MATH-2415, Ordinary and Partial Differential Equations **Name:**
Summer 2023
Problem Set 5
Due July 9, 2023 by midnight

Directions: You can either

(I) Print this sheet and show all work on the sheet itself, or

(II) Use separate paper to work out the problems. The paper must be plain white paper. No notebook paper, no lined paper. Engineering paper is acceptable.

For either selection, clearly show all work that leads to your final answer. Don't make me hunt for your steps! And DON'T turn in **any** scratched out work! Your final product must be clear and legible, or it will be returned, ungraded.

You can scan your work and save the pages as a single pdf. Or you can take pictures of your work, add the pictures to Word or Powerpoint and export the pages to a single pdf. You will submit the pdf to blackboard.

1. Verify that $y_1(x) = e^{-x/3}\cos\frac{\sqrt{2}}{3}x$ and $y_2(x) = e^{-x/3}\sin\frac{\sqrt{2}}{3}x$ are solutions of the differential equation

$$3y'' + 2y' + y = 0 \ \text{ on } (-\infty, \infty)$$

Compute the Wronskian $W(y1, y2)$ and determine if $\{y_1, y_2\}$ forms a fundamental set of solutions.

$$y_1'(x) = -\frac{1}{3}e^{-\frac{x}{3}}x\cos\left(\sqrt{\frac{2}{3}}x\right) - \sqrt{\frac{2}{3}}\,e^{-\frac{x}{3}}x\sin\left(\sqrt{\frac{2}{3}}x\right)$$

$$y_1''(x) = \frac{2}{9}e^{-\frac{x}{3}}x\cos\left(\sqrt{\frac{2}{3}}x\right) + \frac{4}{9}\sqrt{\frac{2}{3}}\,e^{-\frac{x}{3}}x\sin\left(\sqrt{\frac{2}{3}}x\right) - \frac{2}{3}e^{-\frac{x}{3}}x\sin\left(\sqrt{\frac{2}{3}}x\right) - \frac{2}{3}\sqrt{\frac{2}{3}}\,e^{-\frac{x}{3}}x\cos\left(\sqrt{\frac{2}{3}}x\right)$$

$$3\left[\frac{2}{9}e^{-\frac{x}{3}}x\cos\left(\sqrt{\frac{2}{3}}x\right) + \frac{4}{9}\sqrt{\frac{2}{3}}\,e^{-\frac{x}{3}}x\sin\left(\sqrt{\frac{2}{3}}x\right) - \frac{2}{3}e^{-\frac{x}{3}}x\sin\left(\sqrt{\frac{2}{3}}x\right) - \frac{2}{3}\sqrt{\frac{2}{3}}\,e^{-\frac{x}{3}}x\cos\left(\sqrt{\frac{2}{3}}x\right)\right]$$

$$+\,2\left[-\frac{1}{3}e^{-\frac{x}{3}}x\cos\left(\sqrt{\frac{2}{3}}x\right) - \sqrt{\frac{2}{3}}\,e^{-\frac{x}{3}}x\sin\left(\sqrt{\frac{2}{3}}x\right)\right]$$

• $e^{-\frac{x}{3}} x \cos\left(\sqrt{\frac{2}{3}} x\right)$

$\hookrightarrow$ $\frac{4}{3}\sqrt{\frac{2}{3}}\, e^{-\frac{x}{3}} x \sin\left(\sqrt{\frac{2}{3}} x\right) + e^{-\frac{x}{3}} x \cos\left(\sqrt{\frac{2}{3}} x\right) = 0$

$y_1(x)$ is a solution

y_2

$\hookrightarrow$ $-\frac{4}{3}\sqrt{\frac{2}{3}}\, e^{-\frac{x}{3}} x \cos\left(\sqrt{\frac{2}{3}} x\right) + e^{-\frac{x}{3}} x \sin\left(\sqrt{\frac{2}{3}} x\right) = 0$

$y_2(x)$ is a solution

$$W(y_1, y_2) = \left(e^{-\frac{x}{3}} x \cos\left(\sqrt{\frac{2}{3}} x\right)\right)\left(\sqrt{\frac{2}{3}}\, e^{-\frac{x}{3}} x \cos\left(\sqrt{\frac{2}{3}} x\right)\right)$$
$$- \left[-\frac{1}{3} e^{-\frac{x}{3}} x \cos\left(\sqrt{\frac{2}{3}} x\right) - \sqrt{\frac{2}{3}}\, e^{-\frac{x}{3}} x \sin\left(\sqrt{\frac{2}{3}} x\right)\right]\left(e^{-\frac{x}{3}} x \sin\left(\sqrt{\frac{2}{3}} x\right)\right)$$

$\hookrightarrow$

$$W(y_1, y_2) = \frac{2}{3} e^{-\frac{2x}{3}} x \cos^2\left(\sqrt{\frac{2}{3}} x\right) + \frac{2}{3} e^{-\frac{2x}{3}} x \sin^2\left(\sqrt{\frac{2}{3}} x\right)$$

$\hookrightarrow$ $\boxed{W(y_1, y_2) = \frac{2}{3} e^{-\frac{2x}{3}}}$

2. Find the general solution to the following 2nd-order homogeneous differential equations:

a) $y'' + 16y = 0$

b) $4y'' + 2y' + y = 0$

d) $y'' - 4y' + 5y = 0$

a) $m^2 + 16 = 0 \rightarrow (m+4)(m-4) = 0$

$$\boxed{y(x) = C_1 e^{4x} + C_2 e^{-4x}}$$

b) $4m^2 + 2m + 1 = 0 \rightarrow$ roots $= -\frac{1}{4} \pm \frac{\sqrt{3}i}{4}$

$$\boxed{y(x) = c_1 \cos\left(-\frac{1}{4} + \frac{\sqrt{3}i}{4}\right) + c_2 \sin\left(-\frac{1}{4} - \frac{\sqrt{3}i}{4}\right)}$$

d) $m^2 - 4m + 5 = 0 \rightarrow$ roots $= 2 \pm i$

$$\boxed{y(x) = C_1 e^{(2+i)x} + C_2 e^{(2-i)x}}$$

3. Find the solution to the following 2nd-order homogeneous initial value problems:

a) $y'' + 25y = 0, \quad y(0) = 2, \quad y'(0) = -2$

b) $2y'' - 2y' + y = 0, \quad y(0) = -1, \quad y'(0) = 0$

a) $m^2 + 25 = 0 \rightarrow (m+5)(m-5) = 0$

$y(x) = c_1 e^{5x} + c_2 e^{-5x} \rightarrow y(0) = c_1(1) + c_2(1) = 0 \rightarrow c_1 = -c_2$

$y'(x) = 5ce^{5x} + 5ce^{-5x} \rightarrow y'(0) = 5c + 5c = -2 \rightarrow c = -\frac{1}{5}$

$$\boxed{y(x) = -\frac{1}{5}e^{5x} + \frac{1}{5xe}^{-5x}}$$

b) $2m^2 - 2m + 1 = 0 \rightarrow$ roots $\frac{1}{2} \pm \frac{1}{2}i$

$y(x) = c_1 \cos\left(\left(\frac{1}{2} + \frac{1}{2}i\right)x\right) + c_2 \sin\left(\left(\frac{1}{2} - \frac{1}{2}i\right)x\right)$

$y(0) = c_1 = -1$

$c_2 = 1$

$$\boxed{y(x) = -\cos\left(\left(\frac{1}{2} + \frac{1}{2}i\right)x\right) + \sin\left(\left(\frac{1}{2} - \frac{1}{2}i\right)x\right)}$$

4. Find the general solution to the following 2^{nd}-order homogeneous differential equations:

a) $y'' - 18y' + 81y = 0$

b) $2y'' - 16y' + 32y = 0$

c) $y'' = 0$

d) $y'' + 2\sqrt{3}y' + 3y = 0$

a) $m^2 - 18m + 81 = 0 \rightarrow (m-9)^2 = 0$

$$\boxed{y(x) = (C_1 x + C_2)e^{9x}}$$

b) $m^2 - 8m + 16 = 0 \rightarrow (m-4)^2 = 0$

$$\boxed{y(x) = (C_1 x + C_2)e^{4x}}$$

c) $\boxed{y(x) = C_1 x + C_2}$

d) $m^2 + 2\sqrt{3}m + 3 = 0 \rightarrow$ root $-\sqrt{3}$

$$\boxed{y(x) = (C_1 x + C_2)e^{-\sqrt{3}x}}$$

5. Find the solution to the following 2nd-order homogeneous initial value problems:

a) $4y'' + 4y' + y = 0 \qquad y(0) = 1 \qquad y'(0) = 4$

b) $16y'' - 8y' + y = 0 \qquad y(0) = -1 \qquad y'(0) = -2$

c) $y'' + 2\sqrt{5}y' + 5y = 0 \qquad y(0) = 5 \qquad y'(0) = -1$

a) $4r^2e^{rx} + 4re^{rx} + e^{rx} = 0 \rightarrow e^{rx}(4r^2 + 4r + 1) = 0$

$\hookrightarrow 4r^2 + 4r + 1 = 0 \rightarrow (2r+1)^2 = 0$

$y(x) = C_1e^{-\frac{1}{2}x} + C_2xe^{-\frac{1}{2}x}$

$y(0) = C_1 = 1$

$y'(x) = -\frac{1}{2}e^{-\frac{1}{2}x} + C_2e^{-\frac{1}{2}x} - \frac{1}{2}C_2e^{-\frac{1}{2}x}$

$y'(0) = -\frac{1}{2} + C_2 = 4 \rightarrow C_2 = \frac{9}{2}$

$\boxed{y(x) = e^{-\frac{1}{2}x} + \frac{9}{2}xe^{-\frac{1}{2}x}}$

b) $16m^2 - 8m + 1 = 0 \rightarrow (4m-1)^2 = 0$

$y(x) = C_1e^{\frac{1}{4}x} + C_2xe^{\frac{1}{4}x}$

$y(0) = C_1 = -1$

$y'(0) = -\frac{1}{4} + C_2 = -2$

$\hookrightarrow C_2 = -\frac{9}{4}$

$\boxed{y(x) = -e^{\frac{1}{4}x} - \frac{9}{4}xe^{\frac{1}{4}x}}$

c)

$$m^2 + 2\sqrt{5}\,m + 5 = 0 \longrightarrow \overset{\text{root}}{-\sqrt{5}}$$

$$y(x) = C_1 e^{-\sqrt{5}x} + C_2 x e^{-\sqrt{5}x}$$

$$y(0) = C_1 = 5$$

$$\boxed{y(x) = 5e^{-\sqrt{5}x} + (5\sqrt{5} - 1)\,x e^{-\sqrt{5}x}}$$

$$y'(0) = 5\sqrt{5} + C_2 = -1$$

$$\hookrightarrow C_2 = 5\sqrt{5} - 1$$

6. Find the general solution to the following 2nd-order homogeneous differential equations:

a) $y'' + 14y' + 54y = 0$

b) $5y'' + 6y' + \frac{9}{5}y = 0$

c) $y'' + \left(3 - \sqrt{5}\right)y' - 3\sqrt{5}y = 0$

a) $m^2 + 14m + 54 = 0 \rightarrow$ roots $-7 \pm \sqrt{103}$

$$\boxed{y(x) = C_1 e^{(-7+\sqrt{103})x} + C_2 x e^{(-7-\sqrt{103})x}}$$

b) $5m^2 + 6m + \frac{9}{5} = 0 \rightarrow$ roots $\frac{3(\pm\sqrt{2}-1)}{5}$

$$\boxed{y(x) = C_1 e^{\frac{3(+\sqrt{2}-1)}{5}x} + C_2 x e^{\frac{3(-\sqrt{2}-1)}{5}x}}$$

c) $m^2 + (3-\sqrt{3})m - 3\sqrt{5} = 0 \rightarrow$ roots $\sqrt{5}, -3$

$$\boxed{y(x) = C_1 e^{\sqrt{3}x} + C_2 e^{-3x}}$$

7. Use reduction of order to find a second solution to the differential equation. Then give the general solution.

$$2x^2y'' + 3xy' - y = 0 \qquad y_1(x) = \sqrt{x}$$

$y_1'(x) = \frac{1}{2}\sqrt{x}$

$y_2'(x) = v'(x)y_1(x) + v(x)y_1'(x)$

$y_2''(x) = v''(x)y_1(x) + 2v'(x)y_1'(x) + v(x)y_1''(x)$

$y_1''(x) = 2x^2(v''(x)\cdot\sqrt{x}) + 4x^2v'(x)(\frac{1}{2}\sqrt{x}) + 3x\,v(x)(\frac{1}{2}\sqrt{x}) - v(x)\sqrt{x} = 0$

$= xv''(x) + v'(x) + 2v(x) = 0$

$y_2(x) = v(x)\,y_1(x)$

$\boxed{y(x) = C_1y_1(x) + C_2y_2(x)}$

8. Find the general solution of the nonhomogeneous equations using the method of undetermined coefficients:

a) $y'' + 4y' - 2y = 2x^2 - 3x + 6$

b) $y'' - y' + y = 2\sin 3x$

c) $y'' - 2y' - 3y = 4x - 5 + 6xe^{3x}$

a) $m^2 + 4m - 2 = 0 \rightarrow -2 \pm \sqrt{6}$

$\boxed{y(x) = C_1 e^{(-2+\sqrt{6})x} + C_2 e^{(-2-\sqrt{6})x}}$ $\quad y = y(x) + y$

b) $m^2 - m + 1 = 0 \rightarrow \frac{1}{2} \pm \frac{3}{2} i$

$\boxed{y(x) = e^{\frac{1}{2}x}\left[c_1 \cos \tfrac{3}{2}x + C_2 \sin \tfrac{3}{2}x\right]}$

c) $m^2 - 2m - 3 = 0 \rightarrow (m-3)(m+1) = 0$

$\boxed{y(x) = C_1 e^{3x} + C_2 e^{-x}}$