Gage Farmer

Homework 9 - Math 2568 (Autumn 2022)
Prof. Cueto

Due date: Wednesday November 16, 2022 (in class).

The sections and problem numbers refer to the course's textbook (L.W. Johnson, R.D. Riess, J.T. Arnold: *Introduction to Linear Algebra*, 5th edition, Pearson.)

Section	Assigned Problems	Problems to be turned in
§5.7	12, 13, 14, 15, 19, 25, 26, 27	12, 13, 14, 15, 27
§5.9	1, 2, 3, 4, 6, 7, 8, 9, 11, 13, 14, 17, 19, 23	1, 2, 8, 13, 14, 17
§6.2	1, 7, 11, 13, 17, 19, 21, 29	1, 7, 11, 17
§6.3	1, 5, 7, 9, 13, 17, 19, 20, 21, 22, 23, 24	7, 9, 13, 17, 24

Section 5.7

12) $V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+2d \\ b-c \end{bmatrix}$

a) $\underline{T\left(x\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)} = \begin{bmatrix} xa + 2xd \\ xb - xc \end{bmatrix} = \begin{bmatrix} x(a+2d) \\ x(b-c) \end{bmatrix} = \underline{xT\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)}$

b) $R^2 = \begin{bmatrix} a+2d \\ b-c \end{bmatrix}$ $\quad T(V) = R^2$ therefore $T(\theta_V) = \theta_{R^2}$

$N(T) = \left\{ V \text{ in } R^2 : V = \begin{bmatrix} -2d & c \\ c & d \end{bmatrix} \right\}$

c) $\begin{bmatrix} a+2d \\ b-c \end{bmatrix} \rightarrow \left\{ \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

d) $\text{rank}(T) + \text{nullity}(T) = \dim(V) = 4$

$\text{rank}(T) = 2 \quad \text{nullity}(T) = 2$

e) R(T) has the same dimensions and is a subspace of R^2

f) $A = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \quad T(A) = \begin{bmatrix} x \\ y \end{bmatrix}$

$T(A) = v$ and $v \in R(T)$ so $R(T) = R^2$

13) $T(P) = p''(x) \quad T: P_4 \to P_2$

a) $T(1) = \frac{d^2}{dx^2}(1) = 0 \to T(x) = \frac{d^2}{dx^2}(x) = 0$

$\to T(x^2) = \frac{d^2}{dx^2}(x^2) = 2 \to T(x^3) = \frac{d^2}{dx^2}(x^3) = 6x$

$\to T(x^4) = \frac{d^2}{dx^2}(x^4) = 12x^2$

$A = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix} \to$ Basis of $R(T) = \{2, 6x, 12x^2\}$

b) $\dim(P_4) = \text{rank}(T) + \text{nullity}(T) \to 5 = 3 + \text{null}(T)$

$\text{nullity}(T) = 2 \neq 0 \to$ not one to one

c) $\int p(x)\,dx = \int (a_0 + a_1 x + a_2 x^2)\,dx = a_0 x + a_1\left(\frac{x^2}{2}\right) + a_2\left(\frac{x^3}{3}\right) + C = r(x)$

$T(r(x)) = p(x)$ so $R(T) = P_2$

14) $T: P_4 \to P_3 \qquad C = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$

$$C = \begin{bmatrix} 1 & -1 & 2 & -1 & 1 \\ -1 & 3 & -2 & 3 & -1 \\ 2 & -3 & 5 & -1 & 1 \\ 3 & -1 & 7 & 2 & 2 \end{bmatrix} \qquad C^T = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -1 & 3 & -3 & -1 \\ 2 & -2 & 5 & 7 \\ -1 & 3 & -1 & 2 \\ 1 & -1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & -.5 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \quad D^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$p(x) = 1 - x$

$q(x) = x$

$r(x) = x^2$

$s(x) = x^3$

Basis $R(T) = \{p(x), q(x), r(x), s(x)\}$

$\text{nullity}(T) = 1$ not one of one

15) $N(T) = \{a_0 + a_1x + a_2x^2 \in P_2 \mid a_0 + 2a_1 + 4a_2 = 0\}$

$R(T) = \mathbb{R}^1$

27) $V = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad T: V \to V$ by $T(A) = A^T$

a) $T(A) = A^T \qquad T(B) = B^T \qquad T(A+B) = (A+B)^T$

$= A^T + B^T = T(A) + T(B) \qquad T(kA) = (kA)^T = k(A)^T = kT(A)$

T is linear transformation

b) $\text{nullity}(T) + \text{rank}(T) = \underset{4}{\dim(V)}$

$\text{nullity}(T) = 0$ so $\text{rank}(T) = 4$

T is one to one

c) $B, C \in V$ $B = C^T$ so $T(C) = C^T = B$ so $R(T) = V$

Section 5.9

1) $S: P_3 \to P_4$ def by $S(p) = p'(0)$

$S(1) = \frac{d}{dx}(1)\Big|_0 = 0$ $S(x) = \frac{d}{dx}(x)\Big|_0 = 1$ $S(x^2) = \frac{d}{dx}(x^2)\Big|_0 = 0$ $S(x^3) = \frac{d}{dx}(x^3)\Big|_0 = 0$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2) $T: P_3 \to P_4$ def by $T(p) = (x+2)p(x)$

$T(1) = (x+2)(1) = 2 + x$ $T(x) = (x+2)x = 2x + x^2$

$T(x^2) = (x+2)(x^2) = 2x^2 + x^3$ $T(x^3) = (x+2)(x^3) = 2x^3 + x^4$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8) $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ in P_3

a) $S(p) = p'(0) = a_1 + 2a_2(0) + 3a_3(0) = a_1$

$[S(p)]_C = \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $[p]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$

b) $P[p]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = [S(p)]_C$

13) $T: V \to V$ by $T(A) = A^T$

a) $E_{11}^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $E_{12}^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $E_{21}^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $E_{22}^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

b) $Q[A]_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix} = [A^T]_B$

14) $S: P_2 \to P_3$ by $S(p) = x^3p'' - x^2p' + 3p$ $\quad S(1) = 3 \quad S(x) = 3x - x^2$

$S(x^2) = 3x - x^2$

$$S = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

17) $T: P_2 \to \mathbb{R}^3$ by $T(p) = \begin{bmatrix} p(0) \\ 3p'(1) \\ p'(1) + p''(0) \end{bmatrix}$ $\quad T(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad T(x) = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 1 & 4 \end{bmatrix} \qquad T(x^2) = \begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}$$

Section 6.2

1) $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ $\det = (1\cdot 1) - (2\cdot 3) = -5$

7) $\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$ $\det = (4\cdot 1) - (-2\cdot 1) = 6$

11) $\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

$A_{11} = (-1)^{1+1}\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} = (2\cdot 1) - (2\cdot 2) = -2$

$A_{12} = (-1)^{1+2}\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = -((-1\cdot 1) - (3\cdot 2)) = 7$

$A_{13} = (-1)^{1+3}\begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix} = (-1\cdot 2) - (3\ 2) = -8$

$A_{33} = (-1)^{3+3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = (2\cdot 2) - (-1\cdot -1) = 3$

17) $\det(A) = 2\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} = 2(-2) - 1(7) + 3(-8) = -35$

Section 6.3

7) $\det B = 6$

9) $\det B = 3$

13) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 4 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 2 \end{bmatrix}$ $\det A = a_{11}a_{22}a_{33}a_{44} = 1 \cdot 3 \cdot 1 \cdot 2 = 6$

17) $\begin{bmatrix} 2 & 4 & -2 & -2 \\ 1 & 3 & 1 & 2 \\ 1 & 3 & 1 & 3 \\ -1 & 2 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 4 & -8 & -11 \end{bmatrix}$ $\det A = 2 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 4 & -8 & -11 \end{vmatrix} = 2 \cdot 1 \begin{vmatrix} 0 & 1 \\ -8 & -11 \end{vmatrix}$

$= 2((0 \cdot -11) - (1 \cdot -8)) = 16$

24) $B = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 1 & 3 & 4 \end{bmatrix}$

a) $AB = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 1 & 3 & 4 \end{bmatrix} = [2A_1 + 3A_2 + A_3, -A_2 + 3A_3, 4A_3]$

Checks out ✓

b) $\det AB = -1 \cdot 2 \cdot 4 \, |A_1, A_2, A_3| = -8 \det A$

c) $\det B = 2((-1 \cdot 4) - (0 \cdot 3)) = -8$ $\det AB = -8 \det A$

$\det AB = \det A \det B$