Foundations of Lie Theory

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Abstract: These are notes for the course *Foundations of Lie Theory* in Fall 2020. Despite the somewhat confusing name of the course, it was mainly about the theory of algebraic groups. The course covered almost all of the book [OV]. These notes were live-LaTeXed and unedited, thus it is more appropriate to consider them as a syllabus for *Foundations of Lie Theory*. All errors introduced are mine.

Contents

Let k be an algebraically closed field and put $V = k^n$.

Proposition 0.1. Let $X \subseteq V$ be an algebraic set.

(i) The Zariski topological of X is T_1, *id est*, points are closed.

(ii) Any family of closed subsets of X contains a minimal one.

(iii) If $X_1 \supseteq X_2 \supseteq \dots$ is a descending sequence of closed subsection of X, there is an h such that $X_i = X_h$ for $i \geq h$.

(iv) Any open covering of X has a finite subcovering.

Proof. A point in V corresponds precisely to a maximal ideal of the coordinate ring $k[T_1, \dots T_n]$, by the Nullstellensatz we know that it is closed, hence (i) follows. Since k is a field and hence is Noetherian, by the Hilbert's Basis Theorem $k[T_1, \dots, T_n]$ is Noetherian, and using the fact that the algebraic sets in V are bijectively in correspondence to the ideals of $k[T_1, \dots, T_n]$, (ii) and (iii) follow.
To show (iv), we just need to show its closed version, that if $\{I_\alpha\}_{\alpha \in A}$ is a family of ideals such that $\cap_{\alpha \in A} V(I_\alpha) = \varnothing$, there is a finite subset $B \subseteq A$ such that $\cap_{\alpha \in B} V(I_\alpha) = \varnothing$.But the assumption $\cap_{\alpha \in A} V(I_\alpha) = \varnothing$ implies that $V(\cup_{\alpha \in A} I_\alpha) = \varnothing$, that is, $k[T_1, \dots, T_n]$ is generated by $\{I_\alpha\}_{\alpha \in A}$. Hence there are finitely many $I_1, \dots, I_h$ such that 1 lies in $I_1 + \cdots I_h$, implying $\cap_{i=1}^{h} V(I_i) = \varnothing$. $\square$

A topological space with the property (ii) in the above proposition is called **Noetherian**.

Lemma 0.2. A closed subset of a Noetherian space is Noetherian for the induced topology.

X is irreducible if and only if any two non-empty open subsets of X have a non-empty intersection.

Lemma 0.3. Let X be a topological space.

(i) $A \subseteq X$ is irreducible if and only if its closure $\overline{A}$ is irreducible.

(ii) Let $f : X \to Y$ be a continuous map to a topological space Y. If X is irreducible then so is the image of $f(X)$.

Proof. Let A be irreducible. If $\overline{A}$ is the union of two closed subsets A_1 and A_2 then A is the union of the closed subsets $A \cap A_1$ and $A \cap A_2$. By the irreducibility of A, we have, say $A = A \cap A_1$, hence $A \subseteq A_1$ and $\overline{A} \subseteq A_1$, which shows that $\overline{A}$ is irreducible.
Conversely, assume that $\overline{A}$ is irreducible. If A is the union of $A \cap B_1$ and $A \cap B_2$, where B_1, B_2 are closed subsets of X, we have $A = (A \cap B_1) \cup (A \cap B_2) = A \cap (B_1 \cup B_2)$, thus $A \subseteq B_1 \cup B_2$ and $\overline{A} \subseteq B_1 \cup B_2$. By the irreducibility of $\overline{A}$, we have, say $\overline{A} = B_1$. Hence $A \subset B_1$ and $A = A \cap B_1$, the irreducibility of A follows.
For (ii), if $f(X)$ is the union of two closed subsets $f(X) \cap Y_1, f(X) \cap Y_2$, where Y_1, Y_2 are closed subsets of Y. Thus $X = f^{-1}(Y_1) \cup f^{-1}(Y_2)$. By the continuity of f, $f^{-1}(Y_1), f^{-1}(Y_2)$ are closed subsets in X and by the irreducible of X we have $X = f^{-1}(Y_1)$, which says that $f(X) \subseteq Y_1$ and $f(X) = f(X) \cap Y_1$, the irreducibility of $f(X)$ follows. $\square$

Proposition 0.4. Let X be a Noetherian topological space. Then X has finitely many maximal irreducible subsets. These are closed and cover X.

Proof. From Lemma 0.3 we know that maximal irreducible subsets of X are closed.
Next we will show that X has finitely many irreducible closed subsets, and these closed subsets cover X. We argue by *reductio ad absurdum.* Denote by $\mathcal{S}$ the class of closed subsets in X that are not a union of finitely many irreducible closed subsets. If the property of X in question were not true, then $\mathcal{S}$ is not empty since it contains X. Recall that X is Noetherian iff any family of closed subsets of X has a minimal element, thus there is a minimal closed subset A in $\mathcal{S}$, that is, we can find a minimal closed subset A in X that is not a finite union of irreducible closed subsets. Immediately we know that A is reducible. Thus $A = A_1 \cup A_2$ with A_1, A_2 the closed subsets of X. But at least one of A_1, A_2 is not the union of finitely many irreducible closed subsets, otherwise A is not in $\mathcal{S}$. But $A_1 \subset A$ is a contradiction to the minimality of A, which shows that X is a union of finitely minimality irreducible closed subsets.
Now let $X = X_1 \cup \cdots \cup X_n$, where X_i are irreducible and closed. We may assume that there are no inclusions among them. If Y is an irreducible subset of X, then $Y = (Y \cap X_1) \cup \cdots \cup (Y \cap X_n)$. The irreducibility of Y tells us that $Y = Y \cap X_i$ for some i and hence $Y \subseteq X_i$, *id est*, any irreducible subset of X is contained in one of the X_i. This implies that the X_i are the maximal irreducible subsets of X. The proposition follows. □

The maximal irreducible subsets are called the **irreducible components** of X.

Proposition 0.5. A closed subset X of V is irreducible if and only if $\mathcal{I}(X)$ is a prime ideal.

Proof. Let X be irreducible and let $f, g \in k[T_1, \ldots, T_n]$ such that $fg \in \mathcal{I}(X)$, thus we have $X \subseteq V(fg)$ and $X = X \cap V(fg) = X \cap (V(f) \cup V(g)) = (X \cap V(f)) \cup (X \cap V(g))$. The irreducibility says that $X \subset V(f)$, which means that $f \in I(X)$. It follows that $I(X)$ is a prime ideal.
Conversely, assume $I(X)$ is prime and let $X = V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$. If $X \neq V(I_1)$ there is $f \in I_1$ but $f \notin I(X)$. Since $I_1 I_2 \subseteq I_1 \cap I_2$, we have $fg \in I(X)$ for all $g \in I_2$. By the primeness of $I(X)$ we have $I_2 \subseteq I(X)$, whence $X = V(I_2)$. So X is irreducible. □

Recall that a topological space is **connected** if it is not the union of two disjoint proper closed subsets. The following lemmas given some resultson connectedness and the ralation with the notion of irreducibility.

Lemma 0.6. Let X be a Noetherian topological space.

(i) X is a disjoint union of finitely minimality connected closed subsets, its connected components. They are uniquely determined.

(ii) A connected component of X is a union of irreducible components.

Proof. The proof of (i) is similar as the proof of Proposition 0.4, with the modification of the definition of $\mathcal{S}$ to

$$\mathcal{S} = \{\text{closed subsets in } X \text{ that is not a finite disjoint union of connected closed subsets}\}.$$

For (ii), let X^0 be a connected component of X. Note that X^0 is both open and closed in X. Taking the Noether decomposition $X = X_1 \cup \cdots \cup X_s$ of X, with $X_1, \ldots X_s$ the irreducible component of X, we have $X^0 = (X^0 \cap X_1) \cup \cdots \cup (X^0 \cap X_s)$. But $X^0 \cap X_i$ are both open and closed in the X_i, so are their complements, thus $X^0 \cap X_i = X_i$ or $X^0 \cap X_i = \varnothing$, whence (ii) follows. □

Let $X \subseteq V$ be an algebraic set. The restriction of the polynomial functions of S forms a k-algebra isomorphic to $S/I(X)$, which we denote by $k[X]$. This algebra has the following properties:

(i) $k[X]$ is a **finitely generated** k**-algebra**, *id est*, there is a finite subset $\{f_1, \dots, f_r\}$ of $k[X]$ such that $k[X] = k[f_1, \dots, f_r]$.

(ii) $k[X]$ is **reduced**, *id est*, 0 is the only nilpotent element of $k[X]$.

A k-algebra with these two properties is called an **affine** k**-algebra**. Conversely, given an affine k-algebra A, there is an algebraic subset X of some k^r such that $A \simeq k[X]$. For $A \simeq k[T_1, \dots, T_r]/I$, where I is the kernel of the homeomorphism sending the T_i to the generator of A, then A is reduced if and only if I is a radical ideal. We call $k[X]$ the **affine algebra** of X.

If I is an ideal in $k[X]$ let $V_X(I)$ be the set of the $x \in X$ with $f(x) = 0$ for all $f \in I$. If Y is a subset of X let $I_X(Y)$ be the ideal in $k[X]$ of the f such that $f(y) = 0$ for all $y \in Y$. If A is any affine algebra, let $\mathrm{MaxSpec}A$ be the set of its maximal ideals. If X is as before and $x \in X$, then $M_x = I_X(\{x\})$ is a maximal ideal.

Proposition 0.7. 1. The map $x \mapsto M_x$ is a bijection of X onto $\mathrm{MaxSpec}k[X]$, moreover $x \in V_X(I)$ if and only if $I \subseteq M_x$.

2. The closed sets of X are the $V_X(I)$, I running through the ideals of $k[X]$.

Proof. By the mighty Nullstellensatz. □

If $f \in k[X]$ put

$$D_X(f) = D(f) = \{x \in X | f(x) \neq 0\}.$$

This is an open subset, and is the complement of $V(f)$. We have

$$D(fg) = D(f) \cap D(g), D(f^n) = D(f).$$

The $D(f)$ are called **principal open subsets** of X.

Lemma 0.8. 1. If $f, g \in k[X]$ and $D(f) \subseteq D(g)$ then $f^n \in (g)$ for some $n \geq 1$.

2. The principal open sets form a basis of the topology of X.

Proof. $D(f) \subseteq D(g)$ if and only if $V(g) \subseteq V(f)$ if and only if $\sqrt{(f)} \subseteq \sqrt{(g)}$, which implies (i). (ii) is equivalent to saying that every closed subset in X is an intersection of the form $V_X(f)$, this follows because every closed subset $V_X(I)$ corresponds to a radical ideal $I \subseteq k[X]$, which is finitely generated by the Noether property. □

Let F be a subfield of k. We say that F is a **Field of definition** of the closed subset X of $V = k^n$ if the ideal $I(X)$ is generated by polynomials with coefficients in F. In this situation we put $F[X] = F[T]/I(X) \cap F[T]$. Then the inclusion $F[T] \to k[T]$ induces an isomorphism of $F[X]$ onto an F-subalgebra of S and an isomorphism of k-algebras $k \otimes_F F[X] \to k[X]$.

Let $A = k[X]$ be an affine algebra. An F**-structure** on X is an F-subalgebra A_0 of A which is of finite type over F and which is such that the homomorphism induced by multiplication

$$k \otimes_F A_0 \to k[X]$$

is an isomorphism. We then write $A_0 = F[X]$. The set $X(F)$ of F**-rational points** for our given F-structure is the set of F-homomorphisms $F[X] \to F$. More generally, if W is any vector space over k, an F-**structure** on W is an F-vector subspace W_0 of W such that the canonical homomorphism

$$k \otimes_F W_0 \to W$$

is an isomorphism. A subspace W' of W is **defined over** F if it is spanned by $W' \cap W_0$. Then $W' \cap W_0$ is an F-structure on W'.

Let $x \in X$. A k-valued function f defined in a neighborhood U of x is called **regular in** x if there are $g, h \in k[X]$ and an open neighborhood $V \subseteq U \cap D(h)$ of x such that $f(y) = g(y)h(y)^{-1}$ for $y \in V$.

A function f defined in a non-empty open subset U of X is **regular** if it is regular in all points of U. So for each $x \in U$ there exist g_x, h_x with the properties stated above. Denote by $\mathcal{O}_X(U)$ or $\mathcal{O}$ the k-algebra of regular functions in U. The ringed space $(X, \mathcal{O}_X)$ are called **affine** k-**varieties**.

Let $(X, \mathcal{O}_X)$ be an algebraic variety. It follows from the definiions that there is a homomorphism $\phi : k[X] \to \mathcal{O}_X(X)$.

Theorem 0.9. ϕ is an isomorphism.

Proof. Same arguements as in the class. ☐

Lemma 0.10. Let A and B be k-algebras of finite type. If A and B are reduced (resp. integral domains) then the same holds for $A \otimes_k B$.

Proof. Assume that A and B are reduced. Let $\sum_i^n a_i \otimes b_i$ be a nilpotent element of $A \otimes B$. For any homomorphism $h : A \to k$, $h \otimes \mathrm{id}$ is a homomorphism $A \otimes B \to B$. Then $\sum_{i=1}^n h(a_i)b_i$ is a nilpotent element of B, which must be zero since B is integral. But since b_i are linearly independent, all $h(a_i)$ are zero. Since h is arbitrary, a_i lie in all maximal ideals of A. Since $A = k[f_1, \dots, f_n]$, by the Hilbert's basis Theorem, we have all $a_i = 0$, which shows that $A \otimes B$ is reduced.
Next let A and B be integral domains. Let $f, g \in A \otimes B, fg = 0$. write $f = \sum_i a_i \otimes b_i, g = \sum_j c_j \otimes d_j$, the sets $\{b_i\}$ and $\{d_j\}$ being linearly independent. An argument similar to the one just given then shows that $a_i c_j = 0$, from which it follows that f or g equals 0. ☐

Theorem 0.11. Let X and Y be two affine k-varieties.

(i) A product variety $X \times Y$ exists. It is unique upto isomorphism.

(ii) If X and Y are irreducible then so is $X \times Y$.

A **prevariety** over k is a quasi-compact ringed space $(X, \mathcal{O}_X)$ such that any point of X has an open neighborhood U with the property that the ringed space $U, \mathcal{O}|_U$ is isomorphic to an affine k-variety. Such a U is called an **affine open subset** of X. A morphism of prevarieties is a morphism of ringed spaces.

Proposition 0.12. A product of two prevarieties exists and is unique up to isomorphism.

A subset $U \subseteq \mathbb{P}^n$ is defined to be open if $U \cap U_i$ is open in the affine variety for $0 \leq i \leq n$.

Let X be a prevariety, denote by Δ_X the diagonal subset of $X \times X$, and denote by $i : X \to \Delta_X$ the obvious map, and endow Δ_X with the induced topology.

Lemma 0.13. $i : X \to \Delta_X$ defines a homomorphism of topological spaces for any prevariety X.

Proof. We can cover Δ_X by open sets of the form $U \times U$ with U affine open in X. Since being homeomorphism is a local property, we only need to consider the case that X is affine, in which the Lemma follows. □

The prevariety is said to be a **variety** if Δ_X is closed in $X \times X$.

Proposition 0.14. Let X be a variety and Y a prevariety.

(i) If $\phi : Y \to X$ is a morphism, then its graph $\Gamma_\phi = \{(y, \phi(y)) | y \in Y\}$ is closed in $Y \times X$.

(ii) If $\phi, \psi : Y \to X$ are two morphisms which coincide on a dense set, then $\phi = \psi$.

Proof. For (i), consider the continuous map $Y \times X \to X \times X$ sending (y, x) to $(\phi(y), x)$. Then Γ_ϕ is the preimage of Δ_X in $X \times X$, hence is closed. For (ii), let W be a dense set of Y and consider the subset

$$Z = \{y \in Y | \phi(y) = \psi(x)\}$$

of Y. The assumption that ϕ and ψ coincide on W is equivalent to saying $W \subset Z$. But notice that Z is the preimage of Δ_X in $X \times X$ under the continuous map $\phi \times \psi : Y \times Y \to X \times X$, we know that Z is closed. So $Y = \overline{W} \subseteq Z$, implying $Y = Z$, hence the assertion follows. □

Proposition 0.15. (i) Let X be a varieties and let U, V be affine open structures in X. Then $U \cap V$ is an affine open set and the images under restriction of $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$ in $\mathcal{O}_X(U \cap V)$ generate the last algebra.

(ii) Let X be a prevariety and let $X = \cup_{i=1}^m U_i$ be a covering by affine open sets. Then X is a variety if and only if the following holds: for each pair (i, j) the intersection $U_i \cap U_j$

Let $S = k[T_0, \dots, T_n]$ be the polynomial ring in $n + 1$ indeterminates. If I is a proper homogeneous ideal in S, then if $x \in k^{n+1}$ is a zero of I, the same is true for all $ax, a \in k^*$. Hence we can define a set $V^*(I) \in \mathbb{P}^n$ by

$$V^*(I) = \{x^* \in \mathbb{P}^n | x \in V_{k^{n+1}}(I)\}.$$

Proposition 0.16. The closed sets in $\mathbb{P}^n$ coincide with the sets $V^*(I)$, I running through the homogeneous ideals of S.

Lemma 0.17. Let $\phi : X \to Y$ be a morphism of affine varieties and let $\phi^* : k[Y] \to k[X]$ be the associated algebra homomorphism.

1. If ϕ^* is surjective, then ϕ maps X onto a closed subset of Y.

2. ϕ^* is surjective if and only if $\phi(X)$ is dense in Y

Proposition 0.18. i There is a unique irreducible component G^0 of G that contains the identity element e, which is a closed normal subgroup of finite index.

ii G^0 is the unique connected component of G containing e.

iii Any closed subgroup of G of finite index contains G^0.

Proof. Let X, Y be irreducible components of G containing e. If μ and ι are the group multiplication and inversion of G, then $XY = \mu(X \times Y)$ is irreducible and hence its closure $\overline{XY}$ is so, $X = Y = \overline{XY}$[1], so X. is closed under multiplication. ι is topologically a homeomorphism, so X^{-1} is also irreducible and contains e, thus must coincide with X. X is a closed subgroup. Using that inner automorphisms define homeomorphisms [2], we have $xXx^{-1} = X$ for any $x \in X$, so that X is normal. The cosets are components of G, and by some proposition [3] the number of cosets is finite. We have proved (i).
irreducible components are mutually disjoint. It then follows [4] that the irreducible components must coincide with the connected components. This implies (ii).
If H is a closed subgroup of G of finite index, then H^0 is a closed subgroup of finite index of G^0. Then H^0 is both open [5] and closed in G^0. Since G^0 is connected, we have $G^0 = H^0$, which proves (iii). $\square$

[1] *Why?*

[2] *References?*

[3] *Which?*

[4] *From where?*

[5] *Why?*

Lemma 0.19. Let U and V be dense open subsets of G. Then $UV = G$.

Lemma 0.20. Let H be a subgroup of G.

1. The closure $\overline{H}$ is a subgroup of G.
2. If H contains a non-empty open subset of $\overline{H}$ then H is closed.

Proposition 0.21. Let $\phi : G \to G'$ be a homeomorphism of algebraic groups.

1. $\ker \phi$ is a closed normal subgroup of G.
2. $\phi(G)$ is a closed subgroup of G'.
3. If G and G' are F-groups and ϕ is defined over F then $\phi(G)$ is an F-subgroup of G'.
4. $\phi(G^0) = (\phi G)^0$.

Proof. (i) $\ker \phi = \phi^{-1}(e)$, thus is closed. (ii) $\phi(G)$ contains a non-empty open subset of its closure, by Lemma 0.20 (ii) follows.
(iv) $\phi(G^0)$ is a closed subgroup of G' by (ii), which is connected and of finite index [6]. $\square$

[6] *Why?*

Proposition 0.22. Let $\{X_i, \phi_i\}_{i \in I}$ be a family of irreducible varieties together with morphism $\phi_i : X_i \to G$. Denote by H the smallest closed subgroup of G containing the images $Y_i = \phi_i(X_i)$. Assume that all Y_i contain the identity element e.

1. H is connected.
2. There exists an integer $n \geq 0$, $a = (a(1), \dots, a(n)) \in I^n$ and $\epsilon(h) = \pm 1, 1 \leq h \leq n$ such tht $H = Y_{a(1)}^{\epsilon(1)} \dots Y_{a(n)}^{\epsilon(n)}$.
3. Assume, moreover, that G is an F-group, that all X_i are F-varieties and that the morphisms ϕ_i are defined over F. Then H is an F-subgroup.

Proof. Use a lot of previous results. $\square$

Corollary 0.23. 1. Assume that $\{G_i\}_{i\in I}$ is a family of closed, connected, subgroups of G. Then the subgroup H generated by them is closed and connected. There is an integer $n \geq 0$ and $a = (a(1), \ldots, a(n)) \in I^n$ such that $H = G_{a(1)} \ldots G_{a(n)}$.

2. If, moreover, G is an F-group and all G_i are F-subgroups then H is an F-subgroup.

If H and K are subgroups of G, we denote by (H, K) the subgroup generated by the commutators $xyx^{-1}y^{-1}$ with $x \in H, y \in K$.

Corollary 0.24. 1. If H and K are closed subgroups of G one of which is connected, then (H, K) is connected.

2. If, moreover, G is an F-group and H, K are F-subgroups then (H, K) is a connected F-subgroup.

Proof. Assume that H is connected. (i) follows by applying Proposition 0.22 which $I = K$, all X_i being H, with $\phi_i(x) = xix^{-1}i^{-1}$. □

A G-**variety** or a G-**space**, is a variety X on which G acts as a permutation group, the action being given by a morphism of varieties. A **homogeneous space** for G is a G-space on which G acts transitively.

Lemma 0.25. 1. An orbit Gx is open in its closure.

2. There exists closed orbits.

Proof. Gx contains a non-empty open subsection U of its closure. Since Gx is the union of the open sets $gU, g \in G$, (i) follows. It implies that for $x \in X$, the set $S_x = \overline{Gx} - Gx$ is closed. □

From now on we assume that G is a linear algebraic group. Let X be an affine G-space, which $a : G \times X \to X$. We have $k[G \times X] = k[G] \otimes_k k[X]$ and a is given by an algebra homomorphism $a^* : k[X] \to k[G] \otimes k[X]$. For $g \in G, x \in X, f \in k[X]$ define

$$(s(g))f(x) = f(g^{-1}x).$$

Then $s(g)$ is an invertible linear map of the (in general infinite dimensional) vector space $k[X]$ and s is a representation of abstract groups $G \to \mathrm{GL}(k[X])$. The next result will imply that s can be built up from rational representations.

Proposition 0.26. Let V be a finite dimensional subspace of $k[X]$.

(i) There is a finite dimensional subspace W of $k[X]$ which contains V and is stable under all $s(g), g \in G$.

(ii) V is stable under all $s(g)$ if and only if $a^*V \subset k[G] \otimes V$. If this is so, s defines a map $s_V : G \times V \to V$ which is a rational representation of G.

(iii) If, moreover, G is an F-group, X is an F-variety, V is defined over F and a is an F-morphism then in (i) W can be taken to be defined over F.

Now we consider the case that G acts by left or right translationss on itself. For $g, x \in G, f \in$

$k[G]$ define

$$(\lambda(g)f)(x) = f(g^{-1}x), (\rho(g)f)(x) = f(xg).$$

Then λ and rho are representations of G in $\mathrm{GL}(k[G])$. If ι is the automorphism of $k[G]$ defined by inversion, then $\rho = \iota \circ \lambda \circ \iota^{-1}$. The represntaions λ and ρ has trivial kernel.

Theorem 0.27. (i) There is an isomorphism of G onto a closed subgroup of some GL_n.

(ii) If G is an F-group the isomorphism of (i) may be taken to be defined over F.

Lemma 0.28. Let H be a closed subgroup of G. Then

$$H = \{g \in G | \lambda(g)\mathcal{I}_G(H) = \mathcal{I}_G(H)\} = \{g \in G | \rho(g)\mathcal{I}_G(H) = \mathcal{I}_G(H)\}$$

Proof. If $g, h \in H$, $f \in \mathcal{I}_G(H)$ then $(\lambda(g)f)(h) = f(g^{-1}h) = 0$, whence $\lambda(g)f \in \mathcal{I}_G(H)$. On the other hand, if $f \in \mathcal{I}_G(H)$, we have $0 = f(g^{-1}) = (\lambda(g)f)(e)$ for all $f \in \mathcal{I}_G(H)$ and $g \in H$. □

1 The Jordan Decomposition

Let k be an algebraically closed field, V a k-vector space, an endomorphism $\phi \in \mathrm{End}(V)$ is called **locally finite**, if $V = \cup_i V_i$ and $\phi(V_i) \subseteq V_i$, where $V_i \subseteq V$ are finite-dimensional subspaces of V.

In particular, if $\dim V < \infty$, then every $\phi \in \mathrm{End}(V)$ is locally finite.

Definition 1.1. Let U be a finite-dimensional k-vector space, $\phi \in \mathrm{End}(U)$ is called **semi-simple**, if it is diagonalizable, or equivalently, the minimal polynomial of ϕ is separable over k. ϕ is called **nilpotent**, if there exist some $n > 0$ such that ϕ^n, and ϕ is called **unipotent** if $\phi - \mathrm{id}$ is nilpotent.
Let V be a k-vector space, $\phi \in \mathrm{End}(V)$ a locally finite endomorphism, ϕ is said to be **semi-simple**, if $\phi|_W$ is semi-simple for any finite-dimensional ϕ-stable subspace $W \subset V$. ϕ being **nilpotent** or **unipotent** is defined similarly.

Lemma 1.1. Let V and W be finite-dimensional k-vector spaces, V_1, V_2 subspaces of V, $\phi \in \mathrm{End}(V)$. Then

i If ϕ is semi-simple (resp. nilpotent, unipotent) and $\phi(V_i) \subseteq V_i$, so is $\phi|_{V_i} \in \mathrm{End}(V_i)$. Conversely, if $\phi|_{V_i}$ are semi-simple (resp. nilpotent, unipotent), and $V = V_1 + V_2$, then ϕ is semi-simple (resp. nilpotent, unipotent).

ii If ϕ is semi-simple (resp. nilpotent, unipotent), $\phi(V_1) \subseteq V_1$, then so are $\phi|_{V_1}$ and $\phi|_{V/V_1}$.

iii If $\psi \in \mathrm{End}(V)$ and $\phi\psi = \psi\phi$, then $\phi\psi$ is semi-simple (resp. nilpotent, unipotent), if both ϕ and ψ are so.

iv If $\psi \in \mathrm{End}(W)$, ϕ and ψ are both semi-simple, nilpotent, unipotent, then so are $\phi \oplus \psi$ and $\phi \otimes \psi$.

v If $\psi \in \mathrm{End}(W)$, ϕ and ψ are both semi-simple (resp. nilpotent), then so is $\phi \otimes \mathrm{id} + \mathrm{id} \otimes \psi$.

Proposition 1.2. Let V be a k-vector space, $\phi \in \mathrm{End}(V)$ locally finite. Then

$$\phi = \phi_s + \phi_n \tag{1.1}$$

where ϕ_s is semi-simple, ϕ_n is nilpotent, and $\phi_s\phi_n = \phi_n\phi_s$. Moreover,

i The decomposition (1.1) is unique.

ii If $\phi \in \mathrm{GL}(V)$, then ϕ has a unique decomposition

$$\phi = \phi_s\phi_u, \tag{1.2}$$

where ϕ_s semi-simple, ϕ_u unipotent, $\phi_s\phi_u = \phi_u\phi_s$.

iii If $W \subseteq V$ is ϕ-stable, then $\phi_s(W) \subset W, \phi_n(W) \subset W$

$$\phi|_W = \phi_s|_W + \phi_n|_W \tag{1.3}$$

is the Jordan decomposition of $\phi|_W$, and

$$\phi|_{V/W} = \phi_s|_{V/W} + \phi_n|_{V/W} \tag{1.4}$$

is also the Jordan decomposition of $\phi|_{V/W}$.

iv If $\alpha : V \to W$ is a k-linear map, $\psi \in \mathrm{End}(W)$, such that the diagram

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d, "\phi"] & W \arrow[d, "\psi"] \\ V \arrow[r, "\alpha"] & W \end{tikzcd}$$

commutes, then

$$\psi_s \circ \alpha = \alpha \circ \phi_s$$

and

$$\psi_n \circ \alpha = \alpha \circ \phi_n.$$

Proof. For ii, if we have $\phi = \phi_s + \phi_n \in \mathrm{GL}(V)$, then we have $\phi_s \in \mathrm{GL}(V)$. Indeed, since ϕ_n is nilpotent, the expansion of the formal expression

$$(\phi - \phi_s)^{-1}$$

is well-defined. Thus we take

$$\phi = \phi_s + \phi_n = \phi_s(1 + \phi_s^{-1}\phi_n) = \phi_s\phi_u.$$

□

Theorem 1.3. Let G be an algebraic group, $\rho : G \to \mathrm{GL}(k[G])$ be the representation corresponding to the right translation. Then

i There exist $g_s, g_u \in G$, such that $\rho(g)_s = \rho(g_s)$ and $\rho(g)_u = \rho(g_u)$ with $g = g_sg_u = g_ug_s$.

ii If $\phi : G \to G$ is a morphism of algebraic groups, then $\forall g \in G$

$$\phi(g_s) = \phi(g)_s,$$
$$\phi(g_u) = \phi(g)_u.$$

iii If $G = \mathrm{GL}_n(k)$, then $g = g_s g_u$ is the Jordan decomposition of g as in Proposition 1.2

The elements g_s and g_u of (i) are the **semi-simple part** and **unipotent part** of $g \in G$.

Proof. Let $A = k[G]$ and let $m : A \otimes A \to A$ be the homomorphism defined by multiplication of polynomials. Since $\rho(g) : A \to A$ is a homomorphism of algebras, it's easy to verify that

$$m \circ (\rho(g) \otimes \rho(g)) = \rho(g) \circ m$$

Applying Proposition 1.2 (iv) to m, we have

$$\rho(g)_s \circ m = m \circ (\rho(g) \otimes \rho(g))_s = m \circ (\rho(g)_s \otimes \rho(g)_s),$$

which shows that $\rho(g)_s$ is an automorphism of the ring A. Hence $f \mapsto (\rho(g)_s f)(e)$ defines a ring homomorphism $A \to k$, which is further more a point $g_s \in G$. Since $\rho(g)$ commutes with all $\lambda(x), x \in G$, again by Proposition 1.2 (iv) we have $\rho(g)_s$ commuting with all $\lambda(x), x \in G$:

$$\begin{aligned}(\rho(g)_s f)(x) &= (\lambda(x^{-1})\rho(g)_s f)(e) \\ &= (\rho(g)_s \lambda(x^{-1}) f)(e) \\ &= (\lambda(x^{-1}) f)(g_s) \\ &= f(x g_s) \\ &= (\rho(g_s) f)(x),\end{aligned}$$

showing that $\rho(g)_s = \rho(g_s)$.
In a similar way, one can also show that there is a $g_u \in G$ such that $\rho(g)_u = \rho(g_u)$.
The proof of (ii) uses 1.9.1. $\square$

Recall that an abstract group H is **nilpotent** if there is an integer n such that all sommutators equal e. Such a group is solvable.

Corollary 1.4. A unipotent linear algebraic group is nilpotent, hen

Definition 1.2. A **Borel subgroup** of an algebraic group G is a connected closed solvable subgroup of maximal dimension.
A **parabolic subgroup** of G is a closed subgroup of G containing a Borel subgroup.
The **radical** of G is the maximal connected closed solvable normal subgroup of G, and is denoted by $R(G)$.
The **unipotent radical** of G is the unipotent maximal connected closed solvable normal subgroup of G, and is denoted by $R_u(G)$.
If $R(G) = \{ e \}$, then G is called semi-simple.
If $R_u(G) = \{ e \}$, then G is called **reductive**.

Example. Take $G = \mathrm{GL}_n(k)$. The subgroup $B_n(k) \subset \mathrm{GL}_n(k)$ consists of the upper diagonal matrix is a Borel subgroup.

A parabolic subgroup of G consists of the upper block-wise diagonal matrix.
$R(\mathrm{GL}_n(k)) = \mathrm{diag}(\lambda, \ldots, \lambda), \lambda \neq 0$, which implies that $\mathrm{GL}_n(k)$ is not semi-simple.
$R_u(\mathrm{GL}_n(k)) \subseteq R(G)_u = \{ e \}$, which implies that $\mathrm{GL}_n(k)$ is reductive.

Definition 1.3. A **maximal torus** of G is a commutative connected closed subgroup T of G with maximal dimension such that $T_u = \{ e \}$.
A **Cartan subgroup** of G is the centralizer subgroup $Z_G(T)$ of a maximal torus T of G.

Theorem 1.5. Let G be an algebraic group, B be a Borel subgroup of G, then

1. G/B is a projective variety,
2. Every two Borel subgroups of G are G-conjugate.
3. If G is connected, then $N_G(B) = B$
4. G is the union of all Borel subgroups of G, if G is connected.

Proof.
ii Let B', B be two Borel subgroups of G, and let B' act on G/B. Since B is Borel, by i G/B is projective, hence is complete. Again since B' is Borel, thus is connected and solvable. So by Borel's fixed point theorem, there exists an $x \in G$ such that $xB \in G/B$ satisfying

$$b'xB = xB, \forall b' \in B',$$

which implies $x^{-1}B'x \subseteq B$. Since the group $x^{-1}B'x$ is again connected and solvable and of maximal dimension, thus we have $x^{-1}B'x = B$.
iii $N_G(B)$ is a closed subgroup of G, and since B is a normal subgroup in $N_G(B)$, we have $N_G(B)/B$ being an affine variety by the quotient theorem. On the other hand, $N_G(B)/B$ is a projective variety, since B is connected, closed and solvable of maximal dimension in G, it is easy to see that B is connected, closed and solvable of maximal dimension in $N_G(B)$i, which means that B is Borel in $N_G(B)$. $N_G(B)/B$ has finitely many points. And by the fact that $N_G(B)$ is connected, we have $N_G(B)/B = 1$. □

Corollary 1.6. Let G be a connected algebraic group. Then

i There is a one-to-one correspondence between the set of all Borel subgroups of G and G/B viewed as a set.

ii $Z(G) = Z(B)$.

iii Let P be a parabolic subgroup of G. Then P is connected and $N_G(P) = P$

iv Let P and Q be two conjugate parabolic subgroups of G, if $P \cap Q$ is also parabolic, then $P = Q$.

Proposition 1.7. Let G, H be diagonalizable groups and let V be a connected affine k-variety. Assume that

$$\phi : G \times V \to H$$

is a morphism of varieties such that

$$\begin{aligned}\phi_v : G &\to H,\\ g &\mapsto \phi(g, v)\end{aligned}$$

is a morphism of algebraic groups for all $v \in V$. Then

$$\phi_v = \phi_{v'}, \forall v, v' \in V.$$

Proof. $k[G]$ has a k-basis $\chi(G)$, $k[H]$ has a k-basis $\chi(H)$ [7].

$$\phi^* : k[H] \to k[G] \otimes k[V],$$

$\forall X \in \chi(H)$, $\phi^*(X) = \sum_{\lambda \in \chi(G)} \lambda \otimes a_{\lambda,X}$.

Claim 1.8. $a_{\lambda,X}$ is either zero or identity in $k[V]$, and there exists a unique [8] γ, such that

$$a_{\gamma,X} \neq 0.$$

With the Claim, we have

$$\phi^*(X)$$

□

[7] *Why?*

[8] *Not so sure*

Corollary 1.9. Let G be an algebraic group, $H \leq G$ a diagonalizable subgroup. Then

i $N_G(H)^\circ = Z_G(H)^\circ$ [9],

ii $N_G(H)/Z_G(H)$ is finite.

[9] *Meaning of the notation $N_G(H)^\circ$?*

Proof. $N_G(H)$ is a closed subgroup of G, hence is an affine variety. Consider the morphism

$$\begin{aligned}\phi : N_G(H)^\circ \times H &\to H,\\ (x, h) &\mapsto xhx^{-1}.\end{aligned}$$

Fix $x \in N_G(H)^\circ$, $\phi_x : H \to H$ mapping h to xhx^{-1} is a morphism of diagonalizable groups. By Proposition 1.7 [10], we have

$$\phi_x = \phi_e \forall x \in N_G(H)^\circ$$

thus $x \in Z_G(H)$, hence $N_G(H)^\circ \subseteq Z_G(H)^\circ \subseteq N_G(H)^\circ$, implying $N_G(H)^\circ = Z_G(H)^\circ$.

□

[10] *What if $N_G(H)^\circ$ is not connected ?*

Proposition 1.10. Let G be a connected solvable algebraic group. Then

i (G, G) is a connected, closed, unipotent, normal subgroup.

ii G_u is a closed connected nilpotent normal subgroup of G, G/G_u is a torus.

iii Each semi-simple element in G is contained in a (maximal) torus of G.

iv $\forall x \in G$ that is semi-simple, $Z_G(x)$ is connected.

v Two maximal tori of G are G-conjugate.

vi If T is a maximal torus of G, the product map $T \times G_u \to G$ is an isomorphism of varieties, so that $G = T \rtimes G_u$

vii If H is a subgroup, $H = H_s$, then H is contained in a maximal torus. Moreover, $N_G(H) = Z_G(H)$.

Proposition 1.11. Let G be a connected algebraic group, T a maximal torus. Then

i $Z_G(T)$ is a connected nilpotent subgroup of G

ii There exists a $t \in T$, such that t lies in finitely many G-conjugates of $Z_G(T)$

iii The union of Cartan subgroups of G, contains a dense open subset of G

iv Let $S \subseteq G$ be a subtorus, then $Z_G(S)$ is connected.

Proof. i, iv Pick $g \in Z_G(s)$, let B be a Borel subgroup of G containing g [11]. Consider the morphism

$$\begin{aligned} \psi_g : G/B &\to G/B \times G/B, \\ xB &\mapsto (xB, gxB) \end{aligned}$$

$\psi_g^{-1}(\Delta)$ is a closed subvariety in G/B, and S acts on $\phi_g^{-1}(\Delta)$.
Since S is connected solvable, thus by Borel's fixed point theorem, there exists $xB \in \psi_g^{-1}(\Delta)$ that is an S-fixed point. [12] □

[11] *Density theorem*

[12] *Get confused from here. Lecture Note 23.*

2 Lie Algebras of Algebraic Groups

Let G be an algebraic group over k, λ, ρ be the left left translation and right translation action on $k[G]$ [13]

Let $\Omega_G = I/I^2$, where I is the kernel of the multiplication map

$$0 \longrightarrow I \longrightarrow k[G] \otimes k[G] \xrightarrow{m} k[G] \longrightarrow 0 .$$

Then G acts on Ω_G, diagonally via left or right translation. [14]

[13] *On G or $k[G]$*

[14] *Verify that λ, ρ commute with $d : k[G] \to \Omega_G$.*

Theorem 2.1. There exists an isomorphism of $k[G]$-modules

$$\Phi : \Omega_G \xrightarrow{\sim} k[G] \otimes (T_e G)^*$$

such that the following diagrams commute for all $x \in G$

$$\begin{array}{ccc} \Omega_G & \xrightarrow{\Phi} & k[G] \otimes (T_e G)^* \\ \downarrow{\scriptstyle \lambda(x)} & & \downarrow{\scriptstyle \lambda(x) \otimes \mathrm{id}} \\ \Omega_G & \xrightarrow{\Phi} & k[G] \otimes (T_e G)^* \end{array} ,$$

$$\begin{array}{ccc} \Omega_G & \xrightarrow{\Phi} & k[G] \otimes (T_e G)^* \\ \downarrow{\scriptstyle \rho(x)} & & \downarrow{\scriptstyle \rho(x) \otimes \mathrm{Ad}_x^*} \\ \Omega_G & \xrightarrow{\Phi} & k[G] \otimes (T_e G)^* \end{array} ,$$

where $\mathrm{Ad}_x^* : (T_eG)^* \to (T_eG)^*$ is the dual map of $\mathrm{Ad}_x : T_eG \to T_eG$, and for $f \in k[G]$,

$$\Phi(df) = -\sum_i f_i \otimes \delta(g_i)$$

with $\delta(g_i)$ the image of $g_i - g_i(e)$ in $(T_*G)^* \simeq \mathfrak{m}_e/\mathfrak{m}_e^2$.

Proof. Identify $k[G] \otimes_k k[G] \simeq k[G \times G]$ and $m : k[G \times G] \to k[G]$, which is the dual of the diagonal map $\Delta : G \to G \times G$. In particular, I is then the defining ideal of $\Delta(G)$ in $G \times G$.
Consider the morphism

$$\begin{aligned} \psi : G \times G &\to G \times G, \\ (x, y) &\mapsto (x, xy), \end{aligned}$$

thus $\psi^{-1}(\Delta(G)) = G \times \{e\}$. In particular, $\psi^* : k[G] \otimes k[G] \to k[G] \otimes k[G]$ sends I to the defining ideal of $G \times \{e\} \subseteq G \times G$, that is

$$\phi^*(I) \subset k[G] \otimes_k \mathfrak{m}_e$$

which induces a morphism $I/I^2 \to k[G] \otimes_k \mathfrak{m}_e/\mathfrak{m}_e^2$ [15]. This is the morphism Φ in the statement of the theorem.
It remains to check the rest of the equalities. □

[15] *How?*

Recall that $\mathcal{D}_G = \mathrm{Der}_k(k[G], k[G])$ is a Lie algebra. Moreover, if $\mathrm{char} k = p \geq 0$, $\mathcal{D}_G$ is further a p-**Lie algebra**, which is a Lie algebra $\mathfrak{g}$ with the p-operation $[p] : \mathfrak{g} \to \mathfrak{g}, x \mapsto x^{[p]}$ satisfying

1. $(\alpha X)^{[p]} = \alpha^p X^{[p]}$,
2. $\mathrm{ad}_{X^{[p]}} = (\mathrm{ad}_X)^p$ with $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}, Y \mapsto [X, Y]$,
3. $(X + Y)^{[p]} = X^{[p]} + Y^{[p]} + \sum_{i=1}^{p-1} s_i(X, Y)/i$, $s_i(X, Y)$ is the coefficient of t^i in the expansion of $\mathrm{ad}_{tX+Y}^{p-1}(Y)$.

Example. Let k be a field with $\mathrm{char} k = p > 0$, $\mathfrak{gl}_n(k)$ is a p-Lie algebra with the p-operation

$$\begin{aligned} \mathfrak{gl}_n(k) &\to \mathfrak{gl}_n(k), \\ A &\mapsto A^p. \end{aligned}$$

Note that $\mathcal{D}_G = \mathrm{Hom}_{k[G]}(\Omega_G, k[G])$, left translation and right translation act on Ω_G and $k[G]$, thus act on $\mathcal{D}_G$. For all $D \in \mathcal{D}_G$, we define [16]

[16] *Note that elements of the form df generate Ω_G.*

$$(\lambda(x) \cdot D)(df) = \lambda(x) D(\lambda(x)^{-1} \cdot df)$$
$$(\rho(x) \cdot D)(df) = \rho(x) D \rho(x)^{-1}(df)$$

Definition 2.1. The **Lie algebra** of the algebraic group G is defined as

$$L(G) = \{\, D \in \mathcal{D}_G \mid \lambda(x) \cdot D = D, \forall x \in G \,\}.$$

Proposition 2.2. Notations as above.

i $L(G)$ is a subalgebra of $\mathcal{D}_G$

ii $L(G)$ is stabilized under $\rho(x), \forall x \in G$

iii There exists an isomorphism of $k[G]$-modules

$$\Psi : \mathcal{D}_G \xrightarrow{\simeq} k[G] \otimes (TeG)$$

such that the diagrams

$$\begin{tikzcd} \mathcal{D}_G \arrow[r, "\Psi"] \arrow[d, "\lambda(x)"] & k[G] \otimes T_e G \arrow[d, "\lambda(x)\otimes \mathrm{id}"] \\ \mathcal{D}_G \arrow[r, "\Psi"] & k[G] \otimes T_e G \end{tikzcd},$$

$$\begin{tikzcd} \mathcal{D}_G \arrow[r, "\Psi"] \arrow[d, "\rho(x)"] & k[G] \otimes T_e G \arrow[d, "\rho(x)\otimes \mathrm{Ad}_x^*"] \\ \mathcal{D}_G \arrow[r, "\Psi"] & k[G] \otimes T_e G \end{tikzcd}$$

commute.

iv Let $\alpha : \mathcal{D}_G \to T_e G$ be the linear map, $\alpha(D)(f) = D(f)(e)$. Then by restricting α to the Lie subalgebra $L(G) \subseteq T_e G$, we get an isomorphism of k-vector spaces

$$L(G) \xrightarrow{\simeq} T_e G$$

such that $\forall x \in G$,

$$\alpha \circ \rho(x) \circ \alpha^{-1} = \mathrm{Ad}_x.$$

In particular, $\dim L(G) = \dim G$.

v Let H be a closed subgroup of G, with the defining ideal I> Then

$$L(H) = \{ \, D \in L(G) \mid D(I) \subseteq I \, \}$$

vi Let $\phi : G \to H$ be a morphism of algebraic groups. Then

$$(d\phi)_e : T_e G \to T_e H$$

is a morphism of Lie algebras (resp. p-Lie algebras, if $\mathrm{char} k = p > 0$.)

vii The Lie algebra of $\mathrm{GL}_n(k)$ is $\mathfrak{gl}_n(k)$, with the Lie bracket

$$[X, Y] = XY - YX, \forall X, Y \in \mathfrak{gl}_n(k).$$

Proof. [17]
iii Using the isomorphism

$$\mathcal{D}_G := \mathrm{Der}_k(k[G], k[G]) \simeq \mathrm{Hom}_{k[G]}(\Omega_G, k[G])$$

and applying the functor $\mathrm{Hom}_{k[G]}(-, k[G])$ to the diagrams in Theorem 2.1, by the Yoneda Lemma.

□

[17] *I'm not sure I can repeat all the computations in the proof. Lecture Note 24.*

Definition 2.2. X is a k-variety defined over $\mathbb{F}_q$, if X has an affine open cover Y_i, such that $k[Y_i] = \mathbb{F}_q[Y_i] \otimes_{\mathbb{F}_q} k$, where $\mathbb{F}_q[Y_i]$ is a $\mathbb{F}_q$-structure of $k[Y_i]$. Now, the Frobenius map on X is defined, precisely via

$$\begin{aligned} k[Y_i] = \mathbb{F}_q[Y_i] \otimes_{\mathbb{F}_q} k \to \mathbb{F}_q[Y_i] \otimes_{\mathbb{F}_q} k = k[Y_i] \\ \sum_i f_i \otimes \lambda_i \mapsto \sum_i f_i^q \otimes \lambda_i. \end{aligned} \tag{2.1}$$

Proposition 2.3. Let $\mathfrak{g} = T_e G$ be the Lie algebra of G.

i $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is the adjoint representation, which satisfies

$$(\mathrm{Ad}_x X)(f) = \sum_1 f_1(x) X(f_2) f_3(x^{-1}) \tag{2.2}$$

where $(\Delta \otimes \mathrm{id})\Delta(f) = \sum f_1 \otimes f_2 \otimes f_3$.

ii For $m : G \times G \to G$, $dm_{(e,e)} : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$ satisfies

$$dm_{(e,e)}(X, Y) = X + Y, \forall X, Y \in \mathfrak{g}.$$

And for the inversion $\iota : G \to G$, $(d\iota)_e(X) = -X, \forall X \in \mathfrak{g}$.

iii $\mathrm{ad} = (d\mathrm{Ad})_e : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$, such that

$$\mathrm{ad}(X)(Y) = [X, Y].$$

iv (Lang's Theorem) Let $\sigma : G \to G$ be a morphism of varieties. Let $\phi : G \to G$ such that $\phi(x) = \sigma(x)x^{-1}$, then $(d\phi)_e = (d\sigma)_e - \mathrm{id}$. In particular, if G is an algebraic group, defined over $\mathbb{F}_q$, and $F : G \to G$ is the Frobenius morphism, then $\phi(x) = F(x)x^{-1}$ is a surjective map.

Proof. iv Lang's Theorem. Consider the morphism

$$\phi : G \overset{\Delta}{\to} G \times G \overset{\sigma \times \iota}{\to} G \times G \overset{m}{\to} G$$

Claim 2.4. $(d\phi)_x$ is bijective for all $x \in G$.

To see it, consider the following commutative diagram

$$\begin{tikzcd} G \arrow[r, "\rho(x)"] \arrow[d, "\Delta"] & G \arrow[r, "\phi"] & G \\ G \times G \arrow[r, "\sigma \times \iota"] & G \times G \arrow[r, "\rho(\phi(x))"] & G \times G \arrow[u, "m"'] \end{tikzcd}$$

Claim 2.5. Let X be the closure of $\phi(G)$, and let $\phi(x)$ be a simple point in X

Since $(d\phi)_x$ is bijective, we have $\dim X = \dim G$. Assume that G is connected, then $X = G$[18] □

[18] *Why?*

Let G be a complex algebraic group. It is known [19] that G has the unique structure of complex Lie group. To distinguish the topologies, we denote the Lie group by G^{an}

[19] *From where?*

Proposition 2.6. $T_eG = T_eG^{\mathrm{an}}$ and G is irreducible iff G^{an} is connected.

Proof. $T_eG = T_eG^{\mathrm{an}}$ as vector spaces, follows from the construction of the complex manifold structure [20] on G.
The structures of Lie algebras coincide, since $G \hookrightarrow \mathrm{GL}_n(\mathbb{C})$, as both an algebraic group and a Lie group, while the Lie brackets on $\mathrm{Lie}(\mathrm{GL}_n(\mathbb{C}))$ and on $\mathrm{Lie}(\mathrm{GL}_n(\mathbb{C})^{\mathrm{an}})$ coincide.
If G^{an} is connected, take G° as the identity component of G, then G is a disjoint union of irreducible components, each being a G°-coset in G. In particular, G° is both open and closed in the Zariski topology [21], $\implies$ G° is both open and closed in real topology [22] $\implies$ $G^\circ = G^{\mathrm{an}} \implies G = G^\circ$.
Conversely, if G is irreducible, take a Basis $\{X_i\}_{i\in I}$ of $T_eG = T_eG^{\mathrm{an}}$. Let P_i be the one-parameter subgroup of G^{an}, corresponding to X_i. Let G_i be the Zariski closure of P_i in G. Then

Claim 2.7. G_i is a commutative, irreducible, closed algebraic subgroup of G.

Claim 2.8. Every irreducible closed commutative complex algebraic group is connected. In particular, G_i's are connected.

Let $\hat{G}$ be the closed subgroup of G generated by G_i. Note that $X_i \in \mathrm{Lie}(P_i) \subseteq \mathrm{Lie}(G_i^{\mathrm{an}}) \implies \mathrm{Lie}(\hat{G}) \supseteq \mathrm{Lie}(G) \implies \hat{G} = G, G_i^{\mathrm{an}} \subset (G^{\mathrm{an}})^\circ, G^{\mathrm{an}} = \langle G_i^{\mathrm{an}} \,|\, i \in I\rangle \implies G^{\mathrm{an}} \subseteq (G^{\mathrm{an}})^\circ \implies G^{\mathrm{an}}$ is connected. □

[20] *From where?*

[21] *Why?*

[22] *Why?*

Corollary 2.9. If B is a Borel subgroup of G, then B^{an} is a Borel subgroup of G^{an}, which means that B^{an} is a maximal connected solvable closed Lie subgroup of G^{an}.

Proof. By Proposition 2.6, B^{an} is a connected, solvable[23] and closed Lie subgroup of G^{an}. Assume that B' is a Borel subgroup of G^{an} containing B^{an}. Let H be the Zariski closure of B', then one may check that

Claim 2.10. H is also solvable.

Claim 2.11. $B^{\mathrm{an}} \subseteq B' \subseteq H^{\mathrm{an}}, B' \subseteq H^\circ$.

Hence $B' = H^\circ \implies B^{\mathrm{an}} = \overline{B'}$ □

[23] *Why?*

Definition 2.3. A subalgebra $\mathfrak{h}$ of $\mathfrak{g} = \mathrm{Lie}(G)$, is called an **algebraic subalgebra**, if there exists closed algebraic subgroup H of G, such that $\mathrm{Lie}(H) = \mathfrak{h}$.

Similar to the construction of the Malcev closure, we define that for each subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the **algebraic closure** $\mathfrak{h}^a$ is defined to be the algebraic subalgebra of smallest dimension containing $\mathfrak{h}$ of $\mathfrak{g}$.

Theorem 2.12. Let G be a connected complex algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$.

i Let H be an algebraic subgroup of G, generated by connected algebraic subgroups $\{H_\alpha | \alpha \in I\}$. Then $\mathrm{Lie}(H)$ is generated by $\mathfrak{h}_\alpha, \alpha \in I$. In particular, the subalgebra of $\mathfrak{g}$ generated by a family of algebraic subalgebras is algebraic.

ii Let $\mathfrak{h}^a$ be the algebraic closure of $\mathfrak{h}$, then $[\mathfrak{h}^a, \mathfrak{h}^a] = [\mathfrak{h}, \mathfrak{h}]$.

iii If $\mathfrak{h}$ is a commutative (resp. solvable) ideal of $\mathfrak{g}$, then so is $\mathfrak{h}^a$. In particular, any Borel subalgebra is algebraic.

Proof. Let $\hat{\mathfrak{h}}$ be the subalgebra of $\mathfrak{g}$, generated by $\mathrm{Lie}(H_\alpha), \alpha \in I$. Let $\hat{H}^{\mathrm{an}}$ be the Lie subgroup of G^{an}, corresponding to $\hat{\mathfrak{h}}$ corresponding to $\hat{\mathfrak{h}}$ (by Lie group theory). Using exponential map, $\forall x \in \mathfrak{H}_\alpha \subseteq \hat{\mathfrak{h}}$, we have

$$H_\alpha{}^{\mathrm{an}} \subseteq \hat{H}^{\mathrm{an}},$$

hence $H^{\mathrm{an}} \subseteq \hat{H}^{\mathrm{an}}$ which implies $\mathrm{Lie}(H) \subseteq \hat{\mathfrak{h}}$.
On the other hand, since H is generated by H_α, $\mathrm{Lie}(H_\alpha) \subseteq \mathrm{Lie}(H) \implies \hat{\mathfrak{h}} \subset \mathrm{Lie}(H)$, then $\hat{\mathfrak{h}} = \mathrm{Lie}(H)$, and hence $H^{\mathrm{an}} = \hat{H}^{\mathrm{an}}$, which means that $\hat{H}$ is an algebraic group[24].
ii Repeating the proof for Malcev closure. Consider the Lie subgroup

$$H_1 \coloneqq G(\mathfrak{h}, [\mathfrak{h},\mathfrak{h}]) = \{\, x \in G \mid (\mathrm{Ad}_x - \mathrm{id})(\mathfrak{h}) \subseteq [\mathfrak{h},\mathfrak{h}] \,\}.$$

We claim that H_1 is a closed algebraic subgroup of G.
Indeed, consider the adjoint representation

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}),$$

and denote

$$\mathrm{GL}(\mathfrak{h}, [\mathfrak{h},\mathfrak{h}]) = \{\, A \in \mathrm{GL}(g) \mid (A - \mathrm{id})(\eta) \subseteq [\mathfrak{h},\mathfrak{h}] \,\},$$

which is an algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$. And H_1 is an algebraic subgroup of G. The Lie algebra of H_1 is

$$\mathfrak{H}_1 = \{\, X \in \mathfrak{g} \mid \mathrm{ad}_X(\mathfrak{h}) \subseteq [\mathfrak{h},\mathfrak{h}] \,\}$$

$\mathfrak{h} \subseteq \mathrm{Lie}(H_1) \implies \mathfrak{h}^a \subseteq \mathrm{Lie}(H_1) \implies [\mathfrak{h}^a, \mathfrak{h}] \subseteq [\mathfrak{h},\mathfrak{h}]$, continue this type of argument, we get $[\mathfrak{h}^a, \mathfrak{h}^a] \subseteq [\mathfrak{h},\mathfrak{h}]$.
iii If $\mathfrak{h}$ is commutative, then by ii $[\mathfrak{h}^a, \mathfrak{h}^a] = [\mathfrak{h},\mathfrak{h}] = 0$, thus $\mathfrak{h}^a$ is commutative. If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ consider the algebraic subgroup

$$H_3 \coloneqq \{g \in G | (\mathrm{Ad}_g - \mathrm{id})(\mathfrak{g}) \subseteq \mathfrak{h}^a\}$$

$\mathrm{Lie}(H_3) = \{X \in \mathfrak{g} | \mathrm{ad}_X(\mathfrak{g}) \subset \mathfrak{h}^a\}$. Obviously, $\mathfrak{h} \subseteq \mathrm{Lie}(H_3)$ since $[\mathfrak{h},\mathfrak{g}] \subset \mathfrak{h} \subset \mathfrak{h}^a$ □

Theorem 2.13. [OV] Let G be a connected complex algebraic group, with $(G,G) \subseteq G$. Then

1. If H is a complex algebraic group, $\phi : G^{\mathrm{an}} \to H^{\mathrm{an}}$ is a Lie group morphism, then $\phi : G \to H$ is an algebraic group morphism.

2. Every finite-dimensional $\mathbb{C}$-representation of G^{an} is a rational G-module.

In particular, for a connected complex Lie group, with $(G,G) = G$, and a faithful linear representation, there exists a unique algebraic group structure on G, such tht the Lie group structure is induced from this algebraic structure. [25]

Proof. Consider the graph of the morphism

$$\Gamma_\phi = \{\, (x,y) \in G \times H \mid y = \phi(x) \,\} \subset G^{\mathrm{an}} \times H^{\mathrm{an}}$$

[24] *Why?*

[25] G^{an} *is determined by* $\mathrm{GL}(V)^{\mathrm{an}}$.

which is a closed Lie subgroup of $G^{\text{an}} \times H^{\text{an}}$, and $p_1 : \Gamma_\phi \simeq G^{\text{an}}$, where p_1 is the projection to the first factor.
In particular, $\text{Lie}(\Gamma_\phi) \simeq \text{Lie}(G^{\text{an}}) = [\text{Lie}(G^{\text{an}}), \text{Lie}(G^{\text{an}})]$, since $(G^{\text{an}}, G^{\text{an}}) = G^{\text{an}}$. Hence $\text{Lie}(\Gamma)$ is an algebraic subalgebra of $\text{Lie}(G^{\text{an}} \times H^{\text{an}})$. THen Γ_ϕ is an algebraic subgroup of $G \times H$ [26]. Consider

$$\Gamma_\phi \xrightarrow{p_1} G$$

which is a morphism of algebraic groups and is bijective. This implies that p_1 is an isomorphism of algebraic groups. $\square$

[26] *Why Why Why*

The most important representation for a Lie algebra $\mathfrak{g}$ is the adjoint representation.

Theorem 2.14 (Schur's Lemma)**.** Let V be an irreducible $\mathfrak{g}$-representation. Then every $\mathfrak{g}$-morphism $f : V \to V$ is either 0 or isomorphic.

Definition 2.4. A Lie algebra $\mathfrak{g}$ is called **solvable** if $\mathfrak{g}^{[i]} = 0$ for some large enough i, where

$$\begin{aligned}
\mathfrak{g}^[0] &= \mathfrak{g}, \\
\mathfrak{g}^[1] &= [\mathfrak{g}, \mathfrak{g}], \\
\mathfrak{g}^[2] &= [\mathfrak{g}^[1], \mathfrak{g}^[1]], \\
\cdots, \\
\mathfrak{g}^[i] &= [\mathfrak{g}^[i-1], \mathfrak{g}^[i-1]].
\end{aligned}$$

$\mathfrak{g}$ is called **nilpotent**, if $\mathfrak{g}^{(i)} = 0$ for some large enough i, where

$$\begin{aligned}
\mathfrak{g}^{(0)} &= \mathfrak{g}, \\
\mathfrak{g}^{(1)} &= [\mathfrak{g}, \mathfrak{g}], \\
\mathfrak{g}^{(2)} &= [\mathfrak{g}, \mathfrak{g}^{(1)}], \\
\cdots, \\
\mathfrak{g}^[i] &= [\mathfrak{g}, \mathfrak{g}^{(i-1)}].
\end{aligned}$$

If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$, then $\mathfrak{g}$ is called **unipotent**, if there is a basis of V, under which the image of $\mathfrak{g}$ in $\mathfrak{gl}(V)$ is of the form

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

If $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is a subalgebra, we called $\mathfrak{g}$ a **commutative diagonalizable subalgebra** of $\mathfrak{gl}(V)$, if there is a basis of V, under which the elements of $\mathfrak{g}$ are all diagonal.
A finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is called **semi-simple**, if $\text{rad}(\mathfrak{g}) = 0$. $\mathfrak{g}$ is called **simple**, if $\mathfrak{g}$ has no proper ideals.
Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a subalgebra, we call $\mathfrak{g}$ a **reductive** Lie algebra, if $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$, where $\mathfrak{z}$ and $\mathfrak{g}'$ are both ideals of $\mathfrak{g}$, and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is a semi-simple Lie algebra, $\mathfrak{z} = \text{rad}(\mathfrak{g})$ is a commutative diagonalizable subalgebra. (Humphrey's book, p.30.)

Definition 2.5. Given a Lie algebra $\mathfrak{g}$, the **radical** of $\mathfrak{g}$, denoted by $\text{rad}(\mathfrak{g})$ is the largest solvable ideal of $\mathfrak{g}$.

There is a symmetric non-degenerate bilinear form $(-,-) : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}$ defined as

$$(X,Y) = \mathrm{tr} XY.$$

This pair has some basic properties.

i $([X,Y],Z) = (X,[Y,Z]), \forall X,Y,Z \in \mathfrak{gl}_n(\mathbb{C})$.

ii If $\mathfrak{n}$ is an ideal of $\mathfrak{gl}_n(\mathbb{C})$, then $\mathfrak{n}^\perp = \{X \in \mathfrak{gl}_n(\mathbb{C}) | (X,\mathfrak{n}) = 0\}$ is also an ideal.

iii If $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}_n(\mathbb{C})$, then $\mathfrak{z}(\mathfrak{g}) \subseteq [\mathfrak{g},\mathfrak{g}]^\perp$. If the restricted bilinear form $(-,-) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is also non-degenerate. Then

$$\mathfrak{z}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]^\perp \cap \mathfrak{g}.$$

iv If $\mathfrak{g}$ is a commutative and diagonalizable subalgebra of $\mathfrak{gl}_n(\mathbb{C})$, then $(-,-) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is non-degenerate, positive definite over $\mathbb{R}$.

v If $\mathfrak{n} \subset \mathfrak{gl}_n(\mathbb{C})$ is a subalgebra such that $(\mathfrak{n},\mathfrak{n}) = 0$, then $\mathfrak{n}$ is unipotent if $\mathfrak{n}$ is algebraic, and is solvable in general.

vi If $\mathfrak{n}$ is a unipotent ideal of $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$, then $\mathfrak{n} \subset \mathfrak{g}^\perp = \{ x \in \mathfrak{g} \mid (x,\mathfrak{g}) = 0 \}$

vii If $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{C})$ is semi-simple, then $(-,-)$ is non-degenerate.

viii If $\mathfrak{g}$ is a simple Lie algebra, then up to scalar, there is a unique non-degenerate, symmetric, associative bilinear form on $\mathfrak{g}$.

Theorem 2.15. 1. Every semi-simple complex Lie algebra, has a non-degenerate invariant bilinear form. In particular, the Cartan-Killing form on $\mathfrak{g}$ is non-degenerate

2. Let $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$ be an algebraic subalgebra, TFAE

   (a) $\mathfrak{g}$ is reductive

   (b) $(-,-) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is non-degenerate.

Proof. $\Longrightarrow$ Let $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$, where $\mathfrak{z}$ is a commutative diagonal ideal, $\mathfrak{g}'$ is a semi-simple ideal, which implies that $\mathfrak{z} = \mathfrak{z}(\mathfrak{g})$, $\forall z = x + y, x \in \mathfrak{z}, y \in \mathfrak{g}'$. Note that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}'$, $\mathfrak{z}(\mathfrak{g}) \subseteq [\mathfrak{g},\mathfrak{g}]^\perp = (\mathfrak{g}')^\perp$[27]
If $y \neq 0$, since $(-,-) : \mathfrak{g}' \times \mathfrak{g}' \to \mathbb{C}$ is non-degenerate, thus there exists some $y' \in \mathfrak{g}'$ such that $(y,y') \neq 0$

$$(z,y') = (x,y') + (y,y') \neq 0.$$

If y=0, note that $(-,-) : \mathfrak{z} \times \mathfrak{z} \to \mathbb{C}$ is non-degenerate, there is an $x' \in \mathfrak{z}$ such that $(x,x') \neq 0$, which means $(z,x') \neq 0$.
$\Leftarrow$ $\mathfrak{z}(\mathfrak{g}) = \mathrm{Lie}(Z(G)^\circ)$, where G is a closed algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$ such that $\mathfrak{g} = \mathrm{Lie}(G)$, since $\mathfrak{g}$ is algebraic. Now $\mathrm{Lie}(Z(G)^\circ)_u$ is a unipotent ideal of $\mathfrak{g}$. By the non-degeneracy of the invariant form, $\mathrm{Lie}(Z(G)^\circ_u) = 0 \implies Z(G)^\circ_u = \{e\}$, which means that $Z(G)^\circ$ is a torus. Thus $\mathfrak{z}(\mathfrak{g})$ is a commutative diagonalizable ideal of $\mathfrak{g}$. Note that $(-,-) : \mathfrak{z}(\mathfrak{g}) \times \mathfrak{z}(\mathfrak{g}) \to \mathbb{C}$ is non-degenerate, we have

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}]$$

as vector spaces. (Non degenerate assumption $\implies \mathfrak{z}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]^\perp, \mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}] = (0)$

[27] *Are* $\mathrm{rad}(\mathfrak{g})$ *and* $\mathfrak{z}(\mathfrak{g})$ *the same thing?*

To show that $[\mathfrak{g},\mathfrak{g}]$ is semi-simple, note

$$(-,-) : [\mathfrak{g},\mathfrak{g}] \times [\mathfrak{g},\mathfrak{g}] \to \mathbb{C}$$

is also non-degenerate, if $\mathfrak{n}$ is a solvable ideal of $[\mathfrak{g},\mathfrak{g}]$, here we may assume that $\mathfrak{n}$ is algebraic, since the algebraic closure of a solvable ideal is solvable. Let N be a closed algebraic subgroup of (G,G) with $\mathrm{Lie}(N) = \mathfrak{n}$. Then $N_u = \{e\}$, since $\mathrm{Lie}(N_u)$ is a unipotent ideal of $[\mathfrak{g},\mathfrak{g}]$ which implies that $\mathrm{Lie}(N_u) = 0$.
N is a torus, $\mathfrak{n}$ is a commutative diagonalizable ideal, $[\mathfrak{g},\mathfrak{g}]$, $\mathfrak{n}^\perp$ is also ideal, $\mathfrak{n} \cap \mathfrak{n}^\perp = (0)$, $\mathfrak{n} \cap \mathfrak{n}^\perp = (0) \implies [\mathfrak{n},\mathfrak{n}^\perp] \subseteq \mathfrak{n} \cap \mathfrak{n}^\perp = (0)$.

$$[\mathfrak{g},\mathfrak{g}] = \mathfrak{n} \oplus \mathfrak{n}^\perp,$$

$\implies \mathfrak{n} \subseteq \mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}] \implies \mathfrak{n} = 0$, $[\mathfrak{g},\mathfrak{g}]$ is semi-simple. Altogether, $\mathfrak{g}$ is a reductive Lie algebra. □

3 Weights and Root System

Let T be a complex algebraic torus, *id est* $T \simeq \mathbb{C}^* \times \cdots \mathbb{C}^*$. Let $\mathfrak{h} = \mathrm{Lie}(T)$ which is a complex diagonalizable linear Lie algebra.

For a character $\chi : T \to \mathbb{C}^*$, consider the differential map

$$d\chi_e : \mathfrak{h} \to \mathbb{C},$$

thus we know that $d\chi_e \in \mathfrak{h}^*$. We write $\chi(T) = \mathbb{Z}[\chi_1, \ldots, \chi_n]$, a polynomial ring in $\chi_1, \ldots, \chi_n$, $d\chi_1, \ldots, d\chi_n$ makes a basis of $\mathfrak{h}^*$,

$$\chi_i : \mathbb{C}^* \times \cdots \times \mathbb{C}^* \to \mathbb{C}^*,$$
$$(x_1, \ldots, x_n) \mapsto x_i$$

Let $\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}d\chi_1 \oplus \cdots \oplus \mathbb{Z}d\chi_n$, a lattice in $\mathfrak{h}^*$, $\mathfrak{h}^*_{\mathbb{R}} = \mathfrak{h}^*_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R} \hookrightarrow \mathfrak{h}^*$, $\mathfrak{h}_{\mathbb{Z}} = \{x \in \mathfrak{h} | (x, \mathfrak{h}^*_{\mathbb{Z}}) \in \mathbb{Z}\}$, $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$.

Let G be a connected complex algebraic group, $T \subseteq G$ is a torus, $\rho : G \to \mathrm{GL}(V)$, rational representation. By the Jordan decomposition theorem [28], $\rho(T)$ is a subgroup, consisting only semi-simple elements

$$V = \bigoplus_{\chi \in \chi(T)} V_\chi$$

where $V_\chi = \{\, v \in V \mid t \cdot v = \chi(t)v, \forall t \in T \,\}$.

Consider the differential $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of ρ. In this case, we have

$$V = \bigoplus_{\lambda \in \mathrm{Lie}(T)^*_{\mathbb{Z}}} V_\lambda = \{\, v \in V \mid h \cdot v = \lambda(h)v, \forall h \in \mathrm{Lie}(T) \,\}.$$

Let Λ_V be the $\mathbb{R}$-span of $d\chi, \chi \in \chi(T)$ with $V_\chi \neq 0$. V_χ is called a χ-weight space of V.

Claim 3.1. $\Lambda_V = \{\, \lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \lambda(h) = 0, \forall h \in \mathrm{Lie}(T) \cap \ker d\rho \,\}$. In particular, if V is a faithful rational representation, that is, ρ is injective, $\Lambda_V = \mathfrak{h}^*_{\mathbb{R}}$

A more interesting case is the adjoint representation, $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. Here we assume that T is a maximal torus

$$\mathfrak{g} = \mathfrak{g}_0 \oplus (\bigoplus_{\alpha \in R(G,T) \subset \chi(T)} \mathfrak{g}_\alpha)$$

[28] *precise statement of the Jordan decomposition theorem?*

called the root space decomposition of $\mathfrak{g}$, where

$$\mathfrak{g}_0 = \{ x \in \mathfrak{g} \mid \mathrm{Ad}_t x = x, \forall t \in T \} = \{ x \in \mathfrak{g} \mid [h, x] = 0, \forall h \in \mathfrak{h} \subset \mathrm{Lie}(T) \},$$
$$\mathfrak{g}_\alpha \{ x \in \mathfrak{g} \mid \mathrm{Ad}_t x = \alpha(t)x, \forall t \in T \} = \{ x \in \mathfrak{g} \mid [h, x] = (d\alpha)(h), \forall h \in \mathfrak{h} \},$$

We call $R(G,T)$ the set of **roots** of $\mathfrak{g}$ with respect to T. Also, we call $\{d\alpha | \alpha \in R(G,T)\}$ the roots of $\mathfrak{g}$ and regard $R(G,T) \subseteq \mathfrak{h}^*_{\mathbb{Z}}$, $\mathfrak{g}_\alpha$ is called the α-root space.

Often we are more interested in the reductive case, where there is a non-degenerate invariant form on $\mathfrak{g}$, say $(-,-) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$.

i $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ if $\alpha + \beta \notin R(G,T)$.

ii $\alpha, \beta \in R(G,T), \alpha + \beta \neq 0, (\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0, (\mathfrak{g}_\alpha, \mathfrak{g}_0) = 0$. In particular, $\forall \alpha \in R(G,T) \implies -\alpha \in R(G,T)$, and hence

$$(-,-) : (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \times (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \to \mathbb{C}$$

is non-degenerate.

iii $(-,-) : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathbb{C}$ is non-degenerate.

Proof. i $\forall h \in \mathfrak{h} = \mathrm{Lie}(T), x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \alpha(h)[x, y] + \beta(h)[x, y] = (\alpha + \beta)(h)[x, y]$$

if $\alpha + \beta \in R(G,T) \implies [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ and zero if $\alpha + \beta \notin R(G,T)$
ii $\forall h \in \mathfrak{h}, \forall x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$

$$\alpha(h)(x, y) = ([h, x], y) = -([x, h], y) = -(x, [h, y]) = -\beta(h)(x, y)$$

If $\alpha + \beta \neq 0, \exists h \in \mathfrak{h}$ such that $\alpha(h) \neq -\beta(h) \implies (x, y) = 0,$

□

Remark. By Theorem 2.15, $\mathfrak{g}$ is reductive since we assume that there is a non-degenerate pairing on $\mathfrak{g}$. By product $Z_G(T) = T$, where G is a reductive complex algebraic group. By general theory on algebraic groups, $Z_G(T)$ is connected, which is known as the **Cartan subgroup**, thus $Z_G(T) \simeq T \times Z_G(T)_u$, and moreover

$$\mathfrak{g}_0 = \{ x \in \mathfrak{g} \mid \mathrm{Ad}_t x = x, \forall t \in T \} = \mathrm{Lie}(Z_G(T))$$

$\mathrm{Lie}(Z_G(T)_u)$ is a unipotent ideal of $\mathrm{Lie}(Z_G(T)) = \mathfrak{g}_0, \implies \mathrm{Lie}(Z_G(T)_u) = 0, \implies Z_G(T) = T \implies \mathfrak{g}_0 = \mathfrak{h}$.

Now , $\mathfrak{g}$ is a reductive Lie algebra, with $\mathfrak{h} = \mathrm{Lie}(T)$, a maximal torus subalgebra

$$\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in R(G,T)} \mathfrak{g}_\alpha).$$

Since $(-,-) : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$, $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ as vector spaces, $\alpha \in R(G,T) \subset \mathfrak{h}^*$

$$\mathfrak{h}^*_{\mathbb{R}} \supseteq \mathfrak{h}^*_{\mathbb{Z}} \ni \alpha \to U_\alpha.$$

Introduce the coroot:

$$h_\alpha := \frac{2u_\alpha}{(u_\alpha, u_\alpha)} = \frac{2u_\alpha}{(\alpha, \alpha)}$$

iv $\forall x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$

$$[x, y] = (x, y)u_\alpha = (x, y)(\alpha, \alpha) \cdot \frac{h_\alpha}{2}$$

v $\sum_{\alpha \in R(G,T)} \mathbb{C}\mathfrak{h}_\alpha = \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$, which is the maximal torus subalgebra of $[\mathfrak{g}, \mathfrak{g}]$.

4 Representations of $\mathfrak{sl}_2(\mathbb{C})$

$\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h$, where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h$$

Let $\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ be a finite dimensional representation. Then

i Jordan decomposition theorem [29] implies that $\rho(h)$ is a semi-simple operator

[29] *How?*

ii If V is irreducible, then $\dim V = n + 1$ for some $n \geq 0$ and V has a basis $\{v_0, v_1, \dots, v_n\}$, such that

$$\begin{cases} hv_i &= (n - 2i)v_i \\ ev_i &= (n + 1 - i)v_{i-1} \\ fv_i &= (i + 1)v_{i+1} \end{cases}$$

Proof. By i, $V = \oplus_\lambda V_\lambda, V_\lambda = \{\, v \in V \mid hv = \lambda v \,\}, \lambda \in \mathbb{C}$, V is irreducible, V is generated by any $o \pm v \in V_\lambda$,

$$\begin{aligned} hv &= \lambda v \\ h(ev) &= ([h, e] + eh)v = (2e + eh)v = (\lambda + 2)ev \\ h(fv) &= (\lambda - 2)fv \end{aligned}$$

thus we have $ev \in V_{\lambda+2}$ and $fv = V_{\lambda-2}$. Since $\dim V = n + 1 < \infty$, there exist $r, s \in \mathbb{Z}_+$ such that

$$\begin{aligned} e^r v \neq 0, e^{r+1}v = 0, \\ f^s v \neq 0, f^{s+1}v = 0. \end{aligned}$$

Then $e^r v \in V_{\lambda+2r}$, $f^s V_{\lambda-2s}$. In this case, we write $v_o = e^r v$, and call it a highest weight vector. Then V is generated by v_0

$$V = \mathbb{C}v_0 \oplus \mathbb{C}fv_0 \oplus \mathbb{C}f^2 v_0 \oplus \cdots \oplus \mathbb{C}f^n v_0.$$

Now $f^{n+1}v_0 = 0 \implies ef^{n+1}v_0 = eff^n v_0 = ([e, f] + fe)f^n v_0 = \cdots = (n + 1)$
After some annihilating and creating tricks, we have $\lambda \in \mathbb{Z}$ and $v_0 \in V_n$.
For simplicity, we also write $V(n)$ for this irreducible representation. The actions of h, e, f on the basis $\{v_0, fv_0, \dots, f^n v_0\}$ is exactly given above.

□

Remark. An alternative construction of V_n is given as $\mathbb{C}[x, y]$. Then $\mathfrak{sl}_2(\mathbb{C})$ acts on $\mathbb{C}[x, y]$ via

$$(Af)(x, y) = f(Ax, Ay),$$

where we regard X as $\begin{pmatrix}1\\0\end{pmatrix}$ and Y as $\begin{pmatrix}0\\1\end{pmatrix}$. Let $V(n)$ be the homogeneous component of $\mathbb{C}[x, y]$ of degree n. This is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ isomorphic to the irreducible representation $V(n)$ above. And the dimension of the homogeneous-n component as a complex vector space can be computed as $\binom{n+1}{1}$, where we use the identification used in algebraic geometry

$$\mathrm{Spec}\mathbb{C}[x, y] = \mathrm{Proj}\mathbb{C}[x, y] = \mathbb{P}^1_{\mathbb{C}}.$$

Every finite-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ is **completely reducible**, *id est*, a direct sum of irreducible representations.

Recall, for a representation $\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$, we consider the dual representation V^*, such that $\forall x \in \mathfrak{sl}_2(\mathbb{C}), f \in V^*, v \in V$,

$$(X \cdot f)(v) = -f(X \cdot v).$$

Then it is easy to check that, $V(n)^* \simeq V(n), \forall n \geq 0$. Now we want to show that $\mathrm{Ext}^1(V(m), V(n)) = 0$. When $m \geq n$, any elements in $\mathrm{Ext}^1(V(m), V(n))$ corresponds to a short exact sequence

$$0 \longrightarrow V(n) \longrightarrow V \longrightarrow V(m) \longrightarrow 0.$$

[30].

When $m \leq n$, we have

$$\mathrm{Ext}^1(V(m), V(n)) \simeq \mathrm{Ext}^1(V(n)^*, V(m)^*) \simeq \mathrm{Ext}^1(V(n), V(m)) = 0.$$

To summarize,

i Every finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ has a weight decomposition

$$V = \oplus_{\lambda \in \mathbb{Z}} V_\lambda$$

and is completely reducible.

ii f.d irreducible $\mathfrak{sl}_2(\mathbb{C})$-representation mod isomorphisms bijectively corresponds to $\mathbb{Z}_{\geq 0}$.

iii $\dim V_\lambda = \dim V_{-\lambda}$ in i.

Let $\rho : \mathfrak{sl}_2 \to \mathfrak{gl}(V)$ be a representation. Since $\mathrm{SL}_2(\mathbb{C})$ is simply-connected as a Lie group, there exists a lift $\tilde{\rho} : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(V)$ of ρ, such that $(dr\tilde{h}o)_e = \rho$.

[30] *I can't understand the following argument. Lecture Note 28*

5 Root System of Reductive Lie Algebras

G connected, reductive complex algebraic group. $T \subseteq G$ is a maximal torus, $\mathfrak{g} = \mathrm{Lie}(G), \mathfrak{h} = \mathrm{Lie}(T)$.

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in R(G,T)} \mathfrak{g}_\alpha).$$

Claim 5.1. For all $\alpha \in R(G, T), \dim \mathfrak{g}_\alpha = 1$ and $\mathfrak{h}_\alpha \in \mathfrak{h}_{\mathbb{Z}}$ and that $c_\alpha \in R(G, T)$ implies $c = \pm 1$.

Proof. $\forall \alpha in R(G,T), \mathfrak{g}_\alpha \neq 0$. Take $x_\alpha \in \mathfrak{g}_\alpha, y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, Y_\alpha] = 1/2(x_\alpha, y_\alpha)(\alpha,\alpha)h_\alpha = h_\alpha$. Then

$$\begin{cases} [h_\alpha, X_\alpha] = \alpha(h_\alpha)X_\alpha = 2X_\alpha \\ [h_\alpha, y_\alpha] = -\alpha(h_\alpha)y_\alpha = -2y_\alpha \\ [x_\alpha, y_\alpha] = h_\alpha \end{cases}$$

id est, $\mathbb{C}x_\alpha \oplus \mathbb{C}y_\alpha \oplus \mathbb{C}h_\alpha \simeq \mathfrak{sl}_2(\mathbb{C})$.
Let $\mathfrak{g}^{(\alpha)} = \mathbb{C}x_\alpha \oplus \mathbb{C}y_\alpha \oplus \mathbb{C}h_\alpha$, then $[\mathfrak{g}^{(\alpha)}, \mathfrak{g}^{(\alpha)}] = \mathfrak{g}^{(\alpha)}$, *id est*, $\mathfrak{g}^{(\alpha)}$ is an algebraic subalgebra of $\mathfrak{g}$, *id est*, there exists a connected closed algebraic subgroup $G^{(\alpha)} \subseteq G$ such that $\mathfrak{g}^{(\alpha)} = \mathrm{Lie}(G^{(\alpha)})$. Using the fact that $\mathrm{SL}_2(\mathbb{C})$ is simply connected[31], there are morphisms of Lie groups

$$\phi : \mathrm{SL}_2(\mathbb{C}) \twoheadrightarrow G^{(\alpha)} \hookrightarrow G$$

which is a morphism of algebraic groups. By a general fact[32] that

$$(d\phi)_e(h) - h_\alpha \in \mathfrak{h}_{\mathbb{Z}}.$$

Now, for $\alpha \in R(G,T)$, if $c\alpha \in R(G,T)$, then $h_\alpha, h_{c\alpha} \subset \mathfrak{h}_{\mathbb{Z}}$, so

$$\alpha\left(\frac{2u_{c\alpha}}{(c\alpha, c\alpha)}\right) = \alpha\left(\frac{2cu_\alpha}{c^2(u_\alpha, u_\alpha)}\right) = \frac{1}{c}\alpha(h_\alpha) = \frac{2}{c} \in \mathbb{Z},$$

and similarly $(c\alpha)(h_\alpha) \in \mathbb{Z}$, which implies $2c \in \mathbb{Z}$. So we have $c = \pm 1, \pm\frac{1}{2}, \pm 2$. □

[31] *What's the use of simply connectedness? Precisely.*

[32] *What general fact it is?*

Let $\tilde{\mathfrak{g}}_\alpha = \{ x \in \mathfrak{g}_\alpha \mid (x, y_\alpha) = 0 \}$ and

$$\mathcal{M} = \begin{cases} \tilde{\mathfrak{g}} \oplus \mathfrak{g}_{2\alpha}, & \text{if } 2\alpha \in R(G,T) \\ \tilde{\mathfrak{g}}_\alpha, & \text{if } 2\alpha \notin R(G,T). \end{cases}$$

Claim 5.2. $\mathcal{M}$ is an $\mathfrak{sl}_2$-representation via the isomorphism $\mathfrak{sl}_2(\mathbb{C}) \simeq \mathbb{C}x_\alpha \oplus \mathbb{C}y_\alpha \oplus \mathbb{C}h_\alpha$.

Proof. If $2\alpha \in R(G,T)$

$$\begin{aligned} [x_\alpha, \mathfrak{g}_{2\alpha}] &\subseteq \mathfrak{g}_{3\alpha} = 0, \\ [x_\alpha, \tilde{\mathfrak{g}}_\alpha] &\subset \mathfrak{g}_{2\alpha}, \\ [\mathfrak{h}_\alpha, \mathcal{M}] &\subset \mathcal{M} \end{aligned} \tag{5.1}$$

take $z \in \tilde{\mathfrak{g}}_\alpha$, $[y_\alpha, z] = -\frac{1}{2}(z, y_\alpha)(\alpha,\alpha)h_\alpha = 0$, $[y_\alpha, \mathfrak{g}_{2\alpha}] \in \tilde{\mathfrak{g}}_\alpha$ since $([y_\alpha, \mathfrak{g}_{2\alpha}], y_\alpha) = -(\mathfrak{g}_{2\alpha}, [y_\alpha, y_\alpha]) = 0$.
If $2\alpha \notin R(G,T)$,

$$\begin{cases} [x_\alpha, \tilde{\mathfrak{g}}_\alpha] = 0 \\ [h_\alpha, \tilde{\mathfrak{g}}_\alpha] \subset \tilde{\mathfrak{g}}_\alpha \\ [y_\alpha, \tilde{\mathfrak{g}}_\alpha] = 0 \end{cases}$$

Moreover, the weights in $\mathcal{M}$ is $2, 4$ if $2\alpha \in R(G,T)$ and is 2 if $2\alpha \notin R(G,T)$.
By the representation theory [33] of $\mathfrak{sl}_2$, we have $\mathcal{M} = 0$. As a result, $\dim \mathfrak{g}_\alpha = 1$ and $c\alpha \in R(G,T)$ imply $c = \pm 1$. □

[33] *How?*

Claim 5.3. $\forall \alpha, \beta \in R(G,T)$ and $\alpha \neq \beta$, there are $p, q \in \mathbb{N}$ such that

$$\beta - p\alpha, \beta - (p-1)\alpha, \cdots, \beta, \beta + \alpha, \cdots, \beta + q\alpha$$

belongs to $R(G,T)$. Moreover

$$(\beta + \mathbb{Z}[\alpha]) \cap R(G,T) = \text{the } \alpha \text{ string through } beta$$

and $p - 1 = \beta(h_\alpha) \in \mathbb{Z}$.

Claim 5.4. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}h_\alpha$,

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \begin{cases} 0 & \alpha + \beta \notin R(G,T) \\ \mathfrak{g}_{\alpha+\beta} & \alpha + \beta \in R(G,T). \end{cases} \tag{5.2}$$

6 Jacobson-Morozov Theorem

[OV, pp. 150]

Theorem 6.1. Assume that $\mathfrak{g}$ is semi-simple, then for any unipotent $0 \neq X \in \mathfrak{g}$, there is a semi-simple element H and a unipotent element Y, such that

$$[X,Y] = H, [H,X] = 2X, [H,Y] = -2Y.$$

That is, (H,X,Y) is an $\mathfrak{sl}_2(\mathbb{C})$-triple.

Proof. $X \in \mathfrak{g}$ non-zero unipotent element. Let $N = \{ g \in G \mid \mathrm{Ad}_g X \in \mathbb{C}X \}$, which is an algebraic subgroup of G. Let T_1 be a maximal torus of N, then $\mathfrak{g}_1 = \mathrm{Lie}(T_1) \hookrightarrow \mathrm{Lie}(N) \subset \mathfrak{G}$.

By Lemma 6.3 T_1 is non-trivial. Then we consider the decomposition of $\mathfrak{g}$ with respect to T_1-weight spaces

$$\mathfrak{g} = \mathfrak{g}_0 \oplus (\oplus_{\alpha \in R(G,T_1)} \mathfrak{g}_\alpha).$$

By the definition of N, we know $(X \in \mathfrak{g}_\alpha$ for some $\alpha \in T(G,T_1)$. To finish the proof, we claim that $X \notin [\tilde{\mathfrak{g}}_\alpha, X]$, by Lemma 6.4.

If $X \in [\tilde{\mathfrak{g}}_0, X]$ *id est*, $X = [Z,X]$ for some $Z \in \tilde{\mathfrak{g}}_0$. Let

$$Z = Z_s + Z_n$$

be the Jordan decomposition, thus we have $[Z_s, X] = X$ and $[Z_n, X] = 0$.

Note that $Z \in \tilde{\mathfrak{g}}_0$ implies $[Z, \mathfrak{h}_1] = [Z_s, \mathfrak{h}_1] = [Z_n, \mathfrak{h}_1] = 0$ [34]

By the maximality of T_1 in N, we have $Z_s \in \mathfrak{h}_1$. Since $\mathfrak{h}_1 \times \mathfrak{h}_1 \to \mathbb{C}$ is non-degenerate, there is an $H \in \mathfrak{h}_1$ such that $(Z_s, H) = 0$.

As a result, $(Z,H) = (Z_s,H) + (Z_n,H) \neq 0$, which contradicts to the choice of Z. ☐

[34] *Very important! The definition of* $\tilde{\mathfrak{g}}_0$ *and* $R(G,T_1)$.

Lemma 6.2. For $X \in \mathfrak{g}$, $\ker(\mathrm{ad}_X) = [\mathfrak{g}, X]^\perp$ with respect to the Cartan-Killing form.

Lemma 6.3. For any non-zero unipotent $X \in \mathfrak{g}$, there is a semi-simple element $H \in \mathfrak{g}$, such that $[H,X] = X$.

Lemma 6.4. Let T_1 be a torus of G (not necessarily maximal). Then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus (\oplus_{\alpha \in R(G,T_1)} \mathfrak{g}_\alpha),$$

where $\mathfrak{g}_0 = \{X \in \mathfrak{g} | [X, H] = 0, \forall H \in \mathrm{Lie}(T_1)\}$.

7 Root Systems

Let E be a finite-dimensional $\mathbb{R}$-vector space, with an inner product $(-,-) : E \times E \to \mathbb{R}$. For a non-zero $\alpha \in E$, let $H_\alpha \coloneqq \{ x \in E \mid (x, \alpha) = 0 \}$, called the α-**reflection hyperplane**. Let $s_\alpha \in \mathrm{GL}(E)$, defined as

$$\begin{aligned} s_\alpha(x) &= x - 2(x, \frac{\alpha}{\|\alpha\|})\frac{\alpha}{\|\alpha\|} \\ &= x - 2(x, \frac{\alpha}{(\alpha, \alpha)})\alpha \\ &= x - (x, \alpha^\vee)\alpha, \end{aligned}$$

where $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$.

Definition 7.1. [OV, pp. 153] A subset $\Phi \subseteq E$, is called a **reduced root system**, if the followings are satisfied

i $0 \notin \Phi, |\Phi| < \infty$,

ii $\forall \alpha \in \Phi, s_\alpha(\Phi) = \Phi$,

iii $\forall \alpha, \beta \in \Phi, (\beta, \alpha^\vee) \in \mathbb{Z}$,

iv $\forall \alpha \in \Phi, c \in \mathbb{R}, c\alpha \in \Phi$ iff $c = \pm 1$.

If iv is not satisfied, it is called a **root system** in [OV].

Let $E' = \mathrm{span}_\mathbb{R}(\Phi)$. Then Φ is a root system in E'. Let W be the subgroup of $\mathrm{GL}(E)$ generated by $s_\alpha, \alpha \in \Phi$, called the **Weyl group** of the root system. $W \simeq W|_{E'}$.

Example. Let G be a reductive[35] complex algebraic group, and $T \subseteq G$ a maximal torus, $R(G,T) \subset \mathfrak{h}_\mathbb{Z}^* \hookrightarrow \mathfrak{h}_\mathbb{R}^* = \mathfrak{h}_\mathbb{Z}^* \otimes_\mathbb{Z} \mathbb{R}$. It's easy to check that $R(G,T)$ is a (reduced) root system in $\mathfrak{h}_\mathbb{R}^*$. Moreover, $\mathrm{span}_\mathbb{R}(R(G,T)) = \mathfrak{h}_\mathbb{R}^*$ iff $\mathfrak{G}$ is semi-simple. Indeed,

$$\mathrm{span}_\mathbb{R}(R(G,T)) = \{ \lambda \in \mathfrak{h}_\mathbb{R}^* \mid \lambda(\mathfrak{z}(\mathfrak{g})) = 0 \}.$$

[35] *The same as reduced?*

Definition 7.2. Let $(\Phi_1, E_1), (\Phi_2, E_2)$ be two root systems. A **morphism** from (Φ_1, E_1) to (Φ_2, E_2) is an $\mathbb{R}$-linear map $f : E_1 \to E_2$ such that

1. $f(\Phi_1) \subseteq \Phi_2$,
2. $\forall \alpha, \beta \in \Phi_1, \langle \alpha, \beta \rangle_{E_1} = \langle f(\alpha), f(\beta) \rangle_{E_2}$.

Denote $\mathrm{Aut}(\Phi)$ as the automorphism group of (Φ, E).

A root system (Φ, E) is called **indecomposable**, if there are no non-empty root systems (Φ_1, E) and (Φ_2, E), such that $(\Phi_1, \Phi_2) = 0$ and $\Phi = \Phi_1 \cup \Phi_2$.

The dual root system of (Φ, E) is the root system $(\Phi^\vee, E^*)$, where E^* is the dual vector space

of E, and
$$\Phi^\vee = \{ \alpha^\vee \mid \alpha \in \Phi \}$$
and $\alpha^\vee \in E^*, \alpha^\vee(x) = (x, \alpha^\vee) \in \mathbb{R}, \forall x \in E$.

Properties of root system.

i Let θ be the angle between two roots $\alpha, \beta \in \Phi$, *id est*
$$\cos\theta = \frac{(\alpha,\beta)}{\|\alpha\|\,\|\beta\|}.$$
Then
$$\langle \alpha,\beta\rangle \langle \beta,\alpha\rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)}\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = 4\cos^2\theta \in \mathbb{Z}.$$
If $\theta \geq \frac{\pi}{2}$ and $\|\beta\| \geq \|\alpha\|$.

ii Let $\alpha, \beta \in \Phi$ be two non-parallel. Then

(a) $(\alpha,\beta) > 0 \implies \alpha - \beta \in \Phi$

(b) $(\alpha,\beta) < 0 \implies \alpha + \beta \in \Phi$

iii Let $\alpha, \beta \in \Phi$ be two non-parallel roots. Then there are $p, q \in \mathbb{N}$ such that the α-string through β belongs to Φ.

7.1 Weyl Chamber and Simple Root System

Assume that Φ spans E, Let H_α be the α-reflection hyperplane,
$$E\backslash \cup_{\alpha\in\Phi} H_\alpha = \{s \in E | (x,\alpha) \neq 0, \forall \alpha \in \Phi\}.$$
This is a disjoint union of convex cones, and each connected component is called a **Weyl chamber**.

Definition 7.3. A subset $\Delta \subseteq \Phi$ is called a **simple root system** if

i The roots in Δ are $\mathbb{R}$-linearly independent

ii $\forall \alpha \in \Phi$, either $\alpha \in \mathbb{Z}_{\geq 0}\Delta$ or $\alpha \in \mathbb{Z}_{\leq 0}\Delta$.

Theorem 7.1. For a root system $\Phi \subset E$ with $\mathrm{rank}\Phi = \dim E$. Then a simple root system exists.
More precisely, there exists a bijection
$$\text{Wely chambers} \xleftrightarrow{1:1} \text{Simple root systems.}$$

Proof. Let C be a Weyl chamber, $\overline{C}$ the closure of C, $\overline{C}\backslash C = H_{\alpha_1} \cup \cdots \cup H_{\alpha_n}$ for some $\alpha_1, \ldots, \alpha_n \in \Phi$[36]. Notice tht $H_\alpha = H_{-\alpha}, H_\alpha = H_{c\alpha}, \forall c \in \mathbb{R}$.
We may assume that n is minimal, and $C \subset H_{\alpha_i}^+$. In particular,
$$C = H_{\alpha_1}^+ \cap \cdots \cap H_{\alpha_n}^+ = \{x \in E | (x, \alpha_i) > 0, \forall i\}.$$

Claim 7.2. $\forall i \neq j, (\alpha_i, \alpha_j) \leq 0$.

Claim 7.3. $\alpha_1, \ldots, \alpha_n$ are $\mathbb{R}$-linearly independent, and make an $\mathbb{R}$-basis of E

[36] *Why?*

Claim 7.4. For each positive root α, *id est*, C

□

Definition 7.4. Let $\Delta \subseteq \Phi$ be a simple root system, $\alpha \in \Phi$ is called a **positive root** if $\alpha \in \mathbb{Z}_{\geq 0}\Delta$, denoted as $\alpha > 0$. α is called a **negative root** if $\alpha \in \mathbb{Z}_{\leq 0}\Delta$, denoted as $\alpha < 0$.

Remark. Decomposition of semi-simple Lie algebras into simple Lie algebras corresponds to the decomposition of root systems into indecomposable simple root systems.

Let G be a reductive complex algebraic group, $T \subset G$ a maximal torus,

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in R(G,T)} \mathfrak{g}_\alpha),$$

$R(G,T) \subseteq \mathfrak{h}^*_{\mathbb{Z}}$, and $(R(G,T), \mathfrak{h}^*_{\mathbb{R}})$ is a root system, though $\mathrm{rank} R(G,T) \leq \dim \mathfrak{h}^*_{\mathbb{R}}$. Let Δ be a simple root system of $R(G,T)$ and write

$$\mathfrak{n}^+ = \sum_{\alpha \in R(G,T), \alpha > 0} \mathfrak{g}_\alpha, \mathfrak{n}^- = \sum_{\alpha \in R(G,T), \alpha < 0} \mathfrak{g}_\alpha$$

and

$$\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+,$$

then $\mathfrak{b}^+$ is a Borel subalgebra of $\mathfrak{g}$. Hence, we have

Weyl chambers $\xleftrightarrow{1:1}$ simple root systems $\xrightarrow{\theta}$ Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ $\xleftrightarrow{1:1}$ Borel subgroups of G containing T.

Theorem 7.5. θ is also a bijection.

Proof. Given a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ containing $\mathfrak{h}$, then we have

$$\mathfrak{b} = \mathfrak{h} \oplus (\oplus_{\alpha \in R} \mathfrak{g}_\alpha), R \subset R(G,T)$$

and

$$\mathfrak{n} = \oplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

thus $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}] = \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ □

7.2 Weyl groups

Let $\Phi \subset E$ be a root system the W the Weyl group of Φ. For any $s_\alpha \in W$, we have $s_\alpha H_\beta = H_{s_\alpha(\beta)}$, so W acts on the set $E \setminus \cup_{\alpha \in \Phi} H_\alpha$, *id est*, W acts on the set of Weyl chambers.

Let C be a Weyl chamber, we say that the hyperplane H_α is a **wall** of C if $\overline{C} \cap H_\alpha \neq (0)$, and we write $C \subset H_\alpha^+$ if $(\lambda, \alpha) > 0$, for all $\lambda \in C$.

To memorize the orientation of H_α, we sometimes write P_α instead of H_α, then we know that

$$C = P_{\alpha_1}^+ \cap \cdots \cap P_{\alpha_n}^+.$$

where $P_{\alpha_1}, \ldots, P_{\alpha_n}$ are walls of C. We say two Weyl chambers C_1, C_2 are **separated** by a hyperplane P_α, if $C_1 \subseteq P_\alpha^+, C_2 \subseteq P_\alpha^-$ or $C_1 \subseteq P_\alpha^-, C_2 \subseteq P_\alpha^+$.

For two Weyl chambers, C_1, C_2, define

$$d(C_1, C_2) = \#\{\text{non-oriented hyperplanes separating } C_1 \text{ and } C_2\}$$

Lemma 7.6. Let P_α be the wall of the Weyl chamber C. Then

$$d(C, s_\alpha C) = 1.$$

Corollary 7.7. Let $C_1 \neq C_2$ be two Weyl chambers, there exists a wall P_α of C_2, such that

$$d(C_1, C_2) = d(C_1, s_\alpha C_2) + 1.$$

Theorem 7.8. [OV, pp. 162] Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a simple root system of the root system Φ. Then $W = \langle s_{\alpha_1}, \ldots, s_{\alpha_n} \rangle$, and for each $\beta \in \Phi$ there is a $w \in W$ such that $w(\beta) \in \Delta$ or $\frac{1}{2}w(\beta) \in \Delta$.

Proof.

□

Definition 7.5. Let $w \in W$, if there is an expression

$$w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_k}}, \alpha_{i_j} \in \Delta$$

such that k is minimal among all such expressions, in this case k is called the **length** of w, denoted by $l(w)$, the expression above is called a **reduced expression**.

Example. A_2-root system, $\Delta = \{\alpha, \beta\}, \Phi = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$.

$$W = \langle s_\alpha, s_\beta \rangle, s_\alpha^2 = 1, s_\beta^2 = 1, s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta.$$

$w = s_\alpha s_\beta s_\alpha$ is a reduced expression and $l(w) = 3$.

Proposition 7.9. Let $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_k}} \in W$ be reduced expression. Then

$$\{\alpha_{i_k}, s_{\alpha_{i_k}}(\alpha_{i_{k-1}}), \ldots, s_{\alpha_{i_k}} \cdots s_{\alpha_{i_2}}(\alpha_{i_1})\}$$

are the only possible positive roots $\alpha \in R^+$ such that $w(\alpha) < 0$. In particular, the hyperplanes C_0 and $w^{-1}C_0$ Are separated exactly by $\{H_\alpha\}$ with α in the list above.

Theorem 7.10. The Weyl group acts simply-transitively on the set of Weyl chambers. Let C_0 be the fundamental Weyl chamber with respect to a simple root system Δ. Then

i For each $\lambda \in C_0$, $\mathrm{Stab}_W(\lambda) = \{1\}$.

ii For $\lambda \in \overline{C_0} \backslash C_0$, $\mathrm{Stab}_W(\lambda) = \langle s_\alpha \mid (\lambda, \alpha^\vee) = 0, \alpha \in R^+ \rangle$

iii For any $\lambda \in E$, the W-orbit of λ intersects $\overline{C_0}$ at a single point.

Let G be a reductive connected complex algebraic group, $T \subset G$ a maximal torus, $N_G(T) = \{g \in |gTg^{-1} = T\}, Z_G(T) = T, N_G(T)^\circ = Z_G(T), W'' = N_G(T)/N_G(T)^\circ = N_G(T)/T$, called the Weyl group of the pair (G, T), denoted by $W(G, T)$.

For each $n \in N_G(T)$, consider the adjoint action $\mathrm{Ad}_n : \mathfrak{h}_\mathbb{Z} \to \mathfrak{h}_\mathbb{Z}$, which extends to a morphism $\mathrm{Ad}_n : \mathfrak{h}_\mathbb{R} \to \mathfrak{h}_\mathbb{R}$. Then we get [37]

$$\nu :$$

[37] *Lecture Note 31. I don't know wether* $W \hookrightarrow \mathrm{GL}(\mathfrak{h}_\mathbb{R})$ *is invertible.*

Theorem 7.11. Let W'' be the image of ν. Then

1. $W'' \simeq N_G(T)/T$
2. $W = W''$.

Theorem 7.12. Let (Φ_1, E_1) and (Φ_2, E_2) be two root systems with $\mathrm{rank}\Phi_1 = \mathrm{rank}\Phi_2$. Let Δ_1 be a simple root system of Φ_i, $\theta : \mathrm{span}_{\mathbb{R}}(\Phi_1) \to \mathrm{span}_{\mathbb{R}}(\Phi_2)$ be a linear isomorphism, such that $\theta(\Delta_1) \subseteq \Phi_2$ and

$$(\alpha, \beta)_1 = (\theta(\alpha), \theta(\beta))_2, \forall \alpha, \beta \in \Delta_1.$$

If Φ_1 is a reduced root system , then $\theta(\Phi_1)$ is a root subsystem of Φ_2. If Φ_1 and Φ_2 are both reduced, then $\theta(\Phi_1) = \Phi_2$, $\theta(\Delta_1)$ is a simply root system of Φ_2.

Given a root system $\Phi \subseteq E$ with Δ a simple root system. The associated **Cartan matrix** is the matrix

$$C = (c_{ij}) \in M_n(\mathbb{Z})$$

with $c_{ij} = {\alpha_i}^\vee(\alpha_j) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}$

$$a^2 + b^2 = c^2 \tag{7.1}$$

References

[OV] Arkadij L. Onishchik and Ernest B. Vinberg. *Lie Groups and Algebraic Groups.* Springer Series in Soviet Mathematics. Springer Berlin Heidelberg.