Gage Farmer

Homework 6 - Math 2568 (Autumn 2022)

Prof. Cueto

Due date: Friday October 21, 2022 (in class).

The sections and problem numbers refer to the course's textbook (L.W. Johnson, R.D. Riess, J.T. Arnold: *Introduction to Linear Algebra*, 5th edition, Pearson.)

Section	Assigned Problems	Problems to be turned in
§3.4	1, 9, 11, 16, 17, 23, 25, 29, 33, 36	1, 11, 17, 23, 29
§3.5	1, 3, 6, 13, 15, 20, 22, 26, 29, 31, 35	6, 13, 20, 26, 35
§3.6	1, 5, 7, 9, 11, 13, 17, 19	1, 5, 11, 13, 19

Bonus Problem: Consider the following three vectors in $\mathbb{R}^4$:

$$\mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ -2 \\ 5 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 9 \end{bmatrix}.$$

Can you find a system of homogeneous linear equations with solution space exactly equal to the subspace of $\mathbb{R}^4$ spanned by the three vectors. What happens if we want a system where $\mathbf{u}$ is a solution, but not the other two?

Section 3.4

1) $x_1 + x_2 - x_3 = 0$ $\qquad x_1 = -x_2 + x_3 = -x_4 + x_3$

$x_2 - x_4 = 0$ $\qquad x_2 = x_4$

$$\begin{bmatrix} -x_4 + x_3 \\ x_4 \\ x_3 \\ x_4 \end{bmatrix} \qquad x_3\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$W = \{(1,0,1,0), (-1,1,0,1)\}$

11) a) $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 5 & 8 & -2 \\ 1 & 1 & 2 & 0 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & 2 & 1 \\ 0 & -1 & -1 & 1 \end{bmatrix} \xrightarrow[R_3 + R_2]{-R_2,\ R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -3 & 0 \end{bmatrix}$

$\xrightarrow[R_1 - 7R_3,\ R_2 + 2R_3]{-\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \longrightarrow \begin{array}{l} x_1 + x_3 + x_4 = 0 \\ x_2 + x_3 - x_4 = 0 \end{array} \qquad \begin{array}{l} x_1 = -x_3 - x_4 \\ x_2 = -x_3 + x_4 \end{array}$

$\begin{bmatrix} -x_3 - x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} \qquad x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$$N(A) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

c) Let $x_3 = 1 \quad x_4 = 0 \longrightarrow \begin{array}{l} x_1 = -1 - 0 = -1 \\ x_2 = -1 + 0 = -1 \end{array}$

$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} -A_1 - A_2 + A_3 = 0 \\ A_3 = A_1 + A_2 \end{array}$

Let $x_3 = 0 \quad x_4 = -1 \longrightarrow \begin{array}{l} x_1 = 0 + 1 = 1 \\ x_2 = 0 - 1 = -1 \end{array}$

$$-A_1 + A_2 + A_4 = 0$$
$$A_4 = A_1 - A_2$$

$\{A_1, A_2\}$ is a basis where $A_3 = A_1 + A_2$ and $A_4 = A_1 - A_2$

d) $\{[1\ 0\ 1\ 1], [0\ 1\ 1\ -1]\}$

17)

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 5 & 8 & -2 \\ 1 & 1 & 2 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 3 & 8 & 2 \\ -1 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$W = \{(1\ 3\ 1)^T, (0\ -1\ -1)^T\}$

23) a)

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 7 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_3 + 3x_4 = 0$

$x_2 + x_3 - x_4 = 0$

x_3 / x_4 are free

$x_3 = 1 \quad x_4 = 1 \longrightarrow x_1 = -1 \quad x_2 = -1$

$-V_1 - V_2 + V_3 = 0 \rightarrow V_3 = V_1 + V_2$

$x_3 = 0 \quad x_4 = 1 \longrightarrow x_1 = -3 \quad x_2 = -1$

$-3V_1 - V_2 + V_4 = 0 \rightarrow V_4 = 3V_1 + V_2$

$\rightarrow aV_1 + bV_2 = 0$

b)

$$a\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = 0 \qquad \begin{bmatrix} a + 2b \\ 2a + 5b \\ a \end{bmatrix} = 0$$

$\{[1\ 2\ 1]^T, [2\ 5\ 0]^T\}$ is a basis for $Sp(S)$

29)
$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ \end{array}$$
$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x_1 = 0$

$x_2 = 0$

$x_3 = 0$

$S = \{v_1 \ v_2 \ v_3\}$ is a solution

$$\begin{array}{ccc} v_1 & v_2 & v_4 \\ \end{array}$$
$$\begin{bmatrix} 1 & -1 & -2 \\ 2 & -1 & -4 \\ 1 & 1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x_1 = 0$

$x_2 = 0$

$x_4 = 0$

$S = \{v_1 \ v_2 \ v_4\}$ is a solution

$$\begin{array}{ccc} v_1 & v_3 & v_4 \\ \end{array}$$
$$\begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & -4 \\ 1 & 7 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 0 \\ 0 & 8 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$S = \{v_1 \ v_3 \ v_4\}$ is a solution

$$\begin{array}{ccc} v_2 & v_3 & v_4 \\ \end{array}$$
$$\begin{bmatrix} -1 & -1 & -2 \\ -1 & 1 & -4 \\ 1 & 7 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 6 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$S = \{v_2 \ v_3 \ v_4\}$ is <u>not</u> a solution

$\{v_1 \ v_2 \ v_3\}$ + $\{v_1 \ v_2 \ v_4\}$, $\{v_1 \ v_3 \ v_4\}$ is the basis

Section 35

(6) $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$ $\quad \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

Linearly Dependent

13) $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix} \right\}$ $\quad \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Not a basis

20)

$x_1 - x_2 = 0 \qquad x_1 = x_2 = 2x_3 = 2x_4$

$x_2 - 2x_3 = 0 \qquad x_2 = 2x_3 = 2x_4$

$x_3 - x_4 = 0 \qquad x_3 = x_4$

$\begin{bmatrix} 2x_4 \\ 2x_4 \\ x_4 \\ x_4 \end{bmatrix} \qquad x_4 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$

$\dim(W) = 1$

26) $A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 4 & 2 & 4 \\ 2 & 1 & 5 & -2 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 4 & 1 \\ 2 & 2 & 5 \\ 0 & 4 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -2 & 1 \\ 0 & 4 & -2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \end{bmatrix}$

$[1\ 2\ 2]^T, [0\ 2\ -1]^T$

Nullity(A) = 4 $\quad$ Rank(A) = 2

35) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \underline{0} & \underline{0} & \underline{0} \end{bmatrix}$ ← Linearly Dependent

Section 3.6

1) $u_1^T u_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$

$u_1^T u_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = -1 + 2 - 1 = 0$ ← Orthogonal

$u_2^T u_3 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = 1 + 0 - 1 = 0$

5) $u_1^T u_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} = 2 + 2 - 4 = 0$

$u_1^T u_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a + b + c = 0$

$u_2^T u_3 = \begin{bmatrix} 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 2a + 2b - 4c = 0$

$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

<u>Orthogonal if $a + b = 0$ & $c = 0$</u>

11) $u_1^T V = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = a_1 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad a_1 = 3$

$u_2^T V = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = a_2 \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad a_2 = 0$

$u_3^T V = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = a_3 \begin{bmatrix} -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \quad a_3 = 0$

$\underline{V = 3u_1}$

13) $u_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$

$u_1^T u_2 = u_1^T w_2 + a u_1^T u_1$

$0 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + a \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$0 = 2 + a \qquad a = -2$

$u_2 = w_2 - 2u_1 \quad = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$u_1^T u_3 = u_1^T w_3 + b u_1^T u_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$0 = 1 + b \qquad b = -1$

$$u_2^T u_3 = u_2^T w_3 + c u_2^T u_2 = \sim$$

$$0 = 2 + 3c \qquad c = -\frac{2}{3}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1/3 \\ -2/3 \\ 0 \\ 1/3 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1/3 \\ -2/3 \\ 0 \\ 1/3 \end{bmatrix}$$

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19) $$\begin{bmatrix} 1 & -2 & 1 & -5 \\ 2 & 1 & 7 & 5 \\ 1 & -1 & 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 1 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 2x_2 + x_3 - 5x_4 = 0$$
$$x_2 + x_3 + 5x_4 = 0$$

$$a\begin{bmatrix} -3 \\ -1 \\ 10 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

★ I didn't want to write the rest of the work, it was a lot of stuff ★

O. basis of null space = $\left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7/11 \\ -27/11 \\ -6/11 \\ 1 \end{bmatrix} \right\}$

O. basis of range space = $\left\{ \begin{bmatrix} 1 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 45/82 \\ 1 \\ 5/82 \end{bmatrix} \right\}$