MATH 2415 – Ordinary and Partial Differential Equations
Lecture 11 notes

10.4 Even and Odd Functions

We now consider the Fourier series for the special cases of even and odd functions, which are important cases that frequently arise in applications. Let's first review these types of functions

Even and odd functions, and some of their properties

- A function is **even** if $f(-x) = f(x)$, and a function is **odd** if $f(-x) = -f(x)$
- The sum (difference) and product (quotient) of two even functions are even
- The sum (difference) of two odd functions is odd; the product (quotient) of two odd functions is even
- The sum (difference) of an odd function and an even function is neither even nor odd; the product (quotient) of an odd function and an even function is odd
- If f is even, then

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

- If f is odd, then

$$\int_{-L}^{L} f(x)dx = 0$$

Fourier Cosine Series

If f is an **even** periodic function with period $2L$, and f and f' are continuous on $-L \leq x < L$ we can use the even and odd properties of the cosine and sine functions to obtain

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad n = 0,1,2,\ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0 \qquad n = 1,2,\ldots$$

The Fourier series of $f(x)$ is then

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

This is called a **Fourier cosine series**

Fourier Sine Series

If f is an **odd** periodic function with period $2L$, and f and f' are continuous on $-L \leq x < L$ we can use the even and odd properties of the cosine and sine functions to obtain

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0 \qquad n = 0,1,2,\ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \; n = 1,2,\ldots$$

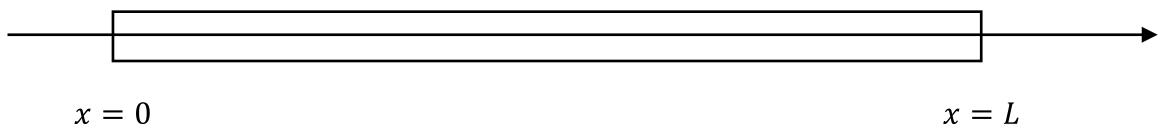
The Fourier series of $f(x)$ is then

$$f(x) = \sum_{i=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

This is called a **Fourier sine series**

10.5 Separation of Variables, Heat Conduction in a Rod

We will now see how to solve the partial differential equation governing heat conduction through a thin rod. We will assume the rod has length L and is lying along the x-axis such that one end is at the origin



We also assume that sides of the rod are insulated and that the rod is so thin that the temperature u in the rod depends only on the position x and time t, $u = u(x,t)$. Under these assumptions, the temperature obeys the following partial differential equation:

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad 0 < x < L \qquad t > 0$$

The constant α^2 is called the **thermal diffusivity** which depends on the material from which the rod is made. The thermal diffusivity is defined as

$$\alpha^2 = \frac{\kappa}{\rho s},$$

where κ is the thermal conductivity, ρ is the mass density, and s is the specific heat of the material. Values of α^2 for common materials are given in Table 10.5.1 of your text (reproduced on the last page of these notes).

We will usually write the PDE for this heat conduction problem as

$$\alpha^2 u_{xx} = u_t \qquad 0 < x < L \qquad t > 0$$

We assume that at time t=0, we know the temperature distribution

$$u(x,0) = f(x) \qquad 0 < x < L$$

and that the temperature at the ends of the rod are fixed at zero (we’ll consider the more general case later)

$$u(0,t) = 0 \qquad u(L,t) = 0 \qquad t > 0$$

We therefore seek a solution $u(x,t)$ that satisfies the **PDE**

$$\alpha^2 u_{xx} = u_t \qquad 0 < x < L \qquad t > 0$$

the **initial condition**

$$u(x,0) = f(x) \qquad 0 < x < L$$

and the **boundary conditions**

$$u(0,t) = 0 \quad u(L,t) = 0 \qquad t > 0$$

This is an **initial value problem** in time, and a two-point **boundary value problem** in space. The PDE is linear and homogeneous, which suggest that a superposition of solutions that satisfy the boundary conditions can be used to satisfy the initial condition (see the theorem on the superposition principle in the Lecture 08 notes)

Separation of Variables

We would like to find a **nontrivial** solution to the PDE that satisfies the initial and boundary conditions. The basic assumption that we make for the form of the solutions is

$$u(x,t) = X(x)T(t)$$

where X is a function that depends only on the spatial variable x, and T is a function that depends only on the temporal variable t.

If we substitute this trial solution into the PDE we get

$$\alpha^2 X''T = XT'$$

where the primes indicate ordinary differentiation either with respect to x in the case of $X(x)$, or with respect to t in the case of $T(t)$.

If we divide both sides by $\alpha^2 XT$, we have

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}$$

The left-hand side depends only on x and the right-hand side depends only on t, so we have separated the independent variables

- The only way for this equation to hold for all x and all t is if both sides are equal to the **same constant**!

We call this **separation constant** $-\lambda$, so we have

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda$$

This results in two linear homogeneous ODEs with constant coefficients, one for X and one for T:

$$X'' + \lambda X = 0$$

$$T' + \alpha^2 \lambda T = 0$$

We can solve these equations for any value of λ that satisfy the ODEs, but only certain values of λ produce solutions that satisfy both the ODEs *and* the boundary conditions.

Let's consider the boundary conditions to find that values of λ that yield nontrivial solutions to our problem:

$$u(0,t) = X(0)T(t) = 0$$

$$u(L,t) = X(L)T(t) = 0$$

For a nontrivial solution, we must have

$$X(0) = 0$$

$$X(L) = 0$$

The boundary-value problem for X(x) is then

$$X'' + \lambda X = 0 \qquad X(0) = 0 \qquad X(L) = 0$$

This is the same **eigenvalue problem** we solved in Lecture 14 (we used $y(x)$ instead of $X(x)$, but otherwise the problem is identical). We found that the eigenfunctions and eigenvalues (i.e., the nontrivial solutions) are

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad n = 1,2,3,\ldots$$

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad n = 1,2,3,\ldots$$

Now that we know the allowed values of λ that yield nontrivial solutions (i.e., the eigenvalues λ_n), we can substitute these values into the equation for T

The ODE for T is then

$$T' + \frac{n^2\pi^2\alpha^2}{L^2}T = 0$$

We can solve this equation for $T(t)$:

Multiplying this solution by the nth solution to the BVP, $X_n(x)$, we have

$$u_n(x,t) = e^{-n^2\pi^2\alpha^2 t/L^2}\sin\left(\frac{n\pi x}{L}\right) \qquad n = 1,2,3,\ldots$$

The functions $u_n(x,t)$ are called **fundamental solutions** of the heat conduction problem.

Each $u_n(x,t)$ satisfies the PDE and the boundary conditions, but will not in general satisfy the initial condition

$$u(x,0) = f(x) \qquad 0 < x < L$$

How do we proceed? Our study of initial value problems in the context of ODEs and our study of Fourier series provides the way: form a linear combination of the fundamental solutions.

Since we have an infinite number of solutions now ($n = 1,2,3,\ldots$), we try

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2}\sin\left(\frac{n\pi x}{L}\right)$$

To satisfy the initial condition, we must have

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

But this is just the Fourier sine series for $f(x)$. The coefficients c_n are given by the Euler-Fourier formula

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

In summary, the solution to the IVP-BVP heat conduction problem

$$\alpha^2 u_{xx} = u_t \qquad 0 < x < L \qquad t > 0$$

$$u(x,0) = f(x) \qquad 0 < x < L$$

$$u(0,t) = 0 \qquad u(L,t) = 0 \qquad t > 0$$

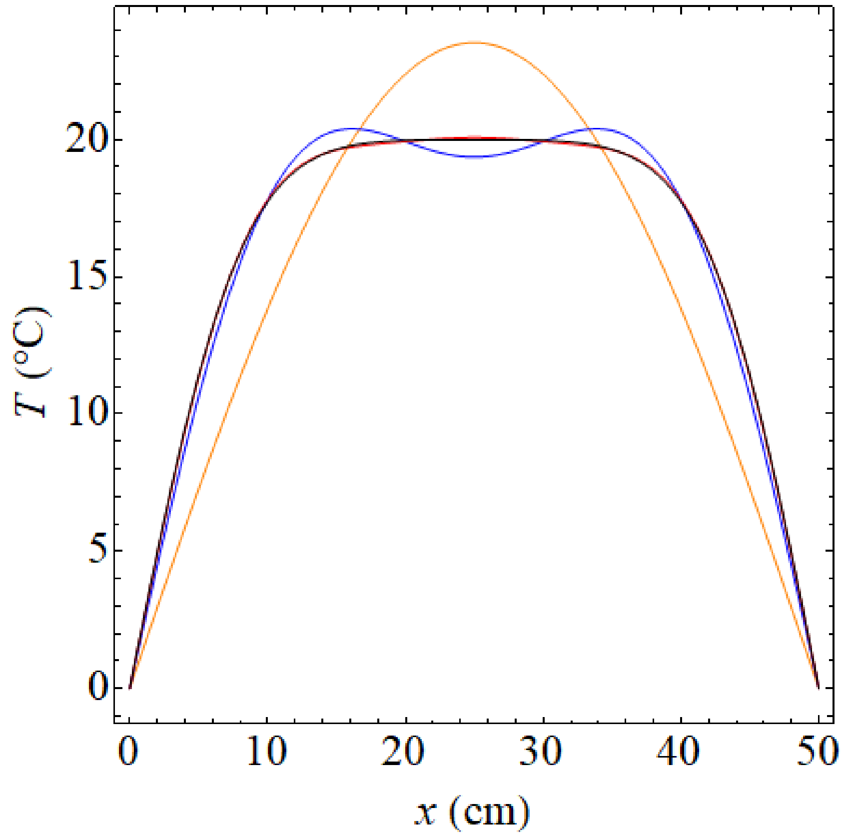
is the function

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

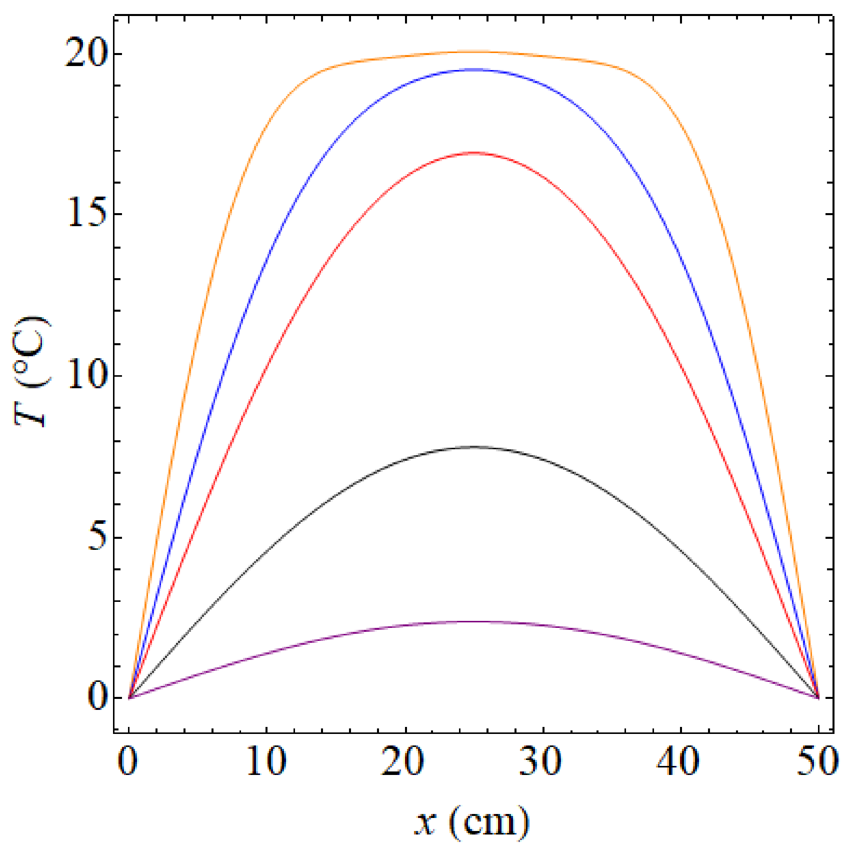
where the coefficients c_n are given by

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

- **Example:** Find the temperature at any time in a copper-aluminum alloy rod 50 cm long, if the initial temperature in the rod is a uniform 20 °C and the end temperatures are fixed at 0 °C. Let $\alpha^2 = 1 \text{ cm}^2/\text{s}$.



0
5
10
15
20
T (°C)
0
10
20
30
40
50
x (cm)



0
5
10
15
20
T (°C)
0
10
20
30
40
50
x (cm)

- **Example:** Solve the heat conduction equation subject to the following IC and BCs:

$$u_{xx} = u_t$$

$$u(x,0) = -2x^2 + 20x$$

$$u(0,t) = 0$$
$$u(10,t) = 0$$

TABLE 10.5.1 Values of the Thermal Diffusivity for Some Common Materials

Material	α^2 (cm^2/s)
Silver (99.9% pure)	1.6563
Gold	1.27
Copper (at 25°C)	1.11
Silicon	0.88
Aluminum	0.8418
Iron	0.23
Air (at 300K)	0.19
Cast Iron	0.12
Steel (1% carbon)	0.1172
Steel (stainless 310 at 25°C)	0.03352
Quartz	0.014
Granite	0.011
Brick	0.0038
Water	0.00144
Wood (yellow pine)	0.00082

10.6 Other Heat Conduction Problems

Now we consider a related heat conduction problem.

Nonhomogeneous Boundary Conditions

Instead of the ends of the rod being held at zero temperature, what if they are held at two different fixed temperatures T_1 and T_2? Now the problem we need to solve is

$$\alpha^2 u_{xx} = u_t \qquad 0 < x < L \qquad t > 0$$

$$u(x,0) = f(x) \qquad 0 < x < L$$

$$u(0,t) = T_1 \qquad u(L,t) = T_2 \qquad t > 0$$

In the long time limit, $t \to \infty$, we expect on physical grounds that the temperature will approach a steady-state solution that is independent of time (we saw that in the case of homogeneous BCs in the last section, $u \to 0$ everywhere in the rod in this limit due to the decaying exponential functions in the solution).

Since the ends of the rods are held at different temperatures, it can be shown (see your text for more details) that the steady-state solution for $u(x,t)$ linearly interpolates between these temperatures at ends of the rod.

The solution $u(x,t)$ to the heat conduction equation in this case is formed by combining the linear steady-state solution and the transient solution given by a Fourier series:

$$u(x,t) = (T_2 - T_1)\frac{x}{L} + T_1 + \sum_{i=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

with the Fourier coefficients

$$c_n = \frac{2}{L}\int_0^L \left(f(x) - (T_2 - T_1)\frac{x}{L} - T_1\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

[Note that as $t \to \infty$, the terms in the Fourier series decay exponentially to zero, leaving the only the linear part $(T_2 - T_1)\frac{x}{L} + T_1$]

- **Example:** Find the steady-state solution, and set up the boundary value problem that determines the transient solution for the following problem:

$$u_{xx} = u_t \qquad 0 < x < 30 \qquad t > 0$$

$$u(x,0) = 60 - 2x \qquad 0 < x < 30$$

$$u(0,t) = 20 \qquad u(30,t) = 50 \qquad t > 0$$