MATH-2415, Ordinary and Partial Differential Equations **Name:**
Summer 2023
Problem Set 4
Due July 2, 2023 by midnight

Directions: You can either

(I) Show all your work on the pages of the assignment itself, or

(II) Use separate paper to work out the problems. The paper must be plain white paper. No notebook paper, no lined paper. Engineering paper is acceptable.

For either selection, **clearly show all work that leads to your final answer**. Don't make me hunt for your steps! And DON'T turn in **any** scratched out work! Your final product must be clear and legible, or it will be returned, ungraded. **Please submit your work to Blackboard as a single pdf file**.

1. The population of a certain community is known to increase at a rate proportional to the number of people present at any time. If the population has doubled in 5 years, how long will it take the population to triple? How long will it take for the population to quadruple?

$$\frac{\log(2)}{5} = 0\ 138629$$

$$\frac{\log(3)}{0.138629} = \boxed{7.92481 \text{ years to triple}}$$

$$\frac{\log(4)}{0.138629} = \boxed{10 \text{ years to quadruple}}$$

2. The differential equation

$$\frac{dP}{dt} = (k\cos t)P$$

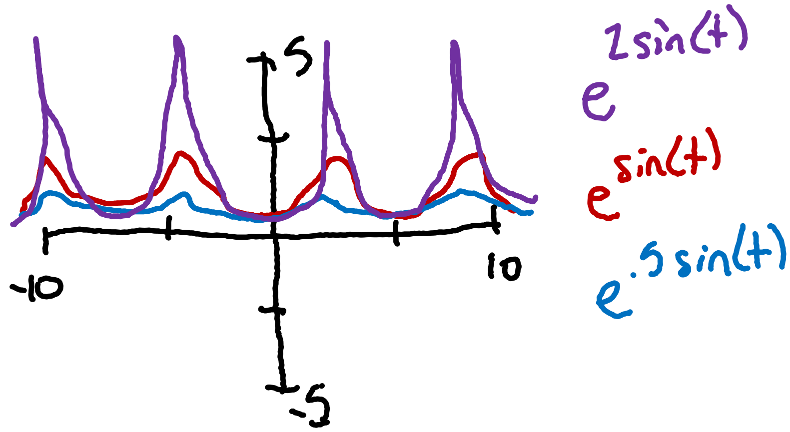
where k is a positive constant, is often used as a model for populations that undergo yearly seasonal fluctuations. Solve for $P(t)$, assuming $P(0) = P_0$. Sketch a graph of $P(t)/P_0$ for three different choices of k.

$$\frac{dP}{P} = k\cos t\, dt \quad \int \frac{1}{P}\, dP \rightarrow = k\int \cos t\, dt$$

$$\hookrightarrow \ln(P) = k\sin t + C \rightarrow e^{\ln(P)} = Ce^{k\sin t}$$

$$\hookrightarrow P = Ce^{k\sin t} \rightarrow P_0 = Ce^{\sin 0} \quad P_0 = Ce^{0} P_0 \rightarrow C = P_0$$

$$\hookrightarrow \boxed{P = P_0 e^{k\sin t}}$$



3. Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days $x(4) = 50$.

$$\frac{dx}{dt} = kt(1000-t) \rightarrow x(t) = \int kt(1000-t)\,dt$$

$$\hookrightarrow x(t) = -333.33kx^3$$

$$x(4) = -333.33k(4^3) = 40$$

$$\hookrightarrow k = -0.001875$$

$$x(t) = 0.625x^3 \rightarrow x(6) = 0.625(6^3) = \boxed{135 \text{ infected after 6 days}}$$

4. Determine if the following first-order differential equation is homogeneous or not, and solve it:

$$x^2 \frac{dy}{dx} = 3xy + y^2$$

$\boxed{\text{It is non-homogeneous}}$

$$x^2 \frac{dy}{dx} - 3xy = y^2 \rightarrow \frac{dy}{dx} - \frac{3y}{x} = \frac{y^2}{x^2}$$

$$\hookrightarrow \frac{dy}{y^2} = \frac{dx}{x} - \frac{3}{x^2} \rightarrow \int \frac{dy}{y^2} = \int \frac{dx}{x} - \frac{3}{x^2}$$

$$\hookrightarrow -\frac{1}{y} = \ln x + \frac{3}{x} + C \rightarrow 1 = -y\left(\ln x + \frac{3}{x} + c\right)$$

$$\hookrightarrow \boxed{y(x) = \frac{-1}{\ln|x| + \frac{3}{x} + c}}$$

5. Find the general solution to the following 2^{nd}-order homogeneous differential equations:

a) $4y'' + y' = 0$

b) $y'' + 9y' = 0$

c) $y'' - y' - 6y = 0$

a) $4\lambda^2 + \lambda = 0$ $\quad \frac{-1 \pm \sqrt{1}}{8} \to -\frac{1}{4}, \; 0$

$$\boxed{y(x) = C_1 e^{-\frac{1}{4}x} + C_2}$$

b) $\lambda^2 + 9\lambda = 0$ $\quad \frac{-9 \pm \sqrt{81}}{2} \to -9, \; 0$

$$\boxed{y(x) = C_1 e^{-9x} + C_2}$$

c) $\lambda^2 - \lambda - 6 = 0$ $\quad \frac{1 \pm \sqrt{-23}}{2} \to 0.5 + 2.398i, \; 0.5 - 2.398i$

$$\boxed{y(x) = e^{0.5x}\left[C_1 \cos(2.398x) + C_2 \sin(2.398x)\right]}$$

6. Find the solution to the following 2nd-order homogeneous initial value problems:

a) $y'' + 16y' = 0, \quad y(0) = 2, \quad y'(0) = -2$

b) $y'' + 6y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 3$

a) $\lambda^2 + 16\lambda = 0$ $\quad \dfrac{-16 \pm 16}{2} \to -16,\ 0$

$y(x) = c_1 e^{-16x} + c_2 \qquad y'(x) = -16 c_1 e^{-16x}$

$y'(0) = -16 c_1 e^{0} = -2 \qquad c_1 = \frac{1}{8} \qquad y(0) = \frac{1}{8} e^{-16x} + c_2 = 2$

$$\boxed{y(x) = \frac{1}{8} e^{-16x} + \frac{15}{8}}$$

b) $\lambda^2 + 6\lambda + 5 = 0$ $\quad \dfrac{-6 \pm \sqrt{6^2 + 4(5)}}{2} \to -5,\ -1$

$y(x) = c_1 e^{-5x} + c_2 e^{-x} \to y'(x) = -5 c_1 e^{-5x} - c_2 e^{-x}$

$y'(0) = -5 c_1 e^{0} - c_2 e^{0}$

$$\to \boxed{y(x) = e^{-3x}(3e^{x} - 2)}$$

7. Determine the longest interval in which the following initial value problem is certain to have a unique solution:

$$(x-3)y'' - (x-3)(\tan x)y = 1 \qquad y(\pi) = 1 \qquad y'(\pi) = 2$$

The longest interval would be $(\pi - \delta, \pi + \delta)$ where $\delta > 0$ and does not contain $x = 3$

8. Find the Wronskian of the following set of solutions. Do these solutions form a fundamental set of solutions on the given interval?

$$\{x, xe^x\}, \qquad x > 0$$

$W(x, xe^x)(x) = x(xe^x)' - x'(xe^x)$

$\hookrightarrow = x(e^x + xe^x) - xe^x \dashrightarrow = xe^x + x^2e^x - xe^x$

$\hookrightarrow$ $\boxed{W = x^2e^x \leftarrow \text{Is nonzero for } \underline{\text{all}}\ x \text{ in the interval}}$