MATH-2415, Ordinary and Partial Differential Equations **Name:**
Summer 2023
Problem Set 1
Due June 4, 2023 by midnight

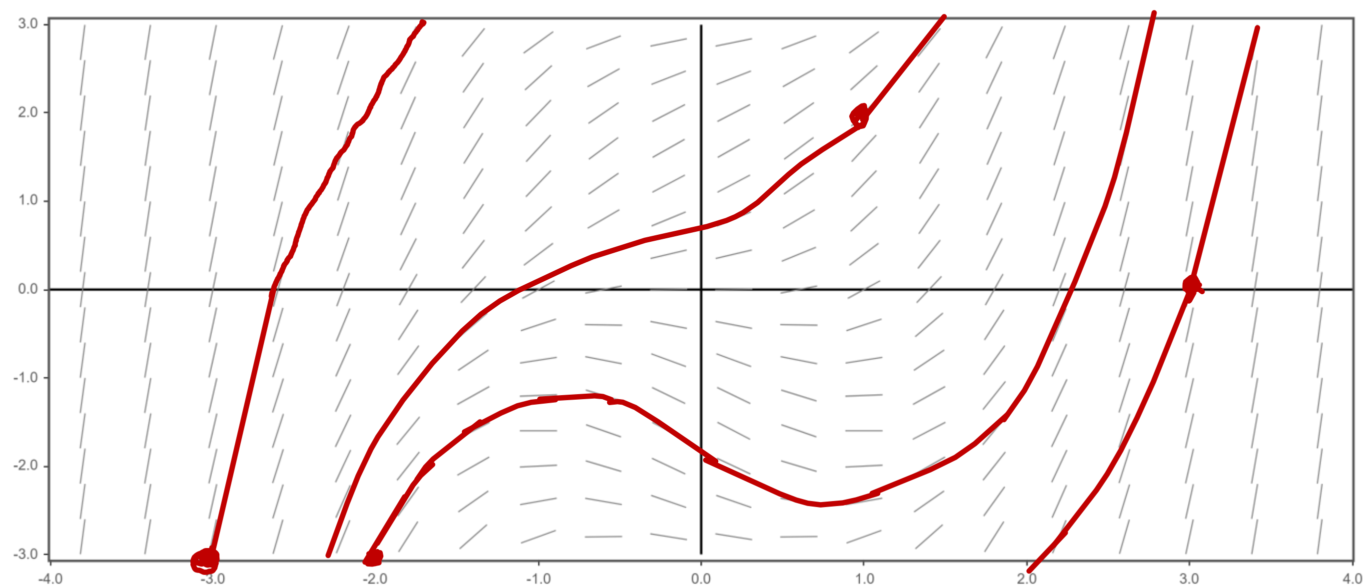
Directions: You can either

(I) Show all your work on the pages of the assignment itself, or

(II) Use separate paper to work out the problems. The paper must be plain white paper. No notebook paper, no lined paper. Engineering paper is acceptable.

For either selection, **clearly show all work that leads to your final answer**. Don't make me hunt for your steps! And DON'T turn in **any** scratched out work! Your final product must be clear and legible, or it will be returned, ungraded. **Please submit your work to Blackboard as a single pdf file**.

1. For the direction field plotted below, sketch the solution corresponding to each of the initial conditions $(-3,-3), (-2,-3), (1,2), (3,0)$. Do two or more solution curves ever cross at a given point? Explain what this implies about the uniqueness of a solution corresponding to a given initial condition [We will discuss the existence and uniqueness of solutions to differential equations in more detail later].



No

2. In class we solved the following differential equations governing the velocity and position of an object falling through the atmosphere near the surface of the earth,

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \qquad\qquad \frac{dx}{dt} = 49\left(1 - e^{-t/5}\right)$$

subject to the initial conditions $v(0) = 0$ and $x(0) = 0$. In solving these equations, we evaluated *indefinite* integrals which gave an arbitrary integration constant in each case. We then used the given initial conditions to find the values of these constants.

Instead of using indefinite integrals, solve these equations by evaluating *definite* integrals with the appropriate initial condition as the lower integration limit, and the variable v, x, or t as the upper integration limit (remember, with definite integrals you don't need to include arbitrary integration constants). Show that you get the same results we found in class.

$$\frac{dv}{dt} = \frac{49-v}{5} \rightarrow \frac{dv}{49-v} = -\frac{dt}{5} \rightarrow \int_0^v \frac{dv}{49-v} = -\frac{1}{5}\int_0^t dt$$

use substitution

$$u = 49-v \quad du = \frac{du}{dv}dv = dv \rightarrow \int_{-49}^{49-v} \frac{du}{u} = -\frac{1}{5}\int_0^t dt$$

$$\rightarrow \ln|u|\Big|_{-49}^{49-v} = -\frac{1}{5}t\Big|_0^t \rightarrow \ln|49-v| - \ln|49| = -\frac{t}{5}$$

$$\rightarrow \ln\left|\frac{49-v}{-49}\right| = -\frac{t}{5} \rightarrow \frac{49-v}{-49} = e^{-t/5} \rightarrow 49-v = -49\left(e^{-t/5}\right)$$

$$\rightarrow \boxed{v(t) = 49\left(1-e^{-t/5}\right)}$$

$$\frac{dx}{dt} = 49\left(1-e^{-t/5}\right) \rightarrow \int_0^x dx = 49\int_0^t 1-e^{-t/5}\,dt \rightarrow x\Big|_0^x = 49\left(t+5e^{-t/5}\right)\Big|_0^t$$

$$\hookrightarrow x-0 = 49\left(t+5e^{-t/5}\right) - 49\left(0+5e^{0}\right) \rightarrow x = 49\left(t+5e^{-t/5}\right) - 49(5)$$

$$\hookrightarrow \boxed{x = 49t + 245e^{-t/5} - 245}$$

3. In the previous problem, the mass of the object was 10 kg and the drag coefficient was 2 kg/s. More generally, the differential equation describing the motion of an object falling through the atmosphere is

$$m\frac{dv}{dt} = mg - \gamma v$$

[see the class notes for a derivation of this equation].

a) Solve this more general equation for the velocity of the object as a function of time, using the initial condition $v(0) = v_0$. Your final answer will contain the symbols m, g, γ, v_0, and t.

b) Substitute the values $m = 10$ kg, $\gamma = 2$ kg/s, $g = 9.8$ m/s², and $v_0 = 0$ into your result from a) and verify your result agrees with your solution from problem 2.

use substitution

a) $\frac{dv}{dt} = g - \frac{\gamma}{m}v \rightarrow \frac{dv}{g - \frac{\gamma}{m}v} = dt \rightarrow u = g - \frac{\gamma}{m}v \quad du = -\frac{m}{\gamma}dv$

$\hookrightarrow -\frac{m}{\gamma}\int \frac{du}{u} = \int dt \rightarrow -\frac{m}{\gamma}\ln|u| = t + C \rightarrow \ln|u| = -\frac{m}{\gamma}t + C$

$\hookrightarrow u = Ce^{-\frac{m}{\gamma}t} \rightarrow g - \frac{\gamma}{m}v = Ce^{-\frac{m}{\gamma}t} \rightarrow \frac{\gamma}{m}v = g - Ce^{-\frac{m}{\gamma}t} \rightarrow v = \frac{mg}{\gamma} - Ce^{-\frac{m}{\gamma}t}$

$\hookrightarrow V_0 = \frac{mg}{\gamma} + V_0 e^{0} \rightarrow C = V_0 - \frac{mg}{\gamma} \rightarrow V(t) = \frac{mg}{\gamma} + V_0 e^{-\frac{\gamma}{m}t} - \frac{mg}{\gamma}e^{-\frac{\gamma}{m}t}$

$\hookrightarrow \boxed{V(t) = V_0 e^{-\frac{\gamma}{m}t} + \frac{mg}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right)}$

b) $v(t) = 0e^{-\frac{1}{5}t} + \frac{10(9.8)}{2}\left(1 - e^{-\frac{1}{5}t}\right) \rightarrow \boxed{V(t) = 49\left(1 - e^{-\frac{1}{5}t}\right)}$