MATH 2415 – Ordinary and Partial Differential Equations
Lecture 09 notes

Partial Differential Equations and Fourier Series

We now begin our study of partial differential equations, which are differential equations involving more than one independent variable. We focus on the method of **separation of variables**, where we replace a PDE by a set of ODEs which we solve for a set of initial or boundary conditions. The PDE solution is then a sum (usually an infinite series) of the ODE solutions. In many cases this will be a series involving sines and cosines which is called a **Fourier series**. We start with discussing **boundary value problems** for ODEs.

10.1 Two-Point Boundary Value Problems

Up to now we have discussed initial value problems, where we specify an ODE along with the value of the function (and in the case of second-order ODEs, the value of the first derivative) at a given point x_0 (or t_0). For example,

$$y'' + p(t)y' + q(t)y = g(t) \qquad y(t_0) = y_0 \qquad y'(t_0) = y_0'$$

In physical applications of differential equations, we often know the value of the dependent variable y or its derivative y' at two different points x_0 and x_1 instead. For example,

$$y'' + p(x)y' + q(x)y = g(x) \qquad y(x_0) = y_0 \qquad y(x_1) = y_1$$

We call this a **two-point boundary value problem** with **boundary conditions**

$$y(x_0) = y_0 \qquad y(x_1) = y_1$$

The solution of the boundary value problem is a function $y = y(x)$ that satisfies the ODE on the interval $x_0 < x < x_1$ and take on the values y_0 and y_1 at the endpoints of the interval.

We classify boundary value problems (BVP) as follows

- If $g(x) = 0$, and y_0 and y_1 are also zero the BVP is called **homogeneous**
- Otherwise, the BVP is **nonhomogeneous**

The main difference between IVPs and BVPs is the number of possible solutions. We have seen that IVPs are certain to have unique solution on an interval where the coefficients are continuous. Under similar conditions, BVPs may have no solution, one solution, or infinitely many solutions (similar to systems of linear algebraic equations).

Let’s do some examples to illustrate this behavior.

- **Example:** Solve the given BVP

$$y'' + 2y = 0 \qquad y(0) = 1 \qquad y(\pi) = 0$$

- **Example:** Solve the given BVP

$$y'' + y = 0 \qquad y(0) = 1 \qquad y(\pi) = a$$

When we discussed IVPs, we saw that nonhomogeneous ODEs have a corresponding homogeneous DE. The nonhomogeneous BVP

$$y'' + p(x)y' + q(x)y = g(x) \qquad y(x_0) = y_0 \qquad y(x_1) = y_1$$

has a corresponding homogeneous problem:

$$y'' + p(x)y' + q(x)y = 0 \qquad y(x_0) = 0 \qquad y(x_1) = 0$$

Every homogeneous problem of this type has the solution $y(x) = 0$ for all x. We call this a **trivial solution**. We generally want to know if there are also nontrivial solutions for a given homogeneous problem

- **Example:** Solve the given BVP

$$y'' + 2y = 0 \qquad y(0) = 0 \qquad y(\pi) = 0$$

- **Example:** Solve the given BVP

$$y'' + y = 0 \qquad y(0) = 0 \qquad y(\pi) = 0$$

If we examine these example problems, we notice that

- if the nonhomogeneous BVP has a unique solution then the corresponding homogeneous problem has only the trivial solution

- if the nonhomogeneous BVP has either no solution or infinitely many solutions, the corresponding homogeneous problem has nontrivial solutions

Eigenvalue Problems

We now formalize these results by posing the BVP in terms of an **eigenvalue problem**. Let's consider the following problem:

$$y'' + \lambda y = 0 \qquad y(0) = 0 \qquad y(\pi) = 0$$

The values of λ for which the BVP has **nontrivial** solutions are called **eigenvalues**, and the nontrivial solutions to the ODE are called **eigenfunctions**

Like the characteristic equations we have seen before, we need to consider three possible cases: $\lambda < 0, \lambda = 0, \lambda > 0$

If we assume $y = e^{rx}$, we find the characteristic equation $r^2 + \lambda = 0$. To simplify the rest of the analysis, it is customary to let $\lambda = \mu^2$ so that we don't have square roots in the resulting equations

Case 1: $\lambda > 0$

Changing from λ to μ^2, we have

$$y'' + \mu^2 y = 0 \qquad y(0) = 0 \qquad y(\pi) = 0$$

Case 1: $\lambda < 0$

$$y'' - \mu^2 y = 0 \qquad y(0) = 0 \qquad y(\pi) = 0$$

Case 3: $\lambda = 0$

$$y'' = 0 \qquad y(0) = 0 \qquad y(\pi) = 0$$

Summary of results:

Summary of Two-Point Boundary Value Problems and Eigenvalue Problems

Examples

1. Solve the following BVP: $y'' + 4y = 0 \qquad y(0) = 0 \qquad y(\pi) = 0$

2. Solve the following BVP: $y'' + 4y = 0 \qquad y(0) = 0 \qquad y(\pi) = 2$

3. Solve the following BVP: $y'' + 4y = 0 \qquad y(0) = 0 \qquad y'(\pi) = 3$

10.2 Fourier Series

We will see later that when we solve some important PDE problems, the answers will be expressed as an infinite sum of sines and/or cosines, depending on the boundary conditions. These types of sums are called **Fourier series**

The series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

defines a function f on the set of points where the series converges. We say that this series is the **Fourier series** of f. We often say that we "expand f in sines and cosines" when we express f as its Fourier series.

Periodicity of Sines and Cosines

Definition: A function f is **periodic** with period $T > 0$ if the domain of f contains $x + T$ whenever it contains x, and if

$$f(x + T) = f(x)$$

for every value of x

- **Example:**

Note that any integer multiple of T is also a period of f. The smallest value of T for which $f(x + T) = f(x)$ holds is called the **fundamental period** of f

Now consider two periodic functions f and g with the same period T. Any linear combination of these functions is also periodic with period T, and the product fg is also periodic with period T

The sum of any finite number of functions with period T, or a convergent infinite series of functions with period T, is also periodic with period T

What about the sine and cosine functions in the Fourier series?

We know that the period of $\sin x$ and $\cos x$ is $T = 2\pi$, and the period of $\sin \alpha x$ and $\cos \alpha x$ is $T = 2\pi/\alpha$

If we let $\alpha = n\pi/L$, we have

$$T = \frac{2\pi}{n\pi/L} = \frac{2L}{n}$$

Also note that since every integral multiple of a period is also a period, each function in the Fourier series has a common period $2L$

So how do we find the coefficients a_0, a_n, and b_n in the Fourier series?

We use the concept of **orthogonality** of functions. This is a generalization of the orthogonality of vectors.

First consider the inner product (or dot product) of two vectors $\mathbf{r} = (r_1, r_2, \dots, r_n)$ and $\mathbf{p} = (p_1, p_2, \dots, p_n)$

$$(\mathbf{r}, \mathbf{p}) = \sum_{i=1}^{n} r_i p_i$$

We say the vectors are **orthogonal** if their inner product equals zero

We can generalize this idea to the inner product of two *functions* $u(x)$ and $v(x)$. The **inner product** of two real-valued functions u and v on the interval $\alpha \leq x \leq \beta$ is defined as

$$(u, v) = \int_{\alpha}^{\beta} u(x)v(x)dx$$

If the inner product of u and v on the interval $\alpha \leq x \leq \beta$ is zero, then u and v **orthogonal** on this interval

A *set* of functions is **mutually orthogonal** if each distinct pair of functions in the set is orthogonal

The functions $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$ for $n = 1,2,3,\ldots$ form a set of mutually orthogonal set of functions on the interval $-L \leq x \leq L$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 0, \qquad \text{for all } m, n$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$

- **Example:** Derive the first equation above

The Euler-Fourier Formulas

Now we exploit the orthogonality of the sine and cosine functions to find the coefficients a_n and b_n in the Fourier series for $f(x)$, assuming the series converges for all x on $-L \leq x \leq L$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

The formulas for a_n and b_n are called the Euler-Fourier formulas for the coefficients in the Fourier series

- **Example:** Determine the Fourier coefficients for the function $f(x)$:

$$f(x) = \begin{cases} -x & -2 \le x < 0 \\ x & 0 < x < 2 \end{cases}$$

$$f(x+4) = f(x)$$

- **Example:** Determine the Fourier coefficients for the function $f(x)$, and graph 3 periods of this function:

$$f(x) = \begin{cases} 0 & -3 < x < -1 \\ 1 & -1 < x < 1 \\ 0 & 1 < x < 3 \end{cases}$$

$$f(x+6) = f(x)$$

We may also be interested in how fast a Fourier series converges, or how many terms are needed to achieve a particular level of error (if we truncate the series at a finite number of terms, it will not represent the function exactly). [Also see example 3 in Section 10.2 of your text for details.]