}_p) = \operatorname*{colim}_{\mathfrak{U}} 0 = 0$$

□

Exercise 28

Show that $H^1(X; \mathbb{Z}_X) = 0$.

Proof. Let $\mathfrak{U}$ be an open cover of X, and $(a_{ij}) \in Z^1(\mathfrak{U}; \mathbb{Z}_X)$ be a 1-cocycle. We want to show that (a_{ij}) is in fact a coboundary, that is, there is $(a_i) \in C^0(\mathfrak{U}; \mathbb{Z}_X)$ such that

$$a_{ij} = a_i - a_j$$

on each $U_i \cap U_j$. To find such (a_i), we observe that $\mathbb{Z}_X$ is a subsheaf of $\mathbb{C}_X$. Since $H^1(X; \mathbb{C}_X) = 0$, $(a_{ij}) \in Z^1(\mathfrak{U}; \mathbb{Z}_X) \subseteq Z^1(\mathfrak{U}; \mathbb{C}_X)$ is in fact a coboundary, thus there is some $(c_i) \in C^1(\mathfrak{U}; \mathbb{C}_X)$ with

$$a_{ij} = c_i - c_j$$

on each $U_i \cap U_j$. Note that a_{ij} are all integer-valued, then

$$\exp 2\pi \mathrm{i} a_{ij} = 1,$$

implying that

$$\exp 2\pi \mathrm{i} c_i = \exp 2\pi \mathrm{i} c_j$$

on each $U_i \cap U_j$. So all $\exp 2\pi \mathrm{i} c_i$ determine a global section b such that

$$b|_{U_i} = \exp 2\pi \mathrm{i} c_i$$

for all U_i. Since X is connected and $\mathbb{C}_X$ is locally constant, the set of global sections $\Gamma(X, \mathbb{C}_X)$ is isomorphic to $\mathbb{C}$. Thus b *a fortiori* an element of $\mathbb{C}^*$. Take $c \in \mathbb{C}$ such that

$$\exp 2\pi \mathrm{i} c = b,$$

then we consider all

$$a_i := c_i - c.$$

Since $\exp 2\pi \mathrm{i} a_i = \exp 2\pi \mathrm{i} c_i \exp -2\pi \mathrm{i} c = bb^{-1} = 1$, (a_i) is an element of $C^0(\mathfrak{U}; \mathbb{Z}_X)$. Also note that

$$a_{ij} = c_i - c_j = (c_i - c) - (c_j - c) = a_i - a_j,$$

showing that (a_i) is our desired 0-cocycle. □

Exercise 29

Prove that the cohomological sequence induced by the short exact sequence is exact.

Proof. Given a short exact sequence

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0 \tag{27}$$

of sheaves, we are asked to show that the induced sequence

$$0 \to H^0(X;\mathcal{F}) \xrightarrow{\alpha} H^0(X;\mathcal{G}) \xrightarrow{\beta} H^0(X;\mathcal{H}) \xrightarrow{\sigma} H^1(X;\mathcal{F}) \xrightarrow{\alpha^1} H^1(X;\mathcal{G}) \xrightarrow{\beta^1} H^1(X;\mathcal{H}) \tag{28}$$

is exact. First of all, since we have

$$C^0(X;\mathcal{F}) = \Gamma(X;\mathcal{F})$$

by definition, and the functor $\Gamma(X;-)$ is left exact, we have that

$$0 \to H^0(X;\mathcal{F}) \xrightarrow{\alpha} H^0(X;\mathcal{G}) \xrightarrow{\beta} H^0(X;\mathcal{H})$$

is exact. What we need is to show the exactness of the rest of (28).

Before doing so, let us first reall the construction of the connecting morphism $\sigma : H^0(X;\mathcal{H}) \to H^1(X;\mathcal{F})$. Take any $h \in H^0(X;s\mathcal{H})$, if $\mathfrak{U}$ is fine enough, $\beta|_{U_i} : \mathcal{G}(U_i) \to \mathcal{H}(U_i)$ is surjective so there is a cochain $C^0(\mathfrak{U};\mathcal{G})$ making

$$\beta(g_i) = h|_{U_i}. \tag{29}$$

Hence $\beta(g_j - g_i) = 0$ on $U_i \cap U_j$. By the exactness of (27) there exists $(f_{ij}) \in C^1(\mathfrak{U};\mathcal{F})$ such that

$$\alpha(f_{ij}) = g_i - g_j \tag{30}$$

On $U_i \cap U_j \cap U_k$ we have

$$\alpha(f_{ij} + f_{jk} - f_{ik}) = g_i - g_j + g_j - g_k - (g_i - g_k) = 0,$$

again by the exactness of (27) we have $f_{ij} + f_{jk} - f_{ik} = 0$, so $(f_{ij}) \in Z^1(\mathfrak{U};\mathcal{F})$ is a cocycle. Now we take $\sigma(h)$ to be the cohomology class represented by (f_{ij}). σ is well-defined, as was checked in class.

$\operatorname{im}\beta \subseteq \ker\sigma$. If $g \in H^0(X;\mathcal{G})$ and $h = \beta(g)$. To construct $\sigma(h)$, we may choose the local lifts of h in (29) to be $g_i = g|_{U_i}$. Thus $\alpha(f_i j) = g|_{U_i} - g|_{U_j} = 0$ on $U_i \cap U_j$. Since $\alpha|_{U_i \cap U_j}$ is injective, we have $f_{ij} = 0$ and by definition $\sigma(h) = 0$, showing that $h \in \ker\sigma$.

$\ker\sigma \subseteq \operatorname{im}\beta$. Suppose $h \in \ker\sigma$. Since $\sigma(h) = 0$, we assume that it is represented by a coboundary $(f_{ij}) \in Z^1(\mathfrak{U};\mathcal{F})$, with $f_{ij} = f_i - f_j$, where $(f_i) \in C^0(\mathfrak{U};\mathcal{F})$. Let $\tilde{g}_i := g_i - \alpha(f_i)$, we have $\tilde{g}_i = \tilde{g}_j$ on $U_i \cap U_j$. Thus $\tilde{g}_i$ are restriction of some global section $g \in H^0(X;\mathcal{G})$. On each U_i we have $\beta(g)|_{U_i} = \beta(\tilde{g}_i) = \beta(g_i) - \beta(\alpha(f_i)) = h|_{U_i}$, showing that $h = \beta(g)$. Thus $h \in \operatorname{im}\beta$.

$\operatorname{im}\sigma \subseteq \ker\alpha^1$ Since $\sigma(h)$ is represented by (f_{ij}), $\alpha(\sigma(h))$ is represented by $(\alpha(f_{ij})) \in Z^1(\mathfrak{U};\mathcal{G})$. But by (30), $(\alpha(f_{ij}))$ is exact, so $\alpha(\sigma(h)) = 0$.

$\ker\alpha^1 \subseteq \operatorname{im}\sigma$. Suppose $\xi \in \ker\alpha^1$ is represented by the cocycle $(f_{ij}) \in Z^1(\mathfrak{U};\mathcal{F})$. Since $\alpha^1(\xi) = 0 \in H^1(X;\mathcal{G})$, there exists a cochain $(g_i) \in C^0(\mathfrak{U};\mathcal{G})$ such that $\alpha(f_{ij}) = g_i - g_j$ on $U_i \cap U_j$. This implies

$$0 = \beta(\alpha(f_{ij})) = \beta(g_j) - \beta(g_i).$$

Hence there exists $h \in H^0(X;\mathcal{H})$ such that $h|_{U_i} = \beta(g|_{U_i})$. By the construction of σ, we have $\sigma(h) = \xi$, showing that $\xi \in \operatorname{im}\sigma$.

$\operatorname{im}\alpha^1 \subseteq \ker\beta^1$ Since α^1, β^1 are induced by $\alpha\beta$, their composition $\beta^1 \circ \alpha^1$ is induced by $\beta \circ \alpha = 0$, thus is also zero.

$\ker \beta^1 \subseteq \operatorname{im} \alpha^1$. Suppose $\eta \in \ker \beta^1$ is resented by cocycle $(g_{ij}) \in Z^1(\mathfrak{U}; \mathcal{G})$. Since $\beta^1(\eta) = 0 \in H^1(X; \mathcal{H})$, there is a cochain $(h_i) \in C^0(\mathfrak{U}; \mathcal{H})$ such that $\beta(g_{ij}) = h_i - h_j$. Now suppose that the cover $\mathfrak{U}$ is fine enough that $\beta|_{U_i} : \mathcal{G}(U_i) \to \mathcal{H}(U_i)$ is surjective for all $U_i \in \mathfrak{U}$. So we can find $(g_i) \in C^0(\mathfrak{U}; \mathcal{G})$ such that $\beta(g_i) = h_i$. Let $\tilde{g}_{ij} = g_{ij} - g_i + g_j$, we see that $(\tilde{g}_{ij})$ and (g_{ij}) represent the same element in $H^1(X; \mathcal{G})$, and $\beta(\tilde{g}_{ij}) = 0$. Then there exists $(f_{ij}) \in C^1(\mathfrak{U}; \mathcal{F})$ such that $\alpha(f_{ij}) = \tilde{g}_{ij}$. If we call the cohomology class of $(f_{ij}) \in H^1(X; \mathcal{F})$ to be ξ, we have $\alpha^1(\xi) = \eta$, completing the proof. □

Exercise 30

Assume that K is a canonical divisor of X, then $\mathcal{O}_K \simeq \Omega$ and $\mathcal{O} \simeq \Omega_{-K}$ as sheaves over X.

Proof. Since K is canonical, so there is a meromorphic 1-form $\omega \in \mathcal{M}^1(X)$ making $K = (\omega)$. We claim that multiplying by ω induces the isomorphisms

$$\begin{aligned} \mathcal{O}_K &\to \Omega, \\ f &\mapsto f\omega, \end{aligned} \tag{31}$$

and

$$\begin{aligned} \mathcal{O} &\to \Omega_K, \\ g &\mapsto g\omega. \end{aligned} \tag{32}$$

It's obvious to see that the two above maps are well-defined maps between sheaves. The proofs for (31) and (32) are identical, so we only show the first one. It suffices to prove (31) is isomorphic stalk-wise. To see this, we assume that U is a small enough open subset, and $\xi \in \Omega(U)$ is a holomorphic 1-form, so that we can expand ξ as

$$\xi(z) = p(z)dz$$

with $p(z)$ a holomorphic function, and expand ω as

$$\omega = q(z)dz$$

with $q(z)$ a meromorphic function satisfying $(q) = K$ on U. Then we have

$$\xi = \frac{p}{q}\omega$$

on U, with

$$\frac{p(z)}{q(z)} \in \mathcal{O}_K(U).$$

This shows that (31) is surjective. For injectivity of (31), suppose if there is some $f \in \mathcal{O}_K(U)$ such that $f\omega = 0$ on U. In local coordinates, the last condition means that

$$f(z)q(z) = 0$$

for all $z \in U$. But this is ridiculous, since f is meromorphic and q is holomorphic by assumption, they have only discrete zeros hence so does fq. So we must have $f = 0$ since $q \neq 0$ by assumption. This shows that (31) is injective, compeleting the proof. □

Exercise 31

(a) Show that there exist non-zero meromorphic vector fields on X.

(b) Show that
$$\deg\theta = 2-2g$$
holds for any non-zero meromorphic vector field θ on X.

Proof. **(a)** We claim that there exist non-trivial meromorphic 1-forms on X. The easiest way to see this is by the previous Exercise, in which we have proved that there is a sheaf isomorphism $\Omega_{-K} \simeq \mathcal{O}$. Note that $\mathcal{O}$ admits non-trivial sections, so does Ω_{-K}. Thus we can pick a non-trivial section $\omega \in \Omega_{-K}$, which is a meromorphic 1-form.

Let $P := \{ p_1, \ldots, p_r \}$ be the set of all poles and zeroes of ω. We claim that there exists a meromorphic vector field θ such that $\omega(\theta) = 1$ identically on X. Indeed, for any $x \in X$, if $x \notin P$, ω is holomorphic (since x is not a pole) and non-vanishing (since x is not a zero) around any sufficiently small coordinate neighborhood (U, z) of x, with $z(x) = 0$. Suppose that $\omega = f(z)dz$ on (U, z), where $f(z)$ is a non-vanishing holomorphic function of z. Then locally we can choose $\theta = \frac{1}{f}\partial$ on U. Under local coordinate transformation $z \mapsto w$, we have

$$\frac{1}{f(z)}\frac{\partial}{\partial z} = \frac{1}{f(z(w))}\frac{\partial w}{\partial z}\frac{\partial}{\partial w} = \frac{1}{f(z(w))\frac{\partial z}{\partial w}}\frac{\partial}{\partial w} = \frac{1}{f(w)}\frac{\partial}{\partial w},$$

where the last equation holds since the transformation law of f under $z \mapsto w$ is

$$f(z)\frac{\partial z}{\partial w} = f(w),$$

by the fact that ω is a coordinate-free object. Thus θ is also a coordinate free object on $X \backslash P$.

When $x \in P$, then we assume that $x = p_i$. In sufficiently small coordinate neighborhoods (U_i, z_i) with $z_i(p_i) = 0$, ω has the form

$$\omega = z_i^{k_i} dz_i$$

with $k_i := \mathrm{ord}_{p_i}\omega \in \mathbb{Z}$. Then locally we may take $\theta = z_i^{-k_i}\partial_{z_i}$, by similar argument as above, θ is invariant under local coordinate trans form. Thus θ is a well-defined meromorphic field on X, with exactly poles and zeroes $p_1, \ldots, p_r$, among which p_i is a zero of θ iff it is a pole of ω and *vice versa*.

(b) Let $P = \{ p_1, \ldots, p_r \}$ and (U_i, z_i) be as in **(a)**. If $\mathrm{ord}_{p_i}\omega = k_i$, then in (U_i, z_i) θ has the expression
$$\theta = z_i^{-k_i}\frac{\partial}{\partial z_i}.$$
Set
$$u(z_i) := \frac{\theta}{\|\theta\|} = z_i^{-k_i},$$
the index of θ at p_i is defined to be the mapping degree of $u : S^1 \to S^1$. So it's easy to see that

$$\mathrm{index}_{p_i}\theta = -k_i.$$

Let $\{p_1, \ldots, p_r\}$ to be the set of all zeroes or poles of θ, with each p_i of degree m_i, thus

$$\deg\theta := \sum_{i=1}^{r}\mathrm{index}_{p_i}\theta = -\sum_{i=1}^{r}k_i = -\deg(\omega)$$

But as (ω) is a cannonical divisor of X, $\deg(\omega) = 2g-2$. So

$$\deg\theta = -\deg(\omega) = 2-2g,$$

completing the proof. $\square$

Exercise 32

Try to deduce the Gauss-Bonnet formula.

Proof. Since X is a compact Riemann surface, we can endow X a Riemannian metric g, and find isothermal coordinate charts $\{(U_i, z_i = x_i + \mathrm{i}y_i)\}$ under which g has local expressions

$$g = \lambda_i(dx_i^2 + dy_i^2) = \lambda_i|dz_i|^2,$$

for each $z_i \in U_i$. Now let θ be a meromorphicmeromorphic vector field, and let $p_1, \ldots, p_r$ be the poles of θ and $q_1, \ldots, q_s$ be the zeros of θ. By taking refinements of the isothermal coordinates $\{(U_i, z_i = x_i + \mathrm{i}y_i)\}$, we may assume that each U_i contains at most one of the zeroes or poles of θ, in addition that

$$z_i(p_j) = 0, j = 1, \ldots, r,$$

or

$$z_i(q_k) = 0, k = 1, \ldots, s.$$

We are now going to calculate the Gauss curvature K of g.
Recall that for a general surface parametrized by u, v with metric

$$h = Edu^2 + Gdv^2,$$

the Gauss curvature K_h of h can be calculated by

$$K_h = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}}\right).$$

In our case, locally we take $E = G = \lambda_i$ on each U_i, to get

$$\begin{aligned} K &= -\frac{1}{2\sqrt{\lambda_i^2}}\left(\frac{\partial}{\partial x_i}\left(\frac{(\lambda_i)_{x_i}}{\sqrt{\lambda_i^2}}\right) + \frac{\partial}{\partial y_i}\left(\frac{(\lambda_i)_{y_i}}{\sqrt{\lambda_i^2}}\right)\right) \\ &= -\frac{1}{2\lambda_i}\left(\frac{\partial^2}{\partial x_i^2}(\log\lambda_i) + \frac{\partial^2}{\partial y_i^2}(\log\lambda_i)\right) \\ &= -\frac{1}{2\lambda_i}\Delta\log\lambda_i \\ &= \frac{\mathrm{i}}{\lambda_i}\frac{\partial^2}{\partial\bar{z}_i\partial z_i}\log\lambda_i. \end{aligned}$$

Also, the volume from dVol locally looks like:

$$\mathrm{dVol} = \sqrt{|\det g|}dx_i \wedge dy_i = \lambda_i dx_i \wedge dy_i = \frac{\mathrm{i}\lambda_i}{2}dz_i \wedge d\bar{z}_i.$$

Thus

$$\begin{aligned} K\mathrm{dVol} &= -\frac{1}{2}\frac{\partial^2}{\partial\bar{z}_i\partial z_i}\log\lambda_i dz_i \wedge d\bar{z}_i \\ &= -\frac{1}{2}\bar{\partial}\partial\log\lambda_i \\ &= -\frac{1}{2}d(\partial\log\lambda_i). \end{aligned} \tag{33}$$

Now define

$$\psi := g(\theta, \theta),$$

then ψ is a non-vanishing smooth function on $X\backslash\{p_1,\ldots,p_r,q_1,\ldots,q_s\}$. Moreover, if on each U_i

$$\theta = \theta_i \frac{\partial}{\partial z_i}$$

on each U_i, the local expression of ψ on U_i reads

$$\psi = \lambda_i |\theta_i|^2.$$

Since $\log|\theta|$ is holomorphic on $X\backslash\{p_1,\ldots,p_r,q_1,\ldots,q_s\}$, we have the replacement

$$K\mathrm{dVol} = -\frac{1}{2}d(\partial \log \lambda_i) = -\frac{1}{2}d(\partial \log \psi) \tag{34}$$

of $K\mathrm{dVol}$ on $X\backslash\{p_1,\ldots,p_r,q_1,\ldots,q_s\}$. The final thing worth mentioning is that on each U_i, ψ can be written as

$$\psi = \rho_i z_i^{2m_i}, \tag{35}$$

where ρ_i is a smooth function and m_i is the degree of θ at $z_i = 0$.

Denote by $D_\epsilon(x)$ the disk of radius ϵ centered at $x \in X$, and set

$$D_\epsilon := D_\epsilon(p_1) \cup \cdots \cup D_\epsilon(p_r) \cup D_\epsilon(q_1) \cup \cdots \cup D_\epsilon(q_s),$$

we are now at the heart of the proof. By (33), we have

$$\int_X K\mathrm{dVol} = \lim_{\epsilon\to 0}\int_{X\backslash D_\epsilon} K\mathrm{dVol}, \tag{36}$$

since $K\mathrm{dVol}$ is smooth on each $D_\epsilon(p_i)$ or $D_\epsilon(q_j)$. Plugging (34) into the right hand side of (36), we have

$$\begin{aligned}
\int_X K\mathrm{dVol} &= \lim_{\epsilon\to 0}\int_{X\backslash D_\epsilon} K\mathrm{dVol} \\
&= -\frac{1}{2}\lim_{\epsilon\to 0}\int_{X\backslash D_\epsilon} d(\partial\log\psi) \\
&= \frac{1}{2}\lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \partial\log\psi \qquad \text{(by Stokes' Theorem)} \\
&= \frac{1}{2}\lim_{\epsilon\to 0}\Big(\sum_{i=1}^{r}\int_{\partial D_\epsilon(p_i)}\partial\log\psi + \sum_{j=1}^{s}\int_{\partial D_\epsilon(q_j)}\partial\log\psi\Big) \\
&= \frac{1}{2}\lim_{\epsilon\to 0}\Big(\sum_{i=1}^{r}\int_{\partial D_\epsilon(p_i)}\partial\log(\rho_i z_i^{2m_i}) + \sum_{j=1}^{s}\int_{\partial D_\epsilon(q_j)}\partial\log(\rho_j z_j^{2m_j})\Big) \qquad \text{(by (35))} \\
&= \lim_{\epsilon\to 0}\Big(\sum_{i=1}^{r} m_i\int_{\partial D_\epsilon(p_i)}\frac{dz_i}{z_i} + \sum_{j=1}^{s} m_j\int_{\partial D_\epsilon(q_j)}\frac{dz_j}{z_j}\Big) \\
&= \sum_{i}^{r} m_i + \sum_{j}^{s} m_j \qquad \text{(by the Residue Theorem)} \\
&= \deg\theta.
\end{aligned}$$

By **Exercise 4** we have

$$\deg\theta = 2 - 2g = \chi(X),$$

finally

$$\chi(X) = \int_X K\mathrm{dVol},$$

which is the very Gauss-Bonnet formula. □

Exercise 33

Let X be a compact Riemann surface of genus g and $D \in \mathrm{Div}(X)$:

(a) If $\deg D < 0$, then $h^0(X, \mathcal{O}_D) = 0$.

(b) If $\deg D = 0$, then $h^0(X, \mathcal{O}_D) = 0$ or 1, and $h^0(X, \mathcal{O}_D) = 1$ if and only if D is a principal divisor.

(c) If $\deg D > 2g - 2$, then $h^1(X, \mathcal{O}_D) = 0$.

(d) If $\deg D = 2g - 2$, then $h^1(, X, \mathcal{O}_D) = 0$ or 1 and $h^1(X, \mathcal{O}_D) = 1$ if and only if D is a canonical divisor.

Proof. **(a)** If $h^0(X, \mathcal{O}_D) := \dim H^0(X; \mathcal{O}_D) \neq 0$, there is some non-zero $f \in H^0(X; \mathcal{O}_D) = \Gamma(X; \mathcal{O}_D)$. By definition of $\mathcal{O}_D$,

$$(f) + D \geq 0,$$

hence

$$\deg f > -\deg D > 0.$$

But this is ridiculous, since f is meromorphic, $\deg f = 0$ is always true.

(b) If there are any non-zero $f \in H^0(X; \mathcal{O}_D)$, we have

$$(f) \geq -D = 0,$$

hence

$$\deg f \geq 0.$$

When $\deg f > 0$, such f doesn't exist, so $H^0(X; \mathcal{O}_D) = 0$ hence $h^0(X, \mathcal{O}_D) = 0$.

When $\deg f = 0$, we claim that $(f) + D = 0$. Otherwise, $(f) > -D$ and $\deg f > 0$, a contradiction. Thus in this case, all $f \in H^0(X; \mathcal{O}_D)$ are equivalent to D, and $H^0(X; \mathcal{O}_D)$ is the $\mathbb{C}$-vector space spanned by D, implying

$$h^0(X, \mathcal{O}_D) = 1.$$

Conversely, if D is principal, there is some meromorhic function g making $D = (g)$. Then all $f \in H^0(X; \mathcal{O}_D)$ are such f that

$$(f) + D = (f) + (g) = 0,$$

showing that $h^0(X, \mathcal{O}_D) = 1$.

(c) Let K be the canonical divisor of X. We know that there is an isomorphism

$$\mathcal{O}_{K-D} \simeq \Omega_{-D}$$

for any $D \in \mathrm{Div}(X)$. Thus

$$H^0(X; \Omega_{-D}) \simeq H^0(X; \mathcal{O}_{K-D}).$$

On the other hand, we have

$$H^1(X; \mathcal{O}_D)^* \simeq H^0(X; \Omega_{-D})$$

by Serre's duality. Thus

$$H^1(X; \mathcal{O}_D)^* \simeq H^0(X; \mathcal{O}_{K-D})$$

and

$$h^1(X, \mathcal{O}_D) = h^0(X, \mathcal{O}_{K-D}). \tag{37}$$

Since $\deg K = 2g-2$, $\deg(K-D) = \deg K - \deg D < 0$, by assumption. Applying **(a)** we have

$$h^1(X, \mathcal{O}_D) = h^0(X, \mathcal{O}_{K-D}) = 0.$$

(d) In this case, we have $\deg(K-D) = 0$. Note (37) still holds, we can apply **(b)** to the sheaf $\mathcal{O}_{K-D}$, then the assertion follows.

□

Exercise 34

Let X be a compact Riemann surface of genus g, show that

(a) Any two $\xi, \eta \in \mathcal{M}^n(X)\backslash\{0\}$ are linearly equivalent.

(b) For any $\xi \in \mathcal{M}^n(X)\backslash\{0\}$, we have $\deg(\xi) = n(2g-2)$.

(c)

$$\dim \Omega^n(X) = \begin{cases} (2n-1)(g-1), & n \geq 2, g \geq 2, \\ 0, & g = 0, \\ 1, & g = 1. \end{cases}$$

Proof. **(a)** For any $\xi, \eta \in \mathcal{M}^n(X)$ and $p \in X$, suppose that they can be written as

$$\xi = f(z)dz^n,$$
$$\eta = g(z)dz^n,$$

with $f, g \in \mathcal{M}(U)$ in any coordinate neighborhood (U, z) of p. Clearly, $h(z) := f(z)/g(z) \in \mathcal{M}(U)$. Suppose that (V, w) is another neighborhood coordinate of p, and the coordinates are related holomorphically by $z = z(w)$. So on $U \cap V$, we have

$$\frac{f(w)}{g(w)} = \frac{(\frac{\partial z}{\partial w})^n f(z(w))}{(\frac{\partial z}{\partial w})^n g(z(w))} = \frac{f(z)}{g(z)}.$$

This shows that there exists a meromorphic h such that $\xi = h\eta$, by the gluing property of the sheaf $\mathcal{M}$. Thus $(\xi) = (\eta) + (h)$, as desired.

(b) Take $0 \neq \theta \in \mathcal{M}^1(X)$, then we define $\Theta \in \mathcal{M}^n(X)$ by

$$\Theta = \theta \otimes \cdots \otimes \theta,$$

which is not identically zero. By **(a)**, any $\xi \in \mathcal{M}^n(X)$ is linearly equivalent to Θ, so

$$(\xi) = (\Theta) + (f)$$

for some meromrphic function f. But since $\deg(f) = 0$, we have

$$\deg(\xi) = \deg(\Theta) = n \deg(\theta) = n(2g-2),$$

as (θ) is a canonical divisor.

(c) Via the isomorphism of sheaves

$$\Omega \simeq \mathcal{O}_K,$$

we have

$$\Omega^n \simeq \mathcal{O}_K \otimes \cdots \otimes \mathcal{O}_K \simeq \mathcal{O}_{nK}.$$

Then

$$\dim \Omega^n(X) = \dim H^0(X;\Omega^n) = \dim H^0(X;\mathcal{O}_{nK}) = h^0(X,\mathcal{O}_{nK}),$$

via which we may prove the assertion applying the Riemann-Roch Theorem. Note that $\deg(nK) = n\deg K = n(2g-2)$, thus there are three cases:

(i) $g=0$, $\deg(nK) = -2n < 0$, then by **Exercise 1(a)**, $\dim \Omega^n(X) = h^0(X,\mathcal{O}_{nK}) = 0$.

(ii) $g=1, \deg(nK)=0$. Since nK is principal, by **Exercise 1(b)** $\dim \Omega^n(X) = h^0(X,\mathcal{O}_{nK}) = 1$.

(iii) $g \geq 2, n > 1, \deg(nK) > 2g-2$. By Riemann-Roch,

$$h^0(X,\mathcal{O}_{nK}) - h^1(X,\mathcal{O}_{nK}) = \deg(nK) + 1 - g = n(2g-2)+1-g = (2n-1)(g-1).$$

By the Vanishing Theorem, $h^1(X,\mathcal{O}_{nK}) = 0$, so $\dim \Omega^n(X) = h^0(X,\mathcal{O}_{nK}) = (2n-1)(g-1)$.

In summary, we have

$$\dim \Omega^n(X) = \begin{cases} (2n-1)(g-1), & n \geq 2, g \geq 2, \\ 0, & g = 0, \\ 1, & g = 1, \end{cases}$$

completing the proof. □

Exercise 35

Let Ω^{-1} be the sheaf of holomorphic vector fields on X and K be a canonical divisor of X. Show that Ω^{-1} and $\mathcal{O}_{-K}$ are isomorphic as sheaves.

Proof. Since K is canonical, we have $K = (\omega)$ for some meromorphic 1-form ω. Then we have a morphism $\phi : \Omega^{-1} \to \mathcal{O}_K$, given by contraction with ω:

$$\begin{aligned} \phi(U) : \Omega^{-1}(U) &\to \mathcal{O}_K(U), \\ v &\mapsto \omega(v). \end{aligned}$$

To show that $\phi : \Omega^{-1} \to \mathcal{O}_K$ is an isomorphism of sheaves, it suffices to show that ϕ_x is an isomorphism of stalks for all x. Indeed, in any small enough open subset $U_x \simeq \mathbb{C}$ containing x, we can write v as $v = f(z)\frac{\partial}{\partial z}$ with $f(z)$ holomorphic and $\omega = g(z)dz$ with $g(z) \in \mathcal{O}_K(U_x)$. Thus if $\omega(v) = f(z)g(z)$ is identically zero on U, there must be $f(z) = 0$ on U since $g(z) \neq 0$. This shows that ϕ_x is injective. For surjectivity, suppose that $h(z) \in \mathcal{O}_K(U)$, we take $v := \frac{h(z)}{g(z)}\frac{\partial}{\partial z}$, which is a holomorphic vector field on U_x and satisfies $\omega(v) = h(z)$. This shows that ϕ_x is surjective. So we are done. □

Exercise 36

Show that on $U \cap V$ we have

$$\Phi(z)dz^{\frac{g(g+1)}{2}} = \Psi(w)dw^{\frac{g(g+1)}{2}},$$

or equivalently

$$\Phi(z) = \Psi(w)(\frac{\partial w}{\partial z})^{\frac{g(g+1)}{2}}.$$

Proof. First we write down how ϕ_i and ψ_i are related by the coordinate transformation:

$$\phi_i(z) = \frac{\partial w}{\partial z}\psi_i(w), 1 \leq i \leq g.$$

The relation between $\phi_i^{(j)}$ and $\psi_i^{(j)}$ needs more attention. For the first few j's, we have

$$\begin{aligned}
\phi_i'(z) &= \frac{\partial^2 w}{\partial z^2}\psi_i(w) + (\frac{\partial w}{\partial z})^2\psi_i'(w),\\
\phi_i^{(2)}(z) &= \frac{\partial^3 w}{\partial z^3}\psi_i(w) + 3\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}\psi_i'(w) + (\frac{\partial w}{\partial z})^3\psi_i^{(2)}(w),\\
&\vdots
\end{aligned}$$

Taking derivatives inductively, we have

$$\phi_i^{(j)} = (\frac{\partial w}{\partial z})^{j+1}\psi_i^{(j)}(w) + \sum_{k=0}^{j-1} a_k\psi_i^{(k)}(w), \tag{38}$$

a_k being polynomial of $\partial w/\partial z, \partial^2 w/\partial z^2, \ldots, \partial^{j-k+1}w/\partial z^{j-k+1}$. One should keep in mind that the terms in (38) lower than $\psi_i^{(j)}(w)$ have no contribution to $\Phi(z)$.

Now we are ready to calculate $\Phi(z)$:

$$\begin{aligned}
\Phi(z) &= \det\begin{pmatrix} \phi_1(z) & \phi_2(z) & \cdots & \phi_g(z)\\ \phi_1'(z) & \phi_2'(z) & \cdots & \phi_g'(z)\\ \vdots & \vdots & & \vdots\\ \phi_1^{(g-1)}(z) & \phi_2^{(g-1)}(z) & \cdots & \phi_g^{(g-1)}(z)\end{pmatrix}\\
&= \det\begin{pmatrix} \frac{\partial w}{\partial z}\psi_1(w) & \frac{\partial w}{\partial z}\psi_2(w) & \cdots & \frac{\partial w}{\partial z}\psi_g(w)\\ (\frac{\partial w}{\partial z})^2\psi_1'(w) & (\frac{\partial w}{\partial z})^2\psi_2'(w) & \cdots & (\frac{\partial w}{\partial z})^2\psi_g'(w)\\ \vdots & \vdots & & \vdots\\ (\frac{\partial w}{\partial z})^g\psi_1^{(g-1)}(w) & (\frac{\partial w}{\partial z})^g\psi_2^{(g-1)}(w) & \cdots & (\frac{\partial w}{\partial z})^g\psi_g^{(g-1)}(w)\end{pmatrix}\\
&= (\frac{\partial w}{\partial z})(\frac{\partial w}{\partial z})^2\cdots(\frac{\partial w}{\partial z})^g \det\begin{pmatrix} \psi_1(w) & \psi_2(w) & \cdots & \psi_g(w)\\ \psi_1'(w) & \psi_2'(w) & \cdots & \psi_g'(w)\\ \vdots & \vdots & & \vdots\\ \psi_1^{(g-1)}(w) & \psi_2^{(g-1)}(w) & \cdots & \psi_g^{(g-1)}(w)\end{pmatrix}\\
&= (\frac{\partial w}{\partial z})^{1+2+\cdots+g}\Psi(w)\\
&= (\frac{\partial w}{\partial z})^{\frac{g(g+1)}{2}}\Psi(w).
\end{aligned}$$

Thus

$$\Phi(z)dz^{\frac{g(g+1)}{2}} = \Psi(w)(\frac{\partial w}{\partial z})^{\frac{g(g+1)}{2}}(\frac{\partial z}{\partial w})^{\frac{g(g+1)}{2}}dw = \Psi(w)dw^{\frac{g(g+1)}{2}}$$

□

Exercise 37

Let N be the number of Weierstrass points of X, without counting multiplicity, show that

$$2(g+1) \leq N \leq (g+1)g(g-1).$$

Moreover $N = 2(g+1)$ if and only if the gaps of X at any Weierstrass point are $1, 2, \ldots, 2g-1$; and $N = (g+1)g(g-1)$ if and only if the gaps of X at any Weierstrass point are $1, 2, \ldots, g-1, g+1$.

Proof. As each $\mathrm{Wt}(p_i)$ is the multiplicity of the Weierstrass point p_i, and we have the equality

$$\mathrm{Wt}(p_1) + \mathrm{Wt}(p_2) + \cdots + \mathrm{Wt}(p_N) = (g-1)g(g+1), \tag{39}$$

if we know the upper and lower bounds of each $\mathrm{Wt}(p_i)$, then we can determine the upper and lower bounds of N from (39). By definition,

$$\mathrm{Wt}(p_i) = \sum_{j=1}^{g}(n_j(p_i) - j).$$

Since p_i is Weierstrass, $n_g(p_i) \geq g+1$. Other restrictions on the gaps $n_j(p_i)$ are that

$$n_1(p_i) < n_2(p_i) < \cdots < n_g(p_i),$$

and

$$n_j(p_i) \leq 2g - 1$$

for all j.

Hence if we let

$$\begin{aligned} n_1(p_i) &= 1, \\ n_2(p_i) &= 2, \\ &\vdots \\ n_{g-1}(p_i) &= g-1, \\ n_g(p_i) &= g+1, \end{aligned}$$

$\mathrm{Wt}(p_i)$ reaches its minimal value

$$\mathrm{Wt}(p_i) = (1-1) + (2-2) + \cdots + (g-1-g+1) + (g+1-g) = 1.$$

If we take

$$\begin{aligned} n_1(p_i) &= 1, \\ n_2(p_i) &= 3, \\ n_3(p_i) &= 5 \\ &\vdots \\ n_{g-1}(p_i) &= 2g-3, \\ n_g(p_i) &= 2g-1, \end{aligned}$$

$\mathrm{Wt}(p_i)$ reaches its maximal value

$$\begin{aligned} \mathrm{Wt}(p_i) &= (1-1) + (3-2) + (5-3) + \cdots + (2g-3-g+1) + (2g-1-g) \\ &= 0 + 1 + 2 + \cdots + g - 1 \\ &= \frac{1}{2}g(g-1). \end{aligned}$$

Thus we get an estimation of $\mathrm{Wt}(p_i)$:

$$1 \leq \mathrm{Wt}(p_i) \leq \frac{g(g-1)}{2}. \tag{40}$$

Combining (39), there is

$$N \leq (g+1)g(g-1) \leq \frac{N}{2}g(g-1),$$

or equivalently

$$2(g+1) \leq N \leq (g+1)g(g-1),$$

completing the proof. □

Exercise 38

Let X be a compact Riemann surface of genus $g \geq 2$, and $f : X \to \mathbb{P}^1$ be a holomorphic map of degree 2. Show that

(a) A point $p \in X$ is a Weierstrass point if and only if p is a branched point of f.

(b) The number of branched points of f is $2(g+1)$.

Proof. **(a)** If p is a branched point of f and let $q = f(p)$, then we claim that the q is the only point in $f^{-1}(q)$. Indeed, if there were other $p_1, \dots, p_r inf^{-1}(q)$, we have

$$2 = \deg f = k_p + \sum_{i=1}^{r} k_{p_i} \geq 2 + \sum_{i=1}^{r} k_{p_i},$$

by the very definition of mapping degree of f. But this is ridiculous, since all k_{p_i} are positive integers. So our claim holds. Thus p is the unique pole of the meromorphic function $g = \frac{1}{f-f(p)}$, of order 2. We know that 2 is not a gap of X at p. Another important observation is that if m, n are not gaps of X at p, so is not $m+n$. Back to our case, we know that $2, 4, 6, \dots, 2g$ are not gaps. Plus the fact that there are exactly g gaps at p and all of them are $\leq 2g-1$, the gaps at p are exactly $1, 3, 5, \dots, 2g-1$. Obviously $n_g = 2g-1 > g$, hence p is a Weierstrass point.

Conversely, if p is a Weierstrass point, then X admits a meromorphic function f with a unique pole p, of order $\leq g$. Assume that f can be expand as

$$f = \frac{1}{z^k}, k > 0.$$

Under a coordinate transformation $w = \frac{1}{z}$ we have

$$f = w^k$$

showing that p is indeed a branched point.

(b) By **(a)** we just need to count the number of Weierstrass points on X. The weight at each Weierstrass point is

$$\mathrm{Wt}_X(p) = \sum_{i=1}^{g}(2i-1-i) = \frac{g(g-1)}{2}.$$

On the other hand,

$$\sum_{p \in X} \mathrm{Wt}_X(p) = (g-1)g(g+1),$$

thus $N = 2(g+1)$ as desired. □

Exercise 39

Let X be a compact Riemann surface of genus g and $D \in \mathrm{Div}(X)$. If $\deg D \geq 2g+1$, then $\phi_D : X \to \mathbb{P}_n$ is a holomorphic embedding, where $n = h^0(X, \mathcal{O}_D) - 1$.

Proof. To prove the theorem, we need to prove a couple of lemmas.

Lemma 4. In the same setting as in the statement of the theorem, given any point $p \in X$, there exists a meromorphic function $f \in \mathcal{O}_D(X)$ such that $f(p) \neq 0$.

Proof of Lemma 4. We just need to show that $\mathcal{O}_{D-p} \subsetneq \mathcal{O}_D$. By Riemann-Roch, we have

$$\begin{aligned} h^0(X, \mathcal{O}_{D-p}) - h^1(X, \mathcal{O}_{D-p}) &= \deg D - 1 + 1 - g = \deg D - g, \\ h^0(X, \mathcal{O}_D) - h^1(X, \mathcal{O}_D) &= \deg D + 1 - g. \end{aligned}$$

Since $\deg D \geq 2g + 1 > 2g - 2$ and $\deg D - 1 \geq 2g > 2g - 2$, $h^1(X, \mathcal{O}_{D-p}) = h^1(X, \mathcal{O}_D) = 0$ by the vanishing theorem. Thus $h^0(X, \mathcal{O}_D) - h^0(X, \mathcal{O}_{D-p}) = 1$, showing that $\mathcal{O}_{D-p} \subsetneq \mathcal{O}_D$, as desired. □

Lemma 5. In the same setting as in the statement of the theorem, given any two distinct points $p, q \in X$, there exists a meromorphic function $f \in \mathcal{O}_D(X)$ such that $f(p) \neq 0$ whilst $f(q) = 0$.

Proof of Lemma 5. It suffices to show that there is an f such that $f \in \mathcal{O}_{D-q}(X)$ but $f \notin \mathcal{O}_{D-p-q}(X)$. To see this, we need to show that $\mathcal{O}_{D-p-q} \subsetneq \mathcal{O}_{D-p}$. Again by Riemann-Roch,

$$\begin{aligned} h^0(X, \mathcal{O}_{D-p}) - h^1(X, \mathcal{O}_{D-p}) &= \deg D - 1 + 1 - g = \deg D - g, \\ h^0(X, \mathcal{O}_{D-p-q}) - h^1(X, \mathcal{O}_{D-p-q}) &= \deg D - 2 + 1 - g = \deg D - 1 - g. \end{aligned}$$

Since $\deg(D - p) \geq 2g > 2g - 2$ and $\deg(D - p - q) \geq 2g - 1 > 2g - 2$, $h^1(X, \mathcal{O}_{D-p}) = h^1(X, \mathcal{O}_{D-p-q}) = 0$ by the vanishing theorem. So $h^0(X, \mathcal{O}_{D-p}) - h^0(X, \mathcal{O}_{D-p-q} = 1$, as desired. □

Now we begin to prove the theorem. By assumption, $h^0(X, \mathcal{O}_D) = n + 1$. We can pick a basis $\phi_0, \phi_1, \ldots, \phi_n$ for the $\mathbb{C}$-space $\mathcal{O}_D(X)$. Then we define $\phi_D : X \to \mathbb{P}^n$ by

$$\begin{aligned} \phi_D : X &\to \mathbb{P}^n, \\ x &\mapsto [\phi_0(x) : \phi_1(x) : \cdots : \phi_n(x)], \end{aligned}$$

which is holomorphic by construction. Then we claim that

Claim 6. ϕ_D is injective.

Proof of Claim 6. We argue by contradiction. Suppose there are distinct $p, q \in X$ such that $\phi_D(p) = \phi_D(q)$. Then by definition we have

$$[\phi_0(p) : \cdots : \phi_n(p)] = [\phi_0(q) : \cdots : \phi_n(q)],$$

or equivalently there exists a non-zero constant $\lambda \in \mathbb{C}$ such that

$$\phi_i(p) = \lambda \phi_i(q) \tag{41}$$

for all $0 \leq i \leq n$. By Lemma 5, there exists $f \in \mathcal{O}_D(X)$ such that $f(p) \neq 0$ but $f(q) = 0$. Writing $f = c_0\phi_0 + \cdots c_n\phi_n$ with $c_0, \ldots, c_n \in \mathbb{C}$, we have

$$f(p) = c_0\phi_0(p) + \cdots + c_n\phi_n(p) \neq 0$$

and

$$f(q) = c_0\phi_0(q) + \cdots + c_n\phi_n(q) = 0.$$

But by (41), we have

$$\begin{aligned} f(p) &= c_0\phi_0(p) + \cdots + c_n\phi_n(p) \\ &= \lambda(c_0\phi_0(q) + \cdots + c_n\phi_n(q)) \\ &= \lambda \cdot 0 \\ &= 0 \end{aligned}$$

a contradiction. So ϕ_D must be injective, as desired.

□

Thus ϕ_D is a bijection from X to $\phi_D(X)$. We need to show that $\phi_D(X)$ is closed in $\mathbb{P}^n$. This is easy. Since $\phi_D : X \to \mathbb{P}^n$ is continuous and X is compact, $\phi_D(X)$ is a compact subset of $\mathbb{P}^n$. Since $\mathbb{P}^n$ is Hausdorff, all its compact subsets are closed, so is $\phi_D(X)$.

Finally, we have to show that $d\phi_D$ is non vanishing on X. We again argue by contradiction. By Lemma 4, we can choose the basis $\phi_0, \ldots, \phi_n$ such that $\phi_0(p) \neq 0$. Then in a coordinate neighborhood (U, z) of p, $\phi_D : X \to \mathbb{P}^n$ can be viewed locally as

$$
\begin{aligned}
\tilde{\phi}_D : U &\to \mathbb{C}^n, \\
z &\mapsto (\frac{\phi_1(z)}{\phi_0(z)}, \cdots, \frac{\phi_n(z)}{\phi_0(z)})
\end{aligned}
$$

Suppose that $d\phi_D$ vanishes at $p \in X$, then

$$
\begin{aligned}
(d\phi_D)(p) &= (d\tilde{\phi}_D)(p) \\
&= (\frac{\phi_1'(0)\phi_0(0) - \phi_1(0)\phi_0'(0)}{(\phi_0(0))^2}, \cdots, \frac{\phi_n'(0)\phi_0(0) - \phi_n(0)\phi_0'(0)}{(\phi_0(0))^2}) \\
&= (0, \ldots, 0),
\end{aligned}
$$

implying that

$$
\phi_i'(0) = \frac{\phi_i(0)\phi_0'(0)}{\phi_0(0)}
$$

for all $1 \leq i \leq n$. By Lemma 5, we can find f such that $\mathrm{ord}_p f = 1$, which implies that $f'(p) \neq 0$. By the virtue of the expansion $f = c_0\phi_0 + \cdots + \cdots c_n\phi_n$ the coordinate neighborhood (U, z), we have

$$
\begin{aligned}
0 \neq f'(0) &= c_0\phi_0'(0) + \cdots + c_n\phi_n'(0) \\
&= c_0\phi_0'(0) + \frac{\phi_0'(0)}{\phi_0(0)} c_1\phi_1(0) + \cdots + \frac{\phi_0'(0)}{\phi_0(0)} c_n\phi_n(0) \\
&= \frac{\phi_0'(0)}{\phi_0(0)} (c_0\phi_0(0) + c_1\phi_1(0) + \cdots + c_n\phi_n(0)) \\
&= 0,
\end{aligned}
$$

a contradiction. So $d\phi_D$ must not vanish at p. That's how the theorem has been proved.

□