Predicate Logic

The weakness of propositional logic:
Some propositions have a very large or even infinite number of cases to check in order to decide if they are true or false.

"Every student in this class has studied calculus"
There are 40 "variables" I need to check to figure out if this is true or false.

"For every non-negative integer n, we have that $n^2 + n + 41$ is prime"
To decide this is false I only need to find one integer for which this is not true, but to decide it is true I need to check an infinite number of cases!

Examples:

"Every map can be colored with 4 colors so that adjacent regions have different colors"
What does this even mean? Need to know exactly what is a map, how many maps are there, what is a region.

Predicates

These problems lead us to predicate (first order) logic.

Definition 0.1. *A **predicate** is a statement whose truth value depends on the value of one or more variables.*

Example: "n is a perfect square."

Like propositions we often name predicates with a letter, but also with a function-like notation to indicate the variables:

Example:

$$P(n) = \text{"}n \text{ is a perfect square"}$$
$$P(n) = \begin{cases} \text{True} & \text{If } n \text{ is a perfect square} \\ \text{False} & \text{Otherwise} \end{cases}$$

The collection of values that can be plugged into a predicate is called the **universe of discourse**.
In theory, the universe of discourse can include everything; people, stars, potatoes, abstract concepts like numbers, emotions, predicates or functions, ect.
We will be restricting ourselves to **first-order logic**, basically meaning that our universe of discourse includes only individual entities, not groups of entities or relations (this is getting technical).

The kind of statements will want to make later on will be about whether or everything in our universe of discourse has a property, or just some things have a property, or nothing has a property, and so on.

We will make these statements using some combination of predicates and two "quantifiers".

Quantifiers

Our first quantifier is for when we want to talk about a property everything shares:
UNIVERSAL QUANTIFIER: $\forall$

Definition 0.2. *Let x be a variable and $P(x)$ be a predicate.*
Then we write

$$\forall x P(x)$$

to say that $P(x)$ is true for everything in our universe of discourse.

Our second quantifier is for when we want to say there is at least one thing that has the property:
EXISTENTIAL QUANTIFIER: $\exists$

Definition 0.3. *Let x be a variable and $P(x)$ be a predicate.*
Then we write

$$\exists x P(x)$$

to say that $P(x)$ is true for at least one thing in our universe of discourse.

We won't stop here, we are not quite at the notation mathematicians and computer scientists like to use.

We often want to limit our universe of discourse, in particular we often want to just make a statement about integers, or real numbers, or functions.

Rather than hiding this limit in some external context we want to encode it in the statements we make. We will do this primarily through defining **sets**.

In a dedicated logic course there would be many steps to take to get to this; since we don't have much time, we will skip a few steps and informally introduce sets now (we will have a more formal discussion of sets in later lectures).

Definition 0.4. *A **set** is a collection of distinct objects, called **elements**.*
To indicate that x is an element of set S, we write $x \in S$.

The kind of statements we will want to make will be about whether or not elements of a set have some property, or about whether or not the set has some property.

Examples:
"The cardinality of $\mathbb{N}$ is equal to the cardinality of $\mathbb{Q}$"

"For all a and b in $\mathbb{R}$, unless $a = b$ there exists a c in $\mathbb{R}$ such that $a < c < b$"

"There are numbers in $\mathbb{N}$ which are only divisible by themselves or 1"

The Typical Use of Quantifiers

Outside of dedicated logic classes/research, this is how you will typically see these quantifiers used:
UNIVERSAL QUANTIFIER: $\forall$

Definition 0.5. *Let x be a variable and $R(x), P(x)$ be predicates. Then we write*

$$\forall R(x), P(x)$$

when we really mean

$$\forall x(R(x) \Rightarrow P(x))$$

EXISTENTIAL QUANTIFIER: $\exists$

Definition 0.6. *Let x be a variable and $R(x), P(x)$ be predicates. Then we write*

$$\exists R(x), P(x)$$

when we really mean

$$\exists x(R(x) \wedge P(x))$$

Free vs Bound Variables

As we build larger statements from smaller ones, it is important to make sure that all variables are **bound** by some quantification.

For example, consider this:

$$\exists p \in P, \exists q \in P, (n = p + q)$$

Here we know where p and q come from, but n is undefined - and don't let convention trick you, just because we typically use n to mean an integer doesn't mean it always will be an integer.

We say here that p and q are **bound**, but n is **free**. A statement can't have a true/false value unless all of the variables are bound.

Order of Operations, Interaction with $\neg$

For order of operations, the typically (but not universal) convention is like this:

$$\neg, \wedge, \vee, \forall, \exists, \Rightarrow, \iff$$

Sometimes we will want to work with the negation of some statement, i.e.

$$\neg \forall x \in S, P(x)$$

This can be a difficult statement to work with.

It is sometimes preferable to "push" the negation to apply directly to the predicate $P(x)$, and then we can apply logical equivalencies to simplify it further, much like how in arithmetic it is convenient to cancel out factors of -1.

If the original statement is "Everything in this set has this property", what is the negation of that (what statement will always have the opposite truth value?)

$$\neg \forall x \in S, P(x)$$

becomes:

$$\exists x \in S, \neg P(x)$$

or in other words, "There is at least one thing in the set that does not have this property".

We can do the a similar thing if we are negating an existential statement:

$$\neg \exists x \in S, P(x)$$

becomes

$$\forall x \in S, \neg P(x)$$

And this applies recursively to any further quantifiers used in $P(x)$.