Algebra II

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ABSTRACT: These are live-TeXed notes for *Algebra II* in Spring 2021. The main reference for this course was [Wei]. The notes are incomplete and unedited. All errors introduced are mine.

Contents

1 Complexes

Proposition 1.1. Small filtered colimits in $\mathbf{Set}$ are exact. In other words, for all small filtered category $\mathbf{I}$ and finite category $\mathbf{J}$, and functor $F : \mathbf{I} \times \mathbf{J} \to \mathbf{Set}$, the natural map

$$\operatorname*{colim}_{i\in\mathbf{I}} \lim_{j\in\mathbf{J}} F(i,j) \to \lim \operatorname{colim} F(i,j)$$

is a bijection.

Proof. Conf. Weizhe Zheng's Notes. □

Theorem 1.2 (Freyd-Mitchell)**.** Let $\mathbf{A}$ be a small abelian category, then there exists a ring R, and a fully faithful exact functor $F : \mathbf{A} \to R - \mathbf{Mod}$.

Proof. See Weibel. □

Lemma 1.3. Let A be an abelian category, and $S \subseteq \mathrm{ob}\mathbf{A}$ be a non-empty set of objects.There is a full small abelian subcategory $\mathbf{B}$ of $\mathbf{A}$ containing S.

Proof. Define inductively a sequence $(A_n)_{n\geq 0}$ of subcategory of $\mathbf{A}$ as follows. Let $\mathbf{A_0}$ be the full subcategory whose objects are these in S. Given $\mathbf{A_n}$, let $\mathbf{A_{n+1}}$ be the full subcategory of $\mathbf{A}$ consisting of the objects in $\mathbf{A_n}$ together with

- a single representation for the kernel and cokernel in $\mathbf{A}$ for every morphism in $\mathbf{A_n}$
- A single representation of every finite product in $\mathbf{A}$ of objects in $\mathbf{A_n}$

Then $\mathbf{A_n}$ are small full subcategory of $\mathbf{A}$, let $\mathbf{B} = \cup_{n\geq 0}\mathbf{A_n}$, then $\mathbf{B}$ is a small and full abelian subcategory. □

Proposition 1.4. Let $\mathbf{A}$ be an abelian category. Consider a commutative diagram

$$\begin{tikzcd}
 & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y''
\end{tikzcd}$$

Then there exists an exact sequence

$$\ker u' \to \ker u \to \ker u'' \to \operatorname{coker} u' \to \operatorname{coker} u \to \operatorname{coker} u''.$$

If moreover f is a monomorphism then so is $\ker u' \to \ker u$, if g' is an epimorphism, then so is $\operatorname{coker} u \to \operatorname{coker} u''$.

Proof. By the Freyd-Mitchell Embedding Theorem and the previous Lemma, we may replace $\mathbf{A}$ by $R - \mathbf{Mod}$. Then one can prove the proposition by diagram chasing (left as an exercise). □

Corollary 1.5. Let $0 \to X' \to X \to X'' \to 0$ be a short exact sequence, in $\mathbf{A}$, TFAE

1. f admits are traction, *i.e.* there exists $r : X \to X'$ such that $r \circ f = \mathrm{id}_{X'}$
2. g admits a section $s : X'' \to X$ such that $g \circ s = \mathrm{id}_{X''}$
3. The sequence splits naturally.

Definition 1.1. A short exact sequence satisfying the above conditions is said to be **split**.

Corollary 1.6. Let $\mathbf{A}$ be an abelian category. Consider a commutative diagram in $\mathbf{A}$

$$\begin{tikzcd} X^0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & X^2 \arrow[r] \arrow[d] & X^3 \arrow[r] \arrow[d] & X^4 \arrow[d] \\ Y^0 \arrow[r] & Y^1 \arrow[r] & Y^2 \arrow[r] & Y^3 \arrow[r] & Y^4 \end{tikzcd}$$

with exact rows. If u^0 is an epi and u^4 is anono, and u^1, u^3 are isomorphisms, then u^2 is an isomorphism.

Definition 1.2. $\mathbf{A}$ add cat A **(cochain) complex** in $\mathbf{A}$ consists of $X = (X^n, d^n)_{n\in\mathbb{Z}}$, where $X^n \in \mathbf{A}$ and $d^n : X^n \to X^{n+1}$ is a morphism in $\mathbf{A}$ called **differential**, such that $d^{n+1} \circ d^n = 0$. A morphism of cochain complexes $X \to Y$ is a collection of morphism where $(f^n)_{n\in\mathbb{Z}}$ where $f^n : X^n \to Y^n$ in $\mathbf{A}$ and $d_Y^n \circ f = f^{n+1} \circ d_X^n$.

Let $C(\mathbf{A})$ be the category of $\mathbf{A}$, if $\mathbf{A}$ is additive, $C(\mathbf{A})$ is also additive. If $\mathbf{A}$ is an abelian category, then so is $C(\mathbf{A})$. If $f : X \to Y$ is in $C(\mathbf{A})$, then $(\ker f)^n = \ker f^n$ and $(\operatorname{coker} f)^n = \operatorname{coker} f^n$.

Definition 1.3. Let $\mathbf{A}$ be an abelian category, and $X \in C(\mathbf{A})$ we define $Z^n(X) = \ker d^n$, $B^n(X) = \operatorname{im} d^{n-1}$ and $H^n(X) = \operatorname{coker} B^n(X) \hookrightarrow Z^n(X)$, and call them the **cocycle, coboundary, cohomology objects** of deg n.

The functors $Z^n, B^n, H^n : C(\mathbf{A}) \to \mathbf{A}$ are additive.

Example (de Rham cohomology).

Definition 1.4. A complex X is said to be **cyclic** if $H^n(X) = 0, \forall n \in \mathbb{Z}$. A morphism of complexes $X \to Y$ is called a **quasi-isomorphism** if $H^n(f) : H^n(X) \to H^n(Y)$ is an isomorphism for all $n \in \mathbb{Z}$.

We have the following operations on complexes.

1. $\forall X \in C(\mathbf{A}), \forall m \in \mathbb{Z}$, let $X[m] \in C(\mathbf{A})$ be the complex with $X[m]^n = X^{m+n}$ and $d^n[m] : X[m]^n \to X[m]^{n+1}$ is given by $(-1)d^{m+n}$. $H^n(X[m]) = H^{n+m}(X)$
2. $\forall n \in \mathbb{Z}, X \in C(\mathbf{A})$, we define $\tau^{\leq n}X = \cdots \to X^{n-1} \to \ker d^n \to 0 \to \cdots$. And $\tilde{\tau}^{\leq n}X = (\cdots \to X^{n-1} \to X^n \to \operatorname{im} d^n \to 0 \to \cdots$, $\tau^{\geq n}X = (\cdots \to 0 \to \operatorname{coker} d^{n-1} \to X^{n+1} \to X^{n+2} \to \cdots$ and $\tilde{\tau}^{\geq n} = (\cdots \to 0 \to \operatorname{im} d^{n-1} \to X^n \to X^{n+1} \to \cdots)$.
3. direct sum
4. tensor product $X, Y \in C(\mathbf{A})$, $(X \otimes Y)^n = \bigoplus_{i+j=n}(X^i \otimes Y_j)$, $d^n(a \otimes b) = d_X a \otimes b + (-1)^{\deg a} d_Y b$

We have $\tau^{\leq n}X \to \tilde{\tau}^{\leq n}X \to X \to \tilde{\tau}^{\geq n}X \to \tau^{\geq n}X$ in $C(\mathbf{A})$. and exact sequences

$$0 \to \tilde{\tau}^{\leq n-1} \to \tau^{\leq n}X \to H^n(X[-n] \to 0 \tag{1.1}$$

Theorem 1.7. If $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $C(\mathbf{A})$, then there are natural maps $\partial : H^n(Z) \to H^{n+1}(X)$ and a cohomology long exact sequence

$$\cdots \to H^{n-1}(Z) \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(Z) \to \cdots$$

Similar results hold for chain complexes.

Proof. We only show the statement for cohomology. Consider

□

Remark. If we have a commutative diagram of short exact sequences, we then get a commutative diagram of long exact sequences.

Corollary 1.8. Let be commutative diagram of complexes with exact rows. If two of u, v, w are quasi-isomorphisms, then so is the third one.

Definition 1.5. Let $f : X \to Y$ be a morphism in $C(\mathbf{A})$. The **mapping cone** $\mathrm{Cone}(f)$ of f is the complex $\mathrm{Cone}(f)^n = X^{n+1} \oplus Y^n$ with the differential

$$d^n = \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix}$$

And we can verify that $\mathrm{Cone}(f)$ is indeed a cochain complex.

By construction, we have a short exact sequence in $C(\mathbf{A})$

$$0 \to Y \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} X[1] \to 0,$$

which induces a cohomology long exact sequence

$$\cdots \to H^{n-1}(X[1]) \xrightarrow{\delta} H^n(Y) \to H^n(\mathrm{Cone}(f)) \to H^n(X[1]) \to \cdots$$

Proposition 1.9. Via the isomorphism $H^{n-1}(X[1]) \simeq H^n(X)$, the connecting morphism δ can be identified with $H^n(f) : H^n(X) \to H^n(Y)$.

Proof. By the snake lemma, left as exercise. □

Corollary 1.10. A morphism $f : X \to Y$ is a quasi-isomorphism in $C(\mathbf{A})$ iff $\mathrm{Cone}(f)$ is acyclic.

Proposition 1.11. Consider a short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $C(\mathbf{A})$, then the map $\phi = (0, g) : \mathrm{Cone}(f) \to Z$ is a quasi-isomorphism.

Proof. We have a short exact sequence

$$0 \to \mathrm{Cone}(\mathrm{id}_X) \xrightarrow{\psi} \mathrm{Cone}(f) \xrightarrow{\phi} Z \to 0$$

where ψ is associated to the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$\square$

Since $\mathrm{Cone}(\mathrm{id}_X)$ is acyclic, the long exact sequence implies that ϕ is a quasi-isomorphism.

Definition 1.6. A morphism $f : X \to Y$ in $C(\mathbf{A})$ is homotopic to zero, if for all $n \in \mathbb{Z}$, there is a morphism $s^n : X^n \to Y^{n-1}$ such that $f^n = s^{n+1}d_X^n + d_Y^{n-1}s^n$.
Two morphisms $f, g : X \to Y$ are homotopic if $f - g$ is homotopic to zero.
An object $X \in C(\mathbf{A})$ is homotopic zero if id_X is homotopic to zero.
We say $f : X \to Y$ is a **homotopy equivalence**, if there exists $g : Y \to X$, such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X.

Proposition 1.12. If $f : X \to Y$ is a homotopic to zero, then $H^n(f) : H^n(X) \to H^n(Y)$ is 0. If f, g are homotopic , then $H^n(f) = H^n(g)$.

Proof. It suffices to show the first assertion. If $f = d_Y s + s d_X$, the restriction of f to $Z^n(X)$ is equal to $d_Y s$, composited with $Z^n(Y) \to H^n(Y)$ is zero, implying $H^n(f) = 0$. $\square$

A homotopy equivalence is a quasi-isomorphism.

Definition 1.7. The **homotopy category** $K(\mathbf{A})$ of an abelian category $\mathbf{A}$ is defined as $\mathrm{ob}K(\mathbf{A}) = \mathrm{ob}C(\mathbf{A})$, as well as

$$\mathrm{Hom}_{K(\mathbf{A}}(X, Y) = \mathrm{Hom}_{C(\mathbf{A})}(X, Y)/Ht(X < Y$$

where $Ht(X, Y) = \{\, f : X \to Y \mid f \text{homotopic to} 0 \,\}$.

Definition 1.8. $\mathbf{A}$ an abelian category. A **double complex** in $\mathbf{A}$ consists of objects $(X^{i,j})_{i,j\in\mathbb{Z}}$ of $\mathbf{A}$, and differentials $d_I^{i,j} : X^{i,j} \to X^{i+1,j}, d_{II}^{i,j} : X^{i,j} \to X^{i,j+1}$ satisfying

$$d_I^2 = 0, d_{II}^2 = 0, d_I d_{II} = d_{II} d_I.$$

We denote $C^2(\mathbf{A})$ as the category of double complexes of $\mathbf{A}$.

$F_I, F_{II} : C^2(\mathbf{A}) \to C(\mathbf{C}(A))$, which are isomorphic functors. $X \in C^2(\mathbf{A})$, set $H_I(X)^{i,j} = \ker d_I^{i,j} / \mathrm{im}\, d_I^{i-1,j}$ and $H_{II}(X)^{i,j}$ similarly.

Definition 1.9. $X \in C^2(\mathbf{A})$ we define two complexes in $\mathbf{A}$

$$(\mathrm{tot}_\oplus X)^n = \oplus_{i+j=n} X^{i,j}$$

if coproducts exist, and

$$(\mathrm{tot}_\times X)^n = \prod_{i+j=n} X^{i,j}$$

if products exist, with differentials for $i + j = n$ the composition $X^{i,j} \to (\mathrm{tot}_\oplus X)^n \xrightarrow{d^n} (\mathrm{tot}_\oplus X)^{n+1}$ is given by $d_I^{i,j} + (-1)^i d_{II}^{i,j}$; the composition $(\mathrm{tot}_\times X)^{n-1} \xrightarrow{d^{n-1}} (\mathrm{tot}_\times X)^n \to X^{i,j}$ is given by $d_I^{i-1,j} + (-1)^i d_{II}^{i,j-1}$.
A double complex X is is **biregular** if for all $n \in \mathbb{Z}$, $X^{i,j} = 0$ for all but finitely many pairs (i, j) with $i + j = n$. Let $C_r^2(\mathbf{A})$ be the full subcategory of biregular complexes.

For $X \in C_r^2(\mathbf{A})$, we have $\mathrm{tot}_\oplus \xrightarrow{\simeq} \mathrm{tot}_\times$, simply denoted by $\mathrm{tot}X$.

Example. $f^* : X^* \to Y^*$ a morphism in $C(\mathbf{A})$. Consider the double complex $Z^{*,*}$ with $Z^{-1,*} = X^*, Z^{0,*} = Y^*, Z^{i,*} = 0 i \neq -1, 0$, with differentials $f^j : Z^{-1,j} \to Z^{0,j}$, then we have an isomorphism in $C(\mathbf{A})$

$$\mathrm{tot}(Z^{*,*}) \simeq \mathrm{Cone}(f^*).$$

Let $\mathbf{A}, \mathbf{A}', \mathbf{A}''$ be abelian categories, and $F : \mathbf{A} \times \mathbf{A}' \to \mathbf{A}''$ be additive in each variable, then F extends to a functor $F : C(\mathbf{A}) \otimes C(\mathbf{A}') \to C^2(\mathbf{A}'')$ additive in each variable.

$X \in C(\mathbf{A}), Y \in C(\mathbf{A}')$, $C^2(F)(X, Y)$ is defined by

$$C^2(F)(X, Y) = F(X^i, Y^j)$$

with $d_I^{i,j} = F(d_X^i, \mathrm{id}_{Y^j}), d_{II}^{i,j} = F(\mathrm{id}_{X^i}, d_Y^j)$.

Example. R a ring. $- \otimes_R - : \mathbf{Mod} - \mathbf{R} \otimes \mathbf{R} - \mathbf{Mod} \to \mathbf{Ab}$ additive in each variable, thus extends to $- \otimes_R - : ($

Exercise. Prove Thm 1.5.12.

Exercise. Prove the snake lemma for $\mathbf{A} = \mathbf{R} - \mathbf{Mod}$.

Exercise. $\mathbf{A}$ Abelian category, consider a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & X \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X'' \arrow[d] \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Y'' \arrow[d] \\
0 \arrow[r] & Z \arrow[r] & Z' \arrow[r] & Z''
\end{tikzcd}$$

Assume that all columns are exact. If the second and third rows are exact, show that the first row is also exact.

Exercise. Prove Proposition 1.6.8.

Definition 1.10. Let $\mathbf{C}$ be a category. An objective P of $\mathbf{C}$ is called **projective** if given a morphism $f : P \to Y$ and an epimorphism $u : X \to Y$ in $\mathbf{C}$, there exists $g : P \to X$ such that $f = ug$

$$\begin{tikzcd}
 & P \arrow[d, "f"] \arrow[dl, dotted, "g"'] \\
X \arrow[r, "u"'] & Y.
\end{tikzcd}$$

An **injective** object $I \in \mathbf{C}$ is defined dually.

Proposition 1.13. $\mathbf{A}$ abelian category. $P \in \mathbf{A}$, TFAE

1. P projective
2. $\mathrm{Hom}_{\mathbf{A}}(-, P) : \mathbf{A} \to \mathbf{Ab}$ is exact
3. every short exact sequence $0 \to A \to B \to P \to 0$ splits.

Proposition 1.14. $P \in \mathbf{R}-\mathbf{Mod}$, TFAE

1. P projective
2. P is a direct summand of some free R-module.

Proof. Let $F(A)$ be the free R-module over the set underlying an R-module A, so there is a surjection $\pi : F(A) \to A$. If A is projective, then there is a morphism $i : A \to F(A)$ making the diagram

$$\begin{tikzcd} & A \arrow[dl, dotted, "i"'] \arrow[d, "\mathrm{id}_A"] & \\ F(A) \arrow[r, "\pi"'] & A \arrow[r] & 0 \end{tikzcd}$$

commute, *id est*, $\pi \circ i = \mathrm{id}_A$, showing that A is a direct summand of $F(A)$.
Conversely, if A is a direct summand of some free R-module F, then there exists $i : A \to F$ and $\pi : F \to A$ such that $\pi \circ i = \mathrm{id}_A$. Given any test diagram

$$\begin{tikzcd} & A \arrow[d, "f"] & \\ X \arrow[r] & Y \arrow[r] & 0 \end{tikzcd},$$

we hope to find some $g : A \to X$ lifting $f : A \to Y$. This can be done by consider the composition $f \circ \pi : F \to Y$, which can be lifted by some $h : F \to X$, since F is free. Then we let $g = h \circ i$, and g indeed lifts f. $\square$

Example. Over $\mathbb{Z}$, the notions of projectivity and freedom are equivalent. In fact, any submodule of a free $\mathbb{Z}$-module is again free. Furthermore, in $\mathbb{Z} - \mathbf{Z}$, the notion of projectivity and freedom and torsion-freedom are all equivalent.

Theorem 1.15 (Baer's criterion)**.** R a ring. $I \in R - \mathbf{Mod}$, then I is injective iff for any left ideal $\mathfrak{a} \subseteq I$, every morphism $\mathfrak{a} \to I$ can be extended to $R \to I$.

Proof. $\Longrightarrow$ clear.
Conversely, let $f : A \to I$ and $u : A \hookrightarrow B$ be an injection. We view $A \subseteq B$ and u as the inclusion. Consider

$$S = \{\, f' : A' \to I, A \subseteq A' \subseteq B, f \mid_A = f \,\}$$

ordered by inclusion. By Zorn's lemma, there is a maximal element of S given by $f' : A' \to I$ extending f.

Claim 1.16. $A' = B$.

$\square$

If not take $b \in B \backslash A'$ and consider $A' + Rb$. Then set

$$\mathfrak{a} = \{\, r \in R \mid rb \in A' \,\},$$

which is a left ideal of R and $f_0 : \mathfrak{a} \to I, r \mapsto f'(rb)$ is an R-module homomorphism, implying f_0 extends to $\tilde{f}_0 : R \to I$, let $u = \tilde{f}_0(1)$ then for all $r \in R$, $\tilde{f}_0(r) = r\tilde{f}_0(1) = ru$. We can define

$$A' + Rb \to I, a + rb \mapsto f'(a) + ra$$

is well defined, since if $rb \in A' \cap Rb$, then $r \in \mathfrak{a}$ and $f'(rb) = f_0(r) = ru$. This gives a contradiction to the choice of f'.

Recall that if M is an R-module, it is **divisible** if for all $m \in M, r \in R\backslash\{0\}$, then there exits $x \in M$ such that $m = rx$, iff $\forall r \in R\backslash\{0\}, rM = M$.

Proposition 1.17. Let R be an integral domain (possibly non-commutative), then any injective R-module is divisible. If R is a PID (left ideal) or a Dedekind domain, the converse also holds.

Proof. Omitted. See Zheng's notes. □

Example. $\mathbb{Q}/\mathbb{Z}, \mathbb{Q}$ are injective $\mathbb{Z}$-modules. If $R = \mathbb{Z}[x] \subset K$, where K is the fraction field of R, then K/R is divisible, but not injective. (exercise).

Definition 1.11. $M \in R-\mathbf{Mod}$ is **flat** if $- \otimes_R M : \mathbf{Mod} - R \to \mathbf{Ab}$ is exact. Similarly, $N \in \mathbf{Mod} - R$ is flat if $N \otimes_R - : R - \mathbf{Mod} \to \mathbf{Ab}$ is exact.

Projective modules are flat. The converse is not true in general. $\mathbb{Q}$ is flat over $\mathbb{Z}$, but not projective.

$M \in R-\mathbf{Mod}$, $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a right R-module, $r \in R, (fr)(a) = f(ra), \forall a \in M$.

Proposition 1.18.

1. $M \in R - \mathbf{Mod}$ is flat iff $M^* \in \mathbf{Mod} - R$ is injective
2. $M \in R-\mathbf{Mod}$ is flat iff $\forall \mathfrak{a} \subseteq R$, the natural map $\mathfrak{a} \otimes_R M \to \mathfrak{a}M$ is an isomorphism of abelian groups.

Proof. Exercise. □

Remark. We have $M \hookrightarrow M^{**}$, which may not be an isomorphism.

Theorem 1.19 (Lazard). Every flat R-module is a filtered colimit of free R-modules.

Lemma 1.20. A $\mathbb{Z}$-module is flat iff it is torsion-free.

Proof. Apply the previous proposition. □

Over $\mathbb{Z}$,

$$\text{projective modules} = \text{free modules} \subsetneq \text{flat modules} = \text{torsion free}$$

$$\text{injective modules} = \text{divisible modules}$$

$\mathbb{Q}, \mathbb{Q}/\mathbb{Z}$ are injective, but not projective. $\mathbb{Q}$ is flat, $\mathbb{Q}/\mathbb{Z}$ are not flat.

Recall that $M \in R = \mathbf{Mod}$ is called finitely presented if there exists an exact sequence

$$R^m \to R^n \to M \to 0, m, n \in \mathbb{N}$$

Proposition 1.21. Any finitely presented flat module M is projective.

Proof. Let $A \to B \to 0$ be an exact sequence in $R - \mathbf{Mod}$, we need to show $\mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, M) \to 0$ is exact. It is enough to show that $0 \to \mathrm{Hom}_R(M, B)^* \to \mathrm{Hom}_R(M, A)^*$ is exact. Since M is finitely presented, $A^* \otimes_R M \xrightarrow{\sim}$

$\mathrm{Hom}_R(M,A)^*, f\otimes m \mapsto (g\mapsto f(g(m)))$ is an isomorphism.
Since $0 \to B^* \to A^*$ is exact and M is flat, $0\to B^*\otimes M \to A^*\otimes M$ is exact, and $B^*\otimes \simeq \mathrm{Hom}_R(M,B)^*, A^*\otimes_R M \simeq \mathrm{Hom}_R(M,A)^*$. ☐

We come back to a general abelian category $\mathbf{A}$.

Definition 1.12. $M\in\mathbf{A}$. A left resolution of M is a chain complex P_* with $P_i=0$ for $i<0$, together with a map $P_* \xrightarrow{\epsilon} M$ such that the complex $\cdots\to P_2\to P_1\to P_0\xrightarrow{\epsilon} M\to 0$ is exact.
It is a **projective resolution** if each P_i is projective.
P_* a complex of projectives with $P_i=0, i<0$, then a map $\epsilon : P_0\to M$ gives a resolution of M iff $\epsilon : P_*\to M$ is a quasi-isomorphism of chain complexes.

Proposition 1.22. Every R-module M has a projective resolution.

Proof. Take a projective P_0 and a surjection $\epsilon : P_0\to M$ and set $M_0 \ker\epsilon$. Inductively, given M_{n-1}, take a projective P_n and say $P_n \overset{\epsilon_n}{\twoheadrightarrow} M_{n-1}$. Set $M_n=\ker\epsilon_n$ and let d_n be the composition $P_n\to M_{n-1}\to P_{n-1}$. Then (P_*,d_*) is a projective resolution of M. ☐

$\mathbf{A}$ abelian category. We say $\mathbf{A}$ has enough projectives if $\forall A\in\mathbf{A}$, there is an epimorphism $P\to A$ with P a projective object in $\mathbf{A}$.

Theorem 1.23. Let $P_*\xrightarrow{\epsilon} M$ be a chain complex with P_i projective, and $f' : M\to N$ a map in $\mathbf{A}$. Then for every resolution $Q_*\to N$ of N, there is a chain map $f : P_*\to Q_*$ lifting f' in the sense $\eta f = f'\epsilon$. Then chain map f is unique up to chain homotopy.

Proof. abcdefg. ☐

Corollary 1.24. Projective resolutions are unique up chain homotopy.

Lemma 1.25 (horseshoe)**.** Suppose given a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\epsilon"] & A' \arrow[r] \arrow[d, "i_A"] & 0 \\
 & & & & A \arrow[d, "\pi_A"] & \\
\cdots \arrow[r] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\epsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

where the column is exact and the rows are projective resolutions. Set $P_n=P_n'\oplus P_n''$. Then the P_n resemble to form a projective resolution of A, and the right hand column lifts to an exact sequence of complexes

$$0\to P'\to P\to P''\to 0$$

Proof. exercise. ☐

Proposition 1.26. $F : \mathbf{A}\to\mathbf{B}$ functor between two abelian categories.
If $F\dashv G$ and G is right exact, then F carries projective objects to projective objects.

If $G \dashv F$ and G is left exact, then F carries injective objects to injective objects.

Proof. We only prove the second assertion. Let $I \in \mathbf{A}$ be an injective object, we must show that $\mathrm{Hom}_B(-, FI)$ is exact. Given a monomorphism $f : B \to B'$ in $\mathbf{B}$. Since $G \dashv F$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{B}}(B', FI) \arrow[r, "f*"] \arrow[d] & \mathrm{Hom}_{\mathbf{B}}(B, FI) \arrow[d, "\simeq"] \\ \mathrm{Hom}_{\mathbf{A}}(GB', I) \arrow[r, "(Gf)^*"] & \mathrm{Hom}_{\mathbf{A}}(GB, I) \end{tikzcd}$$

Since G is left exact, and I is injective, $(Gg)^*$ is epimorphic, so f^* is epimorphism, $\mathrm{Hom}_B(-, FI)$ is exact. $\square$

Proposition 1.27. Every R-module is a submodule of an injective module.

Proof. Step 1, Assume that $R = \mathbb{Z}$, let M be a $\mathbb{Z}$-module, choose a free $\mathbb{Z}$-module F and a surjection $F \to M$ with kernel K, write $F = \mathbb{Z}^{(I)}$ and $D = \mathbb{Q}^{(I)}$ then $M = F/K \subseteq D/K$ and D/K is divisible thus is injective.
Step 2, For general R, the functor $\mathrm{Hom}_{\mathbb{Z}}(R, -)$ admits a left adjoint, the restriction of scalars, which is exact. So $\mathrm{Hom}_{\mathbb{Z}}(R, -)$ carries injectives to injectives. Let M be an R-module, by step 1, $M \hookrightarrow I$ is an injection of abelian groups with I injective. Then $\mathrm{Hom}_{\mathbb{Z}}(R, I)$ is an injective R-module, and we have

$$M \simeq \mathrm{Hom}_R(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, I).$$

$\square$

Definition 1.13. Let $\mathbf{A}$ be an abelian category and $A \in \mathbf{A}$, a right resolution of A is a cochain complex I^* with $I^i = 0, i < 0$ and a map $A \to I^0$ such that the complex

$$0 \to A \to I^0 \xrightarrow{d} I^1 \to \cdots$$

is exact. It is an injective resolution if each I^i is injective.

An abelian category $\mathbf{A}$ has enough injectives if for all $A \in \mathbf{A}$ there are I^* and a monomorphism $A \to I^*$. In particular, $R - \mathbf{Mod}$ has enough injectives.

Theorem 1.28. Let $A \to I^*$ be a cochain complex with I^* injective, and $f' : A' \to A$ a map in $\mathbf{A}$. Then for any resolution $A' \to J^*$, there is a cochain map $f : J^* \to I^*$ lifting f'. The map f is unique up to cochain homotopy equivalence. In particular, injective resolutions are unique up to homotopy.

Theorem 1.29. X a topological space. The category $\mathbf{Shv}(X)$ has enough injectives.

Proof. Firstly, for all $x \in X$, consider the stalk functor $(-)_x : \mathbf{Shv}(X) \to \mathbf{Ab}, \mathcal{F} \mapsto \mathcal{F}_x$. This is an exact functor. Moreover, $\mathcal{F} = 0$ iff $\mathcal{F}_x, \forall x \in X$.
Then for all $x \in X$ and for all $A \in \mathbf{Ab}$, consider the skyscraper sheaf $x_* A \in \mathbf{Shv}(X)$. Then $x_* : \mathbf{Ab} \to \mathbf{Shv}(X)$ is a functor, and $(-)_x \dashv x_*$. So x_* carries injective abelian groups to injective sheaves in $\mathbf{Shv}(X)$.
Finally $\forall \mathcal{F} \in \mathbf{Shv}(X)$ for each $x \in X$, take an injective $\mathcal{F}_x \hookrightarrow I_x$ with I_x some injective abelian group.Then we get $\mathcal{F} \to x_*\mathcal{F}_x \to x_*\mathcal{I}_x$, and let $\mathcal{I} = \prod_{x \in X} x_* I_x$, which in turn

induces $\mathcal{F} \to \mathcal{I}$. The map $\mathcal{F} \to \mathcal{I}$ is an injection since it is so at each stalk. □

Exercise. $\mathbf{A}$ an abelian category. $X \in C(\mathbf{A})$ is called split if there exist $s^n : X^{n+1} \to X^n$ such that $d^n s^n d^n = d^n, \forall n \in \mathbb{Z}$. Prove that X is split iff id_X is homotopic to zero.

Exercise. $\mathbf{A}$ an abelian category.

1. Show that a cochain complex P^* is a projective object in $C(\mathbf{A})$ iff it is split exact complex of projectives in $\mathbf{A}$.
2. Show that if $\mathbf{A}$ has enough projectives so does $C(\mathbf{A})$.

Exercise. Let $m \geq 2$ be an integer and $R = \mathbb{Z}/m\mathbb{Z}$. Show that R is an injective R-module while $\mathbb{Z}/d\mathbb{Z}$ is not an injective R-module when $d|m$ and $p|\gcd(d, \frac{m}{d})$ for some prime p.

2 Derived Functors

2.1 Delta Functors and Derived Functors

Definition 2.1. A covariant cohomological δ-functor from $\mathbf{A}$ to $\mathbf{B}$ is a collection of additive functors $T^n : \mathbf{A} \to \mathbf{B}, n \geq 0$, together with morphisms $\delta^n : T^n(C) \to T^{n+1}(A)$ defined for each short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathbf{A}$. The following two conditions are imposed:

(i) For each short exact sequence as above, there exists a long exact sequence

$$0 \to T^0(A) \to T^0(B) \to T^0(C) \xrightarrow{\delta^1} T^1(A) \to T^1(B) \to T^1(C) \to \cdots \to T^n(C) \xrightarrow{\delta^n} T^{n+1}(A) \to \cdots$$

(ii) For every morphism of short exact sequences from $0 \to A' \to B' \to C' \to 0$ to $0 \to A \to B \to C \to 0$, there is a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & T^n(C') \arrow[r, "\delta^n"] \arrow[d] & T^{n+1}(A') \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & T^n(C) \arrow[r, "\delta^n"] & T^{n+1}(A) \arrow[r] & \cdots \end{tikzcd}$$

between the induced long exact sequences.

A morphism $S \to T$ of δ-functors is a system of natural transformations $S^n \to T^n$ that commute with δ.
Similarly we may define a covariant homological δ-functor from $\mathbf{A}$ to $\mathbf{B}$. Which is a collection of additive functor $T_n : \mathbf{A} \to \mathbf{B}, n \geq 0$, together with $\delta_n : T_n(C) \to T_{n-1}$ defined for every short exact sequence $0 \to A \to B \to C \to 0$ with similar imposed conditions as above.

We immediately know that T^0 is left exact and T_0 is right exact by definition.

Example. Cohomology gives a cohomological δ-functor $H^* : C^{\geq 0}(\mathbf{A}) \to \mathbf{A}$.
$n \in \mathbb{N}, T_0 : \mathbf{Ab} \to \mathbf{Ab}, A \mapsto A/nA, T_1 : \mathbf{Ab} \to \mathbf{Ab}, A \mapsto A[n]$. Consider the short exact

sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \times n} & & \downarrow{\scriptstyle \times n} & & \downarrow{\scriptstyle \times n} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

and apply the snake lemma, we have

$$0 \to A/nA \to B/nB \to C/nC \to A[n] \to B[n] \to C[n] \to 0.$$

Definition 2.2. A cohomology δ-functor T is universal if given any other δ-functor S and $\alpha_0 : T^0 \to S^0$, there exists a unique morphism $\alpha^n : T^n \to S^n$ of δ-functors extending α^0. Similarly we can define universal homological δ-functor. A universal δ-functor T with given $T^0 = F$ is exists is unique.

Definition 2.3. An additive functor $F : \mathbf{A} \to \mathbf{B}$ is effaceable if for all $A \in \mathbf{A}$ there exists a monomorphism $u : A \to I$ such that $F(u) = 0$. It is coeffaceable if $\forall A \in \mathbf{A}$, there exists an epimorphism $u : P \to A$ such that $F(u) = 0$.

Theorem 2.1. Let $T = (T^i)$ be a cohomological δ-functor from $\mathbf{A}$ to $\mathbf{B}$. If T^i is effaceable for each $i > 0$ then T is universal. If $T = (T_i)$ is a homological δ-functor such that T_i is coeffaceable for each $i > 0$, then T is universal.

Proof. We only prove the first assertion. Let S be a δ-functor from $\mathbf{A}$ to $\mathbf{B}$ and $\alpha^0 : T^0 \to S^0$ a morphism of functors, we have to show that there exists a morphism of δ-functors $(\alpha^n) : T \to S$. We construct α^n by induction. Suppose we have $\alpha^i, i \leq n$ compatible with δ in degree $\leq n$. $\forall A \in \mathbf{A}$ by assumption there is a monomorphism $u : A \to I$ such that $T^{n+1}(u) = 0$. Let $M = \operatorname{coker} u$ then $0 \to A \to I \to M \to 0$ is exact.

$$\begin{array}{ccccccc} T^n(I) & \longrightarrow & T^n(M) & \longrightarrow & T^{n+1}(A) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ S^n(I) & \longrightarrow & S^n(M) & \longrightarrow & S^{n+1}(A) & & \end{array}$$

with top row exact. We define $\alpha_A^{n+1} : T^{n+1}(A) \to S^{n+1}(A)$ to be the unique morphism making the above right square commute.
We check α_n^{n+1} does not depend on u. Indeed, if $u' : A \to I^1$ with $T^{n+1}(u') = 0$. Let $M' = \operatorname{coker} u^1$ and $\alpha_A^{n+1} : T^{n+1}(A) \to S^{n+1}(A)$, we have to show that $\alpha_{A,u}^{n+1} = \alpha_{A,u'}^{n+1}$. Let $I'' = I \coprod_A I'$ be he pushout of u and u'. Then the induced $u'' : A \to I''$ is also monomorphic and $T^{n+1}(u'') = 0$. Let $M'' = \operatorname{coker} u''$ and $\alpha_{A,u''}^{n+1}$. It suffices to check that $\alpha_{A,n}^{n+1} = \alpha_{A,u'}^{n+1} = \alpha_{A,u''}^{n+1}$. Consider the commutative diagram.
We have also to check that $\alpha^{n+1} : T^{n+1} \to S^{n+1}$ is a natural transformation. If $A \to A'$ a morphism in $\mathbf{A}$, , then there is a commutative diagram

$$\begin{array}{ccc} T^{n+1}(A) & \longrightarrow & S^{n+1}(A) \\ \downarrow & & \downarrow \\ T^{n+1}(A') & \longrightarrow & S^{n+1}(A') \end{array},$$

we left this as an exercise. □

$\mathbf{A}, \mathbf{B}$ abelian categories, and $F : \mathbf{A} \to \mathbf{B}$ right exact functor. Assume that $\mathbf{A}$ has enough injectives.

Definition 2.4. The left derived functors $L_iF : \mathbf{A} \to \mathbf{B}$ of F are defined as follows: $\forall A \in \mathbf{A}$, choose a projective resolution $P_* \to A$ and set

$$LiF(A) = H_i(F(P_*)).$$

F is right exact implies that $L_0F(A) \simeq F(A)$.

Lemma 2.2. The objects $L_iF(A)$ of $\mathbf{B}$ are well defined.

Proof. To do. □

Corollary 2.3. If A is projective then A is F-acyclic, that is, $L_iF(A) = 0, i > 1$.

Lemma 2.4. If $A \xrightarrow{f} A'$ is a morphism in $\mathbf{A}$, then there exists natural maps

$$L_iF(f) : L_iF(A) \to L_iF(A').$$

Lemma 2.5. Each L_iF is an additive functor from $\mathbf{A}$ to $\mathbf{B}$.

Theorem 2.6. The left derived functors $(L_iF)_{i\geq 0}$ form a universal δ-functor.

Proof.
It suffices to show $(L_iF)_{i\geq 0}$ form a homological δ-functor. Then since $\mathbf{A}$ has enough projectives. By some corollary, L_iF is coeffaceable, by thm 2.1.6 L_iF is universal. □

Given a short exact sequence $0 \to A' \to A \to A'' \to 0$, choose a projective resolution $P' \to A'$ and $P'' \to A''$. By the Horseshoe Lemma, there is a projective resolution $P \to A$ fitting into a short exact sequence $0 \to P' \to P \to P'' \to 0$ in $C(\mathbf{A})$. The fact that P''_n is projective implies that each short exact sequence $0 \to P'_n \to P_n \to P''_n \to 0$ split. Since F is additive , $0 \to F(P'_n) \to F(P_n) \to F(P''_n) \to 0$ is split exact in $\mathbf{B}$, so $0 \to F(P') \to F(P) \to F(P'') \to 0$ is short exact, from which we have a long exact sequence

$$\cdots \to L_iF(A') \to L_iF(A) \to L_iF(A'') \xrightarrow{\delta_i} L_{i-1}F(A') \to \cdots$$

Given a commutative diagram in $\mathbf{A}$ and projective resolutions $\epsilon' : P' \to A', \epsilon'' : P'' \to A'', \eta' : Q' \to B', \eta'' : Q'' \to B''$. By the Horseshoe Lemma, we get projective resolutions $\epsilon : P \to A, \eta : Q \to B$ with $P = P' \oplus P'', Q = Q' \oplus Q''$. By thm 1.7.15, there are liftings $\tilde{f}' : P' \to Q', \tilde{f}'' : P'' \to Q''$ respectively. It suffices to show that there exists a lifting $\tilde{f} : P \to Q$ of f which makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & P'' \arrow[r] \arrow[d] & 0 \\
0Q' \arrow[r] & Q \arrow[r] & Q'' \arrow[r] & 0 &
\end{tikzcd}$$

commutes. We construct $\gamma_n : P'' \to Q'$ and $\tilde{f} : P \to Q$ of the form

$$\tilde{f}_n = \begin{pmatrix} \tilde{f}'_n & \gamma_n \\ 0 & \tilde{f}''_n \end{pmatrix}$$

inductively. For $n = 0$, we need $f\epsilon = \eta\tilde{f}_0$

On P_0', this becomes $f'\epsilon' = \eta'\tilde{f}_0'$, which indeed holds. On P_0'', this requires $f\epsilon|_{P_0''} = \eta(\gamma_0 + \tilde{f}_0'') = \eta'\gamma_0 + \eta|_{Q_0''}\tilde{f}_0''$. Let $\beta = f(\epsilon|_{P_0''}) - (\eta|_{Q_0''})\tilde{f}'' : P_0'' \to B$.

If β factors through B', the projectivity of P_0'' tells us the existence of γ_0 such that $\eta'\gamma_0 = \beta$. So we only need to check $\pi_B\beta = 0$, which is true.

$$\pi_B f(\epsilon|_{P_0''}) - \pi_B(\eta|_{Q_0''})\tilde{f}_0'' = f''\pi_A(\epsilon|_{P_0''}) - \pi_B(\eta|_{Q_0''})\tilde{f}_0'' = 0$$

The general case is left as an exercise (see Weibel p.47)

Conversely, if $T = (T_i)_{i\geq 0}$ is a universal homological δ-functor, then $T_i \simeq L_iT_0, \forall i \geq 0$.

Remark. If $0 \to M \to P \to A \to 0$ exact with P projective or F-acyclic, Then $L_iF(A) \simeq L_{i-1}F(M), i \geq 2$ and $L_1F(A) = \ker(F(M) \to F(P))$. More generally, if

$$0 \to M_m \to P_m \to \cdots \to P_0 \to A \to 0$$

is exact, then $L_iF(A) \simeq L_{i-m-1}F(M_m), i \geq m+2$. $L_{m+1}F(A) = \ker(F(M_m) \to F(p))$.

If $P \to A$ is an F-acyclic resolution of A, then $L_iF(A) \simeq H_i(F(P))$.

$\mathbf{A}, \mathbf{B}$ abelian categories, $F : \mathbf{A} \to \mathbf{B}$ left exact functor. Assume $\mathbf{A}$ has enough injectives.

Definition 2.5. The right derived functor $(R_iF)_{i\geq 0}$ of F are defined as follows. For all $A \in \mathbf{A}$, choose any injective resolution $A \to I^*$ and set $R^iF(A) = H^i(F(I^*))$.

Theorem 2.7. The objects $R^F(A)$ are independent of the choice of injective resolutions. $(R^iF)_{i\geq 0}$ form a universal cohomological δ-functor and $R^iF(I) = 0$ for $i \geq 1$ and I^* injective.

Proof. We may view F as (covariant) right exact functor

$$F^{\mathrm{op}} : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}^{\mathrm{op}}$$

and $\mathbf{A}^{\mathrm{op}}$ has enough projectives. I^* becomes a projective resolution of A in $\mathbf{A}^{\mathrm{op}}$, so $R^iF(A) = (L_iF^{\mathrm{op}})^{\mathrm{op}}(A)$. Then all results about right exact functors apply to left exact functor. □

Example. X topological space. $F = \Gamma(X, -) : \mathbf{Shv}(X) \to \mathbf{Ab}$ left exact, $\mathcal{F} \in \mathbf{Shv}(X)$, $H^i(X, \mathcal{F}) := R^i\Gamma(\mathcal{F})$.

$F : \mathbf{A} \to \mathbf{B}$ contravariant left exact functor, which can be viewed as a covariant left exact functor $F : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$. If $\mathbf{A}$ has enough projectives, we can define the right derived functor $R^iF(A) = H^i(F(P_*))$, where $P_* \to A$ is a projective resolution.

2.2 Tor

$A \in \mathbf{Mod} - R, B \in R - \mathbf{Mod}$, we have two ways to define $\mathrm{Tor}^R(A, B) \in \mathbf{Ab}$. The first is defined as the left derived functor $L_i(- \otimes_R B)(A)$ and the second is defined as $L_i(A \otimes_R -)(A)$.

Theorem 2.8. We have a natural isomorphism

$$\mathrm{Tor}_i^R(A, B) \simeq \overline{\mathrm{Tor}_i^R(A, B)}.$$

Proof. Tensoring P_* and $Q_* \to B \to 0$ gives a map of double complexes $P_* \otimes Q_* \to P_* \otimes B$. Applying the totalization functor $\mathrm{tot}_\oplus$, we get

$$f : \mathrm{tot}_\oplus(P_* \otimes Q_*) \to \mathrm{tot}(P_* \otimes B) = P_* \otimes B.$$

Claim 2.9. f is a quasi-isomorphism.

Proof. It suffices to show that $\mathrm{cone}(f)$ is acyclic. Now $\mathrm{Cone}(f) \simeq \mathrm{tot}_\oplus(P_* \otimes Q'_*)$, where Q'_* is $Q_* \to B \to 0$. The double complex $P_* \otimes Q'_*$ is visually looks like

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[l] \arrow[d] & \\
P_0 \otimes Q_1 \arrow[d] & P_1 \otimes Q_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes Q_0 \arrow[d] & P_1 \otimes Q_0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes B & P_1 \otimes B \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

since P_i is projective so is flat, we have each column exact in the above diagram. By Lemma 2.2.2 $\mathrm{tot}_\oplus(P_* \otimes Q'_*)$ is acyclic, thus we proved the claim. □

Similarly, the morphism $\mathrm{tot}_\oplus(P_* \otimes Q_*) \to A \otimes Q_*$ is a quasi-isomorphism.

$$H_*(\mathrm{tot}_\oplus(P_* \otimes Q_*) \simeq H_*(A \otimes Q_*) = \overline{\mathrm{Tor}_*^R(A, B)}$$

□

Lemma 2.10. Let C be a double complex, then $\mathrm{tot}_\otimes(C)$ is an acyclic chain complex assuming one of the following conditions

i C is an upper half-plane with exact columns.

ii C is a right half-plane with exact rows.

$\mathrm{tot}_\oplus(C)$ is an acyclic complex, assuming either of the following

i C is an upper half-plane chain complex with exact rows.

ii C is a right half-plane with exact columns

Proof. See Weibel p.60 . □

Corollary 2.11. If R is commutative, then $\mathrm{Tor}_i^R(A, B) \simeq \mathrm{Tor}_i^R(B, A)$.

Proof. Choose a projective resolution $P_* \to A$, then $\mathrm{Tor}_i^R(A, B) = H_i(P_* \otimes B)$, since R is commutative, there is a natural morphism $P_* \otimes B \simeq B \otimes P_*$. By Thm 2.2.1, we have $\mathrm{Tor}_i^R(B, A) = \overline{\mathrm{Tor}_i^R(B, A)} = H_i(B \otimes P_*) \simeq H_i(P_* \otimes B)$. □

Proposition 2.12. If $\mathbf{A}$ has enough projectives and arbitrary direct sum exists in $\mathbf{A}$. Let $F : \mathbf{A} \to \mathbf{B}$ be an additive functor which admits a right adjoint. Then for many set $\{A_i\}_{i \in I}$

of objects in $\mathbf{A}$, we have

$$L_*F(\oplus_{i\in I}A_i) \simeq \oplus_{i\in I}L_*F(A_i).$$

Proof. If $P_i \to A_i$ are projective resolutions then so is $\oplus P_i \to \oplus A_i$. Hence

$$L_*F(\oplus A_i) = H_*(F(\oplus P_i)) = H_*(\oplus F(P_i)) \simeq H_*(F(P_I)) = \oplus L_*F(A_i)$$

□

Corollary 2.13.

$$\mathrm{Tor}_i^R(\oplus A_i, B) \simeq \oplus \mathrm{Tor}_i^R(A_i, B).$$

Lemma 2.14. Let $\{A_i\}_{i\in I}$ be a direct system of left R-modules, $A = \mathrm{colim}\, A_i$. Then there exist projective resolutions P_i of A_i forming a direct system such that $P = \mathrm{colim}\, P_i$ is a projective resolution of A.

[1]

$R^n \lim = 0, n \geq 2$ for $I = (N, \leq)$

[1] *Tue. Apr. 20*

Proof. Let $B_i \subseteq Z_i \subset C_i$ be the cocycles and coboundaries, and similarly $B^* \subseteq Z^* \subseteq C^*$. THen we have a short exact sequence

$$0 \to Z_i^{q-1} \to C_i^{q-1} \to B_i^q \to 0, \forall q > 0,$$

which implies $Z^q = \lim Z_i^q, \lim^1 B_i^q = 0$, and

$$0 \to B_i^q \to \lim B_i^q \to \overset{1}{\lim} Z_i^{q-1} \to 0$$

exact.
On the other hand, the exact sequence

$$0 \to B_i^q \to Z_i^q \to H^q(C_i^*) \to 0$$

induces an isomorphism $\lim^1 Z_i^q \simeq \lim^i H^q(C_i^*)$ and

$$0 \to \lim B_i^q \to \lim Z_i^q = Q^q \to \lim H^q(C_i^*) \to 0$$

are exact. Consider the filtration

$$0 \subseteq B^q \subseteq \lim B_i^q \subseteq Z^q \subseteq C^q,$$

we have an exact sequence

$$0 \to \overset{1}{\lim} H^{q-1}(C_i^*) \to H^q(C^*) \to \lim H^q(C_i^*) \to 0$$

□

Corollary 2.15. Let $A \in R - \mathbf{Mod}$ which is the union of submodules

$$A_0 \subset A_1 \subset \cdots \subset A_i \subset \cdots.$$

Then for any $B \in R - \mathbf{Mod}$, and $q \geq 1$, we have an exact sequence

$$0 \to \lim^1 \mathrm{Ext}_R^{q-1}(A_{i-1}, B) \to \mathrm{Ext}_R^q(A_i, B) \to \lim \mathrm{Ext}_R^q(A_i, B) \to 0.$$

Corollary 2.16. Take an injective resolution $B \to E^*$ then we get an inverse system

$$\cdots \to \mathrm{Hom}_R(A_{i+1}, E^*) \to \mathrm{Hom}_R(A_i, E^*) \to \cdots \to \mathrm{Hom}_R(A_0, E^*),$$

since each E^q is injective, $\mathrm{Hom}_R(A_{i+1}, E^q) \to \mathrm{Hom}_R(A_i, E^q)$ is surjective, the above system satisfies the ML condition. The result. follows from Thm 2.4.7.

Example. $\mathbb{Z}_{p^\infty} = \cup_i p^{-i}\mathbb{Z}/\mathbb{Z} \simeq \mathrm{colim}\, p^{-1}\mathbb{Z}/\mathbb{Z}$. The short exact sequence

$$0 \to \lim^1 \mathrm{Hom}(\mathbb{Z}/p^i\mathbb{Z}, B) \to \mathrm{Ext}_\mathbb{Z}^1(\mathbb{Z}_{p^\infty}, B) \to \lim_i B/p^iB \to 0.$$

1. If B is torsion free, then $\mathrm{Ext}_\mathbb{Z}^1(\mathbb{Z}_{p\infty}, B) \simeq \lim_i B/p^iB$ since the first term vanishes.
2. If B is finitely generated $\mathbb{Z}$-module, each $\mathrm{Hom}(\mathbb{Z}/q^i\mathbb{Z}, B)$ is finitely generated $\mathbb{Z}/p^i\mathbb{Z}$-module, thus is also finitely generated. $\mathrm{Hom}(\mathbb{Z}/p^i\mathbb{Z}, B)$ satisfies the ML condition. $\mathrm{Ext}_\mathbb{Z}^1(\mathbb{Z}/p^\infty, B) \simeq B/p^iB$.

Exercise. Compute $\mathrm{Ext}_\mathbb{Z}^1(\mathbb{Z}[1p], \mathbb{Z}), \mathrm{Ext}_\mathbb{Z}^1(\mathbb{Q}, \mathbb{Z})$.

Exercise. Check that in Def 2.3.10 $\theta([\xi]) = \partial(\mathrm{id}_B)$ is well defined, where $\partial : \mathrm{Hom}(A, A) \to \mathrm{Ext}_R^1(A, B)$

Exercise. Weibel Ex 3.3.2, 3.6.1.

3 Group Homology and Cohomology

Let G be a group, and $G-\mathbf{Mod}$ be the category of left G-modules. Consider the covariant functor $(-)^G : G - \mathbf{Mod} \to \mathbf{Ab}$, $(-)_G : G - \mathbf{Mod} \to \mathbf{Ab}$, we can verify that $(-)^G$ is left exact and $(-)_G$ is right exact.

Definition 3.1. For all $A \in G - \mathbf{Mod}$,

$$H^n(G, A) := R^n(-)^G(A)$$
$$H_n(G, A) := L_n(-)_G(A).$$

We define the augmentation map $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ by

$$\epsilon : \mathbb{Z}G \to \mathbb{Z}, \quad g \mapsto 1$$

for any $g \in G$. The **augmentation ideal** $\mathfrak{J} = \ker \epsilon$ is a free $\mathbb{Z}$-module with basis $\{g - 1\}$, so we have and exact sequence

$$0 \to \mathfrak{J} \to \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where $\mathbb{Z}$ is equipped with the trivial G-action. We can verify that $A^G = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$ and $A_G \simeq \mathbb{Z} \otimes_{\mathbb{Z}G} A$ and thus $H^*(G; A) \simeq \mathrm{Ext}_{\mathbb{Z}G}^*(\mathbb{Z}, A)$ and $H_*(G; A) \simeq \mathrm{Tor}_*^{\mathbb{Z}G}(G, A)$.

Proposition 3.1. (i) If G is infinite, then $H^*(G; \mathbb{Z}G) = 0$.

(ii) If G is finite, then $H_n(G; \mathbb{Z}G) = \mathbb{Z}N$ when $n = 0$ and 0 when $n \geq 1$, where $N = \sum_{g \in G} g \in \mathbb{Z}G$ is the norm element.

Proof. Let $x = \sum_g c_g g \in (\mathbb{Z}G)^G$, then $hx = x, \forall h \in G$. the function $c : G \to \mathbb{Z}, g \mapsto c_g$ is constant and thus $c_g = 0$, since x is a finite sum and G is infinite.
In G is finite, the above arguments also make sense. $x \in \mathbb{Z}N$ if $x \in (\mathbb{Z}G)^G$ thus $H^n(G; \mathbb{Z}G) = \mathbb{Z}N$. To show that $H^n(G; \mathbb{Z}G) = 0, \forall n \geq 1$, we apply the following lemmas. Then $\mathbb{Z}G = \mathrm{Ind}_H^G \mathbb{Z} \simeq \mathrm{coInd}_H^G \mathbb{Z}, H = \{1\}$, so

$$H^n(G; \mathbb{Z}G) = H^n(H; \mathbb{Z}) = 0, \forall n \geq 1.$$

□

Lemma 3.2 (Shapiro's Lemma).

$$H_*(G; \mathrm{Ind}_H^G(A)) \simeq H_*(H; A),$$
$$H^*(G; \mathrm{Coind}_H^G(A)) \simeq H^*(H; A).$$

Proof. We prove the case of cohomology only. Let $P_* \to \mathbb{Z}$ be a free resolution in $G - \mathbf{Mod}$, then $H^n(G, \mathrm{coInd}_H^G A) = H^n(\mathrm{Hom}_{\mathbb{Z}G}(P_*, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A))$. Since $\mathrm{Hom}_{\mathbb{Z}}(P_*, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)) \simeq \mathrm{Hom}_{\mathbb{Z}H}(P_* \otimes_{\mathbb{Z}H} \mathbb{Z}G, A) \simeq \mathrm{Hom}_{\mathbb{Z}H}(P_*, A)$ and $P_* \to \mathbb{Z}$ is a projective resolution $H - \mathbf{Mod}$(Exercise). So $H^n(\mathrm{Hom}_{\mathbb{Z}H}(P_*, A)) = H^n(H; A)$.

□

Lemma 3.3. Let $H \subseteq G$ be a subgroup. If the index $[G : H]$ is finite, then there is an isomorphism

$$\mathrm{Ind}_H^G(A) \simeq \mathrm{Coind}_H^G(A).$$

.

Proof. See Weibel Lemma 6.3.4. □

Lemma 3.4. Let G be any group (finite or infinite) , then

$$H^1(G; \mathbb{Z}) \simeq \mathfrak{J}/\mathfrak{J}^2 \simeq G/[G, G]. \tag{3.1}$$

Proof. From the short exact sequence $0 \to \mathfrak{J} \to \mathbb{Z}G \to \mathbb{Z} \to 0$, we get

$$0 = H_1(G; \mathbb{Z}G) \to \mathrm{Hom}_1(G, \mathbb{Z}) \to \mathfrak{J}_G \to (\mathbb{Z}G)_G \to \mathbb{Z} \to 0,$$

Since $(\mathbb{Z}G)_G \to \mathbb{Z}$ is an isomorphism, so

$$H_1(G; \mathbb{Z}) \simeq \mathfrak{J}_G = \mathbb{Z} \otimes_{\mathbb{Z}G} \mathfrak{J} \simeq (\mathbb{Z}G/\mathfrak{J}) \otimes_{\mathbb{Z}G} \mathfrak{J} \simeq \mathfrak{J}/\mathfrak{J}^2.$$

We are left to show that $J/J^2 \simeq G/[G, G]$. Consider $\phi : J \to G/[G, G]$, then this is a group homomorphism. As a $\mathbb{Z}G$-module, $\sum_{g \in G} c_g(g - 1) \to \prod_g \bar{g}^G\ J^2$ is generated by $(g - 1)(g' - 1), g, g' \in G$

□

Proposition 3.5. Let A be a trivial G-module.

(i) $H_0(G;A) = A, H_1(G;A) \simeq G/[G,G]\otimes_{\mathbb{Z}} A$. For $n \geq 2$, there are split exact sequence $0 \to H_n(G;\mathbb{Z}) \otimes A \to H_n(G;A) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(G;\mathbb{Z}), A) \to 0$

(ii) $H^0(G;A) = A, H^1(G;A) \simeq \mathrm{Hom}_{\mathbf{Grp}}(G,A)$, for $n \geq 2$ there are split exact sequence $0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(G;\mathbb{Z}), A) \to H^n(G;A) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(G;\mathbb{Z}), A) \to 0$.

Proof. A trivial G-module, $A \simeq \mathbb{Z}\otimes_{\mathbb{Z}} A \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A)$ as left G-module, Take a projective resolution $P_* \to \mathbb{Z}$ in $\mathbf{Mod} - \mathbb{Z}G$, then $H_*(G;A) \simeq H_*(P_* \otimes_{\mathbb{Z}G} A)$ and $P_* \otimes_{\mathbb{Z}G} A \simeq (P_* \otimes_{\mathbb{Z}G} \mathbb{Z} \otimes_{\mathbb{Z}} A$ then the proposition follows from Thm 2.4.2, 2.4.3. ☐

Example. $G = C_m$ finite cyclic group of order m, $N = 1 + \sigma + \cdots + \sigma^{m-1}$

$$\cdots \to \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

G infinite cyclic group. $\mathbb{Z}G = \mathbb{Z}[t,t^{-1}]$ and $0 \to \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \to \mathbb{Z} \to 0$ a free resolution of $\mathbb{Z}$

$$H_n(G;A) = \mathrm{Hom}_n(G,A) = 0, \forall n \geq 2$$
$$H_0(G;A) \simeq H^1(G;A) \simeq A_G, H^0(G;A) \simeq \mathrm{Hom}_1(G,A) \simeq A^G$$

G free group, J is a free $\mathbb{Z}G$-module with the basis $\{x-1, x \in X\}$, see Weibel Prop.6.2.6.
$G = \mathrm{Gal}(L/K)$ finite Galois group.

G a group, $n \geq 0$, $G^{n+1} = G \times G \times \cdots \times G$ ($n+1$ copies) viewed as a G-set, by the action $g(g_0,\ldots,g_n) = (gg_0,\ldots,gg_n)$. Consider $d_i : G^{n+1} \to G^n, (g_0,\ldots,g_n) \mapsto (g_0,\ldots,\hat{g}_i,\ldots g_n)$. Its easy to verify that $d_i d_j = d_{j-1} d_i, i < j$. Let $B_n(G) = \mathbb{Z}[G^{n+1}]$, d_i induce $d_i : B_n(G) \to B_{n-1}(G)$ for $n \geq 1$. Let $d = \sum_{i=0}^n (-1)^i d_i : B_n(G) \to B_{n-1}(G)$, hence $(B_*(G), d)$ is a complex and $\epsilon d = 0$.

Theorem 3.6. $\cdots \to B_2(G) \to B_1(G) \to B_0(G) \to \mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}$, called the **bar resolution** of G.

$\phi \in \mathrm{Hom}_G(B_n(G), A)$, $d\phi$ is given by

$$d\phi(g_1,\ldots,g_{n+1}) = g_1\phi(g_2,\ldots,g_{n+1}) + \sum_{i=1}^n (-1)^i \phi(\ldots,g_i g_{i+1},\ldots) + (-1)^{n+1}\phi(g_1,\ldots,g_n)$$

Example. When $n = 1$, $Z(G,A) = \{\, \phi : G \to A \mid \phi(gg') = g\phi(g') + \phi(g), \forall g;, g \in G \,\}$, and $B(G,A) = \{\, \phi : G \to A \mid \exists a \in A, \phi(g) = ga - a, \forall g \in G \,\}$. When $n = 2$, a 2-cocycle is a function $\phi : G \times G \to A$ such that $g\phi(g', g'') - \phi(gg', g'') + \phi(g, g'g'') - \phi(g', g'') = 0$, and ϕ is a 2-coboundary if there exists some $\beta : G \to A$ such that $\phi(g,g') = g\beta(g') - \beta(gg') + \beta(g), \forall g, g' \in G$.

If A is an abelian group, recall that a group extension of G by A is a short exact sequence $0 \to A \to E \xrightarrow{\pi} G \to 1$. This induces an action of G on A: $\forall g \in G, a \in A, ^g a := \tilde{g} a \tilde{g}^{-1}$ where $\tilde{g} \in E$ such that $\pi(\tilde{g}) = g$.

Example. $E := A$semi-direct product$G$ gives an extension of G by A: $(a,g)(b,h) = (a + gb, gh), \forall a, b \in A, g, h \in G$. The split extension of G by A.
$A = C_3, G = C_2$, G acts on A by $\gamma a = -a$, the extension of G by A exists uniquely, $0 \to C_3 \to D_3 \to C_2 \to 1$.

Theorem 3.7. There is a bijection $H^2(G;A) \simeq \{$ extensions of G by $A \} / \sim$, two extensions E, E' of G by A satisfy $E \sim E'$ in the natural way.

Proof. See Weibel 6.6. $\square$

Example. $H^2(C_2; C_3) = 0$. $\mathbb{Z}C_2 = \mathbb{Z}[\sigma]/\sigma^2 - 1$

Example. $H \subseteq G$ a subgroup, $\rho : H \to G$ inclusion, which induces a restriction of cohomologies $H^n(G;A) \to H^n(H;A)$. If $H \subseteq G$ a normal subgroup, $A \in G - \mathbf{Mod}$, A^H is a G/H-module, $G \to G/H$ $A^H \to A$, $\mathrm{Inf} : H^n(G/H; A^H) \to H^n(G;A)$. Similarly, we have $H_n(G'; \rho^* A) \to H_n(G;A)$ and $H_n(G'; A') \to H_n(G;A)$.
corestriction and coinvariant.

Proposition 3.8. If $A = \mathbb{Z}$, $H = G$, the conjugation by g^i induces the identity on $H^n(G;\mathbb{Z})$ and $H_n(G;\mathbb{Z})$.

Proof. Exercise. $\square$

If $H \subseteq G$ is normal, the conjugation induces an action of G/H on $H^n(H;\mathbb{Z})$ and $H_n(H;\mathbb{Z})$.

Proposition 3.9. Let $H \subset G$ be a normal subgroup and $A \in G - \mathbf{Mod}$, the sequence

$$0 \to H^1(G/H; A^H) \overset{\mathrm{Inf}}{\to} H^1(G;A) \overset{\mathrm{Res}}{\to} H^1(H;A)$$

is exact.

Proof. Let $\phi : G/H \to A^H$ be a 1-cocycle, such that $\mathrm{Inf}([\phi]) = 0$, thus there is some $a \in A$ such that $\tilde{\phi}(g) = ga - a, \forall g \in G, \tilde{\phi} : G \to G/H \to A^H \hookrightarrow A$. Since $\tilde{\phi}(g) = \tilde{\phi}(gh), \forall h \in H$, $ga - a = gha - a \implies a = ha, \forall h \in H, \implies a \in A^H, \implies [\phi] = 0 \in H^1(G/H; A^H)$.
Let $\phi : G \to A$ be a 1-cocycle, such that $\mathrm{Res}([\phi]) = 0, \implies \phi(h) = ha - a, \forall h \in H$ for some $a \in A$. $\square$

Assume that $H \subseteq G$ is a normal subgroup of finite index m. $A \in G - \mathbf{Mod}$

$$\begin{aligned} N_{G/H} : A^H &\to A^G \\ a &\mapsto \sum_{g \in G/H} ga \end{aligned}$$

$$\begin{aligned} N_{G/H} : A_G &\to A_H \\ [a] &\mapsto [\sum_{G/H} ga] \end{aligned}$$

$N_{G/H} : H^n(H;A) \to H^n(G;A), H^n(G;A) \to H_n(H;A)$, transfer map.

Lemma 3.10. $N_{G/H} \circ \mathrm{Res} = m$, $\mathrm{CoRes} \circ N_{G/H} = m$.

Proof. Exercise. $\square$

Theorem 3.11. Let G be a finite group with m elements. Then for all $n \geq 1$, and $A \in G - \mathbf{Mod}$, $H^n(G;A)$ and $H_n(G;A)$ are $\mathbb{Z}/m\mathbb{Z}$-modules.

Proof. Take $H = \{1\}$. $H^n(G;A) \overset{\text{Res}}{\to} H^n(1;A) \to \overset{N_{G/H}}{\to} H^n(G;A)$ is equal to multiplication by m. For $n \geq 1, H^n(1;A) = 0$, $H^n(G;A)$ annihilated by m, similarly for $H_n(G;A)$. ☐

Corollary 3.12. If G is a finite group and A a finite $\mathbb{Z}G$-module, then $H_n(G;A)$ and $\mathrm{Hom}_n(G,A)$ are finite abelian groups.

Exercise. G finite group, $H^1(G;\mathbb{Z}) = 0$ and $H^2(G;\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbf{Grp}}(G, \mathbb{C}^*)$

Exercise. Weibel Exercises 6.1.8, 6.2.4, 6.7.1, 6.7.7

4 Spectral Sequences

4.1 Basic Definitions

$\mathbf{A}$ abelian category satisfying AB4 and AB4*, $a \in \mathbb{N}$.

Definition 4.1. A cohomology spectral sequence $E = (E_r^{p,q}$ in $\mathbf{A}$ starting on page a consisting of

1. an object $E_r^{p,q}, p, q, r \in \mathbb{Z}, r \geq a$
2. a morphism $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}, \forall p, q, r \in \mathbb{Z}$ such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$.
3. an isomorphism $\alpha_r^{p,q} : E_r^{p,q} \simeq \ker d_r^{p,q} / \mathrm{im}\, d_r^{p-r,q+r-1}, \forall p, q, r \in \mathbb{Z}, r \geq a$.

A morphism $f : E \to E'$ between two spectral sequences is a family of maps $f_r^{p,q} : E_r^{p,q} \to E_r'^{p,q}$ with $d_r' f_r = f_r d_r$.

Lemma 4.1. Let $f : E \to E'$ be a morphism of spectral sequences such that there is some r making $f_r^{p,q} : E_r^{p,q} \to E_r'^{p,q}$ is an isomorphism for all $p, q \in \mathbb{Z}$. Then $f_\infty^{p,q} : E_\infty^{p,q} \to E_\infty'^{p,q}$ is an isomorphism.

Proof. Exercise. ☐

Definition 4.2. A spectral sequence is **regular** is for all p, q, the differentials $d_r^{p,q} = 0$ for all r iff $Z_\infty^{p,q} = Z_r^{p,q}$ for large r. It is **coregular** if the differentials $d_r^{p-r,q+r-1} = 0$ for large r. It is **biregular** if it is both regular and coregular.

$A \in \mathbf{A}$, recall that a filtration on A is a sequence of subobjects of A

$$\cdots \supseteq F^0 A \supseteq F^1 A \supseteq \cdots F^p A \supseteq \cdots .$$

It is **separated** if $\cap_p F^p A = 0$ and exhaustive if $\cup_p F^p A = A$.

Definition 4.3. We say the spectral sequence $(E_r^{p,q})$ **weakly converges** to given objects $H^n \in \mathbf{A}$ if each H^n has a filtration

$$H^n \supseteq \cdots \supseteq F^p H^n \supseteq \cdots$$

with isomorphisms

$$E_\infty^{p,q} \simeq \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}, \forall p, q \in \mathbb{Z}.$$

We say that the spectral sequence $(E_r^{p,q})$ **abuts** to (H^n) if is weakly converges to H^n and the filtration on H^n is separated and exhausted.
We say that the spectral sequence $(E_r^{p,q})$ **converges** to (H^n) if it is regular, abuts to H^n and $\varinjlim_p H^n/F^pH^n, \forall n$, denoted as $E_r^{p,q} \implies H^n$.

Definition 4.4. The spectral sequence $(E_r^{p,q})$ **degenerates** at page r if for all $s \geq r, d_s = 0$
$(E_r^{p,q})$ is bounded below if $\forall n, \exists s = s(n)$ such that $E_a^{p,q} = 0$ for all $p+q=n$ and $p > s$ ($\implies$ regular). It is **bounded** if $\forall n$ there exist only finitely many $E_a^{p,q} \neq 0$ with $p+q=n$ ($\implies$ biregular).

If $E_r^{p,q}$ is bounded, then we have

$$\forall p, q \in \mathbb{Z}, \exists r_0 \textbf{such that } E_r^{p,q} \simeq E_{r_0}^{p,q} \forall r \geq r_0 \implies E_\infty^{p,q} \simeq E_{r_0}^{p,q}.$$

If $E_r^{p,q}$ weakly converges to H^n, then the filtration on H^n is finite, that is, F^pH^n stabilizes when $p \to \pm\infty$.

If $E_r^{p,q}$ bounded below, it converges to H^n whenever it abuts to H^n.

Example. A first quadrant spectral sequence is bounded. If it converges to H^n, then each H^n has a finite filtration of length $n+1$: $H^n = F^0H^n \supseteq F^1H^n \supseteq \cdots F^pH^n \supseteq \cdots F^{n+1}H^n = 0$.
Each $E_{r+1}^{0,n}$ is a subobject of $E_r^{0,n}$, implying $E_a^{0,n}$. $H^n \to E_\infty^{0,n} \hookrightarrow E_a^{0,n}$.
Similarly, we have $E^{n,0} \twoheadrightarrow E_\infty^{n,0} \hookrightarrow H^n$, the compositions are called **edge morphisms**.

Example. $(E_r^{p,q})$ such that $E_2^{p,q} = 0$ except $p = 0, 1$. If it abuts to H^* , we have an exact sequence

$$0 \to E_2^{1,n-1} \to H^n \to E_2^{0,n} \to 0.$$

4.2 Constructions and Examples

$C^* \in C(\mathbf{A})$ with a decreasing filtration $(F^pC^*)_{p\in\mathbb{Z}})$

$$\partial : C^n \to C^{n+1}$$

satisfies $\partial(F^pC^n) \subseteq F^pC^{n+1}, \forall p \in \mathbb{Z}$, so ∂ induces $F^pC^n/F^{p+1}C^n \to F^pC^{n+1}/F^{p+1}C^{n+1}$.

We want to construct $(E_r^{p,q}$ with $E_0^{p,q} = F^pC^{p+q}/F^{p+1}C^{p+1}$ and $E_\infty^{p,q}$ approximate $H^{p+q}(C^*)$.

Lemma 4.2. $D_r^{p,q} = \partial(A_{r-1}^{p-r+1,q+r-2}$ ($\implies B^{p,q} = \partial(Z_{r-1}^{p-r+1,q+r-2}$)
$A_r^{p,q} \cap F^{p+1}C^{p+q} = A_{r-1}^{p+1,q-1}$ ($\implies Z_r^{p,q} = A_r^{p,q}/A_{r-1}^{p+1,q-1}$)
$D_r^{p,q} \cap F^{p+1}C^{p+q} = D_{r+1}^{p+1,q-1}$
The map $\partial : A_r^{p,q} \to D_{r+1}^{p+r,q-r+1}$ induces an isomorphism

Proof. Exercise. $\square$

Theorem 4.3. Suppose that the filtration (F^pC^*) is bounded below and exhausted, then the spectral sequence $(E_r^{p,q})$ is bounded below and converges to $H^*(C^*)$.

Definition 4.5. A spectral sequence collapses at $E_r, r \geq 2$, if there is exactly one non-zero row or column in $E_r^{p,q}$. ($\implies$ If $E_r^{p,q} \implies H^n$, then $H^n \simeq E_r^{p,q}$, p, q are unique such that $n = p+q$.

Theorem 4.4. Let P_* be a bounded below chain complex of projective R-module, then for all $M \in R$-**Mod**, there is a convergent spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_R^p(H_q(P), M) \Rightarrow H^{p+q}(\mathrm{Hom}_R(P_*, M))$$

Theorem 4.5. Corollary 3.2.5

Definition 4.6. A homological spectral sequence starting on page E^a in $\mathbf{A}$ consists of

1. objects $E^r_{p,q}, \forall p, q, r \in \mathbb{Z}, r \geq a$
2. differentials $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ such that $d^r_{p,q} d^r_{p+r,q-r+1} = 0$
3. $E^{r+1}_{p,q} \ker d^r_{p,q} / \operatorname{im} d^r p + r, q - r + 1$

Theorem 4.6. Let P_* be a bounded below complex of flat R-modules, and $M \in R$-**Mod**, then there is a boundly convergent spectral sequence

$$E^2_{p,q} \mathrm{Tor}_1^R(H_1(P_*), M) \Rightarrow H_{p+q}(P_* \otimes_R M).$$

Lemma 4.7 (Cartan-Eilenberg Resolution)**.** Assume that $\mathbf{A}$ has enough injective resolution. Then any cochain complex has a fully injective resolution. If C is acyclic, then we may choose. a full injective resolution $I^{*,*}$ with each row acyclic.

Theorem 4.8 (The Grothendieck Spectral Sequence)**.** Let $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{C}$ be additive left exact functors between abelian categories, where $\mathbf{A}, \mathbf{B}$ have enough injectives and $\mathbf{C}$ is cocomplete. Suppose that F sends injectives to G- acyclics. Then, for all $A \in \mathbf{A}$, there is a convergent first quadrant spectral sequence E starting on page 0, such that $E_2^{p,q} = R^p G(R^q F(A)) \implies R^{p+q}(GF)(A)$. The exact sequence of low terms is

$$0 \to R^1 G(FA) \to R^1(GF)(A) \to G(R^1 F(A)) \to (R^2 G)(FA) \to R^2(GF)(A) \to 0.$$

Example. $R \to S$ ring homomorphism. Then for all $A \in S$-**Mod**. There exists a first quadrant spectral sequence $E_2^{p,q} = \mathrm{Ext}_S^p(A, \mathrm{Ext}_R^q(S, B)) \implies \mathrm{Ext}_R^{p+q}(A, B)$. In particular, if S is projective as an R-module, then $\mathrm{Ext}_S^n(A, \mathrm{Hom}_R(S, B)) \simeq \mathrm{Ext}_R^n(A, B)$. In this case, we take $\mathbf{A} = R - \mathbf{Mod}, \mathbf{B} = S - \mathbf{Mod}, \mathbf{C} = \mathbf{Ab}, F = \mathrm{Hom}_R(S, -), G = \mathrm{Hom}_S(A, -), G \circ F = \mathrm{Hom}_S(A, \mathrm{Hom}_R(S, -)) \simeq \mathrm{Hom}_R(A \otimes_S S, -) \simeq \mathrm{Hom}_R(A, -)$. F admits a left adjoint which is exact, so F sends injectives to injectives.
$R \to S$ ring homomorphism, $A \in S$-**Mod**, $B \in (R, S)$-**Mod** such that $\mathrm{Ext}^i_R(A, \mathrm{Hom}_S(B, E)) = 0$. Then for all $C \in R$-**Mod**, there is a first quadrant spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_S^p(A, \mathrm{Ext}_R^q(B, C)) \implies \mathrm{Ext}_R^{p+q}(B \otimes_S A, C).$$

If B is projective as an R-mod, then $\mathrm{Ext}_S^n(A, \mathrm{Hom}_R(B, C)) \simeq \mathrm{Ext}_R^n(B \otimes_S A, C)$

Exercise. Prove Lemma 3.1.4.

Exercise. Let $\{E_r^{p,q}\}$ is a bounded spectral sequence of R-modules, and $E_r^{p,q} \implies H^n$. Assume $\forall p, q \in \mathbb{Z}$, $E_r^{p,q}$ is a finitely generated R-module, prove that each H^n is also finitely generated.

Exercise. Suppose $E_2^{p,q} = 0$ unless $q = 0$ or n for some $n \geq 2$. Prove that there is a long exact sequence

$$\cdots \to H^{p+n} \to E_2^{p,n} \to E_2^{p+n+1,0} \to H^{p+n+1} \to E_2^{p+1,n} \to E^{p+n+2,0} \to \cdots$$

Exercise. Weibel exercises. 5.6.1, 5.6.4.

Theorem 4.9. $B \in R$-**Mod**, $B \to C^*$ a resolution (not necessarily inject). Then for all $A \in R$-**Mod**, there is a spectral sequence $E_1^{p,q} = \mathrm{Ext}_R^q(A, C^p) \Rightarrow \mathrm{Ext}_R^{p+q}(A, B)$.

Remark. Another important construction of spectral sequence comes from the theory of exact couples. See Weibel 5.9 for some details.

5 Derived Categories

5.1 Triangulated Categories and Derived Categories

$\mathbf{A}$ an abelian category, $C(\mathbf{A})$ is also abelian. The homotopy category $K(\mathbf{A})$, we want to construct the derived category $D(\mathbf{A})$. There are two approaches to the derived category $D(\mathbf{A})$

1. localization in $K(\mathbf{A})$
2. stable ∞-categories.

We sketch the classical localization construction. For more details, see Zheng's notes. Or Gelfand-Manin. Or Kashiwara-Shapira.

$X, Y \in C(\mathbf{A})$ and $f : X \to Y$, recall that we have an exact sequence

$$0 \to Y \to \mathrm{cone}\, f \to X[1] \to 0,$$

and for all exact sequence $0 \to X \to Y \to Z \to 0$ in $C(\mathbf{A})$, $\mathrm{cone}\, f \to Z$ is quasi-isomorphic. A **triangle** in $K(\mathbf{A})$ is a sequence $X \to Y \to Z \to X[1]$ and a morphism of triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \arrow[r] \arrow[d] & X[1] \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}.$$

A triangle is called **distinguished** if it is isomorphic to $X \xrightarrow{f} Y \to \mathrm{cone}\, f \to X[1]$ for some $f : X \to Y$.

Let $\mathbf{C}$ be an additive category and $T : \mathbf{C} \to \mathbf{C}$ an automorphism of $\mathbf{C}$. A triangle in $\mathbf{C}$ is a sequence of morphism

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX = X[1],$$

and morphisms of triangles are defined in the natural way.

Example. The triangle $X \to Y \to Z \to TX$ is isomorphism to (*). but $X \xrightarrow{-f} Y \xrightarrow{-g} Z \xrightarrow{-h} TX$ is not isomorphic to (*) in general.

Definition 5.1. A triangle category is an additive category $\mathbf{C}$ endowed with an automorphism T and a family of a triangle called distinguished triangle, satisfying

1. A triangle isomorphic to a distinguished triangle is a distinguished triangle
2. $X \xrightarrow{\mathrm{id}} X \to 0 \to TX$ is a distinguished triangle

3. for all $f : X \to Y$, there is a distinguished triangle $X \xrightarrow{f} Y \to Z \to TX$

4. $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is distinguished iff $Y \xrightarrow{g} Z \xrightarrow{h} TX \xrightarrow{-Tf} TY$ is distinguished.

5. Given two distinguished triangles, there is an extension $Z \to Z'$

6. octahedron axiom.

Remark. TR4 can be deduced from TR0, 1, 2, 5. See Zheng Prop 2.2.18. The extension in TR4 is not unique in general.

Theorem 5.1. $\mathbf{A}$ an abelian category, $K(\mathbf{A})$ with the distinguished triangle is a triangulated category.

Proof. See Gelfand-Manin p.246-250. □

Definition 5.2. A triangulated functor $F : (\mathbf{C}, T) \to (\mathbf{C}', T')$ of triangulated categories is an additive functor that satisfies $F \circ T = T' \circ F'$ and sends distinguished triangles to distinguished triangles.
A triangulated subcategory $\mathbf{C}' \subseteq \mathbf{C}$ is a subcategory $\mathbf{C}'$ of $\mathbf{C}$ which is triangulated and the inclusion $i : \mathbf{C}' \to \mathbf{C}$ is a triangulated functor.
$(\mathbf{C}, T)$ a triangulated category and $\mathbf{A}$ is an abelian category. $F : \mathbf{C} \to \mathbf{A}$ is a cohomology functor if for all distinguished triangle $X \to Y \to Z \to TX$ in $\mathbf{C}$, the sequence $FX \to FY \to FZ$ is exact in $\mathbf{A}$.

Proposition 5.2. If $X \xrightarrow{f} Y \xrightarrow{g} Z \to TX$ is a distinguished triangle, then $gf = 0$.
$\forall W \in \mathbf{C}$, the functor $\mathrm{Hom}_{\mathbf{C}}(W, -)$ and $\mathrm{Hom}_{\mathbf{C}}(-, W)$ are cohomological.

Proposition 5.3. Consider a morphism of distinguished triangles $X \to Y \to Z \to TX$ and $X' \to Y' \to Z' \to TX'$. If the leftmost and the middle vertical morphisms are isomorphisms, then so is the right most one.

Proof. Apply $\mathrm{Hom}(W, -)$ to the diagram in the above proposition and use Yoneda lemma. □

Corollary 5.4. $\mathbf{C}' \subset \mathbf{C}$ full triangulated subcategory.
A triangle $X \to Y \to Z \to TX$ in $\mathbf{C}'$ and assume that it is distinguished in $\mathbf{C}$, then it is distinguished in $\mathbf{C}'$.
$X \to Y \to Z \to TX$ distinguished triangle in $\mathbf{C}$ with X, Y in $\mathbf{C}'$, then there exists $Z' \in \mathbf{C}'$ and an isomorphism $Z \simeq Z'$.

$\mathbf{C}$ a category, S a family of morphisms in $\mathbf{C}$.

Proposition 5.5. There is a category $\mathbf{C}_S$ together with a functor $Q : \mathbf{C} \to \mathbf{C}_S$ such that

1. $\forall s \in S$, $Q(s)$ is an isomorphism

2. for all functor $F : \mathbf{C} \to \mathbf{D}$, such that $F(s)$ is an isomorphism for all $s \in S$, there exists functor $F_S : \mathbf{C}_S \to \mathbf{D}$ and an isomorphism $F \simeq F_S Q$.

Example. $\mathbf{A}$ abelian category, $K(\mathbf{A}) = C(\mathbf{A})_S$, $S = \{$ homotopy equivalences $\}$.

Definition 5.3. S is called a right multiplicative system if it satisfies

1. $\forall x \in \mathbf{C}, \mathrm{id}_x \in S$
2. $\forall f, g \in S$, if gf exists, then $gf \in S$.
3. $\forall f : X \to Y$ and $s : X \to X'$ with $s \in S$, there exists $t : Y \to Y'$ with $t \in S$ and $gs = tf$
4. $\forall f, g : X \to Y$, if there exists $s \in S$ and $s : W \to X$ such that $fs = gs$, then there exists $t \in S$ and $t : Y \to Z$ such that $tf = tg$.

Assume that S is a right multiplicative system. Then $(\mathbf{C}_S, Q)$ admits a simpler description

1. $\mathrm{ob}\mathbf{C}_s = \mathbf{C}$
2. $X, Y \in \mathbf{C}_S, \mathrm{Hom}_{\mathbf{C}_S}(X, Y) = \mathrm{colim}_{(Y \to Y') \in S^Y} \mathrm{Hom}_{\mathbf{C}}(X, Y')$

where S^Y is a filtered category.

$\mathbf{C}$ category. $N \subset \mathrm{ob}\mathbf{C}$.

Definition 5.4. N is called a null system if

1. $0 \in N$
2. $X \in N \iff TX \in N$
3. If $X \to Y \to Z \to TX$ is a distinguished triangle and $X, Y \in N$, then $Z \in N$.

Set $S(N) = \{\, f : X \to Y \mid f embedded into a distinguished triangle X \to Y \to Z \to TX with Z \in N \,\}$.

Proposition 5.6. $S(N)$ is a multiplicative system in $\mathbf{C}$.

Write $\mathbf{C}/N = \mathbf{C}_{S(N)}$.

Proposition 5.7. $(\mathbf{C}, T)$ triangulated category, N null system. T induces an automorphism $T : \mathbf{C}/N \to \mathbf{C}/N$. $\mathbf{C}/N$ is a triangulated category by taking distinguished triangles as those isomorphic to the images a distinguished triangle in $\mathbf{C}$ under $Q : \mathbf{C} \to \mathbf{C}/N$, then Q is a triangulated functor. If $X \in N$, then $Q(X) = 0$. $\forall F : \mathbf{C} \to \mathbf{D}$ of triangulated category such that $FX = 0$ for all $X \in N$, then F factors through Q.

Definition 5.5. The derived category $D(\mathbf{A}) \coloneqq K(\mathbf{A})/N$.

References

[Wei] Charles A. Weibel. *An Introduction to Homological Algebra.* Cambridge University Press, 1 edition.