CSE 2321 Foundations I Spring, 2024 Dr. Estill

Homework 7 Due: Friday, March 22

Theorem 1 (Master Theorem)**.**
Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function $f : \mathbb{N} \to \mathbb{R}^+$, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n).$$

Then $T(n)$ has the following asymptotic bounds:

1. *if $f(n) \in O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$,*
2. *if $f(n) \in \Theta(n^{\log_b a})$ then $T(n) \in \Theta(n^{\log_b a} \log n)$, and*
3. *if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$ and if $af(n/b) \leq df(n)$ for some constant $d < 1$ and all sufficiently large n, then $T(n) \in \Theta(f(n))$.*

Use the Master Theorem above to solve the following recurrences when possible. If you need to confirm the regularity condition ($af(n/b) \leq df(n)$ for $d < 1$), work should be shown, but otherwise answers are all that is needed. (Note that not every blank needs to be filled in in every problem.):

(20 points each)

1.) $T(n) = T(n/3) + c$
$f(n) = c$ versus $n^{\log_b a} = n^3$
Which is growing faster: $f(n)$ or $n^{\log_b a}$? $n^{\log_b a}$
Which case of the Master Theorem does that potentially put us in? Case 1
What can you conclude? $T(n) = \theta(n^3)$

2.) $T(n) = T(n/3) + c \log_2 n$
$f(n) = \log_2 n$ versus $n^{\log_b a} = n^3$
Which is growing faster: $f(n)$ or $n^{\log_b a}$? - $n^{log_b a}$
Which case of the Master Theorem does that potentially put us in? Case 1
What can you conclude? - $T(n) = \theta(n^3)$

3.) $T(n) = 4T(n/2) + cn$
$f(n) = cn$ versus $n^{\log_b a} = n^2$
Which is growing faster: $f(n)$ or $n^{\log_b a}$? $n^{\log_b a}$
Which case of the Master Theorem does that potentially put us in? Case 1
What can you conclude? $T(n) = \theta(n^2)$

4.) $T(n) = 4T(n/2) + cn^3$

$f(n) = cn^3$ versus $n^{\log_b a} = n^2$

Which is growing faster: $f(n)$ or $n^{\log_b a}$? $f(n)$

Which case of the Master Theorem does that potentially put us in? Case 3

If you're potentially in case one or three, is it possible to find an epsilon which makes either $f(n) \in O(n^{\log_b a - \varepsilon})$ (if you're in case one) or $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ (if you're in case three) true? Choose one or show an inequality. $n^3 <= 2dn^3$

If you're potentially in case three and there is an ε, try to find a constant $d < 1$ such that $af(n/b) \leq df(n)$ for large enough n's. $d = 1/2$

What can you conclude? $T(n) = \theta(n^3)$

5.) $T(n) = 2T(n/6) + \sqrt{n}$

$f(n) = n^{0.500}$ versus $n^{\log_b a} = n^{0.387}$

Which is growing faster: $f(n)$ or $n^{\log_b a}$? $f(n)$

Which case of the Master Theorem does that potentially put us in? Case 3

If you're potentially in case one or three, is it possible to find an epsilon which makes either $f(n) \in O(n^{\log_b a - \varepsilon})$ (if you're in case one) or $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ (if you're in case three) true? Choose one or show an inequality. $2f(n/6) <= d\sqrt{n}$

If you're potentially in case three and there is an ε, try to find a constant $d < 1$ such that $af(n/b) \leq df(n)$ for large enough n's. $d = 0.816$

What can you conclude? $T(n) = \theta(\sqrt{n})$