Riemann Surfaces

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ABSTRACT: These are unedited and live-TeXed notes for *Lectures on Riemann Surfaces* in Spring 2021. All errors introduced are mine.

Contents

1 Riemann-Hurwitz Theorem

Theorem 1.1. Let X, Y be Riemann surfaces, $f : X \to Y$ be non-constant holomorphic map. Consider $p \in X, q = f(p)$, then there exist local coordinates z on X near p with $z(p) = 0$ and w on Y near q, such that f is represented by

$$w = z^k,$$

where $k \geqslant 1$ is independent of the choices of z, w.

Proof. Let z' be a coordinate on X near p with $z'(p) = 0$, and w on Y near q with $w(q) = 0$. Then under z', w is represented as $z' \mapsto w = a_k z'^k + a_{k+1} z'^{k+1} + \cdots$. Rewrite the formula as

$$w = z'^k (a_k + a_{k+1} z' + \cdots),$$

we may take a holomorphic function $h(z')$ near $z' = 0$ such that $h^k = a_k + a_{k+1} z' + \cdots$.

Let $z = z' h(z')$, then

$$\frac{dz}{dz'} = h(0) + \cdots \neq 0$$

hence z is a local coordinate on X near p. Under z, w, f is given by $w = z^k$.

We can view k as follows. For any sufficiently small neighborhood U of p in X, there exists a neighborhood V of q in Y, such that for any $q' \in V \backslash \{q\}$, $f^{-1}(q)$ has precisely k points. Which implies (???) that k is independent of z and w. □

Definition 1.1. (i) k is called the **multiplicity** of f at p, denoted by $m_f(p)$

(ii) $b_f(p) := k - 1$ is called the **ramification number** of f at p.

(iii) If $b_f(p) \geqslant 1$, then p is called a ramification point of f.

Definition 1.2. Let $f \in \mathcal{M}(X)$ be non-constant, $p \in X$.

1. If $f(p) = 0$ then there exists a local coordinate z near p, such that $z(p) = 0$ and $f(z) = z^k$. We call p a **zero** of multiplicity p.

2. If $f(p) = \infty$ then there exists a local z such that $f(z) = z^{-k}$. We call p a **pole** of multiplicity k.

3. $$\mathrm{ord}_f(p) = \begin{cases} k, p \text{is a zero of multiplicity} k \\ -k, \text{pole} \\ 0 \end{cases}$$

Assume that X is a compact Riemann surface, let $f \in \mathcal{M}(X)$ be non-constant, let $p_1, \ldots, p_r$ be zeros of f of multiplicity $m_1, \ldots, m_r$, $q_1, \ldots, q_s$ be poles of multiplicity $n_1, \ldots, n_s$. Consider the formal sum

$$(f) = \sum_{p \in X} \mathrm{ord}_f(p) \cdot p,$$

and let $\mathrm{Div}(X)$ be the free abelian group generated by the points in X. $\mathrm{Div}(X)$ is called the

divisor group of X.

Assume $f, g \in \mathcal{M}(X)$ and $(f) = (g)$, and we soon know that

$$h = \frac{f}{g} \in \mathcal{O}(X).$$

Since X is compact, h is constant and hence f, g are the same up to a constant.

f is uniquely determined by (f), up to a constant divisor.

Question: Given $D \in Div(X)$, can we find $f \in \mathcal{M}(X)$ such that $D = (f)$?

Definition 1.3 (covering space)**.** Let X, Y be two topological spaces, a continuous map $\pi : X \to Y$ is called a **covering map** if $\forall y \in Y$, there exists a neighborhood V of Y and a family of disjoint open subsets $\{U_i\}_{i\in I}$ of X, such that $f^{-1}(V) = \coprod_{i\in I} U_i$ and $f|_{U_i} \to V$ is a homeomorphism.

Example. $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering map.

$$z \mapsto z^n$$

$$\theta \mapsto e^{2\pi i\theta}$$

Definition 1.4 (ramified covering)**.** Let X, Y be topological surfaces, $f : X \to Y$ be a continuous map. f is called a **branched covering** if there exists a discrete subset $A \subseteq X$ such that

1. $f|_{X\backslash A} : X\backslash A \to Y\backslash f(A)$ is a covering map.
2. $\forall a \in A$, there exists a coordinate z of X near a and a coordinate w of Y near $f(a)$ such that f is represented by $z \mapsto w = z^k$ locally, where k is independent of the choice of w, z

We can define the multiplicity, ramification number of a ramified covering as in the holomorphic case.

Definition 1.5. Let $f : X \to Y$ be a ramified covering, and assume that X, Y are connected. For any $y \in Y$, we define

$$\deg f = \sum_{x\in f^{-1}(y)} m_f(x),$$

which is called the **mapping degree** of f. And define $b_f = \sum_{x\in X} b_f(x)$ is called the total **branching number** of f.

Exercise. The definition of $\deg f$ is independent of the choice of y.

Exercise. Let X, Y be two Riemann surfaces, and $f : X \to Y$ be a proper map, then f is a branched covering. In particular, for any $y_1, y_2 \in Y$ we have $\#f^{-1}(y_1) = \#f^{-1}(y_2)$ counting multiplicity.

By the above exercise, we know the number of zeros and the numbers of poles of a meromorphic function f on a compact Riemann surface X are equal.

$\deg D = 0$ is a necessary condition for the existence of $f \in \mathcal{M}(X)$, but is not sufficient.

Let X be a compact surface (with boundary) and T be a triangulation of X, then the number

$$\chi(T) = V(T) - E(T) + F(T)$$

is called the **Euler characteristic** of X, where $V(T), E(T), F(T)$ are the number of the vertices, edges and faces of T respectively.

Remark. The Euler characteristic is independent of the choices of T.

Exercise. Show that if X is a compact surface of genus g, then $\chi(X) = 2 - 2g$.

Proposition 1.2. Let X, Y be closed surfaces, $f : X \to Y$ be a covering map of degree d, then

$$\chi(X) = d\chi(Y).$$

Proof. Let T be a triangulation of Y, by refining T, we may assume that any triangle Δ of T has a neighborhood $V \subseteq Y$, such that $f^{-1}(V) = \coprod_{1 \leqslant i \leqslant d} U_i$ and $f|_{U_i} : U_i \to V$ is a homomorphism for all i. Then $\tilde{T} = f^{-1}(T)$ is a triangulation of X. It's obvious that

$$V(\tilde{T}) = dV(T), E(\tilde{T}) = dE(T), F(\tilde{T}) = dF(T),$$

so the proposition follows. $\square$

Theorem 1.3 (Riemann-Hurwitz). Assume X, Y are closed surfaces and $f : X \to Y$ is a branched cover of degree d, then

$$\chi(X) = d\chi(Y) - b_f,$$

where b_f is the total branch number of f.

Proof. Let $A \subseteq X$ be a finite subset such that $f|_{X \setminus A} \to Y \setminus f(A)$ is a covering map. Choose a triangulation T of Y such that $f(A)$ consists of some vertices of T. We assume that T is fine enough. $\tilde{T} = f^{-1}(T)$ is a triangulation of X. One can see that

$$V(\tilde{T}) = dV(T) - b_f, E(\tilde{T}) = dE(T), F(\tilde{T}) = dF(T),$$

thus the theorem follows. $\square$

If $g(X), g(Y)$ be the genera of X, Y, by the Riemann-Hurwitz formula, we get

$$g(X) = d(g(Y) - 1) + 1 + \frac{b_f}{2}.$$

In particular, b_f is an even number.

Corollary 1.4. Let X, Y be two compact Riemann surfaces, $f : X \to Y$ be a non-constant holomorphic map

1. $g(X) \geqslant g(Y)$.
2. If $g(X) = g(Y) = 1$, then f is a covering map.
3. If $g(X) = g(Y) > 1$, then f is biholomorphic.

Exercise. Let X be a compact Riemann surface of genus g, and $f : X \to \mathbb{P}^1$ be a holomorphic map of degree 2, compute the number of branching points of f.

The following proposition is the one of the foundations of this course. [Don, pp.45, Corollary 1.]

Proposition 1.5. Let X be a compact Riemann surface. If there is a meromorphic function on X having exactly one pole, and that pole has order 1, then X is equivalent to the $\mathbb{P}^1$.

Proof. Let $f : X \to \mathbb{P}^1$ be the given meromorphic function. It is proper, since X is compact. By assumption, the mapping degree of f is 1. [1] This means that for any $y \in \mathbb{P}^1$ there is exactly one point x in $f^{-1}(y)$. Thus f is a bijection. We next need to show that f is a homeomorphism, this suffices to show that $f^{-1} : \mathbb{P}^1 \to X$ is continuous, namely, $(f^{-1})^{-1}(C) = f(C)$ is a closed subset of $\mathbb{P}^1$ given any closed subset $C \subseteq X$. Indeed, since X is compact, so is C as it is a closed subset of a compact space. $f(C)$ is compact in $\mathbb{P}^1$ since it is the continuous image of a compact subset. Since $\mathbb{P}^1$ is Hausdorff, $f(C)$ is closed, as desired. By some lemma in [Don], the inverse f^{-1} is holomorphic. □

[1] *Note that we define* $\deg_y f := \sum_{x \in f^{-1}(y)} k_x$, *whether* $x \in f^{-1}(y)$ *is a ramification point or not doesn't matter.*

Proposition 1.6. The number of zeroes is equal to the number of poles for any meromorphic function on a Riemann surface (not necessarily compact), counting multiplicity.

Proof. Let f be a meromorphic function on a Riemann surface X. f corresponds to a holomorphic map $f : X \to \mathbb{P}^1$. Since the degree of f is the same for every point, in particular, for 0 and ∞. Thus the number of preimages of 0 counted with multiplicities is the same as the number of preimages of ∞ counted with multiplicities. □

2 Calculus on Riemann Surfaces

2.1 Vector Fields and Differential Forms

Let X be a Riemann surface, $U \subseteq X$ an open subset of X, denote $C^\infty(U)$ by the space of complex valued smooth functions on U.

For $p \in X$, the space of germs of smooth functions at p is defined as

$$C^\infty_{X,p} = \Big(\coprod_{p \in U} C^\infty(U)\Big)/\sim$$

the equivalence relation $\sim$ is defined as follows: $f \sim g$ if there exists an open subset $W \subset U \cap V$ such that $f|_W = g|_W$.

Definition 2.1. A **tangent vector** of X at p is a $\mathbb{C}$-linear map $v : C^\infty_{X,p} \to \mathbb{C}$ satisfying the Leibniz rule

$$v(fg) = v(f)g(p) + f(p)v(g).$$

Denote T_pX as the linear space spanned by all tangent vectors of X at p.

Let $z = x + iy$ be a local coordinate near p. For $f \in C^\infty_{X,p}$, define

$$\frac{\partial}{\partial x}|_p f = \frac{\partial f}{\partial x}(p),$$
$$\frac{\partial}{\partial y}|_p f = \frac{\partial f}{\partial y}(p).$$

Then $\frac{\partial}{\partial x}|_p, \frac{\partial}{\partial y}|_p \in T_pX$.

Exercise. Show that $\frac{\partial}{\partial x}|_p \frac{\partial}{\partial y}|_p$ is a basis of T_pX over $\mathbb{C}$.

Let

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} = i\frac{\partial}{\partial y})$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

Consider the complex structure on X, then the coordinate transformations are holomorphic, hence we get $\frac{\partial w}{\partial \bar{z}} = \frac{\partial \bar{w}}{\partial z} = 0$, so we have

$$\frac{\partial}{\partial z} = \frac{\partial w}{\partial z}\frac{\partial}{\partial w}$$
$$\frac{\partial}{\partial \bar{z}} = \frac{\partial \bar{z}}{\partial \bar{z}}\frac{\partial}{\partial \bar{w}}.$$

It makes sense to define

$$T^{(1,0)}X = \mathbb{C}\{\frac{\partial}{\partial z}\}$$
$$T^{(0,1)}X = \mathbb{C}\{\frac{\partial}{\partial \bar{z}}\},$$

and we have the decomposition

$$T_pX = T_p^{(1,0)}X \oplus T_p^{(0,1)}X.$$

Conjugate on T_pX

$$\overline{\alpha\frac{\partial}{\partial z} + \beta\frac{\partial}{\partial \bar{z}}} = \bar{\alpha}\frac{\partial}{\partial \bar{z}} + \bar{\beta}\frac{\partial}{\partial z}.$$

By definition, we have

$$T_p^{(0,1)}X = \overline{T_p^{(0,1)}}X.$$

Let X be a Riemann surface and $p \in X$, the **cotangent space** of X at p is the dual space of T_pX, and is denoted by

$$T_p^*X := (T_pX)^*.$$

We have already known that $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ and $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\}$ are two bases of T_pX, given the local coordinate $z = x + iy$. Thus we may define $dx, dy \in T_p^*X$ as the dual to $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, and let

$$dz = dx + idy,$$
$$d\bar{z} = dx - idy.$$

Then $\{dz, d\bar{z}\}$ is a basis of T_p^*X, dual to $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\}$.

Assume that $w = u + iv$ is another local coordinate of p, then $\{du, dv\}$ and $\{dw, d\bar{w}\}$ are two bases of T_p^*X. The transformation laws of these bases under the coordinate change are

$$
\begin{aligned}
dw &= \frac{\partial w}{\partial z} dz + \frac{\partial w}{\partial \bar{z}} d\bar{z} = \frac{\partial w}{\partial z} dz \\
d\bar{w} &= \frac{\partial \bar{w}}{\partial z} dz + \frac{\partial \bar{w}}{\partial \bar{z}} d\bar{z} = \frac{\partial \bar{w}}{\partial \bar{z}} d\bar{z}.
\end{aligned}
$$

So it makes sense to define

$$
\begin{aligned}
(T_p^* X)^{(1,0)} &= \mathbb{C}\{dz\}, \\
(T_p^* X)^{(0,1)} &= \mathbb{C}\{d\bar{z}\}.
\end{aligned}
$$

It's easy to check that

$$
\begin{aligned}
T_p^* X = (T_p^* X)^{(0,1)} &\oplus (T_p^* X)^{(0,1)}, \\
(T_p^* X)^{(0,1)} &= \overline{(T_p^* X)^{(1,0)}}.
\end{aligned}
$$

2.2 Vector Fields on Riemann Surfaces

The **tangent bundle** of a Riemann surface X is defined to be

$$
TX := \coprod_{p \in X} T_p X.
$$

Definition 2.2. A **vector field** on X is a map $\theta : X \to TX$ such that $\theta(p) \in T_p X, \forall p \in X$.
A vector field θ is **smooth** if under any local coordinate $z = x + iy$, it can be locally represented as

$$
\theta = a(z) \frac{\partial}{\partial z} + b(z) \frac{\partial}{\partial \bar{z}},
$$

where a, b are locally smooth functions.
Similarly, θ is **holomorphic** if locally

$$
\theta = a(z) \frac{\partial}{\partial z},
$$

where a is locally a holomorphic function.

The above definitions are independent of choice of local coordinates.

Example. Holomorphic vector fields on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.
Let U be the neighborhood of $\{0\}$, with coordinate z, and V be the neighborhood of $\{\infty\}$ with coordinate $w = \frac{1}{z}$. Locally, a holomorphic vector field θ on X may be represented as

$$
\begin{aligned}
\theta|_U &= f(z) \frac{\partial}{\partial z}, \\
\theta|_V &= g(w) \frac{\partial}{\partial w},
\end{aligned}
$$

where f, g are holomorphic functions.
For any $z \in U \cap V$, we have

$$
\theta = f(z) dz = g(\frac{1}{z}) \frac{\partial z}{\partial w} \frac{\partial}{\partial z} = -z^2 g(\frac{1}{w}) \frac{\partial}{\partial z},
$$

implying

$$f(z) = -z^2 g(\frac{1}{z}). \tag{2.1}$$

We expand both f, g as

$$\begin{aligned} f(z) &= a_0 + a_1 z + a_2 z^2 + \cdots, \\ g(w) &= b_0 + b_1 w + b_2 w^2 + \cdots. \end{aligned}$$

By (2.1), we have

$$a_0 + a_1 z + a_2 z^2 + \cdots = -b_0 z^2 - b_1 z - b_2 \cdots,$$

showing that

$$\begin{cases} a_j = b_j = 0, j \geqslant 3 \\ b_0 = -a_2, b_1 = -a_1, b_3 = -a_0. \end{cases}$$

Denote L by the space of all homomorphic vector fields on $\mathbb{P}^1$, then

$$L = \left\{ (a_0 + a_1 z + a_2 z^2)\frac{\partial}{\partial} \;\middle|\; a_0, a_1, a_2 \in \mathbb{C} \right\}.$$

In particular, $\dim L = 3$.

Remark. Assume that θ is a holomorphic vector field on a Riemann surface X, $p \in X$. Let X be a coordinate near p, and represent θ near p as $\theta = a(z)\frac{\partial}{\partial z}$. Then the multiplicity of zero of θ at p is defined to be that of the function $a(z)$ at $z = 0$.

Exercise. Compute the number of zeros of any holomorphic vector fields on $\mathbb{P}^1$, counting multiplicity.

Exercise. Give a description of all holomorphic vector fields on an elliptic curve $\mathbb{C}/\Gamma$.

Claim 2.1. Let X be a compact Riemann surface of genus $\geqslant 2$. Then X has no non-zero holomorphic vector fields.

Exercise. Let L be the space of the holomorphic vector fields on $\mathbb{P}^1$ with the operation of Lie bracket of vector fields, then L is a Lie algebra over $\mathbb{C}$. Show that L and $\mathfrak{sl}(2,\mathbb{C})$ are isomorphic Lie algebras, where

$$\mathfrak{sl}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{C}, a + d = 0 \right\}.$$

2.3 Differential Forms

$$T^*X = \coprod_{p \in X} T_p^* X$$

is called the cotangent bundle of X.

Let

$$\begin{aligned} (T^*X)^{(1,0)} &:= \coprod_{p \in X} (T_p^* X)^{(1,0)} \\ (T^*X)^{(0,1)} &:= \coprod_{p \in X} (T_p^* X)^{(0,1)}. \end{aligned}$$

Definition 2.3. A **1-form** on X is a map $\xi : X \to T^*X$ such that $\xi(p) \in T_p^*X, \forall p \in X$. ξ is called of **type** $(1,0)$ (or $(0,1)$) if $\xi(p) \in (T_p^*X)^{(1,0)}$ (or $\xi(p) \in (T_p^*X)^{(0,1)}$).
A 1-form is called **smooth** if locally it can be written as

$$\xi = a(z)dz + b(z)d\bar{z},$$

with a, b smooth functions.
ξ is called **holomorphic** if locally

$$\xi = \alpha(z)dz,$$

where α is a holomorphic function.

Exercise. There are no non-zero holomorphic 1-forms on $\mathbb{P}^1$.

In general, we can define **meromorphic** 1-forms on X, which may locally be represented as

$$\alpha(z)dz,$$

where α is a meromorphic function. Denote $\mathcal{M}^1(X)$ as the space of meromorphic 1-forms on X. In parallel to the case of meromorphic functions, we can define the multiplicity of zeros and poles of a meromorphic 1-form at a given point p.

Assume that X is compact with genus g and $\xi \in \mathcal{M}^1(X)\backslash\{0\}$, the divisor of ξ is defined as

$$(\xi) = \sum_{p \in X} \text{ord}_\xi(p)p.$$

We have

1. If $f \in \mathcal{M}(X)\backslash\{0\}, \xi \in \mathcal{M}^1(X)\backslash\{0\}$, then $(f\xi) = (f) + (\xi)$
2. If $\xi, \eta \in \mathcal{M}^1(X)$, then $(\frac{\xi}{\eta}) = (\xi) - (\eta)$.

Recall that for all $f \in \mathcal{M}(X)$, we have

$$\deg f = \sum_{p \in X} \text{ord}_f(p) = 0.$$

So for any $\eta, \xi \in \mathcal{M}^1(X)$, we have

$$\deg(\frac{\xi}{\eta}) = \deg \xi - \deg \eta = 0,$$

showing that

$$\deg \xi = \deg \eta,$$

which should be an intrinsic quantity of X.

Let V be a vector space of dimension n. The wedge product $V \wedge V$ is defined to be

$$V \wedge V = \{u_1 \wedge v_1 + \cdots u_r \wedge v_r | r \geqslant 1, u_i, v_i \in V\}/\sim,$$

where $\sim$ is generated by

$$u \wedge v = -v \wedge u.$$

A **quadratic form** on V is a bilinear map

$$\begin{aligned} Q : V \times V &\to \mathbb{C} \\ (u, v) &\mapsto Q(u, v). \end{aligned}$$

Q is called **anti-symmetric** if $Q(u, v) = -Q(v, u), \forall u, v \in \mathbb{C}$.

> **Example.** Let $f, g \in V^*$, define
>
> $$Q(u, v) = f(u)g(v) - f(v)g(u),$$
>
> then Q is an anti-symmetric bilinear form on V.

Exercise. Define $\sigma : V^* \wedge V^* \to L$ to be

$$\sigma(f \wedge g)(u, v) = f(u)g(v) - f(v)g(u),$$

where

$$L = \{ \text{anti-symmetric forms on } V \}.$$

Show that σ is a well-defined linear isomorphism from $V^* \wedge V^*$ to L.

Exercise. Assume that $\{ e_1, \dots, e_n \}$ is a basis of V, then

$$\{ e_i \wedge e_j \mid 1 \leqslant i < j \leqslant n \}$$

is a basis of $V \wedge V$. In particular, $\dim V \wedge V = \frac{n(n-1)}{2}$.

Let X be a Riemann surface and $p \in X$. Consider

$$\bigwedge_p^2 X = T_p^*X \wedge T_p^*X.$$

Assume that $z = x + iy$ is a coordinate near p, then $\{ dx, dy \}$ or $\{ dz, d\bar{z} \}$ are a basis of T_p^*X. So $dx \wedge dy$ or $dz \wedge d\bar{z}$ are two bases of $\bigwedge_p^2 X$. Let $\wedge^2 X = \coprod_{p \in X} \wedge_p^2 X$.

> **Definition 2.4.** A 2-form on X is a map $\mu : X \to \wedge^2 X$ such that $\mu(p) \in \wedge_p^2 X, \forall p \in X$. A 2-form μ is called **smooth** if under local coordinate $z = x + iy$, it can be represented as
>
> $$\mu = a(x, y) dx \wedge dy,$$
>
> where $a(x, y)$ are smooth functions of x, y.

Let $\xi, \eta \in \Omega^2(X)$, we can define $\xi \wedge \eta \in \Omega^2(X)$ in the obvious way.

$$\Omega^*(X) = \Omega^{(0,0)}(X) \oplus \Omega^{(0,1)}(X) \oplus \Omega^{(1,0)}(X) \oplus \Omega^{(1,1)}(X)$$

Let X be a Riemann surface and $f \in C^\infty(X)$, we can define $df \in \Omega^1(X)$ as follows: for $p \in X, v \in T_pX$, $df(p) \in T_p^*X$ with

$$df(p)(v) = v(f),$$

where we view $f \in C^\infty_{X,p}$. Locally, if $z = x + iy$ is some coordinate, then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

Assume $\xi \in \Omega^1(X)$, $d\xi$ is defined as follows. Locally, in some neighborhood U

$$\xi|_U = a(x,y)dx + b(x,y)dy,$$

then

$$d\xi|_U = (\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y})dx \wedge dy,$$

or

$$d\xi|_U = d\alpha \wedge dz + d\beta \wedge d\bar{z} = (\frac{\partial \beta}{\partial z} - \frac{\partial \alpha}{\partial \bar{z}})dz \wedge d\bar{z}.$$

One can show that $d\xi$ is independent of the choice of coordinates. A simple formula

$$d(f\xi) = df \wedge \xi + fd\xi, f \in C^\infty(X), \xi \in \Omega^1(X).$$

Denote $\pi_1 : \Omega^1 \to \Omega^{(1,0)}$ and $\pi_1 : \Omega^1 \to \Omega^{(0,1)}$ to be the projections, we may define

$$\partial f = \pi_1 \circ df,$$
$$\bar{\partial} f = \pi_2 \circ df.$$

By definition,

$$df = \partial f + \bar{\partial} f.$$

Let $z = x + iy$ be a local coordinate, then locally

$$\partial f = \frac{\partial f}{\partial z}dz$$

and

$$\bar{\partial} = \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

Assume that $\xi \in \Omega^1(X)$, we have

$$\partial \xi = \frac{\partial \beta}{\partial z}dz \wedge d\bar{z}\partial\xi = -\frac{\partial \alpha}{\partial \bar{z}}dz \wedge d\bar{z}.$$

So far we have defined differential operators

$$d : \Omega^*(X) \to \Omega^{*+1},$$
$$\partial : \Omega^{*,*} \to \Omega^{*+1,*}$$
$$\bar{\partial} : \Omega^{*,*} \to \Omega^{*,*+1}.$$

on $\Omega^*(X)$

Let X, Y be two Riemann surfaces, $f : X \to Y$ be a smooth map. Then f induces a $\mathbb{C}$-algebra homomorphism $f^* : C^\infty_{Y,q} \to C^\infty_{X,p}, \phi \mapsto \phi \circ \phi$. f^* induces a linear map $df_p : T_pX \to T_qY$ such that

$$df_p(v)(\phi) = v(\phi \circ f), \phi \in C^\infty_{Y,q},$$

and further induces a linear map

$$f^* : T_q^*Y \to T_p^*X.$$

Locally, assume $z = x + iy$ be a coordinate on X near p, $w = u + iv$ be a coordinate on Y near q,

$$df_p(\frac{\partial}{\partial z}) = \frac{\partial w}{\partial z}\frac{\partial}{\partial w} + \frac{\partial \bar{w}}{\partial z}\frac{\partial}{\partial \bar{w}}$$
$$df_p(\frac{\partial}{\partial \bar{z}}) = \frac{\partial w}{\partial \bar{z}}\frac{\partial}{\partial \bar{w}}$$

and

$$f^*(dw) = \frac{\partial w}{\partial z}dz + \frac{\partial w}{\partial \bar{z}}d\bar{z}$$
$$f^*(d\bar{w}) = \frac{\partial \bar{w}}{\partial z}dz + \frac{\partial \bar{w}}{\partial z}d\bar{z}.$$

If f is holomorphic, then df_p and f^* preserves types.

Some properties of pullback of differential forms

1. $f^*(\xi \wedge \eta) f^*(\xi) \wedge f^*(\eta)$

2. $f^*(\Omega^i(X)) \subseteq \Omega^i(X)$
3. $f^*(d\xi) = df^*(\xi)$
4. If f is holomorphic, $f^*(\Omega^{(p,q)}(X) \subset \Omega^{(p,q)}$,
5. If f is holomorphic, $f^*(\bar{\partial}\xi) = \bar{\partial} f^*(\xi)$.

2.4 Integration on Riemann Surfaces

Integration of functions: assume that $f \in C^\infty(X)$ and $p_1, \dots, p_n \in X$, $a_1, \dots, a_n \in \mathbb{C}$, then the integration of f on the formal sum $a_1p_1 + \cdots a_np_n$ is defined to be

$$\int_{a_1p_1+\cdots+a_np_n} f = \sum_{i=1}^{n} a_i f(p_i).$$

Integration of 1-forms: assume that ξ is a 1-form and $\gamma : [a, b] \to X$ is a smooth curve, then we define

$$\int_\gamma \xi = \int_a^b \gamma^*(\xi).$$

In general, if $\gamma_1, \dots, \gamma_1$ are smooth curves on X, $a_1, \dots, a_n \in \mathbb{C}$, then

$$\int_{a_1\gamma_1+\cdots+a_n\gamma_n} \xi \coloneqq \sum_{i=1}^{n} a_i \int_{\gamma_i} \xi.$$

Integration of 2-forms: assume that μ is a 2-form on X and $\text{supp}(\mu)$ is compact. If $\text{supp}(\mu) \in U$ for some coordinate (U, z) of X, and $\mu = a(x, y)dx \wedge dy$, then we define

$$\int_X \mu = \int_U \mu = \int_U a(x, y)dx \wedge dy.$$

In general, we choose a locally finite coordinate chart $\{(U_i, z_i)\}_{i\in I}$ of X and choose a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to the open cover $\{U_i\}_{i\in I}$, then we define

$$\int_X \mu = \int_X \sum_i \rho_i \mu = \sum_{i\in I} \int_{U_i} \rho_i \mu$$

Remark. Assume that D is a relatively compact open subset in X, and μ is a 2-from possibly compactly supported, then $\int_D \mu$ can be defined similarly.

Theorem 2.2 (Stokes)**.** Assume that D is a relatively compact open set of X, which has piecewise smooth boundary. Let ξ be a 1-form, then we have

$$\int_{\partial D} \xi = \int_D d\xi.$$

3 Topology of Riemann Surfaces

3.1 Monodromy Theorem

Lemma 3.1. Assume that $\xi \in \Omega^1(D)$ where $D = \{z \in \mathbb{C} | |z| < 1\}$. Then there exists some $f \in C^\infty(D)$ such that $df = \xi$.

Proof. Define a function f on D as

$$f(z) = \int_{L_{az}} \xi,$$

where $a \in D$ is a given point and L_{az} is the line segment from a to z.
By Green's Formula, one could show that $f \in C^\infty(D)$ and $df = \xi$. □

Lemma 3.2. Assume $\xi \in \Omega^1(X)$ is closed, and $\gamma : [0, 1] \to D$ is a piece-wise smooth curve, then

$$\int_\gamma \xi = f(\gamma(1)) - f(\gamma(0))$$

depends only on $\gamma(0)$ and $\gamma(1)$, where $f \in C^\infty(X)$ satisfies $df = \xi$.

Proof. By the previous Lemma, there is an $f \in C^\infty(D)$ such that $df = \xi$. We may assume that γ is smooth, then

$$\int_\gamma \xi = \int_\gamma df = \int_0^1 df(\gamma(t)) = f(\gamma(1)) - f(\gamma(0)).$$

□

Definition 3.1. Assume $\xi \in \Omega^1(D)$ is closed and $\gamma : [0, 1] \to D$ be a continuous curve, then we define

$$\int_\gamma \xi := f(\gamma(1)) - f(\gamma(0)).$$

By the previous lemma, this definition makes sense.

Lemma 3.3. Let X be a Riemann surface, $\gamma_0 : [0,1] \to X$ be a continuous curve. Then there is an $\epsilon > 0$ such that for any two piece-wise smooth curves $\gamma, \gamma' : [0,1] \to X$ with the same start points and end points as of γ_0, and

$$d(\gamma(t), \gamma_0(t)) + d(\gamma'(t), \gamma_0(t)) < \epsilon, t \in [0,1]$$

where d is an arbitrary distance on X that is compatible with the topology of X,

$$\int_\gamma \xi = \int_{\gamma'} \xi$$

holds for any $\xi \in \Omega^1(X)$.

Proof. We choose a partition $0 = t_0 < t_1 < \cdots t_n = 1$ of $[0,1]$, and disks $D_0, D_1, D_2, \ldots, D_{n-1}$ such that $\gamma_0([t_0, t_1]) \subseteq U_0, \cdots, \gamma_0([t_{n-1}, t_n]) \subseteq U_{n-1}$. Then if ϵ is sufficiently small and γ, γ' satisfies the above condition we also have

$$\gamma([t_{i-1}, t_{i+1}]) \subseteq U_i, \gamma'([t_{i-1}, t_{i+1}]) \subseteq U_i.$$

Note that

$$\int_\gamma \xi = \sum_{i=0}^{n-1} \int_{\gamma|_{[t_i, t_{i+1}}} \xi,$$

similarly for $\int_{\gamma'} \xi$. By the above lemma, we have

$$\begin{aligned}
\int_{\gamma|_{[t_0,t_1]}} \xi &= \int_{\gamma_{[t_0,t_1]}+C_0} \xi, \\
\int_{\gamma|_{[t_1,t_2]}} \xi &= \int_{\gamma'|_{[t_1,t_2]}-C_0+C_1} \\
&\vdots \\
\int_{\gamma|_{[t_{n-1},t_n]}} \xi &= \int_{\gamma'|_{[t_{n-1},t_n]}-C_{n-1}}.
\end{aligned}$$

Taking sum of both sides of the above equalities, we have

$$\int_\gamma \xi = \int_{\gamma'} \xi.$$

$\square$

Definition 3.2. Assume that $\gamma_0 : [0,1] \to X$ is a continuous curve and $\epsilon > 0$ as in the above lemma, one could find a piece-wise smooth curve $\gamma : [0,1] \to X$ such that

$$d(\gamma(t), \gamma_0(t)) < \frac{\epsilon}{2}, t \in [0,1]. \tag{3.1}$$

For some closed $\xi \in \Omega^1(X)$, we define the integral

$$\int_{\gamma_0} \xi := \int_\gamma \xi.$$

Remark. Assume $\gamma_0' : [0,1] \to X$ be another continuous curve such that

$$d(\gamma_0'(t), \gamma_0(t)) < \frac{\epsilon}{2}, t \in [0,1]$$

we have

$$\int_{\gamma_0} \xi = \int_{\gamma_0'} \xi.$$

Theorem 3.4 (Monodromy Theorem)**.** Assume $\gamma, \gamma' : [0,1] \to X$ are two continuous curves with the same start and end points. If $\gamma \sim \gamma'$, then

$$\int_\gamma \xi = \int_\gamma' \xi$$

for all closed $\xi \in \Omega^1(X)$.

3.2 The Poincaré Duality and de Rham Isomorphism

2

Characterization of exact forms of integrals. Recall that

2 *Thu. Apr. 15*

Lemma 3.5. $\xi \in \Omega^1(X)$, then ξ is exact iff $\int_\gamma \xi = 0$ for all $[\gamma] \in H_1(X;\mathbb{Z})$.

Proof. If $\xi = df$ is exact, and $\gamma : [0,1] \to X$ is a closed curve, then we have $\int_\gamma = \int_\gamma df = f(\gamma(0)) - f(\gamma(1)) = 0$.
Conversely, we assume that $\int_\gamma \xi = 0$ for all $\gamma \in H_1(X;\mathbb{Z})$. Fix a $z_0 \in X$ and define a function f on X by $f(z) = \int_\gamma' \xi$, where γ' is a curve from z_0 to z. By assumption, $f(z)$ is independent of the choice of γ. One can show that $f \in C^\infty(X)$ and $df = \xi$. □

By the above lemma, we can define the following bilinear function

$$Q : H^1(X;\mathbb{C}) \times H_1(X;\mathbb{C}) \to \mathbb{C},$$
$$([\xi],[\gamma]) \mapsto \int_\gamma \xi.$$

Given $[\xi] \in H^1(X;\mathbb{C})$ if $Q([\xi],[\gamma]) = 0$ for all $[\gamma] \in H_1(X;\mathbb{C})$ then $[\xi] = 0$.

Our aim is to show that Q is non-degenerate.

Let X be a Riemann surface and γ be a smooth, simply connected curve in X. We then construct a closed 1-form η_γ as follows.

1. Choose a tabular neighborhood T of γ in X, and choose an orientation preserving diffeomorphism from T to $\mathbb{R} \times \gamma$.
2. Choose a function $\rho : \mathbb{R} \to \mathbb{R}$, such that

   $$\rho(t) = \begin{cases} 0, t > \epsilon \text{ or } t < 0, \\ 1, 0 \leqslant t \leqslant \frac{\epsilon}{2} \end{cases}$$

   and is smooth on $\mathbb{R} \backslash \{0\}$.
3. Define

   $$\eta_\gamma = \begin{cases} d\rho, t > 0, \\ 0, t \leqslant 0, \end{cases}$$

implying that η_γ is a smooth 1-form on $\mathbb{R}$.

We can naturally view η_γ as a 1-form on $\mathbb{R}\times\gamma$. It's also clear that $d\eta_\gamma = 0$ and $\operatorname{supp}\eta_\gamma \subset [0,\epsilon]\times\gamma$. By adjusting T via diffeomorphisms , to make $T \simeq \mathbb{R}\times\gamma$, we may view η_γ as a closed subset on X. We call η_γ the **Poincaré dual** of γ.

Lemma 3.6. Assume X, γ, η_γ as above. Then

$$\int_X \xi\wedge\eta_\gamma = \int_\gamma \xi$$

holds for any closed 1-form ξ on X.

Proof. By noting that $\operatorname{supp}\eta_\gamma \subset [0,\epsilon]\times\gamma$, we may identify T and $\mathbb{R}\times\gamma$. Thus

$$\begin{aligned}\int_X \xi\wedge\eta_\gamma &= \int_{[0,\epsilon]\times\gamma} \xi\wedge\eta_\gamma = \int_{[0,\epsilon]\times\gamma} \xi\wedge d\rho \\ &= \int_{[0,\epsilon]\times\gamma} d(-\rho\wedge\xi) = \int_{\{0\}\times\gamma} \rho\xi - \int_{\{\epsilon\}\times\gamma} \rho\xi \\ &= \int_{\{0\}\times\gamma} \xi = \int_\gamma \xi.\end{aligned}$$

□

Let X be a compact Riemann surface, and $\mu \in \Omega^2(X)$. If μ is exact, we have

$$\int_X \mu = 0.$$

Hence we can define a bilinear form on $H^1(X;\mathbb{C})$

$$\begin{aligned}W : H^1(X;\mathbb{C}) \times H^1(X;\mathbb{C}) &\to \mathbb{C}, \\ ([\xi],[\eta]) &\mapsto \int_X \xi\wedge\eta.\end{aligned}$$

Theorem 3.7 (Poincaré Duality)**.** Let X be a compact Riemann surface. Then the bilinear form

$$\begin{aligned}W : H^1(X;\mathbb{C}) \times H^1(X;\mathbb{C}) &\to \mathbb{C} \\ ([\xi],[\eta]) &\mapsto \int_X \xi\wedge\eta\end{aligned}$$

is non-degenerate.

Proof. We need to show that for a given closed ξ, if

$$\int_X \xi\wedge\eta = 0$$

holds for all closed η, then ξ is exact.
It suffices to show

$$\int_\gamma \xi = 0,$$

for all closed curve γ in X. Since there exists simply connected curves $\gamma_1,\ldots,\gamma_k$ in X

such that γ and $\gamma_1 + \cdots + \gamma_k$ are homologous. So we may assume that γ is a smooth simply connected curve.
Let η_γ be the Poincaré dual of γ. By Lemma 0.12, we have

$$\int_\gamma \xi = \int_X \xi \wedge \eta_\gamma = 0.$$

□

Remark. By the Poincaré duality, we obtain a linear isomorphism

$$\sigma : H^1(X;\mathbb{C}) \to H^1(X;\mathbb{C})^*,$$
$$\sigma([\eta])([\xi]) \mapsto \int_X \xi \wedge \eta.$$

$[\eta_\gamma] \in H^1(X;\mathbb{C})$ is completely determined by γ.
Generally, for any oriented compact manifold X of dimension n and an oriented submanifold of dimension r $M \subset X$, we can construct a closed $(n-r)$ form η_M on X such that

$$\int_M \xi = \int_X \xi \wedge \eta_M, \forall \xi \in Z^1(M)$$

$[\eta_M] \in H^{n-r}(X;\mathbb{R})$ is determined by M and is called the Poincaré dual of M in X. η_M is given by the Thom class of the normal bundle of M in X.

Recall that if X is a Riemann surface of genus g, then

$$H_1(X;\mathbb{Z}) = \bigoplus_{i=1}^{g} (\mathbb{Z}[\alpha_i] \oplus \mathbb{Z}[\beta_i]).$$

Lemma 3.8. Let X, α_i, β_i be as above and $\eta_{\alpha_i}, \eta_{\beta_i}$ be the Poincaré dual of α_i, β_i. Then we have

$$\int_{\alpha_i} \eta_{\alpha_j} = 0 = \int_{\alpha_j} \alpha_i$$

and

$$\int_{\beta_j} \eta_{\alpha_i} = \delta_{ij} = -\int_{\alpha_j} \eta_{\beta_i}.$$

Proof. [3] Let α_i' be a small perturbation of α_i such that $\alpha_i' \cap \alpha_i = \varnothing$. We may assume that the supports of $\eta_{\alpha_i'}$ and η_{α_i} lie in a neighborhood, which is small enough that

$$\mathrm{supp}\eta_{\alpha_i} \cap (\alpha_i' \cup (\cup_{j\neq i}\alpha_j) \cup (\cup_{j\neq i}\beta_j)) = \varnothing.$$

Then

$$\int_{\alpha_j} \eta_{\alpha_i} = 0, \int_{\beta_j} \eta_{\alpha_i} = 0, j \neq i.$$

Similarly, we have

$$\int_{\beta_j} \eta_{\beta_i} = 0, \forall i, j, \int_{\beta_j} \eta_{\beta_i} = 0, j \neq i.$$

Denote by $p = \beta_i \cap \alpha_i$, we choose a point $q \in \beta_i$ so as $t(q) = \epsilon$. We furthermore

[3] *Tue. Apr. 20*

take assumptions that $q = \beta_i(t_0), p = \beta_i(t_1), t_0 < t_1$. So

$$\int_{\beta_i} \eta_{\alpha_i} = \int_{\beta_i|_{[t_0,t_1]}} d\rho = \rho(\beta_i(t_1)) - \rho(\beta_i(t_0)) = 1.$$

Similarly, we can show that

$$\int_{\alpha_i} \eta_{\beta_i} = -1,$$

where the negative value of the integral represents the opposite orientations of α_i and β_i. □

Theorem 3.9. The bilinear form

$$\begin{aligned} Q : H^1(X;\mathbb{C}) \times H_1(X;\mathbb{C}) &\to \mathbb{C} \\ ([\xi],[\gamma]) &\mapsto \int_\gamma \xi \end{aligned}$$

is non-degenerate.

Hence we get a linear isomorphism

$$\tau : H_1(X;\mathbb{C}) \to H^1(X;\mathbb{C})^*$$

Proof. We have already seen that for any given $[\xi] \in H^1(X;\mathbb{C})$, provided $Q([\xi],[\gamma]) = 0$ holds for all $[\gamma] \in H_1(X;\mathbb{C})$, then $[\xi] = 0$ in $H^1(X;\mathbb{C})$.
On the other hand, for any given $[\gamma] \in H_1(X;\mathbb{C})$ such that $Q([\xi],[\gamma]) = \int_\gamma \xi = 0$, we need to show $[\gamma] = 0$ in $H_1(X;\mathbb{C})$. Since

$$H_1(X;\mathbb{C}) = \bigoplus_{i=1}^{g} (\mathbb{C}[\alpha_i] \oplus \mathbb{C}[\beta_i],$$

we may write $[\gamma]$ as

$$[\gamma] = \sum_{i=1}^{g} (r_i[\alpha_i] + s_i[\beta_i]), r_i, s_i \in \mathbb{C}.$$

So

$$Q([\eta_{\alpha_i}],[\gamma]) = \int_\gamma \eta_{\alpha_i} = s_i,$$

and

$$Q([\eta_{\beta_i}],[\gamma]) = \int_\gamma \eta_{\beta_i} = -r_i.$$

By assumption, $r_i = s_i = 0, \forall i$, so $[\gamma] = 0$ must hold in $H_1(X;\mathbb{C})$, which completes the proof. □

Remark.

$$\begin{tikzcd} H^1(X;\mathbb{C}) \arrow[rr, "\sigma"] & & H^1(X;\mathbb{C})^* \\ & H_1(X;\mathbb{C}) \arrow[ur, "\tau"] & \end{tikzcd},$$

where σ comes from Poincaré duality and τ comes from de Rham isomorphism.
Denote $p \coloneqq \sigma^{-1} \circ \tau$ and inspect it more closely. We denote by $p([\gamma]) = \eta_\gamma$, then

$\sigma(\eta_\gamma) = \tau([$
$gamma])$, *id est*,

$$\int_X \xi \wedge \eta_\gamma = \int_\gamma \xi, \forall [\xi] \in H^1(X;\mathbb{C}).$$

So if γ is a simply closed curve, η_γ is just the Poincaré duality of γ, defined as before. Generally, the notion of Poincaré duality in the non-compact case can be defined similarly.

3.3 Duality for $H^0(X;\mathbb{C})$ and $H^2(X;\mathbb{C})$

.

Theorem 3.10. Let X be a Riemann surface and $\mu \in \Omega^2(X)$ a compactly supported 2-form. Then $\int_X \mu = 0$ iff $\mu = d\xi$ for some $\xi \in \Omega^1(X)$ with compact support.
In particular, if X is compact, then we get a linear isomorphism

$$\begin{aligned} H^2(X;\mathbb{C}) &\to \mathbb{C} \\ [\mu] &\mapsto \int_X \mu. \end{aligned}$$

Therefore we have the duality

$$H^2(X;\mathbb{C}) \simeq \mathbb{C} = H^0(X;\mathbb{C})^*.$$

Exercise. Let $I = \{ (x,y) \in \mathbb{R}^2 \mid 0 < x, y < 1 \}$ and $\mu \in \Omega^2(X), \operatorname{supp}\mu \subseteq cI$ for some $c \in \mathbb{C}$. SHow that if $\int_I \mu = 0$, there exists some $\xi \in \Omega^1(I)$ with compact support , such that $\mu = d\xi$.

Exercise. Use the above exercise to show the last theorem for general non-compact Riemann surfaces.

Remark. The Poincaré duality holds for general smooth manifolds.

3.4 Intersection Theory on Riemann Surfaces

Recall that we have constructed a linear isomorphism

$$\begin{aligned} p : H_1(X;\mathbb{C}) &\xrightarrow{\sim} H^1(X;\mathbb{C}) \\ [\gamma] &\mapsto \eta_\gamma. \end{aligned}$$

The bilinear form W on $H^1(X;\mathbb{C})$, defined by

$$W([\xi],[\eta]) = \int_X \xi \wedge \eta,$$

induces a bilinear form I on $H_1(X;\mathbb{C})$ via p.
Our motivation is to determine the geometric meaning of $I(\gamma_1, \gamma_2)$.

Definition 3.3. Let X be a Riemann surface, and γ_1, γ_2 be simple, smooth and closed curves in X. We say that γ_1, γ_2 are **transverse**, if for any $p \in \gamma_1 \cap \gamma_2$ we have

$$T_pX = T_p\gamma_1 \oplus T_p\gamma_2.$$

If γ_1, γ_2 are transverse, and $p \in \gamma_1 \cap \gamma_2$. The **intersection index** of γ_1, γ_2 is defined to be

$$\operatorname{ind}_p(\gamma_1, \gamma_2) = 1,$$

if $T_p\gamma_1$ and $T_p\gamma_2$ given the orientation of X. Otherwise, we define

$$\operatorname{ind}_p(\gamma_1, \gamma_2) = -1.$$

The **intersection number** of γ_1 and γ_2 is defined as

$$\gamma_1 \cdot \gamma_2 \coloneqq \sum_{p \in \gamma_1 \cap \gamma_2} \operatorname{ind}_p(\gamma_1, \gamma_2).$$

Remark. $\gamma_1 \cap \gamma_2$ is a finite set if γ_1, γ_2 are transverse.

Example. Let $\mathbb{T}$ be a torus and γ_1, γ_2 be the well-known basis of $H_1(\mathbb{T}; \mathbb{C})$. THen

$$\gamma_1 \cdot \gamma_2 = 1.$$

Example. Let X be a compact Riemann surface of genus g. Obviously,

$$\begin{cases} \alpha_i \cdot \alpha_j = 0, i \neq j \\ \alpha_i \cdot \beta_j = \delta_{ij} = -\beta_j \cdot \alpha_i. \end{cases}$$

Theorem 3.11. Let X be a Riemann surface, and γ_1, γ_2 be two transversal smooth simply closed curves in X. Then

$$\gamma_1 \cdot \gamma_2 = \int_{\gamma_2} \eta_{\gamma_1},$$

where η_{γ_1} is the Poincaré dual of γ_1.

If X is compact, $\int_{\gamma_2} \gamma_1$ depends only on $[\gamma_1] \in H_1(X; \mathbb{Z})$, hence the intersection number $\gamma_1 \cdot \gamma_2$ depends only on $[\gamma_1], [\gamma_2] \in H_1(X; \mathbb{Z})$.

Exercise. Prove the theorem.

From the property of Poincaré duality[4], we have

[4] *Thu. Apr. 22*

$$\int_{\gamma_2} \eta_{\gamma_1} = \int_X \eta_{\gamma_1} \wedge \eta_{\gamma_2}.$$

If X is compact, we have a linear isomorphism

$$\begin{aligned} p : H_1(X; \mathbb{C}) &\to H^1(X; \mathbb{C}) \\ [\gamma] &\mapsto \eta_\gamma. \end{aligned}$$

Hence

$$\gamma_1 \cdot \gamma_2 = \int_X p([\gamma_1]) \wedge p([\gamma_2]) = \int_X \eta_{\gamma_1} \wedge \eta_{\gamma_2}$$

Assume that X is a compact Riemann surface of genus g, a basis $\{\alpha_i, \beta_i\}_{i=1}^g$ of $H_1(X; \mathbb{C})$ is called **canonical** if

$$\begin{aligned} \alpha_i \cdot \alpha_j &= 0 = \beta_i \cdot \beta_j, \\ \alpha_i \cdot \beta_j &= \delta_{ij} = -\beta_j \cdot \alpha_i. \end{aligned}$$

Theorem 3.12. Assume $\{\alpha_i, \beta_i\}_{i=1}^g$ is a canonical basis of $H_1(X;\mathbb{Z})$. For any closed $\xi, \eta \in \Omega^1(X)$, we have

$$\int_X \xi \wedge \eta = \sum_{i=1}^{g} (\int_{\alpha_i} \xi \int_{\beta_i} \eta - \int_{\alpha_i} \eta \int_{\beta_i} \xi.$$

Exercise. Show the above theorem.

Corollary 3.13 (Riemann's First Bilinear Relation)**.** Assume that η, ξ are two holomorphic 1-forms, then

$$\sum_{i=1}^{g} (\int_{\alpha_i} \xi \int_{\beta_i} \eta - \int_{\alpha_i} \eta \int_{\beta_i} \xi = 0$$

Proof. Since $\xi \wedge \eta = 0$. □

4 Harmonic Analysis on Riemann Surfaces

4.1 Weyl's Lemma

Generalized functions. Let $D \subseteq \mathbb{C}$ be a domain and $C_c^\infty(D)$ be the space of compactly supported smooth functions on D. Next we are going to endow $C_c^\infty(D)$ with a topology.

For $f_n \in C_c^\infty(D)$ and $f \in C_c^\infty(D)$ we say $f_n \to f$ if

i there exists a compact subset $K \subseteq D$ such that $\mathrm{supp} f_n \subset K$ and $\mathrm{supp} f \subset K$.

ii $\forall i, j \geqslant 0$, $\frac{\partial^{i+j} f_n}{\partial x^i \partial y^j} \to \frac{\partial^{i+j} f}{\partial x^i \partial y^j}$ uniformly on K.

$C_c^\infty(D)$ equipped with the above topology is denoted by $\mathcal{D}(D)$.

Definition 4.1. A **generalized function** on D is a continuous linear functional $f : \mathcal{D}(D) \to \mathbb{C}$.

Denote the space of all generalized functions on D by $\mathcal{D}(D)^*$. We often use different notations for $f(\phi)$, such as (f, ϕ) or $\int_D f\phi$.

Definition 4.2. Assume that $f \in \mathcal{D}(D)^*$, we may define **generalized partial derivatives** $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \in \mathcal{D}(D)$ as

$$\int_D \frac{\partial f}{\partial x}\phi := \int_D f\frac{\partial \phi}{\partial x}$$
$$\int_D \frac{\partial f}{\partial y}\phi := -\int_D f\frac{\partial \phi}{\partial y}.$$

Higher order derivatives of f can be defined inductively.

Example. Let $L^1_{loc}(D)$ be the space of Lebesgue locally integrable functions on D. For any $f \in L^1_{loc}(D)$, we define naturally a generalized function on D as

$$\mathcal{D}(D) \to \mathbb{C}, \phi \mapsto \int_D f\phi.$$

Dirac functions

$$\begin{aligned}\delta_z : \mathcal{D}(D) &\to \mathbb{C}, \\ \phi &\mapsto \phi(z).\end{aligned}$$

Definition 4.3. $f \in \mathcal{D}(D)^*$ is said to be **smooth**, if there is some $g \in C^\infty(D)$ such that $f(\phi) = \int_D g\phi, \forall \phi \in \mathcal{D}(D)$.

Theorem 4.1 (Weyl's Lemma)**.** Assume $f \in \mathcal{D}(D)^*$ satisfies $\Delta f = 0$. Then f is smooth and hence a harmonic function.

Proof. Observation: $\forall \phi \in \mathcal{D}(D)$, we need to find $g \in C^\infty(D)$ such that $f(\phi) = \int_D g\phi, \forall \phi \in \mathcal{D}(D)$.
Take $\rho \in C_c^\infty(\Delta_\epsilon), \Delta_\epsilon = \{|z| < \epsilon\}$, $\rho = 1$ near $z = 0$. Let $h = \frac{1}{2\pi}\rho \ln|z|$. So

$$\delta h = \delta_0 + \gamma,$$

where $\gamma \in C_c^\infty(\Delta_\epsilon)$.
Let $D_\epsilon = \{z \in D | d(z, \partial D) > \epsilon\}$. Let $\phi \in \mathcal{D}(D_\epsilon) \subseteq \mathcal{D}(D)$, define

$$\psi(z) = \int_D \phi(z-w)h(w)d\mu_w$$

Since h is integrable and ϕ is smooth, ψ is smooth and

$$\begin{aligned}\Delta\psi(z) &= \int_D \Delta_z\phi(z-w)h(w)d\mu_w \\ &= \Delta_w\phi(z-w)h(w)d\mu_w \\ &= \int_D \phi(z-w)\Delta_w h(w)d\mu_w \\ &= \int_D \phi(z-w)(\delta_0 + \gamma(w))d\phi_w \\ &= \phi(z) + \int_D \phi(z-w)\gamma(w)d\mu_w.\end{aligned}$$

So

$$\begin{aligned}
\int_{D_\epsilon} f\phi &= \int_{D_\epsilon} f(z)\phi(z)d\mu_z \\
&= \int_{D_\epsilon} f(z)(\Delta\psi(z) - \int_D \phi(z-w)\gamma(w)d\mu_w)d\mu_z \\
&= \int_{D_\epsilon} f\Delta\psi - \int f(z)\int \phi(z-w)\gamma(w)d\mu_w d\mu_z \\
&= -\int f(z)(\int \phi(z-w)\gamma(w)d\mu_w)d\mu_z \\
&\quad - \int f(z)(\int \phi(w)\gamma(z-w)d\mu_w)d\mu_z \\
&= \int \phi(w)(\int f(z)\gamma(z-w)d\mu_z)d\mu_w \\
&= \int \phi(w)g(w)d\mu_w \\
&= \int g\phi
\end{aligned}$$

where

$$g(w) = \int \gamma(z-w)f(z)d\mu_z.$$

By similar rationale, $g \in C_c^\infty(D)$, so we have a smooth g satisfying

$$\int_{D_\epsilon} f\phi = \int_{D_\epsilon} g\phi$$

in D_ϵ. Let $\epsilon \to 0$, the theorem follows. □

Similarly, we have

Theorem 4.2. If $f \in L^1_{loc}(D)$ and Δf is smooth, so f is smooth.

Exercise. Assume that $f \in L^1_{loc}$ satisfies $\frac{\partial f}{\partial \bar{z}} = 0$, prove that $f \in C^\infty(\mathbb{C})$ hence $f \in \mathcal{O}(D)$.

Exercise. Assume $f \in L^1_{loc}(\mathbb{R})$ satisfies $\frac{df}{dx} = 0$, prove that $f = c$ almost everywhere on $\mathbb{R}$, where c is a constant.

4.2 Harmonic Forms on Riemann Surfaces

Recall that a smooth function $u : D \to \mathbb{C}$ is **harmonic** if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Exercise. Assume that $f : D \to D'$ is a homomorphic map between two domains in $\mathbb{C}$, and $u : D' \to \mathbb{C}$ is a harmonic function, then $u \circ f : D \to \mathbb{C}$ is a harmonic function on D.

Remark. From the exercise above, we see that the definition of harmonic functions is independent of the choices of local coordinates, hence we can define harmonic functions on Riemann surfaces.
Holomorphic functions are harmonic.
Since Δ is a real operator, the real part and imaginary part of a harmonic function are harmonic.

$\star$-operator is defined as follows

$$\begin{aligned}\star : T_p^*X &\to T_p^*X\\ dx &\mapsto dy\\ dy &\mapsto -dx.\end{aligned}$$

Example. Let u be a real harmonic function on $D \subseteq \mathbb{C}$. A conjugation of u is a harmonic function v on D such that $u + iv \in \mathcal{O}(D)$, which is equivalent to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$dv = \star du.$$

The definition of $\star$-operator is independent of the choice of local coordinates.

Lemma 4.3. Assume that $f \in C^\infty(X)$, then f is harmonic iff $d \star df = 2i\partial\bar{\partial}f = 0$.

Proof. Assume $(U, z = x + iy)$ is a coordinate on U of X, locally

$$d \star df = d \star (\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy) = d(\frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx) = (\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2})dx \wedge dy,$$

hence the lemma follows. □

Definition 4.4. $\xi \in \Omega^1(X)$ is called harmonic if $d\xi = 0$ and $d \star \xi = 0$.

Exercise. Show that $\xi \in \Omega^1(X)$ iff locally ξ is the differential of a harmonic function.

We denote by $\mathcal{H}(X)$ the space of all harmonic 1-forms on X.

Since holomorphic 1-forms are locally differentials of holomorphic functions, thus all holomorphic 1-forms are harmonic.

Theorem 4.4. $\mathcal{H}(X) = \Omega(X) \oplus \overline{\Omega(X)}$,

namely, for any $\xi \in \mathcal{H}(X)$, there are $\alpha, \beta \in \Omega(X)$,such that

$$\xi = \alpha + \bar{\beta}.$$

Proof. Uniqueness is easy to show. We are now to prove the existence.
Assume $U \subseteq X$ is a coordinate disc on X, and assume $\xi|_U = df$ for some harmonic function $f = u + iv$ on U. Take $g_1, g_2 \in \mathcal{O}(U)$ such that $u = g_1 + \bar{g}_1$ and $v = g_2 + \overline{g_2}$, so

$$\xi|_U = d(g_1 + \overline{g_1} + ig_2 + i\overline{g_2} = d(g_1 + ig_2) + d\overline{(g_1 - ig_2)} = \alpha + \bar{\beta}.$$

□

4.3 L^2-Spaces

Let X be a Riemann surface. A **measurable** 1-form on X is defined to be a 1-form with measurable coefficient, *id est*, if $(U, z = x + iy)$ is a coordinate on X, then

$$\xi|_U = adx + bdy,$$

with a, b measurable functions on U.

If ξ is as above, locally

$$\xi \wedge \overline{\star\xi} = (a dx + b dy) \wedge (\overline{a} dy - \overline{b} dx) = (|a|^2 + |b|^2) dx \wedge dy.$$

We define the L^2**-norm**of ξ as

$$\|\xi\|^2 := \int_x \xi \wedge \overline{\star\xi} \leqslant +\infty.$$

Denote by $L_1^2(X)$ the space of measurable 1-forms on X with the L^2-norm, then $L_1^2(X)$ is a Hilbert space with inner product

$$(\xi, \eta) = \int_X \xi \wedge \overline{\star\eta}.$$

Exercise. Show that $(\eta, \xi) = \overline{(\xi, \eta)}$ and $(\star\xi, \star\eta) = (\xi, \eta)$.

Let E be the closure of the space $\{df | C_c^\infty(X)\}$, and $F = \star E$ be the closure of the space $\{\star df | f \in C_c^\infty(X)\}$.

Lemma 4.5. $E \perp F$.

Proof. It suffices to prove that $(df, \star dg) = 0, \forall f, g \in C_c^\infty(X)$. By definition,

$$(df, \star dg) = \int_X df \wedge \overline{\star(\star dg))} = -\int_X df \wedge dg = -\int_X d(f dg) = 0.$$

□

Let $H(X) = E^\perp \cap F^\perp = (E \oplus F)^\perp$. Then we get an orthogonal, decomposition

$$L_1^2 = H(X) \oplus E \oplus F.$$

For $\xi \in L_1^2(X)$, we define $d\xi$ as a linear functional

$$\begin{aligned} d\xi : C_c^\infty(X) &\to \mathbb{C}, \\ \phi &\mapsto \int_X \phi \wedge d\xi = -\int_X d\phi \wedge \xi. \end{aligned}$$

We can view $d\xi$ as generalized function on X.

Lemma 4.6. If $\xi \in E$ is smooth, then there exists $f \in C^\infty(X)$ such that $\xi = df$.

Proof. We will show that

$$\int_\gamma \xi = 0$$

for any closed curve $\gamma \subseteq X$.

$$\int_\gamma \xi = \int_X \xi \wedge \eta_\gamma = -(\xi, \star\eta_\gamma).$$

By definition of E, there exist $f_n \in C_c^\infty(X)$ such that $df_n \to \xi$ in $L_1^2(X)$. We just need to show

$$(df_n, \star\eta_\gamma) = -\int_X df_n \wedge \eta_\gamma = -\int_X d(f_n\eta_\gamma) = 0.$$

□

Lemma 4.7. $\xi \in L_1^2(X)$ then

(i) $\xi \in E^\perp \iff d \star \xi = 0$.

(ii) $\xi \in F^\perp \iff d\xi = 0$.

In particular, $\xi \in \mathcal{H}(X)$ iff $d\xi = 0$ and $d \star \xi = 0$.

Proof. Easy. □

4.4 Hodge Decomposition on Riemann Surfaces

[5] If $\xi \in L_1^2(X)$, then there exists a unique $h \in H$ and $\alpha, \beta \in E$ such that

[5] *Tue. Apr. 27*

$$\xi = h + \alpha + \star\beta.$$

We want to show ξ smooth implies h, α, β smooth.

Theorem 4.8. Assume $\xi \in L_1^2(X)$.

(i) $d\xi = 0, d \star \xi = 0 \implies \xi$ is smooth.

(ii) more generally, if $d\xi, d \star \xi$ are both smooth, then ξ is smooth.

Proof. Let $(U, z = xiy)$ be an arbitrary coordinate on X, we prove that $\xi|_U$ is smooth. Write ξ on U as

$$\xi|_U = p\,dx + q\,dy,$$

where p, q are measurable functions on U. We need to show that p, q are smooth. Given any $\phi \in C_c^\infty(U)$, by assumption

$$\int_U d\phi_x \wedge \xi = 0$$
$$\int_U d\phi_y \wedge \star\xi = 0.$$

And further

$$d\phi_x \wedge \xi = (\phi_{xx}dx + \phi_{xy}dy) \wedge (p\,dx + q\,dy) = (\phi_{xx}q - \phi_{xy}p)dx \wedge dy,$$

and

$$d\phi_y \wedge \star\xi = (\phi_{xy}dx + \phi_{yy}dy) \wedge (p\,dy - q\,dx) = (\phi_{xy}p + \phi_{yy}q)dx \wedge dy.$$

So

$$0 = \int_U d\phi_x \wedge \xi + \int_U d\phi_y \wedge \star\xi = \int_U (\phi_{xx} + \phi_{yy})q = \int_U \Delta\phi q,$$

which shows that $\Delta q = 0$. By Weyl's Lemma, q is smooth.

Exercise. Show that p is smooth.

Thus $\xi|_U$ is smooth.

The second assertion follows from the same argument as in the proof of the first one. □

Theorem 4.9. Let X be a Riemann surface and $\xi \in L^2_1(X)$. If ξ is smooth, then there exists a smooth harmonic form $h \in H$, $f, g \in C^\infty(X)$ such that $\xi = h + df + \star dg$ and $df, dg \in E$.

Proof. From the decomposition

$$L^2_1(X) = H \oplus E \oplus F.$$

There exists $h \in H$ and $\alpha, \beta \in E$ such that

$$\xi = h + \alpha + \star\beta.$$

From the above theorem, we know that h is smooth. On the other hand, $\alpha, \beta \in F^\perp$

$$d\alpha = d\beta = 0$$

showing that $d \star \beta = d\xi$ is smooth. From the above theorem, β is smooth so $\alpha = \xi - h \star \beta$ is smooth.
By some lemma, there exist $f, g \in C^\infty$ such that $\alpha = df, \beta = dg$.

□

Theorem 4.10. Assume that X is a compact Riemann surface, then we have the following orthogonal decomposition

$$\Omega^1(X) = \mathcal{H}(X) \oplus B^1(X) \oplus \star B^1(X).$$

By the above theorem, the map

$$\begin{aligned} \mathcal{H}(X) &\to H^1(X; \mathbb{C}), \\ \xi &\mapsto [\xi]. \end{aligned}$$

So we can identify $H^1(X; \mathbb{C})$ and $\mathcal{H}(X)$. On the other hand, we have seen that

$$\mathcal{H}(X) = \Omega(X) \oplus \overline{\Omega(X)}.$$

Recall that

$$\dim H^1(X; \mathbb{C}) = \dim H_1(X; \mathbb{C}) = 2g.$$

So we get

Theorem 4.11. Let X be a compact Riemann surface of genus g, then we have a natural linear isomorphism

$$H^1(X; \mathbb{C}) \simeq \Omega(X) \oplus \overline{\Omega(X)}.$$

In particular,

$$\dim \Omega(X) = g.$$

Let X be a compact Riemann surface of genus g. We defined the following invariants:

(i) de Rham cohomology $H^0(X;\mathbb{C}), H^1(X;\mathbb{C}), H^2(X;\mathbb{C})$

(ii) Dolbeault cohomology $H^{0,0}(X), H^{0,1}(X), H^{1,0}(X), H^{1,1}(X)$

What we concern is that whether there is a relation

$$H^i(X;\mathbb{C}) = \oplus_{p+q=i} H^{p,q}(X)$$

between the two types of invariants.

Theorem 4.12 ($\partial\bar{\partial}$-Lemma)**.** Assume that X is a Riemann surface (may not be compact) and μ be a 2-form with compact support. If $\int_X \mu = 0$, the n

$$\mu = d \star df = 2i\partial\bar{\partial} f$$

for some $f \in C^\infty(X)$.

Proof. $\int_X \mu = 0$ forces $\mu = d\xi$ for some $\xi \in \mathcal{E}^1(X)$ with compact support. By the above theorem, there exists $h \in \mathcal{H}$, $f, g \in C^\infty(X)$ such that

$$\xi = h + df + \star dg.$$

□

Lemma 4.13. Let X be a compact Riemann surface, then the following map

$$\begin{aligned} \sigma : \overline{\Omega(X)} &\to H^{0,1}(X) \\ \xi &\mapsto [\xi] \end{aligned}$$

is a linear isomorphism.

Proof. Firstly, we show that σ is surjective. Assume that $\xi \in \mathcal{E}^{0,1}(X)$, then

$$d\xi = \partial\xi \in \mathcal{E}^{1,1}(X) \implies$$

$$\int_X \partial\xi = \int_X d\xi = 0.$$

By the $\partial\bar{\partial}$-Lemma, there exists $f \in C^\infty(X)$ such that

$$\partial\xi = \partial\bar{\partial} f$$

so

$$\partial(\xi - \bar{\partial} f) = 0.$$

Let $\alpha := \xi - \bar{\partial} f \in \overline{\Omega(X)}$, we have $\sigma(\alpha) = [\xi]$, so it is surjective.
Secondly, we show that σ is injective. Assume $\xi \in \Omega(X)$ such that $\sigma(\xi) = [\xi] = 0$.

Then $\bar{\xi} = \bar{\partial}\bar{f}$ for some $f \in C^\infty$.

$$\begin{aligned}
\int_X i\xi \wedge \bar{\xi} &= \int_X i\partial f \wedge \bar{\partial}\bar{f} \\
&= i\int_X (\partial f + \bar{\partial} f) \wedge \bar{\partial}\bar{f} \\
&= i\int_X df \wedge \bar{\partial}\bar{f} \\
&= i\int_X d(f\bar{\partial}\bar{f}) - i\int_X d(\bar{\partial}\bar{f}) \\
&= 0 - i\int_X f\partial\bar{\partial} f = 0
\end{aligned}$$

On the other hand, we have seen that

$$H^1(X;\mathbb{C}) = \Omega(X) \oplus \overline{\Omega(X)}, H^{1,0}(X) = \Omega(X), H^{0,1}(X) = \overline{\Omega(X)}.$$

□

Theorem 4.14. Let X be a compact Riemann surface, then we have natural isomorphisms

$$\begin{aligned}
&H^0(X;\mathbb{C}) \simeq H^{0,0}(X;\mathbb{C}) \\
&H^1(X;\mathbb{C}) \simeq H^{(0,1)}(X;\mathbb{C}) \oplus H^{0,1}(X;\mathbb{C}) \\
&H^2(X;\mathbb{C}) \simeq H^{1,1}(X;\mathbb{C}).
\end{aligned}$$

Exercise. Construct a natural linear isomorphism between $H^2(X;\mathbb{C})$ and $H^{1,1}(X)$.

4.5 $H^{0,1}(X)$ and $\bar{\partial}$-Equation on Riemann Surfaces

Problem (the First Cousin Problem)**.** Let a $A \coloneqq \{p_n\}_{n\geqslant 1}$ be a discrete set in X, and let U_n be neighborhoods of p_n such that $U_n \cap U_m = \varnothing$ for $n \neq m$. Given a family of functions $\phi_n \in \mathcal{M}(U_n) \cap \mathcal{O}(U_n \backslash \{p_n\}), n \geqslant 1$, can one find $f \in \mathcal{M}(X) \cap \mathcal{O}(X\backslash A)$ such that $f|_{U_n} - \phi_n \in \mathcal{O}(U_n)$ for all n?

The idea of finding a solution of the above problem is as follows. We first construct a smooth function on $X\backslash A$ with the given principal part near the points in A, and find the required meromorphic function by solving a $\bar{\partial}$-equation. The solvability of $\bar{\partial}$-equation depends on $H^{0,1}(X)$. So the existence of solutions to Problem 4.5 is encoded in the group $H^{p,q}(X)$.

We consider the simplest case $A = \{p\}$ for demonstration use.

Step 1 Let (U, z) be a coordinate neighborhood near p with $z(p) = 0$.

Step 2 Take a smooth function $\rho \in C^\infty(X)$ such that $\rho = 1$ near p and $\operatorname{supp}(\rho)$. Then we define

$$\phi_i \coloneqq \frac{\rho}{z^i}$$

for all $1 \leqslant i \leqslant n$, where n is a sufficiently large integer such that $\dim H^{0,1}(X) < n$.

Step 3 Let $\xi_i \coloneqq \bar{\partial}\phi_i$. Note that $\xi_i = 0$ near p and has support in $\operatorname{supp}(\rho)$, so $\xi_i \in \mathcal{E}^{0,1}(X)$ is a well-defined smooth $(0,1)$-form on X. $[\xi_i] \in H^{0,1}(X)$ denote the image of all ξ_i for $1 \leqslant i \leqslant n$.

Step 4 Since $\dim H^{0,1}(X) < n$, there exists constants $a_1, \ldots, a_n$ not simultaneously vanish such that

$$0a_1[\xi_1] + \cdots + a_n[\xi_n] = [a_1\xi_1 + \cdots + a_n\xi_n] \in H^{0,1}(X).$$

$a_1\xi_1 + \cdots + a_n\xi_n$ is thus $\bar{\partial}$-exact.

Step 5 Therefore there is a smooth function g satisfying $\bar{\partial}g = \sum_{i=1}^n a_i\xi_i = \bar{\partial}(\sum_{i=1}^n a_i\phi_i)$. Hence

$$\bar{\partial}(\sum_{i=1}^n a_i\phi_i - g) = 0$$

holds on $X \backslash \{p\}$.

Step 6 Set

$$f := \sum_{i=1}^n a_i\phi_i - g,$$

then f is holomorphic on $X \backslash \{p\}$, hence is meromorphic on X with principal part

$$\frac{a_n}{z^n} + \cdots + \frac{a_1}{z}$$

at p.

From the above discussion we get the following

Proposition 4.15. Let X be a Riemann surface (may be non-compact) and let $p \in X$ be an arbitrary point, then

1. if $\dim H^{0,1}(X) < n$, then there exists non-constant $f \in \mathcal{O}(X \backslash \{p\}) \cap \mathcal{M}(X)$, such that $\mathrm{ord}_p f \geqslant -n$;
2. if X is compact with genus g, then there exists non-constant $f \in \mathcal{O}(X \backslash \{p\}) \cap \mathcal{M}(X)$, such that $\mathrm{ord}_p f \geqslant -(g+1)$.

Remark. When X is compact, the above proposition can be easily proved by Riemann-Roch. But as there is no restriction on the compactness of X, the above proposition is still non-trivial.

Proposition 4.16. Let ξ be a meromorphic 1-form on a compact Riemann surface X, then

$$\sum_p \mathrm{Res}_p \xi = 0,$$

with p running over the poles of ξ.

Proof. Let $p_1, \ldots, p_r$ be all poles of ξ. Then we choose closed small coordinate disks D_i around p_i and denote $\gamma_i = \partial D_i$, for all $1 \leqslant i \leqslant r$. Then we let $\tilde{X} := X \backslash (\cup_i D_i)$, so

$\partial\tilde{X} = -\gamma_1 - \cdots - \gamma_r$. By the very definition of residues of ξ,

$$\begin{aligned}\sum_{i=1}^{r} \mathrm{Res}_{p_i}\xi &= \sum_{i=1}^{r} \frac{1}{2\pi i}\int_{\gamma_i} \xi \\ &= -\frac{1}{2\pi i}\int_{\partial\tilde{X}} \xi \\ &= -\frac{1}{2\pi i}\int_{\tilde{X}} d\xi \\ &= 0,\end{aligned}$$

where the last equality holds by dimensional reason, since $d\xi$ is a holomorphic 2-form but $\tilde{X}$ is a 1-dimensional complex manifold. □

Theorem 4.17. Let X be a Riemann surface, and let $p_1, \ldots, p_r \in X$ be distinct points in X and $D_1, \ldots, D_r$ be disjoint neighborhoods of $p_1, \ldots, p_r$. Assume $\xi_i \in \Omega(D_i \backslash \{p_i\}) \cap \mathcal{M}^1(D_i)$ satisfy the condition

$$\sum_{i=1}^{r} \mathrm{Res}_{p_i}\xi_i = 0.$$

Then there exists a meromorphic form $\xi \in \Omega(X\backslash\{p_1, \ldots, p_r\}) \cap \mathcal{M}^1(X)$ such that $\xi|_{D_i} - \xi_i \in \Omega(D_i)$ for all $1 \leqslant i \leqslant r$.

5 Sheaf and Sheaf Cohomology

5.1 Basic Definitions and Examples

Definition 5.1. A sheaf $\mathcal{F}$ of abelian groups on a topological space X consists of a family of abelian groups $\{\mathcal{F}(U) | U \subset X \text{ open}\}$ and group morphisms

$$\mathrm{res}^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$$

such that $\mathrm{res}^V_W \circ \mathrm{res}^U_V = \mathrm{res}^U_W : \mathcal{F}(U) \to \mathcal{F}(W)$ for $W \subset V \subset U$ and satisfy gluing properties.

Cohomology groups are presheaves, but not sheaves in general.

Example. Let A be an abelian group. We can define a sheaf A_X on a topological space as follows:

$$A_X(U) := \{ f : U \to A \mid f \text{ is locally constant} \}.$$

For $V \subset U$ open, the map

$$\mathrm{res}^U_V : A_X(U) \to A_X(V)$$

is given by restriction of the functions.
G_X is called the **locally constant sheaf** on X with values in G.

Example. Let X be a topological space, $p \in X$. We can define a sheaf $\mathbb{C}_p$ on X as follows

$$\mathbb{C}_p(U) = \begin{cases} 0, p \notin U \\ \mathbb{C}, p \in U. \end{cases}$$

For $V \subset U$ open

$$\mathrm{res}^U_V = \begin{cases} \mathrm{id}_{\mathbb{C}}, p \in V \\ 0, p \notin V. \end{cases}$$

$\mathbb{C}$ is called the skyscraper sheaf on X.

5.2 Morphism of Sheaves

Given a morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves, for $U \subset X$ open, we define

$$\ker f(U) \coloneqq \ker(f_U : \mathcal{F}(U) \to \mathcal{G}(U)).$$

Exercise. Show that $\ker f$ defined above is a sheaf on X, and $(\ker f)_x = \ker(f_x : \mathcal{F}_x \to \mathcal{G}_x)$.

Remark. We may similarly define $\operatorname{im} f$, but unfortunately, $\operatorname{im} f$ defined in this way may be not a sheaf in general.

Definition 5.2. A sequence of of sheaves

$$\cdots \to \mathcal{F}_{i-1}\mathcal{F}_i \to \mathcal{F}_{i+1} \to \cdots$$

is called exact if

$$\cdots \to (\mathcal{F}_{i-1})_x \to (\mathcal{F}_i)_x \to (\mathcal{F}_{i+1})_x \to \cdots$$

is exact.

Remark. A sequence

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0.$$

is exact iff

(i) $\forall U \subset X$ open, $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.

(ii) $\forall U \subset X$ open, $s \in \mathcal{G}(U)$ $g_U(s) = 0$ iff $s = f_U(s')$ for some $s' \in \mathcal{F}(U)$.

(iii) $\forall U \subset X$ open, $s \in \mathcal{H}(U)$, there exists an open cover $\{U_i\}_{i \in I}$ of u and $s_i \in \mathcal{G}(U_i)$ such that $f_{U_i} = s|_{U_i}, \forall i \in I$.

However, in general $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is not surjective, in particular, $f_X : \mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective.

Example. Consider the morphism of sheaves

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}^*_X \to 0,$$

where $\mathbb{Z}_X$ is the locally constant sheaf with values in $\mathbb{Z}$, $\mathcal{O}$ is the sheaf of holomorphic functions, and $\mathcal{O}^*$ is the sheaf of non-vanishing holomorphic functions. $\exp_U : \mathcal{O}(U) \to \mathcal{O}^*(U)$ is not surjective if

$$U \simeq \left\{ z \in \mathbb{C} \;\middle|\; \frac{1}{2} \in |z| < 1 \right\}.$$

Exercise. Let $h \in C^\infty_c(\Delta)$, show that there exits $f \in C^\infty(\Delta)$ such that $\partial f / \partial \bar{z} = h$,*id est*, $\bar{\partial} f = h d\bar{z}$.

Prove the exactness of the sequences

$$0 \to \mathcal{O} \to \mathcal{E}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1} \to 0,$$

and

$$0 \to \Omega \to \mathcal{E}^{0,1} \xrightarrow{\partial} \mathcal{E}^{1,1} \to 0.$$

5.3 Cohomology of Sheaves

Recall the first Cousin problem: Let X be a Riemann surface, and $\{U_i\}$ an open cover of X. Assume $f_i \in \mathcal{M}(U_i)$ such that $g_{ij} := f_i - f_j \in \mathcal{O}(U_i \cap U_j), \forall i, j$. Can we find $f \in \mathcal{M}(X)$ such that $f - f_i \in \mathcal{O}(U_i)$.

Some observations: $\{g_{ij}\}$ satisfies the following cocycle condition

$$g_{ij} + g_{jk} = g_{ik}.$$

If f is a solution of the above problem, we let $g_i = f - f_i \in \mathcal{O}(U_i), i \in I$. Then

$$g_{ij} = g_j - g_i, \forall i, j.$$

On the other hand, assume there exists $g_i \in \mathcal{O}(U_i)$ such that $g_{ij} = g_j - g_i$, then we have

$$g_j - g_i = f_i - f_j$$

on $U_i \cap U_j$, which is equivalent to

$$f_i + g_i = f_j + g_j.$$

Hence we get $f \in \mathcal{M}(X)$ such that

$$f|_{U_i} = f_i + g_i$$

and $f - f_i = g_i \in \mathcal{O}(U_i)$, so f is a solution of the above problem.

So given data $\{ f_i \in \mathcal{M}(U_i) \}$ with $g_{ij} = f_i - f_j$, the first Cousin problem is solvable iff there are $\{ g_i \in \mathcal{O}(U_i) \}$ such that $g_{ij} = g_j - g_i$.

Let $\mathcal{F}$ be a sheaf on X.

Definition 5.3. Let $\mathfrak{U} = \{ U_i \}$ be an open cover of X. The qth cochain group of $\mathcal{F}$ associated to the open cover $\{ U \}$ is

$$C^q(\mathfrak{U}, \mathcal{F}) := \prod_{(i_0, i_1, \ldots, i_q) \in I^{q+1}} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_q}).$$

If $f \in C^q(\mathfrak{U}, \mathcal{F})$, it is usually written as $f = (f_{i_0 i_1 \ldots i_q})$ with $f_{i_0 i_1 \ldots i_q} \in \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_q})$. Obviously, $C^q(\mathfrak{U}, \mathcal{F})$ is an abelian group.

We define

$$\delta^q : C^q(\mathfrak{U}, \mathcal{F}) \to C^{q+1}(\mathfrak{U}, \mathcal{F})$$

explicitly. For any $f = (f_{i_0 i_1 \ldots i_q}) \in C^q(\mathfrak{U}, \mathcal{F})$, we define

$$(\delta^q f)_{i_0 i_1 \ldots i_{q+1}} = \sum_{r=0}^{q+1} (-1)^r f_{i_0 \ldots \hat{i}_r \ldots i_{q+1}}.$$

It's a straightforward exercise to verify that $\delta^{q+1} \circ \delta^q = 0$ for all $q \geqslant 0$.

Definition 5.4. Let $\mathfrak{U} = \{ U_i \mid i \in I \}$ and $\mathfrak{V} = \{ V_j \mid j \in J \}$ be two open covers of X. We say that $\mathfrak{V}$ **finer** than $\mathfrak{U}$, denoted by $\mathfrak{V} < \mathfrak{U}$, if there is a map $\tau : J \to I$ such that $V_j \subseteq U_{\tau(j)}$ for all $j \in J$. Such a map is called a **refining map**.

Assume that $\mathfrak{V} < \mathfrak{U}$ and $\tau : J \to I$ is a refining map. Then we have a group homomorphism

$$\tau^{\mathfrak{U}}_{\mathfrak{V}} : C^q(\mathfrak{U}, \mathcal{F}) \to C^q(\mathfrak{V}, \mathcal{F})$$

f on the cochains, given by

$$\tau^{\mathfrak{U}}_{\mathfrak{V}}(f) = (f_{\tau(j_0)\dots\tau(j_q)}|_{V_{j_0}\cap\dots V_{j_q}}).$$

for any $f = (f_{i_0\dots i_q})$ Assembling these homomorphisms degree-wise we get cochain map

$$\tau^{\mathfrak{U}}_{\mathfrak{V}} : C^*(\mathfrak{U}, \mathcal{F}) \to C^*(\mathfrak{V}, \mathcal{V})$$

between Čech complexes.

5.4 De Rham-Weil Isomorphism and Dolbeault Isomorphism

Theorem 5.1 (de Rham-Weil)**.** Let X be a Riemann surface, then we have $H^1(X; \mathbb{C}_X) \simeq H^1(X; \mathbb{C})$.

Theorem 5.2 (Dolbeault)**.** Let X be a Riemann surface, then we have the following isomorphism of cohomology groups

$$H^1(X; \mathcal{O}) \simeq H^{0,1}(X),$$
$$H^1(X; \Omega) \simeq H^{1,1}(X).$$

In particular, if X is compact, then $H^1(X; \Omega) \simeq H^2(X; \mathbb{C}) \simeq \mathbb{C}$.

6 Some Results about Compact Riemann Surfaces

6.1 The Riemann-Roch Theorem

Definition 6.1. The **divisor group** $\mathrm{Div}(X)$ is the free abelian group generated by points in X:

$$\mathrm{Div}(X) = \{ m_1p_1 + \dots + m_rp_r \mid r \geqslant 0, m_i \in \mathbb{Z}, p_i \in X \}. \tag{6.1}$$

For a divisor $D = m_1p_1 + \dots + m_rp_r \in \mathrm{Div}(X)$, the **degree** of X is defined to be

$$\deg D = m_1 + \dots + m_r \in \mathbb{Z}.$$

We use the notation $\mathrm{Div}(X)_0$ for the subgroup

$$\mathrm{Div}(X)_0 := \{ D \in \mathrm{Div}(X) \mid \deg D = 0 \}.$$

Definition 6.2. Let $D = m_1p_1 + \dots + m_rp_r \in \mathrm{Div}(X)$. Then

(i) if $m_i \geqslant 0$ for all $1 \leqslant i \leqslant r$, we call D to be an **effective divisor**, which is often denoted by $D \geqslant 0$;

(ii) if there exists a meromorphic function $f \in \mathcal{M}(X)$ such that $D = (f)$, then we say that D is a **principal divisor**. The subgroup of principal divisors on X is often denoted by $\mathrm{Div}(X)_P$;

(iii) two divisors $D, D' \in \mathrm{Div}(X)$ are called **linearly equivalent**, often denoted by $D \sim D'$, if $D - D'$ is principal.

Note that if $D = (f)$ is principal, then $\deg(f) = 0$, since the numbers of poles and zeros of a meromorphic function f on a compact Riemann surface X are equal, counting multiplicity.[6] Thus there is a natural inclusion $\mathrm{Div}(X)_P \subseteq \mathrm{Div}(X)_0$. The quotient group

$$\mathrm{Pic}(X) \coloneqq \mathrm{Div}_0(X)/\mathrm{Div}_P(X)$$

is called by the **Picard group** of X.

[6] *Needs more explanation.*

For a divisor $D \in \mathrm{Div}(X)$, we can associate a sheaf $\mathcal{O}_D$ on X as follows. For any open subset $U \subseteq X$, we define

$$\mathcal{O}_D(U) \coloneqq \{ f \in \mathcal{M}(X) \mid \mathrm{ord}_p f + D(p) \geqslant 0, \forall p \in U \}$$

Theorem 6.1 (Riemann-Roch). Let X be a compact Riemann surface of genus g and $D \in \mathrm{Div}(X)$ be a divisor, then

$$h^0(X, \mathcal{O}_D) - h^1(X, \mathcal{O}_D) = \deg D - g + 1. \tag{6.2}$$

Proof. For the special case $D = 0$, see [For]. ☐

Definition 6.3 (canonical divisors). A divisor $K \in \mathrm{Div}(X)$ is said to be **canonical** if $K = (\xi)$ for some $\xi \in \mathcal{M}^1(X)$.

By some previous theorem [7], there are always meromorphic 1-forms on X, so X always admits canonical divisors.

[7] *To do.*

If ξ_1, ξ_2 are two meromorphic 1-forms, their quotient $f \coloneqq \xi_1/\xi_2$ is a well-defined meromorphic function on X. Thus $(\xi_1) = (f) + (\xi_2)$, showing that the difference of any two canonical divisors is principal, or equivalently, any two canonical divisors are linearly equivalent.

Given a divisor $D \in \mathrm{Div}(X)$, we can associate another sheaf Ω_D to D, as follows. For any open subset $U \subseteq X$, we define

$$\Omega_D(U) \coloneqq \left\{ \xi \in \mathcal{M}^1(U) \;\middle|\; \mathrm{ord}_p(\xi) + D(p) \geqslant 0, \forall p \in U \right\}. \tag{6.3}$$

Theorem 6.2 (Serre duality). Let $D \in \mathrm{Div}(X)$ be a divisor, then we have a linear isomorphism

$$H^1(X; \mathcal{O}_D)^* \simeq H^0(X; \Omega_{-D}) \tag{6.4}$$

of $\mathbb{C}$-linear spaces.

With the help of Serre duality, the Riemann-Roch Theorem 6.1 can be restated as

Corollary 6.3. Let X be a compact Riemann surface of genus g and $D \in \mathrm{Div}(X)$ be a divisor. Then

$$h^0(X, \mathcal{O}_D) - h^0(X, \Omega_{-D}) = \deg D + 1 - g. \tag{6.5}$$

Corollary 6.4. Let X be a compact Riemann surface of genus g and $K \in \mathrm{Div}(X)$ be a canonical divisor. Then

$$\deg K = 2g - 2. \tag{6.6}$$

Proof. By Serre duality we have

$$h^0(X, \mathcal{O}_K) = h^0(X, \Omega) = \dim \Omega(X) = g$$

and

$$h^0(X, \Omega_{-K}) = h^0(X, \mathcal{O}) = \dim \mathcal{O}(X) = 1.$$

So

$$h^0(X, \mathcal{O}_K) - h^0(X, \Omega_{-K}) = g - 1 = \deg K + 1 - g,$$

by Riemann-Roch. Thus $\deg K = 2g - 2$, as desired. □

Corollary 6.5. Assume that X is a compact Riemann surface of genus 0, then X must be isomorphic to $\mathbb{P}^1$.

Proof. Let $p \in X$ be any point of X and let $D = p \in \mathrm{Div}(X)$. By Riemann-Roch,

$$\dim \mathcal{O}_D(X) \coloneqq h^0(X, \mathcal{O}_D) = h^0(X, \Omega_{-D}) + 1 - g + \deg D \geqslant 1 - g + \deg D = 1 - 0 + 1 = 2,$$

showing that $\mathcal{O}_D(X)$ is not empty. Note that $\mathcal{O}_D$ is the sheaf of meromorphic function with exactly one pole p such that $\mathrm{ord}_p f \geqslant -1$. We claim that there must exist some f such that $\mathrm{ord}_p f = -1$. If this were the case, then by Proposition 1.5 X is isomorphic to $\mathbb{P}^1$. Otherwise, all $f \in \mathcal{O}_D$ are holomorphic, thus $\mathcal{O}_D \subseteq \mathcal{O}$. However, as $\dim \mathcal{O}_D(X) \leqslant \dim \mathcal{O}(X) = \dim \mathbb{C} = 1$, we reach a contradiction. So the desired f indeed exists, we are done. □

More generally,

Corollary 6.6. Let X be a compact Riemann surface of genus g, then there exists a holomorphic branched covering $f : X \to \mathbb{P}^1$ of mapping degree $\leqslant g + 1$.

Proof. Just take $D = (g + 1)p$, then follow the lines of the proof of Corollary 6.5. □

Definition 6.4. A compact Riemann surface X is said to be **hyperelliptic** if there exists a holomorphic branch covering $f : X \to \mathbb{P}^1$ of mapping degree 2.

If a compact Riemann surface is of genus 1, then by Corollary 6.6 the branched covering $X \to \mathbb{P}^1$ are of mapping degree $\leqslant 2$. But if there is a branched covering $f : X \to \mathbb{P}^1$ of mapping degree 1, by the proof of Proposition 1.5, X is isomorphic to $\mathbb{P}^1$. This is a contradiction. So all the branched covering $X \to \mathbb{P}^1$ are of mapping degree 2. The argument shows that

Corollary 6.7. All compact Riemann surfaces of genus 1 are hyperelliptic.

Proposition 6.8. Let X be a compact Riemann surface of genus g and $D \in \mathrm{Div}(X)$, then

(i) if $\deg D < 0$, then $h^0(X, \mathcal{O}_D) = 0$;

(ii) if $\deg D = 0$, then $h^1(X, \mathcal{O}_D) = 0$ or 1. $h^1(X, \mathcal{O}_D) = 1$ iff D is principal.

Whilst

(i') if $\deg D > 2g - 2$, then $h^1(X, \mathcal{O}_D) = 0$;

(ii') if $\deg D = 2g - 2$, then $h^1(X, \mathcal{O}_D) = 0$ or 1. $h^1(X, \mathcal{O}_D) = 1$ iff D is canonical.

An immediate corollary of the above proposition is

Corollary 6.9. Let X be a compact Riemann surface of genus g and $D \in \mathrm{Div}(X)$ with $\deg D > 2g - 2$, then

$$h^0(X, \mathcal{O}_D) = \deg D + 1 - g.$$

6.2 Weierstrass Points

Definition 6.5. A **holomorphic differential** of degree n on X is a map

$$\xi : X \to K^n X$$

such that

(i) $\xi(p) \in K^n_p X$ for all $p \in X$;

(ii) if (U, z) is a local coordinate and $\xi|_U = f(z)dz^n$, then $f \in \mathcal{O}(U)$.

The notion of **meromorphic differentials** of degree n can be defined analogously.

Equivalently, a holomorphic differential of degree n is a holomorphic section of the bundle $K^n X$, whilst a meromorphic differential of degree n is a meromorphic section of the bundle $K^n X$. We denote $\Omega^n(X)$ by the space of all holomorphic differentials of degree n on X, and $\mathcal{M}^n(X)$ by the space of all meromorphic differentials of degree n on X. Furthermore, on the space

$$\Omega^*(X) \coloneqq \bigoplus_{n \geqslant 0} \Omega^n(X) \tag{6.7}$$

there is a canonical multiplication induced by the tensor product. Thus $\Omega^*(X)$ is a graded $\mathbb{C}$-algebra, and is called the **canonical ring** of X.

We ask a natural question before we go into the discussion of Weierstrass points on compact Riemann surfaces. Given a point $p \in X$ and an integer $n \geqslant 1$, can we find a meromorphic function f with a unique pole p, such that $\mathrm{ord}_p f = -n$?

Definition 6.6. If the above problem has no solution, namely, given a point $p \in X$, if there doesn't exist a meromorphic function f on X with p as its unique pole and $\mathrm{ord}_p f = -n$, we say that n is a **gap** of X at p.

By definition, $H^0(X; \mathcal{O}_{np})$ is the space of meromorphic functions f on X having a unique pole p such that $\mathrm{ord}_p f \geqslant -n$. Similarly, $H^0(X; \mathcal{O}_{(n-1)p})$ is the space of meromorphic functions having a unique pole p with orders no less than than $-n+1$. Thus

$H^0(X;\mathcal{O}_{(n-1)p}) \subseteq H^0(X;\mathcal{O}_{np})$. If $g \in H^0(X;\mathcal{O}_{np})$ but $g \notin H^0(X;\mathcal{O}_{(n-1)p})$, then $\mathrm{ord}_p g = -n$. Conversely, if $g \in H^0(X;\mathcal{O}_{np})$ satisfies $\mathrm{ord}_p f = -n$, obviously $g \notin H^0(X;\mathcal{O}_{(n-1)p})$. So

Lemma 6.10. n is a gap of X at p iff $h^0(X,\mathcal{O}_{np}) = h^0(X,\mathcal{O}_{(n-1)p})$.

By Riemann-Roch, we have

$$h^0(X,\mathcal{O}_{np}) - h^1(X,\mathcal{O}_{np}) = n + 1 - g,$$

and

$$h^0(X,\mathcal{O}_{(n-1)p}) - h^1(X,\mathcal{O}_{(n-1)p}) = n - 1 + 1 - g = n - g.$$

Substracting one of the above equation from another, we have

$$h^0(X,\mathcal{O}_{np}) - h^0(X,\mathcal{O}_{(n-1)p}) = 1 - (h^1(X,\mathcal{O}_{(n-1)p}) - h^1(X,\mathcal{O}_{np})). \tag{6.8}$$

By Serre duality, the right hand side of (6.8) equals to

$$1 - (h^0(X,\Omega_{-(n-1)p}) - h^0(X,\Omega_{-np})) \leqslant 1,$$

as $\Omega_{-np} \subseteq \Omega_{-(n-1)p} \implies h^0(X,\Omega_{-(n-1)p}) - h^0(X,\Omega_{-np}) \geqslant 0$. So

$$h^0(X,\mathcal{O}_{np}) - h^0(X,\mathcal{O}_{(n-1)p}) \leqslant 1.$$

On the other hand, $h^0(X,\mathcal{O}_{np}) - h^0(X,\mathcal{O}_{(n-1)p}) \geqslant 0$ as $\mathcal{O}_{(n-1)p} \subseteq \mathcal{O}_{np}$, so

$$h^0(X,\mathcal{O}_{np}) - h^0(X,\mathcal{O}_{(n-1)p}) = 0, 1. \tag{6.9}$$

By Lemma 6.10, n is a gap of X at p iff $h^1(X,\mathcal{O}_{(n-1)p}) - h^1(X,\mathcal{O}_{np}) = 1$, iff $h^0(X,\Omega_{-(n-1)p}) - h^0(X,\Omega_{-np}) = 1$, iff there exists a holomorphic 1-form ω with $\mathrm{ord}_p\omega = n-1$. In summary,

Proposition 6.11. n is a gap of X at p iff one of the following conditions holds:

(i) $h^0(X,\mathcal{O}_{np}) = h^0(X,\mathcal{O}_{(n-1)p})$;

(ii) $h^1(X,\mathcal{O}_{(n-1)p}) = h^1(X,\mathcal{O}_{np}) + 1$;

(iii) $\exists\omega \in \Omega(X)$ such that $\mathrm{ord}_p\omega = n-1$.

If $n \geqslant 2g$, then $n-1 > 2g-2$. By Proposition 6.8, $h^1(X,\mathcal{O}_{(n-1)p}) = h^1(X,\mathcal{O}_{np}) = 0$, so n is not a gap of X at p. [8] Thus

[8] *There are always solutions of the problem above Definition 6.6, whence n is sufficiently large.*

Lemma 6.12. If n is a gap of X at p, then $n \leqslant 2g-1$.

Proposition 6.13. Let X be a compact Riemann surface of genus g, then X has exactly g gaps at any point $p \in X$, with all of them less than $2g-1$.

Definition 6.7. Let $p \in X$ be a point and denote by $n_1 < \cdots < n_g$ the gaps of X at p. We say that p is a **Weierstrass point** if $n_g > g$.

If p is not a Weierstrass point of X, then the gaps of X at p are $1, 2, \dots, g$.

If p is a Weierstrass point of X, then there must be some $1 \leqslant n \leqslant g$ such that n is not a gap. So there is a meromorphic function f on X with a unique pole p such that $\operatorname{ord}_p f = -n$.

Definition 6.8. Let X be a compact Riemann surface of genus g and $p \in X$ be a point. Then let $n_1, \dots, n_g$ be the gaps of X at p. The **weight** of X at p is defined to be

$$\mathrm{Wt}_X(p) = \sum_{j=1}^{g} (n_j - j).$$

Immediately, if p is not a Weierstrass point, by definition $\mathrm{Wt}_X(p) = 0$. So only Weierstrass points contribute to the weight of X. The weight $\mathrm{Wt}_X(x)$ of X at a Weierstrass point x counts the "multiplicity" of x.

Our purpose is to count the number of Weierstrass points on X. To do this, we need to introduce a little more concepts.

Definition 6.9. Let $D \subseteq \mathbb{C}$ be a domain, and $\phi_1, \dots, \phi_n \in \mathcal{O}(D)$ be holomorphic functions on D. The **Wronskian** $W(\phi_1, \dots, \phi_n)$ of $\phi_1, \dots, \phi_n$ is defined by

$$W(\phi_1, \dots, \phi_n) \coloneqq \det \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{pmatrix},$$

which is a holomorphic function on D.

Lemma 6.14. For any $f, \phi_1, \dots, \phi_n \in \mathcal{O}(D)$,

$$W(f\phi_1, \dots, f\phi_n) = f^n W(\phi_1, \dots, \phi_n).$$

Lemma 6.15. Let $p \in D$ be a point, and assume that $\phi_1, \dots, \phi_n \in \mathcal{O}(D)$ satisfy $\operatorname{ord}_p \phi_1 < \cdots < \operatorname{ord}_p \phi_n$, then

$$\operatorname{ord}_p W(\phi_1, \dots, \phi_n) = \sum_{j=1}^{n} (\operatorname{ord}_p \phi_j - j + 1).$$

Let X be a compact Riemann surface of genus $g \geqslant 2$. By the Dolbeault isomorphism we know that $\dim \Omega(X) = g$. Pick a basis $\omega_1, \dots, \omega_g$ of $\Omega(X)$. Assume that in any coordinate neighborhood (U, z), $\omega_i|_U = \phi_i(z)dz$ for all $1 \leqslant i \leqslant g$, we set

$$\Phi(z) = W(\phi_1, \dots, \phi_g) \in \mathcal{O}(U).$$

It's easy to verify that

$$\Phi(z) dz^{g(g+1)/2}$$

is invariant under coordinate transformation, thus defines a holomorphic differential on X of degree $g(g+1)/2$. We denote this holomorphic differential by W_X.

Theorem 6.16. Let X be a compact Riemann surface of genus g and W_X be as above, then

(i) for any $p \in X$, $\mathrm{Wt}_X(p) = \mathrm{ord}_p W_X$;

(ii) the number of Weierstrass points on X, counting multiplicity, is

$$\sum_{p \in X} \mathrm{Wt}_X(p) = (g-1)g(g+1).$$

Corollary 6.17. Let X be a compact Riemann surface of genus $g \geqslant 2$, then

(i) X must admit Weierstrass points;

(ii) there exists a holomorphic map $f : X \to \mathbb{P}^1$ of mapping degree $\leqslant g$.

Proof. (i) By Theorem 6.16 (ii),

$$\sum_{p \in X} \mathrm{Wt}_X(p) = (g-1)g(g+1) > 0.$$

Since only Weierstrass points contribute to the left hand side, the assertion follows.
(ii) By (i) X admits a Weierstrass point x. By the argument below Definition 6.7, there exists a meromorphic function f on X holomorphic everywhere except x, with $\mathrm{ord}_x f = -n$. So there exists a holomorphic map $f : X \to \mathbb{P}^1$ of mapping degree $n \leqslant g$. $\square$

Corollary 6.18. All Riemann surfaces of genus 2 are hyperelliptic.

Proof. Let X be a compact Riemann surface of genus 2. Recall that X is hyperelliptic iff there exists a homomorphic branch covering $f : X \to \mathbb{P}^1$ of mapping degree 2. By Corollary 6.17, there exists a holomorphic map $f : X \to \mathbb{P}^1$ of mapping degree $\leqslant 2$. We claim that $\deg f = 2$. Otherwise $\deg f = 1$, then $f : X \to \mathbb{P}^1$ is a biholomorphic map. But X is of genus 2, a contradiction. So our claim holds, hence the corollary holds. $\square$

6.3 Embedding into Projective Spaces

Let X be the compact Riemann surface of genus $g \geqslant 1$, and let $\omega_1, \ldots, \omega_g$ be a basis of $\Omega(X)$.

Definition 6.10. The **canonical mapping** of X is defined to be the following map

$$\begin{aligned} \phi : X &\to \mathbb{P}^{g-1}, \\ x &\mapsto [\omega_1(x) : \cdots : \omega_g(x)]. \end{aligned}$$

Lemma 6.19. Let X be a compact Riemann surface of genus $g \geqslant 1$. Then for any $p \in X$ there exists $\omega \in \Omega(X)$ such that $\omega(p) \neq 0$.

Proof. First note that the elements in $H^0(X; \mathcal{O}_p)$ are constant functions. Otherwise, if there were $f \in \mathcal{O}_p$, f corresponds to a holomorphic map $f : X \to \mathbb{P}^1$. By Proposition 1.5, X is isomorphic to $\mathbb{P}^1$. Thus $g(X) = g(\mathbb{P}^1) = 0$, a contradiction.

By Riemann-Roch

$$h^0(X, \mathcal{O}_p) - h^0(X, \Omega_{-p}) = 1 - h^0(X, \Omega_{-p}) = 1 + 1 - g,$$

hence $h^0(X, \Omega_{-p}) = \Omega_{-p}(X) = g - 1 < g$. This means that there must be some element $\omega \in \Omega(X)$ but not in $\Omega_{-p}(X)$. Thus $\mathrm{ord}_p \omega = 0$, or $\omega(p) \neq 0$. $\square$

Lemma 6.20. If a compact Riemann surface X is not hyperelliptic, then for any different points $p, q \in X$, there exists $\omega \in \Omega(X)$ such that $\omega(p) = 0$ whilst $\omega(q) \neq 0$.

Proof. To using the powerful Riemann-Roch, we reformulate the statement of the Lemma in an adaptable language. In virtue of the proof of the last lemma, we see that $\omega \in \Omega(X)$ such that $\omega(p) = 0$ exits iff $\omega \in \Omega_{-p}(X)$ holds. In the same spirit, $\omega \in \Omega(X)$ satisfying $\omega(p) = 0$ but $\omega(q) \neq 0$ iff $\omega \in \Omega_{-p}(X)$ but $\omega \notin \Omega_{-p-q}(X)$ exists. It suffices to show that $\Omega_{-p-q}(X) \subsetneq \Omega_{-p}(X)$. By Riemann-Roch,

$$\begin{aligned} h^0(X, \mathcal{O}_p) - h^0(X, \Omega_{-p}) &= 1 + 1 - g = 2 - g, \\ h^0(X, \mathcal{O}_{p+q}) - h^0(X, \Omega_{-p-q}) &= 2 + 1 - g = 3 - g. \end{aligned}$$

As $\mathcal{O}_p \simeq \mathbb{C}$ and $\mathcal{O}_{p+q} \simeq \mathbb{C}$ by Proposition 1.6, $h^0(X, \mathcal{O}_p) = h^0(X, \mathcal{O}_{p+q}) = 1$. So $h^0(X, \Omega_{-p}) - h^0(X, \Omega_{-p-q}) = 1$, implying that $\dim \Omega_{-p}(X) > \dim \Omega_{-p-q}(X)$, as desired. $\square$

Lemma 6.21. If a compact Riemann surface X that is not hyperelliptic, then for any $p \in X$, there exists $\omega \in \Omega(X)$ such that $\mathrm{ord}_p \omega = 1$

Proof. It suffices to show that $\Omega_{-2p} \subsetneq \Omega_{-p}$. Again by Riemann-Roch,

$$\begin{aligned} h^0(X, \mathcal{O}_p) - h^0(X, \Omega_{-p}) &= 2 - g, \\ h^0(X, \mathcal{O}_{2p}) - h^0(X, \Omega_{-2p}) &= 3 - g. \end{aligned}$$

Since X is not hyperelliptic, we conclude that there is no constant meromorphic function $f \in \mathcal{O}_{2p}(X)$. Otherwise, suppose there were some non-constant meromorphic function $f \in \mathcal{O}_{2p}(X)$, then f has exactly one pole p of order 2. So f corresponds to a non-constant holomorphic $f : X \to \mathbb{P}^1$ of mapping degree 2, implying that X is hyperelliptic, a contradiction. If $g \neq 0$, then $\mathcal{O}_p(X)$ has only constant sections, by Proposition 1.5. In this case, $h^0(X, \mathcal{O}_p) = h^0(X, \mathcal{O}_{2p}) = 1$, therefore $h^0(X, \Omega_{-p}) = h^0(X, \Omega_{-2p}) + 1$, as desired. $\square$

Theorem 6.22. If a compact Riemann surface X of genus g is not hyperelliptic, then the canonical map $\phi : X \to \mathbb{P}^{g-1}$ is a holomorphic embedding.

Proof.

$\square$

Theorem 6.23. Let X be a compact Riemann surface of genus g and $D \in \mathrm{Div}(X)$. If $\deg D \geqslant 2g + 1$, then $\phi_D : X \to \mathbb{P}^n$ is a holomorphic embedding, where $n = h^0(X, \mathcal{O}_D) - 1$.

Since there are lots of D satisfying the condition $\deg D \geqslant 2g + 1$ on a compact Riemann surface X of genus g, we have the following immediate corollary:

Corollary 6.24. All compact Riemann surfaces are projective.

6.4 Jacobi Varieties, Abel's Theorem and Jacobi's Inversion

In this subsection we assume that X is a compact Riemann surface of genus $g \geqslant 1$, if not otherwise mentioned.

In the rest of the subsection, we always assume that $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ is a canonical basis of $H_1(X; \mathbb{Z})$.

Lemma 6.25. Let $\omega_1, \dots, \omega_g$ be a basis of $\Omega(X)$. Then the following matrix

$$\begin{pmatrix} \int_{\alpha_1} \omega_1 & \cdots & \int_{\alpha_g} \omega_1 \\ \vdots & & \vdots \\ \int_{\alpha_1} \omega_g & \cdots & \int_{\alpha_g} \omega_g \end{pmatrix}_{g \times g}$$

is invertible.

By the above lemma, we may uniquely choose a basis $\omega_1, \dots, \omega_g$ of $\Omega(X)$ such that

$$\begin{pmatrix} \int_{\alpha_1} \omega_1 & \cdots & \int_{\alpha_g} \omega_1 \\ \vdots & & \vdots \\ \int_{\alpha_1} \omega_g & \cdots & \int_{\alpha_g} \omega_g \end{pmatrix}_{g \times g} = I_g,$$

where I_g is the identity matrix of order g.

If we choose $\omega_1, \dots, \omega_g$ as above, then under the canonical basis $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ of $H_1(X; \mathbb{Z})$, the period matrix has the form (I_g, Z), with

$$Z = (Z_1, \dots, Z_g) = \begin{pmatrix} \int_{\beta_1} \omega_1 & \cdots & \int_{\beta_g} \omega_1 \\ \vdots & & \vdots \\ \int_{\beta_1} \omega_g & \cdots & \int_{\beta_g} \omega_g \end{pmatrix}$$

Theorem 6.26. Let Z be the matrix given as above, then

(i) Z is symmetric;

(ii) the imaginary part of Z is positively definite.

Proof. (i) Note that $\omega_i \wedge \omega_j = 0$ for stupid reason, for all $1 \leqslant i, j \leqslant g$. Thus we have

$$\begin{aligned} 0 = \int_X \omega_i \wedge \omega_j &= \sum_{k=1}^{g} \Big(\int_{\alpha_k} \omega_i \int_{\beta_k} \omega_j - \int_{\alpha_k} \omega_j \int_{\beta_k} \omega_i\Big) \\ &= \sum_{k=1}^{g} \Big(\delta_{ik} \int_{\beta_k} \omega_j - \delta_{jk} \int_{\beta_k} \omega_i\Big) \\ &= \int_{\beta_i} \omega_j - \int_{\beta_j} \omega_i \\ &= Z_{ji} - Z_{ij}, \end{aligned}$$

as desired.

(ii) □

Corollary 6.27.

Let $f : X \to Y$ be a proper holomorphic map between Riemann surfaces. If f is a branched covering of mapping degree d, then there exists a discrete subset $A \subseteq Y$ such that

$$f|_{X\backslash f^{-1}(A)} : X\backslash f^{-1}(A) \to Y\backslash A$$

is a covering map of degree d. For $\omega \in \Omega(X)$, we can define $f_*\omega \in \Omega(Y\backslash A)$ as follows. Let V be an open subset in $Y\backslash A$ such that $f^{-1}(V) = \coprod_i^d U_i$, with $f|_{U_i} : U_i \to V$ being homeomorphisms. We define

$$f_*\omega|_V = \sum_{i=1}^{d} (f|_{U_i}^{-1})^*\omega,$$

then $f_*\omega$ is a holomorphic 1-form on $Y\backslash A$.

Definition 6.11. Let $D = n_1p_1 + \cdots + n_rp_r \in \mathrm{Div}(X)$ be a divisor, with $p_1, \ldots, p_r$ being distinct. We say that $f \in C^\infty(X\backslash\{p_1, \ldots, p_r\})$ a smooth solution of D, if

(i) f is not identically zero;

(ii) there exist local coordinate z_i near p_i with $z_i(p_i) = 0$, such that f can be locally written as $f(z) = z^{n_i}h_i(z)$ near p_i, where h_i is a smooth function near p_i with $h_i(0) \neq 0$.

Lemma 6.28. Let f be a smooth solution of $D = n_1p_1 + \cdots + n_rp_r$, and let $\phi \in C^\infty(X)$ be a smooth function on X. If either df or ϕ has compact support, then

$$\frac{1}{2\pi i}\int_X \frac{df}{f} \wedge d\phi = \sum_{i=1}^{r} n_i\phi(p_i).$$

Lemma 6.29. Let $\gamma : [0, 1] \to X$ be a smooth curve with $\gamma(0) = q, \gamma(1) = p$. Then there exists a smooth solution $f \in C^\infty(X\backslash\{q\})$ of $p - q$ such that

$$\frac{1}{2\pi i}\int_X \frac{df}{f} \wedge \xi = \int_\gamma \xi$$

holds for any smooth closed 1-form ξ on X.

Theorem 6.30 (Abel's Theorem)**.** A divisor $D \in \mathrm{Div}(X)_0$ is principal iff $u(D) = 0$.

Theorem 6.31 (Jacobi's Inversion Theorem)**.** The map $u : \mathrm{Div}(X)_0 \to J(X)$ is surjective.

こにちは、世界。

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