MATH 2415 – Ordinary and Partial Differential Equations
Lecture 08 notes

Summary of the Method of Undetermined Coefficients

Last time we discussed how to solve certain second-order linear **nonhomogeneous** ODEs with constant coefficients:

$$ay'' + by' + cy = g(x)$$

We saw that we can write the general solution of this ODE as

$$y(x) = y_c(x) + y_p(x)$$

where $y_c(x)$ is a solution of the corresponding homogeneous ODE

$$ay'' + by' + cy = 0$$

and $y_p(x)$ is any solution of the nonhomogeneous ODE.

If $g(x)$ is a polynomial, an exponential, sines, cosines, or sums and products of these functions, we can use the **method of undetermined coefficients** to find $y_p(x)$ and then form the general solution:

1. First find the complimentary solution $y_c(x)$

2. If $g(x) = g_1(x) + \cdots + g_n(x)$ then form n subproblems,

$$ay'' + by' + cy = g_i(x)$$

where each $g_i(x)$ contains only products of the functions mentioned above

3. For the ith subproblem, assume a particular solution $Y_i(x)$ consisting of the appropriate combination of polynomials, exponentials, sines, and/or cosines. If there is any duplication of the assumed form of $Y_i(x)$ in the solution $y_c(x)$ for the homogeneous equation, then multiply $Y_i(x)$ by x (or x^2 if needed) to ensure the solutions are linearly independent. The $Y_i(x)$ will contain unknown coefficients at this point.

4. Substitute $Y_i(x)$ into the ODE and match the unknown coefficients on one side of the equation to the coefficients on the other side. This will result in a set of algebraic equations you need to solve to find the values of the unknown coefficients.

5. The sum of the $Y_i(x)$ is a particular solution $y_p(x)$ of the nonhomogeneous equation

6. The sum of $y_c(x)$ and $y_p(x)$ is the general solution of the nonhomogeneous equation

7. If there are initial conditions, use them to determine the values of the arbitrary constants in the general solution

Here we summarize the forms we assume for $Y_i(x)$ given different forms of $g_i(x)$:

$g_i(x)$	$Y_i(x)$
$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$	$x^s(A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0)$
$P_n(x) e^{\alpha x}$	$x^s\left(A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0\right) e^{\alpha x}$
$P_n(x) e^{\alpha x} \begin{cases} \sin \beta x \\ \cos \beta x \end{cases}$	$x^s[(A_n x^n + \cdots + A_0) e^{\alpha x} \cos \beta x$ $+(B_n x^n + \cdots + B_0) e^{\alpha x} \sin \beta x]$

Here s is the smallest non-negative integer (0, 1, or 2) to ensure that no term in $Y_i(x)$ appears in $y_c(x)$.

Examples:

3.7 Mechanical and Electrical Vibrations

One of the main applications of second-order linear differential equations with constant coefficients is modelling systems that oscillate or vibrate:

$$ay'' + by' + cy = g(t), \qquad y(0) = y_0, \quad y'(0) = y_0'$$

Note that we are using t as the independent variable here since we are mostly concerned with the motion as a function of time in this section (but spatial oscillations are also possible, in which case we would probably use x)

Examples of relevant systems that can be modelled (at lease approximately) by differential equations of this form include a mass connected to a spring, atoms in crystals, pendulums with small displacements, the flow of current in simple electrical circuits, etc.

We first consider the oscillation of a mass connected to a spring, which is the starting point for understanding vibrations in mechanical systems

- **Example:** A mass weighing 4 N stretches a spring by 2 cm. The mass is then given an additional displacement of 6 cm in the positive direction and released. A drag force acts on the mass equal to 6 N when the velocity of the mass is 3 m/s. Formulate the initial value problem that describes this system.

Now let's consider the special case of **undamped free vibrations**, corresponding to $\gamma = 0$ and $F(t) = 0$:

- **Example:** A mass weighing 10 N stretches a spring by 2 cm. The mass is then given an additional displacement of 2 cm and set into motion with an initial upward velocity of 1 m/s, find the position of the mass as a function of time. Also find the period, amplitude, and phase of the motion.

Now let's consider the special case of **damped free vibrations**, corresponding to $\gamma \neq 0$ and $F(t) = 0$:

- **Example:** If the differential equation governing a mass-spring system is

$$u'' + \frac{1}{8}u' + u = 0$$

determine $u(t)$ if $u(0) = 2$ and $u'(0) = 0$. Assume u is measured in meters and t in seconds. Also, find the quasi-frequency, the quasi-period, the time when the mass first passes through its equilibrium position, and the time τ such that $|u(t)| < 0.1$ for all $t > \tau$.

Electrical Circuits

In addition to mechanical vibrations, we can also model the oscillation of electrical current in some simple circuits using second-order linear ODEs with constant coefficients.

If a circuit has a resistance R, a capacitance C, an inductance L, and current I flows in the circuit, we can use Kirchhoff's second law (in a closed circuit, the impressed voltage E equals the sum of the voltage drops in the rest of the circuit) and known laws for circuit components to formulate our ODE

Component	Voltage drop
resistor	RI
capacitor	Q/C
inductor	$L\,dI/dt$

The charge Q is related to the current through $I = dQ/dt$

3.8 Forced Periodic Vibrations

Here we consider the effects of a periodic external force (sometimes called a **driving force**) on a vibrating mechanical system.

Forced Vibrations with Damping

We will assume the external force is of the form $F(t) = F_0 \cos \omega t$. The ODE governing the motion is then

$$mu''(t) + \gamma u'(t) + ku(t) = F_o \cos \omega t$$

- **Example:** Find the solution of the following IVP that governs a mass-spring system:

$$u'' + u' + \frac{5}{4}u = 3\cos t\,, \qquad u(0) = 2, \qquad u'(0) = 3$$

Forced Vibrations Without Damping

Now let’s consider the special case of forced vibrations when there is no damping, $\gamma = 0$. The ODE governing the system is

$$mu'' + ku = F_0 \cos \omega t$$

- **Example:** Find the solution of the following IVP that governs a mass-spring system:

$$u'' + u = \frac{1}{2}\cos(0.8\,t)\,, \qquad u(0) = 0, \qquad u'(0) = 0$$

- **Example:** Find the solution of the following IVP that governs a mass-spring system:

$$u'' + u = \frac{1}{2}\cos t, \qquad u(0) = 0, \qquad u'(0) = 0$$