Gage Farmer

Homework 10 - Math 2568 (Autumn 2022)
Prof. Cueto

Due date: Wednesday November 30, 2022 (in class).

The sections and problem numbers refer to the course's textbook (L.W. Johnson, R.D. Riess, J.T. Arnold: *Introduction to Linear Algebra*, 5th edition, Pearson.)

Section	Assigned Problems	Problems to be turned in
§4.1	1, 3, 8, 11, 13, 15, 17, 18, 19	1, 3, 11, 13, 17, 18, 19
§4.2	8, 15, 16, 18, 21, 22, 27, 29	18, 21, 22, 27, 29
§4.4	1, 2, 3, 7, 9, 11, 13, 15, 16, 17, 18, 22, 25	1, 3, 9, 15, 16, 18, 22

Section 4.1

1. $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ $\begin{bmatrix} 1-\lambda & 0 \\ 2 & 3-\lambda \end{bmatrix}$ $(1-\lambda)(3-\lambda) = 0$ $\lambda = 1, 3$

$\lambda^2 - 4\lambda + 3 = 0$

$(A - I)x = 0$ $\left(\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ $\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2x_1 + 2x_2 = 0$ $x_1 = -x_2$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\lambda = 1$

$(A - I)x = 0$ $\left(\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$ $-2x_1 = 0$ $\begin{bmatrix} 0 \\ x_2 \end{bmatrix}$

$x_2 \neq 0$ $\lambda = 3$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $(2-\lambda)^2 - 1 = 0$ $\lambda^2 - 4\lambda + 3 = 0$ $\lambda = 1, 3$

$(A - I)x = \left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 - x_2 = 0$ $x_1 = x_2$ $\lambda = 1$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$(A - 3I)x = \left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $-x_1 - x_2 = 0$ $x_1 = -x_2$ $\lambda = 3$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ $(1-\lambda)(3-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = 0$ $\lambda = 2, 2$

$(A - 2I)x = \left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 + x_2 = 0$ $x_1 = -x_2$ $\lambda = 2$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

13. $A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$ $(-2-\lambda)(2-\lambda) + 5 = \lambda^2 + 1 = 0$ $\lambda^2 = -1$

There is no scalar λ such that $(A - \lambda I)$ is singular

17. $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ $(a-\lambda)(d-\lambda) - b^2 = \lambda^2 - (a+d)\lambda + (ad - b^2) = 0$

$D = (a+d)^2 - 4(ad - b^2) = a^2 - 2ad + d^2 + 4b^2 = (a-d)^2 + 4b^2$

Discriminant always positive so λ always real

18 $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ $b \neq 0$ $(a-\lambda)^2 + b^2 = \lambda^2 - (2a)\lambda + (a^2+b^2) = 0$

$D = 4a^2 - 4a^2 - 4b^2 = -4b^2$ discriminant always negative so λ doesn't exist

19. $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $(2-\lambda)(5-\lambda) - 12$

$A^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$ $(2-\lambda)(5-\lambda) - 12$

Same equation = same eigenvalues

Section 4.2

18. $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\det(A) = 1 \times \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= 0 \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$

$\det(A) = 0$

21. $A = \begin{bmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ $\quad x\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} - y\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} = 0$

$x[3+1] - y[2-0] - 1[-2-0] = 4x - 2y + 2 = 0$

$y = 2x - 1$

22. $A = \begin{bmatrix} x & 1 & 1 \\ 2 & 1 & 1 \\ 0 & -1 & y \end{bmatrix}$ $\quad x\begin{vmatrix} 1 & 1 \\ -1 & y \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 0 & y \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix}$

$x[y+1] - [2y-0] - [-2-0] = (x-2)(y+1) = 0$ $\quad x = 2$, $y = -1$

27. $\det(A) = 3$
$\det(B) = 5$
$\det(A^{-1}) = \frac{1}{3}$

$\det(ABA^{-1}) = 5$

29. $\det(A^{-1}B^{-1}A^{2}) = \frac{1}{3} \cdot \frac{1}{5} \cdot 9 = \frac{3}{5}$

Section 4.4

1. $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ $\quad p(t) = (1-t)(3-t)$ $\quad \lambda = 1, 3$
Alg. Mult = 1 for both

3. $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\quad p(t) = (2-t)(2-t) - 1 = t^2 - 4t + 3$ $\quad \lambda = 1, 3$
Alg Mult = 1 for both

9 $A = \begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix}$ $\begin{bmatrix} 3-t & -1 & -1 \\ -12 & -t & 5 \\ 4 & -2 & -1-t \end{bmatrix}$

$\det(A-tI) = (3-t)\begin{vmatrix} -t & 5 \\ -2 & -1-t \end{vmatrix} - (-1)\begin{vmatrix} -12 & 5 \\ 4 & -1-t \end{vmatrix} - (-1)\begin{vmatrix} -12 & -t \\ 4 & -2 \end{vmatrix}$

$\lambda = 2, 1, -1$

$= (3-t)[t^2+t+10] + [12t-8] - [4t+24]$

Alg. Mult = 1 for all

$= -t^3+2t^2+t-2$ $p(t) = -(t-2)(t-1)(t+1)$

15. $Ax = \lambda x$ $x \neq 0$

$A^{-1}(Ax) = A^{-1}(\lambda x) \longrightarrow Ix = \lambda(A^{-1}x) \rightarrow \frac{I}{\lambda}x = A^{-1}x$

$\frac{1}{\lambda}$ is eigenvalue of A^{-1} w/ eigenvector x

16. $\lambda + \alpha$ is eigenvalue of $A + \alpha I$

$Ax + \alpha x = \lambda x + \alpha x \rightarrow (A + \alpha I)x = (\lambda + \alpha I)x$

$\rightarrow (A + \alpha I)x = (\lambda + \alpha)x$

18. a) $q(t) = t^3 - 2t^2 - t + 2$ $q(H) = H^3 - 2H^2 - H + 2I$

$q(H)x = H^3x - 2H^2x - Hx + \frac{2}{x} = (\lambda^3 - 2\lambda^2 - \lambda + 2I)x$ $q(H)x = q(\lambda)x$

b) $A = \begin{bmatrix} -6 & -1 & 2 \\ 3 & 2 & 0 \\ -14 & -2 & 5 \end{bmatrix}$ $B = \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & -2 & -1 \end{bmatrix}$

$$P_A(t) = (-6-t)\begin{vmatrix} 2-t & 0 \\ -2 & 5-t \end{vmatrix} - 3\begin{vmatrix} -1 & 2 \\ -2 & 5-t \end{vmatrix} + (-14)\begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= (-6-t)\left[10-7t+t^2\right] - 3t + 3 + 56 - 28t$$

$$= -t^3 + t^2 + t - 1 = -(t-1)^2(t+1) \quad \lambda = -1, 1$$

$$q(1) = 1^3 - 2(1^2) - 1 + 2 = 0$$

$$q(-1) = (-1)^3 - 2(-1^2) - (-1) + 2 = 0$$

$$P_B(t) = (-2-t)\begin{vmatrix} 1-t & 1 \\ -2 & -1-t \end{vmatrix} - (0) + (-2)\begin{vmatrix} -1 & 0 \\ 1-t & 1 \end{vmatrix} = (-2-t)(t^2-t+2)+2$$

$$= -t^3 - 2t^2 - t - 2 + 2 = t(t+1)^2 \quad \lambda = 0, -1$$

$$q(0) = 0^3 - 2(0)^2 - 0 + 2 = 2$$

$$q(-1) = (-1)^3 - 2(-1^2) - (-1) + 2 = 0$$

22. $A = \begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix}$ $p(t) = -t^3 + 2t^2 + t - 2$

$$A^2 = \begin{bmatrix} 17 & -1 & -7 \\ -16 & 2 & 7 \\ 32 & -2 & -13 \end{bmatrix} \quad A^3 = \begin{bmatrix} 35 & -3 & -15 \\ -44 & 2 & 19 \\ 68 & -6 & -29 \end{bmatrix}$$

$$p(A) = -A^3 + 2A^2 + A - 2$$

$$= -\begin{bmatrix} 35 & -3 & -15 \\ -44 & 2 & 19 \\ 68 & -6 & -29 \end{bmatrix} + 2\begin{bmatrix} 17 & -1 & -7 \\ -16 & 2 & 7 \\ 32 & -2 & -13 \end{bmatrix} + \begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$