Differential Cohomology and Classical Chern-Simons Theory

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Definition 0.1. We say a k-cochain $c \in C^k(M;\mathbb{Z})$ is Λ-*periodic* if for all $z \in Z_k(M;\mathbb{Z})$,

$$\langle c, z\rangle \in \Lambda$$

holds.

Let $\Omega^*_0(M)$ denote the set of Λ-periodic closed differential forms on M.

Definition 0.2 ([CS]). We denote the degree k *differential character* on M by

$$\hat{H}^k(M;\mathbb{R}/\Lambda) = \{\, f \in \mathrm{Hom}_{\mathbb{Z}}(Z_k(M;\mathbb{Z}), \mathbb{Z}/\Lambda) \,\}$$

And we set $\hat{H}^{-1}(M;\mathbb{R}/\Lambda) \coloneqq \Lambda$ and

$$\hat{H}^*(M;\mathbb{R}/\Lambda) = \bigoplus_k \hat{H}^k(M;\mathbb{R}/\Lambda).$$

We can easily check that $\hat{H}^*(-;\mathbb{R}/\Lambda)$ is a contravariant functor assigning a topological space X a graded abelian group $\hat{H}^*(X;\mathbb{R}/\mathbb{Z})$. In fact, it assingns a topological space a graded ring.

Let $f \in \hat{H}^k(M;\mathbb{R}/\Lambda)$, since $Z_k(M;\mathbb{Z})$ is a free abelian group, then there exists a lift $\tilde{f} : Z_k(M;\mathbb{Z}) \to \mathbb{R}$, such that the diagram

$$\begin{tikzcd} & Z_k(M;\mathbb{Z}) \arrow[dl, "\tilde{f}"'] \arrow[d, "f"] \\ \mathbb{R} \arrow[r] & \mathbb{R}/\Lambda \end{tikzcd}$$

commutes. Then we precompose the diagram with $\partial_k : C_{k+1}(M;\mathbb{Z}) \to Z_k(M;\mathbb{Z})$ to get the following commutative diagram

$$\begin{tikzcd} C_{k+1}(M;\mathbb{Z}) \arrow[r, "\partial"] & Z_k(M;\mathbb{Z}) \arrow[dl, "\tilde{f}"'] \arrow[d, "f"] \\ \mathbb{R} \arrow[r] & \mathbb{R}/\Lambda \end{tikzcd},$$

which reads

$$\tilde{f} \circ \partial \mod \Lambda = f \circ \partial.$$

By definition, since $f \in \hat{H}^k(M;\mathbb{R}/\Lambda)$, there is a differential form $\omega \in \Omega^{k+1}(M)$ such that

$$\omega = f \circ \partial = \tilde{f} \circ \partial \mod \Lambda,$$

which is an equation in $C^{k+1}(M;\mathbb{R})$. Then we can tell that the diffrence of ω and $\tilde{f} \circ \partial$ is in $C^{k+1}(M;\Lambda)$, which we denote by

$$c = \omega - \tilde{f} \circ \partial \in C^{k+1}(M;\Lambda). \tag{0.1}$$

We can precompose the above equation by ∂ again, and get

$$0 = \tilde{f} \circ \partial \circ \partial = \omega \circ \partial - c \circ \partial = d\omega - \delta c,$$

since d and δ are duals of ∂ in complexes $\Omega^*(M)$ and $C^*(M;\Lambda)$. The following observation is fundamental in the theory of differential cohomology:

Claim 0.1. A non-vanishing differential form never takes values lying only in a proper subring $\Lambda \subset \mathbb{R}$.

Using the above claim we conclude that ω is closed and has Λ-periods. Since we have $d\omega = \delta c \in C^{k+2}(M;\Lambda)$, for any $\sigma \in C_{k+2}(M;\Lambda)$

$$\int_\sigma d\omega = \langle \delta c, \sigma \rangle = \langle c, \partial\sigma \rangle \in \Lambda,$$

which contradicts the claim. Thus $d\omega = \delta c = 0$. Take any $z \in Z_{k+1}(M;\mathbb{Z})$ and pair it with both sides of Equation 0.1, we have

$$\int_z \omega = \langle c, z \rangle \in \Lambda$$

showing ω is Λ-periodic.

We will now show that ω and $[c]$ is independent on the choice of $\tilde{f}$. Say if $\tilde{f}' \in \mathrm{Hom}_{\mathbb{Z}}(Z_k(M;\mathbb{Z}),\mathbb{R})$ is another lift of f, we have $\tilde{f} = f + g$ for some $g \in \mathrm{Hom}_{\mathbb{Z}}(Z_k(M;\mathbb{Z}),\Lambda)$. By precomposing ∂, we have

$$\omega' - \omega = \tilde{f}' \circ \partial - \tilde{f} \circ \partial + c' - c = g \circ \partial + c' - c \in C^{k+1}(M;\Lambda).$$

Evaluating the last equation on $C_{k+1}(M;\mathbb{Z})$, we get a contradiction to Claim 0.1 and thus $\omega' = \omega, c' = c + \delta g$, which means that ω and $[c]$ is independent on the choice of the lift of $f \in \hat{H}^k(M;\mathbb{R}/\Lambda)$. To sum up, we actually have two well-defined morphisms:

$$\begin{aligned} \mathrm{curv} : \hat{H}^k(M;\mathbb{R}/\Lambda) &\to \Omega_0^{k+1}(M) \\ f &\mapsto \omega, \end{aligned}$$

and

$$\begin{aligned} \mathrm{ch} : \hat{H}^k(M;\mathbb{R}/\Lambda) &\to H^{k+1}(M;\Lambda) \\ f &\mapsto [c]. \end{aligned}$$

Moreover, if we restrict $\tilde{f} \circ \partial = \omega - c$ on $B_{k+1}(M;\mathbb{Z})$, we have $\omega|_{B_{k+1}} = c|_{B_{k+1}}$, which implies ω and c determine the same cohomology class in $H^{k+1}(M;\mathbb{R})$. That is, we have

$$[\omega] = r([c]),$$

or

$$[\mathrm{curv}(-)] = r(\mathrm{ch}(-)).$$

To measure the size of $\hat{H}^k(M;\mathbb{R}/\mathbb{Z})$, we need to fit it in some exact sequences.

Theorem 0.2 ([CS]). There are natural exact sequences

$$0 \longrightarrow H^k(M;\mathbb{R}/\Lambda) \longrightarrow \hat{H}^k(M;\mathbb{R}/\Lambda) \xrightarrow{\mathrm{curv}} \Omega_0^{k+1}(M) \longrightarrow 0$$

$$0 \longrightarrow \Omega^k(M)/\Omega_0^k(M) \longrightarrow \hat{H}^k(M;\mathbb{R}/\Lambda) \xrightarrow{\mathrm{ch}} H^{k+1}(M;\Lambda) \longrightarrow 0$$

Proof. First, we need to show that the morphisms curv, ch are surjective. To show that, we introduce another observation. By definition, $\Omega_0^*(M)$ is the set closed Λ-periodic differential forms on M, and in fact it is the following pullback

$$\begin{array}{ccc} \Omega^*_0(M) & \longrightarrow & \Omega^*_{\mathrm{cl}}(M) \\ \downarrow & & \downarrow \\ H^*(M;\Lambda) & \xrightarrow{r} & H^*(M;\mathbb{R}). \end{array} \tag{0.2}$$

Thus for any $\omega \in \Omega_0^{k+1}(M)$ we can find a class $u \in H^{k+1}(M;\Lambda)$ such that $r(u) = [\omega]$ and for any given class $u \in H^{k+1}(M;\Lambda)$ there is a differential form $\omega \in \Omega_0^{k+1}(M)$ satisfying the same relation, vise versa. Take a represnetative c of u, and find that $\omega - c$ is identically 0 when restricted to $Z_{k+1}(M;\mathbb{Z})$, thus $\omega - c$ is a coboundary in $C_{k+1}(M;\mathbb{R})$ and there is some $\tilde{f} \in C^k(M;\mathbb{R})$ such that $\tilde{f} \circ \partial = \omega - c$. Let $f := \tilde{f} \mod \Lambda$, we have found $f \in \hat{H}^k(M;\mathbb{R}/\Lambda)\mathrm{curv}(f) = \omega$ and $\mathrm{ch}(f) = [c] = u$ hold.
Next, if $f \in \ker \mathrm{curv}$, then pick any lift $\tilde{f}$ of f we have $\tilde{f} \circ \partial = -c$. Since $c \in C^{k+1}(M;\Lambda)$, take both sides modulo Λ we have $\delta f = 0 \in \mathbb{R}/\Lambda$, which says f is closed in $C^k(M;\mathbb{R}/\Lambda)$, thus f defines a cohomology class in $H^k(M;\mathbb{R}/\Lambda)$. Conversely, given a cohomology class $[s] \in H^k(M;\mathbb{R}/\Lambda)$, $s \circ \partial$ is identically zero on $Z_k(M;\mathbb{Z})$ and thus $\mathrm{curv}(s) = 0$.
Finally, if $\mathrm{ch}(f) = 0$ we have $\delta e = c$ for some $e \in C^k(M;\Lambda)$. We thus have $\omega = \tilde{f} \circ \partial + \delta e = \delta(\tilde{f} + e) \in C^{k+1}(M;\mathbb{R})$. The left hand side is a closed Λ-periodic differential form and the right hand side is a real $k+1$-cochain. By the de Rham theorem, there is $\theta \in \Omega^k(M)$ such that $d\theta = \omega$. Then $\delta(\tilde{f} + e - \theta) = 0$ so that $\tilde{f} + e - \theta = z$ for some $z \in Z^k(M;\mathbb{R})$. Again, by the de Rham theorem there is a $\phi \in \Omega^k_{\mathrm{cl}}(M)$ such that $\phi = z$ and $\tilde{f} = \phi + \theta - e$. We get the map $\Omega^k(M) \to \hat{H}^k(M;\mathbb{R}/\mathbb{Z})$ by modoloing Λ and the kernel of this map is $\Omega_0^k(M)$. □

Using the two exact obtained above, we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \frac{H^k(M;\mathbb{R})}{r(H^k(M;\Lambda))} & \longrightarrow & \frac{\Omega^k(M)}{\Omega^k_0(M)} & \xrightarrow{d} & d\Omega^k(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^k(M;\mathbb{R}/\Lambda) & \longrightarrow & \hat{H}^k(M;\mathbb{R}/\Lambda) & \xrightarrow{\mathrm{curv}} & \Omega_0^{k+1}(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ext}(H_k(M;\mathbb{Z}),\Lambda) & \longrightarrow & H^{k+1}(M;\Lambda) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(H_{k+1}(M;\mathbb{Z}),\Lambda) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{0.3}$$

Denote $R^*(M;\Lambda)$ as the following pullback

$$\begin{array}{ccc} R^*(M;\Lambda) & \longrightarrow & \Omega^*_0(M) \\ \downarrow & & \downarrow \\ H^*(M;\Lambda) & \xrightarrow{r} & H^*(M;\mathbb{R}). \end{array}$$

By the diagram (0.3) we have

Claim 0.3. There is an exact sequence

$$0 \longrightarrow \frac{H^k(M;\mathbb{R})}{r(H^k(M;\Lambda))} \longrightarrow \hat{H}^k(M;\mathbb{R}/\Lambda) \xrightarrow{\text{curv}\times\text{ch}} R^*(M;\Lambda) \longrightarrow 0 \tag{0.4}$$

By the Borel theorem (??) , we have

$$H^3(BG;\mathbb{R}) = 0$$

for any compact Lie group G. Taking $\Lambda = \mathbb{Z}$, the short exact sequence (0.4) in this case can be written as

$$0 \longrightarrow \hat{H}^3(BG;\mathbb{R}/\mathbb{Z}) \longrightarrow R^3(BG;\mathbb{Z}) \longrightarrow 0,$$

which implies that the differential character in $\hat{H}^3(BG;\mathbb{R}/\mathbb{Z})$ can be uniquely determined by a differential form and a characteristic class.

We can define the action of the Chern-Simons theory as

$$S = \langle \alpha_A, [M] \rangle$$

where $\alpha_A = \gamma^*\alpha$, and $\alpha \in \hat{H}^3(BG;\mathbb{R}/\mathbb{Z})$ is uniquely determined by the element $(\Omega(F_u), \omega) \in R^3(BG)$.

Definition 0.3. Let M be a smooth manifold. For Λ a subring of $\mathbb{R}$ and for $p \geq 0$, the **smooth Deligne complex** $\Lambda(p)^\infty_D$ is the complex of sheaves:

$$\Lambda(p)_M \xrightarrow{i} \Omega^0_{M,\mathbb{C}}(-) \xrightarrow{d} \Omega^1_{M,\mathbb{C}}(-) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}_{M,\mathbb{C}}(-), \tag{0.5}$$

The hypercohomology groups $H^q(M;\Lambda(p)^\infty_D)$ are called the **smooth Deligne cohomology groups** of M, and are sometimes denoted by $H^p_D(M,\Lambda(p)^\infty)$.

There is a natural homomorphism $\kappa : H^q(M;\Lambda(p)^\infty_D) \to H^q(M;\Lambda(p))$, where the left side is the ordinary singular (Čech) cohomology of M with coefficients group $\Lambda(p)$, since the complex $\Lambda(p)^\infty_D$ projects to the constant sheaf $\Lambda(p)_M$. We can identify the sheaf cohomology $H^p(M;\Lambda(p))$ of the constant sheaf $\Lambda(p)_M$ with the singular cohomology $H^p(M;\Lambda(p))$ because of the following theorem [Bre, Chapter III, Theorem 1.1]

Theorem 0.4. There exist the natural multiplicative transformations of functors (naturality for X as well as $\mathcal{A}$)

$$H^\bullet_\Phi(X;\mathcal{A}) \xrightarrow{\theta} {}_S H^\bullet_\Phi(X;\mathcal{A}) \xleftarrow{\mu^*} {}_\Delta H^*_\Phi(X;\mathcal{A})$$

in which the groups ${}_\Delta H^\bullet_\Phi(X;\mathcal{A})$, and hence μ^* are defined only for locally constant $\mathcal{A}$ and are the classical singular cohomology groups when Φ is paracompactifying. The map μ^* is an isomorphism when X when $\mathcal{A}$ has finitely generated stalks. Both θ and μ^* are isomorphisms when X is locally compact Hausdorff and Φ is paracompactifying. Both natural transformations extend to closed pairs of spaces with the same conclusions.

Note that we have an exact sequence of complexes (definition of exact sequence of complexes of sheaves)

$$0 \longrightarrow \sigma_{\leq p-1}(\Omega^\bullet_{M,\mathbb{C}}(-)) \longrightarrow \Lambda(p)^\infty_D \longrightarrow \Lambda(p) \longrightarrow 0$$

where $\sigma_{\leq p-1}(\Omega^{\bullet}_{M,\mathbb{C}}(-))$ denotes the complex $\Omega^0_{M,\mathbb{C}}(-) \to \cdots \to \Omega^{p-1}_{M,\mathbb{C}}(-)$ obtained by chopping the part of the complex of sheaves $\Omega^{\bullet}_{M,\mathbb{C}}(-)$ in degrees $\geq p$. If M is paracompact, we know the sheaves $\Omega^p_{M,\mathbb{C}}(-)$ are soft (Proof of the claim.) Then the hypercohomology of the complex of sheaves are simply

$$H^q(M; \sigma_{\leq p-1}\Omega^p_{M,\mathbb{C}}(-)) = \begin{cases} H^q_{\mathrm{dR}}(M) \times \mathbb{C}, & q \leq p-2, \\ \Omega^{p-1}_{\mathbb{C}}(M)/d(\Omega^{p-2}_{\mathbb{C}}(M)), & q = p-1 \\ 0, & q \geq p. \end{cases} \tag{0.6}$$

Theorem 0.5. Let M be a smooth paracompact manifold such that the sheaves $\Omega^p_M(-)$ are soft. The smooth Deligne cohomology groups $H^{\bullet}(M; \Lambda(p)^{\infty}_D)$ are as follows:

1. For $q \leq p-1$, the group $H^p(M; \Lambda(p)^{\infty}_D)$ fits in the exact sequence

$$0 \longrightarrow H^{q-1}(M; \Lambda(p)) \longrightarrow H^{q-1}(M; \mathbb{C}) \longrightarrow H^q(M; \Lambda(p)^{\infty}_D) \xrightarrow{\kappa} \mathrm{Tor} H^q(M; \Lambda(p)) \longrightarrow 0 \tag{0.7}$$

2. The group $H^p(M; \Lambda(p)^{\infty}_D)$ fits in the exact sequence

$$0 \longrightarrow \Omega^{p-1}_{\mathbb{C}}(M)/\Omega^{p-1}_{\mathbb{C},0}(M) \longrightarrow H^p(M; \Lambda(p)^{\infty}_D) \xrightarrow{\kappa} H^p(M; \Lambda(p)) \longrightarrow 0. \tag{0.8}$$

3. For $q \geq p+1$, we have

$$H^q(M; \Lambda(p)^{\infty}_D) \cong H^q(M; \Lambda(p)). \tag{0.9}$$

References

[Bre] Glen E Bredon. *Sheaf Theory*, volume 170. Springer Science & Business Media.

[CS] Jeff Cheeger and James Simons. *Differential Characters and Geometric Invariants*, volume 1167 of *Lecture Notes in Mathematics*, pages 50–80. Springer Berlin Heidelberg.