MATH-2415, Ordinary and Partial Differential Equations **Name:** Key

Instructor: Michael Fellinger

Fall 2022

Final Exam

December 15, 2022

Clearly show all work that leads to your final answer.

1. Solve the following IVPs:

a) $xy' + y = 2x\cos x \qquad y(\pi) = 0$

b) $y' = \dfrac{2t}{\sin y} \qquad y(1) = \pi$

a) $y' + \frac{1}{x}y = 2\cos x \rightarrow p(x) = \frac{1}{x} \quad g(x) = 2\cos x$

$\mu(x) = e^{\int p(x)dx} = e^{\int \frac{dx}{x}} = e^{\ln|x|} = |x| = x$ (no abs values needed since IC is $x_0 = \pi$)

$\rightarrow y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)g(x)\,dx + c\right] = \frac{1}{x}\left[2\int x\cos x\,dx + c\right]$ Let $u = x \rightarrow du = dx$, $dv = \cos x\,dx \rightarrow v = \sin x$

$= \frac{1}{x}\left[2x\sin x - 2\int \sin x\,dx + c\right] = \frac{1}{x}\left[2x\sin x + 2\cos x + c\right]$

$\Rightarrow y(x) = \frac{c}{x} + 2\sin x + 2\frac{\cos x}{x}$

IC: $y(\pi) = \frac{c}{\pi} - \frac{2}{\pi} = 0 \Rightarrow c = 2 \Rightarrow \boxed{y(x) = \frac{2}{x} + 2\sin x + 2\frac{\cos x}{x}}$

b) $\frac{dy}{dt} = \frac{2t}{\sin y} \rightarrow \underbrace{\frac{dy}{dt}dt}_{dy} = \frac{2t}{\sin y}dt \rightarrow \sin y\,dy = 2t\,dt$

$\int \sin y\,dy = 2\int t\,dt \rightarrow -\cos y = t^2 + c \rightarrow \cos y = c - t^2$

IC: $y(1) = \pi \rightarrow \cos\pi = c - 1 \rightarrow -1 = c - 1 \Rightarrow c = 0$

$\Rightarrow \boxed{\cos y = -t^2 \quad \text{or} \quad y(t) = \cos^{-1}(-t^2)}$

2. Is the following equation exact? If so, find the general solution.

$$M(x,y) = \frac{\partial \psi(x,y)}{\partial x}$$

$$N(x,y) = \frac{\partial \psi(x,y)}{\partial y}$$

$$\underbrace{(2xe^{3y} + e^x)}_{M(x,y)} + \underbrace{(3x^2e^{3y} - y^2)}_{N(x,y)}\frac{dy}{dx} = 0$$

Exact? $\frac{\partial M}{\partial y} = 6xe^{3y}$ $\quad \frac{\partial N}{\partial x} = 6xe^{3y}$ $\Rightarrow$ Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ODE is exact

$$\psi(x,y) = \int \frac{\partial \psi}{\partial x}dx = \int M(x,y)dx = \int (2xe^{3y} + e^x)dx = x^2e^{3y} + e^x + h(y)$$

$$\frac{\partial \psi}{\partial y} = 3x^2e^{3y} + \frac{dh}{dy} = \underbrace{3x^2e^{3y} - y^2}_{N(x,y)} \Rightarrow \frac{dh}{dy} = -y^2$$

$$h(y) = \int \frac{dh}{dy}dy = -\int y^2dy = -\frac{y^3}{3} + C_1$$

$$\Rightarrow \psi(x,y) = x^2e^{3y} + e^x - \frac{y^3}{3} + C_1$$

Solution: $\psi(x,y) = C_2$

$$\Rightarrow \boxed{x^2e^{3y} + e^x - \frac{y^3}{3} = C}$$

where $C = C_2 - C_1$

3. a) Solve the second-order ODE:

$$y'' - 6y' + 12y = 0$$

b) If the right-hand side equaled $7x^2e^x + 157\sin x$ instead of 0, what form of particular solution would you need for the method of undetermined coefficients (you do not need to solve this nonhomogeneous equation)

a) Assume $y = e^{rx} \rightarrow r^2 - 6r + 12 = 0 \rightarrow r = \frac{6 \pm \sqrt{36 - 4(1)(12)}}{2(1)} = \underset{\lambda}{\underset{\uparrow}{3}} \pm i \underset{\mu}{\underset{\uparrow}{\sqrt{3}}}$

$$\Rightarrow \boxed{y(x) = c_1 e^{3x} \cos\sqrt{3}x + c_2 e^{3x} \sin\sqrt{3}x}$$

b) $\boxed{Y(x) = (Ax^2 + Bx + C)e^x + D\sin x + E\cos x}$

4. Solve the following heat conduction equation with the prescribed initial condition and homogeneous boundary conditions:

$$u_{xx} = u_t \quad \leftarrow \alpha^2 = 1$$

IC

$$u(0,t) = 0 \quad u(20,t) = 0 \quad \leftarrow \text{homogeneous BC's}$$

$$u(x,0) = \begin{cases} 10 - x & 0 \le x \le 10 \\ 0 & 10 \le x \le 20 \end{cases} \quad L = 20$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t/400} \sin\left(\frac{n\pi x}{20}\right)$$

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{20}\int_0^{20} f(x)\sin\left(\frac{n\pi x}{20}\right)dx$$

$$= \frac{1}{10}\left[\int_0^{10}(10-x)\sin\left(\frac{n\pi x}{20}\right)dx + \cancel{\int_{10}^{20}(0)\sin\left(\frac{n\pi x}{20}\right)dx}^{\,0}\right]$$

$$= \int_0^{10}\sin\left(\frac{n\pi x}{20}\right)dx - \frac{1}{10}\int_0^{10} x\sin\left(\frac{n\pi x}{20}\right)dx \quad \leftarrow \text{Let } u = x \quad dv = \sin\left(\frac{n\pi x}{20}\right)dx$$

$$du = dx \quad v = -\frac{20}{n\pi}\cos\left(\frac{n\pi x}{20}\right)$$

$$= -\frac{20}{n\pi}\cos\left(\frac{n\pi x}{20}\right)\Big|_0^{10} - \frac{1}{10}\left[-\frac{20}{n\pi}x\cos\left(\frac{n\pi x}{20}\right)\Big|_0^{10} + \frac{20}{n\pi}\int_0^{10}\cos\left(\frac{n\pi x}{20}\right)dx\right]$$

$$= \cancel{-\frac{20}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \frac{20}{n\pi} + \cancel{\frac{20}{n\pi}\cos\left(\frac{n\pi}{2}\right)} - \frac{2}{n\pi}\cdot\frac{20}{n\pi}\sin\left(\frac{n\pi x}{20}\right)\Big|_0^{10}$$

$$= \frac{20}{n\pi} - \frac{40}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty}\left(\frac{20}{n\pi} - \frac{40}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)e^{-n^2\pi^2 t/400}\sin\left(\frac{n\pi x}{20}\right)}$$

5. Consider the following system of linear first-order ODEs:

$$\mathbf{x}' = \underbrace{\begin{pmatrix} 0 & 1 \\ 16 & 0 \end{pmatrix}}_{A} \mathbf{x}$$

a) Solve the system and describe the behavior of the solutions as $t \to \infty$ for different choices of c_1 and c_2.

b) Convert the system of first-order ODE's into a single second-order ODE and solve it. Verify that the solution is consistent with the solutions for $x_1(t)$ and $x_2(t)$ from part a).

a) Eigenvalues: $\det(A - rI) = \begin{vmatrix} -r & 1 \\ 16 & -r \end{vmatrix} = r^2 - 16 = 0 \rightarrow r^2 = 16 \rightarrow r = \pm 4$

eigenvalues: $r_1 = -4 \quad r_2 = 4$

Eigenvectors:

- $r_1 = -4$: $(A - r_1 I)\vec{\xi}^{(1)} = \vec{0} \rightarrow \begin{pmatrix} 4 & 1 \\ 16 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Augmented matrix: $\left(\begin{array}{cc|c} 4 & 1 & 0 \\ 16 & 4 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 1/4 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow a + \frac{1}{4}b = 0$ Let $b = -4$ $\rightarrow a = 1$

$\Rightarrow \vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \rightarrow \vec{x}^{(1)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4t}$

- $r_2 = 4$: $(A - r_2 I)\vec{\xi}^{(2)} = \vec{0} \rightarrow \begin{pmatrix} -4 & 1 \\ 16 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Augmented matrix: $\left(\begin{array}{cc|c} -4 & 1 & 0 \\ 16 & -4 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & -1/4 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow a - \frac{1}{4}b = 0$ Let $b = 4$ $\rightarrow a = 1$

$\Rightarrow \vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \rightarrow \vec{x}^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4t}$

$$\Rightarrow \vec{x}(t) = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} = \boxed{c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4t}} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

If $c_2 \neq 0$, $x_1(t)$ and $x_2(t)$ will both increase exponentially without bound as $t \to \infty$ regardless of the value of c_1. If $c_2 = 0$, $x_1(t)$ and $x_2(t)$ will decay exponentially to zero as $t \to \infty$

b) $\vec{x}' = \begin{pmatrix} 0 & 1 \\ 16 & 0 \end{pmatrix} \vec{x} \quad \rightarrow \quad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 16 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 16x_1 \end{pmatrix}$

$\rightarrow \quad x_1' = x_2$

$x_2' = 16x_1$

Differentiating the first equation gives $x_1'' = x_2'$. Now substitute x_2' from the second equation into this equation:

$x_1'' = 16x_1$

Let $x_1(t) = e^{rt} \rightarrow r^2 = 16 \rightarrow r = \pm 4 \Rightarrow \boxed{x_1(t) = c_1 e^{-4t} + c_2 e^{4t}}$

But $x_1' = x_2$, so we have $\boxed{x_2(t) = -4c_1 e^{-4t} + 4c_2 e^{4t}}$

Our solution from a) was

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4t} = \begin{pmatrix} c_1 e^{-4t} + c_2 e^{4t} \\ -4c_1 e^{-4t} + 4c_2 e^{4t} \end{pmatrix}$$

or $x_1(t) = c_1 e^{-4t} + c_2 e^{4t}$

$x_2(t) = -4c_1 e^{-4t} + 4c_2 e^{4t}$

This is the same as our solution in b) ✓