On Swan's Theorem

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This essay gives a complete proof of Swan's Theorem, which provides a prominent example of one ever-lasting philosophy of modern mathematics: the geometric properties of an object can be largely determined by the algebraic properties of the functions on this object, and *vice versa*.

In this essay, we first define the $K_0(R)$ group of a ring R, as the group completion of the monoid $\mathrm{Proj}(R)$ of equivalence classes of finitely generated projective modules over R.

Then we introduce the topological K-theory, say $K^0(X)$ of a compact Hausdorff topological space X. The group $K^0(X)$ is defined as the group completion of the monoid $\mathrm{Vect}_{\mathbb{F}}X$, which is the monoid of equivalence classes of the (trivializable) $\mathbb{F}$-vector bundles over X.

Finally, we reach the apex of this essay, Swan's Theorem. The proof of Swan's Theorem is standard and elementary, and we follow the lines of [Ros, pp. 34].

1 Projective Modules

Now let's recall the basics projective modules over a ring R. The word **ring** in this essay means a ring with unit, and the word R-**module** means a left R-module.

Definition 1.1. Let R be a ring. A **projective module** over R means an R-module P satisfies the following lifting property: if $\phi : P \to N$ is any homomorphism of R-modules, and $\psi : M \to N$ is a surjective homomorphism, for R-modules M, N; then there exists a homomorphism $\tilde{\phi} : P \to M$ such that the diagram

$$\begin{tikzcd} & M \arrow[d, "\psi"] \\ P \arrow[ur, dashed, "\exists\tilde{\phi}"] \arrow[r, "\phi"] & N \end{tikzcd}$$

commutes.

There is another equivalent characterization of projective modules:

Theorem 1.1. Let R be a ring. An R-module is projective if and only if it is isomorphic to a direct summand in a free R-module. In particular, a finitely generated R-module is projective if and only if it is isomorphic to a direct summand in R^n for some n.

Proof. This is a standard result. See any textbook of algebra. Or, see [Ros, pp.2] □

By the last theorem, we know that the direct sum of two projective R-modules is again projective, since the direct sum of free modules is again free. So the isomorphism classes of finitely generated projective modules over R form an abelian semigroup $\mathrm{Proj}(R)$, in fact a monoid, with $\oplus$ as the addition operation and with the 0-module as the identity element. In general, $\mathrm{Proj}(R)$ is not a group, but we may force it into a group. The ideal of how to do this is similar to the way $\mathbb{Z}$ is constructed from the additive semigroup of positive integers, or $\mathbb{Q}^*$ is constructed from the multiplicative semigroup of non-zero integers.

2 The Grothendieck Group $K_0(R)$

Theorem 2.1. Let S be a commutative semigroup (not necessarily having a unit). There is an abelian group G (called the **Grothendieck group** or **group completion** of S), together with a semigroup homomorphism $\phi : S \to G$, such that for any group H and homomorphism $\psi : S \to H$, there is a unique homomorphism $\theta : G \to H$ with $\psi = \theta \circ \phi$.

Proof. We define G to be the set of equivalence classes of pairs (x, y) with $x, y \in S$, where $(x, y) \sim (u, v)$ iff there is some $t \in S$ such that

$$x + v + t = u + y + t \in S.$$

We denote by $[(x, y)]$ the equivalence class of (x, y) under the relation $\sim$. Then addition is defined by the rule

$$[(x, y)] + [(x', y')] \coloneqq [(x + x', y + y')].$$

It's easy to check that this addition is well-defined and associative.
Note that for any $x, y \in S$,

$$[(x, x)] = [(y, y)]$$

follows, as $x + y = y + x$. Let 0 be this distinguished element $[(x, x)]$. This is the zero element for G, as for any $x, y, t \in S$, $(x + t, y + t) \sim (x, y)$.
Also, G is a group since

$$[(x, y)] + [(y, x)] = [(x + y, x + y)] = 0.$$

We define $\phi : S \to G$ by

$$\phi(x) \coloneqq [(x + x, x)],$$

and it is easy to see that this is a homomorphism. Note that the image of ϕ generates G as a group, since

$$[(x, y)] = \phi(x) - \phi(y)$$

in G. Given a group H and homomorphism $\psi : S \to H$, the homomorphism $\theta : G \to H$ is defined by

$$\theta([(x, y)]) = \psi(x) - \psi(y).$$

Hence $\psi = \theta \circ \phi$. □

Now we are ready to define $K_0(R)$:

Definition 2.1. Let R be a ring. Then $K_0(R)$ is the Grothendieck group of the semigroup $\mathrm{Proj}(R)$ of isomorphism classes of finitely generated projective modules over R.

3 Topological K-Theory and Swan's Theorem

Definition 3.1. Let X be a compact Hausdorff topological space and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. An $\mathbb{F}$-vector bundle consists of a topological space E and a continuous surjective map $p : E \to X$ with extra structure defined by the following:

(i) each fiber $p^{-1}(x)$ of p for $x \in X$, is a finite-dimensional vector space over $\mathbb{F}$;

(ii) the are continuous maps

$$E \times_p E \to E$$

and

$$\mathbb{F} \times E \to E$$

which restrict to addition and scalar multiplication of $\mathbb{F}$-vector space on each fiber.

We may define the **Whitney sum** on two two vector bundles $E \xrightarrow{p} X$ and $E' \xrightarrow{p'} E$ over X by taking direct sum fiber-wise, denoted by $E \oplus E'$. More precisely

$$E \oplus E' \coloneqq \left\{ (x, x') \,\middle|\, x \in E, x' \in E', p(x) = p'(x') \right\}.$$

For most purpose we want our vector bundles to be **locally trivial**.

Definition 3.2. If X is a compact Hausdorff space, let $\mathrm{Vect}_{\mathbb{F}}(X)$ denote the monoid of isomorphic classes of locally trivial $\mathbb{F}$-vector bundles over X, with the monoid operation induced by the Whitney sum and the 0-element the trivial bundle of rank 0. The **topological K-theory** of X is defined by $K^0_{\mathbb{F}}(X) \coloneqq G(\mathrm{Vect}_{\mathbb{F}}(X))$. Some times this is denoted simply $K(X)$ or $KU(X)$ if $\mathbb{F} = \mathbb{C}$, $KO(X)$ if $\mathbb{F} = \mathbb{R}$. If X is connected, the **reduced** topological K-theory $\tilde{K}^0_{\mathbb{F}}(X) = \ker(\mathrm{rank} : K^0_{\mathbb{F}}(X) \to \mathbb{Z})$.

Theorem 3.1 (Swan's Theorem)**.** Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let X be a compact Hausdorff space, and let $R \coloneqq C(X, \mathbb{F})$ be the ring of continuous $\mathbb{F}$-valued continuous functions on X. If $E \xrightarrow{p} X$ is a locally trivial $\mathbb{F}$-vector bundle over X, let $\Gamma(X, E)$ be the set of continuous sections of p. Observe that $\Gamma(X, E)$ is naturally an R-module. Then $\Gamma(X, E)$ is finitely generated and projective over R arises from this construction. The map $E \mapsto \Gamma(X, E)$ an isomorphism of categories from the category of vector bundles over X to the vector bundles over X to the category of finitely generated projective R-modules. It also induces an isomorphism $K^0(X) \to K_0(R)$.

Proof. Let $E \xrightarrow{p} X$ be a locally trivial $\mathbb{F}$-vector bundle over X and let $\Gamma(X, E)$ to be its R-module of sections. For each $x \in X$, there is an open neighborhood U over which E looks like a trivial bundle $U \times \mathbb{F}^n$. Locally, there exist n constant functions $e_j : U \to \mathbb{F}^n$ determined by the standard basis of $\mathbb{F}^n$. It's clear that these e_j's form a basis of the $C(U, \mathbb{F})$-linear space $\Gamma(U, U \times \mathbb{F}^n)$. Since X is compact, we can cover X by finitely many U_i, such that E is locally trivial on each U_i. Then we choose a partition of unity $\{ \rho_i \}$ subordinate to this covering $\{ U_i \}$. Define $e_{ij} \coloneqq \rho_i e_j$ and we get each e_{ij} supported in U_i. e_{ij}'s can be viewed as global sections of $E \xrightarrow{p} X$, via extension by zero. $\left\{ e_{ij} \right\}$ thus generates $\Gamma(X, E)$ as an R-module, by construction, with $R \coloneqq C(X, \mathbb{F})$. This shows that $\Gamma(X, E)$ is a finitely generated R-module.

Next we show that $\Gamma(X, E)$ is projective. It suffices to show that $\Gamma(X, E)$ is a direct summand of a free R-module of finite rank. Since $\Gamma(X, E)$ is finitely generated as an R- module, we can find its generators $s_j, 1 \leqslant j \leqslant k$, where s_j's may not be e_{ij}'s constructed before. Then consider the trivial bundle $X \times \mathbb{F}^k \xrightarrow{\pi_1} X$, and define a morphism of vector bundles

$$\begin{aligned} \phi : X \times \mathbb{F}^k &\to E, \\ (x, v_1, \ldots, v_k) &\mapsto \sum_{j=1}^{k} v_j s_j(x). \end{aligned}$$

Since the $s_j(x)$'s span $p^{-1}(x)$ for each x, ϕ is surjective on each fiber. We define a subbundle E' of the trivial bundle $X \times \mathbb{F}^k$ by taking kernel of ϕ, or namely, let $E'_x = \ker \phi_x$. E' is indeed a vector bundle, since each fiber E_x is a subspace of the vector space $\mathbb{F}^k$. E' is also trivializable on open subset U where E is trivializable, hence is a locally trivial vector bundle. Now we claim that

Claim 3.2. There is an isomorphism of vector bundles

$$E \oplus E' \simeq X \times \mathbb{F}^k.$$

Proof. It is well-known that there exists an Hermitian metric $\langle -, - \rangle_E$ on E, which can be obtained by gluing the trivial metrics on each open subset by a partition of unity. There is also a canonical Hermitian metric $\langle -, - \rangle_{X \times \mathbb{F}^k}$ on the trivial bundle $X \times \mathbb{F}^k$, and furthermore ϕ is compatible with these metrics. Denote the adjoint ϕ^* of ϕ with respect to these metrics. Since $\phi : X \times \mathbb{F}^k \to E$ is surjective on each fiber, $\phi^* : E \to X \times \mathbb{F}^k$ is injective on each fiber, by adjointness. For any $u \in \Gamma(X, X \times \mathbb{F}^k)$ and $w \in \Gamma(X, E)$, the adjointness relation

$$\langle \phi(v), w \rangle_E = \langle v, \phi^*(w) \rangle_{X \times \mathbb{F}^k}$$

tells us that $v \in \ker \phi = E'$ iff $v \perp \phi^*(E)$. This means $E' \perp \phi^*(E)$, where both sides are viewed as subbundles of $X \times \mathbb{F}^k$. Since ϕ^* is injective on each fiber, $\phi^* : E \to \phi^*(E)$ is bijective on each fiber and hence is an isomorphism of vector bundles. Via the identification $E \simeq \phi^*(E)$, we know that $E \perp E'$. Since

$$X \times \mathbb{F}^k = E \oplus E^{\perp} \simeq E \oplus E',$$

we are done. $\square$

By Claim 3.2 ,

$$\Gamma(X, E) \oplus \Gamma(X, E') \simeq \Gamma(X, E \oplus E') \simeq \Gamma(X, X \times \mathbb{F}^k) \simeq R^k,$$

where the first and the last isomoprhism follow directly by definition. The last equation shows that $\Gamma(X, E)$ is a direct summand of the free R-module R^k, hence is a projective R-module. If we denote by $V(X, \oplus)$ the tensor category of vector bundles over X with $\oplus$ the Whitney sum, then the above argument shows that $\Gamma(X, -) : V(X, \oplus) \to R$-**Proj** is a functor between tensor categories, where R-**Proj** is the tensor category of projective R-modules, equipped with the direct sum of R-modules. Now we work on the other direction, *i.e.*, we need to find a functor $V : R$-**Proj** $\to V(X, \oplus)$, sending each projective R-module to a vector bundle over X. Suppose P is a projective R-module, so we can find an R-module Q and an integer $n \in \mathbb{N}$ such that $P \oplus Q = R^n$. Note that $R^n = C(X, \mathbb{F})^n \simeq C(X, \mathbb{F}^n)$, we may view P as a subset of $C(X, \mathbb{F}^n)$ and define

$$V(P) \coloneqq \left\{ (x, v_1, \ldots, v_n) \in X \times \mathbb{F}^n \;\middle|\; \text{if } \exists s \in P \text{ such that } s(x) = (v_1, \ldots, v_n) \right\}$$

Define $p : V(P) \to X$ to be the projection on to the first factor. We need to show that $V(P) \xrightarrow{p} X$ is indeed a vector bundle. It's easy to see that $V(P)$ is fiber-wise linear, since P is linear. We need only check that $V(P)$ is trivializable. Given $x \in X$, it's possible to choose elements $e^1, \ldots, e^r \in P$ such that $e^1(x), \ldots, e^r(x)$ form a basis for $V(P)_x \coloneqq p^{-1}(x)$, with $p^{-1}(x)$ viewed as a subspace of $\mathbb{F}^n$. Recall that e^i's are vector-valued functions, so we write $e^i = (e^i_1, \ldots, e^i_n)$. Since $e^1(x), \ldots, e^r(x)$ are linearly independent, we can choose $1 \leqslant j_1 < \cdots < j_r \leqslant n$ such that

$$e \coloneqq \det \begin{pmatrix} e^1_{j_1} & e^1_{j_2} & \cdots & e^1_{j_r} \\ \vdots & & & \vdots \\ e^r_{j_1} & e^r_{j_2} & \cdots & e^r_{j_r} \end{pmatrix}$$

is non-zero at x. Since e is continuous, there must exist some neighborhood U of x in which e is non-vanishing. Thus for $\forall y \in U$, $e^1(y), \ldots, e^r(y)$ are linearly independent and form a basis of the vector space $V(P)_y$, showing that $V(P)$ is trivial over such U. Moreover, V maps direct sum to Whitney sum, by definition.

The verification of $V \circ \Gamma(X, -) = \mathrm{id}$ and $\Gamma(X, -) \circ V = \mathrm{id}$ is tautological. Finally, let $X \times \{0\}$ be the trivial rank 0 vector bundle over X, then $\Gamma(X, X \times \{0\}) \simeq C(X, \{0\}) \simeq \{0\}$, which is the trivial rank 0 module. Thus $\Gamma(X, -) : V(X, \oplus) \to R$-**Proj** is an isomorphism of tensor categories.

Passing to equivalence classes, $\Gamma(X, -)$ induces an isomorphism $\mathrm{Vect}_{\mathbb{F}} X \simeq \mathrm{Proj}(R)$ between monoids. By virtue of Theorem 2.1, this furthermore induces an isomorphism $K^0(X) \simeq K_0(R)$.

□

References

[Ros] Jonathan M. Rosenberg. *Algebraic K-Theory and Its Applications*. Number 147 in Graduate Texts in Mathematics. Springer, corr. 2nd print edition.