ECE 2050
Using Phasors to Combine Sinusoids of the Same Frequency

Bradley D. Clymer

Phasor notation is a shorthand that allows the quick addition or subtraction of multiple sinusoids *that have the same frequency*. In electrical and computer engineering, this is especially useful in calculating the steady-state response of a linear time independent system for a single sinusoidal input (also called AC or sinusoidal analysis). In this handout, we first use the fairly complicated method of trigonometric identities to add two sinusoids, then use the Euler expansion to add the same two sinusoids and finally we use phasor notation to combine the sinusoids.

Let us first define two sinusoidal signals that have the same frequency, but different magnitudes and different phase offsets:

$$g_1(t) = A_1 \cos(2\pi f t + \theta_1)$$

and

$$g_2(t) = A_2 \cos(2\pi f t + \theta_2)$$

where f is the cyclic frequency of *both* sinusoids in Hz, and A_1, A_2, θ_1, and θ_2 are constants. Now let us define a signal that is the sum of the two sinusoids:

$$g_3(t) = g_1(t) + g_2(t) = A_3 \cos(2\pi f t + \theta_3)$$

where A_3 and θ_3 are constants and it is easily shown that if the cyclic frequency of $g_1(t)$ and $g_2(t)$ is the same, then this must also be the cyclic frequency of $g_3(t)$.

Solution using trigonometric identities

To find A_3 and θ_3 using trigonometric identities, we can use the relationship

$$A\cos(\alpha+\beta) = A\cos(\alpha)\cos(\beta) - A\sin(\alpha)\sin(\beta)$$

Applying this to the three signals above, we get:

$$g_1(t) = A_1 \cos(2\pi f t)\cos(\theta_1) - A_1 \sin(2\pi f t)\sin(\theta_1)$$

$$g_2(t) = A_2 \cos(2\pi f t)\cos(\theta_2) - A_2 \sin(2\pi f t)\sin(\theta_2)$$

and

$$g_3(t) = A_3 \cos(2\pi f t)\cos(\theta_3) - A_3 \sin(2\pi f t)\sin(\theta_3)$$

Observing the sum of all terms that contain the factor $\cos(2\pi f t)$ we get:

$$A_3 \cos(2\pi f t)\cos(\theta_3) =$$

$$A_1 \cos(2\pi f t)\cos(\theta_1) + A_2 \cos(2\pi f t)\cos(\theta_2)$$

which can be reduced to

$$A_3 \cos(\theta_3) = A_1 \cos(\theta_1) + A_2 \cos(\theta_2)$$

Similarly, observing the sum of all terms that contain the factor $\sin(2\pi f t)$ we get

$$-A_3 \sin(2\pi f t)\sin(\theta_3) =$$

$$-A_1 \sin(2\pi f t)\sin(\theta_1) - A_2 \sin(2\pi f t)\sin(\theta_2)$$

which can be reduced to

$$A_3 \sin(\theta_3) = A_1 \sin(\theta_1) + A_2 \sin(\theta_2)$$

If we know values for A_1, A_2, θ_1, and θ_2 , we can solve for the value of A_3 using the Pythagorean relationship :

$$A_3 = \sqrt{(A_3 \cos\theta_3)^2 + (A_3 \sin\theta_3)^2}$$

where A_3 represents the length of a vector in polar coordinates that has an angle of θ_3 with the positive horizontal axis. Using substitutions for $A_3 \cos\theta_3$ and $A_3 \sin\theta_3$ from above, we get:

$$A_3 = \Big[(A_1 \cos\theta_1 + A_2 \cos\theta_2)^2 + (A_1 \sin\theta_1 + A_2 \sin\theta_2)^2\Big]^{1/2}$$

To find the angle, θ_3, we can apply the trigonometric identity:

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

which in this case is

$$\tan\theta_3 = \frac{A_3 \sin\theta_3}{A_3 \cos\theta_3} = \frac{A_1 \sin\theta_1 + A_2 \sin\theta_2}{A_1 \cos\theta_1 + A_2 \cos\theta_2}$$

Note that using the arctan to find θ_3, we have to pay attention to the sign of both the numerator and denominator to get the assignment of θ_3 in the correct quadrant in the polar description of the vector.

As a demonstration, let's consider an example:

$$g_1(t) = 120\cos(2\pi ft + 25^o)$$

$$g_2(t) = 450\cos(2\pi ft + 150^o)$$

where f is an arbitrary frequency in Hz that is the same for both $g_1(t)$ and $g_2(t)$ and

$$g_3(t) = g_1(t) + g_2(t) = A_3 \cos(2\pi ft + \theta_3)$$

Expanding $g_1(t)$ and $g_2(t)$ and combining terms containing $\cos(2\pi ft)$:

$$\begin{aligned} A_3 \cos\theta_3 &= 120\cos(25^0) + 450\cos(150^o) \\ &= -280.95 \end{aligned}$$

Similarly, combining terms containing $\sin(2\pi ft)$:

$$\begin{aligned} A_3 \sin\theta_3 &= 120\sin(25^o) + 450\sin(150^o) \\ &= 275.71 \end{aligned}$$

The Pythagorean relationship gives us the value of A_3:

$$A_3 = \sqrt{(-280.95)^2 + (275.71)^2} = 393.64$$

and θ_3 has the tangent

$$\tan\theta_3 = \frac{275.71}{-280.95}$$

where the horizontal component is negative and the vertical component is positive (second quadrant in polar space) so

$$\theta_3 = -44.46^o + 180^o = 135.54^o$$

Solution using Euler identities

The Euler identities for complex sinusoids allow a similar approach, but instead of needing to remember the sum angle trigonometric formulas, we can use the properties of complex numbers and powers. The Euler identities are

$$Ae^{j\alpha} = A\cos\alpha + jA\sin\alpha$$

$$A\cos\alpha = \frac{A}{2}e^{j\alpha} + \frac{A}{2}e^{-j\alpha}$$

and

$$A\sin\alpha = \frac{A}{2j}e^{j\alpha} - \frac{A}{2j}e^{-j\alpha}$$

Applying the Euler identities to the signals $g_1(t)$, $g_2(t)$ and $g_3(t)$ we get:

$$g_1(t) = A_1 \cos(2\pi ft+\theta_1)$$
$$= \frac{A_1}{2}e^{j(2\pi ft+\theta_1)} + \frac{A_1}{2}e^{-j(2\pi ft+\theta_1)}$$
$$= \frac{A_1}{2}e^{j\theta_1}e^{j2\pi ft} + \frac{A_1}{2}e^{-j\theta_1}e^{-j2\pi ft}$$
$$g_2(t) = A_2 \cos(2\pi ft+\theta_2)$$
$$= \frac{A_2}{2}e^{j(2\pi ft+\theta_2)} + \frac{A_2}{2}e^{-j(2\pi ft+\theta_2)}$$
$$= \frac{A_2}{2}e^{j\theta_2}e^{j2\pi ft} + \frac{A_2}{2}e^{-j\theta_2}e^{-j2\pi ft}$$

and

$$g_3(t) = g_1(t)+g_2(t) = A_3 \cos(2\pi ft+\theta_3)$$
$$= \frac{A_3}{2}e^{j(2\pi ft+\theta_3)} + \frac{A_3}{2}e^{-j(2\pi ft+\theta_3)}$$
$$= \frac{A_3}{2}e^{j\theta_3}e^{j2\pi ft} + \frac{A_3}{2}e^{-j\theta_3}e^{-j2\pi ft}$$

Grouping terms that contain the factor $e^{j2\pi ft}$:

$$\frac{A_1}{2}e^{j\theta_1}e^{j2\pi ft} + \frac{A_2}{2}e^{j\theta_2}e^{j2\pi ft} = \frac{A_3}{2}e^{j\theta_3}e^{j2\pi ft}$$

which can be reduced to

$$\frac{A_1}{2}e^{j\theta_1} + \frac{A_2}{2}e^{j\theta_2} = \frac{A_3}{2}e^{j\theta_3}$$

where $\frac{1}{2}A_3e^{j\theta_3}$ is a complex number (shown in polar form) that is the sum of two complex numbers (also shown in polar form). Therefore, to find the values for A_3 and θ_3 from the constants A_1, A_2, θ_1, and θ_2, we only need to apply the rules for addition of complex numbers.

As a side note, we can also group the terms contain the factor $e^{-j2\pi ft}$ to get

$$\frac{A_1}{2}e^{-j\theta_1}e^{-j2\pi ft} + \frac{A_2}{2}e^{-j\theta_2}e^{-j2\pi ft} = \frac{A_3}{2}e^{-j\theta_3}e^{-j2\pi ft}$$

which can be reduced to

$$\frac{A_1}{2}e^{-j\theta_1} + \frac{A_2}{2}e^{-j\theta_2} = \frac{A_3}{2}e^{-j\theta_3}$$

but this gives no additional information because $\frac{1}{2}A_3e^{-j\theta_3}$ is the complex conjugate for $\frac{1}{2}A_3e^{j\theta_3}$, $\frac{1}{2}A_1e^{-j\theta_1}$ is the complex conjugate for $\frac{1}{2}A_1e^{j\theta_1}$ and $\frac{1}{2}A_2e^{-j\theta_2}$ is the complex conjugate for $\frac{1}{2}A_2e^{j\theta_2}$. The complex arithmetic for this combination yields exactly the same result as the grouping for the terms that contain the factor $e^{j2\pi ft}$.

Applying the same numerical example used in the trigonometric identity approach,

$$g_1(t) = 120\cos(2\pi ft + 25^o)$$
$$= 60e^{j25^o}e^{j2\pi ft} + 60e^{-j25^o}e^{-j2\pi ft}$$
$$g_2(t) = 450\cos(2\pi ft + 150^o)$$
$$= 225e^{j150^o}e^{j2\pi ft}+225e^{-j150^o}e^{-j2\pi ft}$$

and

$$g_3(t) = g_1(t) + g_2(t) = A_3\cos(2\pi ft + \theta_3)$$
$$= \frac{A_3}{2}e^{j\theta_3}e^{j2\pi ft} + \frac{A_3}{2}e^{-j\theta_3}e^{-j2\pi ft}$$

Grouping terms that contain the factor $e^{j2\pi ft}$:

$$60e^{j25^o} + 225e^{j150^o} = \frac{A_3}{2}e^{j\theta_3}$$

An advanced calculator can automatically convert the polar forms of these complex numbers to cartesian form, add them, then convert the display back to polar form (for simpler calculators this conversion must be done by the user):

$$\frac{A_3}{2}e^{j\theta_3} = 196.82e^{j135.54^o}$$

so $A_3 = 393.64$ and $\theta_3 = 135.54^o$.

Solution using Phasors

The phasor approach to adding sinusoids of the same frequency is very similar to the approach using the Euler identities, but is a little simpler because it takes advantage of the complex conjugate redundancy of the Euler approach, and of the fact that all of the terms in the Euler approach have a (1/2) factor. Both of these shorten the procedure.

When adding sinusoids of the same frequency, the phasor representation is easily determined from the cosine-only form of the sinusoids (also called "laboratory form"). For a signal $g(t) = A\cos(2\pi ft + \theta)$, the *phasor* shorthand is $\tilde{G} = Ae^{j\theta}$, which is a complex number in polar form where A is the magnitude of the sinusoid and θ is the constant phase shift in the cosine-only description of the sinuoid. *The frequency of the sinusoid is not explicit in the phasor shorthand notation, but noted as an aside.*

To combine sinusoids of the *same frequency*, we convert each sinusoid into the phasor shorthand, then use complex arithmetic to add the individual phasor components to get the resulting phasor for the sum. Conversion of the sum phasor back to the cosine-only form of the sinusoidal time function is done by inspection, noting the frequency of the component sinusoids that were summed.

Using phasors with the previously introduced sinusoidal signals:

$$g_1(t) = A_1\cos(2\pi ft + \theta_1) \Rightarrow \tilde{G}_1 = A_1e^{j\theta_1}$$

$$g_2(t) = A_2\cos(2\pi ft + \theta_2) \Rightarrow \tilde{G}_2 = A_2e^{j\theta_2}$$

$$g_3(t) = g_1(t) + g_2(t) = A_3\cos(2\pi ft + \theta_3)$$

where f is the frequency in Hz. Note that to use phasors, this frequency must be the same for the two signals being summed together. The phasor for $g_3(t)$ is then

$$\tilde{G}_3 = A_3e^{j\theta_3} = A_1e^{j\theta_1} + A_2e^{j\theta_2}$$

which can be determined by the sum of the complex numbers $\tilde{G}_1$ and $\tilde{G}_2$. After calculating $\tilde{G}_3$ by complex arithmetic, we can determine the sinusoidal time function for $g_3(t)$ using the calculated magnitude of $\tilde{G}_3$, the calculated angle of $\tilde{G}_3$, and remembering the frequency, f, for $g_1(t)$ and $g_2(t)$ and substituting these into the formula for $g_3(t)$.

To repeat the same numerical example, this time let's use an actual frequency, $f =$ 100 Hz:

$$g_1(t) = 120\cos(200\pi t + 25^o)$$

$$g_2(t) = 450\cos(200\pi t + 150^o)$$

$$g_3(t) = g_1(t) + g_2(t) = A_3\cos(200\pi t + \theta_3)$$

The phasor shorthand for $g_1(t)$ is

$$g_1(t) \Rightarrow \tilde{G}_1 = 120e^{j25^o}$$

and we must remember $f = 100$ Hz. The phasor shorthand for $g_2(t)$ is

$$g_2(t) \Rightarrow \tilde{G}_2 = 450e^{j150^o}$$

and we must remember $f = 100$ Hz. The phasor shorthand for $g_3(t)$ is

$$g_3(t) \Rightarrow \tilde{G}_3 = A_3e^{j\theta_3}$$

and we must remember $f = 100$ Hz. We find the numerical value for $\tilde{G}_3$ by the complex addition of $\tilde{G}_1$ and $\tilde{G}_2$:

$$\tilde{G}_3 = \tilde{G}_1 + \tilde{G}_2 = 120e^{j25^o} + 450e^{j150^o}$$

Again, an advanced calculator can automatically convert the polar forms of these complex numbers to cartesian form, add them, then convert the display back to polar form (for simpler calculators this conversion must be done by the user):

$$\tilde{G}_3 = 393.64e^{j135.54^o}$$

The cosine-only form of the sinusoidal time function can be generated by inspection of the phasor, $\tilde{G}_3$, and remembering the frequency is $f = 100$ Hz:

$$g_3(t) = 393.64\cos\left(200\pi t + 135.54^o\right)$$