MATH-2415, Ordinary and Partial Differential Equations **Name:** Gage F.
Summer 2023
Problem Set 8
Due July 30, 2023 by midnight

Directions: You can either

(I) Print this sheet and show all work on the sheet itself, or

(II) Use separate paper to work out the problems. The paper must be plain white paper. No notebook paper, no lined paper. Engineering paper is acceptable.

For either selection, clearly show all work that leads to your final answer. Don't make me hunt for your steps! And DON'T turn in **any** scratched out work! Your final product must be clear and legible, or it will be returned, ungraded.

You can scan your work and save the pages as a single pdf. Or you can take pictures of your work, add the pictures to Word or Powerpoint and export the pages to a single pdf. You will submit the pdf to Blackboard.

1. Find the Fourier Series for the following function:

$$f(x) = x^2 + 1 \quad \text{if } -1 < x < 1$$

$$f(x+2) = f(x)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{p}\right) + b_n \sin\left(\frac{2\pi n x}{p}\right)$$

$$a_0 = \frac{2}{p}\int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{2}{p}\int_{-L}^{L} f(x)\cos\left(\frac{2\pi n x}{p}\right)dx$$

$$b_n = \frac{2}{p}\int_{-L}^{L} f(x)\sin\left(\frac{2\pi n x}{p}\right)dx$$

$$a_0 = \cancel{\frac{2}{2}}\int_{-1}^{1} x^2\,dx \to = \frac{x^3}{3}\Big|_{-1}^{1} \to \frac{1}{3} - \left(-\frac{1}{3}\right) \to a_0 = \frac{2}{3}$$

$$\frac{a_0}{2} = \frac{1}{3}$$

$$a_n = \frac{2}{2}\int_{-1}^{1} x^2 \cos\left(\frac{\cancel{2}\pi n x}{\cancel{2}}\right)dx$$

☆used calculator for integral☆

$$= \frac{\left(180 \cdot 360 n\pi^2 x\cos(n\pi x) + (n^2\pi^4 x^2 - 64800)\sin(n\pi x)\right)\Big|_{-1}^{1}}{n^3\pi^6}$$

$$a_n = 4\frac{-1^n}{(n\pi)^2}$$

$$b_n = \frac{2}{2}\int_{-1}^{1} x^2 \sin\left(\frac{\cancel{2}\pi n x}{\cancel{2}}\right)dx$$

$$= \frac{-180\cdot\left(\left(n^2\pi^4x^2 - 64800\right)\cdot\cos(n\pi x) - 360\, n\pi^2 x\sin(n\pi x)\right)}{n^3\pi^6}$$

$$b_n = 0$$

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} 4\,\frac{-1^n}{(n\pi)^2}\cos(\pi n x) + \cancel{0\sin(\pi n x)}$$

$$\boxed{f(x) = \frac{1}{3} - \frac{4}{\pi^2}\cos(\pi x) + \frac{1}{\pi^2}\cos(2\pi x) - \frac{4}{9\pi^2}\cos(3\pi x)}$$

2. Find the Fourier series for the periodic function:

$$f(x) = \begin{cases} 0 & \text{if } -1 \leq x < 0 \\ x^2 & \text{if } \quad 0 \leq x < 1 \end{cases}$$

$$f(x+\underset{p}{\underline{2}}) = f(x)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{2}\right)$$

$$a_0 = \frac{2}{2}\int_{-1}^{0} 0\,dx = 0$$

$$\boxed{f(x) = 0 \text{ from } -1 \text{ to } 0}$$

$$a_n = \frac{2}{2}\int_{-1}^{0} 0\cos(\pi n x)\,dx = 0$$

$$a_0 = \frac{2}{2}\int_{0}^{1} x^2\,dx \rightarrow \left.\frac{x^3}{3}\right|_0^1 \rightarrow a_0 = \frac{1}{3}$$

$$\frac{a_0}{2} = \frac{1}{6}$$

$$a_n = \frac{2}{2}\int_0^1 x^2\cos(\pi n x)\,dx \rightarrow$$ ☆ same as problem 1☆

$$a_n = 2\frac{-1^n}{(n\pi)^2}$$

$$\boxed{f(x) = \frac{1}{6} + \sum_{n=1}^{\infty} 2\frac{-1^n}{(n\pi)^2}\cos(n\pi x) \text{ for } 0 \text{ to } 1}$$

3. Solve the heat conduction equation with the prescribed initial and boundary conditions:

$$u_{xx} = u_t$$

$$u(x,0) = \begin{cases} x & \text{if } \ 0 \le x < 20 \\ 40 - x & \text{if } \ 20 \le x \le 40 \end{cases}$$

$$u(0,t) = 0 \qquad u(40,t) = 0$$

$$u(x,t) = \sum D_n \sin\left(\frac{n\pi x}{p}\right) e^{-k\frac{n^2\pi^2}{p^2}t}$$

$$D_n = \frac{2}{40}\left[\int_0^{20} x\sin\left(\frac{n\pi x}{40}\right)dx + \int_{20}^{40}(40-x)\sin\left(\frac{n\pi x}{40}\right)dx\right]$$

$$= \frac{1}{20}\left[\frac{-40x}{n\pi}\cos\left(\frac{n\pi x}{40}\right)\Big|_0^{20} + \frac{40}{n\pi}\int_0^{20}\frac{\cos(n\pi x)}{40}dx + \left(-(40-x)\frac{40}{n\pi}\cos\left(\frac{n\pi x}{40}\right)\right)\Big|_{20}^{40}\right.$$

$$\left. - \frac{40}{n\pi}\int_{20}^{40}\cos\left(\frac{n\pi x}{40}\right)dx\right]$$

$$= \frac{1}{20}\left[\frac{1600}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{1600}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right] = \frac{160}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\boxed{u(x,t) = \sum \frac{160}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{2}\right)e^{-\frac{n^2\pi^2}{1600}t}}$$

4. In class we solved the following heat conduction problem (see page 10 in Lecture 11):

$$u_{xx} = u_t$$

$$u(x,0) = -2x^2 + 20x$$

$$u(0,t) = 0$$
$$u(10,t) = 0$$

We found that the solution is

$$u(x,t) = \frac{800}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} e^{-\frac{n^2\pi^2 t}{100}} \sin\left(\frac{n\pi x}{10}\right)$$

a) Graph this series solution at $t = 0$ along with the initial condition using a tool such as Desmos with the following bounds on your axes: $0 \le x \le 10$ and $0 \le u \le 55$. As we discussed in class, you will need to truncate the sum at some upper limit m, so be sure to include enough terms so that the series solution closely matches the initial condition to the eye. How many terms did you need? Include your graph showing both the truncated series solution $u_m(x,t)$ at $t = 0$ and the initial condition.

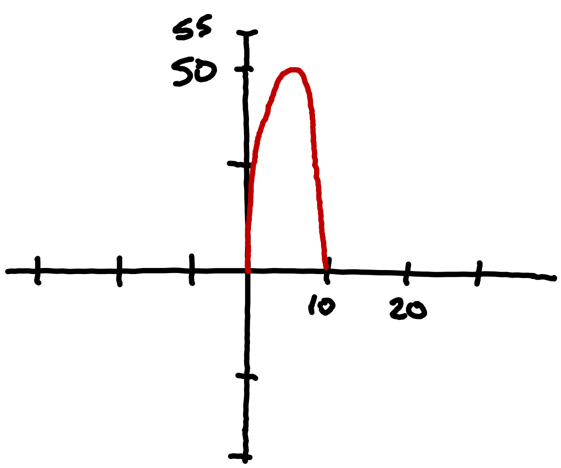
b) Now we will analyze the solution in more detail. Define the error $r_m(x)$ made by truncating the sum at a finite number of terms m as:

$$r_m(x) = |u_m(x,0) - u(x,0)| = \left|u_m(x,0) + 2x^2 - 20x\right|$$

By graphing this error using, e.g. Desmos, find the number of terms m needed such that $r_m(x) < 0.01$ on $0 \le x \le 10$. How many terms did you need? (We did a similar calculation in class. See pages 11 and 12 of Lecture 10)

c) Now truncate the sum at the upper limit you found in part b) and find the temperature in the bar when $t = 27s$ at $x = 5$ (again, you can use Desmos to evaluate the sum for you).

a)



$\boxed{11}$ terms needed

b) Lowest vale of m appears to be $\boxed{31}$

c) $\boxed{u(5,27) = 3.592}$

5. Solve the heat conduction equation with the prescribed initial and boundary conditions (note that these are nonhomogeneous boundary conditions):

$$u_{xx} = u_t$$

$$u(x,0) = \begin{cases} x & \text{if } \ 0 \le x < 15 \\ 30 - x & \text{if } \ 15 \le x \le 30 \end{cases}$$

$$u(0,t) = 10 \qquad u(30,t) = 30$$

$$u(x,t) = \sum f(x) \sin\left(\frac{n\pi x}{p}\right) e^{-k\frac{n^2\pi^2}{p^2}t}$$

$$D_n = \frac{1}{15}\left[\int_0^{15} x \sin\left(\frac{n\pi x}{30}\right)dx + \int_{15}^{30} (30-x)\sin\left(\frac{n\pi x}{30}\right)dx\right]$$

$$= \underbrace{\text{same-ish}}_{\text{as \#3}} = \frac{90}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$t=0$

$$u(0,t) = \sum_{n=1}^{\infty} \frac{90}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi 0}{30}\right)e^{0} + C_1 = 10$$

$$C_1 = 10$$

$$\boxed{u(x,t) = \sum \frac{90}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{30}\right)e^{-\frac{n^2\pi^2}{900}t} + \frac{2}{3}x + 10}$$