Planar Graphs

Here we are just thinking about undirected graphs, but these notions can be applied to directed graphs as well.

We say a graph is **planar** if it has a **planar embedding:** if there is some nice way we can draw the vertices and edges in the plane so that none of the edges cross each other.
It should be easy to draw K_4 without crossing edges, so we can conclude it is planar.
If you spend a little bit of time trying to draw K_5 you will quickly convince yourself that it is not planar, but how can we prove it? We can't say that we checked every way of drawing it, because there are an infinite number of ways to draw it.

To formally prove this, we will need an additional definition. Informally, notice that when we draw a planar graph, this divides the plane into regions. For example, if we draw a simple cycle, it divides the plane into an interior and exterior region. We call these regions "faces".

Thanks to Euler, we have the following formula, which is true for all connected planar graphs. If $v = |V|$, $e = |E|$, and $f =$ the number of faces, then

$$v - e + f = 2.$$

To prove Euler's polyhedral formula, we first need to prove this following fact about graphs:

Theorem 0.1. *If G is a connected graph where all vertices have even degree then G contains an Eulerian cycle.*

Proof. Suppose $G = (V, E)$ is connected and all vertices have even degree. Consider this algorithm:

1. Color all vertices and edges in G white
2. Select $u \in V$. Color u black.

3. Staring from u create a path by traveling as far as possible by crossing white edges. Color each vertex visited black. Color each edge crossed black.

4. If after (3) there is a black vertex x connected to a white edge, set $u = x$ and goto (3)

5. Return all the paths made by (3)

Claim 1: Each path found in (3) is actually a cycle.
The path found in (3) only ends once we reach a vertex but have no white edge to leave on.
Since each vertex has even degree, the only vertex we can end on is the vertex we started on (each "enter" must be paired with a "leave").
Claim 2: The union of all the cycles found in (3) will make a Eulerian cycle.
We color the edges black so we don't reuse them, so an edge can appear in only one cycle found by (3).
Since the graph is connected each edge will be colored black, so the union of all cycles contains all of the edges.
So the union of the cycles found in (3) forms an Eulerian cycle. □

Now we can prove Euler's polyhedral formula:

Theorem 0.2. *If G is a connected planar graph with v vertices, e edges, and f faces, then $v - e + f = 2$.*

Proof. Let $G = (V, E)$ be a connected planar graph.
Case 1: G is Eulerian.
Let R be the number of repeated vertices in an Eulerian cycle (note the starting vertex is repeated.
Then $f = R + 1$, since every repeat "closes" some cycle.
We also have that $R = (e + 1) - v$, since the cycle visits $e + 1$ vertices from start to end.
So $f = R + 1 = (e + 1 - v) + 1 = e - v + 2$.
So $v - e + f = 2$.
Case 2: G is not Eulerian.
Create a new $G' = (V, E')$ where E' has 2 copies of each edge in E.
Let $v = |V|$, $e = |E|$, and f be the number of faces in a planar embedding of G.

Then in G', we have
$v' = |V| = v$
$e' = |E'| = 2|E| = 2e$
$f' = f + e$
All vertices in G' have even degree.
So G' has an Eulerian cycle.
By case 1, we have that

$$v' - e' + f' = 2$$

so

$$v - 2e + f + e = 2$$

Therefore,

$$v - e + f = 2$$

□

Using Euler's polyhedral formula, we can prove the following statements:

Corollary 0.3. *If G is a connected planar graph, then G has a vertex of degree < 6.*

Proof. Let G be a connected planar graph with $v = |V|$, $e = |E|$, an f faces.
The **degree** of a face is the length of a boundary walk.
The sum of the degrees of the faces $= 2e$, since each edge exists in the boundary between to faces.
Each face must have degree ≥ 3, so $3f \leq 2e$ since $3f$ underestimates the sum of the degrees of the faces.
Since G is planar, $e - v + 2 = f$ by Euler's formula, thus

$$3e - 3v + 6 = 3f \leq 2e$$

and so

$$e \leq 3v - 6$$

Case 1: $|V| \leq 2$. Then clearly G has a vertex of degree less than 6.
Case 2: $|V| \geq 3$.
Assume (for contradiction) that the degree of each vertex is ≥ 6.
Then

$$2e = \sum_{v \in V} \deg(v) \geq 6v$$

We have from before that $e \leq 3v - 6$, so $2e \leq 6v - 12$.
So $6v \leq 2e \leq 6v - 12$.
$\rightarrow\leftarrow$
So there must be a vertex with degree < 6.
This ends case 1 and 2, so G has a vertex with degree < 6.

□

Corollary 0.4. *K_5 is not a planar graph.*

Proof. For K_5, we have $v = |V| = 5$ and $e = |E| = 10$.
Assume (for contradiction) that K_5 is planar.
Then $e \leq 3v - 6$ (see previous corollary).
So $10 \leq 3 \cdot 5 - 6 = 9$.
$\rightarrow\leftarrow$.
So K_5 is not planar. □

Corollary 0.5. *$K_{3,3}$ is not a planar graph.*

Proof. For $K_{3,3}$ we have $v = |V| = 6$ and $e = |E| = 9$.
Assume (for contradiction) that $K_{3,3}$ is planar.
All cycles in $K_{3,3}$ are of length ≥ 4, so each face of $K_{3,3}$ has degree ≥ 4.
So $4f \leq 2e$, so $2f \leq e$.
From Euler's formula, we have that $2e - 2v + 4 = 2f$.
Therefore,

$$\begin{aligned} 2e - 2v + 4 &\leq e \\ e &\leq 2v - 4 \\ 9 &\leq 12 - 4 = 8 \end{aligned}$$

$\rightarrow\leftarrow$
So $K_{3,3}$ is not planar. □

Lets discuss the Kuratowski/Wagner theorem:

Theorem 0.6. *A graph is non-planar $\Leftrightarrow$ It contains a $K_{3,3}$ or K_5 minor.*

Definition 0.7. *Let G and H be graphs. We say that H is a minor of G if G can be made into H by applying some sequence of the following steps:*

1. *Delete a vertex*

2. *Delete an edge*

3. *Contract an edge, i.e. replace two vertices u, v connected by an edge with a new vertex u', and connect u' to all vertices that were connected to u and v.*

This is a difficult theorem to prove, so we will not prove it. What we should note is that this surprising theorem tells us about the structural of all planar graphs in terms of two very simple graphs.

Coloring Planar Graphs

One application of planar graphs involves coloring maps of countries. Two countries sharing a border (not a point) must be given different colors. The natural question is, how many colors do I need to color a given map?
Can translate this to a graph problem in this way:

1. Represent each country as a vertex

2. If two countries share a border, connect their vertices with an edge

This is known as the "dual" of our original map. Because the map is planar, the dual must also be planar.
Suppose we want to know the maximum number of colors we will need to color any map?
It is easy to put a lower bound on the maximum number of colors, just consider a map that has K_4 as its dual.

Theorem 0.8. *An planar graph can colored with at most 6 colors.*

Proof By Induction. Will be doing strong induction on the number of vertices.
Base Case:
The planar graph of 1 vertex has max degree 0, and needs only once color.
Induction Step:
Assume that any planar graph with $< k$ vertices can be colored with 6 colors.
Let G be a planar graph with k vertices.
By corollary 0.7 above, G has a vertex with degree < 6.
Pick such a vertex and call it v.
Let $G' = G\ v$, i.e. the graph with v and any edges connected to v removed.
G' must be planar, and G' has $k - 1$ vertices.
So, by the induction hypothesis, G' can be colored with 6 colors.
That coloring can then by copied to G, leaving only v uncolored.
Since v has less than 6 neighbors, the neighbors of v have at most 5 unique colors.
So v can be given the unused 6th color.
Thus, it takes at most 6 colors to color G. $\square$

It is time consuming, but not particularly hard, to upgrade this proof and show we only need at most 5 colors.

In fact, we can even prove that any planar graph only needs 4 colors, but this proof is even more time consuming!
The 4-color proof was done by Kenneth Appel and Wolfgang Haken in 1976. They reduced this problem to check 1482 special cases, and wrote a program to check all of those cases, making this the first major theorem proven with computer assistance. This proof is still considered somewhat controversial, because it is extremely difficult for a human to verify the correctness of the program and the proof.