Assignment

Zhang Chi (201828001207022)
zhangchi2018@itp.ac.cn

Exercise 1

Since the spectrum Spec$A$ of any ring A is quasi-compact, we can consider the finest open cover $\{D(f_i)\}_{i\in I}$ consists of the principal open set with f_i running over A. By quasi-compactness there is a finite index subset $J \subset I$ so that Spec$A$ can be covered by $\{D(f_i)\}_{i\in J}$. Take $W_i = D(f_i) \cap (U \cap V), i \in J$, thus $\{W_i\}_{i\in J}$ is the finest open cover of $U \cap V$ and moreover it is finite, which shows that $U \cap V$ is quasi-compact.

Exercise 2

(a) We claim that the singleton of null ideal $\{(0)\} \subset \mathrm{Spec}\mathbb{Z}[x]$ is open and cannot be expressed as $D(f)$ for any polynomial $f \in \mathbb{Z}[x]$. A set V is closed in Spec$\mathbb{Z}[x]$ if and only if it is algebraic, i.e. there are some $f \in \mathbb{Z}[x]$ such that

$$V = V(f) = \{\, \mathfrak{p} \in \mathbb{Z}[x] \mid f \in \mathfrak{p} \,\}\,.$$

Obviously there is no polynomial contained in (0) other than the null polynomial 0, but on the other hand, we have

$$V(0) = \mathrm{Spec}\mathbb{Z}[x],$$

which shows $\{(0)\}$ is not closed in Spec$\mathbb{Z}[x]$.

Next we show that $\{(0)\}$ can not be expressed in the form $D(f)$ for any f. If there were some $f \in \mathbb{Z}[x]$ such that $\{0\} = D(f)$, then we have

$$\mathrm{Spec}\mathbb{Z}[x] - \{(0)\} = V(f). \tag{1}$$

Although we don't know Spec$\mathbb{Z}[x]$ well, we still know that it at least consists the prime ideals generated by irreducible polynomials. Thus Equation (1) implies that f must at least be the product of all irreducible polynomials in $\mathbb{Z}[x]$, which is impossible.

(b)

Exercise 3

Recall the definition of the set of morphisms of presheaves:

Definition 1. A **morphism of presheaves** $\phi : \mathcal{F} \to \mathcal{G}$ on X, is the the collection of maps $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for all open subsets $U \subset X$, compatible with the restrictions: if $U \hookrightarrow V$ then

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\phi(V)"] \arrow[d, "\mathrm{res}^U_V"] & \mathcal{G}(V) \arrow[d, "\mathrm{res}^U_V"] \\ \mathcal{F}(U) \arrow[r, "\phi(U)"] & \mathcal{G}(U) \end{tikzcd} \,. \tag{2}$$

The set of all morphisms between presheaves $\mathcal{F}, \mathcal{G}$ is denoted by $\mathcal{H}\wr\Updownarrow(\mathcal{F}, \mathcal{G})$.

We are asked to show that, if $\mathcal{F},\mathcal{G}$ are sheaves on X, then the presheaf $\mathcal{H}\wr\Updownarrow(\mathcal{F},\mathcal{G})$ is actually a sheaf. The compatibility of $\mathcal{H}\wr\Updownarrow(\mathcal{F},\mathcal{G})$ with restrictions is given by definition, we just need to verify the gluing property.

Let $U\subset X$ be any open set, and $\{U_i\}_{i\in I}$ be any open cover of U. If we are given maps $\phi_i:\mathcal{F}(U_i)\to\mathcal{G}(U_i), i\in I$ in $\mathcal{H}\wr\Updownarrow(\mathcal{F},\mathcal{G})(\mathcal{U}_\rangle), i\in I$, satisfying

$$\phi_i(U_i\cap U_j)=\phi_j(U_i\cap U_j),\forall i,j\in I, \tag{3}$$

we are expected to find a unique map $\phi:\mathcal{F}(U)\to\mathcal{G}(U)$ in $\mathcal{H}\wr\Updownarrow(\mathcal{F},\mathcal{G})(U)$ satisfying

$$\phi(U_i)=\phi_i,\forall i\in I.$$

Given a section $s\in\mathcal{U}$, $s|_{U_i}$ are sections in $\mathcal{F}(U_i)$ and we denote

$$t_i:=\phi_i(s|_{U_i}),\forall i\in I,$$

which are sections in $\mathcal{G}(U_i)$. But note that

$$\begin{aligned}
t_i|_{U_i\cap U_j}&=\phi_i(s|_{U_i})|_{U_i\cap U_j}\\
&=\phi_i(U_i\cap U_j)(s|_{U_i\cap U_j})\\
&=\phi_j(U_i\cap U_j)(s|_{U_i\cap U_j})\\
&=\phi_j(s|U_j)|_{U_i\cap U_j}\\
&=t_j|_{U_i\cap U_j},\forall i,j\in I
\end{aligned}\tag{4}$$

where the first equality holds by the commutative diagram (2) and the second equality holds by the assumption (3). Equation (4) implies that the local sections $t_i\in\mathcal{G}(U_i)$ defines a unique global section $t\in\mathcal{G}(U)$ such that $t_i=t|_{U_i}$, by the gluing property of the sheaf $\mathcal{G}$. Now we can define the desired map

$$\begin{aligned}\phi:\mathcal{F}(U)&\to\mathcal{G}(U)\\ s&\mapsto t.\end{aligned}$$

The uniqueness of ϕ also follows by the gluing property of the sheaf $\mathcal{G}$.

What we have not shown is $\phi(U_i)=\phi_i$, which holds easily. Since we have $\phi(s)=t$ on U forall $s\in\mathcal{F}(U)$, we can restrict this equation on U_i:

$$\phi(U_i)(s|_{U_i})=\phi(s)|_{U_i}=t|_{U_i}=t_i=\phi_i(s|_{U_i}),\forall s\in\mathcal{F}(U)$$

implying $\phi_i=\phi(U_i)$.

The last thing we need to show is that ϕ is independent of the choice of the open cover $\mathscr{U}=\{U_i\}$ of U. We can modify the definition of ϕ by taking $\phi=\operatorname{colim}_{\mathscr{U}}\phi_i$, which always exists since all open covers of U form a direct system.

One interesting thing worth remarking is the fact $\mathcal{H}\wr\Updownarrow(\mathcal{F},\mathcal{G})$ is a sheaf only depends on that $\mathcal{G}$ is a sheaf, and $\mathcal{F}$ only need to be a presheaf. Thus we in fact have proved a stronger result:

Proposition 1. If $\mathcal{F}$ is a presheaf on X and $\mathcal{G}$ is a sheaf on X, then the presheaf $\mathcal{H}\wr\Updownarrow(\mathcal{F},\mathcal{G})$ is a sheaf on X.

Exercise 4

(a) First we verify that $j_!^{\mathbf{Set}}\mathcal{F}$ is a presheaf. For open sets $S\subset T\subset X$, the following cases exhaust all possibilities:

$$\begin{cases} S\subset T\subset U,\\ S\subset U, T\text{ otherwise},\\ S\text{ and }T\text{ otherwise}.\end{cases}$$

For the first case, $j_!^{\mathbf{Set}}\mathcal{F}(T) = \mathcal{F}(T), j_!^{\mathbf{Set}}\mathcal{F}(S) = \mathcal{F}(S)$, thus we have restriction map $j_!^{\mathbf{Set}}\mathcal{F}(T) \to j_!^{\mathbf{Set}}\mathcal{F}(S)$ inherited from $\mathcal{F}$. For the second case, $j_!^{\mathbf{Set}}\mathcal{F}(S) = \mathcal{F}(S), j_!^{\mathbf{Set}}\mathcal{F}(T) = \varnothing$. There is a unique map $\varnothing \to \mathcal{F}(S)$ since $\varnothing$ is the initial object in category **Set**, which is our restriction map $j_!^{\mathbf{Set}}\mathcal{F}(T) \to j_!^{\mathbf{Set}}\mathcal{F}(S)$. For the third case, the restriction map is actually the unique map $\varnothing \to \varnothing$.

Now we are going to show $j_!^{\mathbf{Set}}\mathcal{F}$ satisfying the gluing property. For any open set $W \subset X$, if $W \subset U$ then $j_!^{\mathbf{Set}}\mathcal{F}(W) = \mathcal{F}(W)$, the gluing property is inherited from $\mathcal{F}$ and we are done. We are only to consider the case that W is not a subset of U. For any open cover $\{W_i\}_{i\in I}$ of W, if there is a index subset $J \subset I$ such that $W_j \subset U, j \in J$ while $W_i \not\subset U, i \in I - J$, we have products $\prod_{i\in I} \mathcal{F}(W_i) = \varnothing$ and $\prod_{i,k\in I} \mathcal{F}(W_i \cap W_k) = \varnothing$ because $\mathcal{F}(W_i) = \varnothing, i \in I - J$ and the product of any set with $\varnothing$ is $\varnothing$. The arguments also hold when $J = \varnothing$. Thus the diagram

$$\mathcal{F}(W) \longrightarrow \prod_{i\in I} \mathcal{F}(W_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(W_i \cap W_j)$$

becomes

$$\varnothing \longrightarrow \varnothing \longrightarrow \varnothing,$$

which is trivially an equilizer diagram. We have showed $j_!^{\mathbf{Set}}\mathcal{F}$ is indeed a sheaf on X.

The stalks of $j_!^{\mathbf{Set}}\mathcal{F}$ is easy to compute. By definition

$$(j_!^{\mathbf{Set}}\mathcal{F})_x = \operatorname*{colim}_{x\subset U} j_!^{\mathbf{Set}}\mathcal{F}(U), \forall U \subset_{\text{open}} X,$$

we have

$$(j_!^{\mathbf{Set}}\mathcal{F})_x = \begin{cases} \mathcal{F}_x, & x \in U, \\ \varnothing, & x \notin U. \end{cases}$$

(b) Since sheafification induces isomorphisms on stalks, to compute the stalks of the sheaf $j_!\mathcal{F}$ it is sufficient to compute the stalks of the presheaf $j_!^{\text{pre}}\mathcal{F}$. Thus

$$(j_!\mathcal{F})_x = \begin{cases} \mathcal{F}_x, & x \in U, \\ 0, & x \notin U. \end{cases} \tag{5}$$

Suppose

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0 \tag{6}$$

is an exact sequence of sheaves on U. To show $j_!$ is exact, we need to show

$$0 \longrightarrow j_!\mathcal{F} \longrightarrow j_!\mathcal{G} \longrightarrow j_!\mathcal{H} \longrightarrow 0 \tag{7}$$

is an exact sequence of sheaves on X. Recall that

Proposition 2. A sequence $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ of morphisms of sheaves on X is exact if and only if, for every $x \in X$, the sequence of groups $\mathcal{A}_x \xrightarrow{f_x} \mathcal{B}_x \xrightarrow{g_x} \mathcal{C}_x$ is exact.

We only need to show the exactness of (7) at the level of stalk:

$$0 \longrightarrow (j_!\mathcal{F})_x \longrightarrow (j_!\mathcal{G})_x \longrightarrow (j_!\mathcal{H})_x \longrightarrow 0. \tag{8}$$

When $x \in U$, (8) becomes

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$$

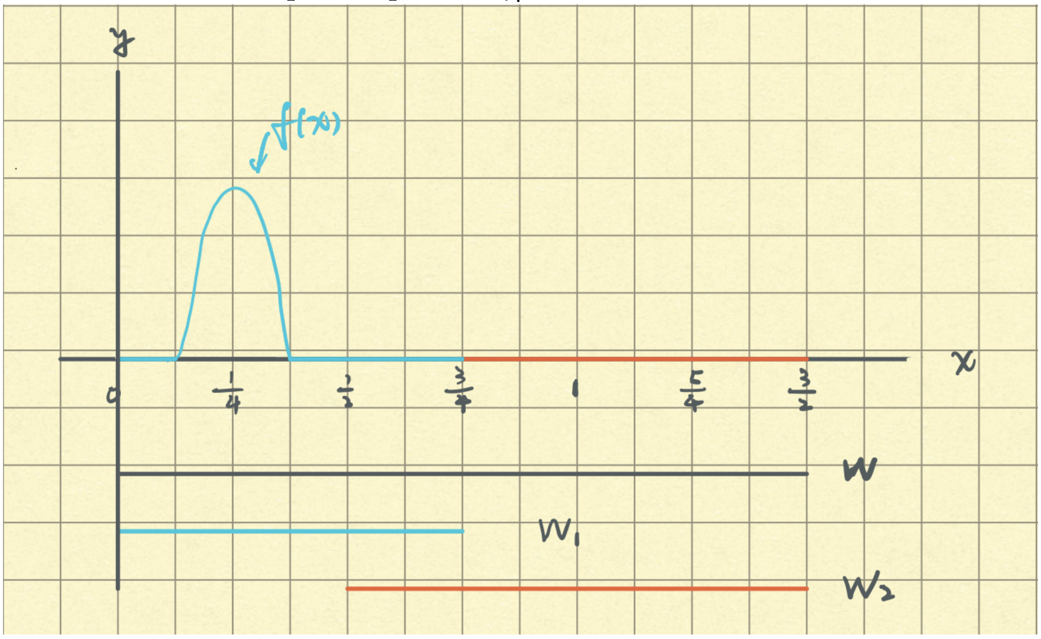
and the exactness is guaranteed by the exactness of the sequence (6), by Proposition 2. When $x \notin U$, the sequence (8) becomes

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0,$$

which is trivially exact.

Now we provide an example in which $j_!^{\text{pre}}\mathcal{F}$ is a presheaf but not sheaf.

Example. Let $X = \mathbb{R}, U = (0,1)$ and $\mathcal{F}$ be the sheaf of smooth functions on the open interval $(0,1)$. For the open subset $W = (0, \frac{3}{2}) \subset \mathbb{R}$, we take the open cover $\{W_1, W_2\}$ of it with $W_1 = (0, \frac{3}{4})$ and $(\frac{1}{2}, \frac{3}{2})$. By definition, $(j_!^{\text{pre}}\mathcal{F})(W) = \{0\}, (j_!^{\text{pre}}\mathcal{F})(W_2) = \{0\}, (j_!^{\text{pre}}\mathcal{F})(W_1) = C^\infty((0, \frac{3}{4}),)$. Take $f \in C^\infty((0, \frac{3}{4}), =)(j_!^{\text{pre}}\mathcal{F})(W_1)$ as the bump function with support $\operatorname{supp} f = [\frac{1}{8}, \frac{3}{8}] \subset W_1$, and take $g \in (j_!^{\text{pre}}\mathcal{F})(W_2) = \{0\}$ to be the only section, 0 function on W_2. Thus on $W_1 \cap W_2 = (\frac{1}{2}, \frac{3}{4})$, we have $f|_{W_1 \cap W_2} = 0 = g|_{W_1 \cap W_2}$. If $j_!^{\text{pre}}\mathcal{F}$ were a sheaf, there must exist a unique section h in $(j_!^{\text{pre}}\mathcal{F})(W)$ such that $h|_{W_1} = f, h|_{W_2} = g = 0$, but there is only one section $0 \in (j_!^{\text{pre}}\mathcal{F})(W) = \{0\}$ and the restriction of 0 to W_1 can never be f, which is a contradiction. So we have found a example of a presheaf $j_!^{\text{pre}}\mathcal{F}$ not a sheaf.



Exercise 5

(a) Recall that in the category **Ring**, a morphism $\phi : A \to B$ is called a **monomorphism** if it is left cancellative, that is, for any object $C \in$ **Ring** and any morphisms $g_1, g_2 : C \to A$, the condition

$$f \circ g_1 = f \circ g_2$$

implies

$$g_1 = g_2.$$

Now we are asked to show that a ring homomorphism $\phi : A \to B$ is left cancellative iff it is injective. We first show the easier necessary part. Suppose $\phi : A \to B$ is injective and $g_1, g_2 : C \to A$ arbitrary homomorphisms from an arbitrary ring C. Then $\phi \circ g_1 = \phi \circ g_2 \iff \phi \circ (g_1 - g_2) = 0 \iff g_1 - g_2 = 0$, where the third implication holds by the injectivity of ϕ. This shows that ϕ is left cancellative.

The sufficient part is harder, so we use *reductio ad absurdum*. Suppose $\phi : A \to B$ is injective, if we can show that ϕ is not left cancellative, *i.e.*, there exists a ring C and exist two distinct morphisms $g_1, g_2 : C \to A$ satisfying $\phi \circ g_1 = \phi \circ g_2$, the proof is done. Since $\phi : A \to B$ is not injective, then $\ker(\phi) \neq \{0\}$ so as there exists some non-zero $a \in \ker(\phi)$. We choose $C = \mathbb{Z}[x]$, and define $g_1 = 0, g_2(x) = a$, then the natural bijection $\mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x], A) \cong \mathrm{Hom}_{\mathbf{Set}}(\{x\}, U(A))$ tells us that g_1 and g_2 are uniquely defined and thus are distinct ring homomorphism, where $U : \mathbf{Ring} \to \mathbf{Set}$ is the forgetful functor. As we have found distinct ring homomorphisms $g_1, g_2 : \mathbb{Z}[x] \to A$ such that $\phi \circ g_1 = \phi \circ g_2 = 0$, the sufficiency is proved.

(b) Suppose $f_X^\flat : \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$ is not an injection, then there exist $a, b \in \mathcal{O}_X$ such that $f_X^\flat(a) = f_X^\flat(b)$. We can get two distinct ring homomorphisms $g_a^\flat, g_b^\flat : \mathbb{Z}[x] \to \mathcal{O}_X(X)$ by mapping x to a and b respectively, and $f_X^\flat \circ g_a^\flat = f_X^\flat \circ g_b^\flat$. But we have

$$\mathrm{Hom}_{\mathbf{Sch}}(Y, \mathrm{Spec}(\mathbb{Z}[x])) \simeq \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x], \mathcal{O}_Y(Y)),$$

so the corresponding morphisms of schemes $Y \to X \to \mathrm{Spec}(\mathbb{Z}[x])$ are equal $g_a \circ f = g_b \circ f$. By the epimorphicity of f, we have $g_a = g_b$, hence $g_a^\flat = g_b^\flat$, a contradiction.

For the second statement, to show is for any closed subset $Z \subset= 0, f(Y) \cap Z \neq \varnothing$. Consider the scheme $X \coprod_{X-Z} X$ obtained by gluing two copies of X along $X = Z$, and the diagram

$$Y \xrightarrow{f} X \underset{j}{\overset{i}{\rightrightarrows}} X \textstyle\coprod_{X-Z} X$$

where i, j are the canonical map induced by the two inclusions $X \to X \coprod X$. If $f(Y) \cap Z = \varnothing$, then $f(Y)$ lies in $X - Z$ so $i \circ f = j \circ f$, which implies $i = j$ by the epimorphicity of f, a contradiction.

(c) Take $\phi : \mathbb{Z} \to \mathbb{Q}$ as the canonical inclusion. Obviously, the corresponding $\mathrm{Spec}\phi$: $\mathrm{Spec}\mathbb{Q} \to \mathrm{Spec}\mathbb{Z}$ is not epimorphic.

Exercise 6

(a) For simplicity denote $X = \mathrm{Spec}B$ and $Y = \mathrm{Spec}A$, and $f = \mathrm{Spec}\phi$.

$\Leftarrow$ If $f : X \to Y$ is scheme-theoretically dominant, then the morphism of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$ on $Y = \mathrm{Spec}A$ is a monomorphism, iff $\mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$ is injective for any $U \subset Y$. Then take $U = \mathrm{Spec}A$, we have an injection $A \to B$.

$\Rightarrow$If $\phi : A \to B$ is an injection, then $\phi_a : A_a \to B_{\phi(a)}$ is also an injection for all $a \in A$, since localization is an exact functor in **Ring**. But, note that $\mathcal{O}_Y(D(a)) \simeq A_a$ and $(f_*\mathcal{O}_X)(D(a)) = \mathcal{O}_X(f^{-1}(D(a))) = \mathcal{O}_X(D(\phi(a))) = B_{f(a)}$, which implies that the morphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective on all principal open subset $D(a)$ of A, hence is injective on all open subsets.

(b) We need to show that f is dense, that is, the image $f(Y)$ intersects with any non-empty open subset $U \subset X$. Since X is a scheme, it can be covered with affine open subsets, so it suffices to show that $f(Y)$ intersects with all non-empty affine open subsets. Suppose this were not true, then there exists a non-empty affine open subset $V = \mathrm{Spec}A$, such that $f(Y) \cap V = \varnothing$. But by assumption f is scheme-theoretically dominant, so on V we have an injection $A = \mathcal{O}_X(V) \to \mathcal{O}_Y(f^{-1}(V)) = \mathcal{O}_Y(\varnothing) = 0$, which implies $A = 0$ and $V = \mathrm{Spec}0 = \varnothing$, a contradiction.

If X is reduced, we want to show that a dominant morphism $f : Y \to X$ induces a monomorphism of sheaves $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$. If it were not true, then there exists an open subset U, such that $f^\sharp(U) : \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U))$ has non-zero kernel. Thus there must exists a smaller open subset $V \subset U$ such that $\ker(f^\sharp(V)) \neq 0$, otherwise $\ker(f^\sharp)$ is 0 at every stalk, a contradiction. So without loss of generality, we may assume that $U = \mathrm{Spec}A$, and let $0 \neq s \in \ker(f^\sharp(U)) \subset \mathcal{O}_X(U) = A$. By assumption A is reduced, so $s^n \neq 0$ holds for any $n \in \mathbb{N}$, hence $A_s = \mathcal{O}_X(D(s)) \neq 0$. Consider the following diagram

$$\begin{tikzcd} A = \mathcal{O}_X(U) \arrow[r, "f^\sharp(U)"] \arrow[d, "\mathrm{res}^U_{D(s)}"] & \mathcal{O}_Y(f^{-1}(U)) \arrow[d, "\mathrm{res}^{f^{-1}(U)}_{f^{-1}(D(s))}"] \\ A_s = \mathcal{O}_X(D(s)) \arrow[r, "0"] & \mathcal{O}_Y(f^{-1}(D(s))) \end{tikzcd},$$

which is commutative, and the vertical arrows are the canonical restriction, the lower horizontal map is 0 since $s|_{D(s)}$ is in the kernel of it. We can see that $\mathcal{O}_Y(f^{-1}(D(s))) \neq 0$, because f is dense and thus $f^{-1}(D(s)) \neq \varnothing$. But for any $t \notin \ker(f^\sharp(U))$, $\mathrm{res}^{f^{-1}(U)}_{f^{-1}(D(s))}(f^\sharp(U)(t)) \neq 0$, a contradiction.

(c) If $(f, f^\flat) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a surjective sheaf-theoretically dominant morphism. Then for any $(g_1, g_1^\flat) : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ and $(g_2, g_2^\flat) : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$, such that

$$g_1 \circ f = g_2 \circ f,$$
$$f^\flat \circ f^{-1} g_1^\flat = f^\flat \circ f^{-1} g_2^\flat$$

we can conclude $g_1 = g_2$ by surjectivity of f, and $f^{-1}g_1^\flat = f^{-1}g_2^\flat$ by injectivity of $f^\flat$. This shows that $(f, f^\flat)$ is epimorphic.

If $f : Y \to X$ is surjective, then it is dense since $\overline{f(Y)} = \overline{X} = X$. If X is reduced, by **(b)** it is sheaf-theoretically dominant, so it is epimorphic by the last paragraph.

Exercise 7

(a) This is purely abstract nonsense. Since we have the adjunction $\Gamma \dashv \mathrm{Spec}$, in which Spec is right adjoint, thus it reserves limits. But $\mathrm{Spec} : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Sch}$ is contravariant, so it maps colimits in **Ring** to limits **Sch**, hence maps epimorphisms to monomorphisms.

(b) If we can show every $f_x : \mathrm{Spec}\kappa(x) \to X$ is a monomorphism, then $f : \coprod_{x \in X} \mathrm{Spec}(\kappa(x)) \to X$ is a monomorphism. Topologically, $f_x : \mathrm{Spec}\kappa(x) \to X$ is clearly an injection. If $g_1, g_2, Y \to \mathrm{Spec}(\kappa(x))$ are any two morphisms of schemes such that $f_x \circ g_1 = f_x \circ g_2$, we have the corresponding commutative diagram ring homomorphisms

$$\mathcal{O}_{X,x} \xrightarrow{f_x^\sharp} \kappa(x) \overset{g_1^\sharp}{\underset{g_2^\sharp}{\rightrightarrows}} \mathcal{O}_Y(Y).$$

So

$$g_1^\sharp \circ f_x^\sharp = g_2^\sharp \circ f_x^\sharp,$$

iff $\mathrm{im} f_x^\sharp \subset \ker(g_1^\sharp - g_2^\sharp)$. But $f_x^\sharp$ is surjective since it is a quotient, so we have $\ker(g_1^\sharp - g_2^\sharp) = \kappa(x) \iff g_1^\sharp - g_2^\sharp = 0$, which shows $f_x : \mathrm{Spec}(\kappa(x)) \to X$ is a monomorphism of schemes.

(c) Consider the affine spectrum defined as

$$X' = \coprod_{p \in \mathbb{Z}} \mathrm{Spec}\mathbb{F}_p,$$

since $\mathbb{F}_p$ is the residue field of $\mathrm{Spec}\mathbb{Z}$ at the point (p), by **(b)** the canonical morphism

$$\coprod_p \mathrm{Spec}\mathbb{F}_p \to \mathrm{Spec}\mathbb{Z}$$

is a monomorphism.

Moreover, it is an epimorphic by **Exercise 6(c)**, since the map on the underlying topological spaces is clearly surjective, and $\mathrm{Spec}\mathbb{Z}$ is reduced because $\mathbb{Z}$ is a domain. Thus, $X' \to \mathrm{Spec}\mathbb{Z}$ is both a monomorphism and an epimorphism, and is a bijection. But the ring morphism

$$\mathbb{Z} \to \prod_{p\in\mathbb{Z}} \mathbb{F}_p$$

is not an isomorphism, since $\prod_{p\in\mathbb{Z}} \mathbb{F}_p$ has cardinality much larger than $\mathbb{Z}$ has.

Exercise 8

(a) It is easy to verify that $\mathfrak{m}_p$ and $\mathfrak{m}_\infty$ are indeed ideals of the ring A. To show their maximality, we need to show that $A/\mathfrak{m}_p$ and $A/\mathfrak{m}_\infty$ are fields. It is obvious that the former is a field, since the map

$$\begin{aligned} A &\to \mathbb{F}_p, \\ (a_q) &\mapsto a_p \end{aligned}$$

is surjective and $\mathfrak{m}_p$ is defined to be its kernel. Thus we have

$$\mathbb{F}_p \cong A/\ker(A \to \mathbb{F}_p) = A/\mathfrak{m}_p.$$

The fact that the $A/\mathfrak{m}_\infty$ is a field needs a little more effort. We claim that

$$A/\mathfrak{m}_\infty \cong \{\, a \in A \mid \mathrm{supp} a = \varnothing \,\} \cup \{\, (0_p) \,\},$$

in which the right hand side is a field, with zero element $(0_p) \in \prod_{p\in\mathcal{P}} \mathbb{F}_p$ in which 0_p the zero element of $\mathbb{F}_p$, and identity $(1_p) \in \prod_{p\in\mathcal{P}} \mathbb{F}_p$ in which 1_p the identity of $\mathbb{F}_p$. If our claim were right, the maximality of $\mathfrak{m}_\infty$ is proved.

Indeed, notice that

$$\mathfrak{m}_\infty := \bigoplus_{p\in\mathcal{P}} \mathbb{F}_p = \left\{ a \in \prod_{p\in\mathcal{P}} \mathbb{F}_p \;\middle|\; \mathrm{supp} a \text{ is finite} \right\},$$

we have

$$\begin{aligned} A/\mathfrak{m}_\infty &= \frac{\left\{ a \in \prod_{p\in\mathcal{P}} \mathbb{F}_p \;\middle|\; \mathrm{supp} a \text{ is finite} \right\} \cup \left\{ a \in \prod_{p\in\mathcal{P}} \mathbb{F}_p \;\middle|\; \mathrm{supp} a \text{ is cofinite} \right\}}{\left\{ a \in \prod_{p\in\mathcal{P}} \mathbb{F}_p \;\middle|\; \mathrm{supp} a \text{ is finite} \right\}} \\ &= \{\, (0_p) \,\} \cup \frac{\left\{ a \in \prod_{p\in\mathcal{P}} \mathbb{F}_p \;\middle|\; \mathrm{supp} a \text{ is cofinite} \right\}}{\left\{ a \in \prod_{p\in\mathcal{P}} \mathbb{F}_p \;\middle|\; \mathrm{supp} a \text{ is finite} \right\}} \\ &= \left\{ a \in \prod_{p\in\mathcal{P}} \mathbb{F}_p \;\middle|\; \mathrm{supp} a = \varnothing \right\} \cup \{\, (0_p) \,\}. \end{aligned}$$

The last equation holds, because every cofinitely supported element equals to an empty supported element, modulo a finitely supported element. Say, given any empty supported element $(a_p) \in \prod_{p\in\mathcal{P}} \mathbb{F}_p$ and any cofinitely supported element $(b_p) \in \prod_{p\in\mathcal{P}} \mathbb{F}_p$, their difference

$$(a_p) - (b_p) = (\dots, 0, a_{p_1}, 0, \dots, 0, a_{p_n}, 0, \dots)$$

is an element in $\bigoplus_{p\in\mathcal{P}} \mathbb{F}_p$, with all components zero but $a_{p_1}, a_{p_2}, \ldots, a_{p_n}$ not zero, where $p_1, p_2, \ldots, p_n$ are the vanishing sites of (b_p). Thus the claim has been proved.

To prove that the map

$$\mathcal{P}^* \to \mathrm{Spec}A,$$
$$p \mapsto \mathfrak{m}_p$$

is a homeomorphism, we need to show it is a bijection and maps closed subsets to closed subsets and pulls back closed subsets to closed subsets.. The map is clearly injective. If the map is surjective, then notice that the singleton $p \in \mathcal{P}^*$ is closed since it is the complement of the cofinite subset $\mathcal{P}^* - \{p\}$, and $\mathfrak{m}_p$ in Spec$A$ is closed under Zariski topology, since $\mathfrak{m}_p = V(\mathfrak{m}_p)$ by maximality. Easy verification shows that finite sum of $\mathfrak{m}_{p_i}$ is also prime. The bijection $p \mapsto \mathfrak{m}_p$ thus maps closed sets to closed sets is and pulls back closed sets to closed sets, hence is homeomorphic. The only thing needs to be verified is that $\mathcal{P}^* \to \mathrm{Spec}A$ is surjective, or equivalently, $\mathfrak{m}_p$ and $\mathfrak{m}_\infty$ contain no other prime ideals than themselves.

(b) According to the previous sub-exercise, we adapt ourselves to the new notation

$$\kappa_\infty := A/\mathfrak{m}_\infty = \left\{ a \in \prod_{p\in\mathcal{P}} \mathbb{F}_p \;\middle|\; \mathrm{supp}a = \varnothing \right\}.$$

We are now to show that κ_∞ is of zero characteristic and its cardinality is continuum. Suppose κ_∞ is of characteristic N for some $N \in \mathbb{N}$, then we have

$$N \cdot (1_p) = (0_p),$$

which means that we have

$$N \cdot 1_p = 0_p \in \mathbb{F}_p$$

for every prime $p \in \mathbb{Z}$. But such N would not exist, since if we choose any prime $q > N$, the equality

$$N \cdot 1_q = 0_q$$

fails in $\mathbb{F}_q$, implying that the characteristic of κ_∞ is 0.

Exercise 9

Since X is quasi-compact it can be covered by a finite affine cover $\{U_i\}_i^n$ with $U_i = \mathrm{Spec}A_i$. Since maximal ideals are in one-to-one correspondence with closed points, we can take a maximal ideal $\mathfrak{m}_1 \in \mathrm{Spec}A_1$. If $\mathfrak{m}_1$ is closed in X, the proof is done. Otherwise, we take $\mathfrak{m}_2$ in the closure of $\mathfrak{m}_1$. Now $\mathfrak{m}_2$ is in some U_i but not in U_1, since were it true, we must have $\mathfrak{m}_2 \in \overline{\mathfrak{m}_1} \cap U_1$ which is a closed subset of $U_1 = \mathrm{Spec}A_1$ but $\mathfrak{m}_1 = V(\mathfrak{m}_1)$ is the closure of $\mathfrak{m}_1$ in U_1, which is a contradiction. If $\mathfrak{m}_2$ is closed, we are done, if not, choose $\mathfrak{m}_3$ in the closure of $\mathfrak{m}_2$ which is also the closure of $\mathfrak{m}_1$, we then have $\mathfrak{m}_3 \in U_3$ not in U_1 and U_2. Do this procedure iteratively, we will finally have a closed point in some U_k because $\{U_i\}_{i=1}^n$ is finite. If the procedure consists of infinitely many steps, we will have closed points that don't belong to any U_i, which is impossible.

Exercise 10

(a) By the universal property of localization. Since The restriction map $A \to \mathcal{O}_X(X_f)$ maps f to the unit of $\mathcal{O}_X(X_f)$, it factors through the localization A_f. It remains to verify that $\phi : A_f \to \mathcal{O}_X(X_f)$ is injective.

(b) Since X has a finite affine cover $\{U_i\}$, X can be finitely covered by $\{D(f_{ij})\}$, where $\{D(f_{ij})\}_j = \{(A_i)_{f_{ij}}\}$ is the principal open subset of $U_i = \text{Spec}A_i$. Since $f \in \mathcal{O}_X(X)$ is a global section, it must agree on each $D(f_{il}) \cap D(f_{jk})$ for all indices i, j, k, l, we have $a_{il}/f_{il}^{m_{il}} = a_{jk}/f_{jk}^{m_{jk}}$. Since the intersections $D(f_{il}) \cap D(f_{jk})$ are also principal, there must be some $D(g_{iljk}) = D(f_{il}) \cap D(f_{jk})$ and $(f_{iljk})^{r_{iljk}}(a_{il}f_{jk}^{m_{jk}} - a_{jk}f_{il}^{m_{il}}) = 0$. Since $D(f_{ij})$ is finite, we can choose r large enough such that $f_{iljk}^r(a_{il}f_{jk}^{m_{jk}} - a_{jk}f_{il}^{m_{il}}) = 0$ and by absorbing f_{ijkl}^r into $f_{il}^{m_{il}}$ and $f_{jk}^{m_{jk}}$ and again choose m large enough by finiteness of $D(f_{ij})$, we finally have $a_{il}f_{jk}^m - a_{jk}f_{il}^m$ for all i, j, k, l. Since $D(f_{il}^m) = D(f_{il})$, thus $D(f_{il}^m)$ cover X and we have

(c) By **(b)** we have seen that $A_{f_i} \xrightarrow{\sim} \mathcal{O}_X(X_f)$ is an isomorphism for all f_i, and the affine opens X_{f_i} satisfy the property $X_{f_i} \cap X_{f_j}$. Then we can use the gluing property of affine schemes $X_{f_i} \hookrightarrow X$, by the uniqueness of gluing, we conclude that X and $\text{Spec}A$ are isomorphic.

Exercise 11

(a)

(b) We first show a simple but quite useful result that

Lemma 3. A finite union of quasi-compact subset is quasi-compact.

Proof. Suppose $X_1, \ldots, X_n$ are all quasi-compact, and let $X = X_1 \cup \cdots \cup X_n$. Suppose $\mathcal{U}$ is a cover of X, it is a cover of each X_i. By quasi-compactness of each X_i, we can find a finite subcover $\mathcal{U}_i$ of $\mathcal{U}$ that covers X_i. So $\cup_{i=1}^n \mathcal{U}_i$ is a finite cover of $\mathcal{U}$ that covers X. □

Now back to our problem. Suppose $f : Y \to X$ a quasi-compact morphism of schemes. And $U \subset X$ is a quasi-compact open subset of X, to show is that $f^{-1}(U)$ is quasi-compact. Since X can be covered by a collection of affine open subsets $\{V_i\}$, and each $V_i = \text{Spec}A_i$ can be covered by principal open subsets, so by the quasi-compactness of U, it can be covered by finitely many $D(g_i)$, with $g_i \in A_i$ for some A_i. If we can show every $f^{-1}(D(g_i))$ is quasi-compact, then $f^{-1}(U) = \cup_i f^{-1}(D(g_i))$ is a finite union of quasi-compact subsets hence is quasi-compact by the lemma above.

Indeed, for any $D(g_i) \subset V_i$ with $g_i \in A_i$, by assumption we have $f^{-1}(V_i) = \cup_j W_{ij} = \cup_j \text{Spec}B_{ij}$. Consider the open subset $f^{-1}(D(g_i)) \cap W_{ij} = D(\phi_j(g_i)) \cap W_{ij} = D(\phi_j(g_i))$, where

$$\phi_j : A_j \to B_{ij}$$

is the corresponding ring homomorphism with respect to the restriction of f on W_{ij}

$$f|_{W_{ij}} : W_{ij} = \text{Spec}B_{ij} \to V_i = \text{Spec}A_i.$$

But notice that $f^{-1}(D(g_i)) = \cup_j(f^{-1}(D(g_i)) \cap W_{ij}) = \cup_j D(\phi_j(g_i)) \simeq \cup_j \text{Spec}(B_{ij})_{\phi_j(g_i)}$, which is a finite union of affine scheme $\text{Spec}(B_{ij})_{\phi_j(g_i)}$. But each $\text{Spec}(B_{ij})_{\phi_j(g_i)}$ is quasi-compact, so is $f^{-1}(D(g_i))$ and $f^{-1}(U)$, by the lemma above.

(c)

(d)

Exercise 12

(a) First we show that every $a \in \mathfrak{m}$ is in the union $\cup_i \mathfrak{p}_i$, where $\mathfrak{p}_i \subset A$ are all height 1 prime ideals of A. Suppose, $a \in \mathfrak{p}$, $\mathfrak{p}$ is some prime ideal. By the Krull's principal ideal theorem, if $\mathfrak{p}$ is any prime ideal over a, then $\mathfrak{p}$ has height ≤ 1. Then we have $a \in \cup_i \mathfrak{p}_i$ since $\mathfrak{p}$ has height ≤ 1. This shows that $\mathfrak{m} \subset \cup_i \mathfrak{p}_i$. Conversely, since A is a local ring, $\mathfrak{m}$ is the unique maximal ideal and we have $\mathfrak{p}_i \subset \mathfrak{m}$ for all $\mathfrak{p}_i$, which implies $\cup_i \mathfrak{p}_i \subset \mathfrak{m}$. Thus we have showed $\mathfrak{m} = \cup_i \mathfrak{p}_i$.

(b) Choose a minimal ideal $\mathfrak{q}$ of height 0, and a maximal ideal $\mathfrak{m}$ containing $\mathfrak{q}$, by the assumption $\dim A \geq 2$, there is some proper prime ideals between them. Choose one point which is not a 0-divisor, by the Krull's principal ideal theorem, it is contained in a prime ideal $\mathfrak{p}_1$ of height 1. Then choose another point that is not a zero divisor in $\mathfrak{m} - \mathfrak{p}_2$, there is some height 1 prime ideal $\mathfrak{p}_2$ containing it, again by Krull's principal ideal theorem. We can do this procedure iteratively, to get a collection of height 1 ideals $\{\mathfrak{p}_i\}$. These prime ideals don't contain each others, since they are all height 1. Suppose that the collection $\{\mathfrak{p}_i\}$ is finitely many, say, has n distinct prime ideals. Then we can use Krull's principal ideal theorem again to get $\mathfrak{p}_{n+1}$ from some elements not a 0-divisor in $\mathfrak{m} - \cup_{i=1}^n \mathfrak{p}_i$, then we must have $\mathfrak{p}_{n+1} = \mathfrak{p}_i$ for some $1 \leq i \leq n$, by assumption. However, since $\mathfrak{p}_{n+1}$ doesn't contain any $\mathfrak{p}_i, 1 \leq i \leq n$, by the prime avoidance lemma, $\mathfrak{p}_{n+1}$ is not contained in $\cap_{i=1}^n \mathfrak{p}_i$, which is a contradiction. Thus we have proved that the collection $\{\mathfrak{p}_i\}$ must have infinitely many elements.

(c) Suppose X is a locally Noetherian scheme of dimension ≥ 2. Then we can find an affine open U of X such that $U = \mathrm{Spec} A$, A a Noetherian ring with $\dim A \geq 2$. Thus by the result of **(b)**, U has infinitely many irreducible closed subset of codimension 1, namely $V(\mathfrak{p}_i)$, which are all distinct. Thus U has infinitely many points, which implies X has infinitely many points.

Exercise 13

(a) Denote the projection maps induced by $f : Y \to X$ aspr_1 and pr_2:

$$\begin{tikzcd} Y \times_X Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

We are asked to show that $\mathrm{pr}_1 : Y \times_X Y \to Y$ is an isomorphism.

Now suppose that $f : Y \to X$ is monomorphic, in this case since $f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$, we have

$$\mathrm{pr}_1 = \mathrm{pr}_2. \tag{9}$$

And consider the commutative diagram below,

$$\begin{tikzcd} Y \arrow[dr, dotted, "\Delta_f"] \arrow[drr, bend left, "\mathrm{id}_Y"] \arrow[ddr, bend right, "\mathrm{id}_Y"'] & & \\ & Y \times_X Y \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, dashed, "\mathrm{pr}_1"] & Y \arrow[d, dashed, "f"] \\ & Y \arrow[r, "f"] & X \end{tikzcd} \tag{10}$$

where Δ_f is the unique map induced by $\mathrm{id}_Y : Y \to Y$. By commutativity of the diagram we have

$$\mathrm{pr}_1 \circ \Delta_f = \mathrm{id}_Y \tag{11}$$

. Then we consider $\Delta_f \circ \mathrm{pr}_1 : Y \times_X Y \to Y \times_X Y$, which is thus uniquely determined by its components. But we have

$$\mathrm{pr}_1 \circ \Delta_f \circ \mathrm{pr}_1 = \mathrm{pr}_1,$$

by Equation (11), and

$$\mathrm{pr}_2 \circ \Delta_f \circ \mathrm{pr}_1 = \mathrm{pr}_1 \circ \Delta_f \circ \mathrm{pr}_1 = \mathrm{pr}_1 = \mathrm{pr}_2$$

by Equation (9) and Equation (11). This shows that $\Delta_f \circ \mathrm{pr}_1$ is uniquely determined by pr_1 and pr_2, then by the universal property we have

$$\Delta_f \circ \mathrm{pr}_1 = \mathrm{id}_{Y \times_X Y}. \tag{12}$$

Thus, we have an isomorphism $\mathrm{pr}_1 : Y \times_X Y \to Y$.

Conversely, suppose that $\mathrm{pr}_1 : Y \times_X Y \to Y$ is an isomorphism. If there is any Z and any morphisms $g_1 : Z \to Y$ and $g_2 : Z \to Y$ such that $f \circ g_1 = f \circ g_2$, there is a unique $g : Z \to Y \times_X Y$ such that the diagram

$$\begin{tikzcd}
Z \arrow[dr, dotted, "g"] \arrow[drr, bend left, "g_2"] \arrow[ddr, bend right, "g_1"'] & & \\
& Y \times_X Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & Y \arrow[d, "f"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd}$$

commutes. Thus we have

$$\begin{aligned} g_1 &= \mathrm{pr}_1 \circ g, \\ g_2 &= \mathrm{pr}_2 \circ g. \end{aligned} \tag{13}$$

If we again can show $\mathrm{pr}_1 = \mathrm{pr}_2$, then we can deduce $g_1 = g_2$ and f is monomorphic follows.

This is indeed the case. Consider the commutative diagram (10) again. Since $\mathrm{pr}_1 : Y \times_X Y \to Y$ is an isomorphism, it has a unique right inverse, this right inverse is exactly $\Delta_f : Y \to Y \times_X Y$. Thus, Equation (11) and Equation (12) all hold and in addition

$$\mathrm{pr}_2 \circ \Delta_f = \mathrm{id}_Y$$

holds. The last equation says that Δ_f has a right inverse, but Δ_f is also an isomorphism since it is the inverse of the isomorphism pr_1, then Δ_f has a unique right inverse, so we have $\mathrm{pr}_1 = \mathrm{pr}_2$. Thus, from Equation (13) tells us that

$$g_1 = \mathrm{pr}_1 \circ g = \mathrm{pr}_2 \circ g = g_2.$$

Since Z and g_1, g_2 is arbitrary, we have showed that $f : Y \to X$ is a monomorphism.

(b) $\phi : k \to B$ is an epimorphism in **Ring** iff the induced $\mathrm{Spec}\phi : \mathrm{Spec}B \to \mathrm{Spec}k$ is monomorphism in the category **Sch**, since Spec is right adjoint hence maps colimits to limits. Then by **(a)**, $\mathrm{Spec}B \to \mathrm{Spec}k$ is a monomorphism iff $\mathrm{Spec}B \times_k \mathrm{Spec}B = \mathrm{Spec}B \otimes_k B \to \mathrm{Spec}B$ is an isomorphism, iff $B \to B \otimes_k B$ is an isomorphism. Thus in the category **Ring**, the pushforward diagram

$$\begin{tikzcd}
k \arrow[r, "\phi"] \arrow[d, "\phi"] & B \arrow[d] \\
B \arrow[r] & B \otimes_k B
\end{tikzcd}$$

can be replaced by the diagram

$$\begin{tikzcd} k \arrow[r, "\phi"] \arrow[d, "\phi"] & B \arrow[d, "\mathrm{id}_B"] \\ B \arrow[r, "\mathrm{id}_B"] & B \end{tikzcd}.$$

For any ring C and ring homomorphism $f_1, f_2 : B \to C$ such that $f_1 \circ \phi = f_2 \circ \phi$, we have the commutative diagram

$$\begin{tikzcd} k \arrow[r, "\phi"] \arrow[d, "\phi"] & B \arrow[d, "\mathrm{id}_B"] \arrow[ddr, bend left, "f_2"] & \\ B \arrow[r, "\mathrm{id}_B"] \arrow[drr, bend right, "f_1"] & B \arrow[dr, dotted, "f"] & \\ & & C \end{tikzcd},$$

with f uniquely determined by f_1 and f_2. But we have

$$f_1 = f \circ \mathrm{id}_B = f = f \circ \mathrm{id}_B = f_2,$$

which shows that $\phi : k \to B$ is a ring epimorphism. Note that if k is a field and B is a ring, any ring homomorphism between them is injective and hence monomorphic, thus $\phi : k \to B$ is an isomorphism.

Thus if $f : Y \to \mathrm{Spec}k$ is a monomorphism iff $Y \times_{\mathrm{Spec}k} Y \to Y$ are isomorphic. If $\{U_i = \mathrm{Spec}A_i\}_{i \in I}$ is an affine open cover of Y, let $\iota_i : U_i \hookrightarrow Y$ be the open immersion of the open affine scheme U_i. Open immersions are monomorphisms Lemma 01L7 , we have monomorphism $f \circ \iota_i : U_i = \mathrm{Spec}A_i \to \mathrm{Spec}k$, since compositions of monomorphisms are monomorphisms. We have showed that $\mathrm{Spec}A_i \overset{f\circ\iota_i}{\to} \mathrm{Spec}k$ are monomorphisms iff $A_i = 0$ or A_i isomorphic to k. If all $A_i = 0$ we have $U_i = \mathrm{Spec}A_i = \varnothing$ and $Y = \cup_i U_i = \varnothing$, otherwise suppose $U_j = \mathrm{Spec}k \neq \varnothing, j \in J$ for some subset $J \subset I$. Hence we have $Y = \cup_{j \in J}\{x_j\}$ where x_j corresponds to the unique point of $\mathrm{Spec}k$, and $f : Y \to \mathrm{Spec}k$ to be monomorphism asks the under lying continuous map between topological spaces is injective, implying $Y = \{x\} = \mathrm{Spec}A_i = \mathrm{Spec}k$ for some $i \in J$. We can conclude that if $f : Y \to \mathrm{Spec}k$ is monomorphic, then $Y = \varnothing$ or f is an isomorphism. Conversely, if $Y = \varnothing$ or Y isomorphic to $\mathrm{Spec}k$, $Y \to \mathrm{Spec}k$ is left cancellative hence is monomorphic.

(c) Suppose $f : Y \to X$ is monomorphism. If $Y = \varnothing$, $\phi : \varnothing \to X$ is indeed an injection on the underlying topological spaces. If $Y \neq \varnothing$, let k be any field and let $t_1 : \mathrm{Spec}k \to Y$ and $t_2 : \mathrm{Spec}k \to Y$ be to different morphisms of schemes, such that $f \circ t_1 = f \circ t_2$. Let y_1 and y_2 be the image of the unique point z in $\mathrm{Spec}k$ under t_1 and t_2. As topological spaces, we have $f(y_1) = f \circ t_1(z) = f \circ t_2(z) = f(y_2)$, but on the other hand we have $y_1 = t_1(x) = t_2(x) = y_2$, since f is left cancellative. This shows that $f : X \to Y$ is an injection on the underlying topological spaces.

If there are any scheme Z and any morphisms of schemes $g_1, g_2 : Z \to Y$satisfying $f \circ g_1 = f \circ g_2$, the monomorphicity of $f : Y \to X$ tells us that on the underlying topological spaces, g_1 and g_2 are the same continuous maps. And restricted to every stalk, we have the commutative diagram of local rings

$$\begin{tikzcd} \mathcal{O}_{X,f(y)} \arrow[r, "f_{*,x}"] \arrow[d, "f_{*,x}"] & \mathcal{O}_{Y,y} \arrow[d, "(f_* g_2^\sharp)_x"] \\ \mathcal{O}_{Y,y} \arrow[r, "(f_* g_1^\sharp)_x"] & \mathcal{O}_{Z,z} \end{tikzcd},$$

where $y = g_1(z) = g_2(z)$. This is in bijection with the commutative diagram of residue fields

$$\begin{array}{ccc} \kappa(f(y)) & \longrightarrow & \kappa(y) \\ \downarrow & & \downarrow \\ \kappa(y) & \longrightarrow & \kappa(z) \end{array},$$

and since g_1, g_2 are arbitrary, $\kappa(f(y)) \to \kappa(y)$ is epimorphic, and thus isomorphic. This shows that the field extension $\kappa(y)/\kappa(f(y))$ is trivial.

Exercise 14

(a) We are asked to show that$X(\phi) : X(K) \to X(L)$ is an injection, which is equivalent to the condition that for any $f_1 : \mathrm{Spec}K \to X$ and $f_2 : \mathrm{Spec}K \to X$, $f_1 \circ \mathrm{Spec}\phi = f_2 \circ \mathrm{Spec}\phi$ implies $f_1 = f_2$. That is, we are asked to show that $\mathrm{Spec}\phi : \mathrm{Spec}L \to \mathrm{Spec}K$ is an epimorphism of schemes.

Since $\phi : K \to L$ is a field embedding, we must have $\ker(\phi) = 0$, since $\ker(\phi)$ is an ideal in K, as ϕ is viewed as a ring homomorphism. Thus $\phi : K \to L$ is a monomorphism in the category **Ring**. In **Ring** $\phi : K \to L$ is monomorphic iff it is an injection, by **Exercise 5(a)**, thus $\mathrm{Spec}\phi : \mathrm{Spec}K \to \mathrm{Spec}L$ is sheaf-theoretically dominant by **Exercise 6(a)**. By **Exercise 6(c)**, any sheaf-theoretically dominant morphism that is surjective on the underlying topological spaces is epimorphic, which is just out case, since the underlying spaces of $\mathrm{Spec}K$ and $\mathrm{Spec}L$ are all $\{\mathrm{pt}\}$, $\mathrm{Spec}\phi$ on them is surjective.

(b) $\Leftarrow$ Suppose $f : X \to Y$ is a morphism of schemes, and for every field K, there is a field extension L/K such that $f(L) : X(L) \to Y(L)$ is surjective. More specifically, $f(L)$ is surjective means for any $y : \mathrm{Spec}L \to Y$, there exists $x : \mathrm{Spec}L \to X$, such that $y = f \circ x$ as morphisms of schemes. For any $y : \mathrm{Spec}L \to Y$, the image of $p \in \mathrm{Spec}L$ under y is just the point y, and the corresponding map of sheaves is $y^\sharp : \mathcal{O}_{Y,y} \to L$, which factors through $\kappa(y) \to L$. And similar things holds for $x : \mathrm{Spec}L \to X$. Thus, $f(L) : X(L) \to Y(L)$ is surjective iff for any $y \in Y$, there exists $x \in X$ such that $f(x) = y$ and the diagram

$$\begin{tikzcd} \kappa(y) \arrow[r, "f_x^\sharp"] \arrow[dr, "y^\sharp"'] & \kappa(x) \arrow[d, "x^\sharp"] \\ & L \end{tikzcd}$$

commutes. Thus $f : X \to Y$ is surjective on the underlying topological spaces.

$\Rightarrow$ Let $f : X \to Y$ be a morphism of schemes that is surjective on the underlying spaces, that is, for any $y \in Y$, there exists some $x \in X$ such that $f(x) = y$. Then for any field K of characteristic p. Since there are no morphisms between fields of different characteristic, we only consider all the point of $x \in X$ such that $\kappa(x)$ is of characteristic p, and take a cardinal number $\aleph$ bigger than all of them.

Choose L to be an algebraically closed field of transcendence degree at least $\aleph$ over K. Then, for any $y : \mathrm{Spec}L \to Y$, we need to find a some $x : \mathrm{Spec}L \to X$ such that $f \circ x = y$. The point x is easy to choose, since $f : X \to Y$ is surjective on the underlying spaces. Now the only question is to find a dashed arrow,

$$\begin{tikzcd} \kappa(y) \arrow[r] \arrow[dr, "f_x^\sharp"] & \mathrm{Spec}L \\ & \kappa(x) \arrow[u, dashed] \end{tikzcd}$$

making the above diagram commute. To do this, we choose a transcendence basis of $\kappa(x)$ over $\kappa(y)$ and embed it to L, which is possible because L is algebraically closed and has transcendence degree larger than any $\kappa(x)$.

(c) $\Rightarrow$ If $f : X \to Y$ is radicial, then it is injective under any base change. Then in particular we have the commutative diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "f'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd},$$

in which $f' : X \times_Y X \to X$ is injective. Since $\Delta_f : X \to X \times_Y X$ is induced by $id_X : X \to X$, then we have

$$f' \circ \Delta_f = \mathrm{id}_X.$$

Since Δ_f is a right inverse of the injection f', then it is surjective under the topological spaces.

$\Leftarrow$Recall the equivalent conditions of $f : X \to Y$ being radicial, it suffices to show that $f(K) : X(K) \to Y(K)$ is injective for any field K. Suppose there are $x_1 : \mathrm{Spec}K \to X$ and $x_2 : \mathrm{Spec}K \to X$ satisfy $f \circ x_1 = f \circ x_2$. By the universal property of $X \times_Y X$, there is a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}K \arrow[rrd, bend left, "x_1"] \arrow[rd, dotted, "x"] \arrow[rdd, bend right, "x_2"'] & & \\ & X \times_Y X \arrow[r, dashed, "f'"] \arrow[d, dashed, "f'"] & X \arrow[d, dashed, "f"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

with $x : \mathrm{Spec}K \to X \times_Y X$ uniquely determined by x_1 and x_2. But $x_1 = f' \circ x = x_2$, which shows that $f(K) : X(K) \to Y(K)$ is injective.

Recall that a morphism $f : X \to Y$ is separated, if Δ_f is a closed immersion. We already learned that Δ_f is an immersion, what left to us is to show that $\Delta_f(X)$ is a closed subset in $X \times_Y X$. But if $f : X \to Y$ is radicial, then Δ_f is surjective and $\Delta_f(X) = X \times_Y X$ as underlying topological spaces, thus the image of Δ_f is closed in $X \times_Y X$.

Exercise 15

(a) Recall that a morphism $f : X \to Y$ of schemes is an immersion iff it is locally closed and the map $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective for all $x \in X$. In our situation, we want to show the map $\pi : E \to X$ is an immersion, so we just need to show π is locally closed topologically and the induced maps $\pi_x^\sharp$ between stalks are surjective. Indeed, the underlying topological space E' of E is the set-theoretic equilizer, by definition. Topologically, E' is the intersection of the closed subsets $g^{-1}(W)$ and $h^{-1}(W)$ in X and thus is closed hence is locally closed. Then, we just need to show that the morphism $\pi^\sharp : \mathcal{O}_X \to \pi_*\mathcal{O}_E$ is surjective stalk-wise.

Since we are considering the maps $\pi_x^\sharp : \mathcal{O}_{X,x} \to \mathcal{O}_{E,(x,y)}$, without loss of generality, we may assume that $X = \mathrm{Spec}A$ and $W = \mathrm{Spec}B$ are affine. Then, the equilizer $E = \mathrm{Spec}A \times_B \mathrm{Spec}A$ is also affine, and the morphism of sheaves reduced to the pushforward diagram in the category **Ring**:

$$\begin{tikzcd} B \arrow[r, "g^\sharp"] \arrow[d, "h^\sharp"] & A \arrow[d] \\ A \arrow[r] & A \otimes_B A \end{tikzcd}.$$

The map $A \to A \otimes_B A$ is surjective, since the preimage of $a \otimes_B a' \in A \otimes_B A$ contains $a \cdot a' \in A$. And since the localization is an exact functor, the map is surjective on the stalk level.

(b) If we could show that $f : Y \to X$ is surjective, then by **Exercise 6(c)**, the scheme-theoretically dominant morphism f is an epimorphism. It suffices to show that for every point $x \in X$, the inverse image $f^{-1}(x)$ is not empty. Consider the closure $\overline{\{x\}}$ of the singleton $\{x\} \subset X$, by assumption, the intersection $f(Y) \cap \overline{\{x\}}$ is not empty. Observe that $f(Y) \cap \overline{\{x\}}$ is a closed subset in X, since $f^{-1}(f(Y) \cap \overline{\{x\}}) = f^{-1}(\overline{\{x\}})$ is closed in Y. If $x \in f(Y) \cap \overline{\{x\}}$, we are done. If not, we can find an open subset $V_x \subset \overline{\{x\}}$ containing x such that $V_x \cap (f(Y) \cap \overline{\{x\}}) = V_x \cap f(Y) = \varnothing$. But by **Exercise 6(b)**, any scheme-theoretically dominant morphism is dominant, that is, $f(Y)$ is dense in X, which means that the intersection of $f(Y)$ with any open subset of X is not empty. Thus we have $V_x \cap f(Y) \neq \varnothing$, which is absurd.

(c) Since A is a local domain of dimension ≥ 2, then we can find a chain of prime ideals $(0) \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{m}$, thus we can infer that Spec$A$ has at least 3 points. Next we consider the morphism of schemes Spec$K \coprod$ Spec$k \to$ Spec$A$. The morphism is induced by $\iota_1 :$ Spec$K \to$ Spec$A$ and $\iota_2 :$ Spec$k \to$ Spec$A$, where ι_1 sends the unique point of Spec$K$ to the point in Spec$A$ corresponding to the prime ideal (0), and ι_2 sends the unique point of Spec$k$ to the point in Spec$A$ corresponding to the maximal ideal $\mathfrak{m}$. The associated morphisms of sheaves are as follows: at the stalk level we have $\iota^\sharp_{1,(0)} : A_{(0)} \xrightarrow{\sim} K$, and $\iota^\sharp_{2,\mathfrak{m}} : A_\mathfrak{m} = A \to k$.

By **Exercise 7(b)** the map $\iota_1 \coprod \iota_2$ is a monomorphism of schemes. We next want to show that $\iota_1 \coprod \iota_2$ is scheme-theoretically dominant and its image intersects with all closed subsets in Spec$A$. A morphism $f : Y \to X$ is scheme-theoretically dominant iff $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is an injection when restricted to every open subset, iff it is an injection at the stalk level. Now, in our case, the map $\iota^\sharp_{1,(0)} : A_{(0)} \xrightarrow{\sim} K$ is an injection since it is an isomorphism of rings, thus $\iota^\sharp_{1,(0)} \coprod \iota^\sharp_{2,\mathfrak{m}}$ is an injection hence $\iota_1 \coprod \iota_2$ is scheme-theoretically dominant. And note that the image of Spec$k$ in Spec$A$ is the unique maximal ideal $\mathfrak{m}$, for any ideal $I \subset A$ we have $I \subset \mathfrak{m}$, which is equivalent to $\mathfrak{m} \in V(I)$. This shows that the image of Spec$K \coprod$ Spec$k$ in Spec$A$ intersects all non-empty closed subsets. Thus by **(b)** we have an epimorphism Spec$K \coprod$ Spec$k \to$ Spec$A$.

But, Spec$K \coprod$ Spec$k \to$ Spec$A$ is not surjective, note there is at least a prime ideal $\mathfrak{p}_1 \subset A$ neither 0 nor maximal by assumption, but the inverse image of this $\mathfrak{p}$ in Spec$K \times$ Spec$k$ is empty. Thus we have constructed a map that is both monomorphic and epimorphic but not surjective.

Exercise 16

(a) Since surjection is a local property, we may assume that $X =$ Spec$A$ is affine. Thus, the closed immersion $i : Z \hookrightarrow X$ corresponds to the morphism Spec$(A/I) \hookrightarrow$ Spec$A$, where $I \subset A$ is an ideal. Moreover the morphism of sheaves $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ becomes the ring homomorphism $A \to A/I$. The sheaf $\mathcal{O}_X \times_{i_*\mathcal{O}_Z} \mathcal{O}_X$ is defined via the following pullback diagram

$$\begin{tikzcd} A \times_{A/I} A \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & A/I \end{tikzcd} .$$

The pullback indeed exists. Note that the ring $A \times_{A/I} A$ can be characterized as

$$A \times_{A/I} A = \{\, (a,b) \in A \times A \mid a - b \in I \,\},$$

and is a ring. The prime ideals in $A \times_{A/I} A$ is of the form $\mathfrak{p}_1 \times \mathfrak{p}_2$, where $\mathfrak{p}_1, \mathfrak{p}_2$ both contain the ideal I.

The natural projections pr_i:$A \times_{A/I} A \to A, i = 1,2$ induces a ring homomorphism $A \times_{A/I} A \to A \coprod A$, furthermore induces morphism of schemes $X \coprod X = \mathrm{Spec}(A \coprod A) \to \mathrm{Spec}(A \times_{A/I} A) = W$. We are asked to show this map is surjective and finite. For surjectivity, given any prime ideal $\mathfrak{p}_1 \times \mathfrak{p}_2 \subset A \times_{A/I} A$, the inverse image is just $(\mathfrak{p}_1, \mathfrak{p}_2) \subset A \coprod A$, thus the map is surjective. For finiteness, we need to show that $A \coprod A$ is a finite $(A \times_{A/I} A)$-algebra, that is, $A \coprod A$ is a finitely generated $(A \times_{A/I} A)$-module. But this is indeed true, since $A \coprod A$ can be generated by the element $(1,1)$ as an $(A \times_{A/I} A)$-module, where $1 \in A$ is the identity.

Intuitively, we can describe the underlying topological space of W as the topological space obtained by gluing two copies of X along the closed subset Z. That is, $\mathrm{sp}(W) = \mathrm{sp}(X) \coprod_{\mathrm{sp}(Z)} \mathrm{sp}(X)$.

(b) By definition we have the exact sequence of sheaves over X

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y.$$

Since $\mathcal{O}_X$ is quasi-coherent, and quasi-coherent sheaves over X form an abelian category, if we can show that $f_*\mathcal{O}_Y$ is quasi-coherent, then we can show $\mathcal{I}$ is quasi-coherent.

To show that $f_*\mathcal{O}_Y$ is quasi-coherent, it suffices to show that on any affine open $\mathrm{Spec}A = U \subset X$, $f_*\mathcal{O}_Y(U) = \mathcal{O}_Y(f^{-1}(U))$ is the kernel or cokernel of quasi-coherent sheaves. Since $f : Y \to X$ is quasi-compact, then $f^{-1}(U)$ is quasi-compact and can be covered by finitely many affine opens, say $f^{-1}(U) = \cup_i^n V_i = \mathrm{Spec}B_i$. Moreover, for each $V_i \cap V_j$, we can find finitely many principal opens $V_{ijk} = \mathrm{Spec}(B_i)_{g_{jk}}$ such that $V_i \cap V_j = \cup_k \mathrm{Spec}(B_i)_{g_{jk}}$, since each $V_i = \mathrm{Spec}B_i$ is quasi-compact. Thus, by the sheaf condition of $\mathcal{O}_Y$, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(f^{-1}(U)) \longrightarrow \prod_i^n \mathcal{O}_Y(V_i) \longrightarrow \prod_{ijk} \mathcal{O}_Y(V_{ijk}) \ .$$

Since the products $\prod_i^n \mathcal{O}_Y(V_i)$ and $\prod_{ijk} \mathcal{O}_Y(V_{ijk})$ are finite and V_i, V_{ijk} are affine, they are quasi-coherent. Thus $f_*\mathcal{O}_Y(U)$ is the kernel of quasi-coherent sheaves hence is quasi-coherent. This shows that $\mathcal{I}$ is also quasi-coherent.

Next we have to show that the closed subscheme Z determined by the ideal $\mathcal{I}$ is the smallest closed subscheme such that $f : Y \to X$ factors through. Note that the closed subschemes of X are in one-to-one correspondence to the quasi-coherent ideal sheaves of $\mathcal{O}_X$. Suppose there is another closed Z' determined by the quasi-coherent ideal sheaf $\mathcal{I}_{Z'}$, such that $f : Y \to X$ factors through, then we have the commutative diagram

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "g"] \arrow[ddr, "h"'] & & X \\
& Z \arrow[ur, "i"'] & \\
& Z' \arrow[uur, "i'"'] &
\end{tikzcd}$$

and the corresponding commutative diagrams of sheaves

$$\begin{tikzcd}
\mathcal{O}_X \arrow[rr, "f^\sharp"] \arrow[dr, "i'^\sharp"] \arrow[ddr, "i^\sharp"'] & & f_*\mathcal{O}_Y \\
& \mathcal{O}_X/\mathcal{I}_{Z'} \arrow[ur, "g^\sharp"] \arrow[d, dashed] & \\
& \mathcal{O}_X/\mathcal{I} \arrow[uur, "h^\sharp"'] &
\end{tikzcd} .$$

The morphism $\mathcal{O}_X/\mathcal{I}_{Z'} \to \mathcal{O}_X/\mathcal{I}$ is induced by the universal property of the quotient $\mathcal{O}_X/\mathcal{I}$. And by the one-to-one correspondence of quasi-coherent ideal sheaves and closed subschemes, we deduce that there is a morphism of schemes $Z \to Z'$, which shows that Z is the smallest ideal meeting the requirement.

(c) $\Leftarrow$ By **Exercise 15(b)**, if f is scheme-theoretically dominant and $f(Y)$ intersects all non-empty closed subsets in X, it is an epimorphism.

$\Rightarrow$ By **Exercise 5(b)** we know that $f(Y)$ intersects with all closed subsets in X, because of the epimorphicity. We are left to show that $f : Y \to X$ is scheme-theoretically dominant. To show this, we just need to show at every point the maps $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ between stalks are injective. This follows from epimorphicity. Suppose there is a scheme W and morphisms of schemes $g_1, g_2 : W \to Y$ such that $f \circ g_1 = f \circ g_2$, then we have

$$\mathcal{O}_{X,fg(w)} \xrightarrow{f^\sharp_{g(w)}} \mathcal{O}_{Y,g(w)} \underset{(f\circ g_1)^\sharp_w}{\overset{(f\circ g_2)^\sharp_w}{\rightrightarrows}} \mathcal{O}_{W,w} \ .$$

Since W and g_1, g_2 are arbitrary, the epimorphicity implies that $(f \circ g_2)^\sharp = (f \circ g_1)^\sharp$, which shows that $\mathcal{O}_{X,fg(w)} \to \mathcal{O}_{Y,g(w)}$ is injective.

Exercise 17

(a) Firstly we have

$$k[x,y]/(y^7 - x^{2020}) \simeq k[t^{2020}, t^7] = A.$$

Since t is the solution of the equation $x^7 - t^7 = 0$, but $t \notin A$, so A is not integrally closed. The integral closure of A in its fractional field $\mathrm{Frac}(A)$ is $\bar{A} = k[t]$. So we have find the normalization $X^\nu = \mathrm{Spec}\bar{A} = \mathrm{Spec}k[t]$ of $X = \mathrm{Spec}A$. To compute the fiber of the normalization $X^\nu \to X$, we need to characterize the prime ideal in X.

Since $y^7 - x^{2020}$ is irreducible in $k[x,y]$, the ideal $(y^7 - x^{2020})$ is prime. So we can view $X = \mathrm{Spec}A = V(y^7 - x^{2020})$ as an irreducible closed subscheme in $\mathrm{Spec}k[x,y]$. And note that $\dim \mathrm{Spec}k[x,y] = 2$, and (0) is prime in $k[x,y]$, hence the prime ideals in $X = V(y^7 - x^{2020})$ are those maximal ideals in $k[x,y]$ containing $(y^7 - x^{2020})$. Since k is algebraically closed, by the Hilbert basis theorem, the maximal ideals in $k[x,y]$ are of the form $(x-a, y-b)$ with $a, b \in k$.

For a prime ideal $\mathfrak{p} \in X$, denote $\kappa(\mathfrak{p})$ as its residue field, then the fiber of $X^\nu \to X$ at $\mathfrak{p}$ is

$$\mathrm{Spec}\bar{A} \times_A \mathrm{Spec}\kappa(\mathfrak{p}) = \mathrm{Spec}(\bar{A} \otimes_A \kappa(\mathfrak{p})) \simeq \mathrm{Spec}\bar{A}_\mathfrak{p}/\mathfrak{p}\bar{A}_\mathfrak{p}. \tag{14}$$

Thus, to tell the reducibility of the fiber at $\mathfrak{p}$, we just need to see the reducibility of the ideal $\mathfrak{p}\bar{A}_\mathfrak{p}$ in $\bar{A}$. Since the A-action on $\bar{A}$ is induced via the natural inclusion

$$A = k[t^{2020}, t^7] \hookrightarrow \bar{A} = k[t],$$

the ideal $\mathfrak{p}\bar{A}_\mathfrak{p}$ corresponds to maximal ideal with the form $\mathfrak{p} = (x-a, y-b) \in k[x,y]$ can be written explicitly

$$\mathfrak{p}\bar{A}_\mathfrak{p} = (t^{2020} - a, t^7 - b).$$

If $a \neq 0, b \neq 0$ or $a = b = 0$, we have

$$\mathfrak{p}\bar{A}_\mathfrak{p} = (at + b),$$

or

$$\mathfrak{p}\bar{A}_\mathfrak{p} = (t^{2020}, t^7) = (t)$$

because 7 and 2020 are coprime and in both cases we have

$$\bar{A}_\mathfrak{p}/\mathfrak{p}\bar{A}_\mathfrak{p} = k,$$

and with any algebraic extension $k \to \bar{k}$ we have

$$k \otimes_{\kappa(\mathfrak{p})} \bar{k} \simeq \bar{k}.$$

Thus, when $a = b = 0$ or $a \neq 0, b \neq 0$, the fiber over $\mathfrak{p}$ is geometrically irreducible.

If one of a, b is 0 while the other is not, say $a \neq 0, b = 0$, we have

$$\mathfrak{p}\bar{A}_{\mathfrak{p}} = (t^{2020} - a, t^7) = (a) = k[t]$$

thus we have

$$\bar{A}_{\mathfrak{p}}/\mathfrak{p}\bar{A}_{\mathfrak{p}} = 0$$

and thus the fiber is $\varnothing$. In this case, the fiber is also geometrically irreducible.

To sum up, the all fibers of the normalization $X^{\nu} \to X$ are geometrically irreducible.

(b) This is more challenging than **(a)**. Firstly, we compute the normalization. We have

$$A = k[x, y, z]/(xy^2 - z^2) \simeq k[u^2, uv, v].$$

However, the solution u of the equation $x^2 - u^2 = 0$ is not in A, but is in $\mathrm{Frac}(A)$: $u = uv/v$. Thus A is not integrally closed, and the integral closure $\bar{A}$ of A in $\mathrm{Frac}(A)$ is

$$\bar{A} = k[u, v].$$

Thus, we have found the normalization $X^{\nu} = \mathrm{Spec}\bar{A}$, and have to compute the fiber of this normalization. To do so, we have to characterize the prime ideals of $A = k[u^2, uv, v]$.

Note that $(xy^2 - z^2)$ is prime in $k[x, y, z]$ since $xy^2 - z^2$ is irreducible. And also note that $\dim \mathrm{Spec} k[x, y, z] = 3$ and $(0) \subset k[x, y, z]$ is also prime, then the prime ideals in A correspond precisely to the prime ideals of height 2 or height 3 (thus is maximal) containing $(xy^2 - z^2)$.

1. If the prime ideal $\mathfrak{p}$ containing $(xy^2 - z^2)$ is of height 2, then we have $\mathfrak{p} = (y, z)$.
2. If the prime ideal $\mathfrak{p}$ containing $(xy^2 - z^2)$ is of height 3, then it is maximal, by the Hilbert basis theorem, $\mathfrak{p} = (x - a, y - b, z - c)$ with $ab^2 - c^2 = 0 \in k$.

Using the same procedure as in **(a)**, we know that the fiber of the normalization $X^{\nu} \to X$ at the point $\mathfrak{p}$ is $\mathrm{Spec}(\bar{A}_{\mathfrak{p}}/\mathfrak{p}\bar{A}_{\mathfrak{p}})$. Since the A-algebra structure on $\bar{A}$ is induced by the natural inclusion

$$A \simeq k[u^2, uv, v] \hookrightarrow k[u, v]$$

We can describe the rings $\bar{A}_{\mathfrak{p}}/\mathfrak{p}\bar{A}_{\mathfrak{p}}$ more explicitly.

If $\mathfrak{p}$ is of height 2 in $k[x, y, z]$, then $\mathfrak{p} = (y, z) \subset k[x, y, z]$. We have

$$\mathfrak{p}\bar{A}_{\mathfrak{p}} = (uv, v) = (v),$$

which is irreducible. For any algebraic extension $k \to \bar{k}$, we have $\bar{A}_{\mathfrak{p}}/\mathfrak{p}\bar{A}_{\mathfrak{p}} \otimes_{\kappa(\mathfrak{p})} \bar{k}$ still a field. Thus in this case the fiber is geometrically irreducible.

If $\mathfrak{p}$ is maximal in $k[x, y, z]$, we have

$$\mathfrak{p}\bar{A}_{\mathfrak{p}} = (u^2 - a, uv - b, v - c).$$

Now, it's time to involve $\mathrm{char}(k)$, the characteristic of the fields k in.

1. If $\mathrm{char}(k) = 2$, then every number $a \in k$ has just one square root, since $\sqrt{a} = -\sqrt{a} \in k$, thus $u^2 - a$ is irreducible hence $\mathrm{Spec}(\bar{A}_{\mathfrak{p}}/\mathfrak{p}\bar{A}_{\mathfrak{p}}) = V(u^2 - a, uv - b, v - c) = V(u^2 - a) \cap V(uv - b) \cap V(v - c)$ is irreducible, viewed as closed subsets in $\mathrm{Spec}\bar{A}_{\mathfrak{p}}$. In this case, the fiber is geometrically irreducible.

2. If $\mathrm{char}(k) \neq 2$, then $u^2 - a$ is reducible since k is algebraically closed by assumption. Thus, we have

$$\mathrm{Spec}(\bar{A}_{\mathfrak{p}}/\mathfrak{p}\bar{A}_{\mathfrak{p}}) = V(u^2 - a, uv - b, v - c) = V(u + \sqrt{a}, uv - b, v - c) \cup V(u - \sqrt{a}, uv - b, v - c)$$

which says the fiber $\mathrm{Spec}(\bar{A}_{\mathfrak{p}}/\mathfrak{p}\bar{A}_{\mathfrak{p}})$ is not irreducible. Hence it is not geometrically irreducible.

To sum up, we conclude that if $\mathrm{char}(k) = 2$, then all fibers of the normalization $X^\nu \to X$ are irreducible and geometrically irreducible, while if $\mathrm{char}(k) \neq 0$, the fibers at maximal ideals are not irreducible, while at non-maximal ideals are irreducible and geometrically irreducible.

Exercise 18

(a) Suppose the morphism $f : Y \to X$ of schemes is injective and closed. Then for any affine open $U \subset X, U = \mathrm{Spec}A$, the set $f(Y) \cap U$ is closed in U, and thus is of the form $V(I) = \mathrm{Spec}(V/I)$ for some ideal $I \subset A$. So by adjusting $A' = A/I$, we may assume that $U = \mathrm{Spec}A$ lies in $f(Y)$.

To show $V = f^{-1}(U)$ is affine, we note that $f|_V$ is topologically a homeomorphism, since $f|_V$ is bijective and the inverse is continuous by the closedness of f. Thus the corresponding morphism of sheaves is an isomorphism, thus we have an isomorphism of schemes $(V, \mathcal{O}_Y|_V) \simeq (U, \mathcal{O}_X|_U)$. So we have $A = \mathcal{O}_X(U) \simeq f_{V,*}\mathcal{O}_Y(U) = \mathcal{O}_Y(V)$, which says that V is also an affine open and $V = \mathrm{Spec}A$ up to an isomorphism.

(b) If $f : Y \to X$ is injective and universally closed, then by **(a)**, it is affine and universally closed. The latter condition is equivalent to f being integral, which already has been showed in class.

Exercise 19

(a) $\Rightarrow$ X is separated iff $X \to \mathrm{Spec}\mathbb{Z}$ is separated, iff the diagonal $\Delta : X \to X \times_{\mathbb{Z}} X$ is a closed immersion. From **Exercise 18(a)** we know that $\Delta : X \to X \times_{\mathbb{Z}} X$ is affine. Choose an affine cover $\{U_i = \mathrm{Spec}A_i\}$ of X, we can see that $U_i \times_{\mathbb{Z}} U_j = \mathrm{Spec}(A_i \otimes_{\mathbb{Z}} A_j)$ is affine thus $\{U_i \times_{\mathbb{Z}} U_j\}$ is an affine cover of $X \times_{\mathbb{Z}} X$. Next note that $\Delta^{-1}(U_i \times_{\mathbb{Z}} U_j) = U_i \cap U_j$, so we have $U_i \cap U_j$ affine for all i, j.

$\Delta : X \to X \times_{\mathbb{Z}} X$ being closed immersion also implies that the morphism $\Delta^\sharp : \mathcal{O}_{X \times_{\mathbb{Z}} X} \to \Delta_*\mathcal{O}_X$ is an epimorphism of sheaves on $X \times_{\mathbb{Z}} X$, which is equivalent to it being surjective when restricted to every open subset V of $X \times_{\mathbb{Z}} X$. When we choose $V = U_i \times_{\mathbb{Z}} U_j$, we have

$$\mathcal{O}_{X \times_{\mathbb{Z}} X}(U_i \times_{\mathbb{Z}} U_j) = A_i \otimes_{\mathbb{Z}} A_j \simeq \mathcal{O}_X(U_i) \otimes_{\mathbb{Z}} \mathcal{O}_X(U_j)$$

and

$$(\Delta_*\mathcal{O}_X)(U_i \times_{\mathbb{Z}} U_j) = \mathcal{O}_X(\Delta^{-1}(U_i \times_{\mathbb{Z}} U_j)) = \mathcal{O}_X(U_i \cap U_j).$$

So the morphism

$$\mathcal{O}_X(U_i) \otimes \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j)$$

is obtained by just restricting $\Delta^\sharp : \mathcal{O}_{X \times_{\mathbb{Z}} X} \to \Delta_*\mathcal{O}_X$ on $V = U_i \times_{\mathbb{Z}} U_j$, thus is surjective.

$\Leftarrow$ We need to show that

$$\mathcal{O}_{X \times_{\mathbb{Z}} X} \to \Delta_*\mathcal{O}_X \tag{15}$$

is an epimorphism of sheaves on $X \times_{\mathbb{Z}} X$. We just need to check the morphism induces surjections on each stalk.

For any $x \in X$, we can find an affine open subset $U_i \times U_j$ in $X \times_{\mathbb{Z}} X$ containing $\Delta(x) = (x, x)$ and such that $U_i \cap U_j$ is affine, by assumption. Since

$$\mathcal{O}_{X \times_{\mathbb{Z}} X}(U_i \times U_j) = \mathcal{O}_X(U_i) \otimes_{\mathbb{Z}} \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j)$$

is surjective, by doing localization, we have

$$\mathcal{O}_{X,x} \otimes_{\mathbb{Z}} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$$

also surjective, because localization is an exact functor and commutes with tensor product, which shows that (15) is surjective at $(x, x) \in X \times_{\mathbb{Z}} X$.

(b) For any $a/(fg)^n \in R_{(fg)}$ with $\deg a = n(\deg f + \deg g)$, we need to show there exist $b \in R_{(f)}$ and $c \in R_{(g)}$ such that $b \otimes c = a/(fg)^n$. Indeed, we can choose

$$b = \frac{a g^{n(\deg f - 1)}}{f^{n(\deg g + 1)}}$$

and

$$c = \frac{f^{\deg g}}{g^{\deg f}}.$$

In this case, we have $\{D_+(f) \simeq \mathrm{Spec} R_{(f)}\}$ as an affine open cover of $X = \mathrm{Proj} R$, which apparently satisfy $D_+(f) \cap D_+(g) = D_+(fg)$. And in addition we have $\mathcal{O}_X(D_+(f)) = R_{(f)}$. Thus by **(a)** $X = \mathrm{Proj} R$ is separated.

Exercise 20

(a) Note that $I = \oplus_{d \geq 0}(\mathfrak{p} \cap R_d)$ is indeed a homogeneous ideal of R, so we just need to show it's prime. For any $x \notin I$ and $y \notin I$, there exist $d_0, e_0 \geq 0$ such that $x_{d_0} \notin I$ and $y_{e_0} \notin I$, thus $x_{d_0} \notin \mathfrak{p}, y_{e_0} \notin \mathfrak{p}$ by definition, which says $x_{d_0} y_{e_0} \notin \mathfrak{p}$ hence $x_{d_0} y_{e_0} \notin I$ hence $xy \notin I$. This shows I is a homogeneous prime ideal.

If $\mathfrak{q}$ is a minimal prime ideal, then we have $J = \oplus_{d \geq 0}(\mathfrak{q} \cap R_d) \subset \mathfrak{q}$. But we have just shown that J is prime, which contradicts the minimality of $\mathfrak{q}$, so we must have

$$J = \oplus_{d \geq 0}(\mathfrak{q} \cap R_d) = \mathfrak{q},$$

showing that $\mathfrak{q}$ is homogeneous.

(b) We denote M as the maximal point of $\mathrm{Proj} R$. We need to show that

$$T \cap \mathrm{Proj} R = M.$$

Immediately, we have $T \cap \mathrm{Proj} R \subset M$, since a maximal ideal in R which is homogeneous is a maximal homogeneous ideal. We show the inverse inclusion by *reductio ad absurdum*. For any maximal point $\mathfrak{p} \in M$, if it were not maximal in R, then there would be some maximal ideal $\mathfrak{m}$ such that $\mathfrak{p} \subset \mathfrak{m}$. Thus we have

$$\mathfrak{p} \cap R_d \subset \mathfrak{m} \cap R_d, \forall d \geq 0,$$

then

$$\mathfrak{p} = \oplus_{d \geq 0}(\mathfrak{p} \cap R_d) \subset \oplus_{d \geq 0}(\mathfrak{m} \cap R_d) =: K$$

since $\mathfrak{p}$ is homogeneous. But by **(a)**, K is prime and is homogeneous by construction thus $K \in M$, which contradicts the maximality of $\mathfrak{p} \in M$. This shows the desired identity.

(c) Recall that all $\mathfrak{p} \in X = \mathrm{Proj}R$, we have $\mathcal{O}_{X,\mathfrak{p}} = R_{(\mathfrak{p})}$, where $R_{(\mathfrak{p})}$ is the degree 0 piece of $S^{-1}R$, where S is the set of all homogeneous polynomials not in $\mathfrak{p}$.

Since R is integrally closed and localization preserves the integral dependence [AM, Proposition 5.3.], we have $S^{-1}R$ integrally closed. We just need to show its degree 0 piece is integrally closed. Otherwise, suppose there were an element $x \in S^{-1}R$ with non-zero degree n, and it were a solution of the equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0,$$

with $a_1, \ldots, a_n$ all degree 0 elements in $S^{-1}R$. But this is impossible, since every term of the above equation is of different degree, so we must have $\deg x = 0$, a contradiction. This shows that $R_{(\mathfrak{p})}$ is integrally closed so by definition $\mathrm{Proj}R$ is normal.

Exercise 21

(a) First note that $(\mathfrak{a} : \mathfrak{b})$ is indeed an ideal of A. Then for all $n \geq 0$, if $x \in (\mathfrak{a} : \mathfrak{b}^n)$, equivalently we have

$$x\mathfrak{b}^n \subset \mathfrak{a},$$

implying

$$x\mathfrak{b}^{n+1} = x\mathfrak{b}^n \cdot \mathfrak{b} \subset \mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a},$$

hence $x \in (\mathfrak{a} : \mathfrak{b}^{n+1})$. Thus we have shown $(\mathfrak{a} : \mathfrak{b}^n) \subset (\mathfrak{a} : \mathfrak{b}^{n+1})$, so we have an ascending chain of ideals

$$(\mathfrak{a} : \mathfrak{b}) \subset (\mathfrak{a} : \mathfrak{b}^2) \subset \cdots \subset (\mathfrak{a} : \mathfrak{b}^n) \subset \cdots . \tag{16}$$

But since A is Noetherian, so the chain (16) must stabilize at some large enough N.

Next we show that $I := (\mathfrak{a} :^\infty \mathfrak{b})$ is a $\mathfrak{b}$-saturated ideal containing $\mathfrak{a}$. Obviously, we have $\mathfrak{a} \subset I$, since $\forall a \in \mathfrak{a}$ we have $a \cdot \mathfrak{b}^n \in \mathfrak{a}$ for all n. We now show that I is $\mathfrak{b}$-saturated, *i.e.*, $(I : \mathfrak{b}) = I$. For any $x \in (I : b)$, $x \cdot \mathfrak{b} \subset I$, so we have $x \cdot b \in (\mathfrak{a} :^\infty \mathfrak{b}), \forall b \in \mathfrak{b}$, hence $x \cdot b \cdot \mathfrak{b}^N \subset \mathfrak{a}, \forall b \in \mathfrak{b}$ with N large enough stabilizing the chain (16), which says $x \cdot \mathfrak{b}^{N+1} \subset \mathfrak{a}$, so $x \in I$ and $(I : \mathfrak{b}) \subset I$. Conversely, if $x \in I = (\mathfrak{a} :^\infty \mathfrak{b})$, we have $x \cdot \mathfrak{b}^N \in \mathfrak{a}$ for N large enough, but this implies $x \cdot b \cdot \mathfrak{b}^{N-1} \subset \mathfrak{a}, \forall b \in \mathfrak{b}$, thus $x \cdot b \in I, \forall b \in \mathfrak{b}$, which shows $I \subset (I : \mathfrak{b})$.

Now we show that I is minimal. Otherwise, suppose there were a $\mathfrak{b}$-saturated ideal J such that $\mathfrak{a} \subset J \subsetneq I$, then there would be some $x \in I$ but $x \notin J$. Then there exists some N large enough such that $x \cdot b^N \subset \mathfrak{a}$, but $\mathfrak{a} \subset J$, so $x \cdot \mathfrak{b}^N \subset J$ implying $x \cdot \mathfrak{b}^{N-1} \subset (J : \mathfrak{b}) = J$, since J is $\mathfrak{b}$-saturated. By doing this procedure iteratively, after Nth steps we have $x \in J$, a contradiction. So we must have $J = I$, showing $I = (\mathfrak{a} :^\infty \mathfrak{b})$ minimal.

(b) We first compute $(\mathfrak{q} :^\infty \mathfrak{b})$.

1. If $\mathfrak{b} \not\subset \sqrt{\mathfrak{q}}$, we first know $\mathfrak{q} \subset (\mathfrak{q} :^\infty \mathfrak{b})$, we just need to show the inverse inclusion. If $x \in (\mathfrak{q} :^\infty \mathfrak{b})$, then there exists n such that $x \cdot \mathfrak{b}^n \subset \mathfrak{q}$. So choose any $b \in \mathfrak{b}$ we have $x \cdot b^n \in \mathfrak{q}$, but $\mathfrak{q}$ is primary, so $x \in \mathfrak{q}$ or $(b^n)^m \in \mathfrak{q}$ for some m. But by assumption $\mathfrak{b} \not\subset \sqrt{\mathfrak{q}}$, so $b^{nm} \notin \mathfrak{q}$ but $x \in \mathfrak{q}$. This shows $(\mathfrak{q} :^\infty \mathfrak{b}) \subset \mathfrak{q}$, implying $(\mathfrak{q} :^\infty \mathfrak{b}) = \mathfrak{q}$.

2. If $\mathfrak{b} \subset \sqrt{\mathfrak{q}}$, it's obvious that $(\mathfrak{q} :^\infty \mathfrak{b}) \subset A$. Now to show the reverse inclusion, we just need to show $1 \in (\mathfrak{q} :^\infty \mathfrak{b})$. Otherwise, suppose $1 \notin (\mathfrak{q} :^\infty \mathfrak{b})$, so for any $n > 0$, there exists $b \in \mathfrak{b}^n$ with $b \notin \mathfrak{q}$. Let $b = b_1 b_2 \cdots b_n$, since $\mathfrak{q}$ is primary, so $b_1 \notin \mathfrak{q}$ and $(b_2 \cdots b_n)^m \notin \mathfrak{q}$ for all $m \geq 0$. Moreover, inductively, this shows that any power of $b_1, b_2, \ldots, b_n$ are not in $\mathfrak{q}$, which contradicts the assumption that $\mathfrak{b} \subset \sqrt{\mathfrak{q}}$. So we have showed $1 \in (\mathfrak{q} :^\infty \mathfrak{b})$ thus $(\mathfrak{q} :^\infty \mathfrak{b}) = A$.

Since A is Noetherian, so every ideal $\mathfrak{a}$ has a primary decomposition. Take a minimal primary decomposition $\mathfrak{a} = \cap_{i=1}^n \mathfrak{q}_i$ of $\mathfrak{a}$, we have

$$(\sqrt{\mathfrak{a}} :^\infty \mathfrak{b}) = \cup_{j\geq 0}(\sqrt{\cap_{i=1}^n \mathfrak{q}_i} : \mathfrak{b}^j) = \cup_{j\geq 0}\{\cap_{i=1}^n(\sqrt{\mathfrak{q}_i} : \mathfrak{b}^j)\} = \cap_{i=1}^n(\sqrt{\mathfrak{q}_i} :^\infty \mathfrak{b}). \tag{17}$$

Since $\mathfrak{q}_i$ is primary, then $\sqrt{\mathfrak{q}_i}$ is prime, thus is also primary. According to our previous calculation, if $\mathfrak{b} \not\subset \sqrt{\sqrt{\mathfrak{q}_i}} = \sqrt{\mathfrak{q}_i}$, that is, $\sqrt{\mathfrak{q}_i} \notin V(\mathfrak{b})$, we have $(\sqrt{\mathfrak{q}_i} :^\infty \mathfrak{b}) = \sqrt{\mathfrak{q}_i}$; if $\sqrt{\mathfrak{q}_i} \in V(\mathfrak{b})$, we have $(\sqrt{\mathfrak{q}_i} :^\infty \mathfrak{b}) = A$. Since the decomposition in (17) is minimal, we have

$$(\sqrt{\mathfrak{a}} :^\infty \mathfrak{b}) = \cap_{\sqrt{\mathfrak{q}_i}\notin V(\mathfrak{b})}\sqrt{\mathfrak{q}_i} = \cap_{\mathfrak{p}\in V(\mathfrak{a})\setminus V(\mathfrak{b})}\mathfrak{p},$$

(c) First we have

$$V_+(I(Y)) = \mathrm{Proj}R \cap V(I(Y))$$

for $V(I(Y)) \subset \mathrm{Spec}R$. Since all closed subsets in $\mathrm{Proj}R$ are of the form

$$V_+(J) = \mathrm{Proj}R \cap V(J),$$

with $J \subset R$ some ideals of R.

Exercise 22

Suppose the ring $R = A[x_0, x_1]$, where (x_0, x_1) is of degree (a, b). Without loss of generality, we may assume that a, b are coprime. Otherwise, if there were a greatest common divisor d of a, b, then any homogeneous polynomial in x_0, x_1 is in R_{dk} for some k, so we have $R = R^{(d)}$. Recall that $\mathrm{Proj}R^{(d)} \simeq \mathrm{Proj}R$, thus we have

$$\mathbb{P}^1_A(a,b) = \mathrm{Proj}R \simeq \mathrm{Proj}R^{(\frac{1}{d})} = \mathbb{P}^1_A(\frac{a}{d}, \frac{b}{d}),$$

which can be reduced to the case of a, b coprime.

Now, since a, b are coprime, the only term can appear in $R^{(ab)}$ are x_0^b and x_1^a, so we have

$$\mathbb{P}^1_A(a,b) = \mathrm{Proj}R \simeq \mathrm{Proj}R^{(ab)} = \mathrm{Proj}(A[x_0^b, x_1^a]) = \mathbb{P}^1_A(ab, ab) \simeq \mathbb{P}^1_A(1,1) = \mathbb{P}^1_A,$$

which proves the statement. The key point is to realize that $\mathrm{Proj}R \simeq \mathrm{Proj}R^{(ab)}$.

Exercise 23

Exercise 24

We know that $\mathrm{Spec}(\mathcal{O}_{X,x})$ are in bijective with the set of irreducible closed subsets of X containing x. And the irreducible components correspond to the minimal points in $\mathrm{Spec}(\mathcal{O}_{X,x})$. We are asked to show that any two irreducible components has no intersection. Suppose it were not true, then there exist two irreducible components X_1, X_2 such that $x \in X_1 \cap X_2$, and $\mathrm{Spec}(\mathcal{O})_{X,x}$ should have two distinct minimal prime ideals, by the correspondence. But by assumption X is locally integral, so $\mathcal{O}_{X,x}$ is a domain and the only minimal ideal is (0), a contradiction. This shows that the irreducible components of X are mutually disjoint.

We know that a topological space X is a union of its irreducible components. We have showed that a locally integral scheme X has disjoint irreducible components $\{X_i\}$, moreover, if they are finitely many, we have

$$X = \coprod_{i=1}^n X_i. \tag{18}$$

We just need to show that each X_i is integral. Recall that a scheme Y is integral iff it is irreducible and reduced, iff it is irreducible and $\mathcal{O}_{Y,y}$ is reduced. In our case, X_i is clearly irreducible, and for any $x \in X_i$, $\mathcal{O}_{X,x}$ is a domain hence is reduced. Thus we have showed that a locally integral scheme is a finite disjoint union of integral schemes.

Exercise 25

(a)

Exercise 26

Exercise 27

Exercise 28

Exercise 29

(a) Let $s : S \to X$ be a section of $f : X \to S$. So we have $f \circ s = \mathrm{id}_S$. Consider the commutative diagram below:

$$\begin{tikzcd}
S \arrow[r, "s"] \arrow[d, "s"] \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[d, "\Delta_f"] & \\
X \arrow[r, "\Delta_f"] \arrow[d, "f"] \arrow[dr, phantom, very near start, "\ulcorner"] & X \times_S X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[d, "f"] \\
S \arrow[r, "s"] & X \arrow[r, "f"] & S
\end{tikzcd} .$$

Since $f : X \to S$ is separated, so $\Delta : X \to X \times_S X$ is a closed immersion. But $s : S \to X$ is the base change of $\Delta_f : X \to X \times_S X$, along $\Delta_f : X \to X \times_S X$, so s is also a closed immersion, since closed immersions are stable under base change.

(b) Since $s^*\mathcal{I} \simeq \mathcal{E}$ iff it holds stalk-wisely, so we may assume that $S = \mathrm{Spec}A$ is affine. In this case, we have $\mathcal{O}_S = \widetilde{A}$ and $\mathcal{E} = \widetilde{M}$ for some A-module M. Then

$$\mathbb{V}(\mathcal{E}) = \underline{\mathrm{Spec}} S^*_{\mathcal{O}_S}(\mathcal{E}) = \mathrm{Spec}(S^*_A(M)),$$

and

$$\mathcal{O}_{\mathbb{V}(\mathcal{E})} = \widetilde{S^*_A(M)}.$$

Since $f : \mathbb{V}(\mathcal{E}) \to S$ is affine hence separated, so by **(a)** $s : S \to \mathbb{V}(\mathcal{E})$ is a closed immersion. The quasi-coherent ideal $\mathcal{I}$ of $\mathcal{O}_{\mathbb{V}(\mathcal{E})}$ corresponding to the closed subscheme $S \subset \mathbb{V}(\mathcal{E})$ can be determined by the exact sequence:

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{V}(\mathcal{E})} \longrightarrow s_*\mathcal{O}_S \longrightarrow 0,$$

which in our case reduces to

$$0 \longrightarrow \mathcal{I} \longrightarrow \widetilde{S^*_A(M)} \longrightarrow s_*\widetilde{A} \longrightarrow 0.$$

When restricted to the principal open subset $D(m)$ with $m \in S^*_A(M)$, the above exact sequence becomes

$$0 \longrightarrow \mathcal{I}|_{D(m)} \longrightarrow (S^*_A(M))_m \longrightarrow A_{s(m)} \longrightarrow 0,$$

but $s : S^*_A(M) \to A$ is induced by the zero morphism $0 : M \to A$

Exercise 30

Exercise 31

Exercise 32

Suppose we have two distinguished triangles

$$T_1 : X_1 \xrightarrow{f_1} Y_1 \to Z_1 \to X_1[1]$$

and

$$T_2 : X_2 \xrightarrow{f_2} Y_2 \to Z_2 \to X_2[1]$$

in a triangulated category $\mathcal{D}$, we consider the morphism

$$X_1 \oplus X_2 \xrightarrow{f_1 \oplus f_2} Y_1 \oplus Y_2 \tag{19}$$

in $\mathcal{D}$. By axiom (TR1) of $\mathcal{D}$, we can extend (19) to a distinguished triangle

$$T : X_1 \oplus X_2 \xrightarrow{f_1 \oplus f_2} Y_1 \oplus Y_2 \to Z \to X_1[1] \oplus X_2[1],$$

and by axiom (TR3) there exists morphisms of distinguished triangles

$$\begin{tikzcd} X_1 \oplus X_2 \arrow[r, "f_1 \oplus f_2"] \arrow[d, "\mathrm{pr}_i"] & Y_1 \oplus Y_2 \arrow[r] \arrow[d, "\mathrm{pr}_i"] & Z \arrow[r] \arrow[d, dashed, "h_i"] & X_1[1] \oplus X_2[1] \arrow[d, "\mathrm{pr}_i"] \\ X_i \arrow[r, "f_i"] & Y_i \arrow[r] & Z_i \arrow[r] & X_i[1] \end{tikzcd} \tag{20}$$

for $i = 1, 2$. Then consider the diagram

$$\begin{tikzcd} X_1 \oplus X_2 \arrow[r, "f_1 \oplus f_2"] \arrow[d, "\mathrm{id}"] & Y_1 \oplus Y_2 \arrow[r] \arrow[d, "\mathrm{id}"] & Z \arrow[r] \arrow[d, "h_1 \oplus h_2"] & X_1[1] \oplus X_2[1] \arrow[d] \\ X_1 \oplus X_2 \arrow[r, "f_1 \oplus f_2"] & Y_1 \oplus Y_2 \arrow[r] & Z_1 \oplus Z_2 \arrow[r] & X_1[1] \oplus X_2[1], \end{tikzcd}$$

which is obtained by summing over the diagrams (20) and hence is commutative, which implies $h_1 \oplus h_2$ is an isomorphism.

Exercise 33

(a) Consider the distinguished triangles

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1] \tag{21}$$

and

$$Y[-1] \xrightarrow{\mathrm{id}_{Y[-1]}} Y[-1] \to 0 \to Y. \tag{22}$$

Since distinguished triangles are stable under clockwise rotations, we can apply the $2\pi/3$ clockwise rotation twice to (22) and get a distinguished triangle

$$0 \to Y \xrightarrow{\mathrm{id}_Y} Y \to 0. \tag{23}$$

We can sum (23) and (21) up, to have

$$X \xrightarrow{i} X \oplus Y \xrightarrow{p} Y \xrightarrow{0} X[1],$$

which is also a distinguished triangle, by **Exercise 32**.

(b) Given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{0} X[1],$$

we can construct another distinguished triangle

$$X \to X \oplus Z \to Z \xrightarrow{0} X[1]$$

by **(a)**. After a counter-clockwise $2\pi/3$ rotation and some shiftings, we get two distinguished triangles

$$Z[-1] \xrightarrow{0} X \xrightarrow{f} Y \xrightarrow{g} Z$$

and

$$Z[-1] \xrightarrow{0} X \to X \oplus Z \to Z.$$

Then there exists a $p : Y \to X \oplus Z$, such that the following diagram

$$\begin{tikzcd}
Z[-1] \arrow[r, "0"] \arrow[d, "\mathrm{id}_{Z[-1]}"] & X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"] & Y \arrow[r, "g"] \arrow[d, dashed, "p"] & Z \arrow[d, "\mathrm{id}_Z"] \\
Z[-1] \arrow[r, "0"] & X \arrow[r] & X \oplus Z \arrow[r] & Z
\end{tikzcd}$$

commutes. The existence of p is based on axiom (TR3) of triangulated category $\mathcal{D}$. Moreover, p is an isomorphism, since $\mathrm{id}_{Z[-1]}, \mathrm{id}_X$ and id_Z are all isomorphisms. Thus we have constructed an isomorphism between the distinguished triangles, of which we deserve.

Exercise 34

Since $\mathcal{A}$ is an abelian category, so the derived category $D(\mathcal{A})$ is a triangulated category. Since we have

$$H^i(\tau^{\leq n} L) = \begin{cases} H^i(L), i \leq n, \\ 0, i > n, \end{cases}$$

and

$$H^i(\tau^{n \geq n+1} L) = \begin{cases} H^i(L), i \geq n+1, \\ 0, i < n+1, \end{cases}$$

the homology long exact sequence

$$\cdots \to H^i(\tau^{\leq n} L) \to H^i(L) \to H^i(\tau^{\geq n+1} L) \to H^{i+1}(\tau^{\leq n} L) \to \cdots$$

induced by the distinguished triangle

$$\tau^{\leq n} L \to L \to \tau^{\geq n+1} L \to \tau^{\leq n} L[1]$$

can be written as

$$\cdots \to H^i(L) \to H^i(L) \to 0 \to H^{i+1}(\tau^{\leq n} L) \to \cdots$$

when $i \leq n$, and

$$\cdots \to 0 \to H^i(L) \to H^i(L) \to 0 \to \cdots$$

when $i \geq n+1$. In the former case, we have

$$H^i h : 0 \to H^{i+1}(\tau^{\leq n} L),$$

and in the latter case, we have

$$H^i h : H^i(L) \to 0.$$

This shows that $H^i h = 0$ forall $i \in \mathbb{Z}$.

Exercise 35

(a) We first show that for any $X \in \mathcal{A}$, there is an F-acyclic object I with a monomorphism $X \to I$. Indeed, if I is F-injective, we have $R^iFI = 0$ for $i > 0$, since we can find a quasi-isomorphism

$$I \xrightarrow{\sim} J^\bullet$$

with

$$J^\bullet = I \to 0 \to 0 \to \cdots,$$

then by definition $R^iFI = H^iF(J^\bullet)$, which equals $F(I)$ if $i = 0$ otherwise equals 0. Thus we can choose an F-injective object as our desired F-acyclic object.

Now suppose we have an exact sequence

$$0 \to A \to B \to C \to 0,$$

in $\mathcal{A}$ with A, B F-acyclic. We want to show that C is also F-acyclic and the sequence

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is also exact in $\mathcal{B}$. Using the long exact sequence of derived functor

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to \cdots,$$

since $R^iF(A) = R^iF(B) = 0$ for $i \geq 1$, we have $R^iF(C)$=0 and the exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to 0,$$

which completes the proof.

(b) For all $X \in \mathcal{A}$, we prove $\mathbb{R}^{N+k}F(X) = 0$ by induction on $k \in \mathbb{N}$.

As $k = 0$, $R^NF(X) = 0$ holds for all $X \in \mathcal{A}$ by assumption.

As for $k = m$, we suppose $R^{N+m}F(X) = 0$. If we can show that $R^{N+m+1}F(X) = 0$, we then finish the proof. Indeed, since $R^{N+m}F(X) = 0$, we can find an F-injective resolution

$$I^\bullet : X \to I^1 \xrightarrow{d^1} I^2 \to \cdots \to I^{N+m-1} \to I^{N+m} \to I^{N+m+1} \to \cdots$$

of X, use which we can compute

$$R^iF(X) = H^iF(I^\bullet). \tag{24}$$

On the other hand,

$$(\tau^{\geq m+1}I^\bullet)[m+1] : \operatorname{im} d^{m+1} \to I^{m+2} \to \cdots \to I^{N-1} \to I^N \to I^{N+1} \to \cdots$$

is an F-injective resolution of $\operatorname{im} d^{m+1}$, use which we can compute

$$R^iF(\operatorname{im} d^{m+1}) = H^iF((\tau^{\geq m+1}I^\bullet)[m+1]) = H^{i+m+1}F(\tau^{\geq m+1}I^\bullet) = H^{i+m+1}F(I^\bullet), i \geq 0.$$

Especially, we can take $i = N$, then

$$0 = R^NF(\operatorname{im} d^{m+1}) = H^{N+m+1}F(I^\bullet) = R^{N+m+1}F(I^\bullet)$$

where the first equality comes form the assumption on N and the last equality holds by (24). This completes the proof.

(c) From the exact sequence

$$X_{N-1} \xrightarrow{d^{N-1}} X_{N-2} \to \cdots \xrightarrow{d^2} X_1 \xrightarrow{d^1} Y \to 0,$$

we have another exact sequence

$$0 \to K \to X_{N-1} \xrightarrow{d^{N-1}} X_{N-2} \to \cdots \xrightarrow{d^2} X_1 \xrightarrow{d^1} Y \to 0, \tag{25}$$

where $K := \ker d^{N-1}$ is the kernel of d^{N-1}.

We can break the long exact sequence (25) into a family of short exact sequences

$$\begin{aligned} 0 \to K \to & X_{N-1} \to \operatorname{im} d^{N-1} \to 0, \\ 0 \to \operatorname{im} d^{N-1} \to & X_{N-2} \to \operatorname{im} d^{N-2} \to 0, \\ & \vdots \\ 0 \to \operatorname{im} d^2 \to & X_1 \to Y \to 0. \end{aligned}$$

By the assumption on the derived functors of $X_1, \dots, X_{N-1}$, for any $k > 0$, we have the following pieces from the induced long exact sequences

$$\begin{aligned} 0 = & R^k F(X_1) \to R^k F(Y) \to R^{k+1} F(\operatorname{im} d^2) \to R^{k+1} F(X_1) = 0, \\ 0 = & R^{k+1} F(X_2) \to R^{k+1} F(\operatorname{im} d^2) \to R^{k+2} F(\operatorname{im} d^3) \to R^{k+2} F(X_3) = 0, \\ & \vdots \\ 0 = & R^{k+N-2} F(X_{N+1}) \to R^{k+N-2} F(\operatorname{im} d^{N-1}) \to R^{k+N-1} F(K) \to R^{k+N-1} F(X_{N+1}) = 0, \end{aligned}$$

which implies that

$$R^k F(Y) \simeq R^{k+1} F(\operatorname{im} d^2) \simeq R^{k+2} F(\operatorname{im} d^3) \simeq \cdots \simeq R^{k+N-2} F(\operatorname{im} d^{N-1}) \simeq R^{k+N-1} F(K).$$

But $k > 0 \implies k + N - 1 \geq N$, we have

$$R^k F(Y) \simeq R^{k+N-1} F(K) = 0$$

by **(b)**, which shows that Y is F-acyclic.

(d) Suppose the complex

$$L^\bullet = \cdots \to L^{n-1} \to L^n \xrightarrow{d^n} L^{n+1} \to \cdots$$

is acyclic, what we need to show is that the complex

$$F(L^\bullet) = \cdots \to F(L^{n-1}) \to F(L^n) \to F(L^{n+1}) \to \cdots$$

is acyclic, which is equivalent to show that

$$H^m F(L^\bullet) = 0$$

for all $m \in \mathbb{Z}$.

Indeed, by **(b)**, we have a fixed $N \in \mathbb{Z}_+$ such that $R^n F(X) = 0$ for all $n \geq N$. In particular, we have $R^n F(Z^{m-(N-1)}) = 0$ for all $n \geq N$, where $Z^{m-(N-1)}$ is the kernel of the differential $d^{m-(N-1)} : L^{m-(N-1)} \to L^{m-N+2}$. On the other hand, we have a natural resolution of $Z^{m-(N-1)}$

$$I^\bullet : 0 \to Z^{m-(N-1)} \to L^{m-N+1} \to L^{m-N+2} \to \cdots \to L^{m-1} \to L^m \to L^{m+1} \to \cdots,$$

which can be used to compute the $R^\bullet F(Z^{m-(N-1)})$. We denote the above resolution as $I^\bullet$. Especially, we have

$$R^N F(Z^{m-(N-1)}) = H^N F(I^\bullet) = H^m F(L^\bullet) = 0,$$

where the last equality shows that $F(L^\bullet)$ is acyclic, since we have chosen arbitrary $m \in \mathbb{Z}$.

Exercise 36

(a) Let $x \in X$ be a closed point and $U = \operatorname{Spec} A$ an affine open neighborhood of X. Thus $Z := X \backslash U$ and $Z' := Z \cup \{x\}$ are closed subsets of X, and can be uniquely endowed with the induced reduced closed subscheme structures, which we also denote as Z and Z'. Let $\mathcal{I}_{Z'}$ and $\mathcal{I}_Z$ be the corresponding ideal sheaves of Z' and Z. Hence the short exact sequence

$$0 \longrightarrow \mathcal{I}_{Z'} \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{I}_{Z'}/\mathcal{I}_Z \longrightarrow 0$$

induces the cohomology long exact sequences

$$0 \to \Gamma(X, \mathcal{I}_{Z'}) \to \Gamma(X, \mathcal{I}_Z) \to \Gamma(X, \mathcal{I}_{Z'}/\mathcal{I}_Z) \to H^1(X; \mathcal{I}_{Z'}) \to \cdots.$$

Since $\mathcal{I}_{Z'}$ is quasi-coherent, by assumption $H^1(X; \mathcal{I}_{Z'}) = 0$ and $\Gamma(X, \mathcal{I}_Z) \to \Gamma(X, \mathcal{I}_{Z'}/\mathcal{I}_Z)$ is surjective. But after a little inspection we find that the sheaf $\mathcal{I}_{Z'}/\mathcal{I}_Z$ is just the skyscraper sheaf with stalks $\kappa(x)$ at x and 0 at other points. Choose a lifting $f \in \Gamma(X, \mathcal{I}_Z)$ of $1 \in \kappa(x)$. Since f is not invertible on Z, we have $X_f \subset U$. Moreover, we note that $X_f = U_{\bar{f}}$, where $\bar{f} = f|_U$, which shows that $X_f = \operatorname{Spec} A_{\bar{f}}$ is affine.

(b) In the proof of **Exercise 9**, we have showed that every quasi-compact scheme X has a closed point. Thus, we let $Y = \cup_{x\text{closed}} X_{f_x}$, where x is a closed point in X and X_{f_x} is the affine open subset constructed as in **(a)** containing x. If $Y \neq X$, then consider $X \backslash Y$, which is closed and then has at least one closed point, (otherwise, if all point are open, it is not closed) a contradiction. Thus we have $Y = X$ and $X = \cup_{x\text{closed}}$. By the quasi-compactness of X, there is a subcover $\{X_{f_i}\}_{i=1}^n$ of $\{X_{f_x}\}_{x\text{closed}}$ such that $X = \cup_{i=1}^n X_{f_i}$.

(c) By **Exercise 10(c)**, to show that X is affine, we just have to show that $f_1, f_2, \cdots, f_n$ generate $\Gamma(X, \mathcal{O}_X)$. Consider the morphism $\alpha : \mathcal{O}_X^n \to \mathcal{O}_X$ of sheaves defined as

$$\begin{aligned} \alpha : \mathcal{O}_X^n(U) &\to \mathcal{O}_X(U), \\ (a_1, a_2, \cdots, a_n) &\mapsto f_1 a_1 + f_2 a_2 + \cdots f_n a_n \end{aligned}$$

for all open subset $U \subset X$. Since $X_{f_1}, X_{f_2}, \ldots, X_{f_n}$ covers X, α is surjective stalk-wisely, thus is an epimorphism of sheaves. Then we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^n \xrightarrow{\alpha} \mathcal{O}_X \longrightarrow 0$$

of sheaves, where $\mathcal{F} := \ker \alpha$. Moreover, $\mathcal{F}$ is quasi-coherent. The first few terms of cohomology long exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{O}_X^n) \to \Gamma(X, \mathcal{O}_X) \to H^1(X; \mathcal{F}) \to \cdots$$

induced by the above short exact sequence, can be written as

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{O}_X^n) \to \Gamma(X, \mathcal{O}_X) \to 0,$$

by the assumption on the quasi-coherent sheaf $\mathcal{F}$. This shows that the maps between global sections

$$\Gamma(X, \mathcal{O}_X^n) \to \Gamma(X, \mathcal{O}_X)$$

induced by α is surjective, which completes the proof.

Exercise 37

Since F is a triangulated functor, so it commutes with the shiftings of degrees and maps distinguished triangles to distinguished triangles. For $X \in D^{\geq 0}(\mathcal{A})$, we consider the distinguished triangle

$$H^0X \to X \to \tau^{\geq 1}X \to (H^0X)[1]$$

in $D^{\geq 0}(\mathcal{A})$. So

$$FH^0X \to FX \to F(\tau^{\geq 1}X) \to (FH^0X)[1]$$

is a distinguished triangle in $D^{\geq 0}(\mathcal{B})$, which induces the long exact sequence

$$0 \to H^0FH^0X \to H^0FX \to H^0F(\tau^{\geq 1}X) \to H^1FH^0X \to H^1FX \to \cdots.$$

After a closer look, we find that

$$H^iF(\tau^{\geq 1}X) = \begin{cases} 0, & i = 0, \\ H^1FH^1X, & i = 1 \\ H^iFX, & i \geq 2, \end{cases}$$

since F is additive and $(\tau^{\geq 1}X)$ is isomorphic to $H^{\geq 1}X$ in $D^{\geq 0}(\mathcal{A})$. Thus from the long exact sequence we have

$$H^0FH^0X \simeq H^0FX$$

and the exact sequence

$$0 \to H^1FH^0X \to H^1FX \to H^1FH^1X \to H^2FH^0X \to H^2FX \to \cdots.$$

Exercise 38

(a) Let $\phi : \mathcal{F} \to \mathcal{F}'$ be a morphism of $\mathcal{G}$-torsors. To show that $\phi : \mathcal{F} \to \mathcal{F}'$ is an isomorphism, we need to show that

$$\phi_x : \mathcal{F}_x \to \mathcal{F}'_x$$

is a bijection at every $x \in X$.

Indeed, ϕ_x is surjective. For any $s_x \in \mathcal{F}_x$ and any $t_x \in \mathcal{F}'_x$, we can find a $g_x \in \mathcal{G}_x$ such that $g_x\phi_x(s_x) = t_x$ since $\mathcal{F}'$ is a $\mathcal{G}$-torsor. On the other hand, we have $t_x = g_x\phi_x(s_x) = \phi_x(g_xs_x)$ since ϕ is $\mathcal{G}$-equivariant, which shows that ϕ_x is surjective.

Now we are to show that ϕ_x is injective. Suppose there are $u_x, v_x \in \mathcal{F}_x$ such that $\phi_x(u_x) = \phi_x(v_x)$, we need to show that $u_x = v_x$. Indeed, there is some $g_x \in \mathcal{G}_x$ such that $u_x = g_xv_x$, then $\phi_x(u_x) = \phi_x(g_xv_x) = g_x\phi_x(v_x) = \phi_x(v_x)$, which implies that $g_x = 1_x$. Hence $u_x = g_xv_x = 1_xv_x = v_x$, this shows ϕ_x is an injection.

(b) Let $\mathcal{F}$ be a $\mathcal{G}$-torsor. Consider the sheaf $\mathbb{Z}[\mathcal{F}]$, which is the associated sheaf to the presheaf corresponding to the assignment

$$U \mapsto \sum_i n_is_i,$$

where $s_i \in \mathcal{F}(U)$ are the local sections. Note that there is a morphism $\sigma : \mathbb{Z}[\mathcal{F}] \to \underline{\mathbb{Z}}$ of sheaves, which can be written out explicitly when restricted to every open subset $U \subset X$:

$$\begin{aligned} \sigma :&(\mathbb{Z}[\mathcal{F}])(U) \to \underline{\mathbb{Z}}(U), \\ &\sum_i n_is_i \mapsto \sum_i n_i, \end{aligned}$$

where $\underline{\mathbb{Z}}$ is the constant abelian sheaf taking values in $\mathbb{Z}$ on X. Now consider the short exact sequences of sheaves

$$0 \longrightarrow \ker\sigma \longrightarrow \mathbb{Z}[\mathcal{F}] \xrightarrow{\sigma} \underline{\mathbb{Z}} \longrightarrow 0,$$

since the local sections of $\ker\sigma$ are of the form $s' - s$, where s', s are local sections of the sheaf $\mathcal{F}$, we can define an epimorphism of sheaves

$$\alpha : \ker\sigma \to \mathcal{G},$$

mapping $s' - s$ to the unique local section g of $\mathcal{G}$ such that $s' = gs$. Take $\mathcal{E}$ to be the pushforward of $\alpha : \ker\sigma \to \mathcal{G}$ along $\ker\sigma \to \mathbb{Z}[\mathcal{F}]$, by some basic homological algebra, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker\sigma \arrow[r] \arrow[d, "\alpha"] & \mathbb{Z}[\mathcal{F}] \arrow[r, "\sigma"] \arrow[d] & \underline{\mathbb{Z}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{E} \arrow[r] & \underline{\mathbb{Z}} \arrow[r] & 0
\end{tikzcd} \tag{26}$$

with rows exact. By the long exact sequence of the rows, we obtain a morphism

$$H^0(X; \mathbb{Z}[\mathcal{F}]) \xrightarrow{\tilde{\sigma}} H^0(X; \underline{\mathbb{Z}}) \xrightarrow{\delta} H^1(X; \mathcal{G})$$

where $\tilde{\sigma}$ is the map between 0th-cohomology groups induced by σ, and δ is the connecting morphism of the long exact sequence induced by the bottom row in (26).

For $[\mathcal{F}] \in \mathrm{Tors}(\mathcal{G})$, we can choose a global section $s \in H^0(X; \mathcal{F}) = \Gamma(X, \mathcal{F})$ satisfying $\tilde{\sigma}(s) = 1 \in H^0(X; \underline{\mathbb{Z}}) = \Gamma(X, \mathbb{Z})$, and define the map

$$\Phi : \mathrm{Tors}(\mathcal{G}) \to H^1(X; \mathcal{G})$$

as

$$[\mathcal{F}] \mapsto (\delta \circ \tilde{\sigma})(s).$$

The above map is well-defined, since if we choose another representative $\mathcal{F}'$ of $[\mathcal{F}]$, we have $s' = gs$ for some $g \in \Gamma(X, \mathcal{G})$. But

$$\tilde{\sigma}(s') = \tilde{\sigma}(s) + \tilde{\sigma}(s' - s) = \tilde{\sigma}(s) = 1,$$

showing that Φ is independent of the choice of $\mathcal{F}$.

Conversely, given $\xi \in H^1(X; \mathcal{G})$, we can define the inverse of Φ as follows. Choose an injective abelian sheaf $\mathcal{I}$ and a monomorphism $\mathcal{G} \to \mathcal{I}$, and denote $\mathcal{Q} = \mathcal{I}/\mathcal{G}$. Then we have the short exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Inspecting the induced long exact sequence, we have a surjection

$$H^0(X; \mathcal{Q}) \to H^1(X; \mathcal{G}),$$

since the cohomology group $H^1(X; \mathcal{I}) = 0$ for the injective sheaf $\mathcal{I}$. Thus we can find a lift $q \in H^0(X; \mathcal{Q})$ of ξ, and let $\mathcal{F} \subset \mathcal{I}$ be the sub sheaf of sections that maps to q. $H^1(\mathcal{U}; \mathcal{G}) \to H^1(X; \mathcal{G})$.

(c) As the hint suggests, for a rank n locally free $\mathcal{O}_X$-module $\mathcal{F}$, we can consider the assignment

$$U \mapsto \mathrm{Isom}_{\mathcal{O}_X^n(U)}(\mathcal{F}(U), \mathcal{O}_X^n(U)), \tag{27}$$

where

$$\mathrm{Isom}_{\mathcal{O}_X^n(U)}(\mathcal{F}(U), \mathcal{O}_X^n(U))$$

denotes the set of $\mathcal{O}_X^n(U)$-module isomorphisms between $\mathcal{O}_X^n(U)$-modules $\mathcal{F}(U)$ and $\mathcal{O}_X^n(U)$. Then we consider the sheafification of the presheaf (27), which is denoted by $\underline{\mathrm{Isom}}_{\mathcal{O}_X^n}(\mathcal{F}, \mathcal{O}_X^n)$. It's easy to see that $\underline{\mathrm{Isom}}_{\mathcal{O}_X^n}(\mathcal{F}, \mathcal{O}_X^n)$ is a $\mathrm{GL}(\mathcal{O}_X^n)$-torsor. Indeed, since $\mathcal{F}$ is locally isomorphic to $\mathcal{O}_X^n$, the stalks of $\underline{\mathrm{Isom}}_{\mathcal{O}_X^n}(\mathcal{F}, \mathcal{O}_X^n)$ are not vanishing. For any $U \subset X$, and any

$$s, t \in \mathrm{Isom}_{\mathcal{O}_X^n(U)}(\mathcal{F}(U), \mathcal{O}_X^n(U)),$$

there exists some $g \in \mathrm{GL}(\mathcal{O}_X^n(U))$ such that $t = gs$. Conversely, if $s \in \mathrm{Isom}_{\mathcal{O}_X^n(U)}(\mathcal{F}(U), \mathcal{O}_X^n(U))$ and $g \in \mathrm{GL}(\mathcal{O}_X^n(U))$, we have $gs \in \mathrm{Isom}_{\mathcal{O}_X^n(U)}(\mathcal{F}(U), \mathcal{O}_X^n(U))$. It is also easy to verify that $\mathcal{F} \mapsto \underline{\mathrm{Isom}}_{\mathcal{O}_X^n}(\mathcal{F}, \mathcal{O}_X^n)$ maps isomorphisms of $\mathcal{O}_X$-modules to isomorphisms to $\mathrm{GL}_n(\mathcal{O}_X)$-torsors. Thus we have defined a map

$$\begin{aligned} \Phi : \mathrm{Loc}_n(\mathcal{O}_X) &\to \mathrm{Tors}(\mathrm{GL}_n(\mathcal{O}_X)), \\ \mathcal{F} &\mapsto \underline{\mathrm{Isom}}_{\mathcal{O}_X^n}(\mathcal{F}, \mathcal{O}_X^n). \end{aligned}$$

What remains is to show that Φ is a bijection. Indeed, for a $\mathrm{GL}_n(\mathcal{O}_X)$-torsor $\mathcal{G}$, we can construct a rank n locally free $\mathcal{O}_X$-module, which is the sheafification of the presheaf

$$U \mapsto \mathcal{O}_X^n(U) \times \mathcal{G}(U)/\mathrm{GL}_n(\mathcal{O}_X)(U),$$

which is clearly a rank n locally free sheaf. Denote

$$\begin{aligned} \Psi : \mathrm{Tors}(\mathrm{GL}_n(\mathcal{O}_X)) &\to \mathrm{Loc}_n(\mathcal{O}_X), \\ \mathcal{G} &\mapsto \mathcal{O}_X^n \times \mathcal{G}/\mathrm{GL}_n(\mathcal{O}_X). \end{aligned}$$

It is easy to verify tht $\Psi \circ \Phi = \mathrm{id}$, which completes the proof.

(d) By definition, we have $\mathrm{Pic}(X, \mathcal{O}_X)$ as the set of isomorphism classes of rank 1 locally free $\mathcal{O}_X$-modules. By **(c)**, we have

$$\mathrm{Pic}(X, \mathcal{O}_X) := \mathrm{Loc}_1(\mathcal{O}_X) \simeq \mathrm{Tors}(\mathrm{GL}_1(\mathcal{O}_X)).$$

On the other hand, we have $\mathrm{GL}_1(\mathcal{O}_X) \simeq \mathcal{O}_X^*$, and by **(b)**, we have

$$\mathrm{Tors}(\mathrm{GL}_1(\mathcal{O}_X)) \simeq \mathrm{Tors}(\mathcal{O}_X^*) \simeq H^1(X; \mathcal{O}_X^*).$$

Thus we have shown that

$$\mathrm{Pic}(X, \mathcal{O}_X) \simeq H^1(X; \mathcal{O}_X^*).$$

Exercise 39

(a) For any $\mathcal{F} \in \mathrm{Shv}(X)$, we first show that there exists a sheaf $\mathcal{I}$ and a monomorphism of sheaves

$$\mathcal{F} \to \mathcal{I},$$

such that $\check{H}^p(\mathfrak{U}; \mathcal{I}) = 0$ holds for all $p > 0$ and all quasi-compact finite covers $\mathcal{U}$. Indeed, we can take

$$\mathcal{I} = \prod_{x \in X} i_{x,*} i_x^{-1} \mathcal{F},$$

with $i_x : x \hookrightarrow X$. We have already known that $\mathcal{I}$ is flabby, thus $\check{H}^p(\mathfrak{U};\mathcal{I}) = 0$ holds for all $\mathfrak{U}$ and $p > 0$, in particular for finite quasi-compact $\mathfrak{U}$'s.

Next suppose we have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0 \tag{28}$$

such that

$$\check{H}^p(\mathfrak{U};\mathcal{F}_i) = 0, i = 1,2 \tag{29}$$

for all quasi-compact open $\mathfrak{U}$'s and $p > 0$, we want to show that

$$\check{H}^p(\mathfrak{U};\mathcal{F}_3) = 0$$

for all quasi-compact open $\mathfrak{U}$'s and $p > 0$ and

$$0 \longrightarrow \Gamma(X,\mathcal{F}_1) \longrightarrow \Gamma(X,\mathcal{F}_2) \longrightarrow \Gamma(X,\mathcal{F}_3) \longrightarrow 0.$$

Indeed, by Inspecting the long exact sequence of Čech cohomology induced by (29)

$$\cdots \to \check{H}^p(\mathfrak{U};\mathcal{F}_2) \to \check{H}^p(\mathfrak{U};\mathcal{F}_3) \to \check{H}^{p+1}(\mathfrak{U};\mathcal{F}_1) \to \cdots,$$

we have $H^p(\mathfrak{U};\mathcal{F}_3) = 0$ for all $p > 0$, by the assumption (29). Finally we take the global sections and have the exact sequence

$$0 \longrightarrow \Gamma(X,\mathcal{F}_1) \longrightarrow \Gamma(X,\mathcal{F}_2) \longrightarrow \Gamma(X,\mathcal{F}_3) \longrightarrow H^1(X;\mathcal{F}_1),$$

but

$$H^1(X;\mathcal{F}_1) = \operatorname*{colim}_{\mathfrak{U}\in\mathrm{Cov}} \check{H}^1(\mathfrak{U};\mathcal{F}_1) = 0,$$

which completes the proof.

(b) If we can show that

(c) By the hint, for any $\mathcal{F}_i$, we can find a flabby sheaf $\mathcal{G}_i$ and a monomorphism $\mathcal{F}_i \to \mathcal{G}_i$. Thus we have the short exact sequence

$$0 \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{G}_i \longrightarrow \mathcal{Q}_i \longrightarrow 0\ ,$$

where the sheaf $\mathcal{Q}_i$ is the cokernel of the monomorphism $\mathcal{F}_i \to \mathcal{G}_i$. Since at the presheaf level, taking sections is an exact functor, and the filtered colimits of abelian groups are exact, so we have a short exact sequence of presheaves

$$0 \longrightarrow \mathrm{colim}_i\, \mathcal{F}_i \longrightarrow \mathrm{colim}_i\, \mathcal{G}_i \longrightarrow \mathrm{colim}_i\, \mathcal{Q}_i \longrightarrow 0\ .$$

Moreover, since the sheafification is an exact functor, the above exact sequence is a short exact of sheaves. Then we consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{colim}_i\, H^p(X;\mathcal{G}_i) \arrow[r] \arrow[d] & \mathrm{colim}_i\, H^p(X;\mathcal{Q}_i) \arrow[r] \arrow[d] & \mathrm{colim}_i\, H^{p+1}(X;\mathcal{F}_i) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^p(X;\mathrm{colim}_i\, \mathcal{G}_i) \arrow[r] & H^p(X;\mathrm{colim}_i\, \mathcal{Q}_i) \arrow[r] & H^{p+1}(X;\mathrm{colim}_i\, \mathcal{F}_i) \arrow[r] & \cdots
\end{tikzcd}.$$

By **(b)**

Exercise 40

(a) By hint, we consider the following filtration of the quasi-coherent $\mathcal{O}_X$-module $\mathcal{F} = 0$

$$0 = \mathcal{I}^n\mathcal{F} \subset \mathcal{I}^{n-1} \subset \cdots \subset \mathcal{I}\mathcal{F} \subset \mathcal{I}^0\mathcal{F} = \mathcal{F}.$$

Let $\mathcal{G}_i = \mathcal{I}^i\mathcal{F}/\mathcal{I}^{i+1}\mathcal{F}, 0 \leq i \leq n$. Since $\mathcal{I}^i\mathcal{F}$ and $\mathcal{I}^{i+1}\mathcal{F}$ are quasi-coherent $\mathcal{O}_X$-modules, $\mathcal{G}_i$ is a natural quasi-coherent $\mathcal{O}_{X_{\text{red}}} = \mathcal{O}_X/\mathcal{I}$-module. Note that the underlying topology spaces X_{red} and X are homeomorphic, so we have $H^m(X;\mathcal{G}_i) = H^m(X_{\text{red}};\mathcal{G}_i)$ for all $0 \leq i \leq n$ and all $m > 0$. However, by assumption on X_{red}, we can use Serre's Theorem on X_{red} and quasi-coherent $\mathcal{O}_{X_{\text{red}}}$-modules $\mathcal{G}_i$, which implies

$$H^m(X;\mathcal{G}_i) = H^m(X_{\text{red}};\mathcal{G}_i) = 0.$$

With this, we consider the short exact sequences

$$0 \longrightarrow \mathcal{I}^{i+1}\mathcal{F} \longrightarrow \mathcal{I}^i\mathcal{F} \longrightarrow \mathcal{G}_i \longrightarrow 0$$

of sheaves on X for all $0 \leq i \leq n$, and the induced cohomology long exact sequences

$$0 = H^{m-1}(X;\mathcal{G}_i) \to H^m(X;\mathcal{I}^{i+1}\mathcal{F}) \to H^m(X;\mathcal{I}^i\mathcal{F}) \to H^m(X;\mathcal{G}_i) = 0,$$

from which we obtain the isomorphisms

$$0 = H^m(X;\mathcal{I}^n\mathcal{F}) \simeq H^m(X;\mathcal{I}^{n-1}\mathcal{F}) \simeq \cdots \simeq H^m(X;\mathcal{F})$$

for all $m > 0$. Again by Serre's Theorem, we conclude that X is affine too.

(b) Let $\mathcal{I}$ be a nilpotent sheaf on X. Since X is Noetherian, there exists large enough N such that $\mathcal{I}^N = 0$. Thus the hypothesis of **(a)** is satisfied, so the assertion follows.

(c) Let $Z_1, Z_2, \ldots, Z_n$ be closed affine subschemes that cover X. Pick Z_1 and consider the ideal sheaf $\mathcal{I}_{Z_1}$ corresponding to it. Since $Z_1 \cap Z_2$ is also a closed subscheme of X, denote the ideal sheaf corresponding to it as $\mathcal{I}_{Z_1\cap Z_2}$, we have the following exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_{Z_1\cup Z_2} \longrightarrow \mathcal{I}_{Z_1} \longrightarrow i_*\mathcal{I}_{Z_1\cap Z_2} \longrightarrow 0,$$

where $i : Z_1 \to X$ is the closed immersion. Inspecting the cohomology long exact sequence

$$\cdots \to H^1(X;\mathcal{I}_{Z_1\cup Z_2}) \to H^1(X;\mathcal{I}_{Z_1}) \to H^1(X;i_*\mathcal{I}_{Z_1\cap Z_2}) \to \cdots$$

associated to the above short exact sequence, we note that

$$H^1(X;i_*\mathcal{I}_{Z_1\cap Z_2}) = R^1\Gamma(X, i_*\mathcal{I}_{Z_1\cap Z_2}) = R^1\Gamma(Z_1, \mathcal{I}_{Z_1\cap Z_2}) = H^1(Z_1;\mathcal{I}_{Z_1\cap Z_2}) = 0,$$

where the last equality comes from applying Serre's Theorem to the assumption on the affiness of Z_1. Then we have a surjection

$$H^1(X;\mathcal{I}_{Z_1\cup Z_2}) \twoheadrightarrow H^1(X;\mathcal{I}_{Z_1}).$$

Next we consider the closed immersion $i' : Z_3 \to X$ and the short exact sequence

$$0 \longrightarrow \mathcal{I}_{Z_1\cup Z_2\cup Z_3} \longrightarrow \mathcal{I}_{Z_1\cup Z_2} \longrightarrow i'_*\mathcal{I}_{(Z_1\cup Z_2)\cap Z_3} \longrightarrow 0\ ,$$

and finally get a surjection

$$H^1(X;\mathcal{I}_{Z_1\cup Z_2\cup Z_3}) \twoheadrightarrow H^1(X;\mathcal{I}_{Z_1\cup Z_2}).$$

Repeating the above procedure, we get a sequence of surjections

$$0 = H^1(X; \mathcal{I}_X) = H^1(X; \mathcal{I}_{Z_1 \cup \cdots \cup Z_n}) \twoheadrightarrow H^1(X; \mathcal{I}_{Z_1 \cup \cdots \cup Z_{n-1}}) \twoheadrightarrow \cdots \twoheadrightarrow H^1(X; \mathcal{I}_{Z_1}),$$

where the first equality comes from the assumption on the reducedness of X. Then

$$H^1(X; \mathcal{I}_{Z_1}) = 0.$$

Since Z_1 is arbitrary, by Serre's Theorem, we conclude that X is affine.

Exercise 41

(a) Since X, Y are integral, they are in particular irreducible. X and Y have exactly one generic point η_X and η_Y respectively, by the bijective correspondence between generic points and irreducible components. Since f maps generic points of X to generic points of Y, the map of stalks $f^\sharp : \mathcal{O}_{Y,\eta_Y} \to (f_*\mathcal{O}_X)_{Y,\eta_Y}$ is in fact a homomorphism of residue fields

$$\kappa(\eta_Y) \to \kappa(\eta_X).$$

By assumption, f is finite, so $\kappa(\eta_X)$ is in fact a finitely generated $\kappa(\eta_Y)$-module, which can be written as

$$\kappa(\eta_X) = \bigoplus_{i=1}^{r} \kappa(\eta_Y) e_i,$$

with e_i the generators. We can then take the coherent sheaf $\mathcal{M}$ on X as

$$\mathcal{M} := \mathcal{O}_X,$$

and the map $\alpha : \mathcal{O}_Y^r \to f_*\mathcal{M}$ as $(f^\sharp)^r$. At η_Y, we have

$$(\mathcal{O}_Y^r)_{\eta_Y} = (\kappa(\eta_Y))^r$$

and $\mathcal{M}_{\eta_Y} = \kappa(\eta_X) = (\kappa(\eta_Y))^r$, which shows that

$$\alpha_{\eta_Y} : (\mathcal{O}_Y^r)_{\eta_Y} \to (f_*\mathcal{M})_{\eta_Y}$$

is indeed an isomorphism.

(b) We apply the functor $\mathrm{Hom}_{\mathcal{O}_Y}(-, \mathcal{F})$ to the morphism

$$\alpha : \mathcal{O}_Y^r \to f_*\mathcal{M},$$

and take

$$\beta := \mathrm{Hom}_{\mathcal{O}_Y}(\alpha, \mathcal{F}) : \mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{M}, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y^r, \mathcal{F}).$$

By assumption, $f : X \to Y$ is finite hence is affine, thus f_* is an equivalence between the category $\mathrm{Coh}(\mathcal{O}_X)$ of quasi-coherent $\mathcal{O}_X$-sheaves to the category $\mathrm{Coh}(f_*\mathcal{O}_X$ of quasi-coherent $f_*\mathcal{O}_X$-sheaves. Since $\mathrm{Coh}(f_*\mathcal{O}_X$ is abelian, so $\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{M}, \mathcal{F})$ is coherent, and we can find some coherent sheaf $\mathcal{G}$ on X such that

$$\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{M}, \mathcal{F}) \simeq f_*\mathcal{G}$$

up to isomorphism, by the fully-faithfulness of the functor f_*. And also note that $\mathcal{F}^r \simeq \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y^r, \mathcal{F})$, thus finally we have

$$\beta : f_*\mathcal{G} \to \mathcal{F}^r,$$

which is an isomorphism at η_Y by the Yoneda Lemma.

(c)

Exercise 42

Since $X \to \mathrm{Spec}k$ is proper, it is separated, of finite type and universally closed. We know that a morphism is integral iff it is separated and of finite type, so we can conclude that the morphism of schemes $X \to \mathrm{Spec}k$ is integral. Meanwhile we have X geometrically reduced, $\mathcal{O}_X(X)$ is an integral domain over k, furthermore is a field. By the assumption on being of finite type, we can infer that $\mathcal{O}_X(X)$ is a finitely generated k-algebra.

Let $\bar{k}$ be the algebraic closure of k. Denote the fiber of $X \to \mathrm{Spec}k$ at $\bar{k}$ as $X_{\bar{k}}$

$$\begin{tikzcd} X_{\bar{k}} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\bar{k} \arrow[r] & \mathrm{Spec}k \end{tikzcd}.$$

Thus we have

$$\mathcal{O}_{X_{\bar{k}}}(X_{\bar{k}}) \simeq \mathcal{O}_X(X) \otimes_k \bar{k}.$$

By assumption, X is geometrically connected and geometrically reduced, hence $X_{\bar{k}}$ is connected and reduced, which implies that $\mathcal{O}_X(X) \otimes_k \bar{k}$ is local and reduced. Since proper morphisms are stable under base change, so $X_{\bar{k}} \to \mathrm{Spec}\bar{k}$ is again proper and $\mathcal{O}_X(X) \otimes_k \bar{k}$ is a finitely generated $\bar{k}$-algebra, thus $\mathcal{O}_X(X) \otimes_k \bar{k} = \bar{k}$. Thus we have $\mathcal{O}_X(X) = k$ by rank reason.

Exercise 43

(a) What we have is the following commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\mathcal{O}_{X,x} \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}.$$

Denote the structure maps as $h : X \to S$ and $g : Y \to S$, we can choose a small enough affine open neighborhood $s \in V \subset S$, such that $U = h^{-1}(V)$ and $W = g^{-1}(V)$ is also affine. Hence we may assume that X, Y and S are all affine schemes, and the corresponding diagram of commutative rings is

$$\begin{tikzcd} A_{\mathfrak{p}} & B \arrow[l, "\phi"'] \\ & C \arrow[u] \arrow[ul] \end{tikzcd}.$$

If we can find some $f \in A/\mathfrak{p}$ and a morphism $\psi : B \to A_f$ of C-algebras, such that

$$\begin{tikzcd} B \arrow[r, "\psi"] \arrow[dr, "\phi"] & A_f \arrow[d] \\ & A_{\mathfrak{p}} \end{tikzcd}$$

commutes, we can take $U = \mathrm{Spec}A_f$, as desired.

By assumption, since $g : Y \to S$ is of finite type, we conclude that B is a finitely generated C-algebra, which means that

$$B = C[x_1, \ldots, x_n]/I,$$

where I is an ideal of the ring $C[x_1, \ldots, x_n]$. We denote

$$\phi(x_i) = \frac{a_i}{s_i} \in A_{\mathfrak{p}}, i = 1, \ldots, n.$$

Exercise 44

From the pullback square,

$$\begin{tikzcd} X \times_S Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

we have the isomorphisms

$$p^*\Omega_{X/S} \simeq \Omega_{X\times_S Y/Y}, \tag{30}$$

which which factors as

$$p^*\Omega_{X/S} \to \Omega_{X\times_S Y/S} \to \Omega_{X\times_S Y/Y}, \tag{31}$$

by virtue of the commutative diagram

$$\begin{tikzcd} X \times_S Y \arrow[r, "\mathrm{id}"] \arrow[d, "q"] & X \times_S Y \arrow[r, "p"] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \arrow[r, "\mathrm{id}"] & S \end{tikzcd}$$

and the functoriality of the Kähler differentials. Likewise, by reverting the roles of X and Y, we have an isomorphism

$$q^*\Omega_{Y/S} \simeq \Omega_{X\times_S Y/X}, \tag{32}$$

which factors as

$$q^*\Omega_{Y/S} \to \Omega_{X\times_S Y/S} \to \Omega_{X\times_S Y/X}.$$

Moreover, we have an exact sequence

$$p^*\Omega_{X/S} \longrightarrow \Omega_{X\times_S Y/S} \longrightarrow \Omega_{X\times_S Y/X} \longrightarrow 0.$$

By (30) and (31), we know that $p^*\Omega_{X/S} \to \Omega_{X\times_S Y/S}$ is a monomorphism, and by (32), the last exact sequence can be written as

$$0 \longrightarrow p^*\Omega_{X/S} \longrightarrow \Omega_{X\times_S Y/S} \longrightarrow q^*\Omega_{Y/S} \longrightarrow 0.$$

Again by the isomorphism (30), we know the above sequence is split exact, thus we have

$$\Omega_{X\times_S Y/S} \simeq p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S},$$

as desired.

Exercise 45

Exercise 46

(a) As the hint suggests, if we want to show that the Noetherian local ring A is regular, it is equivalent to show tht for all A-modules M, N

$$\mathrm{Tor}_n^A(M, N) = 0, n > d \tag{33}$$

holds for some integer d.

By the assumption of flatness on $f : A \to B$, we have$B$ is a faithfully flat A-module, which implies that for any A-module M, the functor $M \mapsto M \otimes_A B$ is an exact functor. Thus, if we can show that

$$\mathrm{Tor}_n^A(M, N) \otimes_A B = 0, n > d,$$

then we can infer that (33) holds. To compute $\mathrm{Tor}_n^A(M, N)$, we take

$$L_\bullet = \cdots L_2 \to L_1 \to L_0 \to M$$

as a free resolution of M, and the nth homology group of the complex $L_\bullet \otimes_A N$. For any $i \geq 0$, we have

$$(L_i \otimes_A N) \otimes_A B = (L_i \otimes_A B) \otimes_B (N \otimes_A B).$$

And since $- \otimes_A B$ is exact, taking homology groups of both sides of the above equation, we have

$$\mathrm{Tor}_n^A(M, N) \otimes_A B = \mathrm{Tor}_n^B(M \otimes_A B, N \otimes_A B).$$

But since B is a regular Noetherian local ring, we have

$$\mathrm{Tor}_n^B(M \otimes_A B, N \otimes_A B) = 0, n > d$$

for B-modules $M \otimes_A B, N \otimes_A B$ and some integer d, which completes the proof.

(b) By our lecture notes, if we want to show that $g : Y \to S$ is smooth, we need to show that g is locally of finite presentation, flat and the fibers of g are smooth. As g is locally of finite presentation by hypothesis, we only need to show that it is flat and having smooth fibers.

For any $x \in X$ we denote $y = f(x) \in Y$ and $s = g(f(x)) \in X$. To show that $g : Y \to S$ is flat, we only need to show that the homomorphism $\mathcal{O}_{S,s} \to \mathcal{O}_{Y,y}$ of local rings is faithfully flat at every s. This is equivalent to show that any sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0 \tag{34}$$

of $\mathcal{O}_{S,s}$-modules is exact iff the sequence

$$0 \longrightarrow M_1 \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \longrightarrow M_2 \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \longrightarrow M_3 \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y} \longrightarrow 0$$

of $\mathcal{O}_{Y,y}$-modules is exact. But as $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is faithfully flat by the assumption on f, the last sequence is exact iff

$$0 \longrightarrow (M_1 \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \longrightarrow (M_2 \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \longrightarrow (M_3 \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \longrightarrow \tag{35}$$

is an exact sequence of $\mathcal{O}_{X,x}$-modules, which is true. Indeed, as we have

$$M_i \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x} = (M_i \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x},$$

and faithfully flat ring homomorphism $\mathcal{O}_{S,s} \to \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ by hypothesis, which shows that the sequence (34) is exact iff the sequence (35) is exact.

Exercise 47

Exercise 48

Let $f : \mathbb{P}^n_k \dashrightarrow X$ be a dominant rational map. Since f is dominant and rational, we can pick an open subset $U \subset \mathbb{P}^n_k$, such that the when restricted to U, $f : U \to X$ is dominant. Since U is an open subset of $\mathbb{P}^n_k$ we have that $U(k)$ is dense in U, since we can always find an $n+1$-tuple $a_1, \ldots, a_{n+1} \in k$ such that $P(a_0, \ldots, a_{n+1}) \neq 0$ for all polynomial $P \in k[x_0, \ldots, x_{n+1}]$. Since $f(U) \subset X$ is dense, and $f(U(k))$ is dense in $f(U)$, by the continuity of f, we deduce that $f(U(k))$ is dense in X. Since $f(U(k)) \subset X(k)$, we have $X(k)$ dense in X.

Exercise 49

To prove the first isomorphism, we take any $K \in D(Y)$, thus we have

$$\begin{aligned} \mathrm{Hom}_Y(K, Rf_* R\mathcal{H}\wr\Updownarrow_X(Lf^*N, M)) &\simeq \mathrm{Hom}_X(Lf^*K, R\mathcal{H}\wr\Updownarrow_X(Lf^*N, M)) \\ &\simeq \mathrm{Hom}_X(Lf^*K \otimes^L_{\mathcal{O}_Y} Lf^*N, M) \\ &\simeq \mathrm{Hom}_X(Lf^*(K \otimes^L_{\mathcal{O}_Y} N), M) \\ &\simeq \mathrm{Hom}_Y(K \otimes^L_{\mathcal{O}_Y} N, Rf_*M) \\ &\simeq \mathrm{Hom}_Y(K, R\mathcal{H}\wr\Updownarrow_Y(N, Rf_*M)) \end{aligned},$$

by the Yoneda Lemma, we have the isomorphism

$$R\mathcal{H}\wr\Updownarrow_Y(N, Rf_*M) \simeq Rf_* R\mathcal{H}\wr\Updownarrow_X(Lf^*N, M).$$

To prove the second isomorphism, suppose that I is a homotopically injective complex and $M \to I$ is a quasi-isomorphism, then we have

$$R\mathrm{Hom}_Y(N, Rf_*M) \simeq \mathrm{Hom}^\bullet_Y(N, f_*(I)) \simeq \mathrm{Hom}^\bullet_X(f^*(N), I) \simeq R\mathrm{Hom}_X(Lf^*N, M),$$

as desired.

Exercise 50

(a) Since $M_1 \boxtimes^L_S M_2 := Lp_1^* M_1 \otimes^L_{\mathcal{O}_X} Lp_2^* M_2$, thus we have

$$\begin{aligned} Rf_*(M_1 \boxtimes^L_S M_2) &= R(f_1 \circ p_1)_*(Lp_1^* M_1 \otimes^L_{\mathcal{O}_X} Lp_2^* M_2) \\ &= Rf_{1*} Rp_{1*}(Lp_1^* M_1 \otimes^L_{\mathcal{O}_X} Lp_2^* M_2) \\ &= Rf_{1*}(M_1 \otimes^L_{\mathcal{O}_{X_1}} Rp_{1*} Lp_2^* M_2) \\ &= Rf_{1*}(M_1 \otimes^L_{\mathcal{O}_{X_1}} Lf_1^* Rf_{2*} M_2) \\ &= Rf_{1*} M_1 \otimes^L_{\mathcal{O}_S} Rf_{2*} M_2. \end{aligned}$$

The third and the last equality come from the projection formulae, and the forth equality comes from the flat base change theorem for the derived category of quasi-coherent sheaves on X_2, which says that for the base change diagram

$$\begin{array}{ccc} X_1 \times_S X_2 & \xrightarrow{p_2} & X_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f_2} \\ X_1 & \xrightarrow{f_1} & S \end{array}$$

with f_1, f_2 quasi-compact and quasi-separated, and f_1 flat, we have an isomorphism in the derived category $D_{\text{qcoh}}(X_2)$

$$Rp_{1*}Lp_2^*(\mathcal{F}) \simeq Lf_1^*Rf_{2*}(\mathcal{F}), \tag{36}$$

for all $\mathcal{F} \in D_{\text{qcoh}}(X_2)$.

(b)

Exercise 51

(a) Let $S = A[x_0, \dots, x_n]$ and by assumption we have $P = \text{Proj}S$. By the canonical exact sequence,

$$0 \longrightarrow \Omega_{P/A} \longrightarrow \mathcal{O}_P(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_P \longrightarrow 0$$

we can express $\bigwedge^p(\mathcal{O}_P(-1)^{\oplus n+1})$ in terms of $\Omega_{P/A}$ and $\mathcal{O}_P$. Observe that we have a filtration

$$\bigwedge^p(\mathcal{O}_P(-1)^{\oplus n+1}) = F^0 \supseteq F^1 \supseteq \cdots F^p \supseteq F^{p+1} = 0,$$

where F^r consists of those elements which have exterior degrees greater than r in components coming from $\Omega_{P/A}$. With some easy verifications we have

$$F^r/F^{r+1} \simeq \wedge^r \Omega_{P/A} \otimes_{\mathcal{O}_P} \wedge^{p-r}\mathcal{O}_P = \Omega^r_{P/A} \otimes_{\mathcal{O}_P} \wedge^{p-r}\mathcal{O}_P.$$

But notice that

$$\wedge^k \mathcal{O}_P = \begin{cases} \mathcal{O}_P, & k = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$

So we have $F^0 = \cdots = F^{p-1} = \Omega^{p-1}_{P/A} \otimes_{\mathcal{O}_P} \mathcal{O}_P = \Omega^{p-1}_{P/A}, F^p = \Omega^p_{P/A}$, and

$$\wedge^p(\mathcal{O}_P(-1)^{\oplus n+1})/\Omega^p_{P/A} = F^{p-1}/F^p = \Omega^{p-1}_{P/A},$$

which can be expressed as an exact sequence

$$0 \longrightarrow \Omega^p_{P/A} \longrightarrow \bigoplus_k \mathcal{O}_P(-p) \longrightarrow \Omega^{p-1}_{P/A} \longrightarrow 0,$$

where $k = \binom{n+1}{p}$. After twisting $\mathcal{O}_P(1)$ m-times, we get an exact sequence of $\mathcal{O}_P$-modules

$$0 \longrightarrow \Omega^p_{P/A}(m) \longrightarrow \bigoplus_k \mathcal{O}_P(m-p) \longrightarrow \Omega^{p-1}_{P/A}(m) \longrightarrow 0. \tag{37}$$

Since cases (ii) and (iii) are Serre dual to each other, we have only to show (i) and (ii).

Exercise 52

(a) As in **(b)**, we take the invertible sheaf $\mathcal{L}$ corresponding to $D = (g+1)P$, we have

$$h^0(\mathcal{L}) = \dim \Gamma(X, D) \geq 2.$$

Since at D we can construct a rational function f having only poles at P, which induces a non-constant map $X \to \mathbb{P}^1_k$, and we may assume that f maps P to $\infty \in \mathbb{P}^1_k$. On the other hand, the gonality of X must less than $\deg f$, by definition. By the formula degree formula $\deg \mathcal{L} = \deg f \deg\{\infty\} = \deg f$, so

$$\text{gon}(X) = \min \left\{ \deg \mathcal{L} \mid h^0(\mathcal{L} \geq 2) \right\}.$$

(b) For any closed point $P \in X$, consider the divisor $D = (g+1)P$. Thus we have, by the Riemann-Roch Theorem

$$l(D) - l(K-D) = \deg(D) + 1 - g = (g+1) + 1 - g = 2,$$

which means that

$$\dim \Gamma(X, D) = 2 + l(K-D) \geq 2,$$

implying that there is a non-constant element $f \in \Gamma(X, D)$.

Since any $f \in \Gamma(X, D)$ induces a morphism $X \to \mathbb{P}^1_k$ of degree at most $\deg D = g+1$, which is the desired morphism.

Exercise 53

(a) For an effective divisor D, we consider any point x in the support of D and the divisor $D - x$. Applying the Riemann-Roch Theorem on $D - x$, we have

$$l(D-x) - l(K-(D-x)) = \deg(D-x) + 1 - g = 2g - 1 + 1 - g = g,$$

where K is the canonical divisor on the curve. But since $\deg(K-(D-x)) = 2g - 2 - 2g + 1 = -1$, we have $l(K-(D-x)) = 0$, which means that

$$l(D-x) = g \geq 0.$$

Thus we have $D - x \simeq D'$ for some effective divisor D', that is, there exists some rational function f on X such that

$$(f) = D - x - D'.$$

(b) Since a Noetherian scheme C is affine iff C_{red} is affine, so we may assume that C is reduced. Since for a reduced Noetherian scheme C, C is affine iff every irreducible component of C is affine, we may assume that C is integral. Consider the normalization $\overline{C}$ of C, if we can show that $\overline{C}$ is affine, then since $\overline{C} \to C$ is finite surjective, then C is affine, by **Exercise 41.**. So we may restrict ourselves to the case where C is smooth.

For a smooth curve C that is not proper, we may consider the compactification $C \to \overline{C}$. In **(a)**, we have constructed a meromorphic function f having pole only at x, where x is inside of the support of an effective divisor D. Take $D = \overline{C} - C$, thus using the result of **(a)**, there exists a meromorphic function f only having poles inside $\overline{C} - C$, which corresponds to a morphism $\phi : \overline{C} \to \mathbb{P}^1_k$ via the inclusion $k(f) \to K(\mathbb{P}^1_k)$. Since any morphism between the smooth curve $\overline{C}$ and $\mathbb{P}^1_k$ is either a constant or a surjective finite morphism, we conclude that $\phi : \overline{C} \to \mathbb{P}^1_k$ is surjective finite. Suppose that ϕ maps $\overline{C} - C$ to $\infty \in \mathbb{P}^1_k$, C is then contained in $\phi^{-1}(\mathbb{A}^1_k)$, by the finiteness of ϕ, C is affine.

Exercise 54

(a) Suppose the hypersurface X is determined by the polynomial f of degree d, and denote the canonical closed immersion by $i : X \to \mathbb{P}^n_k$. Thus, consider the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow i_* \mathcal{O}_X \longrightarrow 0,$$

on $\mathbb{P}^n_k$. From the induced cohomology long exact sequence and the cohomologies of twisted sheaves on $\mathbb{P}^n_k$, we have

$$\begin{aligned}
&H^0(X;\mathcal{O}_X) = H^0(\mathbb{P}^n_k;\mathcal{O}_{\mathbb{P}^n_k}) = k,\\
&H^{n-1}(X;\mathcal{O}_X) = H^n(\mathbb{P}^n_k;\mathcal{O}_{\mathbb{P}^n_k}(-d)) \simeq H^0(\mathbb{P}^n_k;\mathcal{O}_{\mathbb{P}^n_k}(d-n-1))^\vee \simeq \bigoplus_{\binom{d-1}{n}} k,\\
&H^i(X;\mathcal{O}_X) = 0, \text{otherwise},
\end{aligned}$$

and thus the Euler characteristic

$$\chi(\mathcal{O}_X) = 1 + (-1)^{n-1}\binom{d-1}{n}.$$

Since X is determined by single polynomial, thus $\dim X = n-1$ and the arithmetic genus of X reads

$$g_a(X) = (-1)^{n-1}(\chi(\mathcal{O}_X) - 1) = \binom{d-1}{n}.$$

(b) Since $f : X^\nu \to X$ is the normalization of X, we have an exact sequence of abelian sheaves on X,

$$0 \longrightarrow \mathcal{O}_X \longrightarrow f_*\mathcal{O}_{X^\nu} \longrightarrow \textstyle\coprod_{x\in X} \mathcal{O}^\nu_{X,x}/\mathcal{O}_{X,x} \longrightarrow 0. \tag{38}$$

Since X and X^ν are all proper, thus by Chow's Lemma, they are all projective, and we have $H^0(X;\mathcal{O}_X) = k, H^0(X;f_*\mathcal{O}_{X^\nu}) = H^0(X^\nu;\mathcal{O}_{X^\nu}) = k$. Since $\coprod_{x\in X}\mathcal{O}^\nu_{X,x}/\mathcal{O}_{X,x}$ is flabby, we have $H^q(X;\coprod\limits_{x\in X}\mathcal{O}^\nu_{X,x}/\mathcal{O}_{X,x}) = 0$ for all $q \geq 1$. Taking the cohomology long exact sequence of (38), we have an exact sequence

$$0 \to k \to k \to H^0(X;\coprod_{x\in X}\mathcal{O}^\nu_{X,x}/\mathcal{O}_{X,x}) \to H^1(X;\mathcal{O}_X) \to H^1(X^\nu;\mathcal{O}_{X^\nu}) \to 0$$

of k-modules. Since in the case of a curve X over k, we have $g_a(X) = \dim_k H^1(X;\mathcal{O}_X)$ by definition, thus we have

$$g_a(X) = g_a(X^\nu) + \sum_{x\in X}(\mathcal{O}^\nu_{X,x}/\mathcal{O}_{X,x})$$

from the above exact sequence of cohomology groups.

If $g_a(X) = 0$, we have $g_a(X^\nu) = \dim_k(\mathcal{O}^\nu_{X,x}/\mathcal{O}_{X,x}) = 0$, since both of them are positive numbers, by definition. In particular, $\dim_k(\mathcal{O}^\nu_{X,x}/\mathcal{O}_{X,x}) = 0$ tells us that at every point $x \in X$, $\mathcal{O}_{X,x}$ equals to its normalization, which means that X is normal and hence is smooth, because it has no singular points. Thus the genus of X coincides with the arithmetic genus of X, where we can conclude that $g(X) = g_a(X) = 0$ iff $X \simeq \mathbb{P}^1_k$.

Exercise 55

(a)

(b) We prove the assertion by induction on the codimension of X. If $\mathrm{codim}X = 0$, by assumption X is a complete intersection, we have $X = \mathrm{Proj}R = \mathbb{P}^n_k$, hence $H^0(\mathbb{P}^n_k;\mathcal{O}(m)) \to H^0(X;\mathcal{O}_X(m))$ being surjective holds trivially.

For $\mathrm{codim}X \geq 1$, we denote the complete intersection $X = H_1 \cap \cdots \cap X_c$, and consider another complete intersection $Y = H_1 \cap \cdots \cap X_{c-1}$, where we have $\mathrm{codim}Y = \mathrm{codim}X - 1$. So the induction hypothesis holds for the complete intersection $Y \hookrightarrow \mathbb{P}^n_k$, that is, $H^0(\mathbb{P}^n_k;\mathcal{O}(m)) \to H^0(Y;\mathcal{O}_Y(m))$ is surjective and $H^i(Y;\mathcal{O}_Y(m)) = 0$ for all $m \in \mathbb{Z}$ and $0 < i < n-c+1$ (Note that Y is of dimension greater by 1 than X). Let $i : X \to Y$ be the canonical closed immersion. If the polynomial f_c defining the hypersurface H_c is of degree d, then we have an exact sequence

$$0 \longrightarrow \mathcal{O}_Y(m-d) \xrightarrow{\times f_c} \mathcal{O}_Y(m) \longrightarrow i_*\mathcal{O}_X(m) \longrightarrow 0. \tag{39}$$

Now consider the cohomology long exact sequence of (39), since

$$H^1(Y;\mathcal{O}_Y(m-d)) = 0$$

by the induction hypothesis, we have a surjection $H^0(Y;\mathcal{O}_Y(m)) \to H^0(Y;i_*\mathcal{O}_X(m)) = H^0(X;\mathcal{O}_X(m))$, by pre-composing the surjection $H^0(\mathbb{P}^n_k;\mathcal{O}(m)) \to H^0(Y;\mathcal{O}_Y(m))$, we finally have a surjection $H^0(\mathbb{P}^n_k;\mathcal{O}(m)) \to H^0(X;\mathcal{O}_X(m))$. Again by the induction hypothesis we read from the long exact sequence

$$\cdots \to H^i(Y;\mathcal{O}_Y(m-d)) \to H^i(Y;\mathcal{O}_Y(m)) \to H^i(X;\mathcal{O}_X(m)) \to H^{i+1}(Y;\mathcal{O}_Y(m-d)) \to \cdots$$

that

$$H^i(X;\mathcal{O}_X(m)) \simeq H^i(Y;\mathcal{O}_Y(m)) = 0.$$

Since we have

$$H^0(\mathbb{P}^n_k;\mathcal{O}) = R_0 = k,$$

the surjection of k-modules $H^0(\mathbb{P}^n_k;\mathcal{O}) \to H^0(X;\mathcal{O}_X)$ can be written as the surjection of k-modules $k \to H^0(X;\mathcal{O}_X)$, from which we can infer that

$$\mathcal{O}_X(X) = H^0(X;\mathcal{O}_X) = k.$$

Since $\mathcal{O}_X(X)$ has no non-trivial idempotents, X is connected.

(c) Let $X = F_1 \cap F_2$ be a complete intersection, where F_1 is a hypersurface in $\mathbb{P}^3_k$ determined by a polynomial of degree d and F_2 is a hypersurface of degree e. As we take $m = 0$ in **(b)**, we get the 0th cohomology group of $\mathcal{O}_X$,

$$H^0(X;\mathcal{O}_X) = H^0(\mathbb{P}^3_k;\mathcal{O}_{\mathbb{P}^3_k}) = k.$$

Since X is of dimension $3-2=1$, we need to determine the Euler characteristic of $\mathcal{O}_X$ to compute the arithmetic genus.

Denote the canonical closed immersions by $i_1 : F_1 \to \mathbb{P}^3_k$ and $i : X \to F_1$. Using the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^3_k} \longrightarrow i_{1,*}\mathcal{O}_{F_1} \longrightarrow 0$$

of sheaves on $\mathbb{P}^3_k$, we can compute the all the cohomology groups of $\mathcal{O}_{F_1}$,

$$\begin{aligned}
&H^0(F_1;\mathcal{O}_{F_1}) = H^0(\mathbb{P}^3_k;\mathcal{O}_{\mathbb{P}^3_k}) = k,\\
&H^1(F_1;\mathcal{O}_{F_1}) = 0,\\
&H^2(F_2;\mathcal{O}_{F_1}) = H^3(\mathbb{P}^3_k;\mathcal{O}_{\mathbb{P}^3_k}(-d)) = (k[x_0,x_1,x_2,x_3])_{d-4} \simeq \bigoplus_{\binom{d-1}{3}} k,\\
&H^i(F_2;\mathcal{O}_{F_2}) = 0, i \geq 3,
\end{aligned}$$

thus the Euler characteristic of $\mathcal{O}_{F_1}$

$$\chi(\mathcal{O}_{F_1}) = \binom{d-1}{3} + 1.$$

Also, we get the Euler characteristic of $\mathcal{O}_{F_1}(-e)$

$$\chi(\mathcal{O}_{F_1}(-e)) = \binom{d+e-1}{3} - \binom{e-1}{3}$$

by considering the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(-e-d) \longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(-e) \longrightarrow i_{1,*}\mathcal{O}_{F_1}(-e) \longrightarrow 0 \ .$$

Finally, by the exact sequence

$$0 \longrightarrow \mathcal{O}_{F_1}(-e) \longrightarrow \mathcal{O}_{F_1} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0 \ ,$$

we have

$$\chi(\mathcal{O}_{F_1}) = \chi(\mathcal{O}_{F_1}(-e)) + \chi(\mathcal{O}_X), \tag{40}$$

and by the formula of arithmetic genus

$$g_a(X) = 1 - \chi(\mathcal{O}_X) = \chi(\mathcal{O}_{F_1}(-e)) - \chi(\mathcal{O}_{F_1}) + 1 = \binom{d+e-1}{3} - \binom{e-1}{3} - \binom{d-1}{3} = \frac{1}{2}de(d+e-4)+1. \tag{41}$$

References

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