Chapter 8. Fast Convolution

Exercise Solution

Exercise 4. [Solution]:

$s(p) = h(p)x(p), \quad h(p) = h_0 + h_1 p, \quad x(p) = x_0 + x_1 p$

$s(p) = s_0 + s_1 p + s_2 p^2$

$$\begin{array}{llll} \beta = 0, & h(0) = h_0, & x(0) = x_0, & s(0) = h_0\, x_0 \\ \beta = 1, & h(1) = h_0 + h_1, & x(1) = x_0 + x_1, & s(1) = (h_0 + h_1)(x_0 + x_1) \\ \beta = 2, & h(2) = h_0 + 2h_1, & x(2) = x_0 + 2x_1, & s(2) = (h_0 + 2h_1)(x_0 + 2x_1) \end{array}$$

$$\begin{aligned} s(p) &= s(0)\,\frac{(p-1)(p-2)}{(0-1)(0-2)} + s(1)\,\frac{(p-0)(p-2)}{(1-0)(1-2)} + s(2)\,\frac{(p-0)(p-1)}{(2-0)(2-1)} \\ &= s(0) + \left(-\frac{3s(0)}{2} + 2s(1) - \frac{s(2)}{2}\right) p + \left(\frac{s(0)}{2} - s(1) + \frac{s(2)}{2}\right) p^2 \\ &= s_0 + s_1 p + s_2 p^2 \end{aligned}$$

$$\begin{bmatrix} s_0 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \operatorname{diag} \begin{bmatrix} h_0/2 \\ h_0 + h_1 \\ (h_0 + 2h_1)/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

Computation complexity: 3 multiplications, and 6 additions.

Chapter 8.Fast Convolution

Exercise Solution

Exersice 6. [Solution]:

For 3×3 fast convolution using modified Cook-Toom algorithm:

$$\begin{aligned} h(p) &= h_0 + h_1 p + h_2 p^2 \\ x(p) &= x_0 + x_1 p + x_2 p^2 \\ s(p) &= s_0 + s_1 p + s_2 p^2 + s_3 p^3 + h_2 x_2 p^4 \\ s^{'}(p) &= s(p) - h_2 x_2 p^4 \end{aligned}$$

Then

$$\begin{aligned} s^{'}(\beta_0) &= s(\beta_0) \\ s^{'}(\beta_1) &= s(\beta_1) - h_2 x_2 \\ s^{'}(\beta_2) &= s(\beta_2) - h_2 x_2 \\ s^{'}(\beta_3) &= s(\beta_3) - 16 h_2 x_2 \end{aligned}$$

From *Lagrange interpolation formula*,

$$\begin{aligned} s^{'}(p) &= s^{'}(\beta_0) + p[-\frac{1}{2}s^{'}(\beta_0) + s^{'}(\beta_1) - \frac{1}{3}s^{'}(\beta_2) - \frac{1}{6}s^{'}(\beta_3)] \\ &\quad + p^2[-s^{'}(\beta_0) + \frac{1}{2}s^{'}(\beta_1) + \frac{1}{2}s^{'}(\beta_2)] \\ &\quad + p^3[\frac{1}{2}s^{'}(\beta_0) - \frac{1}{2}s^{'}(\beta_1) - \frac{1}{6}s^{'}(\beta_2) + \frac{1}{6}s^{'}(\beta_3)] \\ s(p) &= s^{'}(p) + h_2 x_2 p^4 = s_0 + s_1 p + s_2 p^2 + s_3 p^3 \end{aligned}$$

Finally, we have:

$$\begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ -1 & 2 & -2 & -1 & 2 \\ -2 & 1 & 3 & 0 & -1 \\ 1 & -1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} diag \begin{bmatrix} \frac{h_0}{2} \\ \frac{h_0+h_1+h_2}{2} \\ \frac{h_0-h_1+h_2}{6} \\ \frac{h_0+2h_1+4h_2}{6} \\ h_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \tag{5.1}$$

Build a 4×6 fast convolution using 2×3 & 2×2 fast convolution

$$h(p) = \underbrace{h_0 + h_1 p}_{h_0'} + \underbrace{h_2 p^2 + h_3 p^3}_{h_1' p^2}$$

$$x(p) = \underbrace{x_0 + x_1 p}_{x_0'} + \underbrace{x_2 p^2 + x_3 p^3}_{x_1' p^2} + \underbrace{x_4 p^4 + x_5 p^5}_{x_2' p^4}$$

$$s(p) = h(p)\,x(p) = (h_0' + h_1' p^2)(x_0' + x_1' p^2 + x_2' p^4)$$

plug in the 2×3 fast convolution derived on page 237 of the textbook

$$s(p) = \underbrace{h_0' x_0'}_{a_0 + a_1 p + a_2 p^2} + p^2\Big[\underbrace{(h_0'+h_1')(x_0'+x_1'+x_2')}_{b_0 + b_1 p + b_2 p^2}/2 - \underbrace{(h_0'-h_1')(x_0'-x_1'+x_2')}_{c_0 + c_1 p + c_2 p^2}/2 - \underbrace{h_1' x_2'}_{d_0 + d_1 p + d_2 p^2}\Big]$$

$$+ p^4\Big[-h_0' x_0' + (h_0'+h_1')(x_0'+x_1'+x_2')/2 + (h_0'-h_1')(x_0'-x_1'+x_2')/2\Big]$$

$$+ p^6 h_1' x_2'$$

plug in the 2×2 fast convolution on page 235 to each product above

each takes 3 mult & 3 add

collect the terms to form the formulae for $s_0, s_1 \cdots s_8$

s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
a_0	a_1	a_2						
		b_0	b_1	b_2				
		$-c_0$	$-c_1$	$-c_2$				
		$-d_0$	$-d_1$	$-d_2$				
				$-a_0$	$-a_1$	$-a_2$		
				b_0	b_1	b_2		
				c_0	c_1	c_2		
						d_0	d_1	d_2

total mult

$4 \times 3 = 12$

4: # of 2×2 fast convolution

3: # of mult in each convolution

total add: $\underbrace{4\times 3}_{\text{from 2x2 fast conv.}} + \underbrace{3+2+5+2+3}_{\text{from above columns}} + 6 = 33$

for $x_0'+x_1'+x_2'$ & $x_0'-x_1'+x_2'$

Chapter 9. Algorithmic Strength Reduction in Filters

Exercise Solution

Exercise 1. The 2-parallel filter algorithm is expressed as follows:

$$\begin{bmatrix} Y_0 \\ Y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & z^{-2} \\ 1 & -1 & 1 \end{bmatrix} \text{diag} \begin{bmatrix} H_0 \\ H_0 - H_1 \\ H_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \end{bmatrix}$$

By transposing the post-, pre- and diag matrix, we can obtain another 2-parallel filter structurein Fig.9.1

$$\begin{bmatrix} Y_1 \\ Y_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \text{diag} \begin{bmatrix} H_0 \\ H_0 - H_1 \\ H_1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ z^{-2} & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_0 \end{bmatrix}$$

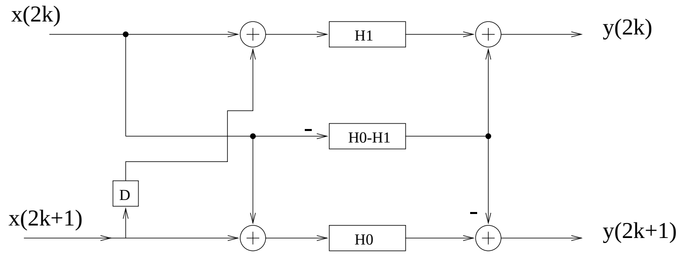

Fig. 9.1 The retimed SFG for Exercise 1.