MATH 2415 – Ordinary and Partial Differential Equations
Lecture 07 notes

3.4 Repeated Roots; Reduction of Order

Let's consider again the characteristic equation

$$ar^2 + br + c = 0$$

for the ODE

$$ay'' + by' + cy = 0$$

The roots of the characteristic equation are given by the quadratic equation:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac = 0$, then we have

$$r_1 = r_2 = -\frac{b}{2a}$$

and both roots give the same solution to the DE:

$$y_1(x) = e^{-bx/2a}$$

- How do we get a second solution that is linearly independent of this one? The general procedure we use is the following:

1. Assume we can write a second solution as $y_2(x) = v(x)y_1(x)$
2. Substitute this guess into the ODE to see if we can find a $v(x)$ that satisfies the equation (this results in an ODE for $v(x)$)
3. Solve this new ODE for $v(x)$
4. Verify that the solutions $y_1(x)$ and $y_2(x)$ are linearly independent by computing the Wronskian, and thus that they form a fundamental set of solutions

Let's carry this procedure out to find $y_2(x)$: $y_1(x) = e^{-bx/2a}$

Assume $y_2(x) = v(x)y_1(x) = v(x)e^{-bx/2a}$ and sub into ODE:

$$y_2' = v'e^{-bx/2a} - \frac{b}{2a}ve^{-bx/2a}$$

$$y_2'' = v''e^{-bx/2a} - \frac{b}{2a}v'e^{-bx/2a} - \frac{b}{2a}v'e^{-bx/2a} + \frac{b^2}{4a^2}ve^{-bx/2a}$$

$$= e^{-bx/2a}\left(v'' - \frac{b}{a}v' + \frac{b^2}{4a^2}v\right)$$

Sub. into the ODE $ay_2'' + by_2' + cy_2 = 0$:

$$\left[a\left(v'' - \frac{b}{a}v' + \frac{b^2}{4a^2}v\right) + b\left(v' - \frac{b}{2a}v\right) + cv\right]e^{-bx/2a} = 0$$

The expression in brackets must be zero since $e^{-bx/2a} \neq 0$

$$av'' + \underbrace{(-b+b)v'}_{=0} + \underbrace{\left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)}_{\frac{-b^2}{4a} + c \,=\, \frac{-b^2+4ac}{4a} \,=\, -\frac{(b^2-4ac)}{4a} \,=\, 0}v = 0$$

since $b^2 - 4ac$ equals zero for repeated real roots

$\Rightarrow \quad av'' = 0$ or $v'' = 0$

$\rightarrow \quad v'(x) = C_4$

$\rightarrow \quad v(x) = C_4x + C_3$

So $y_2(x) = v(x)y_1(x) = (C_4x + C_3)e^{-bx/2a} = C_4xe^{-bx/2a} + C_3e^{-bx/2a}$

general solution: $y(x) = C_5y_1(x) + C_6y_2(x)$

$$= C_5e^{-bx/2a} + C_6\left(C_4xe^{-bx/2a} + C_3e^{-bx/2a}\right)$$

$$= \underbrace{(C_5 + C_6C_3)}_{C_1}e^{-bx/2a} + \underbrace{C_6C_4}_{C_2}xe^{-bx/2a}$$

$$\Rightarrow \boxed{y(x) = C_1e^{-bx/2a} + C_2xe^{-bx/2a}}$$

If we compute $W[e^{-bx/2a}, xe^{-bx/2a}]$ we $e^{-bx/a} \neq 0$ for all x

$\Rightarrow e^{-bx/2a}$ and $xe^{-bx/2a}$ form a fundamental set

- In summary, three cases can arise when solving second-order linear homogeneous ODEs with constant coefficients

$$ay'' + by' + cy = 0$$

1. The roots of the characteristic equation r_1 and r_2 are real but not equal:

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

2. The roots r_1 and r_2 are complex conjugates $\lambda \pm i\mu$:

$$y(x) = c_1 e^{\lambda x} \cos \mu x + c_2 e^{\lambda x} \sin \mu x$$

3. The roots are real and repeated:

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}$$

don't forget the x!

We can apply the procedure we used for repeated roots (called **reduction of order**) to more general cases. If we know one solution $y_1(x)$ to

$$y'' + p(x)y' + q(x)y = 0$$

we can find a second solution by letting $y_2(x) = v(x)y_1(x)$:

$y_2' = v'y_1 + vy_1'$ $\quad y_2'' = v''y_1 + v'y_1' + v'y_1' + vy_1'' = v''y_1 + 2v'y_1' + vy_1''$

Sub into ODE:

$v''y_1 + 2v'y_1' + vy_1'' + p(x)(v'y_1 + vy_1') + q(x)vy_1 = 0$

$y_1v'' + (2y_1' + p(x)y_1)v' + \underbrace{(y_1'' + p(x)y_1' + q(x)y_1)}_{=0 \text{ since } y_1 \text{ satisfies the ODE}}v = 0$

$\Rightarrow y_1v'' + (2y_1' + p(x)y_1)v' = 0$

Now let $u = v' \rightarrow u' = v''$

$\Rightarrow y_1u' + (2y_1' + p(x)y_1)u = 0$

solve this 1st-order ODE for u using, e.g., the method of integrating factors

Then integrate $u = v'$ to get our solution for $v(x)$. Then form $y_2(x) = v(x)y_1(x)$

Finally, $y(x) = c_1y_1(x) + c_2y_2(x)$

Before discussing more general second-order linear ODEs and applications to modelling physical systems, let's discuss a few more things about the Wronskian. Consider the following two theorems about IVPs involving 2nd-order linear homogeneous ODEs:

Theorem: Suppose y_1 and y_2 are two solutions of

$$L[y] = y'' + p(x)y' + q(x)y = 0$$

and there are initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$. It is always possible to choose c_1 and c_2 so that

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

satisfies the differential equation and the initial conditions if and only if the Wronskian $W[y_1, y_2] \neq 0$ at x_0

- **Example:** Solve the IVP

$$y'' - 4y' + 4y = 0 \qquad y(0) = 1 \qquad y'(0) = 1$$

Assume $y = e^{rx}$ $\rightarrow$ $r^2 - 4r + 4 = 0$

$$(r-2)(r-2) = 0$$

$$r_1 = r_2 = 2 \leftarrow \text{repeated real roots}$$

general solution: $y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

don't forget the x!

$$\Rightarrow \quad y(x) = c_1 e^{2x} + c_2 x e^{2x}$$

$$y'(x) = 2c_1 e^{2x} + c_2(e^{2x} + 2xe^{2x})$$

ICs: $y(0) = c_1 = 1$
$y'(0) = 2c_1 + c_2 = 1$

$\Big\}$ $c_1 = 1$, $c_2 = -1$

$$\Rightarrow \quad \boxed{\begin{aligned} y(x) &= e^{2x} - xe^{2x} \\ &= e^{2x}(1-x) \end{aligned}}$$

- **Example:** If $y_1 = e^x$ is a solution of $y'' - y = 0$ on $(-\infty, \infty)$, find a second solution (we know how to solve this already using a simpler method, but we'll see that reduction of order works here as well).

Let $y_2 = v y_1 = v e^x$

$$y_2' = v'e^x + ve^x = (v'+v)e^x$$

$$y_2'' = v''e^x + 2v'e^x + ve^x = (v'' + 2v' + v)e^x$$

Sub into ODE:

$$y_2'' - y_2 = (v'' + 2v' + v)e^x - ve^x = (v'' + 2v')e^x = 0$$

$\Rightarrow \quad v'' + 2v' = 0 \quad$ since $e^x \neq 0$

Reduction of order: Let $u = v' \rightarrow u' = v''$

$$u' + 2u = 0 \qquad p(x) = 2 \quad g(x) = 0$$

$$\mu(x) = e^{\int p(x)dx} = e^{2\int dx} = e^{2x}$$

$$u(x) = \frac{1}{\mu(x)}\left[\int \cancelto{0}{\mu(x) g(x)}\, dx + c_1\right] = \frac{c_1}{\mu(x)} = c_1 e^{-2x}$$

But $v'(x) = u(x) \rightarrow v(x) = \int u(x)dx + c_2 = c_1 \int e^{-2x} dx + c_2 = -\frac{1}{2} c_1 e^{-2x} + c_2$

$$\Rightarrow y_2(x) = v(x) y_1(x) = \left(-\frac{c_1}{2} e^{-2x} + c_2\right) e^x = -\frac{c_1}{2} e^{-x} + c_2 e^x$$

$$y(x) = c_3 y_1(x) + c_4 y_2(x) = c_3 e^x + c_4\left(-\frac{c_1}{2} e^{-x} + c_2 e^x\right)$$

$$= \underbrace{(c_3 + c_4 c_2)}_{K_1} e^x - \underbrace{\frac{c_1 c_4}{2}}_{K_2} e^{-x}$$

$\Rightarrow \quad y(x) = K_1 e^x + K_2 e^{-x} \qquad$ The fundamental set of solutions is e^x and e^{-x}

- **Example:** If $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$, find a second solution on $(0, \infty)$.

Summary of the solutions to second-order linear homogeneous ODEs with constant coefficients:

$$ay'' + by' + cy = 0$$

We assume a solution of the form

$$y = e^{rx}$$

which leads to the **characteristic equation** of the ODE:

$$ar^2 + br + c = 0$$

1. If the roots of the characteristic equation r_1 and r_2 are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

2. If the roots r_1 and r_2 are complex conjugates $\lambda \pm i\mu$, then

$$y(x) = c_1 e^{\lambda x} \cos \mu x + c_2 e^{\lambda x} \sin \mu x$$

3. If the roots are real and repeated, then

$$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

In Case 3 (real repeated roots), we found the second linearly independent solution (i.e. $y_2 = xe^{r_1 x}$) using the method of **reduction of order:**

- Given the solution $y_1 = e^{r_1 x}$, we assume the second solution is of the form $y_2 = v(x)y_1(x)$
- Substitute this assumed form into the ODE, which results in a new **second-order** ODE for $v(x)$
- Make the substitution $u(x) = v'(x)$, so that $u'(x) = v''(x)$. This substitution results in a **first-order** ODE for $u(x)$ (i.e. the order of the ODE is **reduced** from second to first)
- Solve the first-oder ODE for $u(x)$. Then integrate this solution with respect to x to get $v(x)$.
- Finally, form the second solution $y_2 = v(x)y_1(x)$ (you can also verify that $y_1(x)$ and $y_2(x)$ are linearly independent by computing the Wronskian)

We also showed that the method of reduction of order can be used to solve the more general second-order linear homogeneous ODE:

$$y'' + p(x)y' + q(x)y = 0$$

What if we have a second-order linear ODE that is **not** homogeneous (i.e., the right-hand side does not equal zero)?

We will now study a method called the **method of undetermined coefficients** that allows us to solve nonhomogeneous equations in certain cases.

3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

Consider the two second-order linear differential equations, where p, q, and g are continuous on an open interval I:

1. $L[y] = y'' + p(x)y' + q(x)y = g(x)$

2. $L[y] = y'' + p(x)y' + q(x)y = 0$

The first differential equation is nonhomogeneous if $g(x) \neq 0$. The second equation is the corresponding homogeneous version of the first equation. The following two theorems relate the solutions of the two equations, and show how to construct the general solution of the nonhomogeneous equation:

Theorem: If Y_1 and Y_2 are two solutions of equation 1, then their difference is a solution of equation 2. If y_1 and y_2 form a fundamental set of solutions of equation 2, then

$$Y_1(x) - Y_2(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are certain constants

Theorem: If y_1 and y_2 form a fundamental set of solutions of equation 2, then the general solution of equation 1 can be written

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + Y(x)$$

where y_1 and y_2 form a fundamental set of solutions of equation 2, c_1 and c_2 are arbitrary constants, and Y is any solution of equation 1.

The general procedure for solving equation 1

$$L[y] = y'' + p(x)y' + q(x)y = g(x)$$

follows from these theorems:

1. Find the general solution $c_1 y_1(x) + c_2 y_2(x)$ to the corresponding homogeneous equation. This solution is called the **complementary solution**, denoted $y_c(x)$

2. Find any solution $Y(x)$ of the nonhomogeneous equation. This solution is called a **particular solution**, denoted $y_p(x)$

3. Add the complimentary and particular solutions: $y(x) = y_c(x) + y_p(x)$

We can find the complimentary solution using the methods we have already discussed for solving second-order, linear homogeneous ODEs. Two methods for finding the particular solution are the **method of undetermined coefficients** and the method of **variation of parameter**

Method of undetermined coefficients

This method is most useful when $g(x)$ is a polynomial, an exponential, sines, and cosines, or sums and products of these functions, and the coefficients of the linear operator L are constants:

$$ay'' + by' + cy = g(x)$$

1. First find the complimentary solution $y_c(x)$

2. If $g(x) = g_1(x) + \cdots + g_n(x)$ then form n subproblems,

$$ay'' + by' + cy = g_i(x)$$

where each $g_i(x)$ contains only products of the functions mentioned above

3. For the ith subproblem, assume a particular solution $Y_i(x)$ consisting of the appropriate combination of polynomials, exponentials, sines, and/or cosines. If there is any duplication of the assumed form of $Y_i(x)$ in the solution $y_c(x)$ for the homogeneous equation, then multiply $Y_i(x)$ by x (or x^2 if needed) to ensure the solutions are linearly independent

4. The sum of the $Y_i(x)$ is a particular solution $y_p(x)$ of the nonhomogeneous equation

5. The sum of $y_c(x)$ and $y_p(x)$ is the general solution of the nonhomogeneous equation

6. If there are initial conditions, use them to determine the values of the arbitrary constants in the general solution

- **Example:** Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2x}$$

- **Example:** Find a particular solution of

$$y'' - 3y' - 4y = 2\sin x$$

- **Example:** Find a particular solution of

$$y'' - 3y' - 4y = -8e^{x} \cos 2x$$

- **Example:** Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2x} + 2\sin x - 8e^{x}\cos 2x$$

- **Example:** Find the general solution of

$$y'' - 3y' - 4y = 2e^{-x}$$

- **Example:** Find the general solution of

$$y'' + 4y' - 2y = 2x^2 - 3x + 6$$