Practice Sheet Review

2.) FUNCTION $P_2(A[a,\ldots,b])$

```
n ← b − a + 1
IF b ≤ a THEN RETURN(1)
half ← ⌊(b−a+1)/2⌋
x ← 1
WHILE x ≤ n^2 DO
    total ← 0
    FOR i ← 1 TO n
        total ← total + A[i]
    x ← 3x
r = P2(A[a,...,half]) + P2(A[half+1,...,b]) + 1
RETURN(r)
```

q ← half

- Base Case? $b \le a \Rightarrow b-a \le 0 \Rightarrow n \le 1$
- # recursive calls? 2
- What size input? $\frac{n}{2}$
- non-recursive work? $n\log_3(n)$

k	X
0	1
1	3
2	3^2
⋮	⋮
k	$x = 3^k$

$3^k \le n^2 \Rightarrow k \le \log_3 n^2$

$$\sum_{k=1}^{\log_3(n^2)} \sum_{i=1}^{n} C = Cn\log_3(n^2)$$

$$T_2(n) = 2T_2\left(\frac{n}{2}\right) + n\log_3(n)$$

$T_2(\text{input}) = (\text{input})\log_3(\text{input}) + 2T_2\left(\frac{\text{input}}{2}\right)$

$$T_2(n) = n\log_3(n) + 2T_2\left(\frac{n}{2}\right)$$
$$= n\log_3(n) + 2\left(\frac{n}{2}\log_3\left(\frac{n}{2}\right) + 2T_2\left(\frac{n}{2^2}\right)\right)$$
$$= n\log_3(n) + n\log_3\left(\frac{n}{2}\right) + 2^2 T_2\left(\frac{n}{2^2}\right)$$
$$= n\log_3(n) + n\log_3\left(\frac{n}{2}\right) + 2^2\left(\frac{n}{2^2}\log_3\left(\frac{n}{2^2}\right) + 2T_2\left(\frac{n}{2^3}\right)\right)$$
$$= n\log_3(n) + n\log_3\left(\frac{n}{2}\right) + n\log_3\left(\frac{n}{2^2}\right) + 2^3 T_2\left(\frac{n}{2^3}\right)$$
$$= \ldots \approx n\sum_{k=0}^{\lg n}\log_3\left(\frac{n}{2^k}\right) + 2^{\lg n}T_2(1)$$

$\frac{n}{2^m} = 1$

$m = \lg n$

$2^{\lg n} = n$

$$n\sum_{k=0}^{\lg n}\log_3\left(\frac{n}{2^k}\right)$$

$$n\sum_{k=0}^{\lg n}\log_3\left(\frac{n}{2^k}\right) \le n\sum_{k=0}^{\lg n}\log_3 n = n\log_3(n)\lg(n)$$

$$n\sum_{k=0}^{\lg n}\log_3\left(\frac{n}{2^k}\right) \ge n\sum_{k=0}^{\lg(\sqrt{n})}\log_3\left(\frac{n}{2^k}\right) \ge n\sum_{k=0}^{\lg\sqrt{n}}\log_3\left(\frac{n}{2^{\lg\sqrt{n}}}\right)$$

$$\ge n\sum_{k=0}^{\lg\sqrt{n}}\log_3(\sqrt{n}) = n\log_3(\sqrt{n})\lg(\sqrt{n}) = \frac{1}{4}n\log_3(n)\lg(n)$$

Guess: $T_2(n) \in \Theta(n \log^2 n)$

$$T_2(n) = \frac{n \lg n}{\lg 3} + 2T_2\left(\frac{n}{2}\right)$$

O-work

Assume that for $n_0 \le k < n$, $T_2(k) \le ak(\lg k)^2$ $(a>0)$

$$T_2(n) = \frac{n \lg n}{\lg 3} + 2T_2\left(\frac{n}{2}\right) \le \frac{n \lg n}{\lg 3} + 2a\frac{n}{2}\left(\lg \frac{n}{2}\right)^2$$

$$= \frac{n \lg n}{\lg 3} + an(\lg n - 1)^2 = \frac{n \lg n}{\lg 3} + an(\lg n)^2 - 2an\lg n + an$$

WANT

$$= an(\lg n)^2 + \left(\frac{1}{\lg 3} - 2a\right)n\lg n + an \le an(\lg n)^2$$

$$\left(\frac{1}{\lg 3} - 2a\right)\cancel{n}\lg n + a\cancel{n} \le 0$$

$$\left(\frac{1}{\lg 3} - 2a\right)\lg n + a \le 0$$

$$\frac{1}{\lg 3} - 2a \le 0 \Rightarrow a \ge \frac{1}{2\lg 3}$$

For $n \ge 2$ and $a \ge \frac{1}{2\lg 3}$

$$\left(\frac{1}{\lg 3} - 2a\right)\lg n + a \le \left(\frac{1}{\lg 3} - 2a\right)\lg 2 + a \le 0$$

$$a \ge \frac{1}{\lg 3} \ge \frac{1}{2\lg 3}$$

~~$T_2(1) \le a \cdot 1(\lg 1)^2 = 0$~~

$T_2(2) \le a \cdot 2(\lg 2)^2 = 2a$

$T_2(3) \le a \cdot 3(\lg 3)^2 = 3(\lg 3)^2 a$

$$a \ge \max\left\{\frac{1}{\lg 3}, \frac{T_2(2)}{2}, \frac{T_2(3)}{3(\lg 3)^2}\right\}$$

Ω-work

Assume $T_2(k) \ge bk(\lg k)^2$ for $2 \le k < n$ $(b>0)$

$$T_2(n) = \frac{n \lg n}{\lg 3} + 2T_2\left(\frac{n}{2}\right) \ge \frac{n \lg n}{\lg 3} + 2b\frac{n}{2}\lg\left(\frac{n}{2}\right)^2$$

$$= \frac{n \lg n}{\lg 3} + bn(\lg n - 1)^2 = \frac{n \lg n}{\lg 3} + bn(\lg n)^2 - 2bn\lg n + bn$$

WANT

$$= bn(\lg n)^2 + \left(\frac{1}{\lg 3} - 2b\right)\lg n + bn \ge bn(\lg n)^2$$

$$\left(\frac{1}{\lg 3} - 2b\right) \ge 0 \Rightarrow b \le \frac{1}{2\lg 3}$$

$$T_2(2) \geq b\,2\,(\lg 2)^2 \Rightarrow b \leq \frac{T_2(2)}{2}$$

$$T_2(3) \geq b\,3\,(\lg 3)^2 \Rightarrow b \leq \frac{T_2(3)}{3(\lg 3)^2}$$