## **Homework 7 - Math 2568 (Autumn 2022)**

Prof. Cueto

**Due date:** Friday October 28, 2022 (in class).

The sections and problem numbers refer to the course's textbook (L.W. Johnson, R.D. Riess, J.T. Arnold: *Introduction to Linear Algebra*, 5th edition, Pearson.)

| Section | Assigned Problems | Problems to be turned in |
|---|---|---|
| §3.7 | 1, 2, 3, 4, 5, 7, 8, 10, 11, 15, 18, 19, 20, 21, 23, 25, 29, 33, 35, 36, 37, 41 | 2, 4, 7, 10, 11, 18, 19, 21, 23, 33, 35, 36, 37, 41 |
| §5.2 | 1, 2, 3, 6, 9, 10, 19, 25, 29, 31, 33, 36 | 1, 2, 10, 19, 29, 33 |

Section 3.7

2) $A = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix}$

a) $T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2-2 \\ -6+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

b) $T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-1 \\ -9+3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

$T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

c) $T\left(\begin{bmatrix}2\\0\end{bmatrix}\right) = \begin{bmatrix}1 & -1\\-3 & 3\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}2+0\\-6+0\end{bmatrix} = \begin{bmatrix}2\\-6\end{bmatrix}$

$T\left(\begin{bmatrix}2\\0\end{bmatrix}\right) = \begin{bmatrix}2\\-6\end{bmatrix}$

d) $T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$

4) $\begin{bmatrix}2x_1 - 3x_2\\-x_1 + x_2\end{bmatrix} = \begin{bmatrix}2\\-2\end{bmatrix}$

$2x_1 - 3x_2 = 2 \to x_1 = \frac{3}{2}x_2 + 1$

$-x_1 + x_2 = -2 \to x_2 = x_1 - 2$

$x_1 = \frac{3}{2}(x_1 - 2) + 1 = \frac{3}{2}x_1 - 2$

$-\frac{1}{2}x_1 = -2 \to \boxed{x_1 = 4}$

$x_2 = x_1 - 2 = 4 - 2 = 2$

$x = \begin{bmatrix}4\\2\end{bmatrix}$

7) $\begin{bmatrix}1 & -1\\-3 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$ $\qquad \begin{bmatrix}x_1 - x_2\\-3x_1 + 3x_2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

$x_1 - x_2 = 1 \to x_1 = x_2$

$-3x_1 + 3x_2 = 1 \to \boxed{-3x_1 + 3x_1 = 0 \neq 1}$

10) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 1 \end{bmatrix}$ $\quad u+v = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}$

$$F(u+v) = \begin{bmatrix} (u_1+v_1)+(u_2+v_2) \\ 1 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ 1 \end{bmatrix} + \begin{bmatrix} u_2+v_2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} (u_1+v_1)+(u_2+v_2) \\ 2 \end{bmatrix}$$

Not a linear transformation

11) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_1 x_2 \end{bmatrix}$ $\quad F(u+v) = \begin{bmatrix} (u_1+v_1)^2 \\ (u_1+v_1)(u_2+v_2) \end{bmatrix}$

$$= \begin{bmatrix} u_1^2+v_1^2+2u_1v_1 \\ u_1u_2+u_1v_2+v_1u_2+v_1v_2 \end{bmatrix} = \begin{bmatrix} u_1^2 \\ u_1u_2 \end{bmatrix} + \begin{bmatrix} v_1^2 \\ v_1v_2 \end{bmatrix}$$

Not a linear transformation

18) $W = \left( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ x_2 = x_3 = 0 \right)$ $\quad W = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$T(v) = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a$$

W is a subspace of points on the x plane

V is a point $(a,b,c)$ in $\mathbb{R}^3$

19)

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

a)

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ -x_1 \end{bmatrix} \qquad x_1 = x_2 = 1$$

$$= \begin{bmatrix} 1+2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \qquad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

b)

$$T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 2+(-2) \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \qquad T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}$$

c)

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 3+4 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix} \qquad T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}$$

21)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \left.\begin{aligned} x_1 &= a+b \\ x_2 &= a-b \end{aligned}\right\} \Rightarrow a = \frac{x_1 + x_2}{2}$$

$$b = \frac{x_1 - x_2}{2}$$

$$x = \frac{x_1 + x_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{x_1 - x_2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{x_1 + x_2}{2} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{x_1 - x_2}{2} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + x_2 \\ -\frac{1}{2}x_1 - \frac{1}{2}x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{3}{2}x_1 + \frac{3}{2}x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - 2x_2 \end{bmatrix} \qquad T(x) = \begin{bmatrix} x_1 + x_2 \\ x_1 - 2x_2 \end{bmatrix}$$

23) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\quad x_1 = a + c$

$x_2 = -b - c$

$x_3 = a + b$

$$X = \left(\frac{x_1 + x_2 + x_3}{2}\right)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \left(\frac{x_3 - x_1 - x_2}{2}\right)\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \left(\frac{x_1 - x_2 - x_3}{2}\right)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \end{bmatrix}$$

33) $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$ $\quad u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$

$$T(u+v) = \begin{bmatrix} u_1 + v_1 \\ -(u_2 + v_2) \end{bmatrix} = \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

$$\begin{bmatrix} cu_1 \\ -cu_2 \end{bmatrix} = c\begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} = cT(u)$$

Is a linear transformation

35) $[F+G](u+v) = F(u+v) + G(u+v) = [F+G](u) + [F+G](v)$

$[F+G](cu) + [F+G](cv) = c[F+G](u)$ $\quad$ F + G is a linear transformation

36) a) $(F+G)(x) = \begin{bmatrix} 2x_1 - 3x_2 + x_3 \\ 4x_1 + 2x_2 - 5x_3 \end{bmatrix} + \begin{bmatrix} -x_1 + 4x_2 + 2x_3 \\ -2x_1 + 3x_2 + 3x_3 \end{bmatrix}$

$$= \begin{bmatrix} x_1 + x_2 + 3x_3 \\ 2x_1 + 5x_3 - 2x_1 \end{bmatrix} \qquad (F+G)(x) = \begin{bmatrix} x_1 + x_2 + 3x_3 \\ 2x_1 + 5x_3 - 2x_1 \end{bmatrix}$$

b) $F(x) = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 2 & -5 \end{bmatrix} = A$ $\qquad G(x) = \begin{bmatrix} -1 & 4 & 2 \\ -2 & 3 & 3 \end{bmatrix} = B$

$(F+G)(x) = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 5 & -2 \end{bmatrix} = C$

c) $A+B=\begin{bmatrix}1 & 1 & 3\\ 2 & 5 & -2\end{bmatrix}=C$ $\quad C=A+B$

37) $[aT](u+v)=aT(u)+aT(v)$

$[aT](cu)=c[aT](u)$ Is linear transformation

41) $T(x)=T(x_1e_1)+T(x_2e_2)+\dots+T(x_ne_n)$

$=(T(e_1)+T(e_2)+\dots+T(e_n))x$

$A=T(e_1)+T(e_2)+\dots+T(e_n)$ $\quad T(x)=Ax$ so $A=B$

## Section 5.2

1) $u-2v=\begin{bmatrix}2 & 1 & 3\\ -1 & 1 & 2\end{bmatrix}-\begin{bmatrix}2 & 8 & -2\\ 10 & 4 & 14\end{bmatrix}=\begin{bmatrix}0 & -7 & 5\\ -11 & -3 & -12\end{bmatrix}$

$u-(2v-3w)=\begin{bmatrix}2 & 1 & 3\\ -1 & 1 & 2\end{bmatrix}-\left(\begin{bmatrix}2 & 8 & -2\\ 10 & 4 & 14\end{bmatrix}-\begin{bmatrix}12 & -15 & 33\\ -39 & -3 & -3\end{bmatrix}\right)$

$=\begin{bmatrix}12 & -22 & 38\\ -50 & -6 & -15\end{bmatrix}$

$-2u-v+3w=\begin{bmatrix}-4 & -2 & -6\\ 2 & -2 & -4\end{bmatrix}-\begin{bmatrix}1 & 4 & -1\\ 5 & 2 & 7\end{bmatrix}+\begin{bmatrix}12 & -15 & 33\\ -39 & -3 & -3\end{bmatrix}$

$=\begin{bmatrix}7 & -21 & 28\\ -42 & -7 & -14\end{bmatrix}$

2)

$u - 2v = x^2 - 2 - 2x^2 - 4x + 2$

$= -x(x+4)$

$u - (2v - 3w) = x^2 - 2 - (2x^2 + 4x - 2 - 6x - 3)$

$= -x^2 + 2x + 3$

$-2u + v + 3w = -2x^2 + 4 - x^2 - 2x + 1 + 6x + 3$

$= -3x^2 + 4x + 8$

10) $P = \{p(x) \text{ in } P_2 : p(x) = p(-x) \text{ for all } x\}$ is a vector space ✓

19) Q is a vector space ✓

29) The set F is a vector space ✓

33) The set F(R) is a vector space ✓