Gage Farmer

Homework 6 - Math 2568 (Autumn 2022)
Prof. Cueto
Due date: Friday October 21, 2022 (in class).

The sections and problem numbers refer to the course's textbook (L.W. Johnson, R.D. Riess, J.T. Arnold: *Introduction to Linear Algebra*, 5th edition, Pearson.)

Section	Assigned Problems	Problems to be turned in
§3.4	1, 9, 11, 16, 17, 23, 25, 29, 33, 36	1, 11, 17, 23, 29
§3.5	1, 3, 6, 13, 15, 20, 22, 26, 29, 31, 35	6, 13, 20, 26, 35
§3.6	1, 5, 7, 9, 11, 13, 17, 19	1, 5, 11, 13, 19

Bonus Problem: Consider the following three vectors in $\mathbb{R}^4$:

$$\mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ -2 \\ 5 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 9 \end{bmatrix}.$$

Can you find a system of homogeneous linear equations with solution space exactly equal to the subspace of $\mathbb{R}^4$ spanned by the three vectors. What happens if we want a system where $\mathbf{u}$ is a solution, but not the other two?

Section 3.4

1) $\quad X_1 + X_2 - X_3 = 0 \qquad X_1 = -X_2 + X_3 = -X_4 + X_3$

$\quad X_2 - X_4 = 0 \qquad\qquad X_2 = X_4$

$\begin{bmatrix} -X_4 + X_3 \\ X_4 \\ X_3 \\ X_4 \end{bmatrix} \quad X_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + X_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$W = \{(1,0,1,0), (-1,1,0,1)\}$

11) a) $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 5 & 8 & -2 \\ 1 & 1 & 2 & 0 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-3R_1} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & 2 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix} \xrightarrow[R_3+R_2]{\substack{-R_2 \\ R_1-2R_2}} \begin{bmatrix} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -3 & 0 \end{bmatrix}$

$\xrightarrow[\substack{R_1-7R_3 \\ R_2+2R_3}]{-\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \rightarrow \begin{array}{l} X_1 + X_3 + X_4 = 0 \\ X_2 + X_3 - X_4 = 0 \end{array} \quad \begin{array}{l} X_1 = -X_3 - X_4 \\ X_2 = -X_3 + X_4 \end{array}$

$\begin{bmatrix} -X_3 - X_4 \\ -X_3 + X_4 \\ X_3 \\ X_4 \end{bmatrix} \quad X_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + X_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad N(A) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

c) Let $X_3 = 1 \quad X_4 = 0 \rightarrow X_1 = -1 - 0 = -1$
$X_2 = -1 + 0 = -1$

$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} -A_1 - A_2 + A_3 = 0 \\ A_3 = A_1 + A_2 \end{array}$

Let $X_3 = 0 \quad X_4 = -1 \rightarrow X_1 = 0 + 1 = 1$
$X_2 = 0 - 1 = -1$

$-A_1 + A_2 + A_4 = 0$
$A_4 = A_1 - A_2$

$\{A_1, A_2\}$ is a basis
where $A_3 = A_1 + A_2$
and $A_4 = A_1 - A_2$

d) $\underline{\left\{\begin{bmatrix}1 & 0 & 1 & 1\end{bmatrix}, \begin{bmatrix}0 & 1 & 1 & -1\end{bmatrix}\right\}}$

17)

$$A = \begin{bmatrix}1 & 2 & 3 & -1 \\ 3 & 5 & 8 & -2 \\ 1 & 1 & 2 & 0\end{bmatrix} \qquad A^T = \begin{bmatrix}1 & 3 & 1 \\ 2 & 5 & 1 \\ 3 & 8 & 2 \\ -1 & -2 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 1\end{bmatrix}$$

$$\longrightarrow \begin{bmatrix}1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix} \qquad A = \begin{bmatrix}1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0\end{bmatrix} \qquad u_1 = \begin{bmatrix}1 \\ 3 \\ 1\end{bmatrix} \qquad u_2 = \begin{bmatrix}0 \\ -1 \\ -1\end{bmatrix}$$

$$\underline{W = \left\{(1\ 3\ 1)^T, (0\ -1\ -1)^T\right\}}$$

23)

a) $S = \left\{\begin{bmatrix}1 \\ 2 \\ 1\end{bmatrix}, \begin{bmatrix}2 \\ 5 \\ 0\end{bmatrix}, \begin{bmatrix}3 \\ 7 \\ 1\end{bmatrix}, \begin{bmatrix}1 \\ 1 \\ 3\end{bmatrix}\right\}$

$$\begin{bmatrix}1 & 2 & 3 & 1 \\ 2 & 5 & 7 & 1 \\ 1 & 0 & 1 & 3\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0\end{bmatrix} \qquad \begin{array}{l} X_1 + X_3 + 3X_4 = 0 \\ X_2 + X_3 - X_4 = 0 \\ X_3 / X_4 \text{ are free}\end{array}$$

$X_3 = 1 \quad X_4 = 1 \quad \longrightarrow \quad X_1 = -1 \quad X_2 = -1$

$\qquad -V_1 - V_2 + V_3 = 0 \quad \rightarrow \quad V_3 = V_1 + V_2$

$X_3 = 0 \quad X_4 = 1 \quad \longrightarrow \quad X_1 = -3 \quad X_2 = -1$

$\qquad -3V_1 - V_2 + V_4 = 0 \quad \rightarrow \quad V_4 = 3V_1 + V_2 \qquad \Big\} \rightarrow \underline{aV_1 + bV_2 = 0}$

b) $a\begin{bmatrix}1 \\ 2 \\ 1\end{bmatrix} + b\begin{bmatrix}2 \\ 5 \\ 0\end{bmatrix} = 0 \qquad \begin{bmatrix}a + 2b \\ 2a + 5b \\ a\end{bmatrix} = 0$

$$\underline{\left\{\begin{bmatrix}1 & 2 & 1\end{bmatrix}^T, \begin{bmatrix}2 & 5 & 0\end{bmatrix}^T\right\} \text{ is a basis for } Sp(s)}$$

29)

$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ \end{array}$$
$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$X_1 = 0$
$X_2 = 0$ $S = \{V_1, V_2, V_3\}$ is a solution
$X_3 = 0$
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$$\begin{array}{ccc} v_1 & v_2 & v_4 \\ \end{array}$$
$$\begin{bmatrix} 1 & -1 & -2 \\ 2 & -1 & -4 \\ 1 & 1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$X_1 = 0$
$X_2 = 0$ $S = \{V_1, V_2, V_4\}$ is a solution
$X_4 = 0$
————————————————————

$$\begin{array}{ccc} v_1 & v_3 & v_4 \\ \end{array}$$
$$\begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & -4 \\ 1 & 7 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 0 \\ 0 & 8 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$S = \{V_1, V_3, V_4\}$ is a solution
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$$\begin{array}{ccc} v_2 & v_3 & v_4 \\ \end{array}$$
$$\begin{bmatrix} -1 & -1 & -2 \\ -1 & 1 & -4 \\ 1 & 7 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 6 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$S = \{V_2, V_3, V_4\}$ is **not** a solution

$\{V_1, V_2, V_3\} + \{V_1, V_2, V_4\}, \{V_1, V_3, V_4\}$ is the basis
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Section 35

6) $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$ $\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

Linearly Dependent

13) $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix} \right\}$ $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Not a basis

20) $\begin{matrix} x_1 - x_2 = 0 \\ x_2 - 2x_3 = 0 \\ x_3 - x_4 = 0 \end{matrix}$ $\begin{matrix} x_1 = x_2 = 2x_3 = 2x_4 \\ x_2 = 2x_3 = 2x_4 \\ x_3 = x_4 \end{matrix}$ $\begin{bmatrix} 2x_4 \\ 2x_4 \\ x_4 \\ x_4 \end{bmatrix}$ $x_4 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$

$\dim(W) = 1$

26) $A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 4 & 2 & 4 \\ 2 & 1 & 5 & -2 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 4 & 1 \\ 2 & 2 & 5 \\ 0 & 4 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 4 & -2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}^T, \begin{bmatrix} 0 & 2 & -1 \end{bmatrix}^T$

Nullity$(A) = 4$ Rank$(A) = 2$

35)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ ← Linearly Dependent

Section 3.6

1)

$$u_1^T u_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

$$u_1^T u_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = -1 + 2 - 1 = 0 \quad \leftarrow \text{Orthogonal}$$

$$u_2^T u_3 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = 1 + 0 - 1 = 0$$

5)

$$u_1^T u_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} = 2 + 2 - 4 = 0$$

$$u_1^T u_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a + b + c = 0$$

$$u_2^T u_3 = \begin{bmatrix} 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 2a + 2b - 4c = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Orthogonal if $a + b = 0$ & $c = 0$

11) $u_1^T v = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = a_1\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $a_1 = 3$

$u_2^T v = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = a_2\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $a_2 = 0$

$u_3^T v = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = a_3\begin{bmatrix} -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ $a_3 = 0$

$$\underline{V = 3u_1}$$

13) $u_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$

$u_1^T u_2 = u_1^T w_2 + a u_1^T u_1$

$0 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + a\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$0 = 2 + a$ $a = -2$

$u_2 = w_2 - 2u_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$u_1^T u_3 = u_1^T w_3 + b u_1^T u_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$0 = 1 + b$ $b = -1$

$$u_2^T u_3 = u_2^T w_3 + c u_2^T u_2 = \underbrace{}$$

$$0 = 2 + 3c \qquad\qquad c = -\frac{2}{3}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1/3 \\ -2/3 \\ 0 \\ 1/3 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1/3 \\ -2/3 \\ 0 \\ 1/3 \end{bmatrix}$$

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19) $\begin{bmatrix} 1 & -2 & 1 & -5 \\ 2 & 1 & 7 & 5 \\ 1 & -1 & 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 1 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$x_1 - 2x_2 + x_3 - 5x_4 = 0 \qquad\qquad a\begin{bmatrix} -3 \\ -1 \\ 10 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$x_2 + x_3 + 5x_4 = 0$$

* I didn't want to write the rest of the work, it was a lot of stuff *

O. basis of null space $= \left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7/11 \\ -27/11 \\ -6/11 \\ 1 \end{bmatrix} \right\}$

O. basis of range space $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 45/82 \\ 1 \\ 5/82 \end{bmatrix} \right\}$