Introduction to Free BV Theories

Zhang Chi

November 1, 2019

Abstract

Notes of BV formalism and factorization algebras. Largely dependent on [Gwi12].

1 Some Notations

Lines in magenta are things that I'm not very sure, or things I believe myself should pay attention to.

Let E be a smooth vector bundle over a manifold M, denote its dual bundle by $E^\vee$, we take the following notations through out the context:

$$
\begin{aligned}
&E^! \coloneqq E^\vee \otimes \mathrm{Dens}_M, \\
&\mathscr{E}(U) \coloneqq \Gamma(M, E) = \{\text{ smooth sections of } E \text{ on open set } U \subset M\ \} \\
&\mathscr{E}_c(U) \coloneqq \Gamma_c(M, E) = \{\text{ compactly supported smooth sections of } E \text{ on } U\ \} \\
&\mathscr{E}^!(U) \coloneqq \Gamma(M, E^!), \\
&\mathscr{E}^!_c(U) \coloneqq \Gamma_c(M, E^!), \\
&\mathscr{E}(U)^\vee \coloneqq \{\text{ smooth linear functionals on } \mathscr{E}(U)\ \}.
\end{aligned}
$$

Thanks to the following lemma, we can always replace $\mathscr{E}(U)^\vee$ by $\mathscr{E}^!_c(U)$ in the free BV theories.

Lemma 1.1. *The inclusion map $\mathscr{E}^!_c(U) \hookrightarrow \mathscr{E}(U)^\vee$ is a cochain homotopy equivalence of differentiable cochain complexes.*

This lemma is actually a spectial case of the general Atiyah-Bott lemma. See [CG16, pp.304] for details.

2 An Introduction to Free BV Theories on (Closed) Manifolds

Definition 2.1. A *Pois0 algebra*$(A, d, \{-,-\})$ is a commutative dg algebra (A, d) with a Poisson braket $\{-,-\}$ of cohomology degree 1. Explicitly, the braket is a degree 1 bilinear map $\{-,-\} : A \otimes A \to A$ such that

1. $\{x,y\} = -(-1)^{(|x|+1)(|y|+1)}\{y,x\}$
2. $d\{x,y\} = \{dx,y\} + (-1)^{|x|}\{x,dy\}$
3. $x, yz = x, yz + (-1)^{(|x|+1)|y|}y\{x,z\}$

Definition 2.2. A *Beilinson-Drinfeld (BD) algebra* $(A, d, \{-,-\})$ is a commutative graded algebra A, flat as a module over $\mathbb{R}[[\hbar]]$, equipped with a degree 1 Poisson braket such that

$$d(ab) = (da)b + (-1)^{|a|}a(db) + \hbar\{a,b\}$$

Setting $\hbar = 0$, we observe that the above BD algebra $(A, d, \{-,-\})$ becomes a Pois0 algebra. Thus we have the following definition:

Definition 2.3. A *BV quantization* of a Pois0 algebra$A$ is a BD algebra A^q such that $A^q_{\hbar=0} = A$

Definition 2.4. A *free BV field theory* on a manifold M consists the following data:

1. a finite-rank, $\mathbb{Z}-$graded vector bundle (or supervector bundle)E on M;
2. a vector bundle map $\langle -,- \rangle_{loc} : E \otimes E \to \mathrm{Dens}_M$ that is fiberwise nondegenerate, antisymmetric and of cohomology degree -1; this local pairing induces a pairing on compactly-supported sections

   $$\langle -,- \rangle : \mathscr{E}_c \otimes \mathscr{E}_c \to \mathbb{C}$$

   by pairing and integrating;
3. a differential operator $Q : \mathscr{E} \to \mathscr{E}$ of cohomology degree 1 such that
   - $(\mathscr{E}, Q)$ is an elliptic complex;
   - Q is skew self adjoint with respect to the pairing, i.e., $\langle s_0, Qs_1 \rangle = -(-1)^{|s_0|}\langle Qs_0, s_1 \rangle$.

The global sections of a free field theory provides a -1-symplectic vector space, and hence we can try to apply the framework we've developed to it. Our first step is to obtain a well-behaved Pois0 algebra of functions on the fields.

Definition 2.5. The *global classical observables* of the free theory $(M, \mathscr{E}, Q, \langle -,- \rangle)$ are the commutative dg algebra $\mathrm{Obs}^{\mathrm{cl}} := (\mathrm{Sym}\mathscr{E}^!_c, Q)$

The induced degree 1 Poisson braket on the commutative dg algebra $\mathrm{Obs}^{\mathrm{cl}}$ is determined as follows. Because we have a degree -1 pairing $\langle -,- \rangle_{loc} : E \otimes E \to \mathrm{Dens}_M$ which is fiberwise nondegenerate, we then obtain a pairing

$$\langle -,- \rangle^!_{loc} : E^! \otimes E^! \to \mathrm{Dens}_M$$

having cohomology degree 1. By this pairing and integration, we thus obtain a degree 1 antisymmetric pairing

$$\{-,-\} : \mathscr{E}_c^! \otimes \mathscr{E}_c^! \to \mathbb{C}$$

With the natural pairing $\{-,-\}$ on $\mathrm{Obs}^{\mathrm{cl}}$ at hand, we can construct a natural operator Δ, called the *BV operator*. Setting $\Delta \ = 0$ on $\mathrm{Sym}^{\leq 1}\mathscr{E}^!$ and

$$\Delta(xy) = \{x, y\}$$

for $x, y \in \mathscr{E}_c^!$. Then we can extend to all of $\mathrm{Sym}\mathscr{E}_c^!$ by recursively applying the equation

$$\Delta(x, y) = (\Delta x)y + (-1)^{|x|}x(\Delta y) + \{x, y\}$$

Now we move to the quantum observables.

Remark. In order to using the Formal Hodge Theorem, we are forced to pick a hermitian metric h on the vector bundles $E_i$so that we obtain an h-adjoint Q^* and hence an h-Laplacian $D = [Q, Q^*]$. Whenever we speak of a free theory $(\mathscr{E}, Q, \langle -,- \rangle)$, it should be understood that we have made a choice of hermitian metric once and forall.

Definition 2.6. A *contraction* (or *strong deformation retraction*) consists of the following data:

1. a pair of complexes (V, d_V) and (W, d_W)
2. a pair of cochain maps $\pi : V \to W$ and $\iota : W \to V$.

This data must satisfy:

1. W is a retract of V, so $\pi \circ \iota = 1_W$,
2. η is a chain homotopy between 1_V and $\iota \circ \pi$, so

$$\iota \circ \pi - 1_V = d_V \circ \eta + \eta \circ d_V$$

3. the *side conditions*

$$\eta^2 = 0, \quad \eta \circ \iota = 0, \text{and } \pi \circ \eta = 0.$$

We draw this data as

$$(W, d_W) \underset{\iota}{\overset{\pi}{\leftrightarrows}} (V, d_V) \circlearrowleft \eta$$

Theorem 2.1 (Homotopy Perturbation Lemma)**.** *Given a small perturbation δ of a contraction, there is a new contraction*

$$(W, d_W + \delta_W) \underset{\tilde{\iota}}{\overset{\tilde{\pi}}{\leftrightarrows}} (V, d_V + \delta) \circlearrowleft \tilde{\eta}$$

where

$$\begin{aligned}\delta_W &= \pi \circ (1_V - \delta\eta)^{-1} \circ \delta \circ \iota, \\ \tilde{\iota} &= \iota + \eta \circ (1_V - \delta\eta)^{-1} \circ \delta \circ \iota \\ \tilde{\pi} &= \pi + \pi \circ (1_V - \delta\eta)^{-1} \circ \delta \circ \iota \\ \tilde{\eta} &= \eta + \eta \circ (1_V - \delta\eta)^{-1} \circ \delta \circ \eta\end{aligned}$$

Proposition 2.2. *Given a contraction*

$$\begin{tikzcd} (W, d_W) \arrow[r, shift right, "\iota"'] & (V, d_V) \arrow[l, shift right, "\pi"'] \arrow[loop right, "\eta"] \end{tikzcd},$$

there is a natural contraction on the associated symmetric algebras

$$\begin{tikzcd} (\mathrm{Sym}W, d_W) \arrow[r, shift right, "\mathrm{Sym}\iota"'] & (\mathrm{Sym}V, d_V) \arrow[l, shift right, "\mathrm{Sym}\pi"'] \arrow[loop right, "\mathrm{Sym}\eta"] \end{tikzcd}$$

The inner product allows us to introduce adjoint operators Q^* on $\mathscr{E}$ of degree -1, and thus to obtain the Laplacian $D = [Q, Q^*]$. Let $\mathcal{H} = \ker D$ denote the harmonic sections. There is an orthogonal projection map

$$\pi : \mathscr{E} \to \mathcal{H}$$

onto this closed subspace. There is a parametrix for D, which we denote by G, and satisying

$$1_{\mathscr{E}} = \pi + GD = \pi + DG.$$

Theorem 2.3 (Formal Hodge Theorem)**.** *For $(\mathscr{E}, Q)$ an elliptic complex with inner products,*

1. *there is an orthogonal decomposition*

$$\mathscr{E} = \mathcal{H} \oplus QQ^*(\mathscr{E}) \oplus Q^*Q(\mathscr{E})$$

2. *the following commutation relations hold:*
   - $1_{\mathscr{E}} = \pi + DG = \pi + GD$
   - $\pi G = G\pi = \pi D = D\pi = 0$
   - $QD = DQ$ *and* $Q^*D = DQ^*$
   - $QG = GQ$ *and* $Q^*G = GQ^*$

3. $\dim \mathcal{H}$ *is finite and there is a canonical isomorphism*

$$H^i(\mathscr{E}) \cong \mathcal{H} \cap \mathscr{E}^i$$

In short, we obtain a homotopy retraction

$$(\mathcal{H}, 0) \underset{\iota}{\overset{\pi}{\leftrightarrows}} (\mathscr{E}, Q) \circlearrowleft Q^* G$$

Note that $(\mathscr{E}_c^!, Q)$ is an elliptic complex, by the Formal Hodge Theorem, we have a contraction

$$(\mathcal{H}, 0) \underset{\iota}{\overset{\pi}{\leftrightarrows}} (\mathscr{E}, Q) \circlearrowleft \eta$$

by Proposition 2.2, we obain a replacement of classical observables

$$(\mathrm{Sym}\mathcal{H}, 0) \underset{\mathrm{Sym}\iota}{\overset{\mathrm{Sym}\pi}{\leftrightarrows}} (\mathrm{Sym}\mathscr{E}^!, Q) \circlearrowleft \mathrm{Sym}\eta$$

We now construct BV quantization for free fields.

Definition 2.7. The *global quantum observables* of the free theory $(M, \mathscr{E}, Q, \langle -, - \rangle)$ are the dg vector space $\mathrm{Obs}^q \coloneqq (\mathrm{Sym}(\mathscr{E}_c^!)[\hbar], Q + \hbar\Delta)$.

Again using the Homotopy Perturbation Lemma, we have a replacement of the quantum observables:

$$(\mathrm{Sym}(\mathcal{H})[\hbar], D) \underset{\tilde{\iota}}{\overset{\tilde{\pi}}{\leftrightarrows}} \mathrm{Obs}^{\mathrm{q}} \circlearrowleft \mathrm{Sym}\eta$$

We say that for an observable $\mathcal{O} \in \mathrm{Obs}^{\mathrm{q}}$, its expectation value $\langle \mathcal{O} \rangle$ is its image in $H^*\mathrm{Obs}^{\mathrm{q}}$.

Remark. The BV operator is defined on sections of the polyvector field $\Lambda^\bullet TM$, which is often denoted the by "shifted tangent bundle" $T[1]M$. A 1-vector field $\theta(x) \in T[1]M$ is of degree 1, partially because our isomorphism by contraction of a top form and a polyvector

$$\Lambda^k TM \otimes \Omega^n(M) \cong \Omega^{n-k},$$

which is expected to be a chain isomorphism. If the differential forms are sections of a "shifted cotangent bundle" $T^*[-1]M$, then the top form must be of degree $-n$. In order to make the differentials and the chain isomorphism commutative, we must assaign $\theta(x)$ cohomology degree 1. Anyway, inorder to get the degrees correctly, the degrees of a 1-form and 1-vector field must be opposite in the BV formalism.

Since polyvector fields are coordinates of the shifted cotangent bundle $T^*[-1]M$, people always view the BV operator Δ as an oprerator defined on the sheaf of rings of functions on the manifold $T^*[-1]M$. That is, Δ is an operator on $\mathscr{O}(T^*[-1]M)$.

Maybe any manifold which has a de Rham cohomology has a BV algebra. Just inteprete de Rham cohomology as a BV cohomology, via the polyvector fields. So it is not surprising that the modular spaces of Riemann surfaces has a BV structure.

A polyvector field is often interpreted physically as an anti-field.

3 Factorization Algebras: Essential Tools and Backgrounds

Definition 3.1. A *prefactorization algebra* $\mathcal{F}$ on M with values in $\mathbf{C}$, where $\mathbf{C}$ is a symmetric monoidal category, consists of the following data:

1. for each open $U \subset M$, there is an object $\mathcal{F}(U) \in \mathbf{C}$;
2. for each inclusion $\iota : U \to V$, there is a morphism $\iota : \mathcal{F}(U) \to \mathcal{F}(V)$;
3. for any finite collection $U_1, \dots, U_k$ of mutually disjoint opens inside an open V, there isa morphism
$$\mathcal{F}(U_1, \dots, U_k; V) : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_k) \to \mathcal{F}(V)$$
that is equivariant under reordering of the opens;
4. the natural coherences or associativities among these structure maps, e.g., if $U_1, U_2 \subset V$ with $U_1 \cap U_2 = \varnothing$, then we have a commuting diagram
$$\begin{tikzcd} \mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \arrow[r] \arrow[dr] & \mathcal{F}(V) \arrow[d] \\ & \mathcal{F}(W) \end{tikzcd}$$
encoding the transitivity of inclusion of opens;
5. $\mathcal{F}(\varnothing) = \mathbb{1}_{\mathbf{C}}$

Remark. In a more categorical way, a prefactorization algebra on M taking values in a symmetric monoidal category $\mathbf{C}$ is a functor of multicategories from Disj_M to $\mathbf{C}$. See [CG16, pp.40]

Definition 3.2. An open cover $\mathfrak{U} = \{U_i\}_i$ of an open set U is a *Weiss cover* if for any finite set of points $\{ x_1, \dots, x_n \}$ in U, there is an open set $U_i \in \mathfrak{U}$ such that $\{x_1, \dots, x_n\} \subset U_i$

The Weiss cover define a Grothendieck topology on $\mathrm{Opens}(M)$, we call it the *Weiss topology* of M.

Definition 3.3. A cover $\mathfrak{U}$ of M *generates* the Weiss cover $\mathfrak{V}$ if every open $V \in \mathfrak{V}$ is given by a finite disjoint union of opens U_α from $\mathfrak{U}$.

Let Φ be a precosheaf on M with values in cochain complexes in an additive category $\mathbf{C}$. Let $\mathfrak{U}$ be a cover of some open subset U of M. There is a natural map from the 0th-simplex of the Čech complex $C(\mathfrak{U}, \Phi)$ to $\Phi(U_i)$ given by the sum of the structure maps $\Phi(U_i) \to \Phi(U)$.

(Some Abstract Nonsense. Let $\mathbf{C}$ be a multicategory whose underlying category is a Grothendieck abelian category. Then there is a natural multicategory whose underlying category is $\mathrm{Ch}(\mathbf{C})$, the category of cochain complexes in $\mathbf{C}$ in which the weak equivalences are quasi-isomorphisms.)

Definition 3.4. A (homotopy) *factorization algebra* is a prefactorization algebra $\mathcal{F}$ on M valued in $\mathrm{Ch}(\mathbf{C})$, with the property that for every open set $U \subset M$ and Weiss cover $\mathfrak{U}$ of U, the natural map

$$C(\mathfrak{U}, \mathcal{F}) \to \mathcal{F}$$

is a quasi-isomorphism. That is, $\mathcal{F}$ is a homotopy cosheaf with respect to the Weiss topology.

Theorem 3.1. *For every cosheaf of dg vector space $\mathcal{F}$, the precosheaf* $\mathrm{Sym}\mathcal{F}$ *is a factorization algebra. Moreover,* Sym *defines a functor from cosheaves to factorization algebras.*

Proof. It is straightforward to show that $\mathrm{Sym}\mathcal{F}$ is a prefactorization algebra. For any finite collection of disjoint open sets $U_1, \ldots, U_k$, there is an isomorphism

$$\mathcal{F}(U_1 \cup \cdots \cup U_k) \cong \mathcal{F}(U_1) \oplus \cdots \oplus \mathcal{F}(U_k).$$

And invoke the monoidal nature of the functor Sym, there is a canonical isomorphism

$$\mathrm{Sym}\mathcal{F}(U_1 \cup \cdots \cup U_k) \cong \mathrm{Sym}\mathcal{F}(U_1) \otimes \cdots \otimes \mathrm{Sym}\mathcal{F}(U_k).$$

If the opens U_i are all contained in open V, we get the structure map

$$\mathrm{Sym}\mathcal{F}(U_1) \otimes \cdots \otimes \mathrm{Sym}\mathcal{F}(U_k) \cong \mathrm{Sym}\mathcal{F}(U_1 \cup \cdots \cup U_k) \to \mathrm{Sym}\mathcal{F}(V) \qquad (1)$$

from the inclusion map of the cosheaf $\mathcal{F}$:

$$\mathcal{F}(U_1 \cup \cdots \cup U_k) \to \mathcal{F}(V).$$

Now we are to verify the locality axiom. Let U be an open set and let $\mathfrak{U} = \{ U_i \mid i \in I \}$ be a Weiss cover of U.

First observe that the structure map is simply the direct sum of its components

$$\mathrm{Sym}^m \mathcal{F}(U_1 \cup \cdots \cup U_k) \to \mathrm{Sym}^m \mathcal{F}(U),$$

thus it suffices to verify the locality axiom for all m, which is equivalent to show that

$$\mathrm{Sym}^m \mathcal{F}(U) = \mathrm{coker}(\bigoplus_{i,j \in I} \mathrm{Sym}\mathcal{F}(U_i \cap U_j) \to \bigoplus_{i \in I} \mathrm{Sym}\mathcal{F}(U))$$

for all m.

Another important observation is that there exists a canonical isomorphism

$$\mathcal{F}(U)^{\otimes m} \cong \Delta^*(\mathcal{F}^{\boxtimes m})(U),$$

where $\mathcal{F}^{\boxtimes m}$ is the cosheaf on U^m obtained as the external of $\mathcal{F}$ with itself m times, $\Delta : U \to U^m$ is the canonical diagonal map, $\Delta^*(\mathcal{F}^{\boxtimes m})$ is the pullback of the cosheaf $\mathcal{F}^{\boxtimes m}$. Thus it is enough to show that

$$\Delta^*(\mathcal{F}^{\boxtimes m})(U) = \mathrm{coker}(\bigoplus_{i,j \in I} \Delta^*(\mathcal{F}^{\boxtimes m})(U_i \cap U_j) \to \bigoplus_{i \in I} \Delta^*(\mathcal{F}^{\boxtimes m})(U_i)).$$

This is trivial, because $\Delta^*(\mathcal{F}^{\boxtimes m})$ is the pullback of a cosheaf, thus is a cosheaf on U $\square$

Remark. In the proof of Theorem 3.1, only the fact that $\Delta^*(\mathcal{F}^{\boxtimes m})$ is a cosheaf on U is used, and there is nothing about the Weiss cover. Maybe the definition of factorization algebra doesn't need the concept of Weiss cover at all, as Wang Minghao once said.
Costello and Gwilliam have used the properties of the Weiss cover in their proof of this theorem. They reasoned that since $\mathfrak{U} = \{ U_i \mid i \in I \}$ is a Weiss cover, any finite points $x_1, \dots, x_m$ there exists an open $U_i \in \mathfrak{U}$ such that $\{x_1, \dots, x_m\} \subset U_i$, and thus $\{ (U_i)^m \mid i \in I \}$ is a cover of U^m. Then using the fact $(U_i \cap U_j)^m = (U_i)^m \cap (U_j)^m$ (if exist, limits commute with limits).
But their argument is superfluous. Since we are not studying the cosheaf on U^m, but rather the cosheaf on the diagonal, otherwise we are illegal to talk about such words as "isomorphism of cosheaves".

Definition 3.5. A *Pois0 factorization algebra* (respectively, BD factorization algebra) $\mathcal{F}$ is a prefactorization algebra taking values in the symmetric monoidal category of Pois0 algebras such that $\mathcal{F}$ is a factorization algebra wen we forget down to the category **dgVect**.

Definition 3.6. The *classical observables* of the free theory $(M, \mathscr{E}, Q, \langle -, - \rangle)$ are the commutative algebra

$$\mathrm{Obs}^{\mathrm{cl}} : U \mapsto (\mathrm{Sym}\mathscr{E}_c^!(U), Q).$$

Lemma 3.2. *The classical observables* $\mathrm{Obs}^{\mathrm{cl}}$ *is a Pois0 factorization algebra. Moreover, the inclusion* $\mathrm{Obs}^{\mathrm{cl}} \hookrightarrow \mathscr{O}(\mathscr{E})$ *is an open-wise continuous homotopy equivalence. Thus we an represent the observables* $\mathscr{O}(\mathscr{E})$ *by the commutative algebra* $\mathrm{Sym}\mathscr{E}_c^!$.

Proof. By compactness, the inverse of the symplectic pair exsists, which is the canonical Poisson structure. By Theorem 3.1, $\mathrm{Obs}^{\mathrm{cl}}$ is a factorization algebra. By Lemma 1.1, the homotopy equivalence holds. $\square$

Definition 3.7. The *quantum observables* of the free theory $(M, \mathscr{E}, Q, \langle -, - \rangle)$ are the factorization algebra

$$\mathrm{Obs}^{\mathrm{q}} : (\mathrm{Sym}(\mathscr{E}_c^!(U))[\hbar], Q + \hbar\Delta).$$

Lemma 3.3. *The quantum observables* $\mathrm{Obs}^{\mathrm{q}}$ *is a BD factorization algebra.*

Proof. Consider the filtration of the prefactorization algebra

$$F^k\mathrm{Obs}^{\mathrm{q}} = \mathrm{Sym}^{\leq k}(\mathscr{E}_c^!)[\hbar]. \tag{2}$$

And note that the differential $Q+\hbar\Delta$ preserves this filtration. For any open U and its cover $\mathfrak{U}$ ($\mathfrak{U}$ to be Weiss or not? I don't know.), we have the associated Čech complex $\check{C}(\mathfrak{U},\mathrm{Obs}^{\mathrm{q}})$ and a canonical map obtained by the structure maps

$$\check{C}(\mathfrak{U},\mathrm{Obs}^{\mathrm{q}}) \to \mathrm{Obs}^{\mathrm{q}}(U). \tag{3}$$

Applying the filtration (2) we have chosen to Eq (3), we get a map between spectral sequences. The 0th page is

$$\check{C}(\mathfrak{U},\mathrm{Obs}^{\mathrm{cl}}\otimes\mathbb{C}[\hbar]) \to \mathrm{Obs}^{\mathrm{q}}(U)\otimes\mathbb{C}[\hbar],$$

which is a quasi-isomorphism, as $\mathrm{Obs}^{\mathrm{cl}}$ is a factorization algebra. By a corollary of the Eilenberg-Moore Comparision Theorem (if the maps $\mathrm{Gr} f_n : F_nC/F_{n-1}C \to F_nD/F_{n-1}D$ are quasi-isomorphisms for all n, then f is a quasi-isomorphism), the map above is a quasi-isomorphism. □

Theorem 3.4 (Central theorem of free field quantization)**.** *A free BV theory* $(M,\mathscr{E},Q,\langle -,-\rangle)$ *has a canonical Pois0 factorization algebra of classical observables* $\mathrm{Obs}^{\mathrm{cl}}$ *and a canonical BD factorization algebra of quantum observables* $\mathrm{Obs}^{\mathrm{q}}$.

Aside from dg vector spaces, another important kind of factorization algebras take values in dg Lie algebras. These factorization algebras describe the very concept of locality in field theories such as holomorphic Chern-Simons theories and Kac-Moody vertex algebra. Now let us recall the definition of a dg Lie algebra:

Definition 3.8 (dg Lie algebra)**.** A *differential graded Lie algbra* $L, d, [-,-]$ is the following data:

- a graded vector space $L=\bigoplus L_i$ over a field of characteristic zero;
- a bilinear map $[-,-]: L_i\otimes L_j \to L_{i+j}$ of degree 0;
- a differential of degree -1 $d: L_i \to L_{i-1}$,

satisfying the following axioms:

- $[x,y] = -(-1)^{|x||y|}[y,x]$;
- $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]]$ (view the braket as a differential compatible with itself)
- $d[x,y] = [dx,y] + (-1)^{|x|}[x,dy]$.

Generalization to dg Lie algebras endowed with brakets and differentials of any degree is obvious.

A morphism between dg Lie algebras L, L' is a graded linear map $f: L\to L'$ such that $f([x,y]_L) = [f(x),f(y)]_{L'}$ and $f(d_L x) = d_{L'} f(x)$. Dg Lie algebras and their morphisms defines a category.

The coproducts in the category of dg Lie algebras is not the usual direct sum of the underlying dg vector spaces, but the free product of dg Lie algebras. Thus a cosheaf of the underlying dg vector space is not always a cosheaf of the dg Lie algebra, but a precosheaf of the dg Lie algebra (note that the definition of cosheaves involves coproducts). Now we give these kind of precosheaves a name.

Definition 3.9. A *Lie-structured cosheaf* of vector spaces $\mathfrak{g}$ is a precosheaf of dg Lie algebras that is a cosheaf of dg vector spaces (after applying the forgetful functor $\mathbf{dgLie} \to \mathbf{dgVect}$).

Let $\mathfrak{g}$ be a dg Lie algebra, then

- $\mathfrak{g}^M \coloneqq \Omega_c^*(M) \otimes \mathfrak{g}$ to be the cosheaf of compactly supported, $\mathfrak{g}$-valued de Rham forms on a smooth manifold M;
- define $\mathfrak{g}^{M_{\bar{\partial}}} \coloneqq \Omega_c^{0,*}(M) \otimes \mathfrak{g}$ be the cosheaf of compactly-supported, $\mathfrak{g}$-valued Dolbeault forms on a complexes manifold M.

These are our favorite examples of Lie-structured cosheaves.

The next theorem concerning Lie structured cosheaves will be the main tool to recover a Kac-Moody vertex algebra.

Theorem 3.5. *For every Lie-structured cosheaf $\mathfrak{g}$, applying the functor of Chevalley-Eilenberg chains $C_*\mathfrak{g}$ to each open*

$$U \mapsto (\mathrm{Sym}(\mathfrak{g}(U)[1]), d_{CE})$$

is a factorization algebra in dg vector spaces. We denote this factorization algebra by $C_*\mathfrak{g}$. Moreover C_* defines a functor from Lie-structured cosheaves to factorization algebras.

We will call $C_*\mathfrak{g}$ the *envoloping factorization algebra* of $\mathfrak{g}$.

Proof. Consider the filtration on the prefactorization algebra

$$F^i C_*\mathfrak{g} \coloneqq \mathrm{Sym}^{\leq i}(\mathfrak{g}[1]).$$

We will use the spectral sequence induced by this filtration to show $C_*(\mathfrak{g})$ is a factorization algebra. Note that the Lie braket on the 1st page of the spectral sequences of this filtration is trivial, and thus the differentials d_{CE} is also trivial. The 1st page is simply given by applying the functor Sym on the vector space $\mathfrak{g}[1]$. By Theorem 3.1 the map on the 1st page is an isomorphism. The original map is thus a quasi-isomorphism. $\square$

4 Back to Vertex Algebras

To be continued.

References

[CG16] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory*, volume 1 of *New Mathematical Monographs*. Cambridge University Press, 2016.

[Gwi12] Owen Gwilliam. Factorization algebras and free field theories. 2012.