The Power Spectral Density

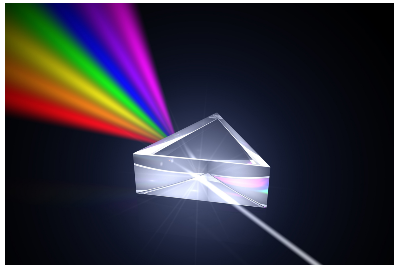
Andreas Jakobsson

The power spectral density

The *power spectral density* (PSD) of a WSS stochastic process is defined as

$$\phi_y(\omega) = \sum_{-\infty}^{\infty} r_y(k) e^{-i\omega k}$$

over the frequencies $-\pi < \omega \leq \pi$.



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over the frequencies $-\pi < \omega \leq \pi$. The inverse transform recovers $r_y(k)$

$$r_y(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_y(\omega) e^{i\omega k} d\omega$$

It is worth noting that

$$r_y(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_y(\omega) d\omega$$

which, for a zero-mean process, measures the power of y_t, i.e.,

$$r_y(0) = E\{|y_t|^2\}$$

The PSD is *real-valued* and *non-negative*. For a real-valued process, the PSD is *symmetric*, whereas it is *non-symmetric* for a complex-valued process.

Example:
As a white noise is uncorrelated, $r_x(k) = \sigma_x^2 \delta_K(k)$, where $\delta_K(k)$ is the Kronecker delta. Thus, the PSD of a white noise is

$$\phi_x(\omega) = \sum_{-\infty}^{\infty} r_x(k) e^{-i\omega k} = \sigma_x^2$$

The power spectral density

Example:
Consider a sinusoidal process

$$y_t = A\cos(\omega_0 t + \phi) + w_t$$

where w_t is an AWGN. The ACF of y_t is then

$$r_y(k) = \frac{A^2}{2}\cos(\omega_0 k) + \sigma_w^2 \delta_K(k)$$

This implies that the PSD of y_t is

$$\phi_y(k) = \frac{A^2}{4}\delta_D(\omega - \omega_0) + \frac{A^2}{4}\delta_D(\omega + \omega_0) + \sigma_w^2$$

where $\sigma_D(\omega)$ is the Dirac delta, satisfying

$$f(a) = \int_{-\infty}^{\infty} f(x)\delta_D(x-a)dx$$

Estimating the power spectral density

Under the weak assumption that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=-N}^{N} |k|\,|r_y(k)| = 0$$

the PSD can be expressed equivalently as

$$\phi_y(\omega) = \lim_{N\to\infty} E\left\{ \frac{1}{N} \left| \sum_{t=1}^{N} y_t e^{-i\omega t} \right|^2 \right\}$$

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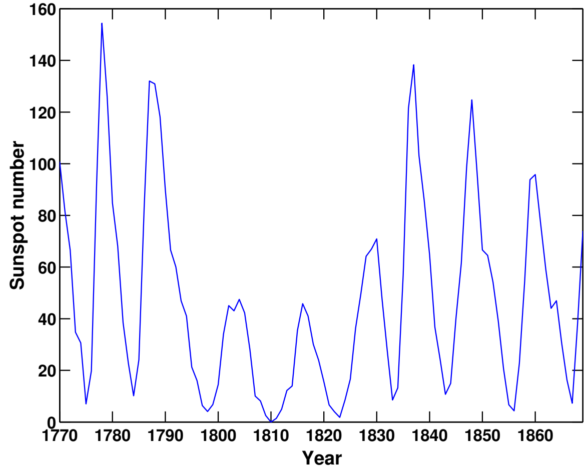
This suggests two natural estimators, namely as the *periodogram*

$$\hat{\phi}_y^p(\omega) = \frac{1}{N} \left| \sum_{t=1}^{N} y_t e^{-i\omega t} \right|^2$$

and the *correlogram*

$$\hat{\phi}_y^c(\omega) = \sum_{k=-N+1}^{N-1} \hat{r}_y(k) e^{-i\omega k}$$

Note that $\hat{r}_y(k)$ should be the *biased* estimator.



The annual number of sunspots

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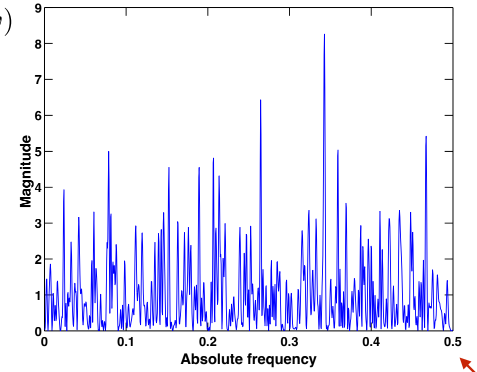
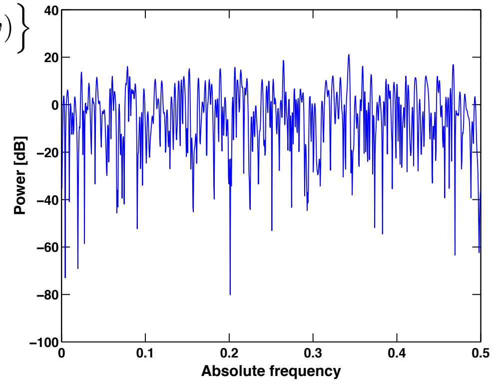
These estimators are equivalent, and are *asymptotically* unbiased, but not consistent. Notably, the variance of the estimate is about the same as the true spectrum squared.

$\hat{\phi}_y^p(\omega)$

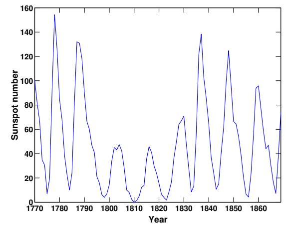


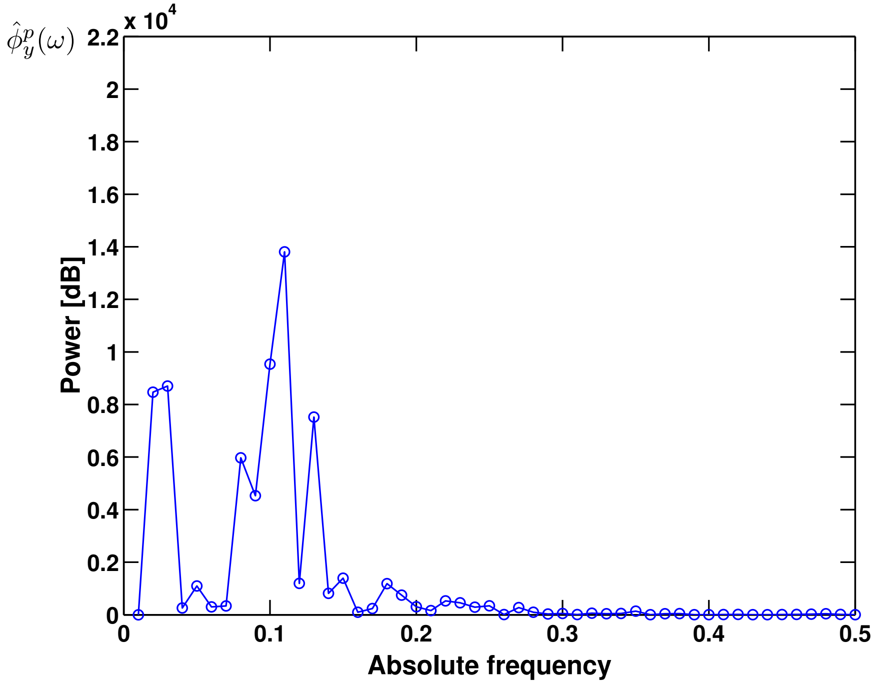
$10\log\left\{\hat{\phi}_y^p(\omega)\right\}$

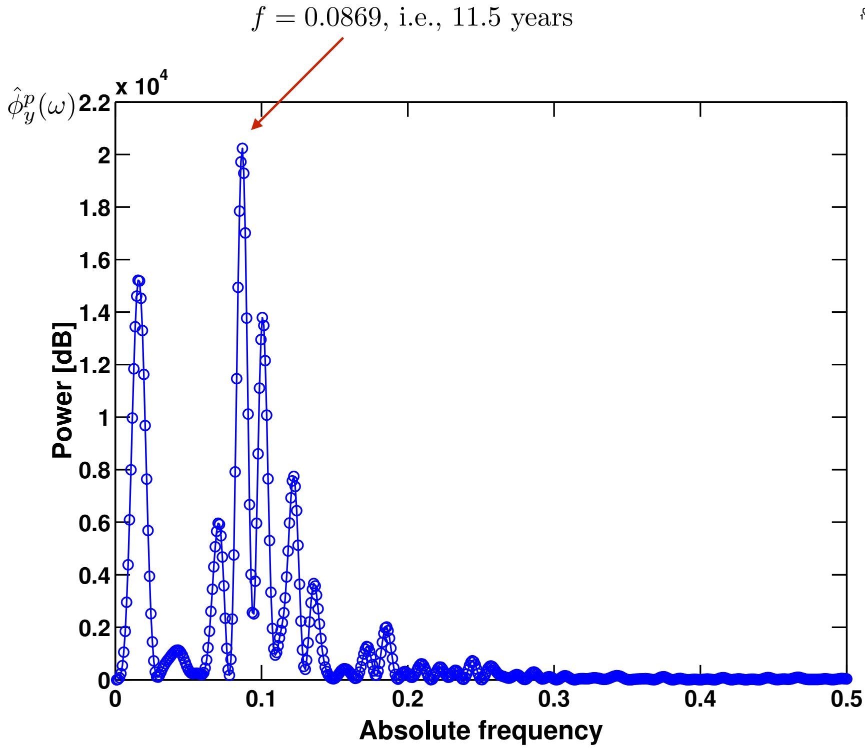


$\omega = 2\pi f$

Zero-padding







Generally, it is preferable to *zero-padd* the data before computing the periodogram estimate. Adding zeros does obviously not yield new information as such, but it interpolates the spectral estimate, yielding an estimate on a finer grid, often revealing features that are not otherwise visible in the estimate.

Windowing

It is often preferable to use some other time or lag window, for the *periodogram*

$$\hat{\phi}_y^p(\omega) = \frac{1}{N}\left|\sum_{t=1}^{N} y_t e^{-i\omega t}\right|^2 = \frac{1}{N}\left|\sum_{t=-\infty}^{\infty} v_t y_t e^{-i\omega t}\right|^2$$

and the *correlogram*

$$\hat{\phi}_y^c(\omega) = \sum_{k=-N+1}^{N-1} \hat{r}_y(k) e^{-i\omega k} = \sum_{k=-\infty}^{\infty} w_k \hat{r}_y(k) e^{-i\omega k}$$

where the *time* and *lag* windows, v_t and w_k, take the value 1 in the given range, and zero otherwise.

Such windows widens the main lobe, but lowers the sidelobes.

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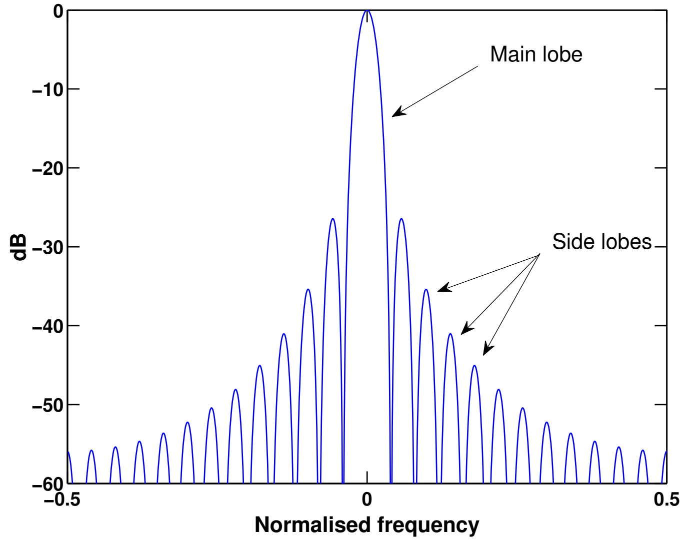

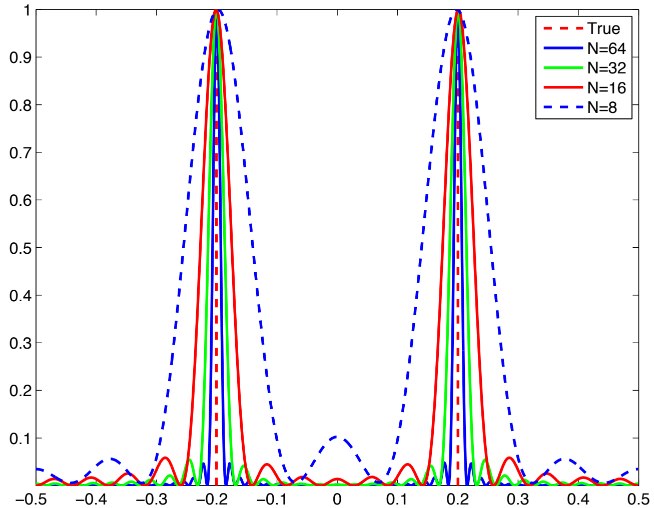

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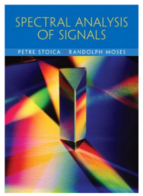
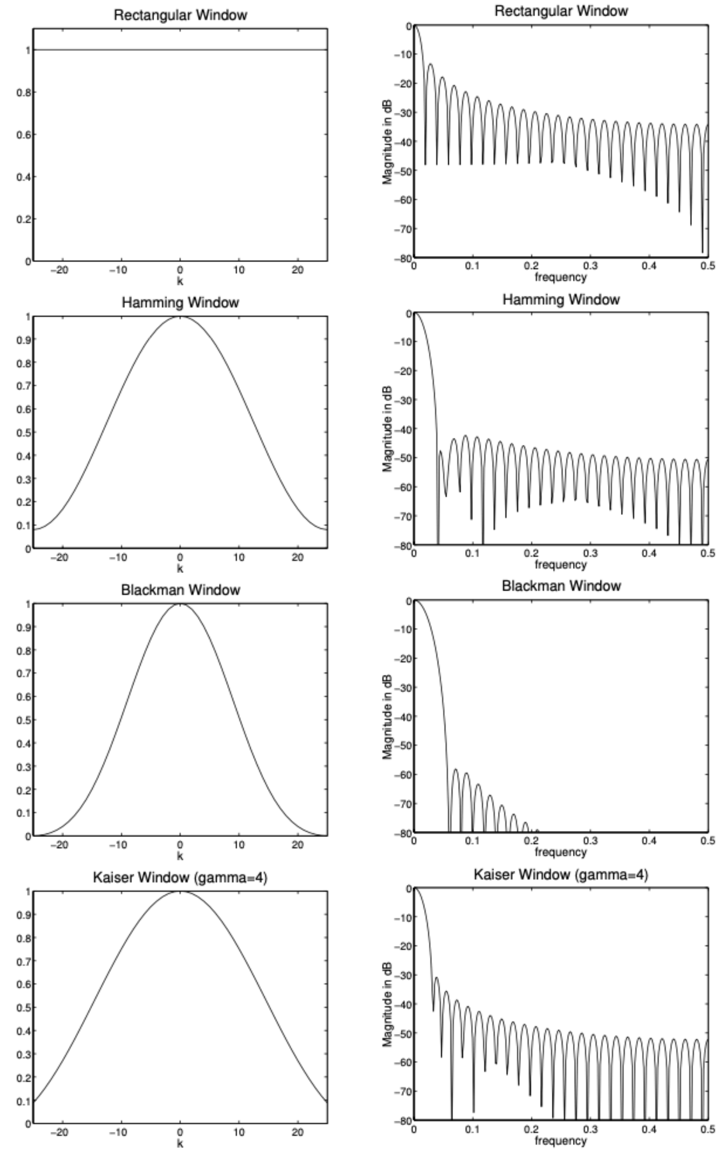
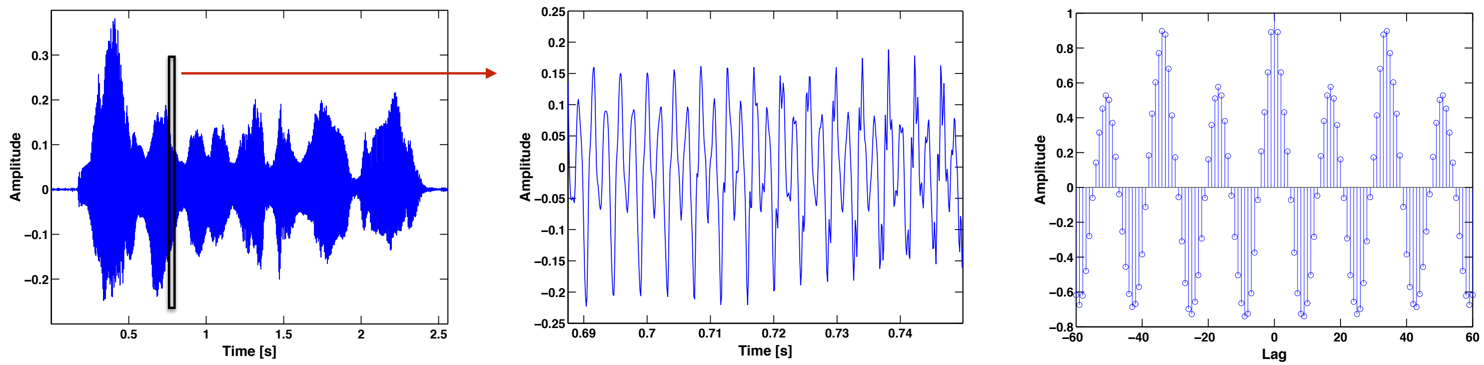
Window Name	Defining Equation	Approx. Main Lobe Width (radians)	Sidelobe Level (dB)
Rectangular	$w(k) = 1$	$2\pi/M$	-13
Bartlett	$w(k) = \frac{M-k}{M}$	$4\pi/M$	-25
Hanning	$w(k) = 0.5 + 0.5\cos\left(\frac{\pi k}{M}\right)$	$4\pi/M$	-31
Hamming	$w(k) = 0.54 + 0.46\cos\left(\frac{\pi k}{M-1}\right)$	$4\pi/M$	-41
Blackman	$w(k) = 0.42 + 0.5\cos\left(\frac{\pi k}{M-1}\right) + 0.08\cos\left(\frac{\pi k}{M-1}\right)$	$6\pi/M$	-57

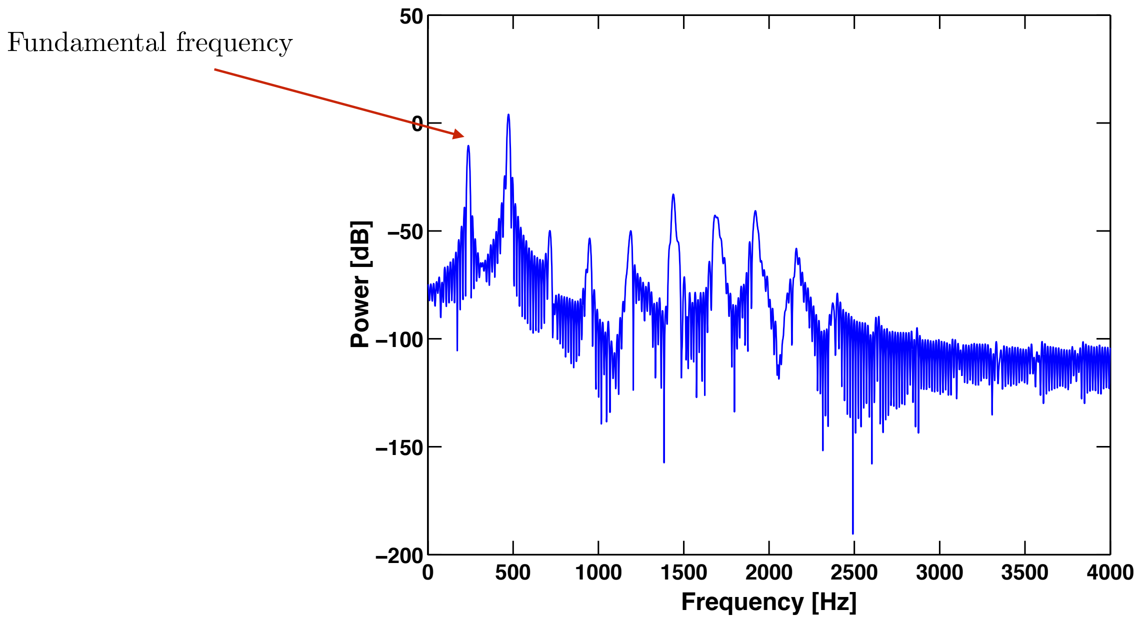


You can read more in the excellent textbook by Stoica and Moses (2005).

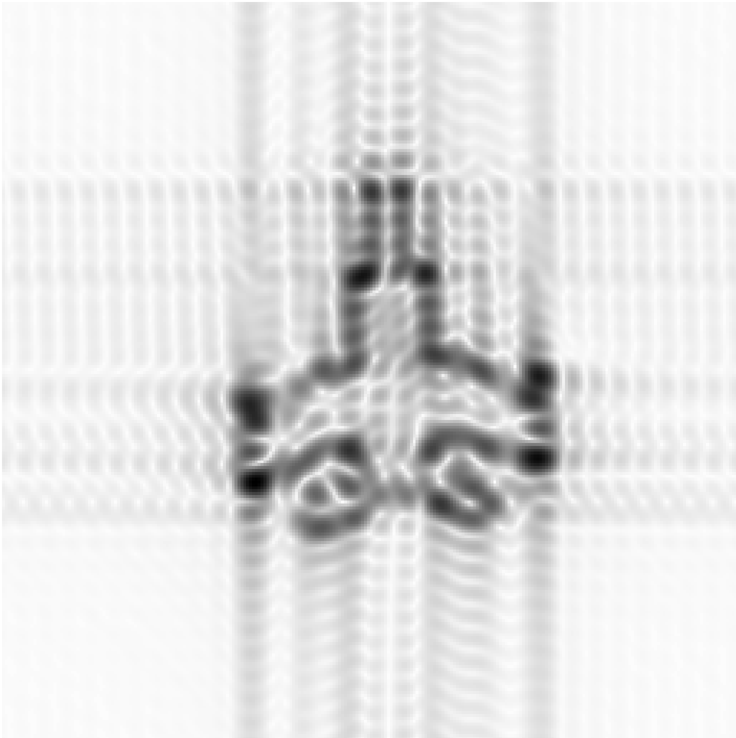


Voiced speech

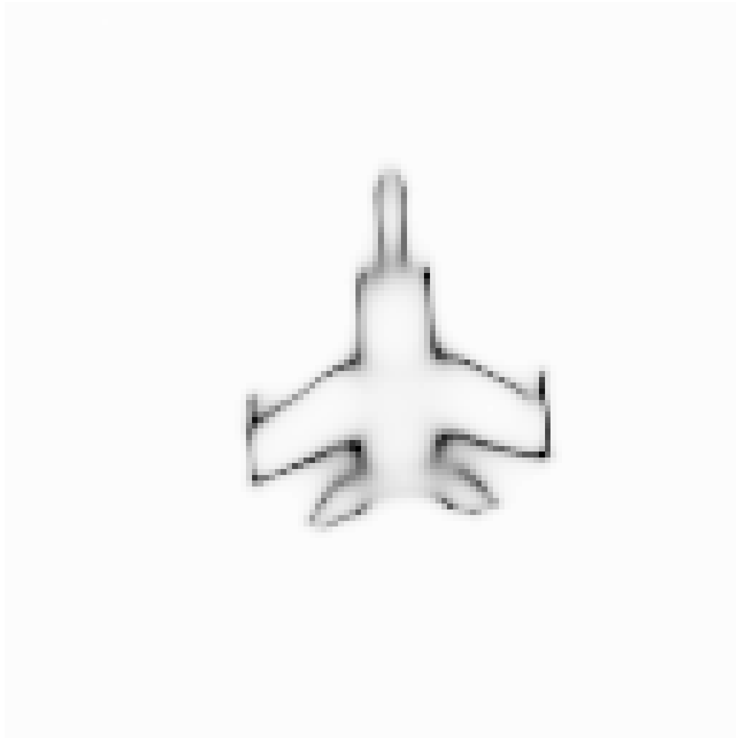




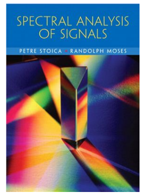
High-resolution estimators



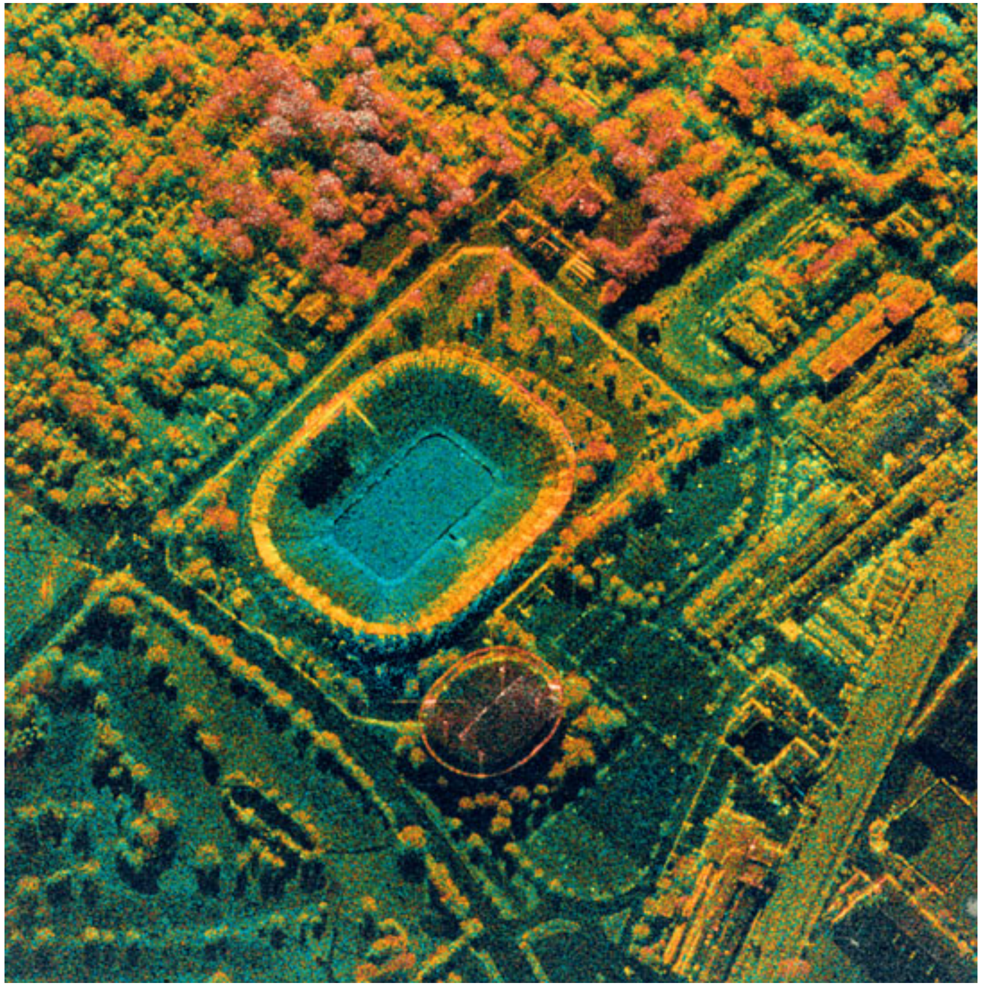
Periodogram

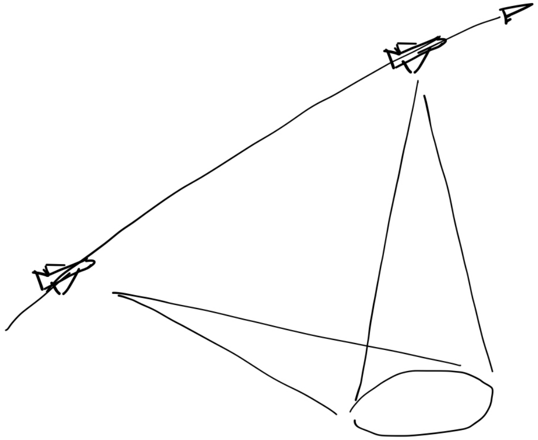


Capon



Synthetic aperture radar





The spectrogram