Introduction to Bordism Theory

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1 The Pontrjagin–Thom Construction

The word "manifold" means a compact, smooth manifold with or without boundary.

The *smash product* between two pointed spaces (X, x_0) and (Y, y_0) with x,y_0 distinguished base points is the quotient of $X \times Y$ under the identifications $(x, y_0) \sim (x_0, y)$ for all $x \in X, y \in Y$. the smash product is usually denoted $X \wedge Y$.

One can think X and Y are subsets in $X \times Y$, identified with $X \times \{ y_0 \}$ and $\{ x_0 \} \times Y$. They intersect in $X \times Y$ at one point (x_0, y_0), thus the set generated by the relation $\sim$ in $X \times Y$ can be identified with the connected sum $X \vee Y$, then we have the usually seen definition of the smash product of X and Y

$$X \wedge Y = \frac{X \times Y}{X \vee Y}. \tag{1.1}$$

The smash product $X \wedge Y$ depends on the choice of base points, unless both X and Y are homogeneous.

We can think the smash product to be an analog as the tensor product in the category of abelian groups, in a suitable category of pointed topological spaces. For abelian groups A, B, $A \otimes B$ (co)represents the functor

$$\mathrm{Bil}(A \times B; -). \tag{1.2}$$

Similarly, for pointed spaces X, Y, $X \wedge Y$ corepresents the functor

$$\mathrm{BasePre}(X \times Y, -)$$

Definition 1.1. Given a topological space X, the *unreduced suspension* of X is defined as

$$SX = (X \times I)/\sim,$$

where the relation $\sim$ is generated by

$$(x_1, 0) \sim (x_1, 0) \text{ and } (x_1, 1) \sim (x_2, 1), \forall x_1, x_2 \in X.$$

Definition 1.2. If X is a pointed space with base point x_0, the *reduced suspension* of X is defined as

$$\Sigma X = (X \times I)/(X \times \{ 0 \} \cup X \times \{ 1 \} \cup \{ 1 \} \cup \{ x_0 \} \times I)$$

If we take

Using their bare definitions, we have the useful relation between smash product and reduced suspension,

Lemma 1.1.

$$\Sigma X = S^1 \wedge X.$$

Definition 1.3. A *framing* of a submanifold Y of a manifold M is a trivialization of the normal bundle of Y. If W is a framed manifold of $M \times I$, $\dim W = \dim Y + 1$, then the two framed submanifold obtained by intersecting W with $M \times \{ 0 \}$ and $M \times \{ 1 \}$ are said to be *framed bordant*.

A framed submanifold defines a *collapse map* $M \to S^n \cong \mathbb{R}^n \cup \{ \infty \}$, by sending the point (p, v) in the normal bundle of Y to v and all points outside the normal bundle of Y to ∞.

Definition 1.4. A *trivialization* of a vector bundle $E \to B$ is a specific bundle isomorphism $E \cong B \times \mathbb{R}^n$.
A *framing* of a vector bundle $E \to M$ is a homotopy class of the trivializations of $E \to M$. Here two trivializations are *homotopic* means that there is a path of trivializations joining the two.

For a real vector bundle over a point, i.e., a vector space, choosing a framing is the same thing as choosing an orientation, since $\mathrm{GL}(n; \mathbb{R})$ has only 2 components.

Definition 1.5. A *normal framing* of a submanifold V of M is a homotopy class of trivializations of the normal bundle $\nu(V)$. Two normally framed submanifold V_0, V_1 of M are said to be *normally framed bordant*, if there exist a normally framed submanifold $W \subset M \times I$ so that the intersection of W with $M \times \{ 0 \}$ and $M \times \{ 1 \}$ are V_0 and V_1, with the identification $M = M \times \{ 0 \} = M \times \{ 1 \}$.
We let $\Omega^{\mathrm{fr}}_{k-n,M}$ denote the bordism classes of normally framed submanifolds of M. where $k - n$ is the dimension of submanifolds, k the dimension of M, and n the codimension of submanifolds.

Definition 1.6. If $E \to B$ is any vector bundle over a CW-complex B with metric then the *Thom space* denoted as $\mathrm{Th}(E)$ of $E \to B$ is the quotient $D(E)/S(E)$, where $D(E)$ denotes the unit disk bundle of E and $S(E) \subset D(E)$ denotes the unit sphere bundle.

Lemma 1.2.

1. If $E \to B$ is a vector bundle, then the Thom space of $E \oplus \epsilon$ is the reduced suspension of the Thom space of E, that is,
$$\mathrm{Th}(E \oplus \epsilon) \cong \Sigma \mathrm{Th}(E). \tag{1.3}$$
2. A vector bundle morphism
$$\begin{tikzcd} E \arrow[r] \arrow[d] & E' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$
preserving the metric on each fiber induces a map of Thom spaces $\mathrm{Th}(E) \to \mathrm{Th}(E')$.

Proof. Note that there exists an $O(n)$-equivariant homeomorphism
$$D^{n+1} \to D^n \times I, \tag{1.4}$$
which can be intuitively interpreted as deforming a closed ball into a solid cylinder. This homeomorphism induces a homeomorphism
$$S^n \to S^{n-1} \times \cup D^n \times \{0, 1\}$$
by restricting to the boundary of D^{n+1}.
Fiberwisely, the homeomorphism (1.4) induces a homeomorphism between the spaces

$D(E \oplus \epsilon)/S(E \oplus \epsilon)$ and

$$(D(E) \times I)/(S(E) \times I \cup D(E) \times \{0, 1\}).$$

By Definition 1.2, $D(E \oplus \epsilon)/S(E \oplus \epsilon)$ is homeomorphic to the reduced suspension of $D(E)/S(E)$.
The second statement is clear. □

The group homomorphism $G_n \to O(n)$ induces an action of G_n on $\mathbb{R}^n$. We can use this induced action to form the universal vector bundle over BG_n

$$\begin{tikzcd} EG_n \times_{G_n} \mathbb{R}^n \arrow[d] \\ BG_n \end{tikzcd} .$$

Let us denote this universal vector bundle by $V_n \to BG_n$. The unit sphere and disk bundles of this vector bundles are defined, by our assumption that G_n maps to $O(n)$[1].

Functoriality gives vector bundle morphisms (which are linear injections on fibers)

$$\begin{tikzcd} V_n \arrow[r] \arrow[d] & V_{n+1} \arrow[d] \\ BG_n \arrow[r] & BG_{n+1}. \end{tikzcd}$$

Let

$$MG_n \coloneqq \mathrm{Th}(V_n). \tag{1.5}$$

Theorem 1.3. The fiberwise injection $V_n \to V_{n+1}$ extends to a metric preserving bundle map $V_n \oplus \epsilon \to V_{n+1}$ which is an isomorphism on each fiber, and hence defines a map

$$k_n : SMG_n \to MG_{n+1}.$$

Thus $\mathbf{MG} = \{ MG_n, k_n \}$ is a spectrum, called the *Thom spectrum*.
Moreover, the bordism group $\Omega_n^G(X)$ is isomorphic to $H_n(X; \mathbf{MG})$.

Proof. First we show that $\{MG_n\}$ form a spectrum. We are given the diagram for all n

$$\begin{tikzcd} G_n \arrow[r] \arrow[d] & G_{n+1} \arrow[d] \\ O(n) \arrow[r, hook] & O(n+1). \end{tikzcd}$$

Since the inclusion $O(n) \hookrightarrow O(n+1)$ is of the form

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

the pullback of the bundle $V_{n+1} \to BG_{n+1}$ along the map $BG_n \to BG_{n+1}$ is isomorphic to the bundle $V_n \oplus \epsilon \to BG_n$. Thus we have a morphism of vector bundles that

[1] *Any real vector bunble, over a paracompact base space, has a metric by the existence of local trivializations and partitions of unity. What does that have to do with wether G_n maps to $O(n)$ or not?*

is an isomorphism when restricted to each fiber

$$\begin{array}{ccc} V_n \oplus \epsilon & \longrightarrow & V_{n+1} \\ \downarrow & & \downarrow \\ BG_n & \longrightarrow & BG_{n+1} \end{array}.$$

By Lemma 1.2, this vector bundle morphism induces maps between Thom spaces

$$\mathrm{Th}(V_n \oplus \epsilon) \cong \Sigma \mathrm{Th}(V_n) \to \mathrm{Th}(V_{n+1}),$$

which we take as structure maps

$$k_n : \Sigma MG_n \to MG_{n+1}.$$

Now we have shown that the Thom spectrum **MG** is indeed a spectrum.
Now we show how to establish the isomorphism

$$\Omega_n^G(X) \to \lim_{l\to\infty} \pi_{n+l}(X_+ \wedge MG_l).$$

We now show how to construct the forward collapse map

$$c : \Omega_n^G(X) \to \lim_{l\to\infty} \pi_{n+l}(X_+ \wedge MG_l). \tag{1.6}$$

Choose $[W, f, \gamma_G] \in \Omega_n^G(X)$. If we set l large enough, we can do the embedding $W \hookrightarrow S^{n+l}$. Let $\nu(W)$ be the normal bundle of W in S^{n+l} and $D \subset \nu(W) \subset S^{n+l}$ be the disk bundle of the normal bundle. Let $\gamma_{G\,l} : W \to BG_l$ be the component of the map $\gamma_G : W \to BG$, $\gamma_{G\,l}^*(V_l)$ is isomorphic to $\nu(W)$, since the normal bundle can be reduced to a G_l-vector bundle, by definition. Then the map

$$D = D(\nu(W)) \to D(V_l) \to MG_l \tag{1.7}$$

is a well defined, and we define the map

$$h : S^{n+l} \to MG_l \tag{1.8}$$

by taking everything outside D to the base point of MG_l, and in D being the map (1.7). And similarly, we define a map $\tilde{f} : S^{n+l} \to X$ by

$$\tilde{f}(p) = \begin{cases} \infty, & p \notin W, \\ f(p), & p \in W. \end{cases} \tag{1.9}$$

Then we compose the map

$$\tilde{f} \times h : S^{n+l} \to X \times MG_l$$

with the quotient

$$X \times MG_l \to X_+ \wedge MG_l$$

to give a map

$$\tilde{f} \wedge h : S^{n+l} \to X$$

. Thus we have constructed a map

$$\begin{aligned} c : \Omega_n^G(X) &\to \lim_{l\to\infty} \pi_{n+l}(X_+ \wedge MG_l) \\ [W, f, \gamma_G] &\mapsto \tilde{f} \wedge h. \end{aligned}$$

The verification of the well-definedness of c is tedious.
Next we define the reverse of the collapse map c. A key observations is that $X \times BG_l$ avoids the base points of X_+ and MG_l, thus $X \times BG_l \subset X_+ \wedge MG_l$ is regular. Given a map

$$\alpha : S^{n+l} \to X_+ \wedge MG_l,$$

we can homotope it to a map α' by transversality, such that

$$W := (\alpha')^{-1}(X \times BG_l)$$

is smooth in S^{n+l}. Moreover, $X \times BG_l$ has a tubular neighborhood in $X_+ \times MG_l$, which is isomorphic to the pullback bundle

$$\begin{tikzcd} \mathrm{pr}_2^*(V_l) \arrow[r] \arrow[d] & V_l \arrow[d] \\ X \times BG_l \arrow[r, "\mathrm{pr}_2"] & BG_l \end{tikzcd}$$

and the pullback of $\mathrm{pr}_2^*(V_l)$ along α' is the normal bundle with G_l structure of W in S^{n+l}. Finally, the composite of the map $\alpha' : W \to X \times BG_l$ and the projection $\mathrm{pr}_1 : X \times BG_l \to X$ is the desired singular manifold $W \to X$. □

Theorem 1.4 (Thom-Pontrjagin)**.** For any smooth compact k-manifold M there is an isomorphism

$$c : \Omega^{\mathrm{fr}}_{k-n,M} \to [M, S^n], k \geq n.$$

The forward map is the Thom-Pontrjagin collapse, and the inverse map is the inverse image of a regular value.

2 Stable Homotopy Groups and Stable Pontrjagin–Thom Theorem

Proof. For $V \subset M$ a $k - n$ dimensional submanifold, choose a tubular neighborhood $V \subset U \subset M$, which is isomorphic to the normal bundle $\nu(V)$ of the embedding $V \hookrightarrow M$. Thus V defines a map $\phi : M \to S^n$

$$\phi(x) = \begin{cases} v, & x = (p, v) \in \nu(V), \\ \infty, & x \notin \nu(V) \end{cases}$$

where we use the identification $S^n = \mathbb{R}^n \cup \{\infty\}$. It is independent of the choice of V and the tubular neighborhood U. If $V_0, V_1 \subset M$ are framed bordant via $X \subset M \times I$, we can choose a tubular neighborhood of Xin $M \times I$, and construct collapse map

$$M \times I \to S^n$$

which is a homotopy of collapse maps.

Conversely, if we are given a map

$$f : M \to S^n$$

we define the corresponding class in $\Omega^{fr}_{k-n,M}$ to be

$$f^{-1}(p), p \in S^n,$$

where $p \in S^n$ is a regular value of f and different from ∞. $\square$

Theorem 2.1 (Hopf). Let M be a closed connected manifold of dimension n.

1. If M is orientable, then there is an isomorphism

$$[M, S^n] \cong \mathbb{Z}.$$

   given by the integer degree.

2. If M is not orientable, then there is an isomorphism

$$[M, S^n] \cong \mathbb{Z}_2.$$

   given by the mod 2 degree.

Definition 2.1 (stable homotopy groups). The k*-th stable homotopy group* of a based[2] space X is the colimit

$$\pi^s_k(X) := \operatorname*{colim}_{q\to\infty} \pi_{k+q}(\Sigma^q(X)) = [S^{k+q}, \Sigma^q(X)].$$

In particular, we call the k-th stable homotopy group of S^0 the *stable* k*-stem* and denote it as

$$\pi^s_k := \pi^s_k(S^0)$$

[2] *What if X is unbased?*

If we take M in Theorem 1.4 to be S^{n+l}, the isomorphism becomes

$$\Omega^{\mathrm{fr}}_{l,S^{n+l}} \cong [S^{n+l}, S^n] = \pi_{n+l}(S^n).$$

If we can take $n \to \infty$, then we have an isomorphism between lth framed bordism group in S^∞ and stable l-stem. Which indeed can be done, by the celebrated Freudenthal Suspension Theorem:

Theorem 2.2 (Freudenthal). Suppose that X is an $(n-1)$-connected space, $n \geq 2$. Then the suspension homomorphism

$$\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$$

is an isomorphism if $k < 2n-1$ and an epimorphism if $k = 2n-1$.

Definition 2.2. A *stable tangential framing* of a k-dimensional manifold V is an equivalence class of trivializations of

$$TV \oplus \epsilon^n,$$

where ϵ^n is the trivial bundle $V \times \mathbb{R}^n$. Two trivializations $t_1 : TV \oplus \epsilon^{n_1} \cong \epsilon^{k+n_1}$ and $t_2 : TV \oplus \epsilon^{n_2} \cong \epsilon^{k+n_2}$ are equivalent iff they are homotopic in a stable sense, that is, they are considered equivalent iff there exists some large enough N greater than n_1 and n_2 such that

$$t_1 \oplus \mathrm{id}_{\epsilon^{N-n_1}} : TV \oplus \epsilon^{n_1} \oplus \epsilon^{N-n_1} \cong \epsilon^{k+N}$$

and

$$t_2 \oplus \mathrm{id}_{\epsilon^{N-n_2}} : TV \oplus \epsilon^{n_2} \oplus \epsilon^{N-n_2} \cong \epsilon^{k+N}$$

are homotopic.
Similarly, a *stable normal framing* of a submanifold ν is an equivalent class of trivializations of $\nu \oplus \epsilon^n$ and a *stable framing* of a vector bundle E is an equivalence class of trivializations of $E \oplus \epsilon^n$.

Lemma 2.3. Let $V^k \subset S^n$ be a closed oriented normally framed submanifold of S^n. Then

1. A normal framing $\gamma : \nu \cong \epsilon^{n-k}$ induces a trivialization
$$\bar{\gamma} : TV \oplus \epsilon^{n-k+1} \cong \epsilon^{n+1}.$$
2. A trivialization $\bar{\gamma} : TV \oplus \epsilon \cong \epsilon^{k+1}$ induces a trivialization
$$\nu \oplus \epsilon^{k+1} \cong \epsilon^{n+1}. \tag{2.1}$$

Proof. The inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ has a trivial 1-dimensional normal bundle, which can be framed by choosing the outward unit normal as a basis [3] This shows that the tangent bundle of S^n is stably trivial

$$TS^n \oplus \epsilon \cong \epsilon^{n+1}, \tag{2.2}$$

since $T\mathbb{R}^{n+1}$ is canonically trivialized.
Choose a split (always exists) of the short exact sequence

$$0 \longrightarrow TV \longrightarrow TS^n|_V \longrightarrow \nu \longrightarrow 0,$$

we have the (non-canonical) decomposition

$$TS^n|_V \cong TV \oplus \nu \cong TV \oplus \epsilon^{n-k}.$$

ν is trivial since V is normally framed. After direct summing ϵ and using the stable trivialization of TS^n (2.2), we get

$$\epsilon^{n+1} \cong TS^n|_V \oplus \epsilon \cong TV \oplus \nu \oplus \epsilon \cong TV \oplus \epsilon^{n-k+1},$$

which shows normal framing determines a tangential framing.
Conversely, if we are given a stable tangential framing $TV \oplus \epsilon \cong \epsilon^{k+1}$, we can plug this into the last equation and

$$\epsilon^{n+1} \cong TS^n|_V \oplus \epsilon \cong TV \oplus \nu \oplus \epsilon \cong \nu \oplus \epsilon^{k+1}$$

showing stable tangential framing induces a stable normal framing. □

[3] *Countable noun. Plural form bases.*

Definition 2.3 ([Fre]). An *isotopy* of embeddings $Y \hookrightarrow S^n$ is a **smooth** map

$$I \times Y \to S^n \tag{2.3}$$

so that the restriction to $\{ t \} \times Y$ is an embedding for all $t \in I$. In other words, an isotopy of embeddings is a path of embeddings.

Theorem 2.4. There is a 1-1 correspondence between stable tangential framings and stable normal framings of a manifold V. More precisely:

1. Let $i : V \hookrightarrow S^n$ be an embedding. A stable framing of TV determines stable framing of $\nu(i)$ and conversely.
2. Let $i_1 : V \hookrightarrow S^{n_1}$ and $i_2 : V \hookrightarrow S^{n_2}$ be embeddings. For n large enough There exists a canonical identification

$$\nu(i_1) \oplus \epsilon^{n-n_1} \cong \nu(i_2) \oplus \epsilon^{n-n_2}.$$

A stable framing of $\nu(i_1)$ determines one of $\nu(i_2)$ and vice versa.

Proof. In Lemma 2.3, we studied the special case when the submanifold V is normally framed. The more general case when the submanifold is **stably normally framed** is quite similar, since is some l such that

$$\nu(i) \oplus TV \oplus \epsilon^l \cong \epsilon^{n+l},$$

and by the associativity of $\oplus$ we can prove claim 1.

Claim 2.5. We can choose n large enough so that any two embeddings of V in S^n are isotopic.

Admitting the claim, we conclude that

Corollary 2.6. For n large enough, any self-isotopy is isotopic to the constant isotopy.

For the embeddings $i_1 : V \hookrightarrow S^{n_1}$ and $i_2 : V \hookrightarrow S^{n_2}$, we consider the embeddings $V \xrightarrow{i_1} S^{n_1} \xrightarrow{j_1} S^n$ and $V \xrightarrow{i_2} S^{n_2} \xrightarrow{j_2} S^n$, where j_1, j_2 are equatorial embeddings. Since n large enough, $j_1 \circ i_1$ and $j_2 \circ i_2$ are isotopic, by the last Claim. Then the normal bundles $\nu(j_1 \circ i_1)$ and $\nu(j_2 \circ i_2)$ in S^n are isomorphic. And

$$\nu(j_1 \circ i_1) \cong \nu(i_1) \oplus \epsilon^{n-n_1},$$
$$\nu(j_2 \circ i_2) \cong \nu(i_2) \oplus \epsilon^{n-n_2},$$

thus we have

$$\nu(i_1) \oplus \epsilon^{n-n_1} \cong \nu(i_2) \oplus \epsilon^{n-n_2}.$$

The isomorphism is canonical, since the isotopy from $j_1 \circ i_1$ to $j_2 \circ i_2$ is isotopic to the constant isotopy, by the last corollary. □

Definition 2.4. Two real vector bundles E, F over V are called *stably equivalent* if there exist non-negative integers i, j so that $E \oplus \epsilon^i$ and $F \oplus \epsilon^j$ are isomorphic.

It is easy to show that the stable equivalence is an equivalence relation. Since every smooth compact manifold V can be embedded into S^n for some n via $i : V \to S^n$, we can take the stable equivalence class of the normal bundle $\nu(i)$ of the embedding i. The second part of the last theorem says this class is a well-defined invariant of the compact manifold V independent of the embedding, and we call it the *stable normal bundle* of V.

Corollary 2.7 (stable Pontrjagin-Thom [DK])**.** The stable k-stem is isomorphic to the stably tangentially framed bordism classes of stably tangentially framed k dimensional oriented closed manifolds, that is,

$$\Omega_k^{\mathrm{fr}} \cong \pi_k^s. \tag{2.4}$$

The corollary above has given a bordism description of the stable stems π_k^s. Generally, if X is an arbitrary space, the stable homotopy groups$\pi_k^s(X)$ can be given a bordism description also. In this case one needs to introduce the structure of a map from the manifold to X. Sometimes people call a map from a manifold to X to be a *singular manifold in* X.

Definition 2.5. Let $V_i, i = 1, 2$ be two stably framed k-manifolds and $g_i : V_i \to X, i = 1, 2$ two maps. We say V_1 is *stably framed bordant to* V_2 *over* X if there exists a stably framed bordism W from V_1 to V_2 and a map

$$G : W \to X$$

extending g_1 and g_2.

Let X_+ denote $X \coprod \mathrm{pt}$, the union of X with a disjoint base point. Let $\Omega_k^{\mathrm{fr}}(X)$ denote the stably framed bordism classes of stably framed k-manifolds over X.

Since $S^0 = \mathrm{pt} \coprod \mathrm{pt} = \mathrm{pt}_+$, we can restate Corollary 2.7 in the from

$$\Omega_k^{\mathrm{fr}}(\mathrm{pt}) \cong \pi_k^s(\mathrm{pt}_+). \tag{2.5}$$

The right hand side makes sense, since every manifold maps uniquely into pt, then a manifold is equivalent to the concept of a singular manifold in pt. The stably framed bordism classes of stably framed manifolds are just the stably framed bordism classes of stably framed singular manifolds in pt.

More generally, we can generalize Equation (2.5) to any topological space X.

Theorem 2.8 (general stable Pontrjagin-Thom)**.**

$$\Omega_k^{\mathrm{fr}}(X) \cong \pi_k^s(X_+). \tag{2.6}$$

Proof. The proof is similar to the proofs of Theorem 2.8 and Theorem 1.4. The forward map sending a class of framed k-dimensional manifold to a stable homotopy class is defined via the embedding $W \hookrightarrow S^{k+l}$, provided l large enough. Choose the normal bundle $\nu(W)$, one can define the corresponding collapse maps $S^{n+k} \to X$ and $S^{n+k} \to S^l$. Conversely, if we are given a class $[f] : S^{n+l} \to X_+ \wedge S^l$, take the inverse image of $X \times \{x\}$, where x is the regular map of $\tilde{f}$, and $\tilde{f}$ is the smooth map representing $[f]$ obtained by transversality. $\square$

3 Spectrums and Generalized (Co)homology Theories

The notion of a spectrum measures "stable" phenomena, that is, phenomena which are preserved by suspending.

Definition 3.1. A *spectrum* is a sequence of pairs $\{ K_n, k_n \}$ where the K_n are based spaces and $k_n : \Sigma K_n \to K_{n+1}$ are basepoint preserving maps, where ΣK_n denotes the (reduced) suspension of K_n

Example. The sphere spectrum

$$S = \{ S^n, k_n : \Sigma S^n \cong S^{n+1} \}.$$

Fix a model for $K(\mathbb{Z}, n)$, there exists a sequence of homotopy equivalences

$$h_n : K(\mathbb{Z}, n) \to \Omega K(\mathbb{Z}, n+1).$$

Then h_n defines a map

$$k_n : \Sigma K(\mathbb{Z}, n) \to K(\mathbb{Z}, n+1)$$

via adjunction. In this way we obtain the *Eilenberg-MacLane spectrum*

$$K(\mathbb{Z}) \coloneqq \{ K(\mathbb{Z}, n), k_n \}.$$

Ordinary homology and cohomology are derived from the Eilenberg-MacLane spectrum, as the next theorem indicates.

Theorem 3.1. For any space X,

1. $H_n(X; \mathbb{Z}) = \lim_{l \to \infty} \pi_{n+l}(X_+ \wedge K(\mathbb{Z}, l))$
2. $H^n(X; \mathbb{Z}) = \lim_{l \to \infty} [\Sigma^l(X_+), K(\mathbb{Z}, n+l)]$

Definition 3.2. Let $K = \{ K_n, k_n \}$ be a spectrum. Define the *(unreduced) homology and cohomology with coefficients in the spectrum* K to be the functor taking a space X to the abelian group

$$H_n(X; K) = \lim_{l \to \infty} \pi_{n+l}(X_+ \wedge K_l)$$

and

$$H^n(X; K) = \lim_{l \to \infty} [\Sigma^l(X_+), K_{n+l}]_0,$$

and the *reduced homology and cohomology with coefficients in the spectrum* K to be the functor taking a space X to the abelian group

$$\widetilde{H}_n(X; K) = \lim_{l \to \infty} \pi_{n+l}(X \wedge K_l)$$

and

$$\widetilde{H}^n(X; K) = \lim_{l \to \infty} [\Sigma^l(X), K_{n+l}]_0,$$

and the *homology and cohomology of a pair with coefficients in the spectrum* K to be the functor taking a space X to the abelian group

$$H_n(X,A;K) = \lim_{l\to\infty} \pi_{n+l}((X_+/A_+)\wedge K_l)$$

and

$$H^n(X,A;K) = \lim_{l\to\infty} [\Sigma^l(X_+/A_+), K_{n+l}]_0,$$

Example. By Definition 2.1, the nth stable homotopy group of a space X is

$$\pi_n^s(X) = \lim_{q\to\infty} \pi_{n+q}(\Sigma^q X).$$

Then by Claim **??**

$$\Sigma^q(X) = \Sigma(\cdots\Sigma(X)) = S^1\wedge(\cdots(S^1\wedge X)) = S^q\wedge X,$$

since in our case $\wedge$ is associative[Sma]. Thus we have

$$\pi_n^s(X) = \lim_{q\to\infty} \pi_{n+q}(X\wedge S^q) =: \widetilde{H}_n(X;S),$$

which says the stable homotopy groups $\pi_n^s(X)$ of X are the reduced homology $\widetilde{H}_n(X;S)$ of X with coefficients in the sphere spectrum S.
By Theorem 2.8, the stably framed bordism group $\Omega_n^{\mathrm{fr}}(X)$ satisfies

$$\Omega_n^{\mathrm{fr}}(X) \cong \pi_k^s(X_+) = \widetilde{H}_n(X_+;S) = H_n(X;S),$$

which says it is an unreduced homology theory.

Note that $H_n(\mathrm{pt};K)$ can be non-zero for $n\neq 0$, for example, $H_n(\mathrm{pt};S) = \pi_n^s$. The groups $H_n(\mathrm{pt};K)$ are called the *coefficients* of the spectrum K.

Definition 3.3. Given a $\mathbf{G}$-structure, define the *nth* $\mathbf{G}$*-bordism group of a space* X to be the $\mathbf{G}$-bordism classes of n-dimensional closed manifolds mapping to X with stable $\mathbf{G}$-structures on the normal bundle of an embedding of the manifold in a sphere. Denote this abelian group by

$$\Omega_n^G(X).$$

Thus, a represent of an element in $\Omega_n^G(X)$ has the data $(W,f;\gamma)$, where W is an n-dimensional closed manifold and $f: W\to X$ an continuous map. $\gamma_G: W\to BG$ is a map such that

$$\begin{tikzcd} W \arrow[r, "\gamma_G"] \arrow[dr, "\gamma"'] & BG \arrow[d] \\ & BO \end{tikzcd} \tag{3.1}$$

where $\gamma: W\to BO$ is the classifying map of the stable normal bundle.

Does there exists a spectrum $\mathbf{K}$ for each structure $\mathbf{G}$ so that

$$\Omega_n^G(X) = H_n(X;\mathbf{K}) = \lim_{q\to\infty}(X_+\wedge K_q)?$$

The answer is **yes** and the spectra for bordism theories are called *Thom spectra* $\mathbf{MG}$.

In particular, one can define $\mathbf{G}$-*cobordism* by taking

$$H^n(X; \mathbf{MG}) = \lim_{q \to \infty} [\Sigma^q X_+; MG_{n+q}]_0.$$

References

[DK] James Davis and Paul Kirk. *Lecture Notes in Algebraic Topology*, volume 35 of *Graduate Studies in Mathematics*. American Mathematical Society.

[Fre] Daniel S Freed. Bordism: Old and New.

[Sma] Smash product in nLab.