MATH 2415 – Ordinary and Partial Differential Equations
Lecture 02 notes

Section 1.3: Classification of Differential Equations

We will learn how to solve many different types of differential equations. We need a scheme for classifying differential equations to determine the appropriate solution method for a given problem.

Differential equations are classified according to **type, order,** and **linearity**

<u>Classification by type</u>

- **Ordinary differential equations (ODE)** contain only *ordinary derivatives* of one or more dependent variables with respect to a *single independent variable*

Examples: $\frac{dy}{dx} + 10y = e^x$ $\qquad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0$ $\qquad L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)$

y depends on x

Q depends on t

- **Partial differential equations (PDE)** contain *partial derivatives* of one or more dependent variables with respect to *two or more independent variables*

Examples: $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $\qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$ $\qquad \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$

u depends on y

v depends on x

u depends on x and t

x and t

∂

Note: We sometimes use a subscript notation to denote partial differentiation (e.g, $\frac{\partial u}{\partial y} = u_y$, $\frac{\partial^2 u}{\partial x^2} = u_{xx}$)

$u_y = -v_x$ $\qquad u_{xx} = u_{tt} - 2u_t$ $\qquad \alpha^2 u_{xx} = u_t$

Systems of differential equations

If there is one unknown function (i.e., the dependent variable) we need one differential equation. If there are two or more unknown functions, then we will have a *system of differential equations* (ordinary or partial depending on the number of independent variables)

Example:

$$\begin{cases} \frac{dx}{dt} = ax - \alpha xy \\ \frac{dy}{dt} = -cy + \gamma xy \end{cases}$$

dep

x and y depend on t

<u>Classification by Order</u>

- The **order of a differential equation** (ODE or PDE) is the order of the highest derivative in the equation

Examples: $\frac{dy}{dx} + 10y = e^x$ $\quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0$ $\quad L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)$

ODE 1st order | ODE 2nd order | ODE 2nd order

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $\quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$ $\quad \alpha^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$

PDE 1st order | PDE 2nd order | PDE 2nd order

$(y - x)\,dx + 4x\,dy = 0$

$4x\,dy = (x-y)\,dx \rightarrow 4x\frac{dy}{dx}dx = x - y$

1st order ODE

$y''' + 2e^t y'' + yy' = t^4$

3rd order ODE

The general form of an nth-order ODE is

$$F\left(x, y, y', \dots, y^{(n)}\right) = 0$$

We will assume that we can always solve the equation for the highest derivative $y^{(n)}$,

$$y^{(n)} = f\left(x, y, y', \dots, y^{(n-1)}\right)$$

Classification by Linearity

An ODE $y^{(n)} = f\left(x, y, y', \dots, y^{(n-1)}\right)$ is **linear** if f is a linear function of y, y', y'', …, $y^{(n-1)}$. In other words, we can write the equation in the form

$$a_n(x)\frac{dy^n}{dx^n} + a_{n-1}(x)\frac{dy^{n-1}}{dx^{n-1}} + \cdots a_1(x)\frac{dy}{dx} + a_0(x)\, y = G(x)$$

The most important things to note are

i) The dependent variable y and all its derivatives are of the first degree (i.e., the power of each term involving y is 1)

ii) Each coefficient depends only on the independent variable x

Examples: $(y - x)\, dx + 4x\, dy = 0$ — linear 1st order ODE

$y'' - 2y' + y = 0$ — linear 2nd order ODE

$x^3\frac{d^3y}{dx^3} - 4x\frac{dy}{dx} + 6y = e^x$ — linear 3rd order ODE

If these conditions are not satisfied, the differential equation is **nonlinear**

Examples: $(1 + y)\, y' + 2y = e^x$ — non linear 1st order ODE

$\frac{d^3y}{dx^3} + \sin y = 0$ — non linear 3rd order ODE

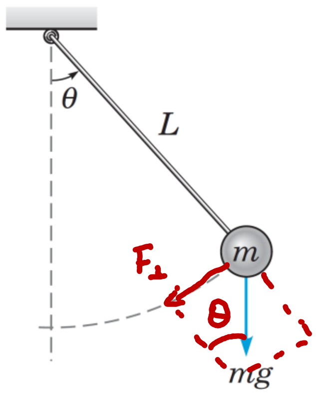
$\frac{d^4y}{dx^4} + 6y^2 = x^4$ — non linear 4th order ODE

$$\begin{cases} \dfrac{dx}{dt} = ax - \alpha xy \\ \dfrac{dy}{dt} = -cy + \gamma xy \end{cases}$$

non linear system of 1st order ODEs

Solving nonlinear DEs can be very difficult, but *sometimes* we can approximate a nonlinear DE by a linear DE. This process is called **linearization**

Examples: The differential equation for the motion of a simple pendulum



$\tau = I\alpha$

τ = Torque
$= F_\perp L$
$= -mg\sin\theta L$

I: Moment of Inertia
$= mL^2$

α : Angular momentum
$= \dfrac{d^2\theta}{dt^2}$

$$-\cancel{m}g\sin\theta\cancel{L} = \cancel{m}L^{\cancel{2}} \cdot \frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

2nd order nonlinear ODE since θ is in sine function

If θ is small, then $\sin\theta \approx \theta$
and we have $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\theta = 0$ ← linear!

This linearized equation (valid for small θ) is very easy to solve

Definition: A **solution** of the *n*th order ODE on the interval $\alpha < x < \beta$ is a function ϕ such that $\phi', \phi', \ldots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)} = f\left(x, \phi, \phi', \ldots, \phi^{(n-1)}\right)$$

for every x in $\alpha < x < \beta$. Here we assume that f is a *real-valued* function, and we are interested in obtaining *real-valued* solutions $y = \phi(x)$.

It can be difficult to *find* the solution to a DE, but it is usually simpler to *verify* that a *given* function is a solution. Just substitute the equation into the DE!

Example: Verify that the function $y = xe^x$ is a solution of the linear equation

$$y'' - 2y' + y = 0$$

on the interval $(-\infty, \infty)$.

1) Find Derivatives

$y' = xe^x + e^x$

$y'' = xe^x + 2e^x$

2) Substitute into ODE

$(xe^x + 2e^x) - 2(xe^x + e^x) + (xe^x) \stackrel{?}{=} 0$

$-\cancel{2xe^x} - \cancel{2e^x} + \cancel{xe^x} + \cancel{2e^x} + \cancel{xe^x} \stackrel{?}{=} 0$

$0 \stackrel{\checkmark}{=} 0$

Example: Verify that $y = x^4/16$ is a solution of the nonlinear equation

$$\frac{dy}{dx} = xy^{1/2}$$

on the interval $(-\infty, \infty)$.

$y' = \frac{x^3}{4}$

$y^{\frac{1}{2}} = \left(\frac{x^4}{16}\right)^{\frac{1}{2}} = \frac{x^2}{4}$

$\frac{x^3}{4} = \frac{x^3}{4}$

0 for all x ✓

Explicit and Implicit Solutions

An **explicit solution** is a solution in which the dependent variable is expressed entirely in terms of the independent variable, $y = \phi(x)$, on some interval. A **trivial solution** is an explicit solution that is zero everywhere in the interval.

An **implicit solution** is a relation of the form $K(x, y) = 0$ on the interval $\alpha < x < \beta$ if there exists at least one function $y = \phi(x)$ that satisfies both the relation and the DE on this interval.

To verify if an implicit solution satisfies an ODE, we usually need to use *implicit differentiation.*

Example: Show that the relation $x^2 + y^2 - 4 = 0$ is an implicit solution of

$$\frac{dy}{dx} = -\frac{x}{y}$$

on $-2 < x < 2$.

Implicit: $\frac{d}{dx} F(y(x)) = \frac{dF}{dy} \cdot \frac{dy}{dx} = \frac{dF}{dy} \cdot y'$

Differentiate it: $\frac{d}{dx}(x^2 + y^2 - 4) = \frac{d}{dx}(0)$

$2x + 2y \cdot \frac{dy}{dx} = 0$

$\hookrightarrow y \frac{dy}{dx} = -x \rightarrow \boxed{\frac{dy}{dx} = -\frac{x}{y}}$ ✓

Example: Show that $y = \sin^{-1} xy$ is an implicit solution of

Method 1:

$$xy' + y = y'\sqrt{1 - x^2y^2}$$

Elim. inv. sine funct. by taking sine of both sides of the solution

↳ $\sin y = xy$ differentiate → $\cos y \cdot y' = y + xy$ → rewrite $\cos y$ as $\sin^2 y + \cos^2 y = 1$

↳ $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2y^2}$

original equation

$$\boxed{xy' + y = y'\sqrt{1 - x^2y^2}}$$

Method 2:

Differentiate the inv sine function instead (problem set 2)

Example: Show that $x + y = \tan^{-1} y$ is an implicit solution of

Method 1:

$$1 + y^2 + y^2y' = 0$$

Differentiate the inv tan funct.

$\frac{d}{dx}(x+y) = \frac{d}{dx}(\tan^{-1} y)$ → $1 + y' = \frac{1}{1+y^2}\, y'$ multiply both sides by $1+y^2$

$(1+y')(1+y^2) = y'$

$1 + y^2 + \cancel{y'} + y^2y' = \cancel{y'}$

$$\boxed{1 + y^2 + y^2y' = 0}$$

Method 2:

Take tangent of both sides and use implicit differentiation

Section 2.1: Linear Differential Equations; Method of Integrating Factors

The general form of an nth-order linear ODE is

$$a_n(x)\frac{dy^n}{dx^n}+a_{n-1}(x)\frac{dy^{n-1}}{dx^{n-1}}+\cdots a_1(x)\frac{dy}{dx}+a_0(x)\,y=G(x)$$

If n = 1, then we have a **first-order linear ODE**

$$a_1(x)\frac{dy}{dx}+a_0(x)\,y=G(x)$$

We can convert this to **standard form** by dividing both sides by $a_1(x)$ (provided $a_1(x)\neq 0$)

$$\frac{dy}{dx}+p(x)\,y=g(x)$$

$p(x)=\frac{a_0(x)}{a_1(x)}$

$g(x)=\frac{G(x)}{a_1(x)}$

Sometimes we can solve a first-order linear ODEs by integrating the equation**.**

Example: Solve the following equation for $y(t)$

$$(4+t^2)\frac{dy}{dt}+2ty=4t$$

First note that $\frac{d}{dt}\left[(4+t^2)y\right]=(4+t^2)\frac{dy}{dt}+2ty$

So we can rewrite the ODE as follows:

$$\frac{d}{dt}\left[(4+t^2)y\right]=4t$$

Integrate w respect to t:

$$\int\frac{d}{dt}\left[(4+t^2)y\right]dt=\int 4t\,dt$$

$$(4+t^2)y+c_1=2t^2+c_2$$

Solving for y, we get general solution:

$$\boxed{y=\frac{2t^2}{4+t^2}+\frac{c}{4+t^2}}$$

$\leftarrow c=c_2-c_1$

We saw another case where we could integrate the equation last lecture in the falling object example.

Usually, we cannot integrate the equation as in the previous example because one side of the equation is not the derivative of y times some other function. In these cases, we can use the **method of integrating factors**.

The main idea is that we can multiply the equation by some function $\mu(x)$ that converts it into an equation we *can* integrate. The function $\mu(x)$ is called the **integrating factor**

Method of Integrating Factors

Consider the first-order linear ODE in standard form, ← Always start in standard form

$$\frac{dy}{dx} + p(x)\,y = g(x)$$

To start, first multiply both sides by $\mu(x)$:

$$\mu(x)\frac{dy}{dx} + p(x)\mu(x)y = \mu(x)g(x)$$

We want the left hand side of this equation to equal $\frac{d}{dx}[\mu(x)y]$ because we can then integrate both sides of the ODE and solve for y. Let's use the product rule to expand this derivative:

$$\frac{d}{dx}[\mu(x)y] = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y$$

Note that the left-hand side of the ODE will equal the derivative of $\mu(x)y$ if

$$\frac{d\mu}{dx} = p(x)\mu(x)$$

We can rearrange this equation and integrate to get an expression for $\mu(x)$:

$$\frac{1}{\mu(x)}\frac{d\mu(x)}{dx} = p(x) \rightarrow \int \frac{d\mu}{\mu} = \int p(x)\,dx$$

$$\ln|\mu(x)| = \int p(x)dx + k$$

where k is an arbitrary constant. We can now exponentiate both sides and set k to zero to get

$$\boxed{\mu(x) = e^{\int p(x)dx}}$$

Putting the pieces together, we get

$$\frac{d}{dx}(\mu(x)y) = \mu(x)g(x)$$

we need these to solve the ODE

Finally, we can integrate both sides and solve for y to get

$$\boxed{y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)g(x)dx + \underline{c}\right)}$$

c inside parenthesis

Note that if we cannot evaluate this integral in terms of elementary functions, the general solution is

$$y(x) = \frac{1}{\mu(x)}\left(\int_{x_0}^{x} \mu(s)g(s)ds \ + c\right)$$

★ Integrating Parts★

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example: Use the method of integrating factors to solve

$$x\frac{dy}{dx} - 4y = x^6 e^x$$

Put into standard form:

$$\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x \qquad p(x) = -\frac{4}{x} \qquad g(x) = x^5 e^x$$

★ Don't fuck up the ± signs★

Integrating factor: $\mu(x) = e^{\int p(x)dx} = e^{-4\int \frac{dx}{x}} = e^{-4\ln|x|} = e^{\ln|x|^{-4}} = |x|^{-4} = x^{-4}$ b/c even power

General Solution: $y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)g(x)dx + C\right] = x^4\left[\int x^{-4}x^5 e^x dx + C\right]$

$= x^4\left[\int x e^x dx + C\right]$ let $u = x$ $\quad du = \frac{du}{dx}dx = dx$

$dv = e^x dx \qquad v = \int e^x dx = e^x$

Integrate by parts

$= x^4\left[xe^x - \int e^x dx + C\right] = x^4\left[xe^x - e^x + C\right] \rightarrow \boxed{y(x) = x^5 e^x - x^4 e^x + Cx^4}$

Example: Use the method of integrating factors to solve

$$\frac{dy}{dx} - 3y = 0 \rightarrow p(x) = -3 \quad g(x) = 0$$

Integrating Factor: $\mu(x) = e^{\int p(x)dx} = e^{-3\int dx} = e^{-3x}$

General Solution $y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)\cancel{g(x)}^{0} dx + c\right] = \frac{c}{\mu(x)} = \frac{c}{e^{-3x}} = ce^{3x}$

$\hookrightarrow \boxed{y(x) = ce^{3x}}$

Check that solution is correct: $\frac{dy}{dx} - 3y = 0$

$$\left.\begin{aligned} y &= ce^{3x} \\ \frac{dy}{dx} &= 3ce^{3x} \end{aligned}\right\} \text{Plug into ODE} \rightarrow 3ce^{3x} - 3ce^{3x} = 0 \checkmark$$

Example: Use the method of integrating factors to solve

$$\left(x^2+9\right)\frac{dy}{dx}+xy=0$$

Example: Use the method of integrating factors to solve the initial value problem

$$x\frac{dy}{dx}+y=2x, \qquad y(1)=0$$

See your text for more example problems!