Propositional Logic

Definition 0.1. *A **proposition** is a statement that is either true or false.*

Examples:
$3 + 7 = 10$

6 - 5 = 11

There are infinitely many twin primes

Operations on propositions

Propositions can be combined to build larger propositions. We will define these operations using truth tables:
$\neg$: "not", negation

p	$\neg p$
T	F
F	T

The rows of this table correspond to a possible truth setting of the propositions, sometimes called a "universe". Each column corresponds to a proposition.
To properly build a truth table, the truth values in each column must be clearly derived from columns to the left of it; when I ask you later on to build truth tables for more complicated propositions **do not** leave out columns or order the columns in a strange way.

$\wedge$: "and", conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$\vee$: "or", disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

$\Rightarrow$: "if _ then _", conditional

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$\iff$: "_ if and only if _", biconditional

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

Definition 0.2. *A statement that is always true no matter the truth value of the statements that occur in it is called a* ***tautology****.*

Example: $A \vee (\neg A)$

A	$\neg A$	$A \vee (\neg A)$	$A \iff A$
T	F	T	T
F	T	T	T

Definition 0.3. *A statement that is always false no matter the truth value of the statements that occur in it is called a* ***contradiction****.*

Example: $A \wedge (\neg A)$

Definition 0.4. *A statement that is neither a tautology nor a contradiction is called a* ***contingency****.*

Example: $A \Rightarrow B$

Logical equivalencies

Definition 0.5. *Two statements P and Q are called* ***logically equivalent*** *(written as $P \equiv Q$) when $P \iff Q$ is a tautology.*

Here are some logical equivalences:

- Double Negation Law: $(\neg(\neg P)) \equiv P$

P	$\neg P$	$\neg(\neg P)$	$(\neg(\neg P)) \iff P$
T	F	T	T
F	T	F	T

- Commutative Laws:
  - $(P \vee Q) \equiv (Q \vee P)$

P	Q	$P \vee Q$	$Q \vee P$	$(P \vee Q) \iff (Q \vee P)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

  - $(P \wedge Q) \equiv (Q \wedge P)$
    I'll leave this truth table as an exercise

- Associative Laws:

- $((P \vee Q) \vee R) \equiv (P \vee (Q \vee R))$

P	Q	R	$P \vee Q$	$Q \vee R$	$(P \vee Q) \vee R$	$P \vee (Q \vee R)$	$((P \vee Q) \vee R) \iff (P \vee (Q \vee R))$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	T

- $((P \wedge Q) \wedge R) \equiv (P \wedge (Q \wedge R))$
  I'll leave this truth table as an exercise

- Distributive Laws:
  - $(P \vee (Q \wedge R)) \equiv ((P \vee Q) \wedge (P \vee R))$
  - $(P \wedge (Q \vee R)) \equiv ((P \wedge Q) \vee (P \wedge R))$
- DeMorgan's Laws:
  - $(\neg(P \vee Q)) \equiv ((\neg P) \wedge (\neg Q))$
  - $(\neg(P \wedge Q)) \equiv ((\neg P) \vee (\neg Q))$
- The conditional and the contrapositive: $(P \Rightarrow Q) \equiv ((\neg Q) \Rightarrow (\neg P))$
- The inverse and the converse (of $P \Rightarrow Q$): $((\neg P) \Rightarrow (\neg Q)) \equiv (Q \Rightarrow P)$
- The conditional and a not/or expression: $(P \Rightarrow Q) \equiv ((\neg P) \vee Q)$

To avoid the tedium of typing out all the "(" and ")", we will introduce the following **Order of Operations**:

$$\neg, \wedge, \vee, \Rightarrow, \iff, \equiv$$

and use parenthesis to override the OoO. For example, we can simplify DeMorgan's Laws as

- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
- $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

Examples for converting English to propositional form

"I am a CS student and I am not taking CSE 2321"

First identify your basic propositions, propositions that cannot be broken down into some combination of smaller propositions

Let P be the proposition "I am a CS student"

and let Q be the proposition "I am taking CSE 2321".

The sentence says "I am NOT...", so we need to combine P and $\neg Q$ somehow, the use of the word "and" here indicates we want $\wedge$.

So this English sentence becomes

$$P \wedge \neg Q$$

To check out work, we should always take a moment to convince ourselves our proposition is true when the English sentence is true, and our proposition is false when our English sentence is false.

"I am a CS student implies I am taking CSE 2321"

We can use the same propositions from before.
Let P be the proposition "I am a CS student"
and let Q be the proposition "I am taking CSE 2321".

Now we need to decide how to combine them.

This might be difficult to do immediately, one way we can make this easier is to come up with English sentences that have the same meaning.

For example, if I rephrase this as
"If I am a CS student then I am taking CSE 2321"

It should be easier to convince ourselves that these have the same meaning (are logically equivalent?), and the "If _ then _" structure makes it clear we

want to use $\Rightarrow$.
So our English sentence becomes

$$P \Rightarrow Q$$

"I am not a CS student unless I am taking CSE 2321"

We can use the same propositions from before.
Let P be the proposition "I am a CS student"
and let Q be the proposition "I am taking CSE 2321".

Figuring out how to combine those propositions is a bit tricky here. Another good approach is to build the truth table, then work backwards to figure out the operations.

P	Q	$P\ ?\ Q$
T	T	T
T	F	F
F	T	T
F	F	T