MATH 2415 – Ordinary and Partial Differential Equations
Lecture 01 notes

Section 1.1: Some Basic Mathematical Models; Direction Fields

The behavior of many processes in the natural world involve relations between physical quantities and the **rates** at which these quantities change. We can create mathematical models of these processes in the form of equations containing functions and their **derivatives**.

Definition: A **differential equation (DE)** is an equation containing the derivatives of one or more dependent variables with respect to one or more independent variables

- **Example:**

  In an algebra course, we solve equations like $x^2 + 7x - 2 = 0$ for the unknown *variable* x

  In a differential equations course, we solve equations like $y'' - 6y' + y = 6$ for the unknown *function* y

Differential equations find application in physics, chemistry, biology, engineering, economics, etc.

Before learning how to classify and solve many types of differential equations, we first consider an example of how differential equations can arise when we formulate mathematical models of processes involving rates

- **Example:** An object falling through the atmosphere near the surface of the earth

  Let's first pick some variables to represent physical quantities

  time: t

  acceleration: $a = \frac{dv}{dt}$

  velocity: $v = \frac{dx}{dt}$

  mass: m

  drag coefficient: $\gamma \leftarrow$ due to air resistance

  acceleration due to gravity: $g = 9.8\ m/s^2$

We can describe the motion of the object using Newton's second law, which relates the acceleration of the object to the net force F acting on it:

$$F = ma$$

Now, remember that the acceleration of an object is the time rate of change of the object's velocity:

$$a = \frac{dv}{dt}$$

We can therefore write Newton's second law as

$$F = m\frac{dv}{dt}$$

To proceed further, we need to determine all the forces acting on the falling object:

> Gravity exerts a downward force equal to the weight of the object, mg
>
> The atmosphere exerts an upward force that we assume is proportional to the object's velocity, γv (this model is usually a good representation for the drag force at low velocities; at higher velocities, γv^2 may be more appropriate)

We can draw a *free-body diagram* of the object showing the directions of the forces to help us set up the equation correctly:

Net Force

$F = mg - \gamma v$

Newton 2

$F = m\frac{dv}{dt}$

$m\frac{dv}{dt} = mg - \gamma v$

Diff eq. for $v(t)$

Combining all the information, we now have a differential equation describing the motion of the falling object:

$$m\frac{dv}{dt} = mg - \gamma v$$

accel = weight - drag

We call m and γ *parameters* since they depend on the particular object we are considering. The acceleration due to gravity g is a physical constant whose value is the same for all objects.

To solve this equation, we need to find a function $v = v(t)$ that satisfies the equation (i.e., when we substitute this solution into the equation, we will get an identity).

Before we attempt to solve this equation, we will investigate what information we can get directly from the equation itself (we will return to this example in a little while)

Direction Fields

A *direction field* is a graph showing the slope of the function $y(t)$ at different points in the (t, y) plane as short line segments

Direction fields are useful for revealing the behavior of solutions to differential equations of the form

$$\frac{dy}{dt} = f(t, y)$$

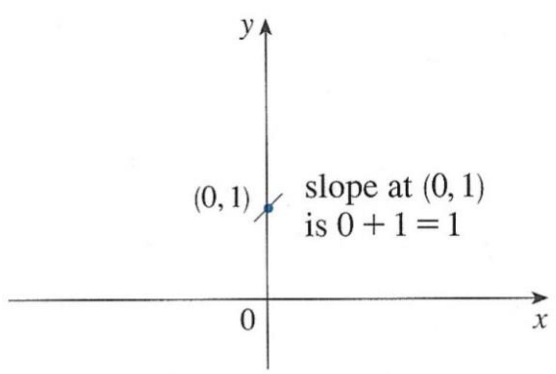
If we draw the direction field using a fine grid of points, we can get a good overall picture of how the solutions behave

- **Example:** Sketch the graph of the solution to

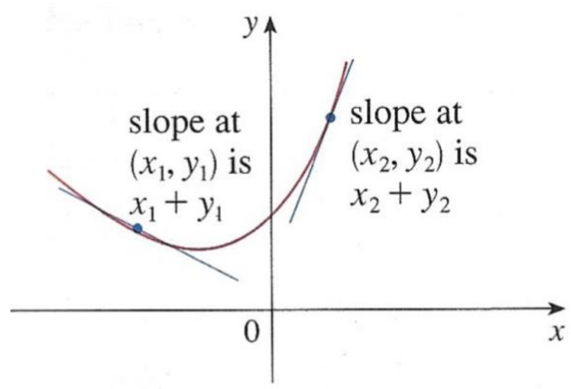
$$y' = x + y \qquad\qquad y(0) = 1$$

[Note that we are using x as the independent variable here]

How can we do this without solving the equation? The equation $y' = x + y$ tells us that the slope at any point (x, y) equals $x + y$

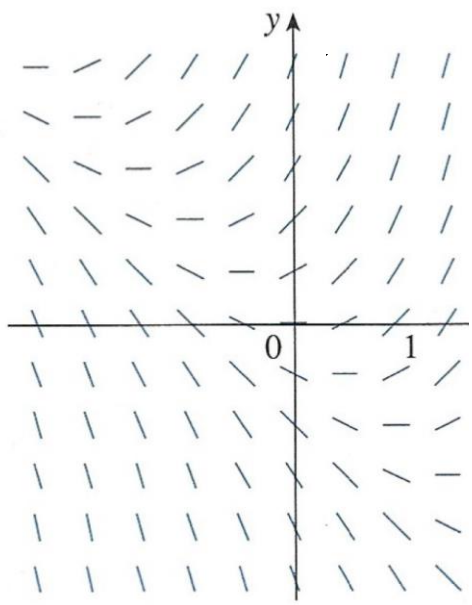


The beginning of the solution curve through (0, 1)

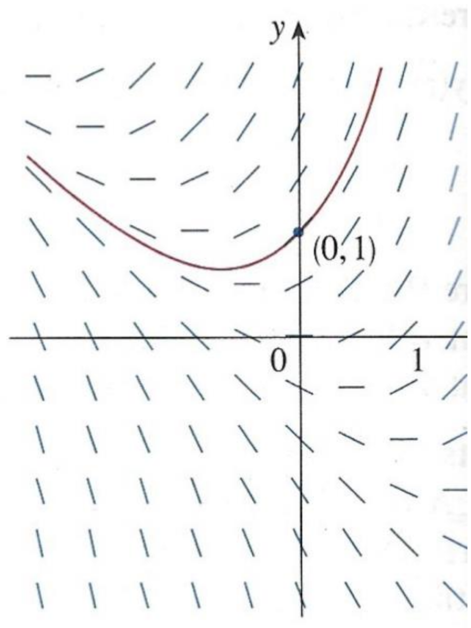


A solution of $y' = x + y$

Now draw similar short line segments at many points (x, y) with slopes $x + y$:



Direction field for $y' = x + y$



The solution curve through (0, 1)

We can sketch the solution curve through (0, 1) by following the direction field and drawing the curve so that it is parallel to nearby line segments

Two useful websites for plotting direction fields:
https://aeb019.hosted.uark.edu/dfield.html
https://homepages.bluffton.edu/~nesterd/apps/slopefields.html

- Now let's return to our example of the falling object and investigate the solutions to the equation of motion using direction fields

$$m\frac{dv}{dt} = mg - \gamma v$$

The value of g is 9.8 m/s^2, and let's use $m = 10$ kg and $\gamma = 2$ kg/s. Plugging these values into the DE gives

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

We can start to build the direction field by picking some values of v (note the right-hand side of the equation is independent of t, so the slopes are all constant for a given value of v):

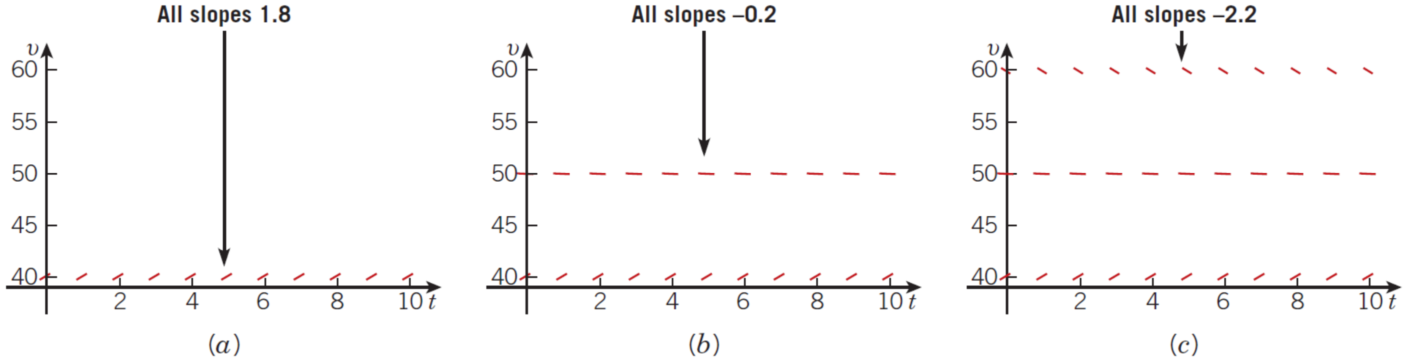


FIGURE 1.1.2 Assembling a direction field for equation (5): $dv/dt = 9.8 - v/5$. (a) when $v = 40$, $dv/dt = 1.8$, (b) when $v = 50$, $dv/dt = -0.2$, and (c) when $v = 60$, $dv/dt = -2.2$.

If we continue this process for more values of v, we have our direction field:

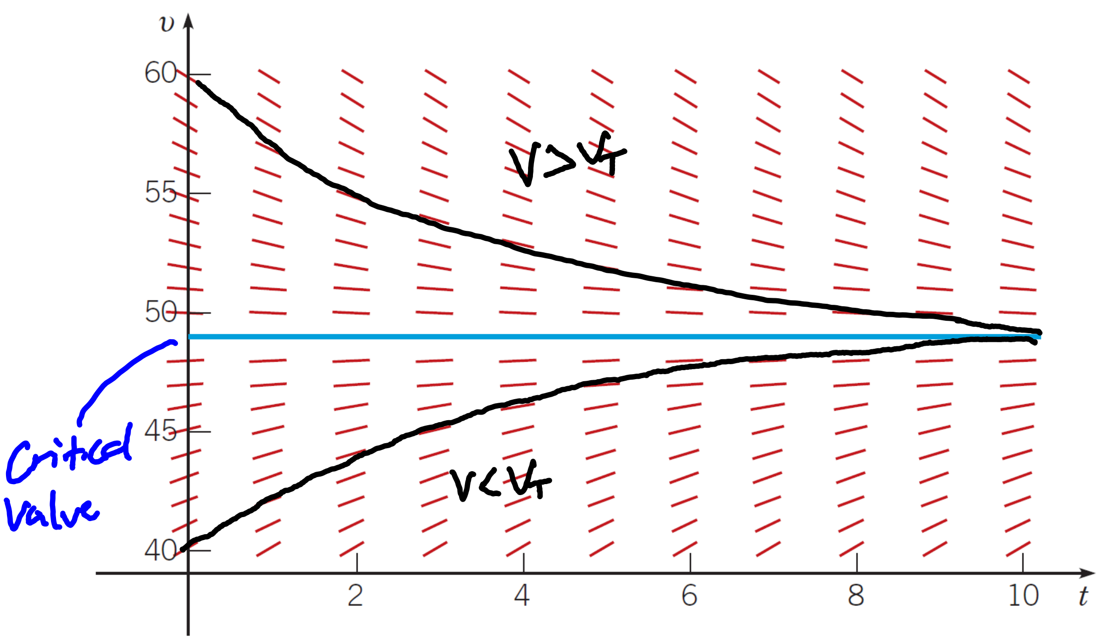


FIGURE 1.1.3 Direction field and equilibrium solution for equation (5): $dv/dt = 9.8 - v/5$.

Our solution $v = v(t)$ is a function whose graph is a curve in the t-v plane

Each line segment is a tangent to one of these solution curves

Note that if v is less than a critical value, all the slopes are positive and the object speeds up as it falls. If v is more than this critical value, the slopes are all negative and the object slows down as it falls

All the solutions approach this critical value as t increases. We call this equilibrium solution the *terminal velocity;* we can solve for it by setting $dv/dt = 0$ in the DE ($v_t = 49$ m/s)

Section 1.2: Solutions of Some Differential Equations

In the last section we developed an equation describing the motion of an object falling through the atmosphere and looked at qualitative solutions to this equation using direction maps.

Now we will solve this equation to obtain the solution $v = v(t)$. In general, this will give a family of solutions for different initial velocities of the object. If we know the *initial conditions* for the problem (i.e., the initial velocity in this problem), we will obtain one solution from this infinite set of all possible solutions.

We get more than one solution in general since we obtain the solution from integration which introduces an arbitrary constant c (we will see this in just a bit)

Definition: An **initial condition (IC)** is the addition condition used to determine c

Definition: An **initial value problem (IVP)** consists of a differential equation together with an initial condition

Our equation is

$$m\frac{dv}{dt} = mg - \gamma v$$

If we again use the values $g = 9.8$ m/s^2, $m = 10$ kg, and $\gamma = 2$ kg/s we have

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

We can rewrite this as

$$\left(\frac{1}{v-49}\right)\frac{dv}{dt} = -\frac{1}{5}$$

One way to solve this equation for $v(t)$ is to use the method of **separation of variables** (we'll say much more about this later). The idea is to isolate one variable (v) on one side of the equation and the other variable (t) on the other side. Start by multiplying the equation by the **differential** dt:

$$\left(\frac{1}{v-49}\right)\frac{dv}{dt}dt = -\frac{1}{5}dt$$

Now, recall that the quantity $\frac{dv}{dt}dt$ is the differential dv. Thus, we have

$$\frac{dv}{v-49} = -\frac{1}{5}dt$$

Now we can integrate both sides of the equation to get

$$\ln|v-49| = -\frac{t}{5} + c$$

Exponentiating both sides and solving for v gives:

$$v(t) = 49 + Ce^{-t/5}$$

Note that v approaches 49 m/s as t becomes large for all values of C

$$\int \frac{dv}{v-49} = -\frac{1}{5}\int dt$$

$$u = v - 49$$

$$du = \frac{du}{dv}dv = dv$$

$$\int \frac{du}{u} = -\frac{t}{5} + c$$

$$\ln|u| = -\frac{t}{5} + c$$

$$\ln|v-49| = -\frac{t}{5} + c$$

$$e^{\ln|v-49|} = e^{-\frac{t}{5} + c}$$

$$|v-49| = e^{c} \cdot e^{-\frac{t}{5}}$$

$$v - 49 = \underbrace{\pm e^{c}}_{C} e^{-\frac{t}{5}}$$

$$v = 49 + Ce^{-\frac{t}{5}}$$

$v(0) = 0$

If our initial condition happens to be $v = 0$ at $t = 0$, we can find a value of C satisfying this condition:

$v(t) = 49 - Ce^{-t/5}$ ← general solution

$v(0) = 49 - C\cancel{e^{0}} = 0$ → $v(t) = 49(1 - e^{-t/5})$ ← particular solution

$C = -49$

(satisfies initial condition

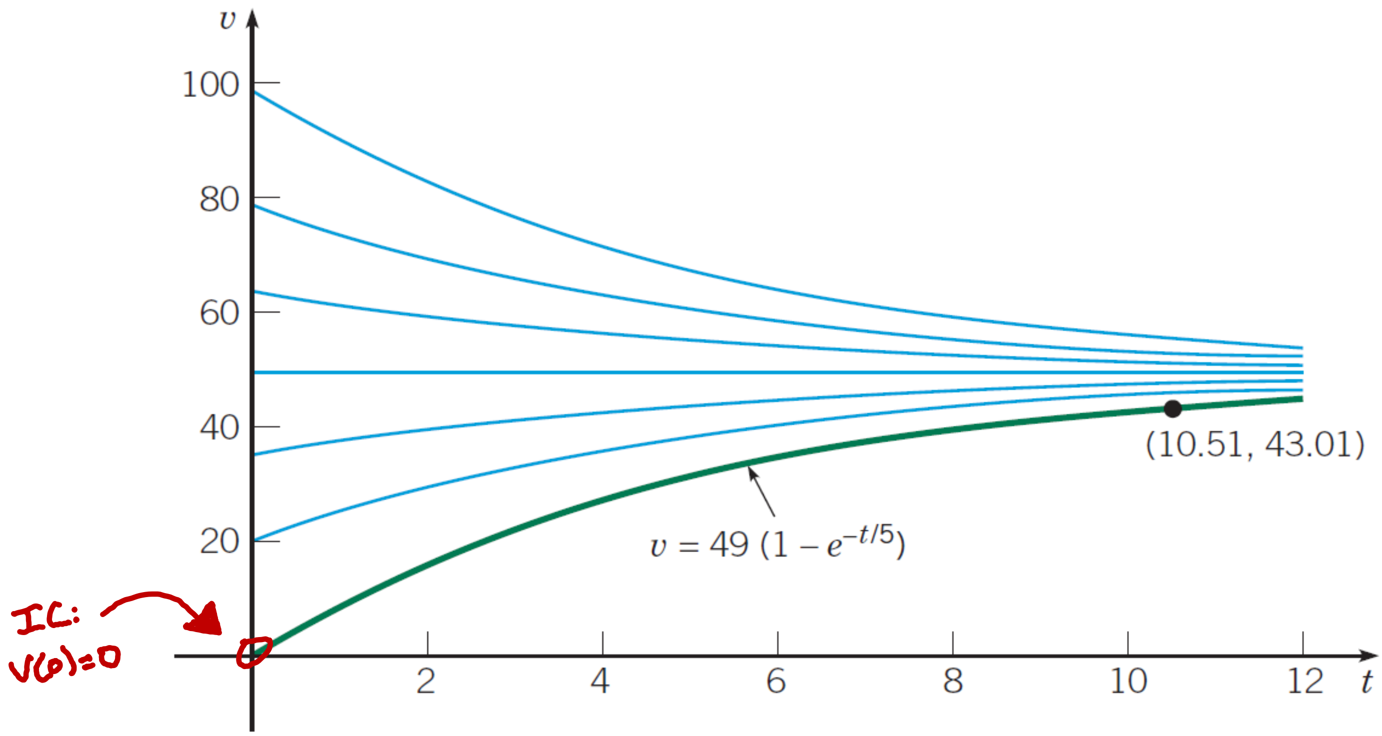


FIGURE 1.2.2 Graphs of the solution (25), $v = 49 + ce^{-t/5}$, for several values of c. The green curve corresponds to the initial condition $v(0) = 0$. The point $(10.51, 43.01)$ shows the velocity when the object hits the ground.

Since $v = dx/dt$ where x is the distance the object has fallen, we can integrate our solution one more time to obtain the height of the object as a function of time $x(t)$

$$v = \frac{dx}{dt} = 49\left(1 - e^{-t/5}\right) \rightarrow \frac{dx}{dt} dt = 49(1 - e^{-t/5})\,dt$$

$$\int dx = \int 49(1 - e^{-t/5})\,dt$$

Upon integration, this gives

$$x(t) = 49t + 245e^{-t/5} + k$$

where k is an integration constant we can determine using an initial condition for x

If our initial condition happens to be $\overbrace{x = 0 \text{ at } t = 0}^{x(0)=0}$, we can find a value of k satisfying this condition:

$x(t) = 49t + 245e^{-t/5} + k$ ← general solution

IC: $x(0) = 49(0) + 245(1) + k = 0$

↳ $k = -245$

$x(t) = 49t + 245e^{-t/5} - 245$ ← particular solution

So, we have

$$x(t) = 49t + 245e^{-t/5} - 245$$

How long does it take for the object to fall 300 m?

$300 = 49t + 245e^{-t/5} - 245$

$\boxed{t = 10.512\text{ s}}$

See your text for another example of mathematical modeling using differential equations (Field Mice and Owls)

Developing a mathematical model can be difficult, but here are some general steps that can be useful:

1. Identify the variables in the problem (both independent and dependent) and assign symbols to identify them (usually letters)

2. Choose the units for each quantity in the problem

3. Identify the underlying principles governing the behavior of the system (e.g., we used Newton's 2nd law to determine the equation of motion of the falling object)

4. Express the principle in step 3 in terms of the variables chosen in step 1

5. Check that your equation is dimensionally consistent

6. More complex problems may require formulating a system of several differential equations