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- Contrapositive Law

  $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$

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$A \oplus B \oplus C \oplus D \oplus E$

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- DeMorgan's Laws

$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$
$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$

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Could we express any Boolean function[1] in terms of these three?

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[1] By this I mean a function that takes as input (zero or more) Boolean variables and outputs a Boolean[2] value.

[2] I.e., true or false.

Let's Make a Function!

Let's try it with a three variable function $f(P, Q, R)$:

P	Q	R	f	g
0	0	0	1	0
0	0	1	1	0
0	1	0	1	1
0	1	1	0	0
1	0	0	0	0
1	0	1	0	0
1	1	0	1	0
1	1	1	0	0

$$g(P,Q,R) \equiv \neg P \wedge Q \wedge \neg R$$

$$f(P,Q,R) \equiv (\neg P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge \neg R)$$

$$\equiv (\bar{P}\bar{Q}\bar{R}) \vee (\bar{P}\bar{Q}R) \vee (\bar{P}Q\bar{R}) \vee (PQ\bar{R})$$

A set of logical connectives (Boolean functions) is called universal if any Boolean function can be expressed using just the members of the set.

$\{\neg, \wedge, \vee\}$ is Universal.

Are There Smaller Universal Sets?

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$$(\neg P \wedge Q \wedge \neg R) \vee (P \wedge (Q \wedge \neg R) \equiv \neg\left(\neg(\neg P \wedge Q \wedge \neg R) \wedge \neg(P \wedge Q \wedge \neg R)\right)$$

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How about $\{\wedge, \vee\}$?

This time we're in trouble. There's just no way to make the $\neg$.

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Remember: $P|Q \equiv \neg(P \wedge Q)$

$\neg A \equiv$

A	$\neg A$	$A|A \equiv \neg(A \wedge A)$
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$$P \wedge (Q \wedge \neg R) \equiv P \wedge (Q \wedge (R|R)) \equiv P \wedge ([Q|(R|R)]|[Q|(R|R)])$$

$$\equiv [P|([Q|(R|R)]|[Q|(R|R)])] \,\Big|\, [P|([Q|(R|R)]|[Q|(R|R)])]$$

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Similarly any proposition that is a conjunction of disjunctions of atomic propositions or negations of atomics is said to be in conjunctive normal form or CNF.

$$(A \vee \neg C \vee D) \wedge (\neg B \vee C \vee D) \wedge \neg D$$

The Satisfiability Problem

While propositional logic is pretty straightforward, that doesn't mean that there's nothing but triviality within. For example, consider the satisfiability problem. Given a proposition in conjunctive normal form, determine whether or not there is a choice of truth values for the atomics that makes the proposition true (i.e. satisfied).

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For example consider the proposition

$$(P \vee Q \vee R) \wedge (\neg P \vee \neg Q) \wedge (\neg P \vee \neg R) \wedge (\neg R \vee \neg Q).$$

Is it satisfiable?