MATH-2415, Ordinary and Partial Differential Equations **Name:**
Summer 2023
Problem Set 2
Due June 11, 2023 by midnight

Directions: You can either

(I) Show all your work on the pages of the assignment itself, or

(II) Use separate paper to work out the problems. The paper must be plain white paper. No notebook paper, no lined paper. Engineering paper is acceptable.

For either selection, **clearly show all work that leads to your final answer**. Don't make me hunt for your steps! And DON'T turn in **any** scratched out work! Your final product must be clear and legible, or it will be returned, ungraded. **Please submit your work to Blackboard as a single pdf file**.

1. For each ODE, state the order and determine if the equation is linear or nonlinear

a. $(1-x)y'' - 4xy' + 5y = \cos x$

b. $\dfrac{dr}{dt} = -\dfrac{k}{r^2}$

c. $y\dfrac{d^3y}{dt^3} + (\sec^2 x)y = e^x$

d. $y' = e^x y - 3x^2$

a) $\boxed{\text{2nd Order, Linear}}$

b) $\boxed{\text{1st Order, Linear}}$

c) $\boxed{\text{3rd Order, Non-Linear}}$

d) $\boxed{\text{1st order, Linear}}$

2. Determine which of the following are solutions to the ODE: $x^2y'' - 3xy' + 4y = 0$
[Note: You do not need to solve the ODE here; just substitute the given solutions into the equation to see if any of the solutions satisfy the ODE]

a. x^2 $\quad 2x \quad 2$

b. $x^2 \ln x$ $\quad 2x\ln x + x \quad 2\ln x + 3$

c. $x^2 + x^2 \ln x$ $\quad 3x + 2x\ln x \quad 6 + 2\ln x$

d. $x^2 + 3x^3$ $\quad 2x + 9x^2 \quad 2 + 18x$

a) $\cancel{2x^2} - \cancel{6x^2} + \cancel{4x^2} = 0$

↳ $\boxed{0=0,\ x^2 \text{ is a solution}}$

b) $\cancel{2x^2\ln x} + \cancel{3x^2} - \cancel{6x^2\ln x} - \cancel{3x^2} + \cancel{4x^2\ln x} = 0$

↳ $\boxed{0=0,\ x^2\ln x \text{ is a solution}}$

c) $6x^2 + \cancel{2x^2\ln x} - 9x^2 - \cancel{6x^2\ln x} + 4x^2 + \cancel{4x^2\ln x} = 0$

↳ $\boxed{x^2=0,\ x^2+x^2\ln x \text{ is } \underline{\text{not}} \text{ a solution}}$

d) $\cancel{2x^2} + 36x^3 - \cancel{6x^2} - 27x^3 + \cancel{4x^2} + 12x^3 = 0$

↳ $\boxed{21x^3=0,\ x^2+3x^3 \text{ is } \underline{\text{not}} \text{ a solution}}$

3. Given the differential equation, $u_{xx} = 4u_y$

a. State the order and the type for the differential equation

b. Verify that $u(x, y) = e^{-36y}\cos 12x - e^{-36y}\sin 12x$ is a solution to this differential equation.

a) $\boxed{\text{2nd order PDE}}$

b) $u_y = -36e^{-36y}\left(\cos(12x) - \sin(12x)\right)$

$$u_x = \frac{-\pi\cos(12x)e^{-36y}}{15} - \frac{\pi\sin(12x)e^{-36y}}{15}$$

$$u_{xx} = \frac{\pi^2\sin(12x)e^{-36y}}{225} - \frac{\pi^2\cos(12x)e^{-36y}}{225}$$

$$\frac{\pi^2\sin(12x)e^{-36y}}{225} - \frac{\pi^2\cos(12x)e^{-36y}}{225} = -144e^{-36y}\left(\cos(12x) - \sin(12x)\right)$$

$\boxed{\text{Not a solution}}$

4. a) In class we showed that $y = \sin^{-1} xy$ is an implicit solution of the ODE $xy' + y = y'\sqrt{1 - x^2y^2}$. We first took the sine of both sides of the given solution to eliminate the inverse sine function, and then used implicit differentiation. Here you will show that this is a solution in a different way: Differentiate both sides of the given solution, using implicit differentiation on the inverse sine function.

b) In class we showed that $x + y = \tan^{-1} y$ is an implicit solution of the ODE $1 + y^2 + y^2y' = 0$. We differentiated both sides of solution, using implicit differentiation on the inverse tangent function. Here you will show that this is a solution in a different way: Take the tangent of both sides of the given solution to eliminate the inverse tangent function, and then used implicit differentiation.

a) $y' = (\sin^{-1}(xy))' = \dfrac{y}{\sqrt{1-x^2y^2}} \rightarrow \quad 0 + y = \dfrac{y\sqrt{1-x^2y^2}}{\sqrt{1-x^2y^2}}$

↳ $\boxed{y = y}$

b) $\tan(x) + \tan(y) = y \rightarrow y' = \dfrac{\sec^2(x)}{\sec^2(y)}$

↳ $1 + y^2 - y^2 \dfrac{\sec^2(x)}{\sec^2(y)} = 0 \rightarrow y^2 \sec^2(x)\cos^2(y) = 1 + y^2$

↳ $y \sec(x)\cos(y) = \sqrt{1+y^2} \rightarrow$ solve w/ trig identities, I just can't figure out which one(s)

5. Solve each first-order linear ODE using the method of integrating factors:

a. $x^2y' - 2xy = 1/x$

b. $\sqrt{x^2+1}\,\dfrac{dy}{dx} + xy = x$

c. $(t\ln t)\dfrac{dy}{dt} + y = \ln t$

a) $y' - \dfrac{2y}{x} = \dfrac{1}{x^3} \rightsquigarrow$ $p(x) = \dfrac{-2}{x}$ $\quad g(x) = \dfrac{1}{x^3}$ $\quad \mu(x) = e^{\int \frac{-2}{x}dx}$ $\quad u = x \quad du = dx$

$\hookrightarrow \mu(x) = e^{-2\int \frac{du}{u}} \rightarrow = e^{-2\ln|u|} = e^{\ln|u|^{-2}} = \dfrac{1}{|u|^2} = \dfrac{1}{x^2}$

$\hookrightarrow y(x) = x^2\left[\int \frac{1}{x^5}dx + c\right] \rightsquigarrow \boxed{y(x) = -\dfrac{5}{x^4} + C}$

b) $\dfrac{dy}{dx} + \dfrac{xy}{\sqrt{x^2+1}} = \dfrac{x}{\sqrt{x^2+1}}$ $\rightsquigarrow$ $p(x) = \dfrac{x}{\sqrt{x^2+1}}$ $\quad g(x) = \dfrac{x}{\sqrt{x^2+1}}$ $\quad \mu(x) = e^{\int \frac{x}{\sqrt{x^2+1}}dx}$ $\quad u = \sqrt{x^2+1} \quad du = \dfrac{x}{\sqrt{x^2+1}}dx$

$\hookrightarrow \mu(x) = e^{\int du} \rightarrow \mu(x) = e^{0} \rightarrow \mu(x) = 1$

$\hookrightarrow y(x) = \int \frac{x}{\sqrt{x^2+1}}dx + C \rightsquigarrow \boxed{y(x) = \dfrac{1}{(x^2+1)^{3/2}}}$

c) $\frac{dy}{dt} + \frac{y}{t\ln t} = \frac{1}{t} \rightarrow$ $p(t) = \frac{1}{t\ln t}$ $\mu(t) = e^{\int \frac{1}{t\ln t} dt}$

$g(t) = \frac{1}{t}$ $u = t\ln t$ $du = \ln t + 1\, dt$

$\hookrightarrow \mu(t) = e^{\ln(\ln(t))} \rightarrow = \ln(t)$

$\hookrightarrow y(t) = \frac{1}{\ln(t)}\left[\int \frac{\ln(t)}{t} dt + C\right] \rightarrow \boxed{y(t) = \frac{1}{2}\ln(t) + C}$

6. Solve each first-order linear initial value problem (IVP) using the method of integrating factor

a. $y' + y = e^x \qquad y(0) = 1$

b. $x^2 \dfrac{dy}{dx} + 3xy = 1 \qquad y(1) = 3$

c. $(\cos x)y' + (\sin x)y = 3 \qquad y(\pi/4) = 1$

a) $p(x) = 1 \qquad \mu(x) = e^{\int dx} = 1$

$g(x) = e^x$

$y(x) = \int e^x\,dx + C = e^x + C$

$y(0) = e^0 + C = 1 + C \rightarrow C = 0 \qquad \boxed{y(x) = e^x}$

b) $\dfrac{dy}{dx} + \dfrac{3y}{x} = \dfrac{1}{x^2} \qquad p(x) = \dfrac{3}{x} \qquad g(x) = \dfrac{1}{x^2} \qquad \mu(x) = e^{\int \frac{3}{x}dx} = e^{3\ln(x)} = e^{\ln(x)^3} = x^3$

$\hookrightarrow y(x) = \dfrac{1}{x^3}\int x\,dx + C \rightarrow y(x) = \dfrac{1}{x^3} + C \qquad y(1) = 1 + C \rightarrow C = 2$

$\boxed{y(x) = \dfrac{1}{x^3} + 2}$

c) $y' + \tan(x)y = \dfrac{3}{\cos(x)} \qquad p(x) = \tan(x) \qquad g(x) = \dfrac{3}{\cos(x)} \qquad \mu(x) = e^{\int \tan x\,dx}$

$\hookrightarrow \mu(x) = e^{\ln(\cos x)^{-1}} \rightarrow \mu(x) = \dfrac{1}{\cos(x)}$

$y(x) = \cos(x)\left[\int \dfrac{3}{\cos(x)^2}\,dx + C\right] \rightarrow y(x) = 3\sin(x) + C$

$y(\frac{\pi}{4}) = 3\sin(\frac{\pi}{4}) + C = \dfrac{3\sqrt{2}}{2} + C = 1 \qquad C = -1.12132034356$

$\boxed{y(x) = 3\sin(x) - 1.12132034356}$

7. Solve each separable first-order ODE:

a. $y' \sin t = y \ln y$

b. $\frac{dy}{dx} = \frac{2xy^2 + x}{x^2 y - y}$:(

c. $y' + 2xy^2 = 0$

a) $\frac{dy}{dt} \sin t = y \ln y \rightarrow dy \sin t = y \ln y \, dt \rightarrow \frac{1}{y \ln y} dy = \frac{1}{\sin t} dt$

$\hookrightarrow \ln|\ln(y)| = \ln|\sin(t)| \rightarrow \ln(y) = \sin(t) \rightarrow \boxed{y = e^{\sin(t)} + C}$

b) $dy = \frac{2xy^2 + x}{x^2 y - y} dx$

pass

c) $\frac{dy}{dx} = -2xy^2 \rightarrow dy = -2xy^2 dx \rightarrow \frac{1}{y^2} dy = -2x dx$

$\hookrightarrow -\frac{1}{y} = -x^2 \rightarrow \frac{1}{y} = x^2 \rightarrow \boxed{y = \frac{1}{x^2} + C}$

8. Find the general solution to each separable first-order ODE, and solve the corresponding IVP:

a. $xy' = y$ $\qquad y(2) = 3$

b. $\cos x \cos y \, dx - \sin x \sin y \, dy = 0$ $\qquad y(\pi/2) = \pi$

c. $(1+y)\dfrac{dy}{dt} = y$ $\qquad y(1) = 1$

a) $x\frac{dy}{dx} = y \rightarrow x\,dy = y\,dx \rightarrow \int \frac{1}{y}dy = \int \frac{1}{x}dx \rightarrow \ln y = \ln x$

$\hookrightarrow y = x + c \rightarrow y = 2 + c = 3 \rightarrow c = 1 \rightarrow \boxed{y(x) = x + 1}$

b) $\cos x \cos y \, dx = \sin x \sin y \, dy \rightarrow \frac{\cos x}{\sin x}dx = \frac{\sin y}{\cos y}dy$

$\hookrightarrow \int \cot x \, dx = \int \tan y \, dy \rightarrow \ln|\sin(x)| = \ln|\cos(y)|$

$\hookrightarrow \sin(x) = \cos(y) \rightarrow y = \cos^{-1}(\sin(x)) + C$

$C = \pi$

$\hookrightarrow y = \underbrace{\cos^{-1}(\sin(\tfrac{\pi}{2}))}_{0} + C \longrightarrow \boxed{y(x) = \cos^{-1}(\sin(x)) + \pi}$

c) $1+y\,dy = y\,dt \rightarrow \int \frac{1+y}{y}dy = dt \rightarrow \ln|y| + y = t + c$

$c = 1$

$\hookrightarrow y + e^{y} = e^{t+c} \rightarrow y = \frac{1}{e^{-t+c}} \rightarrow y = \frac{1}{e^{-t-c}}$

$\boxed{y(x) = \frac{1}{e^{-t-1}}}$