MATH-2415, Ordinary and Partial Differential Equations **Name:**
Summer 2023
Problem Set 7
Due July 23, 2023 by midnight

Directions: You can either

(I) Print this sheet and show all work on the sheet itself, or

(II) Use separate paper to work out the problems. The paper must be plain white paper. No notebook paper, no lined paper. Engineering paper is acceptable.

For either selection, clearly show all work that leads to your final answer. Don't make me hunt for your steps! And DON'T turn in **any** scratched out work! Your final product must be clear and legible, or it will be returned, ungraded.

You can scan your work and save the pages as a single pdf. Or you can take pictures of your work, add the pictures to Word or Powerpoint and export the pages to a single pdf. You will submit the pdf to me via email.

1. A mass weighing 16 N stretches a spring 3 cm. The mass is attached to a viscous damper with a damping constant of 2 N·s/m, and is set into motion from its equilibrium position with a downward velocity of 3 cm/s. Find the position u of the mass at any time t, and determine when the mass first returns to its equilibrium position.

a) $mu'' + \gamma u' + ku = 0 \rightarrow \frac{1}{2}u'' + 2u' + 64u = 0$

$v(0) = 0 \qquad v'(0) = \frac{1}{2}$

$\frac{1}{2}r^2 + 2r + 64 = 0 \rightarrow r^2 + 4r + 128 = 0 \qquad r = -2 \pm 2\sqrt{31}\, i$

$v(t) = e^{-2t}\left[C_1\cos(2\sqrt{31}\,t) + C_2\sin(2\sqrt{31}\,t)\right] \qquad C_1 = 0$

$v'(0) = \frac{1}{2} = 2\sqrt{31}\,C_2 \rightarrow C_2 = \frac{1}{4\sqrt{31}}$

$\boxed{v(t) = \frac{1}{4\sqrt{31}} e^{-2t}\sin(2\sqrt{31}\,t)}$

equilibrium when $2\sqrt{31}\,t = \pi$ so $\boxed{t = 0.2821\,s}$

2. Solve the following IVPs that govern mass-spring systems:

a) $u'' + 2u' + 3u = 6\cos 3t \qquad u(0) = 1, \qquad u'(0) = -1$

b) $u'' + u = \frac{1}{4}\cos 2t \qquad u(0) = 0, \qquad u'(0) = 0$

a) $u'' + 4u = 2\cos 2t \qquad u(0) = 0, \qquad u'(0) = 0$

a) $C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t)$

$u(t) = C_3\cos(3t) + C_4\sin(3t)$

$C_3 = 0 \qquad C_4 = -2$

$-2\sin(3t)$

$$\boxed{u(t) = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t) - 2\sin(3t)}$$

b) $u(t) = C_1\cos(t) + C_2\sin(t)$

$u'' + u = 0$

$C_3\cos(2t) + C_4\sin(2t)$

$C_3 = 0$
$C_4 = 34$

$u_p(t) = \frac{4}{3}\sin(2t)$

$$\boxed{u(t) = C_1\cos(t) + C_2\sin(t) + \frac{4}{3}\sin(2t)}$$

c) $u(t) = C_1\cos(2t) + C_2\sin(2t)$

$C_3\cos(2t) + C_4\sin(2t)$

$C_3 = 1$

$C_4 = 0$

$$\boxed{u(t) = C_1\cos(2t) + C_2\sin(2t) + \sin(2t)}$$

3. Solve the following boundary value problems or show that no solution exists:

a) $y'' - 2y' + 2y = 0, \quad y(0) = 1,\ y'(0) = 0$

b) $y'' - 2y' + 2y = 0, \quad y(0) = 1,\ y(\pi) = -1$

c) $y'' - 2y' + 2y = 0, \quad y(0) = 1,\ y\left(\frac{\pi}{2}\right) = 1$

d) $y'' - 2y' + 2y = 0, \quad y(0) = 0,\ y(\pi) = 0$

4. In class we solved the eigen-problem:

$$y'' + \lambda y = 0 \qquad y(0) = 0 \quad y(\pi) = 0$$

and showed that the nontrivial solutions are given by the eigen-pair (λ_n, y_n) for $n = 1,2,3,\ldots$ where we found:

$$\lambda_n = n^2, \qquad y_n(x) = \sin nx$$

Following that example, find the solutions to the eigen-problem:

$$y'' + \lambda y = 0 \qquad y(0) = 0 \quad y'(\pi) = 0$$

(Be sure to consider positive, negative, and zero eigenvalues like we did in class).

$y(0)=0 \rightarrow c_1 = 0$

$y(x) = c_2 \sin\sqrt{\lambda}x \rightarrow y(0) = c_2\sqrt{\lambda}\cos\sqrt{\lambda}x \rightarrow \sqrt{\lambda} = \frac{(2n-1)}{2}$

$\rightarrow \boxed{y(x) = \frac{x}{2}\sin(2n-1)}$

5. (a) Show that the given functions are orthogonal on the indicated interval.

$$f_1(x) = x^2, \quad f_2(x) = x^3 \qquad [-1,1]$$

(b) Find the constants c_1 and c_2 such that $f_3(x) = x + c_1x^2 + c_2x^3$ is orthogonal to both f_1 and f_2 on the same interval

6. Sketch the graph of the function for three periods:

$$f(x) = \begin{cases} 0 & \text{if } -3 \leq x < -1 \\ 1 - x^2 & \text{if } -1 \leq \mathrm{x} < 1 \\ 0 & \text{if } 1 \leq x < 3 \end{cases}$$

$$f(x + 6) = f(x)$$

7. Show that the following pairs of functions are orthogonal on the interval $[-\pi, \pi]$. (Hint: Use the identity $2\sin A \sin B = \cos(A+B) - \cos(A-B)$)

(a) $\sin mx$, $\cos nx$ for any integers m, n

(b) $\sin mx$, $\sin nx$ for integers $m \neq n$

8. Find the Fourier Series for the following function:

$$f(x) = \begin{cases} 0 & \text{if } -2 \leq x < 0 \\ x & \text{if } 0 \leq x < 2 \end{cases}$$

$$f(x+4) = f(x)$$