Gage Farmer

Homework 3 - Math 2568 (Autumn 2022)

Prof. Cueto

Due date: Friday September 16, 2022 (in class).

The sections and problem numbers refer to the course's textbook (L.W. Johnson, R.D. Riess, J.T. Arnold: *Introduction to Linear Algebra*, 5th edition, Pearson.)

Section	Assigned Problems	Problems to be turned in
§1.6	4, 7, 11, 13, 14, 20, 41, 42	4, 14, 20, 41, 42
§1.7	1, 11, 19, 27, 29, 33, 41, 50, 51, 55	11, 19, 33, 51, 55
§1.9	1, 7, 9, 17, 19, 21, 23, 27, 28, 35, 39, 41	7, 19, 21, 27, 28, 39

Section 1.6

4) $E = \begin{bmatrix} 3 & 6 \\ 2 & 3 \end{bmatrix}$ $F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$EF = \begin{bmatrix} 3+6 & 3+6 \\ 2+3 & 2+3 \end{bmatrix} \to \begin{bmatrix} 9 & 9 \\ 5 & 5 \end{bmatrix}$ $FE = \begin{bmatrix} 3+2 & 6+3 \\ 3+2 & 6+3 \end{bmatrix} \to \begin{bmatrix} 5 & 9 \\ 5 & 9 \end{bmatrix}$

$EF \neq FE$

14) $F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $v = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $v^T F u$

$v^T = [-3 \;\; 3]$

$Fu = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+-1 \\ 1+-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $v^T F u = [-3 \;\; 3] \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = -3(0) + 3(0) = 0$

$\boxed{v^T F u = 0}$

20) $D = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ $v = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$

$||Dv|| = \sqrt{-3^2 + 9^2}$

$Dv = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} -6+3 \\ -3+12 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$

$\boxed{||Dv|| = \sqrt{90}}$

41) $a = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

a) $x^T a = 6$ and $x^T b = 2$ $\quad x^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$

$x_1 + 2x_2 = 6$

$3x_1 + 4x_2 = 2$

$$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 3 & 4 & 2 \end{array}\right] \xrightarrow[R_2 - 3R_1]{} \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & -2 & -16 \end{array}\right] \xrightarrow[R_1 + R_2]{} \left[\begin{array}{cc|c} 1 & 0 & -10 \\ 0 & -2 & -16 \end{array}\right] \xrightarrow[-\frac{1}{2}R_2]{} \left[\begin{array}{cc|c} 1 & 0 & -10 \\ 0 & 1 & 8 \end{array}\right]$$

$$\boxed{x = \begin{bmatrix} -10 \\ 8 \end{bmatrix}}$$

$a + b = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

b) $x^T(a+b) = 12$ $\quad x^T a = 2$

$4x_1 + 6x_2 = 12$

$x_1 + 2x_2 = 2$

$$\left[\begin{array}{cc|c} 4 & 6 & 12 \\ 1 & 2 & 2 \end{array}\right] \xrightarrow[\text{swap}]{} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 4 & 6 & 12 \end{array}\right] \xrightarrow[R_2 - 4R_1]{} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -2 & 4 \end{array}\right] \xrightarrow[R_1 + R_2]{} \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & -2 & 4 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -2 \end{array}\right] \qquad \boxed{x = \begin{bmatrix} 6 \\ -2 \end{bmatrix}}$$

42) $B = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ $\quad C = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

a) $A^T + B = C$ $\quad A^T = C - B$

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}}$$

b) $A^T B = C$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} a+c & 3a+4c \\ b+d & 3b+4d \end{bmatrix}$$

$$\begin{bmatrix} a+c & 3a+4c \\ b+d & 3b+4d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$a = 5$ $c = -3$

$b = 11$ $d = -7$

$A^T = \begin{bmatrix} 5 & -3 \\ 11 & -7 \end{bmatrix}$

$$\boxed{A = \begin{bmatrix} 5 & 11 \\ -3 & -7 \end{bmatrix}}$$

c) $B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $B_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $C_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $C_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$BC^1 = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2+12 \\ 2+16 \end{bmatrix} = \boxed{\begin{bmatrix} 14 \\ 18 \end{bmatrix}}$$

$$B_1^T C = \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2+4 & 3+5 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 8 \end{bmatrix}}$$

$$(BC_1)^T C_2 = \begin{bmatrix} 14 & 18 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 42+90 \end{bmatrix} = \boxed{\begin{bmatrix} 132 \end{bmatrix}}$$

$$\|CB_2\| = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6+12 \\ 12+20 \end{bmatrix} = \begin{bmatrix} 18 \\ 32 \end{bmatrix}$$

$$\sqrt{18^2 + 32^2} = \sqrt{1348} = \boxed{2\sqrt{337}}$$

Section 1.7

11) $u_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \quad u_4 = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} \quad u_5 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$x_1 \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 1 \\ 2 & 4 & 1 \\ -3 & 0 & 0 \end{bmatrix} \xrightarrow[R_3+3R_1]{R_2-2R_1} \begin{bmatrix} 1 & 4 & 1 \\ 0 & -4 & -1 \\ 0 & 12 & 3 \end{bmatrix} \xrightarrow[R_3+3R_2]{R_1+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{4}R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0 \quad x_2 = \frac{1}{4}x_3 \quad x_3 = x_3$

$\boxed{\text{Linearly Dependent} \quad u_4 = 4u_5}$

19) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1+4 & 2+8 \\ 3+8 & 6+16 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 11 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 22 \\ 5 & 10 \end{bmatrix} \xrightarrow{R_2-\frac{5}{11}R_1} \begin{bmatrix} 11 & 22 \\ 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{11}R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$

Singular

$\boxed{X = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$

33) $V_1 = \begin{bmatrix} 1 \\ a \end{bmatrix} \quad V_2 = \begin{bmatrix} b \\ c \end{bmatrix}$

$x_1 + bx_2 = 0$

$ax_1 + cx_2 = 0$

$cx_1 + cbx_2 = 0$

$abx_1 + cbx_2 = 0$

$\rightarrow (c - ab)x_1 = 0$

$(c - ab) = 0$

linearly dependent

$\boxed{c = ab}$

51) $a_1V_1 + a_2(V_1+V_2) + a_3(V_1+V_2+V_3) = 0$

$V_1(a_1+a_2+a_3) + V_2(a_2+a_3) + V_3a_3 = 0$

$\boxed{a_1 = 0 \quad a_2 = 0 \quad a_3 = 0}$

55) $Bx_1 = \theta$

$CBx_1 = C(0)$

$CBx_1 = 0$

$\boxed{CB \text{ is singular}}$

Section 1.9

7) $A = \begin{bmatrix} 0 & 1 & 3 \\ 5 & 5 & 4 \\ 1 & 1 & 1 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $b = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 3 & 1 & 0 & 0 \\ 5 & 5 & 4 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - \frac{1}{5}R_1]{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 5 & 5 & 4 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 & -\frac{1}{5} & 1 \end{array}\right] \xrightarrow[R_2 - 3R_3]{5R_3} \left[\begin{array}{ccc|ccc} 5 & 5 & 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 3 & -15 \\ 0 & 0 & 1 & 0 & -1 & 5 \end{array}\right]$$

$R_1 - 4R_3 \quad R_1 - 5R_2 \quad \frac{1}{5}R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & 11 \\ 0 & 1 & 0 & 1 & 3 & -15 \\ 0 & 0 & 1 & 0 & -1 & 5 \end{array}\right] \quad A^{-1} = \begin{bmatrix} -1 & -2 & 11 \\ 1 & 3 & -15 \\ 0 & -1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -2 & 11 \\ 1 & 3 & 15 \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ -20 \\ 8 \end{bmatrix}$$

$$\boxed{x_1 = 14 \quad x_2 = -20 \quad x_3 = 8}$$

19) $$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 3R_1]{} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -7 & -3 & -3 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + \frac{7}{2}R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -2 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -3 & 3.5 & 0 \end{array}\right]$$

$\frac{1}{2}R_2 \quad R_2 - R_3 \quad 2R_3 \quad R_1 - 2R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 3 & -3 & 0 \\ 0 & 0 & 1 & -6 & 7 & 0 \end{array}\right] \quad \boxed{A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 3 & -3 & 0 \\ -6 & 7 & 0 \end{bmatrix}}$$

21)

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & -3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{} \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & -3 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 - 2R_1 \quad R_3 + R_1 \quad R_4 - R_1$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & -3 & -9 & -2 & -2 & 0 & 1 & 0 \\ 0 & 2 & 5 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & \frac{2}{3} & \frac{2}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & 2 & 5 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_4 + R_2]{R_1 - 2R_2,\ R_3 - 2R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & -3 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 3 & \frac{2}{3} & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & -1 & \frac{2}{3} & -\frac{1}{3} & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 & \frac{1}{3} & 1 \end{array}\right]$$

$R_1 - 3R_3 \quad R_2 + 3R_3 \quad R_4 + 2R_3 \quad -R_3$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{7}{3} & \frac{2}{3} & -3 & \frac{4}{3} & 0 \\ 0 & 1 & 0 & \frac{8}{3} & \frac{2}{3} & 3 & -\frac{5}{3} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 2 & -1 & 2 & -1 & 1 \end{array}\right] \xrightarrow[R_3 + \frac{1}{3}R_4,\ \frac{1}{2}R_4]{R_1 + \frac{7}{6}R_4,\ R_2 - \frac{4}{3}R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{6} & \frac{7}{6} \\ 0 & 1 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{6} & \frac{7}{6} \\ 1 & \frac{1}{3} & \frac{1}{3} & -\frac{4}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}}$$

27)

$$A = \begin{bmatrix} \lambda & 4 \\ 1 & \lambda \end{bmatrix} \qquad \Delta = \lambda^2 - 4 = (\lambda + 2)(\lambda - 2)$$

$\boxed{\text{Inverse for all values of } \lambda \text{ except } \lambda = \pm 2}$

28)

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & -1 & 4 \\ 2 & -3 & \lambda \end{bmatrix} \qquad \Delta = 1\begin{bmatrix} -1 & 4 \\ -3 & \lambda \end{bmatrix} - (-2)\begin{bmatrix} 4 & 4 \\ 2 & \lambda \end{bmatrix} + 3\begin{bmatrix} 4 & -1 \\ 2 & -3 \end{bmatrix}$$

$$= 1(-\lambda + 12) + 2(4\lambda - 8) + 3(-12 + 2)$$

$$= -\lambda + 12 + 8\lambda - 16 - 36 + 6$$

$$= 7\lambda - 34$$

$\boxed{\text{Inverse for all values of } \lambda \text{ except } \lambda = \frac{34}{7}}$

39) $A^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $C^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$

$Q = C^T A^T$

$Q^{-1} = (C^T A^T)^{-1} = (A^T)^{-1} (C^{-1})^T = (A^{-1})^T (C^{-1})^T$

$$Q^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^T \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}^T \rightarrow \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 3 \\ -1+2 & 1+4 \end{bmatrix}$$

$$\boxed{Q^{-1} = \begin{bmatrix} -3 & 3 \\ 1 & 5 \end{bmatrix}}$$