Assignment

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1 Basics of Lie Groups

Exercise 1

(a) By assumption, H is a Lie subgroup of G, which means that H is a subgroup of G and the natural inclusion $i : H \hookrightarrow G$ is an injective immersion. Then we can restrict i on a sufficient small neighborhood U in H containing e, as the restriction $i_U : U \hookrightarrow G$ of the inclusion is also injective and immersive.

For any $h \in U \subset H$, the tangent map

$$d_h i_U : T_h U \to T_h G$$

is of constant rank $m = \dim H$ by assumption, so we can find an open neighborhood $U' \subset G$ containing e and appropriate coordinate systems of H and G, such that the coordinate of U' is

$$(x_1, \dots, x_n) \in \mathbb{R}^n$$

and the coordinate of U is

$$(x_1, \dots, x_m, 0, \dots, 0) \in \mathbb{R}^n.$$

The existence of such U' and appropriate coordinate systems is provided by the Constant Rank Theorem. In this case, we can choose S as a subset of U' characterized by

$$S := \{ (0, \dots, 0, x_{m+1}, \dots, x_n) \} \subset U',$$

this is indeed a (local) submanifold of G, by construction. We can see that $e \in U \cap S$, because in our choice of coordinate systems, we have

$$e = (0, \dots, 0).$$

Then, the multiplication map

$$m : G \times G \to G$$

has an explicit expression when restricted to $S \times U$

$$\begin{aligned} m : S \times U &\to G, \\ (g, h) &\mapsto gh. \end{aligned}$$

if the coordinate of g is

$$g = (0, \dots, 0, g_{m+1}, \dots, g_n)$$

and the coordinate of h is

$$h = (h_1, \dots, h_m, 0, \dots, 0),$$

then the coordinate of gh is just

$$gh = (h_1, \dots, h_m, g_{m+1}, \dots, g_n).$$

So the tangent map of m at (e,e)

$$d_{(e,e)}m : T_{(e,e)}(S \times U) \simeq T_eS \times T_eU \to T_eG \tag{1}$$

is a linear isomorphism, hence there is a neighborhood V of e in G such that

$$m : S \times U \to V$$

is a diffeomorphism, by the Inverse Function Theorem.

Now we compute the dimension of $H \cap S$. For any $h \in H \cap S$, we have

$$\dim(T_hH) + \dim(T_hS) - \dim(T_h(H \cap S)) = \dim T_hG. \tag{2}$$

On the other hand,

$$\dim(T_hH) = \dim(T_eH)$$

and

$$\dim(T_hS) = \dim(T_eS),$$

which holds because S is a submanifold. Equation (2) together with the isomorphism (1) tells us that

$$\dim(T_h(H \cap S)) = 0, \forall h \in H \cap S,$$

which says that $H \cap S$ is of dimension 0 in G, which turns out to be a discrete set, consisting of a union of finitely many points or a union of countably many points.

The last left to us is to show that

$$H \cap V = (H \cap S)U.$$

Note that

$$V \simeq S \times U \simeq SU,$$

where SU is the subset of G generated by elements of the form sh with $s \in S$ and $h \in U$. For any $g \in H \cap V = H \cap SU$, $g = sh \implies s = gh^{-1} \in H$, so $s \in H \cap S$ and $g \in (H \cap S)U$. Conversely, for any $sh \in (H \cap S)U$, $h \in U \subset H \implies sh \in H$, so $sh \in H \cap SU = H \cap V$. This completes the proof.

(b) If H is a closed Lie subgroup of G, this implies that H is a closed submanifold of G. Then we can choose $e \in U \subset H$ as a (local) submanifold of G. Shrinking U and S sufficiently small if necessary, we can infer that $H \cap S = \{e\}$ since they are two submanifolds intersecting transversely and $\dim H \cap S = 0$.

Conversely, if $H \cap S$ has cardinality 1, we can choose the coordinate of H near e as $V \backslash S$, which turns out to be a local submanifold of G. Here V and S has the same definitions as in **(a)**. Moreover the collection $\{h(V\backslash S)\}_{h \in H}$ is an open cover of H whose coordinates are locally of the form

$$(x_1, \dots, x_m, 0, \dots, 0),$$

this shows H is a closed submanifold of G.

Exercise 2

(a) Denote N as the stabilizer of the action $\alpha : G \times X \to X$, and consider the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\alpha"] \arrow[d, "p \times \mathrm{id}_X"'] & X \\ G/N \times X \arrow[ur, "\alpha'"'] & \end{tikzcd}$$

where $p : G \to G/N$ is the quotient map, and α' is defined as

$$\alpha'(g \cdot N, p) = \alpha(g, p)$$

for all $g \cdot N \in G/N$ and $p \in X$. Since N is normal, G/N is a Lie group[OV, Theorem 3, pp. 10], so $\alpha' : G/N \times X \to X$ is a well defined action of abstract group G/N on X, and by definition the diagram is commutative. So,

$$\alpha = \alpha' \circ (p \times \mathrm{id}_X).$$

Since α is smooth and smooth maps are stable under composition, α is smooth.

(b) Take $X = G/H$ which is a smooth manifold [OV, Theorem 3, pp. 10], and consider the natural smooth action

$$\begin{aligned} \alpha : G \times G/H &\to G/H, \\ (g_1, g_2H) &\mapsto g_1g_2H. \end{aligned}$$

We find that the normal closed Lie subgroup N of G acts trivially on the coset G/H, so by **(a)**, there is an induced smooth Lie group action

$$\alpha' : G/N \times G/H \to G/H.$$

Observe that the stabilizer of eH in G/H under the action α' is exactly H/N. By [OV, Theorem 1, pp. 7], H/N is a closed Lie subgroup of G/N.

Exercise 3

Picking an arbitrary point $x \in X$, we want to show the orbit $G \cdot x$ is a closed submanifold of X. Denote the action of G on X as

$$\begin{aligned} \alpha :&G \times X \to X, \\ &(g, x) \mapsto \alpha(g, x), \end{aligned}$$

and denote α_x as

$$\begin{aligned} \alpha_x :&G \to X, \\ &g \mapsto \alpha(g, x). \end{aligned}$$

Note that $G \cdot x$ is the image of G under α_x, and since G is compact and α_x is smooth, $G \cdot x$ is a compact subset in X, hence is closed in X. What left to us is to show is that $G \cdot x$ is a submanifold of X.

To show $G \cdot x$ is a submanifold of X, it suffices to show for any $x' \in G \cdot x$, there is an open neighborhood V of x' in X, such that

$$V \cap G \cdot x = \{ (x_1, \dots, x_k, 0, \dots, 0) \mid (x_1, \dots, x_n) \in U \}$$

where $n = \dim X$ is the dimension of X. In our case, since every $x' \in G \cdot x$ can be obtained by $x' = g \cdot x := \alpha(g, x)$ for some $g \in G$, we just have to check such open neighborhood of x exists.

Since the map $\alpha_x : G \to X$ is of constant rank, so there exist open neighborhoods U_e of $e \in G$ and V_x of $x \in X$, such that α_x is of the form

$$\begin{aligned} \alpha_x :&U_e \to V_x, \\ &(x_1, \dots, x_m) \mapsto (x_1, \dots x_k, 0, \dots, 0), \end{aligned} \tag{3}$$

by the constant rank theorem. Thus we can choose $V = V_x$, which shows that $G \cdot x$ is a submanifold of X.

Exercise 4

(a) $\mathrm{SU}(n)$.

Dimension. The dimension of $\mathrm{SU}(n)$ is defined as the dimension of its Lie algebra $\mathfrak{su}(n)$. We note that an $n \times n$ complex matrix X is an element in $\mathfrak{su}(n)$ iff

$$\begin{aligned} \overline{X}^T + X &= 0, \\ \mathrm{tr} X &= 0. \end{aligned} \tag{4}$$

With the canonical identification

$$\mathrm{Mat}(n; \mathbb{C}) \simeq \mathbb{R}^{2n^2},$$

condition (4) can be identified as $n^2 + 1$ linear equations, so the dimension of $\mathfrak{su}(n)$ is $2n^2 - (n^2 + 1) = n^2 - 1$. Thus the real dimension of $\mathrm{SU}(n)$ is

$$\dim \mathrm{SU}(n) = n^2 - 1.$$

Fundamental Group. When $n = 1$, we have $\mathrm{SU}(1) = (1)$, thus we have

$$\pi_1(\mathrm{SU}(1)) = 0.$$

When $n \geq 2$, consider the map

$$\begin{aligned} p : &\mathrm{SU}(n) \to S^n_{\mathbb{C}}, \\ &A \mapsto Ae_n, \end{aligned}$$

where A is a special unitary $n \times n$ matrix and

$$e_n = (0, \dots, 0, \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}) \in \mathbb{C}^n,$$

and $S^n_{\mathbb{C}}$ is the standard n-sphere in $\mathbb{C}^n$ with respect to the canonical Hermitian metric on $\mathbb{C}^n$. Note that p is surjective, and the elements in the preimage of e_n are precisely of the form

$$A = \begin{pmatrix} A' & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SU}(n),$$

which can be identified as an element $A' \in \mathrm{SU}(n-1)$. Since $\mathrm{SU}(n)$ acts on $S^n_{\mathbb{C}}$ transitively, $S^n_{\mathbb{C}}$ is a homogeneous $\mathrm{SU}(n)$-space and $p : \mathrm{SU}(n) \to S^n_{\mathbb{C}}$ is a fibration with fiber $\mathrm{SU}(n-1)$. Also note that we have the natural isometry

$$S^n_{\mathbb{C}} \simeq S^{2n-1},$$

where S^{2n-1} is the $2n-1$-sphere in $\mathbb{R}^{2n}$ with the canonical Euclidean metric. So the fibration is

$$\mathrm{SU}(n-1) \hookrightarrow \mathrm{SU}(n) \longrightarrow S^{2n-1} .$$

The induced long homotopy exact sequence is

$$\cdots \to \pi_2(S^{2n-1}) \to \pi_1(\mathrm{SU}(n-1)) \to \pi_1(\mathrm{SU}(n)) \to \pi_1(S^{2n-1}) \to \cdots ,$$

where $\pi_2(S^{2n-1}) = \pi_1(S^{2n-1}) = 0$ for $n \geq 2$. Then we have

$$\pi_1(\mathrm{SU}(n)) \simeq \pi_1(\mathrm{SU}(n-1)),$$

but

$$\pi_1(\mathrm{SU}(2)) \simeq \pi_1(S^3) = 0,$$

so we have

$$\pi_1(\mathrm{SU}(n)) = 0$$

for all $n \geq 1$.

(b) $\mathrm{Sp}(n;\mathbb{R})$

Dimension. By definition, $A \in \mathrm{Sp}(n;\mathbb{R})$ iff A is a $2n \times 2n$ real matrix satisfying

$$A^T J A = J,$$

where J is the real $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix}.$$

Take any $2n \times 2n$ matrix $X \in \mathfrak{sp}(n;\mathbb{R})$, we have

$$\exp(tX)^T J \exp(tX) = J.$$

Taking derivative at $t = 0$ of both sides, we have

$$X^T J + JX = 0. \tag{5}$$

Conversely, if X satisfies Equation (5), then by the existence an uniqueness of ODE, we know that $\exp(tX) \in \mathrm{Sp}(n;\mathbb{R})$. So $X \in \mathfrak{sp}(n;\mathbb{R})$ iff X satisfies Equation (5).

Solving Equation (5), we find X is of the form

$$X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix},$$

where A is an $n \times n$ real matrix, B, C are $n \times n$ symmetric real matrix. So we have

$$\dim \mathrm{Sp}(n;\mathbb{R}) = n^2 + \frac{n(n+1)}{2} \cdot 2 = n(2n+1).$$

Fundamental Group. Actually, we have

Claim 1. With the canonical identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $\mathrm{U}(n)$ can be seen as a subgroup of $\mathrm{Sp}(n;\mathbb{R})$, and the inclusion $\mathrm{U}(n) \hookrightarrow \mathrm{Sp}(n;\mathbb{R})$ is a homotopy equivalence.

Proof. We choose the retraction as

$$\begin{aligned} f_t : [0,1] \times \mathrm{Sp}(n;\mathbb{R}) &\to \mathrm{Sp}(n;\mathbb{R}), \\ A &\mapsto A(A^T A)^{-\frac{t}{2}}. \end{aligned}$$

Since $A^T A$ is symmetric positive, thus the expression $(A^T A)^{-t/2}$ makes sense. What remains is to show it's symplectic.

Indeed, for A symplectic, A^T is also symplectic, thus

$$(A^T A)^T J (A^T A) = A^T A J A^T A = A^T J A = J,$$

which means that $P = A^T A$ is symplectic. We need to show that $P^{-t/2}$ is symplectic. Since P is symmetric positive, then its eigenvectors span the whole $\mathbb{R}^{2n}$. Let e_i, e_j be the eigenvectors of P with eigenvalues λ_i, λ_j, we have

$$(e_i)^T J e_j = (e_i)^T P^T J P e_j = (Pe_i)^T J (Pe_j) = \lambda_i \lambda_j (e_i)^T J e_j.$$

By the non-degeneracy of J, we have $(e_i)^T J e_j \neq 0$, thus

$$(\lambda_i \lambda_j) = 1 \implies (\lambda_i \lambda_j)^{-\frac{t}{2}} = 1.$$

So

$$(e_i)^T (P^{-\frac{t}{2}})^T J P^{-\frac{t}{2}} e_j = (P^{-\frac{t}{2}} e_i)^T J (P^{-\frac{t}{2}} e_j) = \lambda_i^{-\frac{t}{2}} \lambda_j^{-\frac{t}{2}} (e_i)^T J e_j = (e_i)^T J e_j.$$

Since $\{e_i\}_{i=1}^n$ spans $\mathbb{R}^{2n}$, the above is equivalent to

$$(P^{-\frac{t}{2}})^T J (P^{-\frac{t}{2}}) = J,$$

which shows $P^{-\frac{t}{2}}$ is symplectic.

It's clear that

$$f_t(A) = A(A^T A)^{-\frac{t}{2}}$$

is symplectic, since it's the multiplication of symplectic matrices A and $(A^T A)^{-\frac{t}{2}}$, which is continuous in t. For any $A \in \mathrm{Sp}(n;\mathbb{R})$, we have $f_0(A) = A \in \mathrm{Sp}(n;\mathbb{R})$ and $f_1(A) = A(A^T A)^{-\frac{1}{2}} \in \mathrm{U}(n)$, thus f_t is indeed a deformation retraction. □

Using the above claim, we have

$$\pi_1(\mathrm{U}(n)) = \pi_1(\mathrm{Sp}(n;\mathbb{R})).$$

What left to us is to compute the fundamental group of $\mathrm{U}(n)$. The argument is similar as in the case of $\mathrm{SU}(n)$. The corresponding fibration is

$$\mathrm{U}(n-1) \hookrightarrow \mathrm{U}(n) \longrightarrow S^{2n-1}$$

and by the induced homotopy long exact sequence we have

$$\pi_1(\mathrm{U}(n)) \simeq \pi_1(\mathrm{U}(n-1)).$$

For all $n \geq 2$. But by definition we have $\mathrm{U}(1) \simeq S^1$, so we have

$$\pi_1(\mathrm{U}(n)) \simeq \pi_1(\mathrm{U}(1)) \simeq \pi_1(S^1) = \mathbb{Z}.$$

Thus we have

$$\pi_1(\mathrm{Sp}(n;\mathbb{R})) = \mathbb{Z}.$$

(c) $\mathrm{SO}(p,q;\mathbb{R})$

Dimension. Using methods as in **(a)** and **(b)**, we have $X \in \mathfrak{so}(p,q;\mathbb{R})$ iff

$$\begin{aligned} X^T I_{p,q} + I_{p,q} X &= 0, \\ \mathrm{tr} X &= 0, \end{aligned} \tag{6}$$

where $I_{p,q}$ is the $(p+q) \times (p+q)$ real matrix

$$I_{p,q} = \begin{pmatrix} \mathrm{id}_p & 0 \\ 0 & -\mathrm{id}_q \end{pmatrix}.$$

We find that X satisfies Equation (6) iff

$$X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where A is a $p \times p$ real antisymmetric matrix, C is a $q \times q$ real antisymmetric matrix, B is a $p \times q$ real matrix. Thus we have

$$\dim \mathrm{SO}(p,q;\mathbb{R}) = \frac{p(p-1)}{2} + \frac{q(q-1)}{2} + pq.$$

Fundamental Group. According to our previous experience, we need to find a geometric object $\mathcal{S}^{p,q}$ that is contractible, such that $\mathrm{SO}(p,q;\mathbb{R}) \to \mathcal{S}^{p,q}$ is a fibration, with fibers which we are familiar with.

With this motivation, let

$$\mathcal{S}^{p,q} = \{ (e,f) \} / \sim$$

with e an oriented orthogonal basis of $\mathbb{R}^p$, f an oriented orthogonal basis of $\mathbb{R}^q$, and orientations of $\mathbb{R}^p$ and $\mathbb{R}^q$ are chosen to be the standard ones. And we define

$$(e,f) \sim (e',f')$$

iff we can find some $A \in \mathrm{SO}(p)$ and $B \in \mathrm{SO}(q)$ such that

$$e = Ae' \text{ and } f = Bf'.$$

Thus we have a fibration

$$\mathrm{SO}(p,q;\mathbb{R}) \to \mathcal{S}^{p,q},$$

and the fiber at the point $[(e,e')] \in \mathcal{S}^{p,q}$ is $\mathrm{SO}(p) \times \mathrm{SO}(q)$, where e and e' are the standard orthogonal bases of $\mathbb{R}^p$ and $\mathbb{R}^q$. If we can show that $\mathcal{S}^{p,q}$ is contractible, then using the homotopy long exact sequence induced by the fibration

$$\mathrm{SO}(p) \times \mathrm{SO}(q) \hookrightarrow \mathrm{SO}(p,q;\mathbb{R}) \longrightarrow \mathcal{S}^{p,q} \ ,$$

we have

$$\pi_1(\mathrm{SO}(p,q;\mathbb{R})) \simeq \pi_1(\mathrm{SO}(p) \times \mathrm{SO}(q)) \simeq \pi_1(\mathrm{SO}(p)) \times \pi_1(\mathrm{SO}(q)). \tag{7}$$

We show the contractibility of $\mathcal{S}^{p,q}$ by induction. $\mathcal{S}^{0,0}$ is trivially contractible, since it is the one-point space. Suppose that $\mathcal{S}^{p-1,q}$ is contractible, then consider the commutative diagram

$$\begin{tikzcd}
\mathrm{SO}(p-1) \times \mathrm{SO}(q) \arrow[r] \arrow[d] & \mathrm{SO}(p) \times \mathrm{SO}(q) \arrow[r] \arrow[d] & \mathbb{R}^p \arrow[d, equal] \\
\mathrm{SO}(p-1,q;\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{SO}(p,q;\mathbb{R}) \arrow[r] \arrow[d] & \mathbb{R}^p \ , \\
\mathcal{S}^{p-1,q} \arrow[r] & \mathcal{S}^{p,q} &
\end{tikzcd}$$

where the rows and columns are all fibrations. By examining the homotopy long exact sequence, we find that the vanishing of homotopy groups of $\mathcal{S}^{p-1,q}$ implies the vanishing of the homotopy groups of $\mathcal{S}^{p,q}$, which completes the induction.

Now using (7), we can compute the fundamental groups of $\mathrm{SO}(p,q;\mathbb{R})$. With

Claim 2.

$$\pi_1(\mathrm{SO}(n)) = \begin{cases} 0, & n = 1, \\ \mathbb{Z}, & n = 2, \\ \mathbb{Z}_2, & n \geq 3. \end{cases}$$

we can show

$\pi_1(\mathrm{SO}(p,q;\mathbb{R}))$	$p = 1$	$p = 2$	$p \geq 3$
$q = 1$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
$q = 2$	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$
$q \geq 3$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

Proof of Claim 2. When $n = 1$, the statement is trivial.

When $n = 2$, we have $\mathrm{SO}(2) \simeq S^1$. So

$$\pi_1(\mathrm{SO}(2)) \simeq \pi_1(S^1) = \mathbb{Z}.$$

When $n \geq 3$, we have $\mathrm{SO}(n) \simeq \mathbb{R}P^n$. So

$$\pi_1(\mathrm{SO}(n)) \simeq \pi_1(\mathbb{R}P^n) = \mathbb{Z}_2.$$

$\square$

Exercise 5

First note that for all $n > 0$, we have

Claim 3. The center $Z(O(n))$ of the orthogonal group $O(n)$ is

$$Z(O(n)) = \{I_n, -I_n\} \simeq \mathbb{Z}_2$$

for all $n \in \mathbb{N}$.

Proof. If $A \in Z(O(n))$, then A must commute with the elementary row switching and column switching matrices E_{ij} and the matrices N_i of the form

$$N_i = \mathrm{diag}\{1, \ldots, 1, -1, 1, \ldots 1\},$$

with only one -1 in the ith diagonal slot.

A commuting with all the E_{ij}'s tells us that A must be of the form

$$A = \begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{pmatrix}$$

, and A commuting with all the N_i's tells us that

$$A = \mathrm{diag}\{a, a, a, \cdots, a\}.$$

But $A^T A = A^2 = \mathrm{diag}\{a^2, a^2, a^2, \cdots, a^2\} = I_n$, so we have $a = \pm 1$ hence we have

$$A = I_n \text{ or } -I_n.$$

□

and

Claim 4. The center $Z(\mathrm{SO}(n))$ of $\mathrm{SO}(n)$ is

$$Z(\mathrm{SO}(n)) = \begin{cases} \mathrm{SO}(2), & n = 2, \\ \{I_n, -I_n\} \simeq \mathbb{Z}_2, & n \text{ even}, \\ \{I_n\}, & n \text{ odd}. \end{cases}$$

Proof. When $n = 2$, the result is easy to check because $\mathrm{SO}(2) \simeq S^1$ is abelian.

When $n \geq 3$, suppose $A \in Z(\mathrm{SO}(n))$, thus A should commute with all matrices E'_{ij}, where E'_{ij} is the matrix switching the ith row and the jth row then inverting the sign of one of the rows. A should also commute with matrices N_{ij} of the form

$$N'_{ij} = \mathrm{diag}\{1 \ldots, 1, -1, 1, \ldots, -1, \ldots 1\},$$

with -1's in the ith and the jth slots. Similarly as the $O(n)$ case, A commuting with all E'_{ij}'s and N'_{ij}'s tells us than A must be of the form

$$A = \mathrm{diag}\{a, a, \ldots, a\},$$

but with the condition $A^T A = I_n$ and $\det A = 1$, we have

$$A = I_n$$

when n is odd and

$$A = \pm I_n$$

when n is even. □

With Claim 3 and Claim 4, we can easily see that when n is odd, the $-I_n \notin \mathrm{SO}(n)$ but $-I_n \in O(n)$, and the isomorphism $\mathbb{Z}_2 \times \mathrm{SO}(n) \simeq O(n)$ can be explicitly constructed as

$$\begin{aligned} \mathbb{Z}_2 \times \mathrm{SO}(n) &\xrightarrow{\sim} O(n) \\ (a, g) &\mapsto ag. \end{aligned}$$

This group homomorphism is injective, since if there are any $a \in \mathbb{Z}_2$ and $g \in G$ such that

$$ag = I_n,$$

we can infer $a = I_n$ by the signs of the determinants of both sides, hence $g = I_n$. The homomorphism is surjective, since for any $h \in O(n)$ but $h \notin \mathrm{SO}(n)$, we have $-I_n h \in \mathrm{SO}(n)$ and

$$(-I_n)(-I_n h) = h.$$

When n is even, the center of $O(n)$ is $\mathbb{Z}_2$, but the center of $\mathbb{Z}_2 \times \mathrm{SO}(n)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathrm{SO}(2)$, so they can not be isomorphic.

Exercise 6

It suffices to show that for any $(x, y) \in \mathbb{R}^2/\mathbb{Z}^2$ with $0 < x, y < 1$, there exists some $m \in \mathbb{Z}$, such that

$$|\{y - \sqrt{2}x - \sqrt{2}m\}| < \epsilon \tag{8}$$

for any real $\epsilon > 0$, where

$$\{x\} := x - [x]$$

denotes the fractional part of $x \in \mathbb{R}$. Since Equation (8) holds modulo integer, we may assume that

$$0 < y - \sqrt{2}x < 1.$$

Let $N = [\frac{1}{\epsilon}]$, we can divide the half interval $[0, 1)$ into $N + 1$ half intervals of length $1/(N + 1)$, and consider the following sequence of $N + 2$ numbers

$$0, \{\sqrt{2}\}, \{2\sqrt{2}\}, \ldots, \{(N+1)\sqrt{2}\}. \tag{9}$$

Note that there must exist an interval of length $1/(N + 1)$ containing at least two elements of the sequence (9), so we denote them as $\{a\sqrt{2}\}$ and $\{b\sqrt{2}\}$ assume that $a < b$ ($a \neq b$ since $\sqrt{2}$ is irrational.). Then we have

$$\{b\sqrt{2}\} - \{a\sqrt{2}\} = (b - a)\sqrt{2} - [b\sqrt{2}] + [a\sqrt{2}] < \frac{1}{N+1} < \epsilon,$$

where we can take $k = b - a$ meanwhile the above equation tells us that we have find a positive integer $k \in \mathbb{Z}$ such that

$$\{k\sqrt{2}\} < \epsilon. \tag{10}$$

With the same k in (10), we have

$$mk\sqrt{2} = m[k\sqrt{2}] + m\{k\sqrt{2}\}$$

for any integer $m \in \mathbb{Z}$, and

$$\{mk\sqrt{2}\} = m\{k\sqrt{2}\} \iff m\{k\sqrt{2}\} < 1.$$

Let M be the largest integer such that

$$M\{k\sqrt{2}\} < 1,$$

then we have

$$\frac{1}{M+1} < \{k\sqrt{2}\} < \frac{1}{M},$$

and

$$1 - \frac{1}{M+1} = \frac{M}{M+1} < \{Mk\sqrt{2}\} < 1.$$

So $M\{k\sqrt{2}\} = \{Mk\sqrt{2}\}$ differs from 1 by less than $1/(M+1)$. But

$$\frac{1}{M+1} < \{k\sqrt{2}\} < \epsilon$$

by (10), therefore each number of the sequence

$$0, \{k\sqrt{2}\}, 2\{k\sqrt{2}\}, \dots, M\{k\sqrt{2}\}$$

is in a subinterval of $[0,1)$ of length $1/(M+1) < \epsilon$. $y - \sqrt{2}x$ must be in one of these subintervals. So there must be some integer $n < M$ such that

$$|\{y - \sqrt{2}x - nk\sqrt{2}\}| = |\{y - \sqrt{2} - \{nk\sqrt{2}\}\}| < \frac{1}{M+1} < \epsilon.$$

This proves that the irrational winding is dense on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Remark. The method using here works for more general irrational windings

$$\begin{aligned} \mathbb{R} &\to \mathbb{T}^2, \\ t &\mapsto (t, \theta t), \end{aligned}$$

where θ are any irrational number.

Exercise 7

We prove the assertion by induction.

For $n = 1$, suppose $\{1, x_1\}$ is $\mathbb{Q}$- linearly independent, which is equivalent to say that x_1 is irrational. Using the same method in **Exercise 6**, we can show that

$$\{x_1\mathbb{Z}\}$$

is dense in $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Conversely, if $\{1, x_1\}$ is $\mathbb{Q}$-dependent, then x_1 is rational, say

$$x_1 = \frac{p}{q}, p, q \in \mathbb{Z} \text{ coprime},$$

then the image of $\{x_1\mathbb{Z}\}$ in $\mathbb{R}/\mathbb{Z}$ is

$$\{0, \frac{p}{q}, \frac{2p}{q}, \dots, \frac{(q-1)p}{q}\},$$

which is not dense, a contradiction.

Assume the assertion holds for $n \in \mathbb{Z}$. Then given a $\mathbb{Q}$- linearly independent set

$$\{1, x_1, \dots, x_n, x_{n+1}\},$$

we want to show that the image of

$$\{nx_1\} \times \{nx_2\} \times \cdots \times \{nx_n\} \times \{nx_{n+1}\}, \forall n \in \mathbb{Z}$$

is dense in $\mathbb{T}^{n+1}$. It's easy to see that there is at least one $x_i \notin \mathbb{Q}$, for simplicity we let $x_{n+1} \notin \mathbb{Q}$. On the other hand, the set

$$\{1, x_1, \dots, x_n\},$$

is $\mathbb{Q}$-independent, otherwise we can find $a_0, \dots, a_n \in \mathbb{Q}$ such that

$$a_0 + a_1x_1 + \cdots + a_nx_n + 0 \cdot x_{n+1} = 0,$$

a contradiction. Then we have

$$\{nx_1\} \times \{nx_2\} \times \cdots \times \{nx_n\}, \forall n \in \mathbb{Z}$$

dense in $\mathbb{T}^n$ by the inductive hypothesis, and

$$\{nx_{n+1}\}, \forall n \in \mathbb{Z}$$

dense in $\mathbb{T}^1$ by $n = 1$ case, thus

$$\{nx_1\} \times \{nx_2\} \times \cdots \times \{nx_n\} \times \{nx_{n+1}\}, \forall n \in \mathbb{Z}$$

is dense in $\mathbb{T}^{n+1}$.

Conversely, suppose

$$\{nx_1\} \times \{nx_2\} \times \cdots \times \{nx_n\} \times \{nx_{n+1}\}, \forall n \in \mathbb{Z}$$

is dense in $\mathbb{T}^{n+1}$. Then

$$\{nx_1\} \times \{nx_2\} \times \cdots \times \{nx_n\}, \forall n \in \mathbb{Z}$$

is dense in $\mathbb{T}^n$. By the inductive hypothesis, the set

$$\{1, x_1, \dots, x_n\}$$

is $\mathbb{Q}$-independent. If the set

$$\{1, x_1, \dots, x_n, x_{n+1}\}$$

is $\mathbb{Q}$-linearly dependent, we can find $b_0, \dots, b_n \in \mathbb{Q}$, such that

$$b_0 + b_1x_1 + \cdots + b_nx_n + x_{n+1} = 0.$$

By the $\mathbb{Q}$-independence, we can see that x_{n+1} is irrational. Again by the inductive hypothesis, we have the image of

$$\{nb_1x_1\} \times \{nb_2x_2\} \times \cdots \times \{nb_nx_n\}, \forall n \in \mathbb{Z}$$

dense in $\mathbb{T}^n$, and

$$\mathbb{Z}[nx_{n+1}], \forall n \in \mathbb{Z}$$

is dense in $\mathbb{T}$, from which we can infer that

$$C := \{nb_1x_1\} \times \{nb_2x_2\} \times \cdots \times \{nb_nx_n\} \times \{nx_{n+1}\}, \forall n \in \mathbb{Z}$$

is dense in $\mathbb{T}^{n+1}$.

Now consider the map

$$\begin{aligned} f : \mathbb{T}^{n+1} &\to \mathbb{T}, \\ (y_1, \dots, y_{n+1}) &\mapsto y_1 + \cdots y_{n+1} \mod \mathbb{Z}, \end{aligned}$$

which is continuous and surjective. Then

$$f(\overline{C}) \subset \overline{f(C)}$$

holds by the continuity of f. But, since C is dense and f is surjective we have

$$\mathbb{T} = f(\mathbb{T}^{n+1}) = f(\overline{C}) \subset \overline{n(b_1x_1 + \cdots b_nx_{n+1} + x_{n+1})} = \overline{-nb_0}, \forall n \in \mathbb{Z}, b_0 \in \mathbb{Q},$$

which is ridiculous, since

$$\overline{-nb_0}, \forall n \in \mathbb{Z}$$

consists a finite abelian group in $\mathbb{T}$, which is not dense.

Exercise 8

We know that a 2×2 matrix $X \in \mathfrak{sl}(2;\mathbb{R})$ iff $\mathrm{tr}X = 0$, because of the formula

$$\det \exp(X) = \exp(\mathrm{tr}X).$$

Denote the two eigenvalues of X by λ_1, λ_2, then we have

$$\lambda_1 + \lambda_2 = \mathrm{tr}X = 0,$$
$$\lambda_1\lambda_2 = \det X.$$

So λ_1, λ_2 are the solutions of the equation

$$x^2 + \det X = 0 \tag{11}$$

Now we characterize the solutions of Equation (11).

(i) $\det X = 0$, thus the two eigenvalues all equal to 0, so X is similar to the matrix

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$$

for some $a \in \mathbb{R}$, and $\exp X$ is similar to the matrix

$$\begin{pmatrix} 1 & e^a \\ 0 & 1 \end{pmatrix},$$

which has eigenvalues 1.

(ii) $\det X < 0$, then X has two distinct eigenvalues $\lambda, -\lambda$ with $\lambda > 0$. So X is similar to the matrix

$$\begin{pmatrix} \lambda & a \\ 0 & -\lambda \end{pmatrix}$$

for some $a \in \mathbb{R}$, so $\exp X$ is similar to the matrix

$$\begin{pmatrix} e^\lambda & b \\ 0 & e^\lambda \end{pmatrix}$$

with some $b \in \mathbb{R}$. $\exp X$ has two distinct eigenvalues $e^\lambda, e^{-\lambda}$.

(iii) $\det X > 0$, then X has two distinct eigenvalues $\mathrm{i}t, -\mathrm{i}t$ with $t \in \mathbb{R}$. So $\exp X$ has two eigenvalues $e^{\mathrm{i}t}, e^{-\mathrm{i}t}$. In this case we need to be a little careful, since the eigenvalues of $\exp X$ could be real.

  (a) $t = 0 \mod 2\pi$, the two eigenvalues of $\exp X$ all equal to 1,
  (b) $t = \pi \mod 2\pi$, the two eigenvalues of $\exp X$ all equal to -1,
  (c) otherwise, the two eigenvalues of $\exp X$ are pure complex numbers $e^{\mathrm{i}t}$ and $e^{-\mathrm{i}t}$.

From the above characterization, we can see that $\exp : \mathfrak{sl}(2;\mathbb{R}) \to \mathrm{SL}(2;\mathbb{R})$ is not surjective, for $\mathrm{tr}(\exp X)$ takes values only in the interval $[-2, \infty)$, indeed

$$\mathrm{tr}(\exp X) = 1 + 1 = 2$$

or

$$\mathrm{tr}(\exp X) = e^\lambda + e^{-\lambda} \geq 2$$

or

$$\mathrm{tr}(\exp X) = e^{\mathrm{i}t} + e^{-\mathrm{i}t} = 2\cos t \in [-2, 2].$$

Then for any $Y \in \mathrm{SL}(2; \mathbb{R})$ with $\mathrm{tr} Y < -2$, say

$$Y = \begin{pmatrix} -3 & 0 \\ 0 & -\frac{1}{3} \end{pmatrix},$$

there is no preimage of Y under the exponential map in $\mathfrak{sl}(2; \mathbb{R})$.

Remark. The result is quite shocking, because $\mathrm{SL}(2; \mathbb{R})$ is a connected and simple Lie group. This shows that even with the condition G is connected, $\exp : \mathfrak{g} \to G$ is not generally a surjection. From this example, we can see the trickiness of the exponential maps.

Exercise 9

We prove the assertion by *reductio ad absurdum*. Suppose every open neighborhood V of $e \in G$ contains a subgroup H of G. So we can choose V small enough such that the closure $\overline{V}$ is compact and the inverse of exponential map

$$\exp^{-1} : V \to \mathfrak{g}$$

is a diffeomorphism between V and its image.

By assumption, we have $H \subset V$ being a subgroup of G. For any $h \in H$, denote the unique preimage of h under the exponential map by X, that is,

$$\exp(X) = h. \tag{12}$$

Denote the cyclic subgroup of H generated by h as $\{h^n\}_{n\in\mathbb{Z}}$. Thus $\{h^n\}_{n\in\mathbb{Z}} \subset H$ and the preimage of $\{h^n\}$ under exp is the abelian subgroup

$$\{nX\}_{n\in\mathbb{Z}}$$

of $\mathfrak{g}$, which is not compact in $\mathfrak{g}$, from which we can infer that $\exp^{-1}(\overline{H})$ is not compact. But the closure $\overline{H}$ of H is compact in G, since it is a closed subset of the compact set $\overline{V}$, which is a contradiction to our assumption that V is diffeomorphic to $\exp^{-1}(V)$.

Exercise 10

The assertion holds when the Lie group G is connected.

Recall that the center Z of a Lie algebra $\mathfrak{g}$ of a Lie group G is defined as

$$Z(\mathfrak{g}) := \{\, \xi \in \mathfrak{g} \mid [\xi, \eta] = 0, \forall \eta \in \mathfrak{g} \,\}.$$

Then if $\xi \in \mathfrak{g}$ satisfying

$$\mathrm{Ad}_g(\xi) = \xi, \forall g \in G,$$

then we can take $g = \exp(\eta), \forall \eta \in \mathfrak{g}$ and take the differential of $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ at $e \in G$, so we have

$$\mathrm{ad}_\eta(\xi) = [\eta, \xi] = 0, \forall \eta \in \mathfrak{g},$$

by the relation $d\mathrm{Ad} = \mathrm{ad}$ at $e \in G$. This implies that

$$\left\{\, \xi \in \mathfrak{g} \;\middle|\; \mathrm{Ad}_g(\xi) = \xi, \forall g \in G \,\right\} \subset Z(\mathfrak{g}).$$

For the converse inclusion, it since G is connected, we can find a smooth path $\gamma(t)$ connecting e and g. Let $\eta \in \mathfrak{g}$ be the initial velocity of path $\gamma(t)$:

$$\eta := \frac{d}{dt}\gamma(t)|_{t=0}.$$

Let

$$h(t) = \mathrm{Ad}_{\gamma(t)}(\xi),$$

for arbitrary $\xi \in Z(\mathfrak{g})$, then

$$\begin{aligned} h(0) &= \mathrm{Ad}_e(\xi) = \xi, \\ \frac{dh(0)}{dt} &= \mathrm{ad}_\eta(\xi) = [\eta, \xi] = 0, \end{aligned}$$

which is a first order ODE. By the existence and uniqueness of first-order ODE, there is a unique solution $h(t)$. Observe that $h(t) = \xi$ solves the equation, so we have

$$h(t) = \xi, \forall t \in [0, 1]$$

In particular, we have

$$h(1) = \mathrm{Ad}_{\gamma(1)} = \mathrm{Ad}_g\xi,$$

which shows the other direction of the inclusion.

Exercise 11

$\Rightarrow$ Given $G_1 \subset G_2$ as inclusion of groups, we want to show that G_1 is a Lie subgroup of G_2, that is, the inclusion $i : G_1 \hookrightarrow G_2$ is an injective immersion. We only need to show it is an injective immersion near $e \in G_1$, since the property of i at arbitrary $g \in G$ can be obtained by left translating the property at e.

By **Exercise 1(a)**, we have shown that for any Lie subgroup $H \subset G$, there is an open neighborhood $e \in U \subset H$ of H and a submanifold $S \subset G$ of G, such that the restriction of the multiplication map $m : G \times G \to G$

$$m : S \times U \xrightarrow{\sim} V$$

is a diffeomorphism, where $V \subset G$ is a sufficiently small neighborhood of G. Apply this result first to the Lie subgroup G_2, we get diffeomorphisms

$$S_2 \times U_2 \xrightarrow{\sim} V_2,$$

where U_2 is a neighborhood of $e \in G_2$, S_2 is the required submanifold of G, V_2 the required open neighborhood of $e \in G$. Then we take

$$U_1 := U_2 \cap G_1,$$

and apply the result to the Lie subgroup G_1, so we have a diffeomorphism

$$S_1 \times U_1 \xrightarrow{\sim} V_1,$$

where the notions S_1 and V_1 are self-evident.

Consider $V = V_1 \cap V_2$ and shrink U_i, S_i small enough, we have the following commutative diagram

$$\begin{tikzcd} U_1 \arrow[r, hook] \arrow[d, "i"] & S_1 \times U_1 \arrow[r, "\simeq"] & V \arrow[d, equal] \\ U_2 & S_2 \times U_2 \arrow[l, two heads] \arrow[r, "\simeq"] & V \end{tikzcd}$$

, so in appropriate coordinate systems, we have

$$\begin{gathered} i : U_1 \to U_2 \\ (x_1, \dots, x_i, 0, \dots, 0) \mapsto (x_1, \dots, x_j, 0, \dots, 0) \end{gathered}$$

which is obviously an injective immersion. This shows that $G_1 \subset G_2$ is a Lie subgroup of G_2. Moreover, takeing the tangent map of $i : U_1 \to U_2$ we find

$$d_e i : \mathfrak{g}_1 = T_e U_1 \to \mathfrak{g}_2 = T_e U_2$$

is an inclusion via local coordinates.

$\Leftarrow$ Conversely, if $\mathfrak{g}_1 \subset \mathfrak{g}_2$, with G_1 connected. Then we can choose a smooth path $\gamma(t)$ connecting e and any $g \in G_1$, such that $\gamma(0) = e$ and $\gamma(1) = g$. Consider the velocity

$$\eta(t) = \frac{d}{dt}\gamma(t) \tag{13}$$

of $\gamma(t)$. Since $T_{\gamma(t)}G_1$ and $\mathfrak{g}_1 = T_e G_1$ can be identified via a left translation of $\gamma(t)$, $\forall t \in [0,1]$, Equation (13) can be viewed as a first order ODE in the linear space $\mathfrak{g}_1$, with the invariance of Equation (13) under left translations. We can furthermore view Equation 13 as an ODE in the linear space $\mathfrak{g}_2$ via the inclusion $\mathfrak{g}_1 \subset \mathfrak{g}_2$ of Lie algebras, and we want to find solutions $\tilde{\gamma}(t)$ satisfying

$$\begin{cases} \tilde{\gamma}(0) = e, \\ \frac{d}{dt}\tilde{\gamma}(t) = \eta(t), \end{cases}$$

in $\mathfrak{g}_2$. By the uniqueness and existence of ODE's of this type, we have $\tilde{\gamma}(t) = \gamma(t)$, which says that $\gamma(t)$ is also a curve in G_2. Thus $\gamma(1) = g \in G_2$, which shows $G_1 \subset G_2$ since $g \in G_1$ is arbitrary. Then by the first statement of this exercise which we have already proved, G_1 is also a Lie subgroup of G_2.

Exercise 12

(a) This is immediate. Since we have

$$[\mathfrak{a}^M, \mathfrak{a}^M] = [\mathfrak{a}, \mathfrak{a}],$$

and by assumption $\mathfrak{a}$ is commutative, so

$$[\mathfrak{a}^M, \mathfrak{a}^M] = [\mathfrak{a}, \mathfrak{a}] = 0,$$

which means $\mathfrak{a}^M$ is commutative.

(b) If we can show that

$$[\mathfrak{a}^M, \mathfrak{b}^M] = [\mathfrak{a}, \mathfrak{b}],$$

then by assumption $[\mathfrak{a}, \mathfrak{b}] \subset \mathfrak{a} \cap \mathfrak{b}$, then we have $[\mathfrak{a}^M, \mathfrak{b}^M] \subset \mathfrak{a} \cap \mathfrak{b}$. The inclusion

$$[\mathfrak{a}, \mathfrak{b}] \subset [\mathfrak{a}^M, \mathfrak{b}^M],$$

is trivial, the non-trivial part is to show that

$$[\mathfrak{a}^M, \mathfrak{b}^M] \subset [\mathfrak{a}, \mathfrak{b}].$$

First consider the subset H_1 of G defined as

$$H_1 := \left\{ g \in G \mid (\mathrm{Ad}_g - \mathrm{id})\mathfrak{b} \subset [\mathfrak{a}, \mathfrak{b}] \right\}.$$

If H_1 is a closed Lie subgroup of G, then its Lie algebra $\mathfrak{h}_1$ is just

$$\mathfrak{h}_1 = \left\{ \xi \in \mathfrak{g} \,\middle|\, \mathrm{ad}_\xi(\mathfrak{b}) \subset [\mathfrak{a}, \mathfrak{b}] \right\}.$$

Obviously, $\mathfrak{a} \subset \mathfrak{h}_1$, thus $\mathfrak{a}^M \subset \mathfrak{h}_1$ by definition. So we have

$$[\mathfrak{a}^M, \mathfrak{b}] \subset [\mathfrak{a}, \mathfrak{b}]. \tag{14}$$

Next consider the subset H_2 of G defined as

$$H_2 := \left\{ g \in G \,\middle|\, (\mathrm{Ad}_g - \mathrm{id})\mathfrak{a}^M \subset [\mathfrak{a}, \mathfrak{b}] \right\}.$$

If H_2 is a closed Lie subgroup of G, then its Lie algebra $\mathfrak{h}_2$ is

$$\mathfrak{h}_2 = \left\{ \eta \in \mathfrak{g} \,\middle|\, \mathrm{ad}_\eta(\mathfrak{a}^M) \subset [\mathfrak{a}, \mathfrak{b}] \right\}.$$

Then we have $\mathfrak{b} \subset \mathfrak{h}_2$, by (14), so $\mathfrak{b}^M \subset \mathfrak{h}_2$ and finally we have

$$[\mathfrak{a}^M, \mathfrak{b}^M] \subset [\mathfrak{a}, \mathfrak{b}],$$

which proves the assertion.

What remains to show is that H_1 and H_2 are indeed closed Lie subgroups of G. To show this, we need to prove a more general

Claim 5 ([OV, Problem 1.25])**.** Let $R : G \to \mathrm{GL}(V)$ be a linear representation of a Lie group G and $U \subset V$ a subspace of V, $W \subset U$ a subspace of U. Then

$$G(U, W) := \{ g \in G \mid (R(g) - \mathrm{id})U \subset W \}$$

is a closed Lie subgroup of G and

$$\mathrm{Lie}(G(U, W)) = \{ \xi \in \mathfrak{g} \mid (dR)_e(\xi)U \subset W \}.$$

Proof. Consider the subgroup $\mathrm{GL}(U, W, V)$ of $\mathrm{GL}(V)$ defined as

$$\mathrm{GL}(U, W, V) = \{ A \in \mathrm{GL}(V) \mid (A - \mathrm{id})U \subset W \},$$

which is indeed a subgroup of $\mathrm{GL}(V)$, since for $A, B \in \mathrm{GL}(U, W, V)$, we have

$$(AB - \mathrm{id})U = (A - \mathrm{id})(B - \mathrm{id})U + (A - \mathrm{id})U + (B - \mathrm{id})U \subset W.$$

Furthermore, if we take $\dim V = n, \dim U = m, \dim W = l$ with $l \le m \le n$, we can choose a basis $\{e_i\}_{i=1}^n$ of V such that $\{e_i\}_{i=1}^l$ is a basis for W and $\{e_i\}_{i=1}^m$ is a basis for U. Then for any $A \in \mathrm{GL}(U, W, V)$, $A - \mathrm{id}$ with respect to the basis $\{e_i\}_{i=1}^n$ is an $n \times n$ invertible matrix of the form

$$A - \mathrm{id} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2m} & a_{2,m+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{l1} & a_{l2} & \cdots & a_{lm} & a_{l,m+1} & \cdots & a_{ln} \\ 0 & 0 & \cdots & 0 & a_{l+1,m+1} & \cdots & a_{l+1,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & a_{n,m+1} & \cdots & a_{nn} \end{pmatrix},$$

which is determined by $(n-l)\times m$ equations

$$\begin{aligned} a_{l+1,1} &= 0, \\ a_{l+1,2} &= 0, \\ &\cdots \\ a_{l+1,m} &= 0, \\ &\cdots \\ a_{n1} &= 0, \\ &\cdots \\ a_{nm} &= 0. \end{aligned} \tag{15}$$

Denote the subset in $\mathrm{Mat}(n;\mathbb{R}) \simeq \mathbb{R}^{n^2}$ defined by Equations (15) as $L(U,W,V)$, it's easy to see that $L(U,W,V)$ is a closed subset in $\mathbb{R}^{n^2}$. Since the topology on $\mathrm{GL}(V)$ is inherited from the standard topology on $\mathbb{R}^{n^2}$, so

$$\mathrm{GL}(U,W,V) = \mathrm{GL}(V) \cap \mathrm{L}(U,W,V),$$

which means $GL(U,W,V)$ is a closed Lie subgroup of $\mathrm{GL}(V)$.

Consider the Lie group homomorphism

$$R : G \to \mathrm{GL}(V),$$

it's clear that

$$R^{-1}(\mathrm{GL}(U,W,V)) = G(U,W).$$

For the G-action on the coset $\mathrm{GL}(V)/\mathrm{GL}(U,W,V)$

$$G \times \mathrm{GL}(V)/\mathrm{GL}(U,W,V) \xrightarrow{R\times \mathrm{id}} \mathrm{GL}(V) \times \mathrm{GL}(V)/\mathrm{GL}(U,W,V) \xrightarrow{\alpha} \mathrm{GL}(V)/\mathrm{GL}(U,W,V)$$

induced by the representation R, where α is the natural $\mathrm{GL}(V)$-action on $\mathrm{GL}(V)/\mathrm{GL}(U,W,V)$. The stabilizer of the Lie group action $\alpha \circ (R \times \mathrm{id})$ at $\mathrm{GL}(U,W,V) \in \mathrm{GL}(V)/\mathrm{GL}(U,W,V)$ is precisely $G(U,W)$, so by the the rest of the claim follows by [OV, Theorem 1, pp.7]. □

In our case, we can take $V = \mathfrak{g}$, $R : G \to \mathrm{GL}(V)$ to be $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. If we take $U = \mathfrak{b}$ and $W = [\mathfrak{a},\mathfrak{b}] \subset \mathfrak{a}\cap\mathfrak{b} \subset \mathfrak{b} = U$, we have

$$H_1 = G(U,W) = G(\mathfrak{b},[\mathfrak{a},\mathfrak{b}]).$$

If we take $U = \mathfrak{a}^M$ and $W = [\mathfrak{a},\mathfrak{b}] \subset \mathfrak{a}\cap\mathfrak{b} \subset \mathfrak{a} \subset \mathfrak{a}^M = U$, then we have

$$H_2 = G(U,W) = G(\mathfrak{a}^M,[\mathfrak{a},\mathfrak{b}]).$$

2 Algebraic Groups

Exercise 13

(a) The underlying topological space of X is the union of the x-axis and the y-axis, which is clearly connected. Since k is algebraically closed, we have $X = \mathrm{Spec} k[x,y]/(xy)$, which is the union of two proper closed subset in the Zariski topology:

$$X = \mathrm{Spec} k[x,y]/xy = V(x) \cup V(y), \tag{16}$$

thus is not irreducible.

(b) Suppose X admits a structure of algebraic group. Since for an algebraic group, the connected components and the irreducible components of X coincide. But this is ridiculous, since we have already seen from **(a)** that X is connected but not irreducible.

Exercise 14

(a) For the group algebra kG of a finite group G, we define the comultiplication map

$$\Delta : kG \to kG \otimes_k kG$$

as the k-linear extension of the map

$$G \to G \otimes_k G,$$
$$g \mapsto g \otimes g.$$

For the counit map

$$e : kG \to k,$$

we define it as the k-linear extension of the map

$$G \to k,$$
$$g \mapsto 1.$$

And for the antipode map

$$\eta : kG \to kG,$$

we define it as the k-linear extension of the inversion of G,

$$G \to G,$$
$$g \mapsto g^{-1}.$$

And finally we denote the natural multiplication on kG as $m : kG \otimes_s kG \to kG$. It's easy to verify that the maps Δ, m, e, η satisfy the following commutative diagrams

$$\begin{tikzcd} kG \arrow[r, "\Delta"] \arrow[d, "\Delta"] & kG \otimes_k kG \arrow[d, "\mathrm{id}\otimes\Delta"] \\ kG \otimes_k G \arrow[r, "\Delta\otimes\mathrm{id}"] & kG \otimes_k kG \otimes_k kG \end{tikzcd},$$

$$\begin{tikzcd} kG \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}"] & kG \otimes_k kG \arrow[d, "\mathrm{id}\otimes e"] \\ kG \otimes_k G \arrow[r, "e\otimes\mathrm{id}"] & kG \end{tikzcd},$$

$$\begin{tikzcd} kG \otimes_k kG \arrow[r, "\eta\otimes\mathrm{id}"] & kG \otimes_k kG \arrow[d, "m"] \\ kG \arrow[u, "\Delta"] \arrow[r, "\epsilon"] \arrow[d, "\Delta"] & kG \\ kG \otimes kG \arrow[r, "\mathrm{id}\otimes\eta"'] & kG \otimes_k kG \arrow[u, "m"] \end{tikzcd},$$

where $\epsilon : kG \to kG$ is the composition of the counit $e : kG \to k$ and the inclusion $\iota : k \to kG$. So we have showed that the group algebra kG is a Hopf algebra.

(b) The comultiplication $\Delta : k[G] \to k[G] \otimes k[G]$ is defined as $(-) \circ \mu$, where $\mu : G \times G \to G$ is the group multiplication and we use the identification $k[G \times G] \simeq k[G] \otimes k[G]$.

The antipode $\iota : k[G] \to k[G]$ is defined via

$$(\eta f)(x) = f(x^{-1}), \forall f \in k[G], x \in G.$$

And the counit $e : k[G] \to k$ is defined via

$$e(f) = f(e), \forall f \in k[G].$$

We also note that there is a natural multiplication $m : k[G] \otimes_k k[G] \to k[G]$ on $k[G]$

$$\begin{aligned} m : k[G] \otimes k[G] &\to k[G], \\ \sum_i f_i \otimes g_i &\mapsto \sum_i f_i g_i. \end{aligned}$$

The verification of the commutativity of the diagrams

$$\begin{tikzcd} k[G] \arrow[r, "\Delta"] \arrow[d, "\Delta"] & k[G] \otimes_k k[G] \arrow[d, "\mathrm{id}\otimes\Delta"] \\ k[G] \otimes_k [G] \arrow[r, "\Delta\otimes\mathrm{id}"] & k[G] \otimes_k k[G] \otimes_k k[G] \end{tikzcd},$$

$$\begin{tikzcd} k[G] \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}"] & k[G] \otimes_k k[G] \arrow[d, "\mathrm{id}\otimes e"] \\ k[G] \otimes_k [G] \arrow[r, "e\otimes\mathrm{id}"] & k[G] \end{tikzcd},$$

$$\begin{tikzcd} k[G] \otimes_k k[G] \arrow[r, "\eta\otimes\mathrm{id}"] & kG \otimes_k k[G] \arrow[d, "m"] \\ k[G] \arrow[u, "\Delta"] \arrow[r, "\epsilon"] \arrow[d, "\Delta"] & k[G] \\ k[G] \otimes kG \arrow[r, "\mathrm{id}\otimes\eta"'] & k[G] \otimes_k k[G] \arrow[u, "m"] \end{tikzcd},$$

is also easy.

(c) For the case when G is a finite group, the vector space spanned by G, which is also denoted by G, is isomorphic to its dual G^*. Thus the morphism obtained by the k-linear extension of

$$\begin{aligned} kG &\to k[G], \\ g &\mapsto g^*, \end{aligned}$$

is an isomorphism between the Hopf algebras.

Exercise 15

(a) If $\mathrm{Sp}_{2n}(k)$ is not connected, then there exists some idempotent $\mathrm{id}_{2n} \neq A \in \mathrm{Sp}_{2n}(k)$, such that $A^2 = A$. Denote the symplectic form as Ω, we have

$$A \Omega A^T = \Omega$$

and
$$A^T \Omega A = \Omega.$$
Thus
$$\begin{aligned} -\mathrm{id}_{2n} &= \Omega^2 \\ &= A\Omega(A^T)^2\Omega A \\ &= A\Omega A^T \Omega A \\ \Omega^2 A &= -A \end{aligned},$$
a contradiction.

(b) Since
$$O(n,k) := \{X | X^T \cdot X = \mathrm{id}\}$$
is the closed subgroup of the algebraic group $GL(n,k)$, we have
$$O(n,k) = V((\det X)^2 - 1).$$
In the case $\mathrm{char}(k) = 2$, note that $-1 = 1 \in k$, thus
$$O(n,k) = V((\det X)^2 - 1) = V((\det X - 1)^2) = V(\det X - 1),$$
which is irreducible, and thus connected.

In the case $\mathrm{char}(k) \neq 2$, we have
$$O(n,k) = V((\det X)^2 - 1) = V((\det X - 1)(\det X + 1)) = V(\det X - 1) \cup V(\det X + 1),$$
which is not irreducible, and thus is not connected. By definition $SO(n,k)$ is the irreducible component containing e in $O(n,k)$, thus is connected.

Exercise 16

(a) Consider the action of $\mathrm{SL}_2(k)$ on k^2. Take $\{e_1, e_2\}$ as a basis of k^2, we find that an element $g \in \mathrm{SL}_2(k)$ iff $ge_1 \in k\{e_1\}$. Also note that $\mathrm{SL}_2(k)$ acts transitively on the moduli of lines through the origin in k^2, thus $\mathrm{SL}_2(k)/H$ is isomorphism to the moduli space oof lines through the origin in k^2, that is, $\mathrm{SL}_2(k)/H \simeq \mathbb{P}^1_k$.

(b) Since the subgroup H of unipotent upper triangular matrices consists of elements of the form
$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},$$
we find that for all $g \in \mathrm{SL}_2(k)$, $g \in H$ iff $ge_1 = e_1$, that is, H is the stabilizer of e_1. Thus, $\mathrm{SL}_2(k)/H$ is isomorphic to the orbit of e_1 under the action of $\mathrm{SL}_2(k)$. The latter is equal to the subset $k^2 - \{0\}$, which is a quasi-projective variety, but is not projective.

Exercise 17

Let H be a unipotent normal subgroup of G, it is unipotent in $\mathrm{GL}_n(k)$, hence the element of H are all unipotent matrices in $\mathrm{GL}_n(k)$. Since the unipotent matrices are of the form
$$\begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & \ddots & * \\ & & & 1 \end{pmatrix},$$

thus in particular for all $h \in H$ we have $he_1 = e_1$, where $e_1 = (1, 0, \dots, 0) \in k^n$. For any $g \in G$, by the assumption that H is normal in G, we have

$$hge_1 = gh'e_1 = ge_1, \forall g \in G, \tag{17}$$

where h' is some element in H. But on the other hand, the irreducibility of the representation $G \hookrightarrow \mathrm{GL}_n(k)$ says that there is no invariant subspace of V under the action of G, so we have $h = \mathrm{id}$ in (17). Since $h \in H$ is arbitrary, we have $H = \{\mathrm{id}\}$.

Exercise 18

By the Characterization Theorem, we know that there is an equivalence between the categories

$$\{\text{diagonalizable algebraic groups over } k\} \simeq \{\text{finitely generated abelian groups without char}k\text{-torsion}\}^{\mathrm{op}}.$$

Since the category on the left hand side is a full subcategory of the category **Group**, in which the injections are monomorphisms and surjections are epimorphisms. Since the equivalence of the categories is contravariant, so it sends monomorphisms to epimorphisms and *vice versa*, as desired.

Exercise 19

(a) First we describe $k[X]$. Since X is a point, we can identify X as the origin of some k^n. Then by the Hilbert Nullstellensatz, we have

$$k[X] = k[T_1, \dots, T_n]/(T_1, \dots, T_n) \simeq k,$$

which is a local ring with the maximal ideal $\mathfrak{m}_X = (0)$. By the isomorphism $T_xX \simeq \mathfrak{m}_x/\mathfrak{m}_x^2$, we have

$$T_xX = (0)/(0)^2 = (0),$$

which is a 0-dimensional k-vector space.

(b) If $X = k^n$, then we have the affine algebra

$$k[X] = k[T_1, \dots, T_n].$$

If $x = (a_1, \dots, a_n) \in X$ is a point in X, by the Nullstellensatz it corresponds to a maximal ideal $\mathfrak{m}_x \subset k[T_1, \dots, T_n]$. By the Hilbert's Basis Theorem, $\mathfrak{m}_x$ is of the form

$$\mathfrak{m}_x = (T_1 - a_1, \dots, T_n - a_n),$$

thus

$$T_xX \simeq \mathfrak{m}_x/\mathfrak{m}_x^2 \simeq k^n.$$

(c) We have

$$k[X] = k[a, b]/ab \simeq k[a] \oplus k[b],$$

and the maximal ideal $\mathfrak{m}_x$ at $x = (0, 0)$ is the image of the ideal $(a, b) \subset k[a, b]$ in $k[X]$, or explicitly

$$\mathfrak{m}_x = (a) \oplus (b).$$

So, the tangent space is

$$T_xX = \mathfrak{m}_x/\mathfrak{m}_x^2 = (a) \oplus (b)/((a) \oplus (b))^2 = (a) \oplus (b)/(a^2) \oplus (b^2) \simeq k \oplus k = k^2.$$

(d) We have

$$k[X] = k[a,b]/(a^2 - b^3),$$

and the and the maximal ideal $\mathfrak{m}_x$ at $x = (0,0)$ is the image of the ideal $(a,b) \subset k[a,b]$ in $k[X]$. Since every element in $\mathfrak{m}_x$ is of the form

$$af(b) + bg(b),$$

where f, g are polynomials in b; and every element in $\mathfrak{m}_x^2$ is thus of the form,

$$abf(b) + b^2 g(b)$$

which leads to

$$T_x X \simeq \mathfrak{m}_x/\mathfrak{m}_x^2 \simeq k$$

as a k-vector space.

Exercise 20

(a) Given a morphism $\phi : k[X] \to k[\epsilon]$ of k- algebras, it corresponds bijectively to a morphism $\mathrm{Spec} k[\epsilon] \to X$ of affine varieties. Since the affine variety $\mathrm{Spec} k[\epsilon]$ has only one point corresponding to the maximal ideal (ϵ), the map is geometrically picking out a point x and evaluating any $k[\epsilon]$-valued function at x. Thus, $\phi : k[X] \to k[\epsilon]$ can uniquely be written as

$$\phi(f) = f(x) + D(f)\epsilon, \forall f \in k[X],$$

where $D : k[X] \to k$ is a k-linear map. Since we have the canonical identification $T_x X = \mathrm{Der}_k(k[X], k)$, what remains to show is that ϕ is a k-algebra homomorphism iff D is a k-derivation. Indeed, for any $f, g \in k[X]$, we have

$$\phi(fg) = (fg)(x) + D(fg)\epsilon = f(x)g(x) + D(fg)\epsilon,$$

and

$$\phi(f)\phi(g) = (f(x) + D(f)\epsilon)(g(x) + D(g)\epsilon) = f(x)g(x) + f(x)D(g)\epsilon + g(x)D(f)\epsilon.$$

The last equality holds since $\epsilon^2 = 0$. Thus we see that $\phi(fg) = \phi(f)\phi(g)$ iff $D(fg) = f(x)D(g) + g(x)D(f)$, which proves the assertion.

(b) Denote the symplectic form as Ω. An element $A \in \mathrm{Sp}_{2n}(k)$ satisfies

$$A\Omega A^T = \Omega. \tag{18}$$

To compute the Lie algebra $\mathfrak{sp}_{2n}(k)$, we have to view A as a polynomial in $k[\mathrm{Sp}_{2n}(k)]$ with value in $k[\epsilon]$, as in **(a)**. Write

$$A = \mathrm{ev}_e + D\delta, \tag{19}$$

where ev_e is the counit $(k[\mathrm{Sp}_{2n}(k)] \to k_e$. Then substitute (19) into (18) and read off the coefficient of δ, we have

$$D\Omega + \Omega D^T = 0. \tag{20}$$

Since

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

$$\mathfrak{sp}_{2n}(k) = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \,\middle|\, B = B^T, C = C^T \right\}.$$

Similarly, take $\Omega = \mathrm{id}_n$, when $n = 2l$

$$\mathfrak{so}_n = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \,\middle|\, B^T = -B, C^T = -C \right\},$$

and when $n = 2l + 1$

$$\mathfrak{so}_n = \left\{ \begin{pmatrix} A & B & D \\ C & -A^T & E \\ -E^T & -D^T & 0 \end{pmatrix} \,\middle|\, B^T = -B, C^T = -C, U, V \in \mathbb{C}^l \right\}.$$

Exercise 21

Identify the Lie algebras $\mathfrak{g}, \mathfrak{h}$ as the left invariant Lie subalgebras $L(G), L(H)$ of the derivations $\mathrm{Der}_k(k[G], k_e)$ and $\mathrm{Der}_k(k[H], k_e)$, and denote $\phi^* : K[H] \to k[G]$ the morphism between algebras induced by $\phi : G \to H$.

For any $X \in \mathfrak{g}$, we need to show that $\mathrm{Ad}_{\phi(g)}(d\phi)_e X$ and $(d\phi)_e \mathrm{Ad}_g X$ are the same element in $\mathfrak{g}$. It suffices to show that for any $f \in k[H]$, we have

$$(\mathrm{Ad}_{\phi(g)}(d\phi)_e X)(f) = ((d\phi)_e \mathrm{Ad}_g X)(f).$$

The left hand side reads

$$(\mathrm{Ad}_{\phi(g)}(d\phi)_e X)(f) = (d(\phi)_e X)(\mathrm{Ad}^*_{\phi(g)} f) = X(\phi^* \mathrm{Ad}^*_{\phi(g)} f),$$

meanwhile the right hand side reads

$$((d\phi)_e \mathrm{Ad}_g X)(f) = (\mathrm{Ad}_g X)(\phi^* f) = X(\mathrm{Ad}^*_g \phi^* f),$$

so we only need to show that

$$\phi^* \mathrm{Ad}^*_{\phi(g)} f = \mathrm{Ad}^*_g \phi^* f$$

in $k[G]$ for all $f \in k[H]$. Take any $x \in G$, we have

$$\begin{aligned} (\phi^* \mathrm{Ad}^*_{\phi(g)} f)(x) &= (\mathrm{Ad}^*_{\phi(g)} f)(\phi(x)) \\ &= f(\phi(g)\phi(x)\phi(g)^{-1}) \\ &= f(\phi(gxg^{-1})) \\ &= (\phi^* f)(gxg^{-1}) \\ &= (\mathrm{Ad}^*_g \phi^* f)(x), \end{aligned}$$

which concludes the proof.

Exercise 22

(a) Since Hilbert's Basis Theorem tells us that the maximal ideals in $k[x, y]$ are of the form $(x - a, y - b), a, b \in k$, so the underlying topological space of G is $\mathrm{Max}(k[x, y])$, and $m : G \times G \to G$ and $\iota : G \to G$ are indeed morphisms of affine varieties, so it remains to show that m and ι are indeed group multiplication and group inversion, that is, to show that m is associative and $m((x, y), \iota(x, y))$ is indeed the identity. For any $(x_1, y_1), (x_2, y_2), (x_3, y_3) ink^2$, we have

$$\begin{aligned} (x_1, y_1)((x_2, y_2)(x_3, y_3)) &= (x_1, y_1)(x_2 x_3, x_3^p y_2 + y_3) \\ &= (x_1 x_2 x_3, x_2^p x_3^p y_1 + x_3^p y_2 + y_3) \end{aligned}$$

and

$$\begin{aligned}((x_1,y_1)(x_2,y_2))(x_3,y_3) &= (x_1x_2, x_2^p y_1 + y_2)(x_3,y_3) \\ &= (x_1x_2x_3, x_2^p x_3^p y_1 + x_3^p y_2 + y_3),\end{aligned}$$

which shows that m is indeed associative.

For any $(x,y) \in G$,

$$(x,y)(x,y)^{-1} = (x,y)(x^{-1}, -x^{-p}y) = (1, x^{-p}y - x^{-p}y) = (1,0),$$

and

$$(1,0)(x,y) = (x,y) = (x,y)(1,0),$$

which shows that ι is indeed the inversion map and $(1,0)$ is the identity.

(b) We identify the Lie algebra $\mathfrak{g}$ as the left invariant Lie subalgebra of the derivations $\mathrm{Der}_k(k[G], k_e)$. Since $k[G] = k[x,y]$, the Lie algebra is spanned by ∂_x, ∂_y over $k[G]$. For all (a,b), we need to find out the derivations that stay invariant under the transformation $(x,y) \mapsto (ax, bx^p + y)$. The Jacobian under this transformation reads

$$\begin{pmatrix} a & 0 \\ pbx^{p-1} & 1 \end{pmatrix},$$

and is furthermore equal to

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

since $\mathrm{char} k = p$. Thus the two left invariant derivations that k-linearly generate $\mathfrak{g}$ are

$$\begin{aligned} D_1 &= x\partial_x, \\ D_2 &= \partial_y. \end{aligned}$$

And it's easy to see that

$$[D_1, D_2] = [x\partial_x, \partial_y] = 0 = [D_1, D_1] = [D_2, D_2].$$

Again by the fact that $\mathrm{char} k = p$, D_1^p and D_2^p are all derivations. By induction, we have

$$\begin{aligned} D_1^p &= D_1 = x\partial_x, \\ D_2^p &= \partial_y^p \end{aligned}$$

and

$$[D_1^p, D_2^p] = 0.$$

Exercise 23

(a) Since $\alpha_v : G \to V$ is a morphism between algebraic varieties, so for any $X \in \mathfrak{g}$, the image $(d\alpha_v)_e(X)$ is thus a tangent vector in $T_v(V)$, which can be canonically identified as $V \simeq T_v(V)$. Since the Lie algebra $\mathfrak{gl}(V)$ can be identified as $\mathrm{End}(V)$, so the expression $(d\rho)_e(X)v$ makes sense, and can also be viewed as an element in V.

Take any $u \in V^*$, if we can show that

$$\langle u, (d\rho)_e(X)v \rangle = \langle u, (d\alpha_v)_e(X) \rangle$$

then we can show $(d\rho)_e(X) = (d\alpha_v)_e(X)$, where $\langle -, - \rangle$ is the canonical pairing between V and V^*. But the left hand side reads that

$$\begin{aligned}\langle u, (d\rho)_e(X)v\rangle &= \langle u, X(g \mapsto \rho(g))v\rangle \\ &= X(g \mapsto \langle u, \rho(g)v\rangle) \\ &= X(g \mapsto \langle u, \alpha_v(g)\rangle) \\ &= \langle u, (d\alpha_v)_e(X)\rangle\,,\end{aligned}$$

which completes the proof.

(b) By definition, we have

$$\mathrm{Stab}_G(v) = \alpha_v^{-1}(v).$$

Consider the map of algebraic varieties

$$\alpha_v : \mathrm{Stab}_G(v) \to \{v\},$$

which induces the map between tangent spaces

$$d(\alpha_v)_e : \mathrm{Lie}(Stab_G(v)) \to \{0\},$$

hence we are done.

Exercise 24

(a) Consider the morphism of algebraic groups

$$\begin{aligned}\phi : G &\to G \\ g &\mapsto F(g)g^{-1}.\end{aligned}$$

Thus by Lang's Theorem, we have $\phi : G \to G$ being surjective. Also note that we have

$$G^F = \ker \phi.$$

To show that G^F has finitely many points, it's sufficient to show that $\ker \phi$ is finite. If it is infinite, then we have

Exercise 25

Let us consider the morphism of varieties

$$\begin{aligned}\phi : &G \to G, \\ &x \mapsto \sigma(x)x^{-1}.\end{aligned}$$

Since ϕ maps B to e,it factors as

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr] & & G \\ & G/B \arrow[ur, "\tilde{\phi}"'] & \end{tikzcd}$$

But since B is a Borel subgroup, it is in particular parabolic in G, thus the quotient variety G/B is projective, and hence complete. Thus the morphism $\tilde{\phi} : G/B \to G$ of G-varieties is constant, so we have $\phi(G) = \tilde{\phi}(G) = e$, since $\phi(B) = e$.

Exercise 26

Since we have the isomorphism

$$G_s \times G_u \xrightarrow{\sim} G,$$

where G_s is the closed connected subgroup of the semi-simple elements of G, and G_u is the closed connected subgroup of the unipotent elements of G. Moreover, since G/G_u is a torus, and by our assumption of semi-simplicity on G, we have $G = G_s = G/G_s$ being a torus.

Exercise 27

Take $Z = Z(G)^0$, the connected component of the center of G. If $Z \nsubseteq H$, we have $ZH \subseteq N_G(H)$. By the unipotency of G, Z is nontrivial and connected, thus $\dim H < \dim N_G(H)$. If $Z \subseteq H$, then we consider the group H/Z in the group G/Z, and the normalizer group $N_{G/Z}(H/Z) = N_G(H)/Z$, by induction on the dimension we close the proof.

Exercise 28

Take $x \in \mathbb{A}^n$. If $x = 0$, the orbits of $\mathrm{GL}_n, \mathrm{D}_n, \mathrm{SL}_n$ are all 0. If $x \neq 0$, the orbit $\mathrm{GL}_n x$ is $\mathbb{A}^n - \{0\}$. The orbit $\mathrm{D}_n x$ is the line determined by x except $\{0\}$.

Exercise 29

Consider the defining action of $G = \mathrm{GL}_2$ on $\mathbb{P}^1$, which is induced from the action of G on $\mathbb{A}^2$.

References

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