Which of the following quantities will have the same measured value independent of the reference frame in which they were measured?

(1) The speed of light in a vacuum

(2) The time interval between two events

(3) The length of an object

(4) 1 & 2 (5) 1 & 3 (6) 2 & 3

A cube of side length **a** is placed in S frame. After correcting for differences in transmission times for light coming from different parts of the cube, what is the volume calculated for the cube in an S' frame that is moving parallel to one side of the cube at a relative speed 0.8c? (c is the speed of light.)

(1) a^3

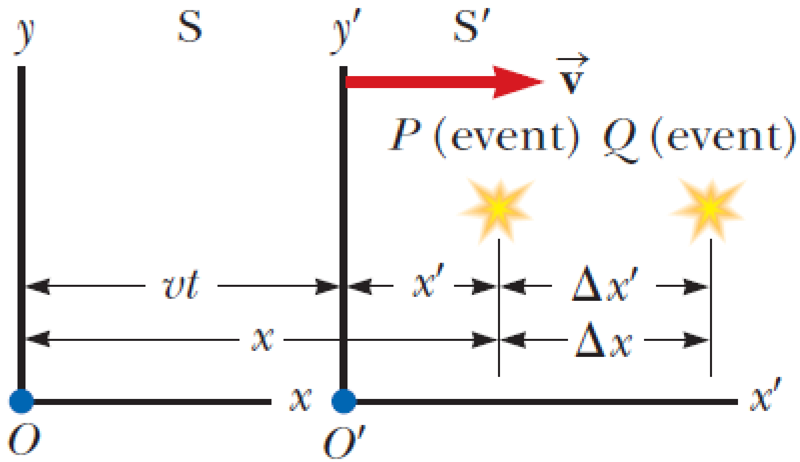
(2) $0.6\ a^3$

(3) $0.8\ a^3$

(4) $0.64\ a^3$

$$L = \frac{L_p}{\gamma}$$

Lorentz Transformation Equations



$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma(t - \frac{v}{c^2}x)$$

$$x = \gamma(x' + vt')$$

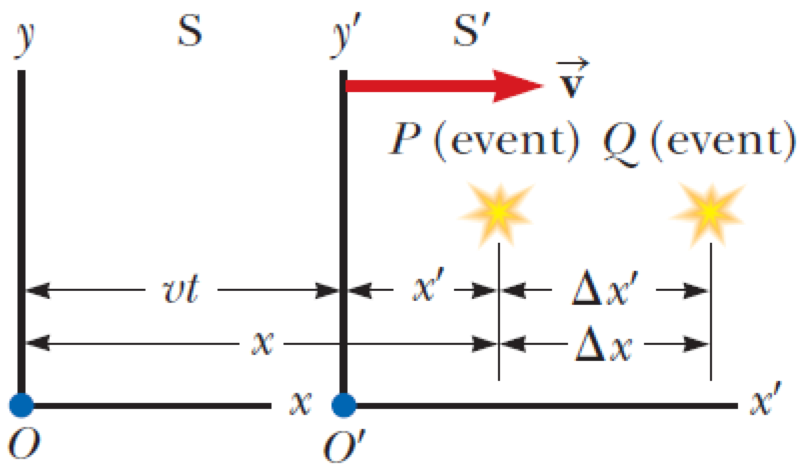
$$y = y'$$

$$z = z'$$

$$t = \gamma(t' + \frac{v}{c^2}x')$$

- Maxwell's equations are invariant under these transformations
- Space and time are related for two different inertial observers
- They are reduced to Galilean transformations when v << c

Lorentz Transformation Equations



$$\Delta x' = \gamma(\Delta x - v\Delta t)$$

$$\Delta y' = \Delta y$$

$$\Delta z' = \Delta z$$

$$\Delta t' = \gamma(\Delta t - \frac{v}{c^2}\Delta x)$$

$$\Delta x = \gamma(\Delta x' + v\Delta t')$$

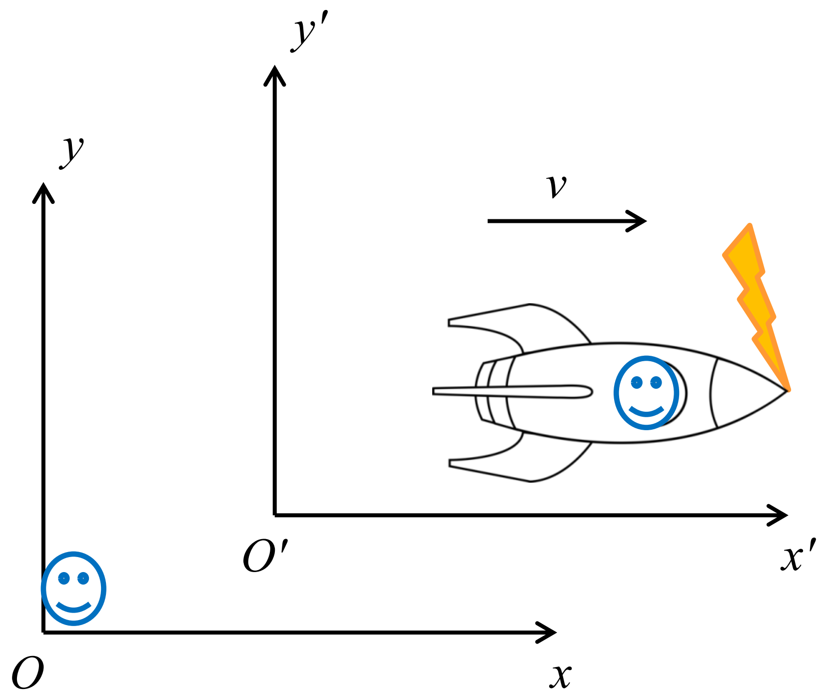
$$\Delta y = \Delta y'$$

$$\Delta z = \Delta z'$$

$$\Delta t = \gamma(\Delta t' + \frac{v}{c^2}\Delta x')$$

Time Dilation using Lorentz Transformation

A rocket is moving at a constant *v* relative to the rest frame *O*. An observer riding in the rocket sees a lightning hits the rocket's nose that lasts for $\Delta t'$.



$$\Delta t' = \Delta t_p$$

$$\Delta x' = 0$$

$$\Delta x = \gamma(\Delta x' + v\Delta t')$$
$$\Delta y = \Delta y'$$
$$\Delta z = \Delta z'$$
$$\Delta t = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)$$

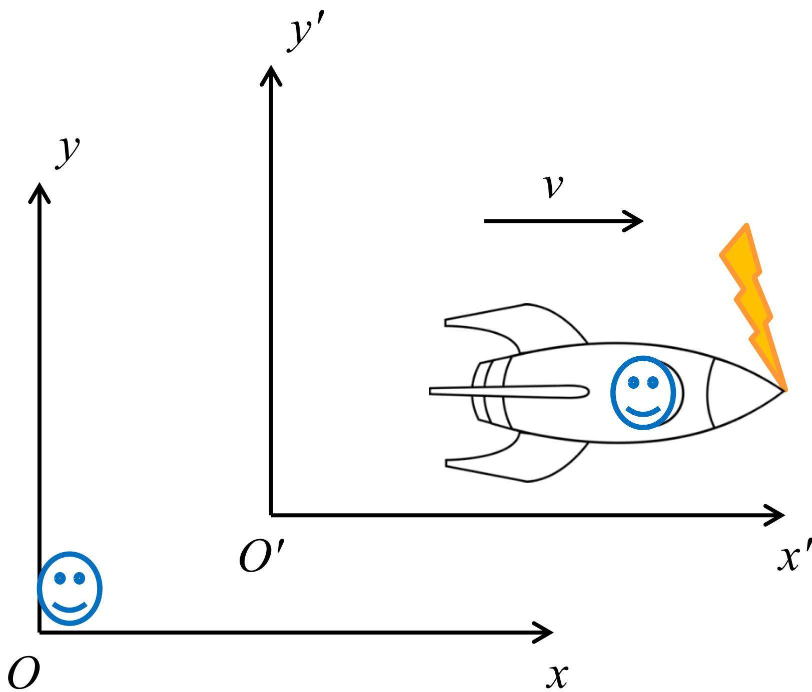
Define a moving frame O' that goes with the rocket. So the rocket in the moving frame O' is stationary.

Time Dilation using Lorentz Transformation

A rocket is moving at a constant v relative to the rest frame O. An observer riding in the rocket sees a lightning hits the rocket's nose that lasts for $\Delta t'$.

Use Lorentz Transformation

$$\boxed{\begin{aligned} \Delta x &= \gamma(\Delta x' + v\Delta t') \\ \Delta y &= \Delta y' \\ \Delta z &= \Delta z' \\ \Delta t &= \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) \end{aligned}}$$



$$\Delta t' = \Delta t_p$$

$$\Delta x' = 0$$

$$\Delta t = \gamma(\Delta t' + 0) = \gamma \Delta t_p$$

Δt measured by the observer in the rest frame O is longer than the proper time – time dilation.

In addition, Δx measured by the observer in the rest frame O is not zero – the lightening starts and ends at different locations in frame O (still on the nose of the rocket).

$$\Delta x = \gamma v \Delta t'$$

Length Contraction using Lorentz Transformation

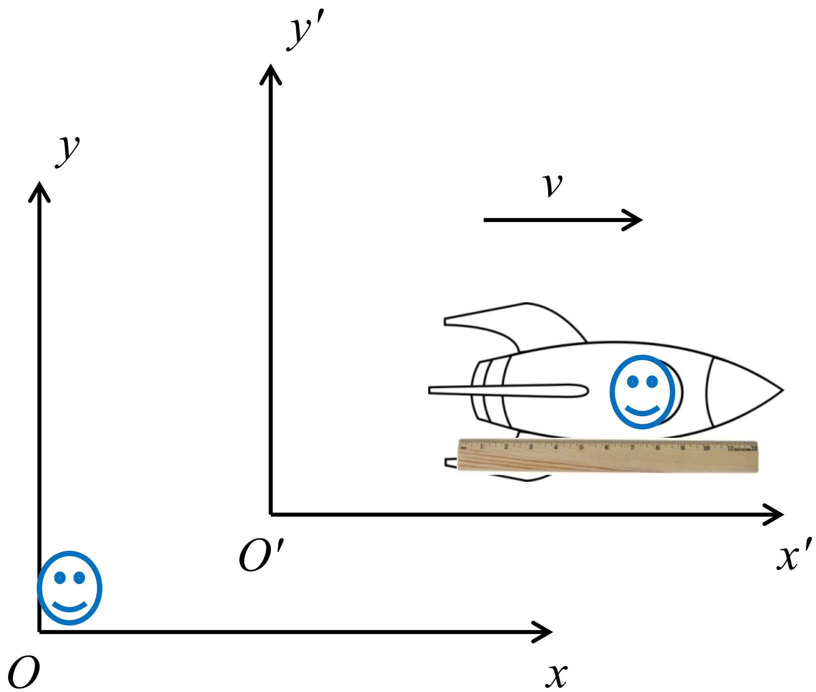
A rocket is moving at a constant v relative to the rest frame O. An observer riding in the rocket carries a ruler and measures its length being $\Delta x'=L'=L_p$.

An observer in the rest frame O also measures the length of the moving ruler being Δx. To measure the length, the beginning and ending points need to be measured together so Δt=0.

$$\Delta t = 0$$

$$\Delta x' = L_p$$

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$
$$\Delta y' = \Delta y$$
$$\Delta z' = \Delta z$$
$$\Delta t' = \gamma(\Delta t - \frac{v}{c^2}\Delta x)$$



What is the length of the ruler measured by the observer in the rest frame O?

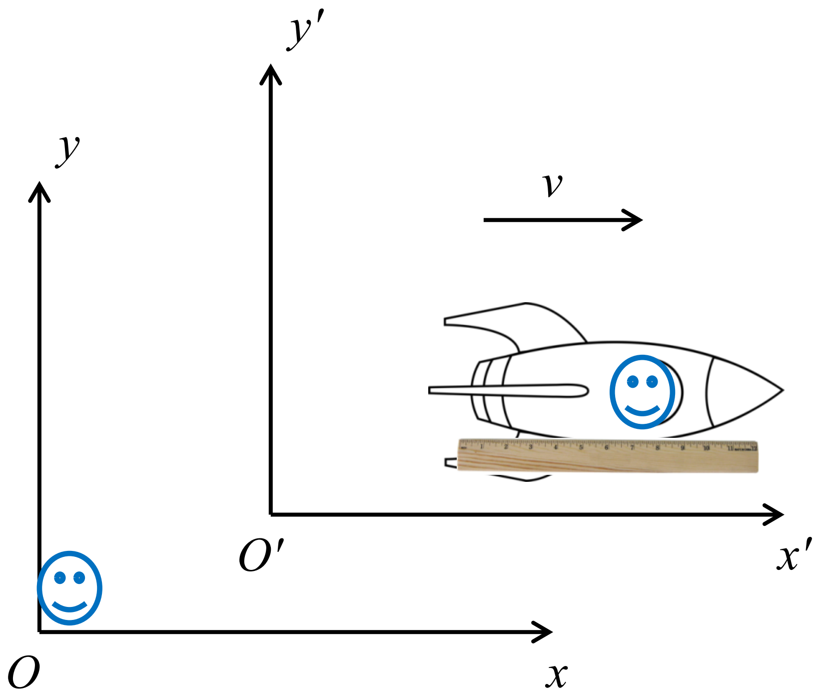
Length Contraction using Lorentz Transformation

What is the length of the ruler measured by the observer in the rest frame O?

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$
$$\Delta y' = \Delta y$$
$$\Delta z' = \Delta z$$
$$\Delta t' = \gamma(\Delta t - \frac{v}{c^2}\Delta x)$$

$$\Delta t = 0$$

$$\Delta x' = L_p$$



$$\Delta x' = \gamma(\Delta x - 0) = \gamma\Delta x = L_p$$

$$\Delta x = \frac{L_p}{\gamma}$$

The length measured in frame O is shorter than its proper length.

Twin Paradox

The Set-up

Twins Mary and Frank at age 30 decide on two career paths: Mary decides to become an astronaut and to leave on a trip 20 lightyears (ly) from the Earth at a speed 0.95c and to return; Frank decides to reside on the Earth.

The Problem

Upon Mary's return, Frank reasons that her clocks measuring her age must run slow. As such, she will return younger. However, Mary claims that it is Frank who is moving and consequently his clocks must run slow.

The Paradox

Who is younger upon Mary's return?

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 3.2$$

$$\Delta t = \frac{20c \times 2}{0.95c} = 42.1 yrs = \gamma \Delta t' \;\Rightarrow\; \Delta t' = \frac{42.1}{3.2} = 13.2 yrs$$

The Resolution

1) Frank's clock is in an **inertial system** during the entire trip; however, Mary's clock is not. As long as Mary is traveling at constant speed away from Frank, both of them can argue that the other twin is aging less rapidly.

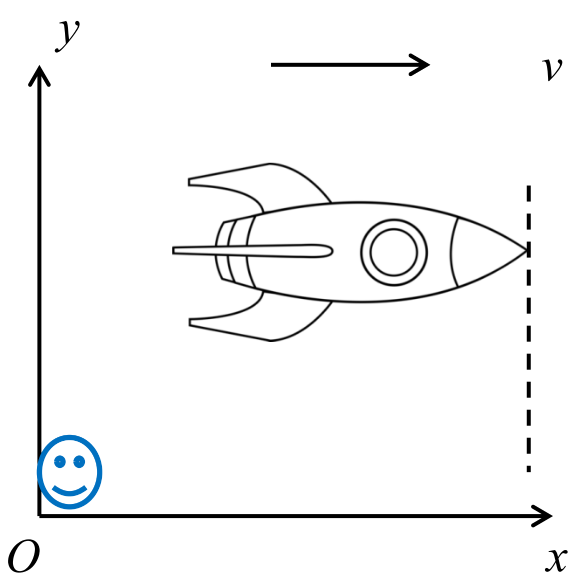
2) When Mary slows down to turn around, she leaves her original inertial system and eventually returns in a completely different inertial system.

3) Mary's claim is no longer valid, because she does not remain in the same inertial system. There is also no doubt as to who is in the inertial system. Frank feels no acceleration during Mary's entire trip, but Mary does.

A rocket with a proper length of 400 m passes by an observer on the Earth. According to the observer, it takes 0.800 μs rocket to pass a fixed point.

What is the speed of the rocket as measured by the Earth-based observer?

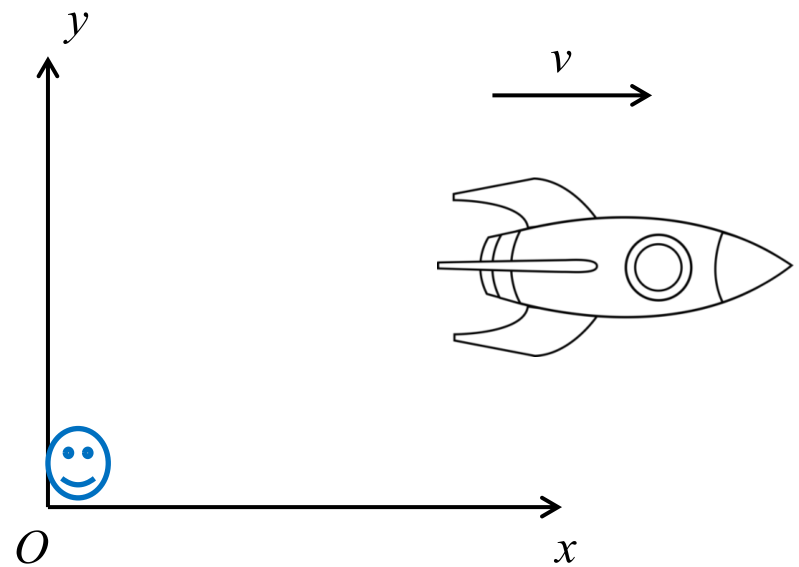
$$\Delta x = \frac{L_p}{\gamma} \qquad \gamma = \frac{1}{\sqrt{1 - v^2 / c^2}}$$



A rocket with a proper length of 400 m passes by an observer on the Earth. According to the observer, it takes 0.800 μs rocket to pass a fixed point.

What is the speed of the rocket as measured by the Earth-based observer?

$$\Delta x = \frac{L_p}{\gamma} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad v = \frac{\Delta x}{\Delta t} = \frac{L_p/\gamma}{0.8\times 10^{-6}s} = \frac{400\sqrt{1-v^2/c^2}}{0.8\times 10^{-6}s}$$



$$\frac{v}{\sqrt{1-v^2/c^2}} = \frac{400}{0.8\times 10^{-6}}$$

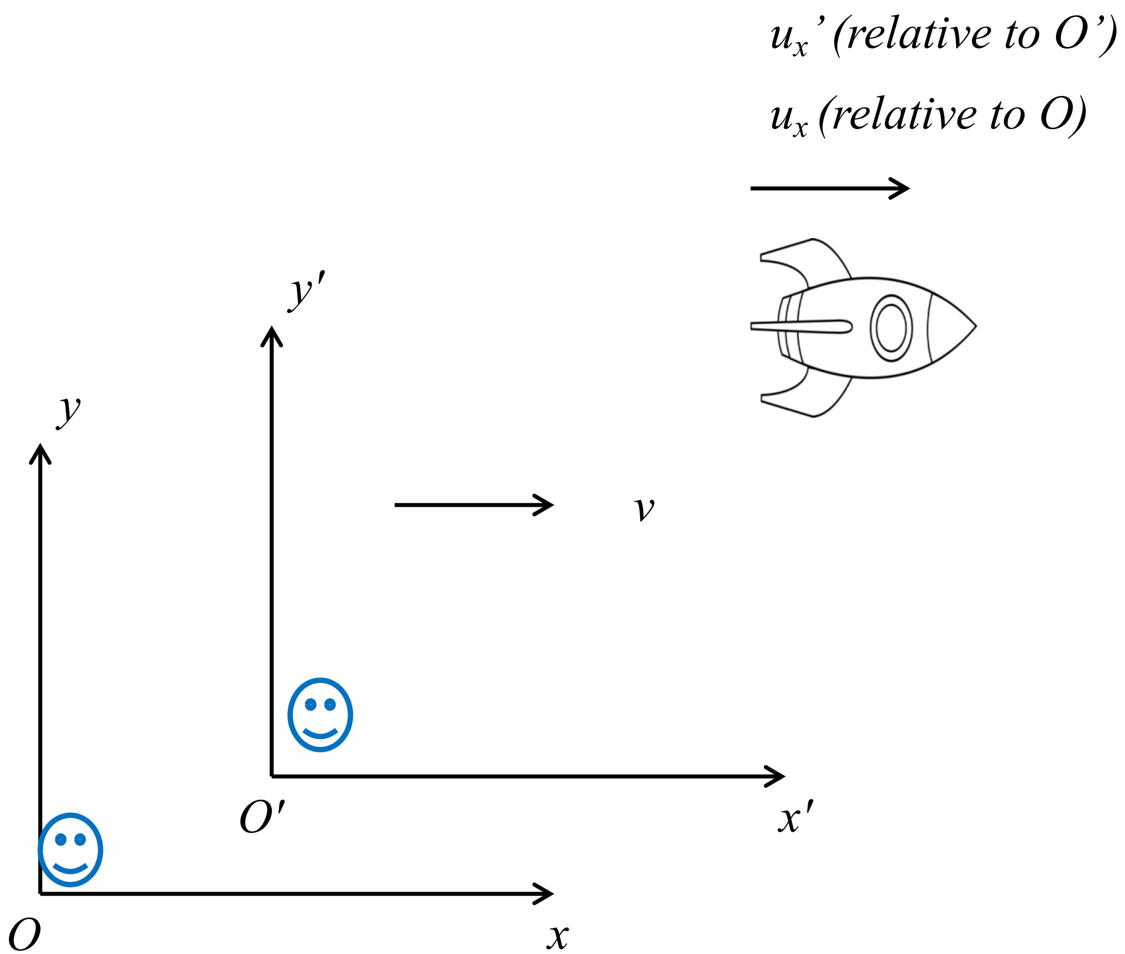
$$\frac{v^2/c^2}{1-v^2/c^2} = \frac{16\times 10^{16}}{0.64}\cdot\frac{1}{c^2} = \frac{100}{36}$$

$$\frac{v^2}{c^2} = \frac{100}{136}$$

$$\frac{v}{c} = \frac{10}{11.66} = 0.857$$

$$v = 0.857c$$

Addition of Velocities



Addition of Velocities

Taking differentials of the Lorentz transformation, relative velocities may be calculated:

$$\boxed{\begin{aligned} dx' &= \gamma(dx - v\,dt) \\ dy' &= dy \\ dz' &= dz \\ dt' &= \gamma\left(dt - \frac{v}{c^2}dx\right) \end{aligned}}$$

$$u'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - v\,dt)}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

$$u'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \qquad u'_z = \frac{dz'}{dt'} = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

$$\boxed{\begin{aligned} dx &= \gamma(dx' + v\,dt') \\ dy &= dy' \\ dz &= dz' \\ dt &= \gamma\left(dt' + \frac{v}{c^2}dx'\right) \end{aligned}}$$

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v\,dt')}{\gamma\left(dt' + \frac{v}{c^2}dx'\right)} = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{v}{c^2}dx'\right)} = \frac{u'_y}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)} \qquad u_z = \frac{dz}{dt'} = \frac{u'_z}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)}$$

The Lorentz Velocity Transformations

In addition to the previous relations, the **Lorentz velocity transformations** for u'_x, u'_y, and u'_z can be obtained by switching primed and unprimed and changing v to $-v$:

$$u'_x = \frac{u_x - v}{1-(v/c^2)u_x}$$

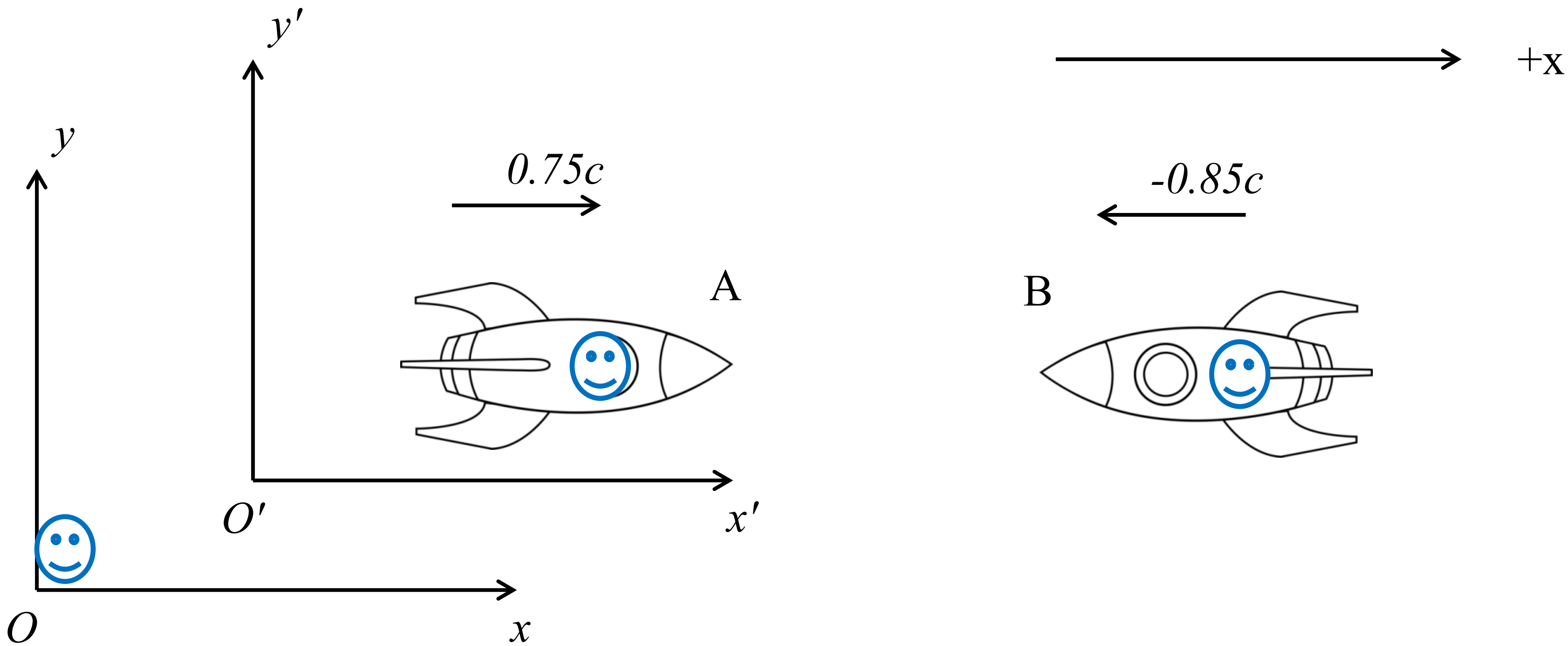
$$u'_y = \frac{u_y}{\gamma\left[1-(v/c^2)u_x\right]}$$

$$u'_z = \frac{u_z}{\gamma\left[1-(v/c^2)u_x\right]}$$

Relative Velocity

Two space ships A and B are moving towards each other. The ground observer measures that A is traveling 0.75c to the right and B is traveling 0.85c to the left.

What is B's velocity measured by A?



$$u'_{Bx} = \frac{u_{Bx} - v}{1 - (v/c^2)u_{Bx}} = \frac{-0.85c - 0.75c}{1 - \dfrac{0.75c}{c^2}(-0.85c)} = \frac{-1.6c}{1 + 0.64} = -0.977c$$