MATH 2415 – Ordinary and Partial Differential Equations:
Lecture 13 notes

7.5 Homogeneous Linear Systems with Constant Coefficients

$\to P(t) = A$

$\vec{g}(t) = \vec{0}$ The system of 1st-order differential equations we seek to solve is

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

matrix contains only real constants

where $\mathbf{A}$ is a constant $n \times n$ matrix containing real numbers (remember, this is just a system of n first-order ODEs written in matrix form)

When we had a single first-order linear ODE with constant coefficients,

$$\frac{dx}{dt} = ax$$

we found the solution $x(t) = ce^{at}$. There are two cases to consider when we consider the long-time behavior of this solution

- If $a < 0$ then all nontrivial solutions approach $x(t) = 0$ as t increases; we call $x(t) = 0$ an **asymptotically stable equilibrium solution**

- If $a > 0$ then all nontrivial solutions move away from the equilibrium solution as t increases; in this case $x(t) = 0$ is unstable

If we have a system of equations, we find equilibrium solutions by setting $\mathbf{x}' = \mathbf{0}$, and solving

$$\mathbf{A}\mathbf{x} = \mathbf{0}$$

We will assume that $\det \mathbf{A} \neq 0$ so $\mathbf{x} = \mathbf{0}$ is the only equilibrium solution. One thing we want to know is if this equilibrium solution is asymptotically stable or unstable.

We proceed by assuming a solution of the form

$$\mathbf{x} = \xi e^{rt}$$

where we need to determine the exponent r and the vector ξ. If we substitute this solution into the system of ODEs $\mathbf{x}' = \mathbf{A}\mathbf{x}$ we get

$$\overbrace{r\xi e^{rt}}^{\vec{x}'} = \overbrace{\mathbf{A}\xi e^{rt}}^{A\vec{x}'} \to A\vec{\xi} = r\vec{\xi}$$

$$= rI\vec{\xi}$$

We can cancel the exponential term from each side to get

$$\mathbf{A}\xi = r\xi \qquad A\vec{\xi} - rI\vec{\xi} = \vec{0}$$

Eigenval Prob $\quad (A - rI)\vec{\xi} = \vec{0}$

Next, we move all terms to the left-hand side and insert the $n \times n$ identity matrix to get

$$(\mathbf{A} - r\mathbf{I})\xi = \mathbf{0}$$

But this is nothing more than the equation that determines the eigenvalues and eigenvectors of **A.**

So the terms we seek, the exponent r and the vector $\boldsymbol{\xi}$, are eigenvalues and eigenvectors of **A**

The vector $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ is a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ if r is an eigenvalue and $\boldsymbol{\xi}$ is an associated eigenvector of the coefficient matrix **A**

For a general $n \times n$ system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, the nature of the eigenvalues and corresponding eigenvectors determines the nature of the solution. There are three cases to consider:

1. All eigenvalues are real and different from each other

2. Some eigenvalues occur in complex conjugate pairs

3. Some eigenvalues, either real or complex, are repeated

Case 1: the n eigenvalues are all real and different

Each eigenvalue has algebraic and geometric multiplicity one. For each eigenvalue r_i there is a real eigenvector $\boldsymbol{\xi}^{(i)}$, and the n eigenvectors $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}$ are linearly independent. There are then n solutions of the system corresponding to these different eigenvalues and eigenvectors:

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t}, \dots, \mathbf{x}^{(n)}(t) = \boldsymbol{\xi}^{(n)} e^{r_n t}$$

The following theorem gives the criteria for linearly independent solutions:

Theorem: Criterion for Linearly Independent Solutions

Let $\mathbf{x}^{(1)} = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, \mathbf{x}^{(1)} = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \dots, \mathbf{x}^{(n)} = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$ be n solution vectors of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ on an interval I. Then the set of solution vectors is linearly independent on I if and only if the **Wronskian**

$$W\left[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\right] = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \neq 0$$

for every t in the interval

We can apply this theorem to our solutions to show that they are linearly independent:

$$W\left[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\right](t) = \begin{vmatrix} \xi_{11}e^{r_1 t} & \cdots & \xi_{n1}e^{r_1 t} \\ \vdots & \ddots & \vdots \\ \xi_{n1}e^{r_1 t} & \cdots & \xi_{nn}e^{r_1 t} \end{vmatrix} = e^{(r_1+\cdots+r_n)t} \begin{vmatrix} \xi_{11} & \cdots & \xi_{n1} \\ \vdots & \ddots & \vdots \\ \xi_{n1} & \cdots & \xi_{nn} \end{vmatrix}$$

Since the exponential is never zero and the eigenvectors are LI, the determinant is nonzero. Therefore, these solutions are LI, and form a fundamental set of solutions.

The general solution to the system is then

$$\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + \cdots + c_n\boldsymbol{\xi}^{(n)}e^{r_n t}$$

If **A** is real and symmetric, then all the eigenvalues $r_1, \dots r_n$ are real. If some of them happen to be repeated, there is still a full set of n linearly independent eigenvectors $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}$ and the general solution above still applies

Case 2: Some eigenvalues occur in complex conjugate pairs

There are still n LI solutions of the form $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)}e^{r_1 t}, \dots, \mathbf{x}^{(n)}(t) = \boldsymbol{\xi}^{(n)}e^{r_n t}$ provided all the eigenvalues are different

Case 3: If an eigenvalue r is repeated, more care is needed in some cases constructing a set of linearly independent solutions.

We will not consider cases 2 and 3 in detail, but you can find more information in the text.

Now let’s do some examples of Case 1.

Example 1

① Find general solution of sys of ODEs and plot $x_1(t)$ and $x_2(t)$ for different ICs

$$\vec{x}' = \underbrace{\begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}}_{A}\vec{x}$$

Eigenvalues:

$$\det(A - rI) = \begin{vmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{vmatrix} = (-3-r)(-2-r) - 2 = r^2 + 5r + 4 = 0$$

$$(r+1)(r+4) = 0$$

$$\underbrace{r_1 = -1 \quad r_2 = -4}_{\text{eigenvalues}}$$

$r_1 = -1$: $(A - r_1 I)\vec{\xi} = \vec{0}$

$$\begin{pmatrix} -3+1 & \sqrt{2} \\ \sqrt{2} & -2+1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$a - \frac{b}{\sqrt{2}} = 0 \to b = \sqrt{2}a \qquad a = 1 \qquad b = \sqrt{2}$$

$$\vec{\xi} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \to \vec{x}^{(1)}(t) = \vec{\xi}^{(1)} e^{r_1 t} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}$$

$r_2 = -4$: $(A - r_2 I)\vec{\xi} = \vec{0}$

$$\begin{pmatrix} -3+4 & \sqrt{2} \\ \sqrt{2} & -2+4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \left(\begin{array}{cc|c} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$c + \sqrt{2}d = 0 \to c = -\sqrt{2}d \qquad d = 1 \qquad c = -\sqrt{2}$$

$$\vec{\xi} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} \to \vec{x}^{(1)}(t) = \vec{\xi}^{(1)} e^{r_2 t} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

general solution: $\vec{x}(t) = c_1\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

Example 2

Solve the IVP $\vec{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \vec{x}$ $\qquad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

general solution

$$x(0) = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-0} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4\cdot 0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} c_1 \\ \sqrt{2} c_1 \end{pmatrix} + \begin{pmatrix} -\sqrt{2} c_2 \\ c_2 \end{pmatrix} \to \begin{pmatrix} c_1 - \sqrt{2} c_2 \\ \sqrt{2} c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad c_1 = \frac{1+\sqrt{2}}{3} \qquad c_2 = \frac{1-\sqrt{2}}{3}$$

Particular solution:

$$\vec{x}(t) = \frac{1+\sqrt{2}}{3} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + \frac{1-\sqrt{2}}{3} \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$x_1(t) = \left(\frac{1+\sqrt{2}}{3}\right) e^{-t} + \left(\frac{-\sqrt{2}+2}{3}\right) e^{-4t}$$

$$x_2(t) = \left(\frac{\sqrt{2}+2}{3}\right) e^{-t} + \left(\frac{1-\sqrt{2}}{3}\right) e^{-4t}$$

graph to see

both approach 0 as $t \to \infty$

Find general solution of $\vec{x}' = \underbrace{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{A} \vec{x}$

found on pg 15 and 16 of lecture 12

$r_1 = 2 \qquad r_2 = r_3 = -1$

$\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \vec{\xi}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$x^{(1)} = \vec{\xi}^{(1)} e^{r_1 t} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}$

$x^{(2)} = \vec{\xi}^{(2)} e^{r_2 t} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}$

$x^{(3)} = \vec{\xi}^{(3)} e^{r_3 t} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$

general solution

$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$

Solve $\vec{x}' = \underbrace{\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}}_{A} \vec{x}$ for gen soln.

Eigenvals

$$\det(A - rI) = \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)(1-r) - 4 = r^2 - 2r - 3 = 0$$

$$(r-3)(r+1)$$

$$r_1 = 3 \quad r_2 = -1$$

$r_1 = 3$: $(A - r_1 I)\vec{\xi} = \vec{0}$

$$\left(\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\rightarrow -2a + b = 0 \rightarrow b = 2a \qquad a = 1 \quad b = 2$

$$\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \vec{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

$r_2 = -1$

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow 2c + d = 0 \quad d = -2c \qquad c = 1 \quad d = -2$

$$\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \vec{x}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}}$$

solution can grow or decay depending on c_1 and c_2