3.2 Solution of Linear Homogeneous Equations; the Wronskian

Now let's continue our discussion of second-order linear homogeneous ODEs

$$ay'' + by' + cy = 0$$

where a, b, and c are constants.

We introduce a linear **differential operator** to aid in our discussion. If $p(x)$ and $q(x)$ are continuous functions on the open interval I, then we define

$$L[y] = y'' + p(x)y' + q(x)y$$

for any function y that is twice differentiable on I.

- **Example:** What is $L[y](x) = y''(x) + p(x)y'(x) + q(x)y(x)$ if $y(x) = \cos x$, $p(x) = \cos x$, and $q(x) = x^4$?

$y(x) = \cos x$

$y'(x) = -\sin x$

$y''(x) = -\cos x$

$$L[y](x) = -\cos x + \cos x\,(-\sin x) + x^4\,(\cos x)$$

$$= \cos x\,(x^4 - \sin x - 1)$$

We can write a general homogeneous second-order linear ODE in terms of L: $y'' + p(x)y' + q(x)y = 0$

$$L[y] = 0$$

We can write an associated initial value problem in terms of L and the initial conditions as

$$\underbrace{L[y] = 0,}_{\text{ODE}} \qquad \underbrace{y(x_0) = y_0, \qquad y'(x_0) = y_0'}_{\text{IC's}}$$

where x_0 is any point in I and y_0 and y'_0 are real numbers.

The following theorem guarantees the existence and uniqueness of the solution to a second-order linear ODE:

Existence and Uniqueness Theorem:

The following initial value problem (IVP)

$$\underbrace{y'' + p(x)y' + q(x)y}_{L[y]=g(x)} = g(x), \qquad y(x_0) = y_0, \qquad y'(x_0) = y_0'$$

where p, q, and g are continuous on an open interval I containing the point x_0, has exactly one solution $y = \phi(x)$ and the solution exists throughout the interval I

Note that this theorem says the IVP has a solution (*existence*), this is the only solution (*uniqueness*), and the solution is defined throughout I where p and q are continuous and ϕ is at least twice differentiable

and g

- **Example:** Find the longest interval in which the solution of the IVP is certain to exist.

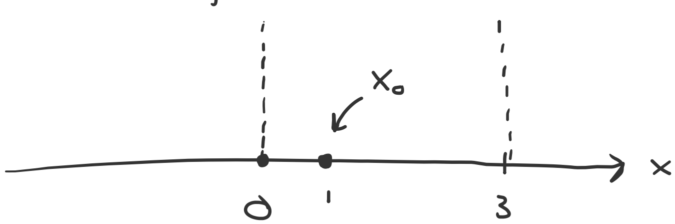
$$(x^2 - 3x)y'' + xy' - (x+3)y = 0, \qquad \underset{\uparrow x_0}{y(1)} = 2, \qquad \underset{\uparrow x_0}{y'(1)} = 1$$

Write in standard form:

$$y'' + \frac{x}{x^2 - 3x} y' - \frac{x+3}{x^2-3x} y = 0$$

$$y'' + \underbrace{\frac{x}{x(x-3)}}_{p(x)} y' - \underbrace{\frac{x+3}{x(x-3)}}_{q(x)} y = 0$$

$p(x)$ and $q(x)$ are discontinuous at $x = 0$ and $x = 3$:



possible intervals where a solution could exist:

$-\infty < x < 0$

$0 < x < 3$

$3 < x < \infty$

The interval $0 < x < 3$ contains $x_0 = 1$

$\Rightarrow$ The longest interval where a solution is guaranteed to exist is

$$\boxed{0 < x < 3}$$

Another useful theorem pertains to a superposition of solutions (i.e., a linear combination)

Principle of Superposition Theorem:

If y_1 and y_2 are two solutions that satisfy the differential equation $L[y] = 0$, then the linear combination

$$y = c_1 y_1 + c_2 y_2$$

is also a solution for any values of the constant c_1 and c_2

Proof:

$$\begin{aligned} L[y] &= L[c_1 y_1 + c_2 y_2] \\ &= (c_1 y_1 + c_2 y_2)'' + p(x)(c_1 y_1 + c_2 y_2)' + q(x)(c_1 y_1 + c_2 y_2) \\ &= c_1 (y_1'' + p(x) y_1' + q(x) y_1) + c_2 (y_2'' + p(x) y_2' + q(x) y_2) \\ &= c_1 \underbrace{L[y_1]}_{=0} + c_2 \underbrace{L[y_2]}_{=0} = 0 \end{aligned}$$

since y_1 and y_2 are solutions of $L[y] = 0$

$\Rightarrow$ The linear combination satisfies $L[y] = 0$ ✓

We now ask if all solutions to $L[y] = 0$ are included in the linear combination $y = c_1 y_1 + c_2 y_2$. First let's see if we can find values for c_1 and c_2 that satisfy our initial conditions:

$$c_1 y_1(x_0) + c_2 y_2(x_0) = y_0$$

$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = y_0'$$

We can solve this system of equations for c_1 and c_2:

$$c_1 = \frac{y_0 y_2'(x_0) - y_0' y_2(x_0)}{y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0)}, \qquad c_2 = \frac{-y_0 y_1'(x_0) + y_0' y_1(x_0)}{y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0)}$$

We can write these solutions in terms of determinants:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(x_0) \\ y_0' & y_2'(x_0) \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} y_1(x_0) & y_0 \\ y_1'(x_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}}$$

The denominator in these equations is called the **Wronskian determinant** W:

$$W[y_1, y_2] = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}$$ ← top row: functions ← bottom row: 1st derivatives

If $W \neq 0$, then the equations for the c_i have a unique solution for any y_0 and y_0'

If $W = 0$, then the numerators must also be zero for there to be a solution and there may be values of y_0 and y_0' for which no solutions exist

Theorem:

If y_1 and y_2 are two solutions that satisfy the differential equation $L[y] = 0$, and there are initial conditions

$$y(x_0) = y_0 \text{ and } y'(x_0) = y_0',$$

then it is always possible to choose constants c_1 and c_2 such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

satisfies the DE and the initial conditions if and only if

$$W[y_1, y_2] \neq 0$$

at x_0

- **Example:**

Find $W[y_1, y_2]$ if $y_1(x) = e^{-2x}$ and $y_2(x) = e^{-3x}$

$$y_1'(x) = -2e^{-2x} \qquad y_2'(x) = -3e^{-3x}$$

$$W[y_1, y_2](x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix}$$

$$= -3e^{-2x}e^{-3x} - (-2e^{-2x}e^{-3x})$$

$$= -3e^{-5x} + 2e^{-5x}$$

$$= -e^{-5x} \neq 0 \quad \text{for all } x$$

More generally, the determinant

$$W[f_1, f_2, \dots, f_n] = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

← functions

1st through $(n-1)$th derivatives of the functions

is called the Wronskian of the functions $f_1, f_2, \dots, f_n$

We say that a set of functions $f_1, f_2, \dots, f_n$ is **linearly dependent** if (LD)

$$c_1 f_1 + c_2 f_2 + \cdots c_n f_n = 0$$

and $c_1, c_2, \dots, c_n$ are not all zero. Otherwise, the functions are **linearly independent** (LI)

If $c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$ and all the c_i's are not zero, then you can one of the functions f_i as a linear combination of the other f_i's: e.g. $f_1 = -\frac{c_2}{c_1} f_2 + \dots - \frac{c_n}{c_1} f_n$

Thus, we can write any one of the functions as a linear combination of the other functions and we say the functions are LD.

Theorem: The set of solutions $y_1, y_2, \dots, y_n$ to a homogeneous linear nth-order DE on an interval I are linearly independent if and only if

$$W[y_1, y_2, \dots, y_n] \neq 0$$

for any x in the interval

Example: Are $y_1 = x$ and $y_2 = x^2 - 1$ LI?

$y_1' = 1 \qquad y_2' = 2x$

$$W[y_1, y_2] = \begin{vmatrix} x & x^2-1 \\ 1 & 2x \end{vmatrix} = x(2x) - (1)(x^2-1) = x^2 + 1 \neq 0 \text{ for all } x$$

$\Rightarrow$ y_1 and y_2 are LI on $(-\infty, \infty)$

In summary, the determinant

$$W[f_1, f_2, \dots, f_n] = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of the functions $f_1, f_2, \dots, f_n$

We discussed a few theorems about 2nd-order linear ODEs and the Wronskian.

We'll state a few more theoerems, and see how they relate to solving 2nd-order *homogeneous* linear ODEs in the cases we did not yet consider (complex roots and repeated roots of the characteristic equation).

Theorem:

If y_1 and y_2 are solutions of the DE $L[y] = 0$, then the two-parameter family of solutions

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

with arbitrary c_1 and c_2 includes every solution of the DE if and only if there is a point x_0 where the Wronskian of y_1 and y_2 is not zero (i.e, if y_1 and y_2 are linearly independent)

We therefore call $y(x) = c_1 y_1(x) + c_2 y_2(x)$ the **general solution** of the DE, and we say the functions y_1 and y_2 are a **fundamental set of solutions.**

Another way to say this is: Any set y_1 and y_2 of LI solutions of $L[y]=0$ on an open interval I is a fundamental set of solutions on that interval. The set of solutions is LI on I if and only if $W[y_1, y_2] \neq 0$ for any x in that interval.

We will use the following theorem when we consider complex roots of the character equation:

Theorem:

If $y = u(x) + iv(x)$ is a complex-valued solution of $L[y] = 0$ (where p and q are continuous real-valued functions), then $u(x)$ and $v(x)$ are also solutions

Verify the second theorem on the previous page:

$$L[y] = y'' + p(x)y' + q(x)y = 0$$

$$\left.\begin{aligned} y(x) &= u(x) + iv(x) \\ y'(x) &= u'(x) + iv'(x) \\ y''(x) &= u''(x) + iv''(x) \end{aligned}\right\} \text{sub into the ODE}$$

$$u'' + iv'' + p(x)(u' + iv') + q(x)(u + iv) = 0$$

$$\underbrace{(u'' + p(x)u' + q(x)u)}_{\text{real part}} + i\underbrace{(v'' + p(x)v' + q(x)v)}_{\text{imaginary part}} = 0$$

A complex number $z = a + ib$ is zero if and only if $a = 0$ and $b = 0$

The real part of our ODE is just $L[u] = 0$ and the imaginary part of our ODE is just $L[v] = 0$

Thus, we see that if $y = u + iv$ is a solution of $L[y] = 0$, then so are u and v.

- **Examples:**

Find $W[y_1, y_2]$ if $y_1(x) = \cos x$ and $y_2(x) = \sin x$

$y_1 = \cos x$ $\quad$ $y_2 = \sin x$

$y_1' = -\sin x$ $\quad$ $y_2' = \cos x$

$$W[y_1, y_2] = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x)$$

$$= \cos^2 x + \sin^2 x = \boxed{1}$$

Since $W \neq 0$, if y_1 and y_2 are solutions of $L[y] = 0$, then $y(x) = c_1 y_1 + c_2 y_2$ will satisfy $L[y] = 0$ and we can find c_1 and c_2 to satisfy any IC's

Find $W[y_1, y_2]$ if $y_1(x) = e^{-2x}$ and $y_2(x) = e^{-3x}$

See earlier in this lecture

3.3 Complex Roots of the Characteristic Equation

When we solve the second-order linear homogeneous ODE with constant coefficients:

$$ay'' + by' + cy = 0$$

we look for a solution of the form $y = e^{rt}$, resulting in the characteristic equation:

$$ar^2 + br + c = 0$$

If we solve for r using the quadratic formula, we find

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac < 0$, the roots of the characteristic equation will be a pair of complex conjugate numbers

$$r_1 = \lambda + i\mu \qquad r_2 = \lambda - i\mu$$

where λ and μ are real numbers. The corresponding solutions of the DE are

$$y_1(x) = e^{(\lambda+i\mu)x} \qquad y_2(x) = e^{(\lambda-i\mu)x}$$

Now recall Euler's formula:

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

We can therefore rewrite these solutions to the DE as

$$y_1(x) = e^{\lambda x}(\cos\mu x + i\sin\mu x) \qquad y_2(x) = e^{\lambda x}(\cos\mu x - i\sin\mu x)$$

Example: Find the general solution of

$$y'' + y' + 9.25y = 0$$

In the previous example, we saw that we could write the general solution in terms of the real and imaginary parts of the complex solutions.

We usually to prefer to work with solutions that are real valued functions, since the DEs we are considering have real coefficients. One of the theorems we considered earlier says that if $y(x) = u(x) + i(v(x)$ is a solution to a 2nd-order linear homogeneous ODE, then the real part u and the imaginary part v are also solutions.

We can choose the real part and imaginary parts of

$$y_1(x) = e^{\lambda x}(\cos \mu x + i \sin \mu x)$$

or the real or imaginary part of

$$y_2(x) = e^{\lambda x}(\cos \mu x - i \sin \mu x)$$

and see if they form a fundamental set of solutions. Let's choose the first option and compute the Wronskian:

$$W\left[e^{\lambda x} \cos \mu x \,, e^{\lambda x} \sin \mu x\right](x) =$$

Since $e^{2\lambda x} \neq 0$, we find that $W \neq 0$ as long as $\mu \neq 0$ (note that if $\mu = 0$, we have the case of repeated roots). Thus, we find that these real and imaginary parts form a fundamental set of solutions if $\mu \neq 0$ and the general solution of our DE is then

$$y(x) = c_1 e^{\lambda x} \cos \mu x + c_2 e^{\lambda x} \sin \mu x$$

- **Example:** Solve the following IVP:

$$16y'' - 8y' + 145y = 0, \qquad y(0) = -2, \qquad y'(0) = 1$$

- **Example:** Find the general solution of the following ODE:

$$y'' + 9y = 0$$

- **Example:** Solve the following IVP:

$$y'' - 6y' + 13y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

Example: Find the general solution of $y'' + 36y = 0$.

Example: Solve the IVP: $y'' + 10y = 0, \quad y(0) = 0, \quad y'(0) = 1$

Example: Find the general solution of $y'' + 2y' + 4y = 0$.

Example: Solve the IVP: $y'' - 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 1$