On Diffeomorphism Groups

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1 Introduction

1.1 Motivation

Had it not been for this assignment, I would never have found the field of studying diffeomorphism groups of smooth manifolds. It is no doubt a profound area in geometric-topology, connecting homotopy theories, K-theories, and bordism theories deeply, as well as other deep theories *e.g.* theory of singularities.

After determined to finish this essay, it was the nLab page [1] that provoked my curiosity. Aside from a concise summary of main results in this area, [1] provides a invaluable reference list for a novice, where I found Hatcher's survey [2].

A glimpse of [2] made me soberly aware that, the computation of diffeomorphism groups could be extremely hard, even for the relatively simple case for spheres. After a short assessment of my topological toolkit and mathematical maturity, I decided to compute $\mathrm{Diff}(S^2)$. Reading a paper older than 60 years old is common for a math student, and usually not bad, at least in my shoes. So I referred to [3] where all these things started.

Smale first computed $\mathrm{Diff}_{\partial,U}(I^2)$, where he used an argument that the flow of a non-vanishing vector field in I^2 must leave I^2. What a *déjà vu*! I had seen this kind of argument several times in different branches in mathematics, and physics. The general theory behind the argument must deserve a name! After two-hours' Googling, I found the correct name of the theory relating singularities of a vector field and asymptotic behaviors of the flow of this vector field, the Poincaré- Bendixson theory, in Alexander Kupers' irreplaceable lecture notes [4] on this topic.

Kupers referred me to [5] for the Poincaré-Bendixson theory. But [5] concerned mainly about foliations. It was not surprising that it just stated the main result of the Poincaré-Bendixson theory and left most proofs in [6]. [6] is an intuitional introduction to dynamic systems, however, with a winding approach to Poincaré-Bendixson theory, which could not be applied directly. Fortunately, in [6], Hirsch and Smale told me that they learned Poincaré-Bendixson from [7]. The last book was exactly what I wanted.

Kupers defined the Whitney C^∞-topology of $C^\infty(X,Y)$ in [4] using sub-bases $\mathcal{N}(\phi,\psi,f,K,\epsilon)$, which was so messy and uninsightful, at least for me. He did so because he followed [8] and [9]. I was not acquainted to the last two good books, and didn't want to define the Whitney C^∞-topology in this way. Fortunately again, as a man never saying die and determined to find a way out, I found an elegant treatment of C^∞-topology in [10], in the language of jet bundles, after another two-hours' Googling.

Up to that time, I had found all references essential for this essay, namely [4], [10], [3] and [7]. So I followed their lines and LaTeXed, during days and nights, offline and online. Finally, I finished this essay before the deadline.

After telling all the stories behind it, I'm going to talk about the mathematics of the essay.

1.2 Overview

In this essay, we discuss the topology of the diffeomorphism group $\mathrm{Diff}(X)$ of a compact manifold X.

In Section 2, we define the Whitney C^∞- topology on the space of smooth mappings $C^\infty(X,Y)$ between smooth manifolds X,Y, in the language of jet bundles. Then we define the Whitney C^∞-topology on $\mathrm{Diff}(X)$ to be the subspace topology induced by the Whitney C^∞-topology on $C^\infty(X,X)$. The main result of this section is

Theorem 1.1. Let X be a compact manifold. Then $\mathrm{Diff}(X)$ is a topological group.

This section is mainly inspired by [10] and [4]. From [10], we learn most of essential preliminaries on the jet bundles; while fill [4], we find the sketch of the proof of the above theorem and fill the proof in detail.

In Section 3, we follow Smale's classical paper [3], to show that

Theorem 1.2 (Smale)**.** $\mathrm{Diff}_{\partial,u}(I^2)$ is contractible.

Using Smale's Theorem and following the lines [4], we can show that

Theorem 1.3. We have that

$$\mathrm{Diff}(S^2) \simeq O(3).$$

These serves as our two examples of the diffeomorphism groups.

In the Appendix A, we introduce the essential analytical tool to prove Smale's theorem: the Poincaré-Bendixson theory. However, this powerful theory of ODE's only holds in the 2-dimensional Euclidean space, largely due to the fact that the Jordan Curve Theorem only holds in the 2-dimension. Hence our proof for Smale's Theorem and the computation for $\mathrm{Diff}S^2$ cannot be generalized to higher dimensional manifolds, which is an obvious drawback.

Appendix A follows mainly [7, Chap. VII], which is an advanced and though exposition on the Poincaré-Bendixson theory.

2 The Whitney C^∞-Topology on Diffeomorphism Groups

Given two smooth manifolds X, Y, there are several ways to define the Whitney C^r-topology on the set $C^\infty(X, Y)$ of smooth mappings. Some definitions are technical and involves the concepts in point-set topology, such as metrizable space and convergence, like those in [8] and [9]; some use more conceptual ways like using jet bundles, as in [10]. We follow the latter way to define the Whitney C^r-topology on $C^\infty(X, Y)$, as it is cleaner and more understandable, at least for me. Of course, all these definitions agree when X is a compact manifold. The main reference for this section is [10].

2.1 Jet Bundles

Definition 2.1. Let X and Y be smooth manifolds, and $p \in X$ be a point. Suppose $f, g : X \to Y$ are smooth maps with $f(p) = g(p) = q$.

(i) f has **first order contact** with g at p if $(Df)_p = (Dg)_p$ as mapping of $T_pX \to T_qY$;

(ii) f has k**th order contact** with g at p if $Df : TX \to TY$ has $(k-1)$st order contact with Dg at every point in T_pX. This is written as $f \sim_{k,p} g$;

(iii) let $J^k(X, Y)_{p,q}$ denote the set of equivalence classes $[f]_{k,p}$ under $\sim_{k,p}$ of mappings $f : X \to Y$ where $f(p) = q$;

(iv) let $J^k(X, Y) := \coprod_{(p,q)\in X\times Y} J^k(X, Y)_{p,q}$ be the disjoint union. An element σ in $J^k(X, Y)$ is called a k-**jet of mappings** from X to Y;

(v) let σ be a k-jet, then there exist $p \in X$ and $q \in Y$ for which σ is in $J^k(X, Y)_{p,q}$, by (iv). p is called the **source** of σ and q is called the **target** of σ. The mapping

$\alpha : J^k(X, Y) \to X$ given by mapping every σ into its source is the **source map** and the mapping $\beta : J^k(X, Y) \to Y$ given by mapping every σ to its target is the **target map**.

Note that given a smooth mapping $f : X \to Y$, there is a canonically defined mapping

$$\begin{aligned} j^k f : X &\to J^k(X, Y), \\ p &\mapsto [f]_{k,p}. \end{aligned} \tag{2.1}$$

We will show that $j^k f(p)$ is just an invariant way of describing the Taylor expansion of f at p up to order k and that $j^k f$ is a *smooth* mapping.

Note that $J^0(X, Y) = X \times Y$, and $j^0 f(p) = (p, f(p))$ is just the graph of f.

Lemma 2.1. Let U be an open subset of $\mathbb{R}^n$ and p be a point in U. Let $f, g : U \to \mathbb{R}^m$ be smooth mappings. Then $f \sim_{k,p} g$ iff

$$\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(p) = \frac{\partial^{|\alpha|} g_i}{\partial x^\alpha}(p) \tag{2.2}$$

for every multi-index α with $|\alpha| \leqslant k$ and $1 \leqslant i \leqslant m$ where f_i and g_i are the coordinate functions determined by f and g respectively, and $x_1, \dots, x_n$ are coordinates on U.

Proof. We proceed by induction on k. For $k = 1$, $f \sim_{1,p} g$ iff $(Df)_p = (Dg)_p$, iff the first partial derivatives of f at p are identical with the first partial derivatives of g at p.

Assume the lemma is true for $k - 1$. We need to show that it is also true for k. By Definition 2.1 (ii), $f \sim_{k,p} g$ for $p \in U$ iff

$$(Df) \sim_{k-1,(p,v)} (Dg) \tag{2.3}$$

for $(p, v) \in T_p U$. So we just have to show that (2.3) indeed holds. To do this, we need the help of local coordinates.

Let $x_1, \dots, x_n, y_1, \dots, y_n$ be the coordinates of $TU = U \times \mathbb{R}^n$. Then $Df : U \times \mathbb{R}^n \to T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m$ is given by

$$(x, y) \mapsto (f(x), \bar{f}_1(y), \dots, \bar{f}_m(y)) \tag{2.4}$$

where

$$\bar{f}_i(y) \coloneqq \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x) y_j. \tag{2.5}$$

Similarly for Dg.

Now let's get back to prove (2.3). By induction hypothesis, (2.3) holds iff

$$\frac{\partial^{|\alpha|} (Df)_i}{\partial z^\alpha}(p, v) = \frac{\partial^{|\alpha|} (Dg)_i}{\partial z^\alpha}(p, v) \tag{2.6}$$

for any multi-index $|\alpha| \leqslant k - 1$ and $1 \leqslant i \leqslant 2m$, where α is any $2n$-tuple and

$z := (x, y)$. Plugging (2.4) and (2.5) into (2.6), we find that (2.6) holds iff

$$\frac{\partial^{|\alpha|+1} f_i}{\partial x^\alpha \partial x_j}(p) = \frac{\partial^{|\alpha|+1} g_i}{\partial x^\alpha \partial x_j}(p)$$

and

$$\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(p) = \frac{\partial^{|\alpha|} g_i}{\partial x^\alpha}(p)$$

hold, where $1 \leqslant i \leqslant m$ and α essentially contains only the first n variables. This completes the proof. $\square$

Corollary 2.2. Let $f, g : U \to \mathbb{R}^m$ be smooth mappings, then $f \sim_{k,p} g$ iff the Taylor expansions of f and g up to order k are identical at p.

Proof. This follows tautologically from the last lemma. $\square$

Lemma 2.3. Let U be an open subset of $\mathbb{R}^n$ and V an open subset of $\mathbb{R}^m$. Let $f_1, f_2 : U \to V$ and $g_1, g_2 : V \to \mathbb{R}^l$ be smooth mappings so that $g_1 \circ f_1$ and $g_2 \circ f_2$ are defined. Let $p \in U$ and suppose that $f_1 \sim_{k,p} f_2$ and $g_1 \sim_{k,q} g_2$ with $q = f_1(p) = f_2(p)$. Then $g_1 \circ f_1 \sim_{k,p} g_2 \circ f_2$.

Proof. We again proceed by induction on k. For $k = 1$, this is just the chain rule, or namely,

$$(D(g_1 \circ f_1))_p = (Dg_1)_q (Df_1)_p = (Dg_2)_q (Df_2)_p = (D(g_2 \circ f_2))_p.$$

Assume that the lemma is true for $k - 1$. Then we have

$$(Dg_1) \circ (Df_1) \sim_{k-1,(p,v)} (Dg_2) \circ (Df_2), \forall (p, v) \in T_p U, \tag{2.7}$$

where we substitute Df_i, Dg_i for $f_i, g_i, i = 1, 2$, and (p, v) for p in the induction hypothesis. However, on the other hand we have $D(g_i \circ f_i) = (Dg_i) \circ (Dg_i), i = 1, 2$. So (2.7) becomes

$$D(f_1 \circ g_1) \sim_{k-1,(p,v)} D(f_2 \circ g_2),$$

which holds iff

$$f_1 \circ g_1 \sim_{k,p} f_2 \circ g_2$$

holds, by Definition 2.1, completing the proof. $\square$

Proposition 2.4. Let X, Y, Z and W be smooth manifolds.

(i) Let $h : Y \to Z$ be smooth, then h induces a mapping $h_* : J^k(X, Y) \to J^k(X, Z)$ defined as follows: let σ be in $J^k(X, Y)_{p,q}$ and let $f : X \to Y$ represent σ; then $h_*(\sigma) := [h \circ f]_{\sim_{k,p}}$ is the equivalence class of $h \circ f$ in $J^k(X, Y)_{p,h(q)}$.

(ii) Let $a : Z \to W$ be smooth. Then $a_* \circ h_* = (a \circ h)_*$ as mappings of $J^k(X, Y) \to J^k(X, W)$ and $(\mathrm{id}_Y)_* = \mathrm{id}_{J^k(X,Y)}$. Thus if h is a diffeomorphism, h_* is a bijection.

(iii) Let $g : Z \to X$ be a smooth diffeomorphism, then g induces a mapping $g^* : J^k(X, Y) \to J^k(Z, Y)$ defined as follows: let τ be in $J^k(X, Y)_{p,q}$ and let $f : X \to Y$

represent τ. Then $g^*(\tau) \coloneqq [f \circ g]_{\sim_{k,g^{-1}(p)}}$ is the equivalence class of $f \circ g$ in $J^k(X,Z)_{g^{-1}(p),q}$.

(iv) Let $a : W \to Z$ be a smooth diffeomorphism. Then $a^* \circ g^* = (g \circ a)^*$ as mappings of $J^k(X,Y) \to J^k(W,Y)$ and $(\mathrm{id}_X)^* = \mathrm{id}_{J^k(X,Y)}$ so that g^* is a bijection.

Proof. (i) By Lemma 2.3, we know that $h_*(\sigma) = [h \circ f]_{\sim_{k,p}}$ is independent of the choice of f, so h_* is well-defined.
(ii) By definition, for any $\sigma \in J^k(X,Y)$ represented by $f : X \to Y$,

$$\begin{aligned}(a_* \circ h_*)(\sigma) &= a_*(h_*(\sigma)) \\ &= a_*([h \circ f]_{\sim_{k,p}}) \\ &= [a \circ (h \circ f)]_{\sim_{k,p}} \\ &= [(a \circ h) \circ f]_{\sim_{k,p}} \\ &= (a \circ h)_*(\sigma),\end{aligned}$$

proving that $a_* \circ h_* = (a \circ h)_*$. In the same spirit,

$$\begin{aligned}(\mathrm{id}_Y)_*(\sigma) &= [\mathrm{id}_Y \circ f]_{\sim_{k,p}} \\ &= [f]_{\sim_{k,p}} \\ &= \sigma,\end{aligned}$$

showing that $(\mathrm{id}_Y)_* = \mathrm{id}_{J^k(X,Y)}$. Finally, if $h : Y \to Z$ is a diffeomorphism, then there exists a smooth map $g : Z \to Y$ such that $g \circ h = \mathrm{id}_Y$ and $h \circ g = \mathrm{id}_Z$. On the space of k-jets, $g_* \circ h_* = (g \circ h)_* = (\mathrm{id}_Y)_* = \mathrm{id}_{J^k(X,Y)}$, and similarly $h_* \circ g_* = \mathrm{id}_{J^k(X,Z)}$, showing that $h_* : J^k(X,Y) \to J^k(X,Z)$ is a bijection.
(iii) follows by the same reason as for (i).
(iv) holds similarly as for (ii). □

Let A^k_n be the $\mathbb{R}$-linear space of polynomials in n indeterminates of degree $\leqslant k$ which have their constant term equal to zero. Choose as coordinates for A^k_n the coefficients of the polynomials. Then A^k_n is isomorphic to some Euclidean space and is, in this way, a smooth manifold. Let $B^k_{n,m} = \bigoplus_{i=1}^m A^k_n$. $B^k_{n,m}$ is also a smooth manifold.

Let U be an open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be smooth. Define $T_k f : U \to A^k_n$ by that $T_k(f)(x_0)$ is the polynomials of degree k given by the first k terms of the Taylor series of f at x_0 after the constant term.

Let V be an open subset of $\mathbb{R}^m$. Then there is a canonical map $T_{U,V} : J^k(U,V) \to U \times V \times B^k_{n,m}$ given by

$$T_{U,V}(\sigma) = (x_0, y_0, T_k f_1(x_0), \dots, T_k f_m(x_0)), \tag{2.8}$$

where $x_0 \coloneqq \alpha(\sigma)$ is the source of σ, and $y_0 \coloneqq \beta(\sigma)$ is the target of σ, and $f : U \to V$ is smooth and represents σ with $f_i : U \to \mathbb{R}$, $1 \leqslant i \leqslant m$ being the coordinate functions of f.

By Corollary 2.2, $T_{U,V}$ is independent of the choice of f, hence is well-defined.

Lemma 2.5. The map $T_{U,V} : J^k(U,V) \to U \times V \times B^k_{n,m}$ defined as above is a bijection.

Proof. We first show that $T_{U,V}$ is surjective, then injective.
If there are two distinct k-jets $\sigma, \tau \in J^k(U, V)$ represented by $f, g : U \to V$ respectively, such that $T_{U,V}(\sigma) = T_{U,V}(\tau)$, then by (2.8), the sources of f, g coincide as well as the targets of f, g, in addition

$$T_k f_i(x_0) = T_k g_i(x_0), 1 \leqslant i \leqslant m$$

holds. By Corollary 2.2, the last condition holds iff $f \sim_{k,p} g$, hence $\sigma = [f]_{\sim_{k,p}} = [g]_{\sim_{k,p}} = \tau$. This concludes the part of injectivity.
For surjectivity, for any

$$(x, y, a_1(x), \ldots, a_m(x)) \in U \times V \times B^k_{n,m}$$

where $a_i(x), 1 \leqslant i \leqslant m$ are polynomials of degree k in $x_1, \ldots, x_n$ without constant terms, we may let

$$f := (a_1, \ldots, a_m) + y$$

and let $\sigma := [f]_{\sim_{k,x}}$. Then $T_{U,V}(\sigma) = (x, y, a_1, \ldots, a_m)$ holds naturally by construction. This shows the surjectivity of $T_{U,V}$. □

Lemma 2.6. Let U and U' be subsets of $\mathbb{R}^n$ and let V and V' be open subsets of $\mathbb{R}^m$. Suppose $h : V \to V'$ and $g : U \to U'$ are smooth mappings with g a diffeomorphism. Then

$$T_{U',V'} \circ (g^{-1})^* \circ h_* \circ T^{-1}_{U,V} : U \times V \times B^k_{n,m} \to U' \times V' \times B^k_{n,m}$$

is a smooth mapping.

Proof. Let $D = (x_0, y_0, f_1(x), \ldots, f_m(x))$ with $f_i \in A^k_n, 1 \leqslant i \leqslant m$ be a point of $U \times V \times B^k_{n,m}$. We define $f : U \to \mathbb{R}^m$ by

$$f(x) := y_0 + (f_1(x - x_0), \ldots, f_m(x - x_0)).$$

Then $f(x_0) = y_0$ and let $\sigma := [f]_{\sim_{k,x_0}}$. Obviously $T_{U,V}(\sigma) = D$. Note that

$$(g^{-1})^*(h_*(\sigma)) = (g^{-1})^*([h \circ f]_{\sim_{k,x_0}}) = [h \circ f \circ g^{-1}]_{\sim_{k,g(x_0)}} = j^k(h \circ f \circ g^{-1})(g(x_0)).$$

by (2.1).
So

$$\begin{aligned}(T_{U',V'} \circ (g^{-1})^* \circ h_* \circ T^{-1}_{U,V})(D) &= (T_{U',V'} \circ (g^{-1})^* \circ h_*)(\sigma)\\ &= T_{U',V'}(j^k(h \circ f \circ g^{-1})(g(x_0)))\\ &= (g(x_0), h(y_0), T_k((h \circ f \circ g^{-1})_1)(g(x_0)), \ldots, T_k((h \circ f \circ g^{-1})_m)(g(x_0)))\end{aligned}$$

where $(h \circ f \circ g^{-1})_i : U' \to \mathbb{R}$ are the coordinate functions of $h \circ f \circ g^{-1} : U' \to \mathbb{R}^m$. To show that this mapping is smooth we need only show that the mapping of $U \times V \times B^k_{n,m} \to A^k_n$ given by $D \mapsto T_k((h \circ f \circ g^{-1})_i)(g(x_0))$ is smooth.
Let $\phi = h \circ f \circ g^{-1}$. Then

$$T_k(\phi_i)(g(x_0)) = \sum_{1 \leqslant |\alpha| \leqslant k} \frac{\partial^{|\alpha|} \phi_i}{\partial x^\alpha}(g(x_0))(x - g(x_0))^\alpha.$$

To show that $D \mapsto T_k(\phi_i)(g(x_0))$ is smooth it is enough to show that $D \mapsto (\partial^{|\alpha|}\phi_i/\partial x^\alpha)(g(x_0))$ is smooth for each multi-index α for which $|\alpha| \leqslant k$. Indeed, since $(\partial^{|\alpha|}\phi_i/\partial x^\alpha)(g(x_0))$ is sums and products of terms of the form

$$\frac{\partial^\beta h_i}{\partial y_j^\beta}(y_0), \frac{\partial^\gamma f_i}{\partial x_j^\gamma}(0) \text{ and } \frac{\partial^\sigma g_i^{-1}}{\partial x_j^\sigma}(g(x_0))$$

where $y_1, \ldots, y_m$ are coordinates on $\mathbb{R}^m$ and h_i, g_i are the coordinate functions determined by h and g, with β, γ and σ being multi-indices. Each of these terms vary smoothly with D, hence $(\partial^{|\alpha|}\phi_i/\partial x^\alpha)(g(x_0))$ varies smoothly with D. □

Theorem 2.7. Let X and Y be smooth manifolds with $n = \dim X$ and $m = \dim Y$. Then

(i) $J^k(X, Y)$ is a smooth manifold with

$$\dim J^k(X, Y) = m + n + \dim(B^k_{n,m}),$$

(ii) $\alpha : J^k(X, Y) \to Y, \beta : J^k(X, Y) \to Y$ and $\alpha \times \beta : J^k(X, Y) \to X \times Y$ are submersions;

(iii) if $h : Y \to Z$ is smooth, then $h_* : J^k(X, Y) \to J^k(X, Z)$ is smooth. If $g : X \to Y$ is a diffeomorphism, then $g^* : J^k(Y, Z) \to J^k(X, Z)$ is a diffeomorphism;

(iv) if $g : X \to Y$ is smooth, then $j^k g : X \to J^k(X, Y)$ is smooth.

Proof. (i) Let U be the domain for a chart ϕ on X and V be the domain for a chart ψ on Y. Let $U' = \phi(U)$ and $V' = \psi(V)$ be the domain for ψ. Then there are bijections $(\phi^{-1})^* \circ \psi_* : J^k(U, V) \to J^k(U', V')$ and $\tau_{U,V} := T_{U',V'} \circ (\phi^{-1})^* \circ \psi_* : J^k(U, V) \to U' \times V' \times B^k_{n,m}$. Give $J^k(X, Y)$ the manifold structure induced by declaring that $\tau_{U,V}$ is a chart. To see this, let $\phi_1, \psi_1, U_1, V_1, U_1', V_1'$ be the data for another chart τ_{U_1,V_1}. Then note that

$$\begin{aligned}\tau_{U_1,V_1} \circ \tau^{-1}_{U,V} &= T_{U_1',V_1'} \circ (\phi_1^{-1})^* \circ (\psi_1)_* \circ (\psi_*)^{-1} \circ \phi^* \circ T^{-1}_{U',V'} \\ &= T_{U_1',V_1'} \circ (\phi_1^{-1})^* \circ \phi^* \circ (\psi_1)_* \circ (\psi_*)^{-1} \circ T^{-1}_{U',V'} \\ &= T_{U_1',V_1'} \circ (\phi \circ \phi_1^{-1})^* \circ (\psi_1 \circ \psi^{-1})_* \circ T^{-1}_{U',V'},\end{aligned}$$

since ϕ^* commutes with ψ_* and $(\psi_1)_*$. Then Lemma 2.6 is applicable to the last mapping, with $g = \phi_1 \circ \phi^{-1} : U' \to U_1'$ and $h = \psi_1 \circ \psi^{-1} : V' \to V_1'$ to show that it is smooth. With coordinates $\{ \tau_{U,V} \}$, $J^k(X, Y)$ is a smooth manifold.
(ii) With the help of local coordinate $\tau_{U,V}$ of $J^k(X, Y)$ and ϕ of X, we can study the local properties of α explicitly, as illustrated in the commutative diagram

$$\begin{tikzcd} J^k(X,Y) \arrow[r, "\alpha"] & X \\ U' \times V' \times B^k_{n,m} \arrow[u, "\tau^{-1}_{U,V}"] \arrow[r, "\phi\circ\alpha\circ\tau^{-1}_{U,V}"] & U'. \arrow[u, "\phi^{-1}"'] \end{tikzcd}$$

Any point $D \in U' \times V' \times B^k_{n,m}$ has the form

$$D = (x_0, y_0, f_1(x), \ldots, f_m(x))$$

with $f_i \in A_n^k$, as in Lemma 2.6. In local coordinates α has the form

$$\begin{aligned}(\phi \circ \alpha \circ \tau_{u,v}^{-1})(D) &= (\phi \circ \alpha \circ (\psi_*)^{-1} \circ \phi^* \circ T_{U',V'}^{-1})(D) \\ &= (\phi \circ \alpha \circ (\psi_*)^{-1} \circ \phi^*)(\sigma) \\ &= (\phi \circ \alpha)([\psi^{-1} \circ f \circ \phi]_{\sim_{k,x_0}}) \\ &= (\phi \circ \alpha \circ j^k(\psi^{-1} \circ f \circ \phi) \circ \phi^{-1})(x_0)\end{aligned}$$

where f and σ is defined as in Lemma 2.6. Note that $\alpha \circ j^k g = \mathrm{id}_X$ for any mapping g from X, from which we conclude $(\phi \circ \alpha \circ \tau_{u,v}^{-1})(D) = (\phi \circ \mathrm{id}_X \circ \phi^{-1})(x_0) = x_0$. Thus locally α is the projection

$$(x_0, y_0, f_1(x), \ldots, f_m(x)) \mapsto x_0,$$

hence is a submersion.
Similarly

$$\begin{aligned}(\psi \circ \beta \circ \tau_{U,V}^{-1}) &= (\psi \circ \beta \circ j^k(\psi^{-1} \circ f \circ \phi) \circ \phi^{-1})(x_0) \\ &= (\psi \circ \psi^{-1} \circ f \circ \phi \circ \phi^{-1})(x_0) \\ &= f(x_0) = y_0\end{aligned}$$

since $\beta \circ j^k g = g$ for any map landing in Y. Thus β is locally the projection $D \mapsto y_0$, hence is smooth and is a submersion. Finally and immediately $\alpha \times \beta : J^k(X, Y) \to X \times Y$ is a submersion.
(iii) Locally, h_* is the map

$$\begin{aligned}h_* : U' \times V' \times B_{n,m}^k &\to U' \times W' \times B_{n,l}^k, \\ (x_0, y_0, f_1(x), \ldots, f_m(x)) &\mapsto (x_0, h(y_0), T_k(h_1 \circ f)(x), \ldots, T_k(h_l \circ f)(x))\end{aligned}$$

where W' is a coordinate neighborhood containing $h(y_0)$ of Z and $h_1, \ldots, h_l$ the coordinate functions of $h : Y \to Z$, with $l := \dim Z$. Since $f_1, \ldots, f_m$ are polynomials in x, h_* is a smooth at D. The proof for that $g^* : J^k(Y, Z) \to J^k(X, Z)$ is smooth follows similarly. As g^* is already a bijection, it is a diffeomorphism.
(iv) We first consider the case when $g : \mathbb{R}^n \to \mathbb{R}^m$. Then $J^k(\mathbb{R}^n, \mathbb{R}^m) \simeq \mathbb{R}^n \times \mathbb{R}^m \times B_{n,m}^k$ and $j^k g$ is given by

$$\begin{aligned}j^k g : \mathbb{R}^n &\to J^k(\mathbb{R}^n, \mathbb{R}^m) \\ x_0 &\mapsto (x_0, g(x_0), T_k g_1(x_0), \ldots, T_k g_m(x_0))\end{aligned}$$

where $g_1, \ldots, g_m$ are coordinate functions of g. Since $T_k g_i$ are just polynomials in of n indeterminates, they are of course smooth functions of x_0. Hence $j^k g : \mathbb{R}^n \to J^k(\mathbb{R}^n, \mathbb{R}^m)$ is smooth. For a general smooth function $g : X \to Y$, with the standard use of the coordinate charts given in (i), one can see that $j^k g : X \to J^k(X, Y)$ is a smooth mapping. □

2.2 The Whitney C^∞-Topology

Definition 2.2. Let X and Y be smooth manifolds.

(i) Denote by $C^\infty(X, Y)$ the set of smooth mappings from X to Y.

(ii) Fix a non-negative integer k. Let U be a subset of $J^k(X, Y)$. Then denote by

$M(U)$ the set
$$\left\{ f \in C^\infty(X,Y) \,\middle|\, j^k f(X) \subseteq U \right\}.$$
It's easy to verify that $M(U \cap V) = M(U) \cap M(V)$.

(iii) The family of sets $\{ M(U) \}$ where U is an open subset of $J^k(X,Y)$ form a basis for a topology on $C^\infty(X,Y)$. This topology is called the **Whitney C^k-topology**. Denote by W_k the subsets of $C^\infty(X,Y)$ in the Whitney C^k-topology.

(iv) The **Whitney C^∞-topology** on $C^\infty(X,Y)$ is the topology given by the basis $W = \bigcup_{k=0}^\infty W_k$.

Remark. The Whitney C^∞-topology on $C^\infty(X,Y)$ generated by W is indeed well-defined. This is because $W_k \subseteq W_l$ whenever $k \leqslant l$. Admitting the last claim up front, for any two basic open subsets $A, B \in W$, by definition $A \subset W_k$ and $B \subset W_l$ for some k, l. We may assume that $k \leqslant l$, then for any larger m such that $k \leqslant l \leqslant m$, $W_k \subseteq W_m$ and $W_l \subset W_m$ hold hence A, B both can be viewed as open subsets in W_m. So $A \cap B$ is an open subset in W_m and there must be some open subset $C \subseteq A \cap B$ of W_m, for example we can pick $C = M(U)$ for some open subset$U \subset J^m(X,Y)$. This shows that W is a well-defined topological basis.
Now we prove the claim. To show that $W_k \subseteq W_l$, it suffices to show
$$\left\{ M(U) \,\middle|\, U \subseteq J^k(X,Y) \text{ open} \right\} \subseteq \left\{ M(V) \,\middle|\, V \subseteq J^l(X,Y) \text{ open} \right\}.$$
Note that there is a canonical mapping $\pi^l_k : J^l(X,Y) \to J^k(X,Y)$, defined by forgetting the jet information of order $> k$. With the help of coordinate charts, π^l_k is locally map,
$$\begin{aligned} U' \times V' \times B^l_{n,m} &\to U' \times V' \times B^k_{n,m}, \\ (x_0, y_0, f_1(x), \dots, f_m(x)) &\mapsto (x_0, y_0, T_k f_1(x), \dots, T_k f_m(x)), \end{aligned}$$
which is a smooth projection since $f_i(x) \in B^l_{n,m}, 1 \leqslant i \leqslant m$ are all polynomials. Hence $\pi^l_k : J^l(X,Y) \to J^k(X,Y)$ is a submersion of manifolds. Let $f \in M(U)$ be a smooth mapping $X \to Y$, hence there are smooth maps $j^k f : X \to J^k(X,Y)$ and $j^l f : X \to J^l(X,Y)$, which locally are
$$\begin{aligned} j^k f : U' &\to U' \times V' \times B^k_{n,m}, \\ x_0 &\mapsto (x_0, f(x_0), T_k f_1(x_0), \dots, T_k f_m(x_0)) \end{aligned}$$
and
$$\begin{aligned} j^l f : U' &\to U' \times V' \times B^l_{n,m}, \\ x_0 &\mapsto (x_0, f(x_0), T_l f_1(x_0), \dots, T_l f_m(x_0)). \end{aligned}$$
Thus there is a commutative diagram
$$\begin{tikzcd} & J^l(X,Y) \arrow[d, "\pi^l_k"] \\ X \arrow[ur, "j^l f"] \arrow[r, "j^k f"] & J^k(X,Y) \end{tikzcd}.$$
If $M(U)$ is a basic open subset containing f with $U \subseteq J^k(X,Y)$, by definition

$j^k f(X) \subseteq U$. By the above diagram, we know that $j^k f \subseteq U$ iff $j^l f(X) \subseteq (\pi^l_k)^{-1}(U)$, hence $f \in M(U)$ iff $f \in M((\pi^l_k)^{-1}(U))$. So $M(U) = M((\pi^l_k)^{-1}(U))$. Moreover, $(\pi^l_k)^{-1}(U) \subseteq J^l(X,Y)$ is an open subset since U is open and π^l_k is continuous, showing that $M((\pi^l_k)^{-1}(U))$ is indeed a basic open subset of the Whitney C^l-topology of $C^\infty(X,Y)$. This concludes the proof of the claim.

Proposition 2.8. Let X and Y be smooth manifolds. The mapping $j^k : C^\infty(X,Y) \to C^\infty(X, J^k(X,Y))$ defined by

$$\begin{aligned} j^k : C^\infty(X,Y) &\to C^\infty(X, J^k(X,Y)), \\ f &\mapsto j^k f \end{aligned} \tag{2.9}$$

is a continuous injection in the Whitney C^∞-topology.

Proof. Let U be an open subset of $J^l(X, J^k(X,Y))$, then $M(U)$ is a basic open set in $C^\infty(X, J^k(X,Y))$. It suffices to show that $(j^k)^{-1}(M(U))$ is an open subset of $C^\infty(X,Y)$. First we define a mapping

$$\alpha_{k,l} : J^{k+l}(X,Y) \to J^l(X, J^k(X,Y))$$

as follows: let σ be a $(k+l)$-jet with source x and let $f : X \to Y$ represent σ. By Theorem 2.7 (iv), $j^k f : X \to J^k(X,Y)$ is a smooth embedding. Define $\alpha_{k,l}(\sigma) := [j^k f]_{\sim_{l,x}} = (j^l(j^k f))(x)$. To see the well-definedness of $\alpha_{k,l}$, we work locally. In local charts, the coordinate of σ is

$$\sigma = (x, f(x), T_{k+l} f_1(x), \dots, T_{k+l} f_m(x)) \in U' \times V' \times B^{k+l}_{n,m}$$

and $j^k f : X \to J^k(X,Y)$ goes by

$$\begin{aligned} j^k f : U' &\to U' \times V' \times B^k_{n,m}, \\ x_0 &\mapsto (x_0, f(x_0), T_k f_1(x_0), \dots, T_k f_m(x_0)). \end{aligned}$$

Thus $\alpha_{k,l}$ locally looks like

$$\begin{aligned} \alpha_{k,l} : U' \times V' \times B^{k+l}_{n,m} &\mapsto U' \times V' \times B^k_{n,m} \times W' \times B^l_{n,p}, \\ (x, f(x), T_k f(x)) &\mapsto (x, f(x), T_k(f)(x), T_l(\mathrm{id}_U), T_l f(x), T_l(T_k f)(x)), \end{aligned} \tag{2.10}$$

where $T_k f :== (T_k f_1, \dots, T_k f_m) : U \to B^k_{n,m}$ and $T_l(T_k f)(x)$ is the Taylor expansion of $T_k f : U' \to B^k_{n,m}$ at $x \in U$ of order $\leqslant l$, with $p := \dim J^k(X,Y) = m + n + \dim B^k_{n,m}$. If $f \sim_{k+l,x} f'$ both represent σ, then $T_k f = T_k f'$, hence $\alpha_{k,l}(\sigma)$ is independent of the choice of f, by (2.10). (2.10) also tells us that $\alpha_{k,l} : J^{k+l}(X,Y) \to J^l(X, J^k(X,Y))$ is a smooth embedding.
Thus $\alpha^{-1}_{k,l}(U)$ is an open subset of $J^{k+l}(X,Y)$, whence $U \subset J^l(X, J^k(X,Y))$ is the open set chosen in the beginning of the proof. By definition of $\alpha_{k,l}$, the diagram

$$\begin{tikzcd} & J^{k+l}(X,Y) \arrow[d, "\alpha_{k,l}"] \\ X \arrow[ur, "j^{k+l} f"] \arrow[r, "j^l(j^k f)"'] & J^l(X, J^k(X,Y)) \end{tikzcd}$$

commutes trivially. We claim that $M(\alpha_{k,l}^{-1}(U)) = (j^k)^{-1}(M(U))$. Indeed, $f \in (j^k)^{-1}(M(U))$ iff $j^k f \in M(U)$ iff $j^l(j^k f)(X) \subseteq U$ iff $(\alpha_{k,l} \circ j^{k+l} f)(X) \subseteq U$ iff $j^{k+l} f(X) \subseteq \alpha_{k,l}^{-1}(U)$ iff $f \in M(\alpha_{k,l}^{-1}(U))$, thus a the claim holds. This shows that $j^k f : C^\infty(X, Y) \to C^\infty(X, J^k(X, Y))$. j^k is an injection follows from the fact that $\alpha \circ j^k f = f$ for any $f \in C^\infty(X, Y)$. □

Proposition 2.8 shows that $C^\infty(X, Y)$ is a subspace of $C^\infty(X, J^k(X, Y))$, where the latter is exactly the space of smooth sections of the fiber bundle $\alpha : J^k(X, Y) \to X$. The map $j^k : C^\infty(X, Y) \to C^\infty(X, J^k(X, Y))$ is sometimes called the **inclusion of holonomic sections into all sections**, where **holonomic sections** refers to the image of j^k. See [4, pp.27], with the settings slightly different.

2.3 Diffeomorphisms Form a Topological Group

Proposition 2.9. Let X, Y and Z be smooth manifolds. Let $\phi : Y \to Z$ be smooth. Then the mapping

$$\begin{aligned} \phi_* : C^\infty(X, Y) &\to C^\infty(X, Z), \\ f &\mapsto \phi \circ f \end{aligned}$$

is a continuous mapping in the Whitney C^∞-topology.

Proof. See [10, pp. 46]. □

Lemma 2.10. Let A, B and P be Hausdorff spaces. Suppose that P is locally compact and paracompact. Let $\pi_A : A \to P$ and $\pi_B : B \to P$ be continuous. Set

$$A \times_P B := \left\{ (a, b) \in A \times B \,\middle|\, \pi_A(a) = \pi_B(b) \right\}$$

and give $A \times_P B$ the topology induced from $A \times B$. Let $K \subset A$ and $L \subset B$ be subsets such that $\pi_A|_K$ and $\pi_B|_L$ are proper. Let U be an open neighborhood of $K \times_P L$ in $A \times_P B$. Then there exists a neighborhood V of K in A and a neighborhood W of L in B such that $V \times_P W \subset U$.

Proof. See [10, pp.47]. □

Proposition 2.11. Let X, Y and Z be smooth manifolds with X compact. Then the mapping

$$\begin{aligned} \mu : C^\infty(X, Y) \times C^\infty(Y, Z) &\to C^\infty(X, Z), \\ (f, g) &\mapsto g \circ f \end{aligned}$$

is continuous.

Proof. To prove the proposition it's enough to show that the preimage of any sub-basic open set of $C^\infty(X, Z)$ is an open subset of $C^\infty(X, Y) \times C^\infty(Y, Z)$. Namely, take $S \subset J^k(X, Z)$, we need to show that $\mu^{-1}(M(S)) \subset C^\infty(X, Y) \times C^\infty(Y, Z)$ is an open subset. To show this, it suffices to show that for any $(f, g) \in \mu^{-1}(M(S))$, we can find $V \in J^k(X, Y)$ and $W \in J^k(Y, Z)$ such that $f \in M(V), g \in M(W)$ and $M(V) \times M(W) \subset \mu^{-1}(M(S))$, as $M(V) \times M(W)$ is a sub-basic open subset of the product $C^\infty(X, Y) \times C^\infty(Y, Z)$. The last condition is equivalent to that if $g \circ f \in M(S), \forall f \in C^\infty(X, Y), \forall g \in C^\infty(Y, Z)$, we can find $V \subseteq C^\infty(X, Y)$ and $W \subseteq C^\infty(Y, Z)$ such that

$M(V)$ contains f and $M(W)$ contains g, and for all other $f' \in M(V), g' \in M(W)$, $f' \circ g'$ lies in $M(S)$.
We claim that the mapping

$$\gamma : J^k(X,Y) \times_Y J^k(Y,Z) \to J^k(X,Z),$$
$$(\sigma,\tau) \mapsto \tau \circ \sigma$$

is continuous. Indeed, consider the following commutative diagram

$$\begin{tikzcd}
J^k(X,Y) \arrow[r, "g_*"] & J^k(X,Z) \\
J^k(X,Y) \times_Y \{g\} \arrow[u, "\simeq"] \arrow[r, "i_g"] \arrow[d] & J^k(X,Y) \times_Y J^k(Y,Z) \arrow[u, "\gamma"'] \arrow[d] \\
\{g\} \arrow[r] & J^k(Y,Z)
\end{tikzcd},$$

in which the lower diagram is a pull-back, and $i_g : J^k(X,Y) \times_Y \{g\} \to J^k(X,Y) \times_Y J^k(Y,Z)$ is the inclusion of the fiber at g, thus is continuous. Thus $g_* = \gamma \circ i_g$ holds. By Theorem 2.7(iii), g_* is smooth, hence continuous. This means that γ must be continuous.
Let $S \subseteq J^k(X,Y)$ be an open subset, and $f \in C^\infty(X,Y), g \in C^\infty(Y,Z)$ be two smooth maps. We assume that $g \circ f \in M(S)$, or equivalently $j^k(g \circ f)(X) \subseteq S$. However, on the other hand, observe that

$$j^k(g \circ f)(X) = \gamma(j^k f(X) \times_Y j^k g(Y))$$

holds by definition. So we have

$$j^k f(X) \times_Y j^k g(Y) \subseteq \gamma^{-1}(S).$$

Since γ is continuous, $\gamma^{-1}(S)$ is an open in $J^k(X,Y) \times_Y J^k(Y,Z)$. We now apply Lemma 2.10 with $A = J^k(X,Y), B = J^k(Y,Z), P = Y, \pi_A = \beta, \pi_B = \alpha, K = j^k f(X), L = j^k g(Y)$ and $U = \gamma^{-1}(S)$, to show the existence of the described V and W. That Lemma 2.10 is applicable follows from the facts that U is open, $\pi_A|_K$ is compact (since X is compact hence$K := j^k f(X)$ is), and $\pi_B|_L$ is proper (since $\pi_B \circ j^k g = \mathrm{id}_Y$).
□

Now let X be compact so that Proposition 2.11 is applicable. We consider the distinguished subset $\mathrm{Diff}(X) \subset C^\infty(X,X)$, which consists of diffeomorphisms of the compact manifold X. We endow $\mathrm{Diff}(X)$ the subspace topology induced by the Whitney C^∞-topology on $C^\infty(X,X)$, so that the natural inclusion $\mathrm{Diff}(X) \hookrightarrow C^\infty(X,X)$ is continuous.

For any diffeomorphism $f : X \to X$, the inverse $f^{-1} : X \to X$ is again a diffeomorphism, hence taking inverse induces a bijection

$$\iota : \mathrm{Diff}(X) \to \mathrm{Diff}(X),$$
$$f \mapsto f^{-1}.$$

The next proposition shows that $\iota : \mathrm{Diff}(X) \to \mathrm{Diff}(X)$ is *a fortiori* a homeomorphism in the Whitney C^∞-topology.

Proposition 2.12. Let X be a compact smooth manifold. Then the inversion mapping

$$\begin{aligned}\iota : \mathrm{Diff}(X) &\to \mathrm{Diff}(X),\\ f &\mapsto f^{-1}\end{aligned}$$

is a homeomorphism in the Whitney C^∞-topology.

Proof. First note that for any diffeomorphism $f \in \mathrm{Diff}(X)$, the image of f along the map

$$j^1 : C^\infty(X,X) \to C^\infty(X, J^1(X,X))$$

lies in the subspace of sections $C^\infty(X, J^{1,\mathrm{inv}}(X,X))$, where $J^{1,\mathrm{inv}}(X,X)$ is the subspace of 1st order jets represented by mappings $f : X \to X$ having invertible differentials. As the natural inclusion $\mathrm{Diff}(X) \hookrightarrow C^\infty(X,X)$, by construction; and the inclusion $j^1 : C^\infty(X,X) \to C^\infty(X, J^1(X,X))$ of holonomic sections is continuous, by Proposition 2.9, their composition $\mathrm{Diff}(X) \hookrightarrow C^\infty(X,X) \xrightarrow{j^1} C^\infty(X, J^1(X,X))$ is again continuous and factors through $\mathrm{Diff}(X) \to C^\infty(X, J^{1,\mathrm{inv}}(X,X))$. The last mapping is hence continuous, and we denote it by $j^1 : \mathrm{Diff}(X) \to C^\infty(X, J^{1,\mathrm{inv}}(X,X))$.
We define a map $\mathrm{inv} : J^{1,\mathrm{inv}}(X,X) \to J^{1,\mathrm{inv}}(X,X)$ by

$$\begin{aligned}\mathrm{inv} : J^{1,\mathrm{inv}}(X,X) &\to J^{1,\mathrm{inv}}(X,X),\\ \sigma &\mapsto [f^{-1}]_{\sim 1,y},\end{aligned}$$

where σ is a 1-jet with source x represented by $f : X \to X$, with $y = f(x)$. In local charts, inv looks like

$$\begin{aligned}\mathrm{inv} : U' \times V' \times B^1_{n,n} &\to V' \times U' \times B^1_{n,n},\\ (x, y, Df(x)) &\mapsto (y, x, D(f^{-1})(y)) = (y, x, (Df)^{-1}(y)),\end{aligned}$$

from which we see that $\mathrm{inv}(\sigma)$ is independent of the choice of f and $\mathrm{inv} : J^{1,\mathrm{inv}}(X,X) \to J^{1,\mathrm{inv}}(X,X)$ is smooth.
Next consider the diagram

$$\begin{array}{ccc}\mathrm{Diff}(X) & \xrightarrow{j^1} & C^\infty(X, J^{1,\mathrm{inv}}(X,X)) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \mathrm{inv}_*} \\ \mathrm{Diff}(X) & \xrightarrow{j^1} & C^\infty(X, J^{1,\mathrm{inv}}(X,X))\end{array},$$

which is commutative by definition. The vertical mapping $\mathrm{inv}_* : C^\infty(X, J^{1,\mathrm{inv}}(X,X)) \to C^\infty(X, J^{1,\mathrm{inv}}(X,X))$ is continuous by Proposition 2.9, as it is induced by the smooth mapping $\mathrm{inv} : J^{1,\mathrm{inv}}(X,X) \to J^{1,\mathrm{inv}}(X,X)$. Thus $\iota : \mathrm{Diff}(X) \to \mathrm{Diff}(X)$ is continuous, as the composition $\iota \circ j^1 = \mathrm{inv}_* \circ j^1$ is. Using the similar method, we see that ι^{-1} is also continuous, completing the proof. □

Proposition 2.11 and Proposition 2.12 show that there is a group structure compatible with the subspace topology on $\mathrm{Diff}(X)$ induced by the Whitney C^∞-topological on $C^\infty(X,X)$. Now we have reached the main result of this section

Theorem 2.13. Let X be a compact manifold. Then $\mathrm{Diff}(X)$ is a topological group.

3 Smale's Theorem

This section mainly follows [3].

Let I^2 be the square in the Euclidean space $\mathbb{R}^2$ with coordinate (x_1, x_2) such that $(x_1, x_2) \in I^2$ iff $0 \leqslant x_1 \leqslant 1$ and $0 \leqslant x_2 \leqslant 1$. Let id_{I^2} be the identity diffeomorphism and $\mathrm{Diff}_{\partial, U}(I^2)$ be the set of diffeomorphisms of I^2 onto I^2 which is which agree with id_{I^2} on some neighborhood of ∂I^2. Although in Section 2.2, we only discussed the diffeomorphism group Diff(X) of a compact manifold X without boundary, we can define the Whitney C^∞- on $\mathrm{Diff}_{\partial, U}(I^2)$ *mutatis mutandis*. This is because that I^2 is compact, thus the key Lemma 2.10 can be applied. Hence $\mathrm{Diff}_{\partial, U}(I^2)$ is a topological group in the Whitney C^∞-topology.

Let $I_1 \subset I^2$ denote the subset $\{ (x_1, x_2) \mid x_1 = 1 \}$, $df_p : T_p I^2 \to T_{f(p)} I^2$ be the tangent map of a diffeomorphism f at $p \in I^2$. Then denote $\mathcal{E}$ the space of diffeomorphisms of I^2 onto I^2 such that if $f \in \mathcal{E}$, then

(i) $f = \mathrm{id}_{I^2}$ on some neighborhood of $\partial I^2 \backslash I_1$, and

(ii) $(df_p)(u_0) = u_0$ for all p in some neighborhood of I_1, where u_0 is the vector $(0, 1)$ in $T_p\mathbb{R}^2 \simeq \mathbb{R}^2 \simeq T_{f(p)}\mathbb{R}^2$.

Note that $\mathrm{Diff}_{\partial, U}(I^2) \subset \mathcal{E}$.

Let $e : I^2 \to u_0$ be the constant map and define $\mathcal{S}$ to be the space with the compact open topology [1] of maps of I^2 into S which agree with e in some neighborhood of ∂I^2, where $S = \mathbb{R}^2 - \{ (0, 0) \}$.

[1] *Unify it with the weak Whitney topology. To do.*

A map $\phi : \mathcal{E} \to \mathcal{S}$ is defined as follows

$$\phi(f)(x_1, x_2) = (df_{f^{-1}(x_1, x_2)})(u_0)$$

Lemma 3.1. There is a homotopy $\phi_s : \mathcal{E} \to \mathcal{S}$ such that for each $f \in \mathcal{E}$,

(i) $\phi_s(f)(x_1, x_2)$ is smooth in s, x_1 and x_2,

(ii) $\phi_0(f) = e$, the constant map,

(iii) $\phi_1(f) = \phi(f)$, and

(iv) $\phi_s(\mathrm{id}_{I^2}) = e$.

Proof. It is well know that $\mathbb{R}^2$ is the universal covering of the punctured plane $S := \mathbb{R}^2 \backslash \{ (0, 0) \}$. Let $\pi : \mathbb{R}^2 \to S$ be the covering map, and let $u \in \pi^{-1}(u_0)$ be a point in the preimage $\pi^{-1}(u_0)$ of u_0. Since $\mathbb{R}^2$ is contractible, let $T_s : \mathbb{R}^2 \to \mathbb{R}^2$ be a deformation retract of $\mathbb{R}^2$ to the point u.
Now we define a homotopy $h_s : \mathcal{S} \to \mathcal{S}$ by $h_s(f)(x) = (\pi \circ T_s \circ \bar{f})(x)$ for any $x \in I^2$. Then it is easily checked that $\phi_s := h_s \circ \phi$ may be taken as our desired homotopy.

$$\begin{tikzcd}
 & \mathbb{R}^2 \arrow[r, "T_s"] \arrow[d, "\pi"] & \mathbb{R}^2 \arrow[dl, "\pi"] \\
I^2 \arrow[ur, dashed, "\bar{f}"] \arrow[r, "f"] & S &
\end{tikzcd}$$

□

Lemma 3.2. There is a homotopy $H_s : \mathcal{E} \to \mathcal{E}$ such that for each $f \in \mathcal{E}$,

(i) $H_s(f)$ is smooth in s, x_1 and x_2,

(ii) $H_0(f) = \mathrm{id}_{I^2}$,

(iii) $H_1(f) = f$, and

(iv) $H_s(\mathrm{id}_{I^2}) = \mathrm{id}_{I^2}$.

Proof. Let $\phi_s(f)(x,y), (x,y) \in I^2$ be considered as a vector field on I^2 as given by the previous lemma, for each $f \in \mathcal{E}$ and $0 \leqslant s \leqslant 1$. Let $P_s(f)(t, x_0, y_0)$ be the integral curve of $\phi_s(f)$ with the initial condition $P_s(f)(0, x_0, y_0) = (x_0, y_0)$.
We define $Q_s(f)(t,y) \coloneqq P_s(f)(t, 0, y)$, that is, $Q_s(f)(t,y)$ is a solution of the equation

$$\begin{cases} x' = \phi_s(f)(x), x \in I^2, \\ x(0) = (0, y). \end{cases}$$

Now suppose integral curve $Q_s(f)(t,y)$ starting at $(0,y)$ which doesn't leave I^2. Thus the closure $\overline{Q_s(f)(t,y)}$ is contained in I^2 since I^2 is closed. Hence $\overline{Q_s(f)(t,y)}$ is compact. By the Poincaré-Bendixson Theorem A.6, $\Omega(Q_s(f)(t,y))$ contains a periodic solution of the equation

$$x' = \phi_s(f)(x), \tag{3.1}$$

which is in addition a Jordan curve. But by Corollary A.7, there is at least a stationary point of the vector field $\phi_s(f)$ in the interior of this periodic solution. Then we reach a contradiction, since $\phi_s(f)(x)$ takes value [2] in $S \coloneqq \mathbb{R}^2 \setminus \{(0,0)\}$ for all $x \in I^2$. So we conclude that there is a t, say $\bar{t}$, with $Q_s(f)(\bar{t}, y)$ meeting I_1.
We denote the above $\bar{t}$ by $\bar{t}(s,f,y)$, then $\bar{t}(s,f,y)$ is smooth in s, y and continuous in f, by the theory of ordinary equations [3].
Let g be the function on $\mathcal{E}$ defined by

$$g(f) = \min\left\{ \frac{\bar{t}(s,f,y)}{1-\bar{t}(s,f,y)}, 1 \;\middle|\; 0 \leqslant s, y \leqslant 1, \bar{t}(s,f,y) < 1 \right\}.$$

Then let η be a continuous function such that $0 < \eta < g$. The existence of such η is assured by the following lemma

Lemma 3.3. Let g be a real lower semi-continuous positive function on a paracompact space X. Then there is a real continuous function h on X such that for all $x \in X$, $0 < h(x) < g(x)$.

Proof. See [11, pp.172]. □

Let γ be a real function on $\mathcal{E} \times \mathbb{R}$, smooth in x such that $\gamma(f,x) = 0$ for x in some neighborhood of 0, $\gamma(f,x) = 1$ for s in some neighborhood of 1 and

$$0 < \frac{d\gamma(f,x)}{dx} < 1 + \eta(f).$$

[2] *A more convicing way to expain this. Optional.*

[3] *More precisely. To do*

Such γ can be obtained by

$$\gamma(f,x) := \int_0^x (1+\eta(f,s))\rho(s)\,ds$$

where $\rho(s)$ is a bump function on $\mathbb{R}$ with $\operatorname{supp}\rho \subset (0,1)$.
Now define $H_s : \mathcal{E} \to \mathcal{E}$ by

$$H_s(f)(x,y) = Q_s(f)(x+\gamma(f,x)(\bar{t}(s,f,y)-1),y).$$

We prove now that $H_s(f) : I^2 \to I^2$ is regular. Note that $H_s(f)$ can be written as the composition $\psi \circ g$ where

$$\begin{aligned} g &: (x,y) \mapsto (x+\gamma(f,x)(\bar{t}(s,f,y)-1),y) = (t,y'),\\ \psi &: (t,y') \mapsto Q_s(f)(t,y'). \end{aligned}$$

By the choice of η it follows that $\frac{\partial t}{\partial x} \neq 0$, hence g is regular.
Now we prove that ψ is regular. Let $\varphi^i(t,x,y), i=1,2$ be the coordinate functions of the integral curve $P_s(f)(t,x,y)$:

$$P_s(f)(t,x,y) = (\varphi^1(t,x,y), \varphi^2(t,x,y)).$$

Also denote $X_i(x,y)$ be the coordinate of the vector field $\phi_s(f)(x,y)$:

$$\phi_s(f)(x,y) = X_1(x,y)\frac{\partial}{\partial x} + X_2(x,y)\frac{\partial}{\partial y}$$

Then $\psi(t,y) = P_s(f)(t,0,y)$. It is sufficient to prove that ψ^{-1} is differentiable. Let $\tau(x,y)$ be the unique time t such that $\varphi^1(t,x,y) = 0$. Then $\psi^{-1}(x,y) = (-\tau(x,y), \varphi^1(\tau(x,y),x,y), \varphi^2(\tau(x,y),x,y))$. The map ψ^{-1} is differentiable if τ is and τ is differentiable if

$$\frac{\partial\varphi^1(t,x,y)}{\partial t}|_{t=\tau(x,y)} \neq 0.$$

But

$$\frac{\partial\varphi^1(t,x,y)}{\partial t}|_{t=\tau(x,y)} = X_1(P_s(f)(\tau(x,y),x,y)) = X_1(0,\phi^2) = 1,$$

where the last equality holds because f is a diffeomorphism that agrees with identity at $\partial I^2 \backslash I_1$.
Then one can check that H_s is our desired homotopy. □

Theorem 3.4. $\mathrm{Diff}_{\partial,U}(I^2)$ is contractible.

Proof. Let $\mathcal{F}$ be the space of diffeomorphisms of I into I which agree with id_I near ∂I. Let $K_s : \mathcal{F} \to \mathcal{F}$ be defined by

$$K_s(f)(t) = f(t)s + t(1-s).$$

We see that $K_s(f)(t)$ is smooth in s and t, and satisfies

$$\begin{aligned} K_0(f)(t) &= t,\\ K_1(f) &= f,\\ K_s(\mathrm{id}_I) &= \mathrm{id}_I. \end{aligned}$$

Let H_s be as in Lemma 3.2. We define $h_s = H_s(h)$ for each $h \in \mathcal{E}$. Let $\bar{h}_s = h_s|_{I_1}$. Let $\beta(t)$ be a smooth function of t such that $\beta(t) = 0$ in a neighborhood of 0, $\beta(t) = 0$ in a neighborhood of 1.
We construct the homotopy $G_s : \mathcal{E} \to \mathcal{E}$ as follows.

$$G_s(h)(t,x) = (t', [K_{\beta(t')}(\bar{h}_s^{-1})](x')),$$

where t' and x' are the t and x components of $h_s(t,x)$, respectively. The map $(t,x) \to (t',x')$ is a diffeomorphism because h_s is. And

$$(t',x') \mapsto (t', [K_{\beta(t')}(\bar{h}_s^{-1})](x'))$$

is a diffeomorphism[4]. So the composition $G_s(h)$ is also a diffeomorphism, and moreover satisfying the conditions

$$\begin{aligned} G_0(h) &= \mathrm{id}_{I^2}, \\ G_1(h) &= h, \end{aligned}$$

[4] *Why?*

for all $h \in \mathcal{E}$. So we have constructed a homotopy G_s connecting id_{I^2} and h. The last thing worth observing is that,

$$\begin{aligned} G_s(\mathrm{id}_{I^2})(t,x) &= (t, [K_{\beta(t)}(\mathrm{id}_{I_1}^{-1})](x)) \\ &= (t, \mathrm{id}_{I_1}^{-1}(x)\beta(t) + x(1-\beta(t))) \\ &= (t, x\beta(t) + x(1-\beta(t))) \\ &= (t,x), \end{aligned}$$

which is equivalent to saying that

$$G_s(\mathrm{id}_{I^2}) = \mathrm{id}_{I^2}.$$

Since $h \in \mathcal{E}$ is identical to id_{I^2} in some neighborhood of ∂I^2, so is $G_s(h)$. Thus we have shown that $G_s(h)$ lies $\mathcal{E}$ for all $s \in [0,1]$ and $h \in \mathcal{E}$. Finally note that $\mathrm{Diff}_{\partial,U}(I^2) \subset \mathcal{E}$, we restrict G_s on $\mathrm{Diff}_{\partial,U}(I^2)$, getting the desired homotopy that contracting $\mathrm{Diff}_{\partial,U}(I^2)$ to $\{\mathrm{id}_{I^2}\}$. □

As an application of Smale's Theorem 3.4, we prove the following theorem, which is the main result of this section.

Theorem 3.5. We have that

$$\mathrm{Diff}(S^2) \simeq O(3).$$

Proof. We prove the theorem by showing that $\mathrm{Diff}(S^2)$ is a product of $O(3)$ and a topological group similar to $\mathrm{Diff}_\partial(D^2)$, where the latter is obviously homeomorphic to the group $\mathrm{Diff}_{\partial,U}(I^2)$. The proof follows the lines of [4, pp. 65].
We start with some standard observations. We consider S^2 as the subspace of $\mathbb{R}^3$ given by $\left\{ x \in \mathbb{R}^2 \;\middle|\; \|x\| = 1 \right\}$. Its tangent space at x may thus be identified with the subspace of $\mathbb{R}^3$ orthogonal to x. This means that the orthogonal frame bundle $\mathrm{Fr}^O(TS^2)$ can be identified by ordered triples of orthonormal vectors in $\mathbb{R}^3$, which admits a freely transitive action of $O(3)$. So we may identify $\mathrm{Fr}^O(TS^2)$ with $O(3)$.
The orthonormal frame bundle $\mathrm{Fr}^O(TS^2)$ is homotopy equivalent to the general

linear frame bundle $\mathrm{Fr}^{GL}(TS^2)$, which consists of ordered triples (x_1, x_2, x_3) with $x_i, i = 1, 2, 3$ points of $\mathbb{R}^3$ such that $x_1 \in S^2$ and x_1, x_2 are linearly independent vectors and orthogonal to x_1. The Gram-Schmidt procedure gives a linear continuous map

$$\begin{aligned} GS : \mathrm{Fr}^{GL}(TS^2) &\to \mathrm{Fr}^{O}(TS^2), \\ (x_1, x_2, x_3) &\mapsto \left(x_1, \frac{x_2}{\|x_2\|}, \frac{x_3 - \langle x_2, x_3 \rangle \frac{x_2}{\|x_2\|^2}}{\left\| x_3 - \langle x_2, x_3 \rangle \frac{x_2}{\|x_2\|^2} \right\|} \right), \end{aligned}$$

which is easily seen to be a homotopy equivalence.
A diffeomorphism $f \in \mathrm{Diff}(S^2)$ acts on $\mathrm{Fr}^{GL}(TS^2)$ by acting on the base S^2. Applying the map $GS : \mathrm{Fr}^{GL}(TS^2) \to \mathrm{Fr}^{O}(TS^2)$ and identifying the orthonormal frame bundle with $O(3)$, we see that there is a map

$$\begin{aligned} \alpha : \mathrm{Diff}(S^2) &\to O(3), \\ f &\mapsto GS(f(e_1), D_{e_1} f(e_2), D_{e_1} f(e_3)), \end{aligned}$$

where e_1, e_2, e_3 are the unit direction vectors of x_1, x_2, x_3 directions, respectively.
On the other hand, there is a map $O(3) \to \mathrm{Diff}(S^2)$ by rotating the sphere, which is a section of the previous map α:

$$\begin{tikzcd} G \coloneqq \ker(g) \arrow[r] & \mathrm{Diff}(S^2) \arrow[r, "\alpha"] \arrow[ddr, "g"] & \mathrm{Fr}^{GL}(TS^2) \arrow[d, "GS"] \\ & & \mathrm{Fr}^{O}(TS^2) \arrow[d, "\simeq"] \\ & O(3) \arrow[uu, "\text{rotation}"] \arrow[r, equal] & O(3) \end{tikzcd} .$$

We conclude that there is a homeomorphism

$$\mathrm{Diff}(S^2) \simeq O(3) \times G,$$

and note that G is the subgroup of $\mathrm{Diff}(S^2)$ consisting of diffeomorphisms which fix e_1 and the frame $(e_2, e_3) \in T_{e_1} S^2$ up to an upper triangular matrix with positive diagonal entries. There is an inclusion of $\mathrm{Diff}_\partial D^2 \simeq \mathrm{Diff}_{\partial, u}(I^2)$ into G by acting on the hemisphere around $-e_1$. The inclusion is in fact a weak equivalence, see [4, Chap. 8]. Thus G contracts to a point, viewed as a subspace of $\mathrm{Diff}_{\partial, u}(I^2)$, by Smale's Theorem 3.4. □

A The Poincaré-Bendixson Theory

Recall that a **Jordan curve** J is defined as a topological image of a circle, or namely, $J : [a, b] \to \mathbb{R}^2$ is a continuous map such that $J(a) = J(b)$ and $J(s) \neq J(t)$ for all $a \leqslant s < t < b$.

Theorem A.1 (Jordan)**.** If J is a plane Jordan curve, then its complement in the plane is the union of two disjoint connected open sets E_1 and E_2, each having J as its boundary.

One of the sets E_1 or E_2 is called the **interior** of J, furthermore the interior of J is simply connected.

Proof. See [12, pp.115] □

Consider a continuous curve $J : [a, b] \to \mathbb{R}^2$. Let $J(t) = (x(t), y(t))$,and let $\eta(x, y) = (\eta^1(x, y), \eta^2(x, y))$ be a vector field on $\mathbb{R}^2$ that doesn't vanish on J. Consider the angle $\varphi = \varphi(t)$ from the positive x direction $(1, 0)$ to $\eta(J(t))$, so that $\cos\varphi = \eta^1/\|\eta\|, \sin\varphi = \eta^2/\|\eta\|$, where $\|\eta\|^2 = (\eta^1)^2 + (\eta)^2$.

We define $\mathrm{ind}_J(\eta)$ by

$$2\pi \mathrm{ind}_J(\eta) := \varphi(b) - \varphi(a). \tag{A.1}$$

In particular, if η is continuously differentiable, we can compute $\mathrm{ind}_J(\eta)$ using the formula

$$2\pi \mathrm{ind}_J(\eta) = \int_a^b \frac{\eta^1\, d\eta^2 - \eta^2\, d\eta^1}{\|\eta\|^2}.$$

Assume that J_1 and J_2 are two plane curves, with the end of J_1 being the start of J_2. We may define the sum $J_1 + J_2$ to be the concatenation of J_1 and J_2. It's easy to check that

$$\mathrm{ind}_{J_1+J_2}(\eta) = \mathrm{ind}_{J_1}(\eta) + \mathrm{ind}_{J_2}(\eta) \tag{A.2}$$

holds directly from the definitions.

The main interest will be in the case that J is a Jordan curve, in which case it will always be assumed that J is positively oriented and $0 \notin J$, it is clear that $\mathrm{ind}_J(\eta)$ is an integer. It is called the **index** of J.

Theorem A.2 (Umlaufsatz)**.** Let $J : [0, 1] \to \mathbb{R}^2$ be a positively oriented Jordan curve of class C^1 and η be the tangent vector field on J so that η is non-vanishing on J. Then $\mathrm{ind}_J(\eta) = 1$.

Proof. For $0 \leqslant s \leqslant t \leqslant 1$, we define $\eta(s, t) := (J(t) - J(s))/\|J(t) - J(s)\|$ if $s \neq t$ or $(s, t) \neq (0, 1)$, and define $\eta(t, t) := J'(t)/\|J'(t)\|$, and $\eta(0, 1) := -\eta(0, 0)$. It's clear that $\eta(s, t)$ is continuous and $\eta(s, t) \neq 0$ for $0 \leqslant s \leqslant t \leqslant 1$.
Suppose that the point $J(0)$ on J is chosen so that the tangent line through $J(0)$ is parallel to the x-axis and no part of J lies below this tangent line. Since the solid triangle defined by vertices $J(0), J(s), J(t)$ is a simply connected area, it is possible to define a continuous function $\varphi(s, t)$ such that $\varphi(0, 0) = 0$ and $\varphi(s, t)$ is an angle from the positive x-direction to $\eta(s, t)$. Then $2\pi \mathrm{ind}_J(\eta) = \varphi(1, 1) - \varphi(0, 0)$, as can be seen by considering $\varphi(t, t)$.
The position of J implies that $0 \leqslant \varphi(0, t) \leqslant \pi$ and that $\varphi(0, 1)$ is an odd multiple of π, hence $\varphi(0, 1) = \pi$. Similarly, a consideration of $\varphi(s, 1) - \varphi(0, 1) = \varphi(s, 1) - \pi$ for $0 \leqslant s \leqslant 1$ shows that $\varphi(1, 1) - \pi = \pi$. Similarly, a consideration of $\varphi(s, 1) - \varphi(0, 1) = \varphi(s, 1) - \pi$ for $0 \leqslant s \leqslant 1$ shows that $\varphi(1, 1) - \pi = \pi$. So $\varphi(1, 1) = 2\pi$ and $2\pi \mathrm{ind}_J(\eta) = \varphi(1, 1) - \varphi(0, 0) = 2\pi$, completing the proof. □

In what follows, $f = (f_1, f_2)$ is continuous vector field on an open plane set U. A point where $f = 0$ is called a **stationary point** and a point where $f \neq 0$ is a **regular point**.

Lemma A.3. Let J_0 and J_1 be two Jordan curves in U which can be deformed into one another in U without passing a stationary point of f. Then $\mathrm{ind}_{J_0}(f) = \mathrm{ind}_{J_1}(f)$.

Proof. By assumption, there exists a continuous map $h(t,s) : [a,b] \times [0,1] \to U$, such that

(i) for a fixed s, $h(t,s) : [a,b] \to U$ is a Jordan curve;

(ii) $h(-,0) = J_0, h(-,1) = J_1$; and

(iii) $f(h(t,s)) \neq 0$ for all $(t,s) \in [a,b] \times [0,1]$.

Then let $j(s)$ be the index of the vector field f around the Jordan curve $h(-,s) : [a,b] \to U$. $j(s)$ is clearly a continuous function of s. However, on the other hand, $j(s)$ is an integer since each $h(-,s)$ is a closed curve in U. This forces $j(s)$ to be constant, thus $j(0) = j(1)$, or $\mathrm{ind}_{J_0}(f) = \mathrm{ind}_{J_1}(f)$. □

Corollary A.4. Let J be a positively oriented Jordan curve in U such that the interior of J is contained in U and that f is non-vanishing and inside J, then $\mathrm{ind}_J(f) = 0$.

Proof. Since the interior of J is simply connected, J can be deformed into a small circle J_1 that is contained in its interior. By assumption, f is non-vanishing inside and on J_1. Since J_1 is sufficiently small, the angle between f and the positive x-direction can be made arbitrary small. On the other hand, $\mathrm{ind}_{J_1}(f)$ is an integer since J_1 is a closed curve. Thus $\mathrm{ind}_{J_1}(f) = 0$. By Lemma A.3, $\mathrm{ind}_J(f) = \mathrm{ind}_{J_0}(f) = 0$. □

Before stating the following powerful theorems, we need to introduce some notations. Let us consider the ordinary differential equation

$$z' = f(z) \tag{A.3}$$

with $z = (x,y) \in \mathbb{R}^2$ and $f(z) = (f_1(z), f_2(z))$ be a continuous vector field on $\mathbb{R}^2$. If the Equation (A.3) has a solution $\gamma_+(t)$ defined on $\mathbb{R}_+$, its set $\Omega(\gamma_+)$ of ω-**limit points** is the set of points z_0 for which there exists a sequence $0 < t_0 < t_1 < \ldots$ such that $t_n \to \infty$ and $\gamma_+(t_n) \to z_0$ as $n \to \infty$.

The next theorem is the foundation of the Poincaré-Bendixson theory, which shows that the stationary points of the vector field f control the asymptotic behavior of the solution γ_+ of (A.3).

Theorem A.5. Let f be a continuous vector field on an open plane set U and let $\gamma_+(t), 0 \leqslant t < \infty$ be a solution of

$$z' = f(z)$$

for $t \geqslant 0$ with a compact closure in U. In addition, suppose that $\gamma_+(t_1) \neq \gamma_+(t_2)$ for $0 \leqslant t_1 < t_2 < \infty$ and that $\Omega(\gamma_+)$ contains no stationary points. Then $\Omega(\gamma_+)$ is the set of points x on a periodic solution $\gamma_p(t)$ of (A.3). Furthermore, if $p > 0$ is the smallest period of $\gamma_p(t)$, then $\gamma_p(t_1) \neq \gamma_p(t_2)$ for $0 \leqslant t_1 < t_2 < p$, that is, the curve $\gamma_p(t), 0 \leqslant t \leqslant p$ is a Jordan curve.

Although Theorem A.5 is deep and powerful, its proof is not much insightful. The proof of Theorem A.5 lies heavily on the Jordan Curve Theorem A.1 and the existence theory of ODE's. The detailed proof can be found in [7, pp. 152], which accounts for a painful reading.

The next theorem is a stronger version of Theorem A.5, shows that the assumption in Theorem A.5 that γ_+ is not self-intersecting can be essentially removed.

Theorem A.6 (Poincaré-Bendixson)**.** Let f be continuous on an open plane set U and let γ_+ a solution of (A.3) for $t \geqslant 0$ with compact closure in U. Then $\Omega(\gamma_+)$ contains a closed periodic orbit γ_p of (A.3) which can reduce to a stationary point.

Proof. See [7, pp. 154] □

However, what we actually use in the proof of Smale's Theorem 3.4 is the following corollary of the Poincaré-Bendixson TheoremA.6.

Corollary A.7. Let $f(x)$ be continuous on an open set U and let γ_p be a solution of (A.3) of period p, whose existence is assured by the Poincaré-Bendixson Theorem A.6. If $\gamma_p(t), 0 \leqslant t \leqslant p$ is in addition a Jordan curve with an interior I contained in U with f non-vanishing on γ_p, then I contains at least one stationary point of f.

Proof. Suppose that I contains no stationary point of f, then f is non-vanishing on and inside γ_p. Thus $\mathrm{ind}_{\gamma_p}(f) = 0$, by Corollary A.4. On the other hand, however, since γ_p is a periodic solution of (A.3), f is the tangent vector field of the Jordan curve γ_p. Thus by the Umlaufsatz A.2, we have $\mathrm{ind}_{\gamma_p}(f) = 1$, a contradiction. □

References

[1] Diffeomorphism group in nLab. [Online]. Available: https://ncatlab.org/nlab/show/diffeomorphism+group

[2] A. Hatcher, "A 50 -Year View of Diffeomorphism Groups," p. 11.

[3] S. Smale, "Diffeomorphisms of the 2-Sphere," vol. 10, no. 4, p. 621.

[4] A. Kupers, *Lectures on Diffeomorphism Groups of Manifolds*. [Online]. Available: http://people.math.harvard.edu/~kupers/teaching/272x/book.pdf

[5] A. Candel and L. Conlon, *Foliations*, ser. Graduate Studies in Mathematics. American Mathematical Society, no. v. 23, 60.

[6] M. W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, ser. Pure and Applied Mathematics; a Series of Monographs and Textbooks. Academic Press, no. v. 60.

[7] P. Hartman, *Ordinary Differential Equations*, 2nd ed., ser. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, no. 38, this SIAM edition is an unabridged, corrected republication of the edition published by Birkhäuser, Boston, Basel, Stuttgart, 1982. The original edition was published by John Wiley & Sons, New York, 1964."–T.p. verso.

[8] M. W. Hirsch, *Differential Topology*, 6th ed., ser. Graduate Texts in Mathematics. Springer, no. 33.

[9] C. T. C. Wall, *Differential Topology*, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, no. 156.

[10] M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, ser. Graduate Texts in Mathematics. Springer US, no. 14. [Online]. Available: http://link.springer.com/10.1007/978-1-4615-7904-5

[11] J. L. Kelley, *General Topology*, ser. Graduate Texts in Mathematics. Springer-Verlag, no. 27, reprint of the ed. published by Van Nostrand, New York in The University series in higher mathematics Includes index.

[12] M. H. Newman, *Elements of the Topology of Plane Sets of Points*, 2nd ed. Cambridge University Press.