Assignment of Algebra II

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Exercise 1

Given functors $\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{G} \mathbf{E}$, if two of F, G and GF are equivalences of categories, then so is the third one.

Proof. Assume that F and G are equivalences of categories, we want to show that GF is also an equivalence. By assumption, there are functors $H : \mathbf{D} \to \mathbf{C}$ and $I : \mathbf{E} \to \mathbf{D}$, such that

$$\begin{aligned} \mathrm{id}_{\mathbf{C}} &\simeq HF, \\ FH &\simeq \mathrm{id}_{\mathbf{D}}, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \mathrm{id}_{\mathbf{D}} &\simeq IG, \\ GI &\simeq \mathrm{id}_{\mathbf{E}}. \end{aligned} \tag{2}$$

Then we compose (2) with H, F, we get

$$HIGF \simeq HF \simeq \mathrm{id}_{\mathbf{C}}.$$

Similarly compose (1) with G, I, we have

$$GFHI \simeq GI \simeq \mathrm{id}_{\mathbf{E}},$$

which shows that $GF : \mathbf{C} \to \mathbf{E}$ is an equivalence.

Now suppose that $F : \mathbf{C} \to \mathbf{D}$ and $GF : \mathbf{C} \to \mathbf{E}$ are equivalences of categories. So there are functors $H : \mathbf{D} \to \mathbf{C}$ and $K : \mathbf{E} \to \mathbf{C}$ such that

$$\begin{aligned} \mathrm{id}_{\mathbf{C}} &\simeq KGF, \\ GFK &\simeq \mathrm{id}_{\mathbf{E}}, \end{aligned} \tag{3}$$

and (1) hold. We already have $G(FK) = \mathrm{id}_{\mathbf{E}}$, *id est*, G has a right inverse FK, what left to us is to show that FK is also a right inverse of G. By composing H to (3) and using 1, we have

$$H \simeq KGFH \simeq KG.$$

Thus,

$$(FK)G = F(KG) \simeq FH \simeq \mathrm{id}_{\mathbf{D}},$$

showing that $G : \mathbf{D} \to \mathbf{E}$ is also an equivalence.

Finally, if we are given $G : \mathbf{D} \to \mathbf{E}$ and $GF : \mathbf{C} \to \mathbf{E}$ being equivalences of categories, then (2) and (3) hold. By composing (3) with I, we have

$$I \simeq IGFK \simeq FK.$$

So

$$F(KG) = (FK)G \simeq IG \simeq \mathrm{id}_{\mathbf{D}},$$

showing that KG is a quasi-inverse of F, compeleting the proof.

□

Exercise 2

Let $\mathbf{C}$ be the following category

$$\begin{tikzcd} x \arrow[loop left] \arrow[r, shift left] & y \arrow[l, shift left] \arrow[loop right] \end{tikzcd}.$$

Let $F : \mathbf{C} \to \mathbf{C}$ be the functor determined by the object map $x, y \to x$. Is F faithful? Is F full?

Proof. Since $\mathrm{Hom}_{\mathbf{C}}(x,x) = \mathrm{Hom}_{\mathbf{C}}(x,y) = \mathrm{Hom}_{\mathbf{C}}(y,x) = \mathrm{Hom}_{\mathbf{C}}(y,y) = \{*\}$, the map $\mathrm{Hom}_{\mathbf{C}}(*,*) \to \mathrm{Hom}_{\mathbf{C}}(F*, F*)$ is bijective. Thus F is fully faithful. □

Exercise 3

Prove that, in the category R-**Mod**, the pullback diagram

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd} \tag{4}$$

is also a pushout with respect to (p_1, p_2) if and only if $(f, g) : A \oplus B \to C$ is surjective.

Proof. Since we are in the category R-**Mod**, we can write things more explicitly. In R-**Mod**, the diagram (4) is a pullback iff P is the R-module defined by

$$P := \{ (a, b) \in A \oplus B \mid f(a) = g(b) \},$$

as a submodule of $A \oplus B$. But this holds iff the sequence

$$0 \longrightarrow P \xrightarrow{p_1 \oplus p_2} A \oplus B \xrightarrow{f-g} C$$

is exact.

Similarly, the diagram (4) is a pushout iff C is the quotient of $A \oplus B$

$$C = (A \oplus B)/\sim$$

with $f(p_1(p)) \sim g(p_2(p))$ for all $p \in P$. But this happens iff the sequence

$$P \xrightarrow{p_1 \oplus p_2} A \oplus B \xrightarrow{f-g} C \longrightarrow 0$$

is exact.

So the pullback diagram (4) is also a pushout iff the exact sequence

$$0 \to P \xrightarrow{p_1 \oplus p_2} A \oplus B \xrightarrow{f-g} C \longrightarrow 0$$

is exact, iff $(f, g) : A \oplus B \to C$ is surjective. □

Exercise 4

The center $Z(\mathbf{C})$ of a category $\mathbf{C}$ is the class of all natural transformations $\alpha : \mathrm{id}_{\mathbf{C}} \to \mathrm{id}_{\mathbf{C}}$, where $\mathrm{id}_{\mathbf{C}}$ is the identity functor on $\mathbf{C}$. Let R be a ring with identity and put $\mathbf{C} = R$-$\mathbf{Mod}$. Prove that there is a bijection $Z(R) \to Z(\mathbf{C})$, where R is the center of R, that is, $Z(R) = \{z \in R | zr = rz, \forall r \in R\}$.

Proof. $Z(R) \subseteq Z(\mathbf{C})$. For any central element $c \in Z(R)$, we can define a natural transformation η_c, defined via $\eta_c(M) : M \to M, x \mapsto cx$ for all R-module M. Indeed, for any morphism $f : M \to N$ of R-modules, we have the commutative diagram

$$\begin{tikzcd} M \arrow[r, "c\cdot(-)"] \arrow[d, "f"'] & M \arrow[d, "f"] \\ N \arrow[r, "c\cdot(-)"] & N \end{tikzcd},$$

thus complete the inclusion.

$Z(R) \supseteq Z(\mathbf{C})$. Suppose we have a natural transformation $\eta : \mathrm{id}_{\mathbf{C}} \to \mathrm{id}_{\mathbf{C}}$, we want to construct a central element of R from η. In particular, R is a right R-module, thus we may consider the commutative diagram

$$\begin{tikzcd} R \arrow[r, "\eta_R"] \arrow[d, "f"'] & R \arrow[d, "f"] \\ R \arrow[r, "\eta_R"] & R \end{tikzcd}, \tag{5}$$

where $f : R \to R$ is an arbitrary morphism between R-modules. However, since $\mathrm{Hom}_R(R, R) \simeq R$, we know that f must be of the form $r \mapsto x \cdot r$ for some $x \in R$. So by the diagram (5), we have

$$\eta_R(f(1)) = f(\eta_R(1)).$$

But the right hand side of the above equation equals $x \cdot \eta_R(1)$, while the left hand side equals $\eta_R(x \cdot 1) = \eta_R(x)$, which shows that $\eta_R(x) = x\eta_R(1)$. It's easy to verify that $\eta_R(1)$ is central. Thus given a natural transformation $\eta : \mathrm{id}_{\mathbf{C}} \to \mathrm{id}_{\mathbf{C}}$, it's we take the central element to be $\eta_R(1)$. □

Exercise 5

Show that a category $\mathbf{C}$ admits small limits iff $\mathbf{C}$ admits equalizers and products.

Proof. $\Rightarrow$ Since equalizers and products are all small limits, this direction is obvious.

$\Leftarrow$ Now suppose that $\mathbf{C}$ is a category admitting small limits. Let $F : \mathbf{J} \to \mathbf{C}$ be any small diagram, the limit $\lim_{\mathbf{J}} F$ exists by assumption. For each morphism f in the indexing category $\mathbf{J}$, there is a commutative diagram in $\mathbf{C}$

$$\begin{tikzcd} & \lim_{\mathbf{J}} F \arrow[dl, "\lambda_{\mathrm{dom} f}"'] \arrow[dr, "\lambda_{\mathrm{cod} f}"] & \\ F(\mathrm{dom} f) \arrow[rr, "F(f)"] & & F(\mathrm{cod} f) \end{tikzcd},$$

where $(\lambda_j)_{j \in \mathbf{J}}$ are the structure maps the limit cone, and $\mathrm{dom} f, \mathrm{cod} f$ are the domain and codomain of the morphism f, respectively. From the diagram above, we know that the defining relations of the limit cone are

$$(Ff) \circ \lambda_{\mathrm{dom} f} = \lambda_{\mathrm{cod} f}, \tag{6}$$

with $f \in \operatorname{mor} \mathbf{J}$ varying. Now consider the diagram

$$\begin{tikzcd} & & F(\operatorname{cod} f) \\ \prod_{j\in \operatorname{ob}\mathbf{J}} Fj \arrow[urr, "\pi_{\operatorname{cod} f}"] \arrow[rr, shift left, "c"] \arrow[rr, shift right, "d"'] \arrow[d, "\pi_{\operatorname{dom} f}"] & & \prod_{f\in \operatorname{mor}\mathbf{J}} F(\operatorname{cod} f) \arrow[u, "\pi_f"'] \arrow[d, "\pi_f"] \\ F(\operatorname{dom} f) \arrow[rr, "Ff"] & & F(\operatorname{cod} f) \end{tikzcd} , \tag{7}$$

where $\pi_f : \prod_{f\in \operatorname{mor}\mathbf{J}} F(\operatorname{cod} f)$ are the projection morphisms defining the product $\prod_{f\in \operatorname{mor}\mathbf{J}} F(\operatorname{cod} f)$, while $\pi_{\operatorname{dom} f} : \prod_{j\in \operatorname{ob}\mathbf{J}} Fj \to F(\operatorname{dom} f)$ are the projection morphisms defining the product $\prod_{j\in \operatorname{ob}\mathbf{J}} Fj$. By the universal property

$$\operatorname{Hom}_{\mathbf{C}}\Big(\prod_{j\in \operatorname{ob}\mathbf{J}} Fj, \prod_{f\in \operatorname{mor}\mathbf{J}} F(\operatorname{cod} f)\Big) \simeq \prod_{f\in \operatorname{mor}\mathbf{J}} \operatorname{Hom}_{\mathbf{C}}\Big(\prod_{j\in \operatorname{ob}\mathbf{J}} Fj, F(\operatorname{cod} f)\Big)$$

of product, the morphisms $c : \prod_{j\in \operatorname{ob} J} Fj \to \prod_{f in \operatorname{mor}\mathbf{J}} F(\operatorname{cod} f)$ and $d : \prod_{j\in \operatorname{ob} J} Fj \to \prod_{f in \operatorname{mor}\mathbf{J}} F(\operatorname{cod} f)$ in (7) are uniquely determined by the morphisms $(\pi_{\operatorname{cod} f})_{f\in \operatorname{mor}\mathbf{J}}$ and $((Ff) \circ \pi_{\operatorname{dom} f})_{f\in \operatorname{mor}\mathbf{J}}$ respectively. Also notice that the diagram (7) can be augmented as

$$\begin{tikzcd} & & & F(\operatorname{cod} f) \\ \lim_{\mathbf{J}} F \arrow[r] & \prod_{j\in \operatorname{ob}\mathbf{J}} Fj \arrow[urr, "\pi_{\operatorname{cod} f}"] \arrow[rr, shift left, "c"] \arrow[rr, shift right, "d"'] \arrow[d, "\pi_{\operatorname{dom} f}"] & & \prod_{f\in \operatorname{mor}\mathbf{J}} F(\operatorname{cod} f) \arrow[u, "\pi_f"'] \arrow[d, "\pi_f"] \\ & F(\operatorname{dom} f) \arrow[rr, "Ff"] & & F(\operatorname{cod} f) \end{tikzcd} , \tag{8}$$

where the morphism $\lim_{\mathbf{J}} F \to \prod_{j\in \operatorname{ob}\mathbf{J}} Fj$ is uniquely defined by the morphisms $\lambda_j : \lim_{\mathbf{J}} F \to Fj, j \in \mathbf{J}$ by the universal property of $\prod_{j\in \operatorname{ob}\mathbf{J}} Fj$. However, the relations (6) tells us that the diagram (8) is commutative, saying that there is a unique morphism $\lim_{\mathbf{J}} F \to \operatorname{Eq}(c, d)$ to the equalizer $\operatorname{Eq}(c, d)$ of c, d. On the other hand, the morphism $\operatorname{Eq}(c, d) \to \lim_{j\in \operatorname{ob}\mathbf{J}} Fj$ uniquely defines a morphism $\operatorname{Eq}(c, d) \to \lim_{\mathbf{J}} F$, by the universal property of $\lim_{\mathbf{J}} J$. By the uniqueness of the limits, $\lim_{\mathbf{J}}$ must be isomorphic to $\operatorname{Eq}(c, d)$.

In summary, we have shown that the diagram

$$\begin{tikzcd} \lim_{\mathbf{J}} F \arrow[r] & \prod_{j\in \operatorname{ob}\mathbf{J}} Fj \arrow[r, shift left, "c"] \arrow[r, shift right, "d"'] & \prod_{j\in \operatorname{mor}\mathbf{J}} F(\operatorname{cod} f) \end{tikzcd}$$

is an equilizer diagram, *id est*, any small limit can be constructed as an equalizer of morphisms between some products, completing the proof. □

Exercise 6

Let $\mathbf{C}$ be a category and $X \in \mathbf{C}$ an object. Prove that the functor $\operatorname{Hom}_{\mathbf{C}}(X, -) : \mathbf{C} \to \mathbf{Set}$ preserves limits.

Proof. Let $F : \mathbf{I} \to \mathbf{C}$ be any small diagram and $\nu : \varprojlim F \to F\mathbf{I}$ be the limit cone. Apply the functor $\operatorname{Hom}_{\mathbf{C}}(X, -)$ to the limit cone $\nu : \varprojlim F \to F\mathbf{I}$, we obtain a cone $\nu_* := \operatorname{Hom}_{\mathbf{C}}(X, \nu) : \operatorname{Hom}_{\mathbf{C}}(X, \varprojlim F) \to \operatorname{Hom}_{\mathbf{C}}(X, F\mathbf{I})$ in $\mathbf{Set}$. We need to show that ν_* is also a limit cone in the category $\mathbf{Set}$.

Given an arbitrary cone $\tau : S \to \operatorname{Hom}_{\mathbf{C}}(X, F\mathbf{I})$ with apex $S \in \mathbf{Set}$ and base $\operatorname{Hom}_{\mathbf{C}}(X, F\mathbf{I})$, the image $\tau(s)$ of each element $s \in S$ is an element in $\operatorname{Hom}_{\mathbf{C}}(X, F\mathbf{I})$, *id est*, a cone $\tau(s) : X \to F\mathbf{I}$

in $\mathbf{C}$ with apex X and base $F\mathbf{I}$. By the universal property of the limit cone $\nu : \varprojlim F \to F\mathbf{I}$, there exists a unique morphism $\eta_s : X \to \varprojlim F$, making the diagram

$$\begin{tikzcd} X \arrow[d, dotted, "\eta_s"'] \arrow[dr, "\tau(s)"] & \\ \varprojlim F \arrow[r, "\nu"] & F\mathbf{I} \end{tikzcd}$$

commute. Since $\eta_s \in \mathrm{Hom}_{\mathbf{C}}(X, \varprojlim F)$, the assaignment $s \mapsto \eta_s$ defines a morphism $\eta : S \to \mathrm{Hom}_{\mathbf{C}}(X, \varprojlim F)$. By construction η is unique. So we have proved the universal property of the cone ν_*, which completes the proof. □

Exercise 7

Let $\phi : F \dashv G$. Then $\eta : \mathrm{id}_{\mathbf{C}} \to GF$ with $\eta_X = \phi(\mathrm{id}_{FX}) : X \to GF(X)$ is a natural transformation. Similarly, $\epsilon = FG \to \mathrm{id}_{\mathbf{D}}$, with $\epsilon_Y : \phi^{-1}(\mathrm{id}_{GY}) : FG(Y) \to Y$, is a natural transformation. Moreover, we have identities of natural transfromations

$$\epsilon F \circ F\eta = \mathrm{id}_F, G\epsilon \circ \eta G = \mathrm{id}_G. \tag{9}$$

Proof. Since $F \dashv G$, so for any $X \in \mathbf{C}$ and $Y \in \mathbf{D}$, we have a natural isomorphism

$$\phi_{XY} : \mathrm{Hom}_{\mathbf{D}}(FX, Y) \to \mathrm{Hom}_{\mathbf{C}}(X, GY).$$

Fixing X, the natural isomorphism

$$\phi_{X-} : \mathrm{Hom}_{\mathbf{D}}(FX, -) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{C}}(X, G(-))$$

is determined by an element of $\mathrm{Hom}_{\mathbf{C}}(X, GFX)$, the image of $\mathrm{id}_{F(X)}$ under the isomorphism $\phi_{X,FX}$. Such an assignment $X \mapsto \eta_X = \phi_{X,FX}(\mathrm{id}_{FX})$ defines a natural transformation η, in which the naturality can be seen in the following way.

To prove that η is natural, we must show that the square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & GFX \arrow[d, "GFf"] \\ X' \arrow[r, "\eta_{X'}"] & GFX' \end{tikzcd}$$

commutes for any $f : X \to X'$ in $\mathbf{C}$. But this is equivalent to the commutativity of the obvious square

$$\begin{tikzcd} FX \arrow[r, "\mathrm{id}_{FX}"] \arrow[d, "Ff"] & FX \arrow[d, "Ff"] \\ FX' \arrow[r, "\mathrm{id}_{FX'}"] & FX' \end{tikzcd}.$$

Dually, fixing $Y \in \mathbf{D}$, the natural isomorphism $\mathrm{Hom}_{\mathbf{C}}(-, GY) \simeq \mathrm{Hom}_{\mathbf{D}}(F(-), Y)$ is determined by an element of $\mathrm{Hom}_{\mathbf{D}}(FGY, Y)$, the preimage of id_{GY} of the isomorphism $\phi_{GY,Y}$, by the Yoneda lemma. The assignment $Y \mapsto \eta_Y := \phi^{-1}_{GY,Y}(\mathrm{id}_{GY})$ defines a natural transformation $\eta : FG \to \mathrm{id}_{\mathbf{D}}$ in a similar way.

Finally we have to verify the identities (9). It suffices to verify them pointwise, since natural transformations are pointwise defined. So for the first identity

$$\epsilon F \circ F\eta = \mathrm{id}_F,$$

we wan to show that for any $X \in \mathbf{C}$, the diagram

$$\begin{tikzcd} FX \arrow[r, "\mathrm{id}_{FX}"] \arrow[d, "F\eta_X"] & FX \arrow[d, "\mathrm{id}_{FX}"] \\ FGFX \arrow[r, "\eta_{FX}F"] & FX \end{tikzcd}$$

commutes. But this is equivalent to the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "\eta_X"] & GFX \arrow[d, "\mathrm{id}_{GFX}"] \\ GFX \arrow[r, "\mathrm{id}_{GFX}"] & GFX \end{tikzcd},$$

which commutes manifestly. The second identity

$$G\epsilon \circ \eta G = \mathrm{id}_G$$

lies on the commutativity of the diagram

$$\begin{tikzcd} GY \arrow[r, "\eta_{GY}"] \arrow[d, "\mathrm{id}_{GY}"] & GFGY \arrow[d, "G\epsilon_Y"] \\ GY \arrow[r, "\mathrm{id}_{GY}"] & GY \end{tikzcd}$$

for all $Y \in \mathbf{D}$, which holds because of the manifestly commutative diagram

$$\begin{tikzcd} FGY \arrow[r, "\mathrm{id}_{FGY}"] \arrow[d, "\mathrm{id}_{FGY}"] & FGY \arrow[d, "\epsilon_Y"] \\ FGY \arrow[r, "\epsilon_Y"] & Y \end{tikzcd}.$$

□

Exercise 8

Prove that

Lemma 1. In an additive category with kernels and cokernels, any morphism $f : A \to B$ factors uniquely as $A \overset{\alpha}{\twoheadrightarrow} \operatorname{coim} f \overset{\gamma}{\to} \operatorname{im} f \overset{\beta}{\hookrightarrow} B$.

Proof. Given any morphism $f : A \to B$ in an additive category $\mathbf{A}$, the kernel $\ker f \overset{f}{\to} A$ and cokernel $B \overset{g}{\to} \operatorname{coker} f$ exist, by assumption. Since $f \circ g = 0$, f must factor through $\operatorname{coker} g = \operatorname{coker} \ker f =: \operatorname{coim} f$ uniquely. Moreover, since any cokernel is an epimorphism, we denote such a factorization by $A \overset{\alpha}{\twoheadrightarrow} \operatorname{coim} f \overset{i}{\to} B$. Now consider the commutative diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & B \arrow[r, "h"] & \operatorname{coker} f \\ & \operatorname{coim} f \arrow[ur, "i"] \arrow[urr, "0"'] & & \end{tikzcd},$$

we have

$$h \circ i \circ \alpha = h \circ f = 0 = 0 \circ \alpha.$$

But note that α is epimorphic, then $(h \circ i) \circ \alpha = 0 \circ \alpha$ implies that

$$h \circ i = 0. \tag{10}$$

But (10) shows that i must factor through $\ker h = \ker \operatorname{coker} f =: \operatorname{im} f$ uniquely, which we denote by $i : \operatorname{coim} f \xrightarrow{\gamma} \operatorname{im} f \overset{\beta}{\hookrightarrow} B$. $\beta : \operatorname{im} f \to B$ is a monomorphism, because $\operatorname{im} f$ is by definition the kernel of h and any kernel is monomorphic. Finally, since every factorization we performed is unique, we are done. □

Exercise 9

Prove that

Lemma 2. Let $\mathbf{A}$ be an abelian category, then

(i) If a morphism is both a monomorphism and an epimorphism, then it is an isomorphism.

(ii) Every monomorphism is the kernel of its cokernel.

(iii) Every epimorphism is the cokernel of its kernel.

(iv) Every morphism $f : A \to B$ can be decomposed as $A \overset{g}{\twoheadrightarrow} \operatorname{im} f \overset{h}{\hookrightarrow} B$ with h monomorphism and g epimorphism.

Proof. (i) By Lemma 1, in the factorization $A \overset{\alpha}{\twoheadrightarrow} \operatorname{coim} f \xrightarrow{\gamma} \operatorname{im} f \overset{\beta}{\hookrightarrow} B$ of $f : A \to B$, if we can show that $\alpha : A \to \operatorname{coim} f$ and $\beta : \operatorname{im} f \to B$ are moth isomorphisms, we can then conclude that $f : A \to B$ is an isomorphism. By assumption, f is monomorphic, then $\ker f = 0$. By definition, $\operatorname{coim} f$ is the cokernel of $0 \to A$, that is, the pushout of the diagram

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d, "0"'] & A \\ A & \end{tikzcd}.$$

It is easy to verify that the diagram

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d, "0"'] & A \arrow[d, "\mathrm{id}_A"] \\ A \arrow[r, "\mathrm{id}_A"] & A \end{tikzcd}$$

is a pushout. So in this case $\operatorname{coim} f \simeq A$ and $\alpha \simeq \mathrm{id}_A$ being an isomorphism.

On the other hand $f : A \to B$ is also epimorphic by assumption, so $\operatorname{coker} f = 0$, and by definition $\operatorname{im} f$ is the kernel of $B \to 0$. The latter is equivalent to the pullback of the diagram

$$\begin{tikzcd} & B \arrow[d, "0"] \\ B \arrow[r, "0"'] & 0 \end{tikzcd},$$

which can be easily verified to be the identity morphism $\mathrm{id}_B : B \to B$. So we have showed that $\operatorname{im} f \simeq B$ and $\beta \simeq \mathrm{id}_B$. Thus the first assertion is proved.

(ii) Since $\mathbf{A}$ is an abelian category, for any monomorphism $f : A \to B$, the morphism $\operatorname{coim} f \to \operatorname{im} f$ is an ismorphism, by the axiom AB2. By the proof of (i), in the factorization

$A \overset{\alpha}{\twoheadrightarrow} \operatorname{coim} f \overset{\gamma}{\to} \operatorname{im} f \overset{\beta}{\hookrightarrow} B$ of f, we have $\operatorname{coim} f = A$ and $\alpha : A \to \operatorname{coim} f$ being isomorphic, by the monomorphicity of f. In this case we have an isomorphism $A \simeq \operatorname{im} f$. Since $\operatorname{im} f := \ker \operatorname{coker} f$, the assertion follows.

(iii) Dually, assume that $f : A \to B$ is epimorphic. Again by the proof of (i) we have an isomorphism $\beta : \operatorname{im} f \to B$. By AB2, $\operatorname{coim} f \to \operatorname{im} f$ is an isomrphism so we have an isomorphism $\operatorname{coim} f \simeq B$. Finally by definition we have $\operatorname{coim} f = \operatorname{coker} \ker f$.

(iv) Again consider the factorization $A \overset{\alpha}{\twoheadrightarrow} \operatorname{coim} f \overset{\gamma}{\to} \operatorname{im} f \overset{\beta}{\hookrightarrow} B$ of any $f : A \to B$. By AB2, $\gamma : \operatorname{coim} f \to \operatorname{im} f$ is an isomorphism, so we take $g := \gamma \circ \alpha$ and $h := \beta$. g is epimorphic since α is epimorphic. $\square$

Exercise 10

Let $\mathbf{C}$ be a category that admits colimits. We say that an object $X \in \mathbf{C}$ is of **finite type** if for all functor $F : \mathbf{I} \to \mathbf{C}$, the natural map

$$\varinjlim \operatorname{Hom}(X, F(-)) \to \operatorname{Hom}(X, (\varinjlim F)(-))$$

is injective, where $\mathbf{I}$ is a directed poset. Show that this definition coincides with the usual definition of finite type modules, when $\mathbf{C} = R\text{-}\mathbf{Mod}$.

Proof. Let I be a directed set and (α_i) a direct system in R-**Mod** indexed by I. We want to show that an R-module M is of finte type iff the map

$$\eta_M : \varinjlim \operatorname{Hom}_R(M, \alpha_i) \to \operatorname{Hom}_R(M, \varinjlim \alpha_i) \tag{11}$$

is injective.

$\Rightarrow$ If M is of finite type, then there exists some $n \in \mathbb{N}$ such that M is the quotient of the R-module R^n. So there must be an exact sequence

$$R^n \to M \to 0 \tag{12}$$

of R-modules. Since the functors $\operatorname{Hom}_R(-, \alpha_i)$ are contravariant left exact functors for all $i \in I$, we get exact sequences

$$0 \to \operatorname{Hom}_R(M, \alpha_i) \to \operatorname{Hom}_R(R^n, \alpha_i) \tag{13}$$

by applying the functors $\operatorname{Hom}_R(-, \alpha_i)$ to the exact sequence (12). Since taking direct limits is an exact functor in the catetory R-**Mod**, we have exact sequences

$$0 \to \varinjlim \operatorname{Hom}_R(M, \alpha_i) \to \varinjlim \operatorname{Hom}_R(R^n, \alpha_i) \tag{14}$$

We can also apply the functor $\operatorname{Hom}_R(-, \varinjlim \alpha_i)$ to the exact sequence (12), to get another exact sequence

$$0 \to \operatorname{Hom}_R(M, \varinjlim \alpha_i) \to \operatorname{Hom}_R(R^n, \varinjlim \alpha_i)$$

. Combining all these up, we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \varinjlim \operatorname{Hom}_R(M, \alpha_i) \arrow[r] \arrow[d, "\eta_M"] & \varinjlim \operatorname{Hom}_R(R^n, \alpha_i) \arrow[d, "\eta_{R^n}"] \\ 0 \arrow[r] & \operatorname{Hom}_R(M, \varinjlim \alpha_i) \arrow[r] & \operatorname{Hom}_R(R^n, \varinjlim \alpha_i) \end{tikzcd} \tag{15}$$

with exact rows, and rows the natural morphisms η_M, η_{R^n}. But look and behold, by the isomorphisms $\varinjlim \operatorname{Hom}_R(R^n, \alpha_i) \simeq \varinjlim(\operatorname{Hom}_R(R, \alpha_i))^n \simeq (\varinjlim \alpha_i)^n$ and $\operatorname{Hom}_R(R^n, \varinjlim \alpha_i) \simeq (\operatorname{Hom}_R(R, \varinjlim \alpha_i))^n \simeq$

$(\varinjlim \alpha_i)^n$, it's easy to check that the vertical natural monomorphism η_{R^n} is *de facto* the identity morphism $\mathrm{id}_{(\varinjlim \alpha_i)^n}$. Thus the diagram (15) tells us that (11) is a monomorphism.

$\Leftarrow$ Suppose that M is not of finite type, we prove that (11) is not injective. Although M is not of finite type, every element of M is contained in a submodule of finite type, say, the submodule generated by itself. Thus we may write M as the direct limit $M = \varinjlim M_i$, with M_i running over all submodules of finte type of M. Note that $(\alpha_i) := (M/M_i)$ also forms a direct system. For each i, let $\pi_i : M \to M/M_i$ be the natural projection, thus (p_i) defines an element in $\varinjlim \mathrm{Hom}_R(M, M/M_i)$. However, as $\varinjlim M/M_i = 0$, η_M is identically zero hence is not injective, as desired. ☐

Exercise 11

Prove that

Theorem 3. Suppose I is a countable set. Let $0 \to A_i \to B_i \to C_i \to 0$ be an exact sequence of inverse systems of R-modules over I. If (A_i, ϕ_{ij}) is Mittag-Leffler, then $0 \to \varprojlim A_i \to \varprojlim B_i \to \varprojlim C_i \to 0$ is exact.

Proof. Since taking limits is left exact, we already have an exact sequence of R-modules

$$0 \to \varprojlim A_i \to \varprojlim B_i \to \varprojlim C_i.$$

What left to us is to show that $\varprojlim B_i \to \varprojlim C_i$ is surjective, since in the cateory R-**Mod** the notions of surjections and epimorphisms coincide. Let $(c_i) \in \varprojlim C_i$. Before doing this, we may view each A_i as a submodule of B_i, and the transition maps of the directed system (B_i) as extensions of the transtion maps ϕ_{ij}. For this reason, we may denote the directed inverse system as (B_i, ϕ_{ij}).

For each $i \in I$, let $E_i := g_i^{-1}(c_i)$, which is non-trivial since $g_i : B_i \to C_i$ is surjective by assumption. Since $E_i \subseteq B_i$ are submodules, the restrictions of $\phi_{ij} : B_j \to B_i$ to $E_j \to E_i$ make (E_i) an inverse system of non-trivial R-modules. If we can show that (E_i) is Mittag-Leffler, then $\varprojlim E_i$ is non empty, and every element of $\varprojlim E_i$ can be viewed as an element of $\varprojlim B_i$ that is the preimage of (c_i), under the map $\varprojlim B_i \to \varprojlim C_i$.

Now we are to show that (E_i) is Mittag-Leffler, namely, for all $i \in I$ there exists $j \geq i$ such that $\phi_{ik}(E_k) = \phi_{ij}(E_j)$ for $k \geq j$. Since $\phi_{ik} = \phi_{ij} \circ \phi_{jk}$, we always have $\phi_{ik}(E_k) = \phi_{ij}(\phi_{jk}(E_k)) \subseteq \phi_{ij}(E_j)$. For the other direction, let $e_j \in E_j$, we need to fine $e_k \in E_k$ such that $\phi_{ij}(e_j) = \phi_{ik}(e_k)$. Let $e'_k \in E_k$ be any other element, and let $e'_j = \phi_{jk}(e'_k)$, then $g_j(e_j - e'_j) = g_j(e_j) - g_j(e'_j) = c_j - c_j = 0$. Hence $e_j - e'_j = a_j \in A_j$ for some a_j. By assumption, since (A_i, ϕ_{ij}) is Mittag-Leffler, there is some $a_k \in A_k$ such that $\phi_{ik}(a_k) = \phi_{ij}(a_j)$. Hence

$$\phi_{ik}(e_k) = \phi_{ik}(e'_k + a_k) = \phi_{ik}(e'_k) + \phi_{ik}(a_k) = \phi_{ij}(e'_j) + \phi_{ij}(a_j) = \phi_{ij}(e'_j + a_j) = \phi_{ij}(e_j),$$

showing that $\phi_{ij}(E_j) \subseteq \phi_{ik}(E_k)$. Thus (E_i) is indeed Mittag-Leffler, which completes the proof. ☐

Exercise 12

Prove the Snake Lemma for $\mathbf{A} = R$-**Mod**.

Proof. Now suppose that in the category R-**Mod** there is a commtative diagram

$$\begin{tikzcd} & X' \arrow[r, "f"] \arrow[d, "u'"] & X \arrow[r, "g"] \arrow[d, "u"] & X'' \arrow[r] \arrow[d, "u''"] & 0 \\ 0 \arrow[r] & Y' \arrow[r, "f'"] & Y \arrow[r, "g'"] & Y'' & \end{tikzcd} \tag{16}$$

with exact rows, we need to show that there is an exact sequence

$$\ker u' \xrightarrow{k'} \ker u \xrightarrow{k} \ker u'' \xrightarrow{\partial} \operatorname{coker} u' \xrightarrow{h'} \operatorname{coker} u \xrightarrow{h} \operatorname{coker} u'' \tag{17}$$

of R-modules.

We first show the existence of the morphisms in (17), then show the exactness of these morphisms. We will do these in the light of the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker u' \arrow[r, dotted, "k'"] \arrow[d, "v'"] & \ker u \arrow[r, dotted, "k"] \arrow[d, "v"] & \ker u'' \arrow[d, "v''"] & \\
 & X' \arrow[r, "f"] \arrow[d, "u'"] & X \arrow[r, "g"] \arrow[d, "u"] & X'' \arrow[r] \arrow[d, "u''"] & 0 \\
0 \arrow[r] & Y' \arrow[r, "f'"] \arrow[d, "w'"] & Y \arrow[r, "g'"] \arrow[d, "w"] & Y'' \arrow[d, "w''"] & \\
 & \operatorname{coker} u' \arrow[r, dotted, "h'"] \arrow[d] & \operatorname{coker} u \arrow[r, dotted, "h"] \arrow[d] & \operatorname{coker} u'' \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd} , \tag{18}$$

with solid arrows being commutative.

Existence. First we consider $f \circ v' : \ker u' \to X$. Because $u \circ (f \circ v') = f' \circ u' \circ v' = 0$, $f \circ v'$ indeed induces a morphism $k' : \ker u' \to \ker u$, by the universal property of $\ker u$. Similarly, there is a morphism $k : \ker u \to \ker u''$ induced by the morphism $g \circ v : \ker u \to X''$.

Dually, we consider the morphism $w \circ f' : Y' \to \operatorname{coker} u$, which satisfies $(w \circ f') \circ u' = w \circ u \circ f = 0$. So by the universal property of $\operatorname{coker} u'$, there is indeed a morphism $h' : \operatorname{coker} u' \to \operatorname{coker} u$. Analogously, $h : \operatorname{coker} u \to \operatorname{coker} u''$ is induced by $w'' \circ g' : Y \to \operatorname{coker} u''$.

The non-trivial part is to show the existence of $\partial : \ker u'' \to \operatorname{coker} u'$. We do this explicitly. Take any $\alpha \in \ker u''$ we define $\delta(a) \in \operatorname{coker} u'$ as follows.

$$\begin{tikzcd}
 & & & \alpha \arrow[d, "v''"] & \\
 & & \gamma \arrow[r, "g"] \arrow[d, "u"] & \beta \arrow[r] \arrow[d, "u''"] & 0 \\
0 \arrow[r] & \eta \arrow[r, "f'"] \arrow[d, "w'"] & \sigma \arrow[r, "g'"] & u''(\beta) & \\
 & \iota & & &
\end{tikzcd}$$

Take $\beta = v''(\alpha) \in X''$. Since $g : X \to X''$ is epimorphic by assumption, we can pick a lift $\gamma \in X$ of β along g. Then take $\sigma = u(\gamma) \in Y$. By the commutativity of (18), we have $g'(\sigma) = g'(u(\gamma)) = u''(g(\gamma)) = u''(\beta) = u''(v''(\alpha)) = 0$. Thus $\sigma \in \ker g' \simeq \operatorname{im} f'$, and take η to be $f'^{-1}(\sigma)$. Finally take $\iota = w'(\eta)$ and define

$$\partial(\alpha) := \iota.$$

All the above steps are natural, except for the choice of $\gamma \in X$. We need to show that $\partial(\alpha)$ is well-defined, that is, independent of the choice of γ.

Suppose we have chosen another lift $\gamma' \in X$ of the same β, thus we have $g(\gamma' - \gamma) = g(\gamma') - g(\gamma) = 0$, so there must be some $\xi \in X'$ such that $f(\xi) = \gamma' - \gamma$. Thus we have

$$\begin{aligned}
w'(f'^{-1}u(\gamma')) &= w'(f'^{-1}u(\gamma + (\gamma' - \gamma))) \\
&= w'(f'^{-1}u(\gamma)) + w'(f'^{-1}u(\gamma' - \gamma)) \\
&= w'(f'^{-1}u(\gamma)) + w'(f'^{-1}u(f(\xi))) \\
&= w'(f'^{-1}u(\gamma)) + w'((f')-1f'(u'(\xi)))' \\
&= w'(f'^{-1}u(\gamma)) + w'(u'(\xi)) \\
&= w'(f'^{-1}u(\gamma))
\end{aligned}$$

showing that $\partial(\alpha) = \iota = w'(f'^{-1}u(\gamma))$ is independent of the choice of γ, as desired.

Exactness. Since we have constructed the dotted arrows in the diagram (18) using universal properties, all arrows in diagram (18) are manifestly commutative, no matter dotted or solid. So we have $v'' \circ (k \circ k') = f \circ g \circ v' = 0$. But seeing that v'' is monomorphic, we have $k \circ k' = 0$, showing that $\operatorname{im} k' \subseteq \ker k$. For any $b \in \ker u$, we have $g(v(b)) = v'(kb) = 0$, thus $v(b) = \ker g = \operatorname{im} f$, there must be some $a \in X'$ such that $f(a) = v(b)$. Then $v(b) = (f \circ v')(v')^{-1}(f(a)) = (v \circ k')(v'^{-1}f(a)) = v(k'(v'^{-1}f(a)))$, showing that $b = k'(v'^{-1}f(a))$, that is, $b \in \operatorname{im} k'$. So we have shown $\ker k \subseteq \operatorname{im} k'$, thus $\operatorname{im} k' = \ker k$.

Next we show that $\operatorname{im} k = \ker \partial$. Again take any $b \in \ker u$, we need to see what $\partial(kb)$ is. By the definition of ∂, we need to find a lift of $v'(kb)$ along g in X', say $v(b)$. Since $g(v(b)) = v'(k(b))$, $v(b)$ is indeed a lift of $v'(kb)$. But we have $u(v(b)) = 0$, as $b \in \ker u$, then $\partial(kb) = w'((f')^{-1}(0)) = 0$, showing that $\operatorname{im} k \subseteq \ker \partial$. Conversely, for any $c \in \ker \partial$, we have $\partial c = 0$ in $\operatorname{coker} u'$, which is amount to say that $\partial c \in \operatorname{im} u'$. So there is some $d \in X'$ such that $\partial c = u'(d)$. Now we find that $f(d)$ is a lift of $v'(c)$, then $c = k(v^{-1}(f(d)))$, showing that c is the image of $v^{-1}(f(d)) \in \ker u$, showing that $\ker \partial \subseteq \operatorname{im} k$, as desired.

The verification of exactness of the rest part of (17) is formally dual to the previous argement, so we don't show it as time is limited. $\square$

Exercise 13

Let **A** be an abelian category, consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & X \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X'' \arrow[d] \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Y'' \arrow[d] \\
0 \arrow[r] & Z \arrow[r] & Z' \arrow[r] & Z''
\end{tikzcd}$$

and assume that all columns are exact. If the second and the third rows are exact, show that the first row is also exact.

Proof. Using the Feyed-Michell Embedding Theorem we may assume the given diagram is in R-**mod** for some ring R. Thus proof of this exercise is essentially contained in the proof of the Snake Lemma, here we tirelessly demonstrate it again. *In principio*, we name all the arrows in the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & X \arrow[r, "k"] \arrow[d, "v"] & X' \arrow[r, "k'"] \arrow[d, "v'"] & X'' \arrow[d, "v''"] \\
0 \arrow[r] & Y \arrow[r, "f"] \arrow[d, "u"] & Y' \arrow[r, "f'"] \arrow[d, "u'"] & Y'' \arrow[d, "u''"] \\
0 \arrow[r] & Z \arrow[r, "g"] & Z' \arrow[r, "g'"] & Z''
\end{tikzcd}.$$

What left us to show is that $\ker k = 0$ and $\operatorname{im} k = \ker k'$.

$\ker k = 0$. If $a \in \ker k$, we have $0 = v'(k(a)) = f(v(a))$. But both f, v are monomorphic by assumption, hence $f \circ v$ is monomorphic and this implies $a = 0$.

$\operatorname{im} k = \ker k'$. By the commutativity of the diagram, we have $v'' \circ k' \circ k = f' \circ f \circ v = 0$. Since v'' is monomorphic by assumption, we have $k' \circ k = 0$, which shows that $\operatorname{im} = \ker k'$. Conversely, for any $b \in \ker k'$, we have $f'(v'(b)) = v''(k'(b)) = 0$, showing that $v'(b)$ is in the image of f. So there is some $c \in Y$ such that

$$f(c) = v'(b). \tag{19}$$

Thus $g(u(c)) = u'(f(c)) = u'(v'(b)) = 0$. But since g is monomorphic, we have $u(c) = 0$, meaning that $c \in \ker u = \operatorname{im} v$. Since v is monomorphic, we take $X \ni d = v^{-1}(c)$ to be the unique preimage. Now (19) tells us that $v'(b) = f(c) = f(v(d)) = v'(k(d))$, implying $b = k(d)$. This shows $\ker k' \subseteq \operatorname{im} k$, completing the proof. □

Exercise 14

Prove that

Proposition 4. Via the isomorphisms $H^{n-1}(X[1]) \simeq H^n(X)$, the connecting form δ can be identified with $H^n(f) : H^n(X) \to H^n(Y)$.

Proof. For any cochain map $f : X \to Y$, there is a short exact sequence of cochain complexes

$$0 \to Y \xrightarrow{i} \operatorname{cone}(f) \xrightarrow{p} X[1] \to 0,$$

where $i(y) = (0, y)$ and $p(x, y) = -x$. By applying the functor $H^*(-)$ to this short exact sequence, we have a long exact sequence

$$\cdots \to H^{n-1}(X[1]) \xrightarrow{\delta^n} H^n(Y) \xrightarrow{H^n(i)} H^n(\operatorname{cone}(f)) \xrightarrow{H^n(p)} H^n(X[1]) \to \cdots.$$

Given the isomorphisms $H^{n-1}(X[1]) \simeq H^n(X)$, the above cohomology long exact sequence can be modified into

$$\cdots \to H^n(X) \xrightarrow{\delta^n} H^n(Y) \to H^n(\operatorname{cone}(f)) \to H^{n+1}(X) \to \cdots.$$

To show that δ^n are the same as $H^n(f)$, we need to inspect the construction of these connecting morphisms. As the diagram

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & Y^n \arrow[u, "d_Y^n"] \arrow[r, "i^n"] & \mathrm{cone}(f)^n = X^{n+1} \oplus Y^n \arrow[u, "d^n_{\mathrm{cone}(f)}"] \arrow[r] & X[1]^n \arrow[u, "d^n_{X[1]}"] \arrow[r] & 0 \\
0 \arrow[r] & Y^{n-1} \arrow[u, "d_Y^{n-1}"] \arrow[r] & \mathrm{cone}(f)^{n-1} = X^n \oplus Y^{n-1} \arrow[u, "d^{n-1}_{\mathrm{cone}(f)}"] \arrow[r, "p^{n-1}"] & X[1]^{n-1} \arrow[u, "d^{n-1}_{X[1]}"] \arrow[r] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

illustrates, δ^n are constructed as follows: pick any cocycle $x \in X[1]^{n-1} = X^n$, then find a lift of x via p^{n-1} in $\mathrm{cone}(f)^{n-1}$, say $(-x, 0)$, then the image $\delta^n([x]) \in Y^n$ is defined to be the cohomology class

$$[(i^n)^{-1}(d^{n-1}_{\mathrm{cone}(f)}(-x, 0))] = [(i^n)^{-1}(0, f^n(x))] = [f(x)],$$

showing that

$$\delta^n([x]) = [f^n(x)] = H^n(f)([x])$$

for all cocyle $x \in X^n$. Hence the proposition holds.

□

Exercise 15

Let $\mathbf{A}$ be an abelian category. $X \in C(\mathbf{A})$ is called **split** if there exist $s^n : X^{n+1} \to X^n$ such that $d^n \circ s^n \circ d^n = d^n, \forall n \in \mathbb{Z}$. Prove that X is split exact iff id_X is homotopic to zero.

Proof. $\Leftarrow$ Suppose that id_X is homotopic to zero, which is equivalent to saying that there exists maps $h^n : X^n \to X^{n-1}, n \in \mathbb{Z}$ such that

$$\mathrm{id}_{X^n} = h^{n+1} \circ d^n + d^{n-1} \circ h^n. \tag{20}$$

Composing (20) with d^n, we have

$$d^n = d^n \circ h^{n+1} \circ d^n + d^n \circ d^{n-1} \circ h^n = d^n \circ h^{n+1} \circ d^n.$$

Taking $s^n := h^{n-1}$, the above equation shows that X^* is split. To show that X^* is split exact, we need to show that X^* is acyclic in addition, namely $H^*(X^*) = 0$. For any $x \in Z^n$, by (20), we have

$$x = \mathrm{id}_{X^n}(x) = h^{n+1}(d^n x) + d^{n-1}(h^n x) = d^{n-1}(h^n x),$$

showing that $x \in B^n$. So we have $H^n(X^*) = 0$, implying that X^* is split exact.

$\Rightarrow$ Assume that X^* is split exact, *id est*, there are $s^n : X^{n+1} \to X^n, n \in \mathbb{Z}$ such that $d^n \circ s^n \circ d^n = d^n$. Consider the short exact sequences

$$0 \longrightarrow Z^n \longrightarrow X^n \xrightarrow{d^n} B^{n+1} \longrightarrow 0 \tag{21}$$

and

$$0 \xrightarrow{i} B^n \longrightarrow Z^n \longrightarrow H^n(X^*) \longrightarrow 0 \tag{22}$$

.

Then consider the morphisms $s^n : X^{n+1} \to X^n$ and $j^n := d^{n-1} \circ s^{n-1}$. For any $d^n x \in B^{n+1}$, we have

$$(d^n \circ s^n)(d^n x) = (d^n \circ s^n \circ d^n)(x) = d^n x,$$

showing that $(d^n \circ s^n)$ is the identity when restricted to B^{n+1}. Thus the short exact sequence 21 splits, hence for any X^n we have

$$X^n \simeq Z^n \oplus B^{n+1}.$$

Similarly, for any $d^{n-1}y \in B^n$, which can be viewed as an element in Z^n via $i : B^n \to Z^n$, we have

$$(j^n \circ i)(d^{n-1}y) = d^{n-1} \circ s^n \circ d^{n-1} = d^{n-1}y,$$

showing that (22) is also split. Thus we have

$$X^n \simeq B^{n+1} \oplus Z^n \simeq B^n n + 1 \oplus B^n \oplus H^n(X^*) = B^{n+1} \oplus B^n, \tag{23}$$

where the last equality comes from the acyclic assumption on X^*. In the light of (23), it's easy to verify that

$$\mathrm{id}_{X^n} = s^n \circ d^n + d^{n-1} \circ s^{n-1},$$

showing that $s^n : X^n \to X^{n-1}$ are the desired chain homotopy making id_X homotopic to zero. □

Exercise 16

Let **A** be an abelian category.

(i) Show that a cochain complex P^* is a projective object in $C(\mathbf{A})$ iff it is a split exact complex of projectives in **A**.

(ii) Show that if **A** has enough projectives, so does $C(\mathbf{A})$.

Proof. (i) $\Rightarrow$ Let P^* be a cochain complex that is projective in the category $C(\mathbf{A})$, we need to show that P is split exact. Since P is projective, $P[1]$ is also projective. Cosider the short exact sequence

$$0 \to P \to \operatorname{cone} \mathrm{id}_P \to P[1] \to 0, \tag{24}$$

which splits since $P[1]$ is projective. Thus we have $\operatorname{cone} \mathrm{id}_P \simeq P[1] \oplus P$ thus $H^*(\operatorname{cone} \mathrm{id}_P) \simeq H^{*+1}(P) \oplus H^*(P)$. But note that $\mathrm{id}_P : P \to P$ is an isomorphism hence a quasi-isomorphism, the cochain complex $\operatorname{cone} \mathrm{id}_P$ is acyclic, so $H^{*+1}(P) \oplus H^*(P) = 0$, from which we know that P is acyclic.

To show that P is split, we have to inspect the split exact sequence (24) closer. Since $P[1]$ is projective, there is a cochain map $s : P[1] \to \operatorname{cone} \mathrm{id}_P$ lifting the identity $\mathrm{id}_{P[1]} : P[1] \to P[1]$. For any $p \in (P[1])^n = P^{n+1}$, since s is lifing the identity, the element $s^n(p) \in (\operatorname{cone} \mathrm{id}_P)^n \simeq P^{n+1} \oplus P^n$ must be of the form $s^n(p) = (-p, -h^n p)$, where $h^n : P^{n+1} \to P^n$ are morphisms in **A**. Moreover, since s is a cochain map, it must commute with the differentials $d_{P[1]}$ and $d_{\operatorname{cone} \mathrm{id}_P}$. Thus we have

$$d^n_{\operatorname{cone} \mathrm{id}_P}(s^n(p)) = s^{n+1}(d^n_{P[1]}(p)).$$

Computing both sides, we have

$$(d_P^{n+1} p, -p + d_P^n(h^n p)) = (-d_P^{n+1}, p - h^{n+1}(d_P^{n+1} p)),$$

thus

$$p = d_P^n(h^n p) + h^{n+1}(d_P^{n+1} p).$$

Since p is arbitrary, the above equation shows that P is split exact.

Finally we need to show that each P^n of P^* is projective in $\mathbf{A}$, namely, for any two objects M, N in $\mathbf{C}$ with an empimorphism $M \xrightarrow{g} N \to 0$, and a morphism $f^n : P^n \to N$, there is an $\tilde{f}^n : P^n \to M$ lifting f^n uniquely. To do this, we consider two associated cochain complexes M^*, N^*, defined as

$$M^* := \cdots \to 0 \to M \to 0 \to \cdots,$$
$$N^* := \cdots \to 0 \to N \to 0 \to \cdots,$$

with M, N in nth degrees and zeroes elsewhere. There is also an epimorphism $M^* \xrightarrow{g^*} N^* \to 0$ in $C(\mathbf{A})$, obtained by extending $M \to N \to 0$ by zeroes, as well as a cochain map $f^* : P^* \to N^*$, obtained by extending $f^n : P^n \to N$ by zeroes. By the projectivity of P^*, there is a unique cochain map $\tilde{f}^*$ lifting f^*:

$$\begin{tikzcd} & P^* \arrow[dl, dotted, "\tilde{f}^*"'] \arrow[d, "f^*"] & \\ M^* \arrow[r, "g^*"] & N^* \arrow[r] & 0 \end{tikzcd}.$$

In particular, in degree n, we have a commutative diagram

$$\begin{tikzcd} & P^n \arrow[dl, dotted, "\tilde{f}^n"'] \arrow[d, "f^*"] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

showing that P^n is projective and compeleting this direction.

$\Leftarrow$ Now suppose that P^* is split exact and with each P^n projective, we want to show that P^* is projective. Since P^* is split exact, there are decompositions for each P^n

$$P^n \simeq B^{n+1} \oplus B^n, \tag{25}$$

by (23) in **Exercise 15**. Immediately we know that B^{n+1} and B^n are both projective, since P^n is. Also observe that d^n is an isomorphism when restricted to B^{n+1}. Indeed, we have $\ker d^n = Z^n \simeq B^n$ by the acyclic assumption, thus we have

$$\operatorname{im} d^n \simeq P^n/\ker d^n \simeq P^n/B^n \simeq B^{n+1},$$

in the light of (25). So we may view the differential $d^n : P^n \to P^{n+1}$ as the composition of the projection $B^{n+1} \oplus B^n \twoheadrightarrow B^{n+1}$ and the inclusion $B^{n+1} \hookrightarrow B^{n+2} \oplus B^{n+1}$:

$$\begin{tikzcd} P^n \arrow[rr, "d^n"] \arrow[d, "\simeq"] & & P^{n+1} \arrow[d, "\simeq"] \\ B^{n+1} \oplus B^n \arrow[r, two heads] & B^{n+1} \arrow[r, hook] & B^{n+2} \oplus B^{n+1} \end{tikzcd}.$$

In this flavor, we may write $P^* = \oplus_{n\in\mathbb{Z}} B(n)$, where $B(n)$ is the complex

$$B(n) := \cdots \to 0 \to B^{n+1} \xrightarrow{d^n} \operatorname{im} d^n \to 0 \to \cdots$$

is projective in $C(\mathbf{A})$, P^* is manifestly projective. But $B(n)$ is indeed projective, since each B^{n+1} is projective and the lifting problem

$$\begin{tikzcd} & B^{n+1} \arrow[dl, dotted] \arrow[d] & \\ X^n \arrow[r] & Y^n \arrow[r] & 0 \end{tikzcd}$$

in $\mathbf{A}$ always has a solution. Thus the lifting problem

$$\begin{tikzcd} & B(n) \arrow[dl, dotted] \arrow[d] & \\ X^* \arrow[r] & Y^* \arrow[r] & 0 \end{tikzcd}$$

in $C(\mathbf{A})$ always has a solution via extension by zeroes, as desired.

(ii) Given any cochain complex in $C(\mathbf{A})$. For any C^n, since $\mathbf{A}$ has enough projectives, we can find a projective module P^n and an epimorphism $g^n : P^n \to C^n$. Now consider the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & P^n \arrow[r, "\mathrm{id}_{P^n}"] \arrow[d, "g^n"] & P^n \arrow[r] \arrow[d, "d^n \circ g^n"] & 0 \arrow[r] & \cdots \\ \cdots \arrow[r] & C^{n-1} \arrow[r] & C^n \arrow[r, "d^n"] & C^{n+1} \arrow[r] & C^{n+2} \arrow[r] & \cdots \end{tikzcd},$$

which is indeed a cochain map $\sigma(n) : P(n) \to C^*$, where $P(n)$ is the cochain complex

$$P(n)^* : \cdots \to 0 \to P^n \overset{\mathrm{id}_{P^n}}{\to} P^n \to 0 \to \cdots .$$

Obviously, we $P(n)^*$ is split exact and each $P(n)^i$ is projective for all $i \in \mathbb{Z}$. Thus by (i) $P(n)^*$ is a projective object in the category $C(\mathbf{A})$. Take $P^* := \oplus_{n\in\mathbb{Z}} P(n)^*$. Thus P^* is the a projective object in $C(\mathbf{A})$ and $P^* \to X^*$ is epimorphic, by construction. Thus the second assertion follows. $\square$

Exercise 17

Let $m \geq 2$ be an integer and $R = \mathbb{Z}/m\mathbb{Z}$. Show that R is an injective R-module while $\mathbb{Z}/d\mathbb{Z}$ is not an injective R-module when $d|m$ and $p|\gcd(d, m/d)$ for some prime p.

Proof. We show that $\mathbb{Z}/m\mathbb{Z}$ is an injective $\mathbb{Z}/m\mathbb{Z}$-module using Baer's criterion. Namely, for any ideal $I \subseteq \mathbb{Z}/m\mathbb{Z}$, we want to show that every $\mathbb{Z}/m\mathbb{Z}$-module morphism $f : I \to \mathbb{Z}/m\mathbb{Z}$ can be extended to a $\mathbb{Z}/m\mathbb{Z}$-module morphism $\tilde{f} : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$.

First of all, note that any proper ideal I of $\mathbb{Z}/m\mathbb{Z}$ is a principal ideal, that is, $I = (\bar{k})$ for some $k \mid m$. We claim that for any $f : I \to \mathbb{Z}/m\mathbb{Z}$, we have $f(I) \subseteq I$. Indeed, for any $x \in \operatorname{im} f$, there is an element $\overline{lk} \in I$ such that $f(\overline{lk}) = \bar{x}$. On the other hand, since $k \mid n$, we have $0 = f(0) = f(\bar{l}\bar{n}) = \bar{s}f(\overline{lk}) = \overline{sx}$, where $n = ks$. So $sx = nt$ for some t, hence $sx = kst \Rightarrow x = kt$. The claim follows.

With the above observation, a morphim $f : I \to \mathbb{Z}/m\mathbb{Z}$ of $\mathbb{Z}/m\mathbb{Z}$-modules is *de facto* a morphism $f : I \to I$. Since I is principal, f must be of the form

$$\begin{aligned} f : I &\to I, \\ \bar{k} &\mapsto c\bar{k}, \end{aligned}$$

with some $c \in \mathbb{Z}/m\mathbb{Z}$. So for any other $r \in \mathbb{Z}/m\mathbb{Z}$, we may define

$$\begin{aligned} \tilde{f} : \mathbb{Z}/m\mathbb{Z} &\to \mathbb{Z}/m\mathbb{Z}, \\ r &\mapsto cr, \end{aligned}$$

which is indeed an extension of f. For now we have proved that $\mathbb{Z}/m\mathbb{Z}$ is an injective $\mathbb{Z}/m\mathbb{Z}$-module. $\square$

Exercise 18

Let A be an abelian group. TFAE:

(i) A is torsion free.

(ii) $\mathrm{Tor}_1^{\mathbb{Z}}(A, -) = 0$.

(iii) $\mathrm{Tor}_1^{\mathbb{Z}}(-, A) = 0$.

Proof. (ii) $\iff$ (iii) Since $\mathbb{Z}$ is commutative, so for any abelian group B, we have

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, B) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(B, A).$$

So if either of $\mathrm{Tor}_1^{\mathbb{Z}}(A, -)$ or $\mathrm{Tor}_1^{\mathbb{Z}}(-, A)$ vanishes, the other must vanish.

(i) $\iff$ (ii) If A is abelian, it is the direct limit of its finitely generated subgroup A_α:

$$A = \varinjlim A_\alpha.$$

Taking the fact that A_α is torsion-free, each A_α is torsion-free, thus is free. Say $A_\alpha = \mathbb{Z}^{n_\alpha}$, for any abelian group B, we have

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, B) \simeq \varinjlim \mathrm{Tor}_1^{\mathbb{Z}}(A_\alpha, B) = \varinjlim \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{n_\alpha}, B) = 0.$$

Conversely, if $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$ vanishes for every abelian group B, we can take $B = \mathbb{Q}/\mathbb{Z}$ in particular. Thus

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, A) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) = 0.$$

But we know that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, A)$ is the torsion subgroup of A, so A is torsion-free. □

Exercise 19

If ${}_rR \neq 0$, all we have is the non-projective resolution

$$0 \to {}_rR \to R \xrightarrow{r} R \to R/rR \to 0.$$

Show that there is a short exact sequence

$$0 \to \mathrm{Tor}_n^R(R/rR, B) \to {}_rR \otimes_R B \to {}_rB \to \mathrm{Tor}_1^R(R/rR, B) \to 0$$

and that $\mathrm{Tor}_n^R(R/rR, B) \simeq \mathrm{Tor}_{n-2}^R({}_rR, B)$ for $n \geq 3$.

Exercise 20

Show that $\mathrm{Tor}_1^R(R/I, R/J) \simeq \frac{I \cap J}{IJ}$ for every right ideal I and left ideal J of R. In particular, $\mathrm{Tor}_1^R(R/I, R/I) \simeq I/I^2$ for every 2-sided ideal I.

Proof. To compute $\mathrm{Tor}_1^R(R/I, R/J)$, we have to use a projective resolution

$$0 \to I \to R \to R/I$$

of the right R-module R/I. By definition, after tensoring R/J, $\mathrm{Tor}_1^R(R/I, R/J)$ is the 1st cohomology of the complex

$$\cdots \to 0 \to I \otimes_R R/J \to R \otimes_R R/J.$$

If we denote by $h : I \otimes_R R/J \to R \otimes_R R/J$, then

$$\mathrm{Tor}_1^R(R/I, R/J) \simeq \ker h.$$

To determine $\ker h$, we consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & R \arrow[r] & R \otimes_R R/J \arrow[r] & 0 \\ 0 \arrow[r] & IJ \arrow[r] \arrow[u, "f"] & I \arrow[r] \arrow[u, "g"] & I \otimes_R R/J \arrow[r] \arrow[u, "h"] & 0 \end{tikzcd}, \tag{26}$$

on which we may apply the Snake Lemma. We thus obtain an exact sequence

$$\cdots \to 0 \to \ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0, \tag{27}$$

since both rows in (26) are exact. However, since f, g are all natural injections, we have

$$\ker f = \ker g = 0,$$

and

$$\operatorname{coker} f = J/IJ, \operatorname{coker} g = R/I.$$

So the exact sequence (26) can be simplified as

$$0 \to \ker h \to J/IJ \to R/I \to \operatorname{coker} h \to 0,$$

from which we can read

$$\ker h = \ker(J/IJ \to R/I) = I \cap J/IJ,$$

showing that $\operatorname{Tor}_1^R(R/I, R/J) \simeq (I \cap J)/IJ$. In particular, if we take $J = I$, we have $\operatorname{Tor}_1^R(R/I, R/I) \simeq (I \cap I)/I^2 = I/I^2$, completing the proof. □

Exercise 21

We saw in the last section if $R = \mathbb{Z}$, a module B is flat iff B is torsion-free. Here is an example of a torsion-free ideal I that is not a flat R-module. Let k be a field and set $R = k[x, y], I = (x, y)R$. Show that $k = R/I$ has the projective resolution

$$0 \to R \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \to k \to 0.$$

Then compute that $\operatorname{Tor}_1^R(I, k) \simeq \operatorname{Tor}_2^R(k, k) \simeq k$, showing that I is not flat.

Proof. Consider the short exact sequence of R-modules,

$$0 \to I \to R \to k \to 0. \tag{28}$$

Since $\operatorname{Tor}_*^R(k, -)$ is a universal delta functor, we obtain a long exact sequence

$$\cdots \to \operatorname{Tor}_2^R(k, R) \to \operatorname{Tor}_2^R(k, k) \to \operatorname{Tor}_1^R(k, I) \to \operatorname{Tor}_1^R(k, R) \to \cdots \tag{29}$$

by applying the functor $\operatorname{Tor}_*^R(k, -)$ to (28). Since R itself is a projective R-module, we have $\operatorname{Tor}_n^R(k, R) = 0, n \geq 1$. Thus we read

$$\operatorname{Tor}_2^R(k, k) \simeq \operatorname{Tor}_1^R(k, I)$$

from (29). In addition $R = k[x, y]$ is commutative,

$$\operatorname{Tor}_2^R(k, k) \simeq \operatorname{Tor}_1^R(k, I) \simeq \operatorname{Tor}_1^R(I, k).$$

Since both R and R^2 are free R-modules, hence are projective, we are left to check that

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow[d_2]{\begin{bmatrix} -y \\ x \end{bmatrix}} R^2 \xrightarrow[d_1]{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow k \longrightarrow 0 \tag{30}$$

is a cochain complex, in order to obtain a projective resolution of k. For any $f \in R = k[x,y]$, we have

$$d_2(f) = \begin{pmatrix} -yf \\ xf \end{pmatrix}$$

and

$$d_1(d_2(f)) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -yf \\ xf \end{pmatrix} = -xyf + xyf = 0,$$

which verifies that (30) is indeed a projective resolution of k. Now we may use (30) to compute $\mathrm{Tor}_2^R(k,k)$.

By definition, $\mathrm{Tor}_2^R(k,k)$ is the 2nd homology group of the complex

$$\cdots \longrightarrow 0 \longrightarrow R \otimes_R k \longrightarrow (R \otimes_R k) \oplus (R \otimes_R k) \longrightarrow R \otimes_R k \longrightarrow k \otimes_R k \ .$$

But the latter is obviously isomorphic to $R \otimes_R k \simeq k$, which leads to

$$k \simeq \mathrm{Tor}_2^R(k,k) \simeq \mathrm{Tor}_1^R(k,I).$$

□

Exercise 22

Compute $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z})$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$.

Proof. (i) By definition of $\mathbb{Z}_{p^\infty}$, there is a short exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \to \mathbb{Z}[\frac{1}{p}] \to \mathbb{Z}_{p^\infty} \to 0. \tag{31}$$

Since $\mathrm{Ext}^*_{\mathbb{Z}}(-, \mathbb{Z})$ is a universal cohomological δ-functor, there is an induced long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \to \cdots .$$

We claim that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) = 0$. Indeed, for any homomorphism $\varphi : \mathbb{Z}[\frac{1}{p}] \to \mathbb{Z}$ of abelian groups, φ is determined uniquely by $\varphi(1) \in \mathbb{Z}$. Let $\varphi(1) = a = p^n q$, where q is an integer such that $p \nmid q$. Now we consider $b := \varphi(\frac{1}{p^{n+1}}) \in \mathbb{Z}$, which satisfies that $p^{n+1} b = a = p^n q$, a contradiction to the assumption that $p \nmid q$ unless $a = 0$. Next we have $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z} \simeq \mathbb{Z}$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0$ since $\mathbb{Z}$ is projective.

Last but not least, since we had already seen that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \simeq (\mathbb{Z}_{p^\infty})^* := \hat{\mathbb{Z}}_p$ in class, the long exact sequence becomes

$$0 \to \mathbb{Z} \to \hat{\mathbb{Z}}_p \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) \to 0,$$

so we have

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) \simeq \hat{\mathbb{Z}}_p / \mathbb{Z}. \tag{32}$$

(ii) To compute $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z})$, we consider another short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0, \tag{33}$$

on which we apply the functor $\mathrm{Ext}^*_{\mathbb{Z}}(-,\mathbb{Z})$ to get a long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},\mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \to \cdots.$$

The above long exact sequence reduces to a short exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},\mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) \to 0, \tag{34}$$

since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) = 0$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0$. On the other hand, we have

$$\mathbb{Q}/\mathbb{Z} = \bigoplus_p \mathbb{Z}_{p^\infty},$$

so we can write $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},\mathbb{Z})$ more explicitly

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},\mathbb{Z}) = \mathrm{Ext}^1_{\mathbb{Z}}(\bigoplus_p \mathbb{Z}_{p^\infty},\mathbb{Z}) \simeq \prod_p \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{p^\infty},\mathbb{Z}) = \prod_p \hat{\mathbb{Z}}_p.$$

Finally from (34) we have

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) \simeq (\prod_p \hat{\mathbb{Z}}_p)/\mathbb{Z}.$$

□

Exercise 23

Check that in Definition 2.3.10 $\Theta([\xi]) = \partial(\mathrm{id}_B)$ is well-defined, where $\partial : \mathrm{Hom}(A,A) \to \mathrm{Ext}^1_R(A,B)$.

Proof. Given two equivalent extensions ξ and ξ' of A by B

$$\begin{tikzcd}
\xi: & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\simeq"] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi': & 0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd},$$

we need to show that $\Theta(\xi) = \Theta(\xi')$.

Indeed, apply the functor $\mathrm{Hom}_R(-,B)$ to the diagram above, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(X',B) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_R(B,B) \arrow[r, "\partial"] \arrow[d, equal] & \mathrm{Ext}^1_R(A,B) \arrow[d, equal] \\
\mathrm{Hom}_R(X,B) \arrow[r] & \mathrm{Hom}_R(B,B) \arrow[r, "\partial"] & \mathrm{Ext}^1_R(A,B)
\end{tikzcd}.$$

By definition $\Theta(\xi) = \partial(\mathrm{id}_B) = \Theta(\xi')$, as desired.

□

Exercise 24

When $R = \mathbb{Z}/m$ and $B = \mathbb{Z}/p$ with $p|m$, show that

$$0 \to \mathbb{Z}/p \xhookrightarrow{\iota} \mathbb{Z}/m \xrightarrow{p} \mathbb{Z}/m \xrightarrow{m/p} \mathbb{Z}/m \xrightarrow{p} \mathbb{Z}/m \xrightarrow{p/m} \cdots$$

is an infinite periodic injective resolution of B. Then compute the groups $\operatorname{Ext}^n_{\mathbb{Z}/m}(A, \mathbb{Z}/p)$ in terms of $A^* = \operatorname{Hom}(A, \mathbb{Z}/m)$. In particular, show that if $p^2|m$, then $\operatorname{Ext}^n_{\mathbb{Z}/m}(\mathbb{Z}/p, \mathbb{Z}/p) \simeq \mathbb{Z}/p$ for all n.

Proof. We have shown that $\mathbb{Z}/m$ is an injective $\mathbb{Z}/m$-module in Exercise 7, so we are left to show that

$$0 \to \mathbb{Z}/p \xhookrightarrow{\iota} \mathbb{Z}/m \xrightarrow{p} \mathbb{Z}/m \xrightarrow{m/p} \mathbb{Z}/m \xrightarrow{p} \mathbb{Z}/m \xrightarrow{p/m} \cdots \tag{35}$$

is indeed a chain complex. Given any $r \in \mathbb{Z}/m$, we have

$$d^{2k+1}(d^{2k}r) = \frac{m}{p}(pr) = mr = 0, k \geq 0,$$

and

$$d^{2k+2}(d^{2k+1}r) = p(\frac{m}{p}r) = mr = 0, k \geq 0,$$

showing that (35) is indeed an injective resolution of the $\mathbb{Z}/m$-module $\mathbb{Z}/p$. Now we can use (35) to compute $\operatorname{Ext}^*_{\mathbb{Z}/m}(A, \mathbb{Z}/m)$.

By definition, $\operatorname{Ext}^*_{\mathbb{Z}/m}(A, \mathbb{Z}/m)$ are the cohomology groups of the cochain complex

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}/m}(A, \mathbb{Z}/p) \xrightarrow[d^{-1}]{\iota^*} \operatorname{Hom}_{\mathbb{Z}/m}(A, \mathbb{Z}/m) \xrightarrow[d^0]{p^*} \operatorname{Hom}_{\mathbb{Z}/m}(A, \mathbb{Z}/m) \xrightarrow[d^1]{(m/p)^*} \cdots . \tag{36}$$

In terms of $A^* := \operatorname{Hom}_{\mathbb{Z}/m}(A, \mathbb{Z}/m)$, we have

$$\begin{aligned}
\operatorname{Ext}^{2k}_{\mathbb{Z}/m}(A, \mathbb{Z}/m) &= \frac{\ker d^{2k}}{\operatorname{im} d^{2k-1}} = \frac{{}_pA^*}{(m/p)A^*}, k \geq 1, \\
\operatorname{Ext}^{2k-1}_{\mathbb{Z}/m}(A, \mathbb{Z}/m) &= \frac{\ker d^{2k-1}}{\operatorname{im} d^{2k-2}} = \frac{{}_{m/p}A^*}{pA^*}, k \geq 1, \\
\operatorname{Ext}^0_{\mathbb{Z}/m}(A, \mathbb{Z}/m) &= \operatorname{Hom}_{\mathbb{Z}/m}(A, \mathbb{Z}/p).
\end{aligned} \tag{37}$$

In particular, if $A = \mathbb{Z}/p$ and $m = p^2q$ for some $q \in \mathbb{Z}$, we claim that

$$A^* = \operatorname{Hom}_{\mathbb{Z}/m}(\mathbb{Z}/p, \mathbb{Z}/m) \simeq \mathbb{Z}/p.$$

Clearly, for any $\bar{k} \in \mathbb{Z}/p$, $\bar{k}$ can be viewed as an element in $\mathbb{Z}/m$, via the natural inclusion $\mathbb{Z}/p \subseteq \mathbb{Z}/m = \mathbb{Z}/qp^2$, which shows that $\mathbb{Z}/p \subseteq \operatorname{Hom}_{\mathbb{Z}/m}(\mathbb{Z}/p, \mathbb{Z}/m)$. Conversely, since the action of $\bar{l} \in \mathbb{Z}/m$ on $\mathbb{Z}/p$ is trivial unless $\bar{l} \neq \overline{ip}, i = 0, 1, \ldots$, $\mathbb{Z}/p$ is in fact a faithful $\mathbb{Z}/p$-module, which shows that $\operatorname{Hom}_{\mathbb{Z}/m}(\mathbb{Z}/p, \mathbb{Z}/m) \subseteq \mathbb{Z}/p$. Thus the action of m/p and p on $A \simeq \mathbb{Z}/p$ is trivial, so the cochain complex (36) becomes

$$0 \to \mathbb{Z}/p \xrightarrow{0} \mathbb{Z}/p \xrightarrow{0} \mathbb{Z}/p \xrightarrow{0} \cdots .$$

Taking cohomology of the cochain complex, we have

$$\operatorname{Ext}^n_{\mathbb{Z}/m}(\mathbb{Z}/p, \mathbb{Z}/p) = \frac{\ker 0}{\operatorname{im} 0} = \mathbb{Z}/p,$$

as desired. □

Exercise 25

Let P be a chain complex and Q a cochain complex of R-modules. As in 2.7.4, from the Hom double cochain complex $\mathrm{Hom}(P,Q) = \{\mathrm{Hom}_R(P_p, Q^q)\}$, and then write $H^*\mathrm{Hom}(P,Q)$ for the cohomology of $\mathrm{Tot}(\mathrm{Hom}(P,Q))$. Show that if each P_n and $d(P_n)$ is projective, there is an exact sequence

$$0 \to \prod_{p+q=n-1} \mathrm{Ext}^1_R(H_p(P), H^q(Q)) \to H^n\mathrm{Hom}(P,Q) \to \prod_{p+q=n} \mathrm{Hom}_R(H_p(P), H^q(Q)) \to 0.$$

Proof. Take $M = H^q(Q)$ in the Universal Coefficient Theorem for cohomology, we have a short exact sequence

$$0 \to \mathrm{Ext}^1_R(H_{p-1}(P), H^q(Q)) \to H^p\mathrm{Hom}_R(P, H^q(Q)) \to \mathrm{Hom}_R(H_p(P), H^q(Q)) \to 0. \qquad (38)$$

By assumption, since each P_n is projective, the functor $\mathrm{Hom}_R(P_p, -)$ is exact, we have

$$\mathrm{Hom}_R(P_p, H^q(Q)) \simeq H^q(\mathrm{Hom}_R(P_p, Q)).$$

Futher, we have

$$\mathrm{Hom}_R(P, H^q(Q)) \simeq \mathrm{Hom}_R(\bigoplus_p P_p, H^q(Q)) \simeq \prod_p \mathrm{Hom}_R(P_p, H^q(Q)) \simeq \prod_p H^q(\mathrm{Hom}_R(P_p, Q)).$$

Taking product of the short exact sequence (38) over all p, q such that $p+q=n$, we have

$$0 \to \prod_{p+q=n-1} \mathrm{Ext}^1_R(H_p(P), H^q(Q)) \to \prod_{p+q=n} H^p\mathrm{Hom}_R(P, H^q(Q)) \to \prod_{p+q=n} \mathrm{Hom}_R(H_p(P), H^q(Q)) \to 0. \qquad (39)$$

The middle term of the above exact sequence reads

$$\begin{aligned}\prod_{p+q=n} H^p(\mathrm{Hom}_R(P, H^q(Q))) &\simeq \prod_{p+q=n} H^p(\prod_p H^q(\mathrm{Hom}_R(P_p, Q))) \\ &\simeq H^p H^q(\bigoplus_{p+q=n} \bigoplus_p \mathrm{Hom}_R(P, Q)) \qquad (40) \\ &\simeq H^n\mathrm{Hom}(P,Q),\end{aligned}$$

where the last isomorphism again uses the assumption that P_p are projective. Finally, substitute (40) into (39), we obtained the desired exact sequence

$$0 \to \prod_{p+q=n-1} \mathrm{Ext}^1_R(H_p(P), H^q(Q)) \to H^n\mathrm{Hom}(P,Q) \to \prod_{p+q=n} \mathrm{Hom}_R(H_p(P), H^q(Q)) \to 0.$$

□

Exercise 26

Let G be a finite group, show that $H^1(G;\mathbb{Z}) = 0$ and $H^2(G;\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbf{Grp}}(G, \mathbb{C}^*)$.

Proof. Since we have seen that $H_1(G;\mathbb{Z}) \simeq G/[G,G]$, we can obtain $H^1(G;\mathbb{Z})$ via the Universal Coefficient Theorem.

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_0(G;\mathbb{Z}), \mathbb{Z}) \longrightarrow H^1(G;\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_1(G;\mathbb{Z}), \mathbb{Z}) \longrightarrow 0.$$

Moreover, $H_0(G;\mathbb{Z}) \simeq \mathbb{Z}$, which is $\mathbb{Z}$-projective, thus $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0$, so

$$H^1(G;\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(G/[G,G], \mathbb{Z}).$$

But since G is finite, so $\mathrm{Hom}_{\mathbb{Z}}(G/[G,G],\mathbb{Z}) = 0$, thus $H^1(G;\mathbb{Z}) = 0$.

Now consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp 2i\pi} \mathbb{C}^* \longrightarrow 0,$$

on which we apply $H^*(G;-)$ to obtain a cohomology long exact sequence

$$\cdots \longrightarrow H^1(G;\mathbb{Z}) \longrightarrow H^1(G;\mathbb{C}^*) \longrightarrow H^2(G;\mathbb{Z}) \longrightarrow H^2(G;\mathbb{C}) \longrightarrow \cdots. \tag{41}$$

Again use the Universal Coefficient Theorem for cohomology to obtain $H^2(G;\mathbb{C})$, we have

$$H^2(G;\mathbb{C}) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_2(G;\mathbb{Z}),\mathbb{C}) = 0,$$

since $\mathrm{Ext}^1_{\mathbb{Z}}(H_1(G;\mathbb{C}),\mathbb{C}) = 0$, by the divisibility of $\mathbb{C}$ and $H_2(G;\mathbb{Z}) = 0$. We read

$$H^2(G;\mathbb{Z}) \simeq H^1(G;\mathbb{C}^*) \simeq \mathrm{Hom}_{\mathbb{Z}}(G/[G,G],\mathbb{C}^*) \simeq \mathrm{Hom}_{\mathbf{Grp}}(G,\mathbb{C}^*).$$

□

Exercise 27

Suppose that each P_p is a finitely generated $\mathbb{Z}G$-module. (For example, this can be done when G is finite.) Show in this case that μ is an isomorphism. Then deduce from the Künneth formula that the cross product fits into a split short exact sequence

$$0 \to \bigoplus_{p+q=n} H^p(G;\mathbb{Z}) \otimes H^q(H;\mathbb{Z}) \xrightarrow{\times} H^n(G\times H;\mathbb{Z}) \to \bigoplus_{p+q=n+1} \mathrm{Tor}_1^{\mathbb{Z}}(H^p(G;\mathbb{Z}),H^q(H;\mathbb{Z})) \to 0. \tag{42}$$

Proof. (i) We first prove that $\mu : \mathrm{Hom}_G(P_*,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{Hom}_H(Q_*,\mathbb{Z}) \to \mathrm{Hom}_{G\times H}(P_* \otimes_{\mathbb{Z}} Q_*,\mathbb{Z})$ is an isomorphism, provided that each P_p is a finitely generated $\mathbb{Z}G$-module. *A fortiori*, it is supposed that $P_* \to \mathbb{Z}$ is a free $\mathbb{Z}G$ resolution of $\mathbb{Z}$. So for each $p \in \mathbb{Z}$ there is some $m_p \in \mathbb{N}$ such that $P_p \simeq (\mathbb{Z}G)^{m_p}$. Then we have $\mathrm{Hom}_G(P_p,\mathbb{Z}) \simeq \mathrm{Hom}_G((\mathbb{Z}G)^{m_p},\mathbb{Z}) \simeq \mathbb{Z}^{m_p}$ as a trvial $\mathbb{Z}G$-module. Fixing p and q, we have an isomorphism

$$\begin{aligned}
\mathrm{Hom}_G(P_p,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{Hom}_H(Q_q,\mathbb{Z}) &\simeq \mathbb{Z}^{m_p} \otimes_{\mathbb{Z}} \mathrm{Hom}_H(Q_q,\mathbb{Z}) \\
&\simeq \bigoplus_{m_p} \mathrm{Hom}_H(Q_q,\mathbb{Z}) \\
&\simeq \mathrm{Hom}_H(\bigoplus_{m_p} Q_q,\mathbb{Z}) \\
&\simeq \mathrm{Hom}_{G\times H}(\mathbb{Z}^{m_p} \otimes_{\mathbb{Z}} Q_q,\mathbb{Z}) \\
&\simeq \mathrm{Hom}_{G\times H}(P_p \otimes_{\mathbb{Z}} Q_q,\mathbb{Z}).
\end{aligned}$$

In this case, for any $f = (f_1,\ldots,f_{m_p}) \in \mathrm{Hom}_G(P_q,\mathbb{Z})$ and $g \in \mathrm{Hom}_H(Q_q,\mathbb{Z})$, we have

$$\mu(f\otimes g) \simeq \mu(f_1 \otimes g,\ldots,f_{m_p}\otimes g) \simeq (f_1 g,\ldots,f_{m_p} g) \in \mathrm{Hom}_H(P_q \otimes_{\mathbb{Z}} Q_q,\mathbb{Z}),$$

showing that

$$\mu^{p,q} : \mathrm{Hom}_G(P_p,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{Hom}_H(Q_q,\mathbb{Z}) \to \mathrm{Hom}_{G\times H}(P_p \otimes_{\mathbb{Z}} Q_q,\mathbb{Z}) \tag{43}$$

is indeed an isomorphism. Then summing (43) over $p,q \in \mathbb{Z}$ satisfying $p+q=n$ for a fixed n we have an isomorphism

$$\mu^n : \bigoplus_{p+q=n} \mathrm{Hom}_G(P_p,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{Hom}_H(Q_q,\mathbb{Z}) \to \mathrm{Hom}_{G\times H}(\bigoplus_{p+q=n} (P_p \otimes_{\mathbb{Z}} .Q_q),\mathbb{Z})$$

for each n. It's easy to check that all μ^n commute with differentials, so we have an isomorphism between cochain complexes

$$\mu : \mathrm{Hom}_G(P_*, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{Hom}_H(Q_*, \mathbb{Z}) \to \mathrm{Hom}_{G\times H}(P_* \otimes_{\mathbb{Z}} Q_*, \mathbb{Z}), \tag{44}$$

as desired.

(ii) First we fix q and find a way to compute the torsion $\mathrm{Tor}_1^{\mathbb{Z}}(H^p(G;\mathbb{Z}), H^q(H;\mathbb{Z}))$. Consider the short exact sequence

$$0 \to d^{p-1}(\tilde{P}^{p-1}) \to Z^p(\tilde{P}^*) \to H^p(G;\mathbb{Z}) \to 0,$$

where the cochain complex $\tilde{P}^* := \mathrm{Hom}_G(P_*, \mathbb{Z})$ with $P_* \to \mathbb{Z}$ a finitely generated free $\mathbb{Z}G$ resolution of $\mathbb{Z}$. By assumption, since each P_p is a free module of finite rank, each $d^{p-1}(\tilde{P}^{p-1}$ is thus projective since it is a direct summand of the free module $\mathrm{Hom}_G(P_p, \mathbb{Z}) \simeq (\mathbb{Z}G)^{m_p}$, using the notation as in (i). So $\mathrm{Tor}_*^{\mathbb{Z}}(d^{p-1}(\tilde{P}^{p-1}, -) = 0$ and we have a short exact sequence

$$0 \to d^{p-1}(\tilde{P}^{p-1}) \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z}) \xrightarrow{\partial^{p-1}} Z^p(\tilde{P}^*) \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z}) \to H^p(G;\mathbb{Z}) \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z}) \to 0,$$

with $\partial^{p-1} := d^{p-1} \otimes \mathrm{id}$. By definition, we have

$$\mathrm{Tor}_1^{\mathbb{Z}}(H^p(G;\mathbb{Z}), H^q(H;\mathbb{Z})) = \ker \partial^{p-1}. \tag{45}$$

To find other terms of the short exact sequence (42), we need to consider of the short exact sequence of cochain complexes

$$0 \to Z^*(\tilde{P}^*) \to \tilde{P}^* \to d(\tilde{P}^*) \to 0,$$

with the differentials of $Z^*(\tilde{P}^*)$ and $d(\tilde{P}^*)$ all zeroes. Again by the assumption on P_*, we have $Z^*(\tilde{P}^*)$ projective since each $Z^p(\tilde{P}$ is a direct summand of the free module $\mathrm{Hom}_G(P_p, \mathbb{Z})$. So we have $\mathrm{Tor}_1^{\mathbb{Z}}(Z^p(\tilde{P}^*), -) = 0$. After tensoring $H^q(H;\mathbb{Z})$, we have another short exact seqeunce of cochain complexes

$$0 \to Z^*(\tilde{P}^*) \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z}) \to \tilde{P}^* \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z}) \to d(\tilde{P}^*) \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z}) \to 0.$$

From the above short exact sequence, we can form a cohomology long exact sequence

$$\cdots \to Z^p(\tilde{P}^*) \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z}) \to H^p(\tilde{P}^* \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z})) \to d^p(\tilde{P}^p) \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z}) \xrightarrow{\partial^p} Z^{p+1}(\tilde{P}^*) \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z}) \to \cdots$$

Thus we have the short exact sequence

$$0 \to \mathrm{im}\, \partial^{p-1} \to H^p(\tilde{P}^* \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z})) \to \ker \partial^p \to 0.$$

Then substitute (45) into the above short exact sequence, we have

$$0 \to H^p(G;\mathbb{Z}) \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z}) \to H^p(\tilde{P}^* \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z})) \to \mathrm{Tor}_1^{\mathbb{Z}}(H^{p+1}(G;\mathbb{Z}), H^q(H;\mathbb{Z})) \to 0.$$

Now summing over all $p, q \in \mathbb{Z}$ such that $p + q = n$ for a fixed n, we have

$$0 \to \bigoplus_{p+q=n} H^p(G;\mathbb{Z}) \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z}) \to \bigoplus_{p+q=n} H^p(\tilde{P}^* \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z})) \to \bigoplus_{p+q=n+1} \mathrm{Tor}_1^{\mathbb{Z}}(H^p(G;\mathbb{Z}), H^q(H;\mathbb{Z})) \to 0.$$

Finally, note that

$$\bigoplus_{p+q=n} H^p(\tilde{P}^* \otimes_{\mathbb{Z}} H^q(H;\mathbb{Z})) \simeq H^n(\tilde{P}^* \otimes_{\mathbb{Z}} \tilde{Q}^*) \overset{\mu}{\simeq} H^n(G \times H;\mathbb{Z}),$$

where $\tilde{Q}^* := \mathrm{Hom}_H(Q_*, \mathbb{Z})$ with $Q_* \to \mathbb{Z}$ a free $\mathbb{Z}H$-resolution of $\mathbb{Z}$. That's how (42) has been proved. □

Exercise 28

Let G be the free group on $\{s, t\}$, and let $T \subseteq G$ be the free group on $\{t\}$. Let $\mathbb{Z}'$ denote the abelian group $\mathbb{Z}$, made into a G-module (and a T-module) by the formulas $s \cdot a = t \cdot a = -a$.

(i) Show that $H_0(G; \mathbb{Z}') = H_0(T; \mathbb{Z}') = \mathbb{Z}/2$.

(ii) Show that $H_1(T; \mathbb{Z}') = 0$ but $H_1(G; \mathbb{Z}') \simeq \mathbb{Z}$.

Exercise 29

Let H be the cyclic subgroup C_m of the cyclic group C_{mn}. Show that the map $\mathrm{cor}_H^G : H_*(C_m; \mathbb{Z}) \to H_*(C_{mn}; \mathbb{Z})$ is the natural inclusion $\mathbb{Z}/m \hookrightarrow \mathbb{Z}/mn$ for $*$ odd, while $\mathrm{res}_H^G : H^*(C_{mn}; \mathbb{Z}) \to H^*(C_m; \mathbb{Z})$ is the natural projection $\mathbb{Z}/mn \to \mathbb{Z}/m$ for $*$ even.

Proof. Since $\mathbb{Z}$ is a trivial $\mathbb{Z}C_{mn}$-module, it is naturally a trivial $\mathbb{Z}C_m$-module by fogetting C_{mn}. Thus a projective C_{mn}-resolution $P_* \to \mathbb{Z}$ of $\mathbb{Z}$ is naturally a projective C_m-resolution of $\mathbb{Z}$. In this case, $\mathbb{Z}_{C_m} \to \mathbb{Z}_{C_{mn}}$ and $\mathbb{Z}^{C_mn} \to \mathbb{Z}^{C_m}$ all coincides with the identity. Moreover, we have already obtained the following results

$$H_n(C_m; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 0, \\ \mathbb{Z}/m, & n \text{ is odd}, \\ 0, & n \text{ is even}, \end{cases}$$

$$H^n(C_m; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & n \text{ is odd}, \\ \mathbb{Z}/m, & n \text{ is even}, \end{cases}$$

in class. So the $\mathrm{cor}_H^G : H_*(C_m; \mathbb{Z}) \to H_*(C_{mn}; \mathbb{Z})$ is the natural inclusion $\mathbb{Z}/m \hookrightarrow \mathbb{Z}/mn$ for $*$ odd; while $\mathrm{res}_H^G : H^*(C_{mn}; \mathbb{Z}) \to H^*(C_m; \mathbb{Z})$ is the natural projection $\mathbb{Z}/mn \to \mathbb{Z}/m$ for $*$ even. $\square$

Exercise 30

Show that the transfer map defined here agrees with the transfer map defined in 6.3.9 using Shapiro's Lemma.

Proof. Since the transfer maps on homology and cohomology are all universal δ functors, we just have to show that $\mathrm{tr} : H_0(G; A) \to H_0(H; A)$ and $\mathrm{tr} : H^0(H; A) \to H^0(G; A)$ coincide with those tr_s defined by Shapiro's Lemma.

Since $A_G = \mathbb{Z} \otimes_{\mathbb{Z}G} A$ and $A_H = \mathbb{Z}G \otimes_{\mathbb{Z}H} A$, the transfer map tr is defined by

$$\begin{aligned} A_G &\to A_H \\ a &\mapsto \sum_{x \in G/H} xa. \end{aligned} \tag{46}$$

But tr_s is defined via

$$\begin{aligned} A &\to \mathrm{Ind}_H^G(A) \\ a &\mapsto \sum_{x \in G/H} x \otimes a \end{aligned}$$

wich *de facto* induces the same map as (46) on A_G. So we have proved that tr and tr_s are the same for homology.

The argument for cohomology follows dually. $\square$

Exercise 31

Prove that

Lemma 5. Let $f : E \to E'$ be a morphism of spectral sequences such that there exists some r and $f_r^{p,q} : E_r^{p,q} \to E_r'^{p,q}$ is an isomorphism for all $p, q \in \mathbb{Z}$. Then $f_\infty^{p,q} : E_\infty^{p,q} \to E_\infty'^{p,q}$ is an isomorphism.

Proof. By definition, the E_{r+1} page of the spectral sequence is the subquotient the the E_r page, *id est*,

$$E_{r+1}^{p,q} := \frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}} = \frac{Z_r^{p,q}}{B_r^{p,q}},$$

where we denote by $Z_r^{p,q} := \ker d_r^{p,q}$ and $B_r^{p,q} := \operatorname{im} d_r^{p-r,q+r-1}$ for simplicity. Now consider the differential

$$d_{r+1}^{p,q} : E_{r+1}^{p,q} \to E_{r+1}^{p+r+1,q-r},$$

which can be viewed as a morphism

$$d_{r+1}^{p,q} : \frac{Z_r^{p,q}}{B_r^{p,q}} \to \frac{Z_r^{p+r+1,q-r}}{B_r^{p+r+1,q-r}}.$$

So we have

$$\begin{aligned} \ker d_{r+1}^{p,q} &= Z_{r+1}^{p,q} / B_r^{p,q}, \\ \operatorname{im} d_{r+1}^{p,q} &= B_{r+1}^{p+r+1,q-r} / B_r^{p+r+1,q-r}. \end{aligned} \tag{47}$$

Note that (47) also gives us a short exact sequence

$$0 \longrightarrow Z_{r+1}^{p,q}/B_r^{p,q} \longrightarrow E_{r+1}^{p,q} \longrightarrow B_{r+1}^{p+r+1,q-r}/B_r^{p+r+1,q-r} \longrightarrow 0.$$

Since $f : E \to E'$ is a morphism of spectral sequences, $f_*^{*,*}$ commute with all $d_*^{*,*}$ and $d_*'^{*,*}$, and thus preserves boundaries and cycles. Moreover we can view $Z_{r+1}^{*,*}$, $B_r^{*,*}$ and $B_{r+1}^{*,*}$ as submodules of $E_r^{*,*}$. Since $f_r^{*,*} : E_r^{*,*} \to E_r'^{*,*}$ is an isomorphism and $f_{r+1}^{*,*}$ are induced by $f_r^{*,*}$ on $E_r^{*,*}$, we have

$$f_{r+1}^{p,q}(Z_{r+1}^{p,q}) = f_r^{p,q}(Z_{r+1}^{p,q}) \simeq Z_{r+1}'^{p,q},$$

Similarly

$$\begin{aligned} f_r^{p,q}(B_r^{p,q}) &\simeq B_r'^{p,q} \\ f_{r+1}^{p+r+1,q-r}(B_{r+1}^{p+r+1,q-r}) = f_r^{p+r+1,q-r}(B_{r+1}^{p+r+1,q-r}) &\simeq B_{r+1}'^{p+r+1,q-r}. \end{aligned}$$

Finally, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Z_{r+1}^{p,q}/B_r^{p,q} \arrow[r] \arrow[d, "\simeq"] & E_{r+1}^{p,q} \arrow[r] \arrow[d, "f_{r+1}^{p,q}"] & B_{r+1}^{p+r+1,q-r}/B_r^{p+r+1,q-r} \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & Z_{r+1}'^{p,q}/B_r'^{p,q} \arrow[r] & E_{r+1}'^{p,q} \arrow[r] & B_{r+1}'^{p+r+1,q-r}/B_r'^{p+r+1,q-r} \arrow[r] & 0, \end{tikzcd}$$

with the leftmost vertical isomorphism induced by $f_r^{p,q}$ and the rightmost vertical isomorphism induced by $f_r^{p+r+1,q-r}$. By the five lemma, $f_{r+1}^{p,q} : E_{r+1}^{p,q} \to E_{r+1}'^{p,q}$ is an isomorphism. Applying the above argument repeatedly, the morphisms $f_n^{p,q} : E_n^{p,q} \to E_n'^{p,q}$ are all isomorphisms.

□

Exercise 32

Let $\{E_r^{p,q}\}$ be a bounded spectral sequence of R-modules, and $E_r^{p,q} \Rightarrow H^n$. Assume $\forall p, q \in \mathbb{Z}$, $E_r^{p,q}$ is a finitely generated R-module, prove that each H^n is also finitely generated.

Proof. Since $E \to H^*$ and E is bounded, we have the isomorphisms

$$0 \neq E_r^{p,n-p} \simeq F^p H^n / F^{p+1} H^n,$$

or equivalently, the short exact sequence

$$0 \to F^{p+1} H^n \to F^p H^n \to E_r^{p,n-p} \to 0 \tag{48}$$

for only finitely many $p \in \mathbb{Z}$. And there is also a Hausdorff and exhausted finite filtration on each H^n

$$H^n = F^0 H^n \supseteq F^1 H^n \supseteq \cdots F^m H^n \supseteq F^{m+1} H^n = 0.$$

Without loss of generality, we may assume that the above descending chain of R-modules is proper, that is

$$H^n = F^0 H^n \supsetneq F^1 H^n \supsetneq \cdots F^m H^n \supsetneq F^{m+1} H^n = 0. \tag{49}$$

So we have

$$F^m H^n \simeq E_r^{m,n-m}$$

by (48). Thus $F^m H^n$ is a finitely generated R-module since $E_r^{m,n-m}$ is, by assumption. Upon this, and the exact sequence

$$0 \to F^m H^n \to F^{m-1} H^n \to E_r^{m-1,n-m+1} \to 0,$$

we know that $F^{m-1} H^n$ is finitely generated. Repeating similar arguments m times, we know that $F^0 H^n \simeq H^n$ is finitely generated, as desired.

□

Exercise 33

Suppose $E_2^{p,q} = 0$ unless $q = 0$ or n for some $n \geq 2$. Prove that there is a long exact sequence

$$\cdots \to H^{p+n} \to E_2^{p,n} \to E_2^{p+n+1,0} \to H^{p+n+1} \to E_2^{p+1,n} \to E_2^{p+n+2,0} \to \cdots.$$

Proof. Since $E_2^{p,q} \neq 0$ only if for $q = 0, n$, the spectral sequence degenerates at page $r = n+1$, or equivalently

$$E_\infty^{p,q} \simeq \cdots \simeq E_{n+2}^{p,q} \simeq E_{n+1}^{p,q}. \tag{50}$$

By assumption, the spectral sequence is convergent to H^*, so we have a finite filtration of H^n

$$H^n = F^0 H^n \supseteq F^1 H^n \supseteq \cdots \supseteq F^n H^n \supseteq F^{n+1} H^n = 0$$

and isomorphisms

$$E_{n+1}^{p,q} \simeq E_\infty^{p,q} \simeq F^p H^{p+1} / F^{p+1} H^{p+q} \tag{51}$$

for $n \in \mathbb{Z}$. So by (51) and (50), we have exact sequences

$$\begin{gathered} \vdots \\ 0 \to E_{n+1}^{p+n+1,0} \to H^{p+n+1} \to E_{n+1}^{p+1,n} \to 0, \\ 0 \to E_n^{p+n,0} \to H^{p+n} \to E_n^{p,n} \to 0, \\ \vdots \end{gathered} \tag{52}$$

for each n. But note that $E_{n+1}^{p+n+1,0}$ is the cokernel of the differential $d_{n+1}^{p,n} : E_n^{p,n} \to E_n^{p+n+1,0}$ and $E_{n+1}^{p+1,n}$ is the kernel of the differential $d_{n+1}^{p+1,n} : E_n^{p+1,n} \to E_n^{p+n+2,0}$. So we may connect the short exact sequences (52) to form a long exact sequence

$$\cdots \to H^{p+n} \to E_n^{p,n} \to E_n^{p+n+1,0} \to H^{p+n+1} \to E_n^{p+1,n} \to E_n^{p+n+2,0} \to \cdots .$$

By analyzing the shape of the differentials of lower pages, we have in addition that

$$E_n^{p,q} \simeq E_{n-1}^{p,q} \simeq \cdots \simeq E_2^{p,q}.$$

So the last long exact sequence becomes

$$\cdots \to H^{p+n} \to E_2^{p,n} \to E_2^{p+n+1,0} \to H^{p+n+1} \to E_2^{p+1,n} \to E_2^{p+n+2,0} \to \cdots ,$$

as desired, by substituting $E_n^{p,q}$ with $E_2^{p,q}$. □

Exercise 34

Give a spectral sequence proof of the Universal Coefficient Theorem for cohomology.

Proof. First let us recall the Universal Coefficient Theorem for cohomology:

Theorem 6. Let P_* be a complex of projective R-modules with the assumption that $d_n(P_n)$ are all projective. Then for very n and every R-module M, there is an exact sequence

$$0 \to \mathrm{Ext}_R^1(H_{n-1}(P), M) \to H^n(\mathrm{Hom}_R(P_*, M)) \to \mathrm{Hom}_R(H_n(P_*), M) \to 0.$$

To prove this theorem, we need to pick an injective resolution $M \hookrightarrow I^*$ of M. Then we form a double complex $C^{*,*}$ with $C^{p,q} := \mathrm{Hom}_R(P_q, I^p)$. Since each I^p is injective, so the functor $\mathrm{Hom}_R(-, I^p)$ is exact, thus

$$H_I^{p,q}(C^{*,*}) := \ker d_{II}^{p,q} / \ker d_{II}^{p,q-1} = H^q(\mathrm{Hom}_R(P_*, I^p)) \simeq \mathrm{Hom}_R(H_q(P_*), I^p).$$

And the E^2 page of the spectral sequence ${}^IE$ follows as

$${}^IE_2^{p,q} = H^p(H_I^{*,q}) = H^p(\mathrm{Hom}_R(H_q(P_*), I^*)) = \mathrm{Ext}_R^p(H_q(P_*), M),$$

by definition of $\mathrm{Ext}_R^p(H_q(P_*), M)$. But by assumption, since $d_n(P_n)$ is projective, the exact sequence

$$0 \to d_{q+1}(P_{q+1}) \to Z_q \to H_q(P_*) \to 0$$

is a projective resolution of $H_q(P_*)$, $\mathrm{Ext}_R^*(H_q(P_*), M)$ is the cohomology of the complex

$$\mathrm{Hom}_R(Z_q, M) \xrightarrow{\delta^0} \mathrm{Hom}_R(d_{q+1}(P_{q+1}), M) \xrightarrow{\delta^1} 0 \to \cdots ,$$

so we have

$${}^IE_2^{p,q} = \mathrm{Ext}_R^p(H_q(P_*), M) = \begin{cases} \mathrm{Hom}_R(H_q(P_*), M), & p = 0, \\ \mathrm{Ext}_R^1(H_q(P_*), M), & p = 1, \\ 0, & p \geq 2. \end{cases}$$

Thus the 2nd page of the spectral seqeuce ${}^{I}E$ looks like

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\mathrm{Hom}_R(H_{q+1}(P_*), M) \arrow[drr] & \mathrm{Ext}^1_R(H_{q+1}(P_*), M) & 0 & \cdots \\
\mathrm{Hom}_R(H_q(P_*), M) \arrow[drr] & \mathrm{Ext}^1_R(H_q(P_*), M) & 0 & \cdots . \\
\mathrm{Hom}_R(H_{q-1}(P_*), M) \arrow[drr] & \mathrm{Ext}^1_R(H_{q-1}(P_*), M) & 0 & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}
$$

Since ${}^{I}E$ has only two non-trivial columns, we have ${}^{I}E^{p,q}_\infty \simeq {}^{I}E^{p,q}_2$ and ${}^{I}E \Rightarrow H^*(\mathrm{Tot}(C^{*,*}))$. By **Exercise 33**, we have a short exact sequence for each q

$$0 \to {}^{I}E_2^{1,q-1} \to H^q(\mathrm{Tot}(C^{*,*})) \to {}^{I}E_2^{0,q} \to 0,$$

which amounts to the exact sequence

$$0 \to \mathrm{Ext}^1_R(H_{q-1}(P_*), M) \to H^q(\mathrm{Tot}(C^{*,*})) \to \mathrm{Hom}_R(H_q(P_*), M) \to 0. \tag{53}$$

Now let us consider the other spectral sequence ${}^{II}E$, which can be computed by the double complex $H^{*,*}_{II}$. We have

$$H^{p,q}_{II}(C^{*,*}) := \ker d_I^{p,q} / \ker d_I^{p-1,q} = H^p(\mathrm{Hom}_R(P_q, I^*)) = \mathrm{Ext}^p_R(P_q, M) \simeq \begin{cases} \mathrm{Hom}_R(P_q, M), & p = 0 \\ 0, & p \geq 1 \end{cases}$$

since each P_q is projective. Thus the 2nd page of ${}^{II}E$ has only one non-trivial column,

$${}^{II}E_2^{p,q} = H^q(H^{p,*}_{II}) = \begin{cases} H^q(\mathrm{Hom}_R(P_*, M)), & p = 0, \\ 0, & p \geq 1. \end{cases}$$

Thus ${}^{II}E$ is convergent to $H^*(\mathrm{Tot}(C^{*,*}))$ by construction and

$$H^n(\mathrm{Tot}(C^{*,*})) \simeq \bigoplus_{p+q=n} {}^{II}E^{p,q}_\infty \simeq {}^{II}E_2^{0,n} = H^n(\mathrm{Hom}_R(P_*, M)).$$

Substituting this into the exact sequence (53), we have

$$0 \to \mathrm{Ext}^1_R(H_{q-1}(P_*), M) \to H^q(\mathrm{Hom}_R(P_*, M)) \to \mathrm{Hom}_R(H_q(P_*), M) \to 0. \tag{54}$$

for each q, as desired. □

Exercise 35

Given rings R and S, let L be a right R-module, M an R-S bimodule, and N a left S-module, so that the tensor product $L \otimes_R M \otimes_S N$ makes sense.

(i) Show that there are two spectral sequences, such that

$$ {}^{I}E^2_{p,q} = \mathrm{Tor}^R_p(L, \mathrm{Tor}^S_q(M,N)), $$
$$ {}^{II}E^2_{p,q} = \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(L,M), N) $$

converging to the same graded abelian group H_*.

(ii) If M is a flat S-module, show that the spectral sequence ${}^{II}E$ converges to $\mathrm{Tor}^R_*(L, M \otimes_S N)$. If M is a flat R-module, show that the spectral sequence ${}^{I}E$ converges to $\mathrm{Tor}^S_*(L \otimes_R M, N)$.

Proof. (i) To show this, we take an $P_* \to L$ to be an R-projective resolution of L and $Q_* \to N$ to be an S-projective resolution. Then we consider the double complex $P_* \otimes_R N \otimes_S Q_*$. The spectral sequence ${}^{I}E$ associated to the filtration ${}^{I}F$ of the total complex $\mathrm{Tot}(P_* \otimes_R M \otimes_S Q_*)$ is easy to compute. We have

$$ {}^{I}E^1_{p,q} = H^v_q(P_p \otimes_R M \otimes_S Q_*) = P_p \otimes_R H^v_q(M \otimes_R Q_*) = P_p \otimes_R \mathrm{Tor}^S_q(M,N) $$

and

$$ {}^{I}E^2_{p,q} = H^h_p(P_* \otimes_R \mathrm{Tor}^S_q(M,N)) = \mathrm{Tor}^R_p(L, \mathrm{Tor}^S_q(M,N)). $$

Since ${}^{I}E$ is first quadrant, it converges to $H_* := H_*(\mathrm{Tot}(P_* \otimes_R M \otimes_S Q_*)$.

Analogously, we have

$$ {}^{II}E^1_{p,q} = H^h_q(P_* \otimes_R M \otimes_S Q_p) = H^h_q(P_* \otimes_R M) \otimes_R Q_p = \mathrm{Tor}^R_q(L,M) \otimes_S Q_p $$

and

$$ {}^{II}E^2_{p,q} = H^v_p(\mathrm{Tor}^R_q(L,M) \otimes_S P_*) = \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(L,M), N), $$

which are the first two pages of the spectral sequence ${}^{II}E$.

(ii) Since M is S-flat, we have $\mathrm{Tor}^S_q(M,N) = 0$ for $q \geq 1$. Hence the spectral sequence ${}^{I}E$ collapses at the 1st page, yielding

$$ H_n(\mathrm{Tot}(P_* \otimes_R M \otimes_S Q_*)) \simeq {}^{I}E^\infty_{n,0} = {}^{I}E^2_{n,0} = H_n(P_* \otimes_R M \otimes_S N) = \mathrm{Tor}^R_n(L, M \otimes_S N). $$

This shows that ${}^{II}E$ converges to $H_*(\mathrm{Tot}(P_* \otimes_R M \otimes_S Q_*)) \simeq \mathrm{Tor}^R_*(L, M \otimes_S N)$, as we have proved in (i) that both ${}^{II}E$ and ${}^{I}E$ converge to H_*. The proof for M being R-flat is similar. □