MATH 2415 – Elementary Differential Equations
Lecture 03 notes

Recap of the Method of Integrating Factors

If we have a first-order linear ODE, → Ordinary Differential Equation

$$a_1(x)\frac{dy}{dx} + a_0(x)\, y = G(x)$$

we can convert it to *standard form* by dividing the equation by $a_1(x)$, provided $a_1(x) \neq 0$:

$$\frac{dy}{dx} + p(x)\, y = g(x)$$

We can solve this equations by first finding the *integrating factor* $\mu(x)$ by exponentiating the integral of $p(x)$:

$$\mu(x) = e^{\int p(x)dx}$$

The *general solution* of our first-order differential equation is then

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)g(x)dx + c\right]$$

We can find a *particular solution* if we have an initial condition, $y(x_0) = y_0$. Once you solve the ODE for the general solution, use the initial condition to solve for the integration constant c. Then substitute this value of c into the general solution to get the particular solution.

So we see that solving this form of differential equation requires evaluating two integrals:

$$\int p(x)dx \qquad \text{and} \qquad \int \mu(x)g(x)dx$$

Another thing to remember is that the solution you find may be valid only for a certain set of values of the independent variable (e.g., values of x for which $a_1(x) \neq 0$).

We will now do some more examples of solving first-order linear ODEs using this method.

Example: Use the method of integrating factors to solve

$$\left(x^2+9\right)\frac{dy}{dx}+xy=0$$

Put into standard form

$$\frac{dy}{dx}+\frac{x}{x^2+9}y=0 \longrightarrow p(x)=\frac{x}{x^2+9} \quad g(x)=0$$

Integrating Factor: $\mu(x)=e^{\int p(x)dx}$ subs $u=x^2+9$

$du=\frac{du}{dx}dx=2x\,dx \rightarrow x\,dx=\frac{du}{2}$

$$\rightarrow \mu(x)=e^{\frac{1}{2}\int\frac{du}{u}}=e^{\frac{1}{2}\ln|u|}=e^{\ln|u|^{\frac{1}{2}}}=|u|^{\frac{1}{2}}=|x^2+9|^{\frac{1}{2}}=(x^2+9)^{\frac{1}{2}}$$

$\leftarrow$ no abs needed since $x^2+9>0$

General Solution: $y(x)=\frac{1}{\mu(x)}\left[\int\mu(x)\cancelto{0}{g(x)}dx+C\right]=\frac{C}{\mu(x)}=\boxed{\frac{C}{(x^2+9)^{1/2}}}$

Example: Use the method of integrating factors to solve the initial value problem

$$x\frac{dy}{dx} + y = 2x, \qquad y(1) = 0$$

Example: Solve the initial value problem (IVP)

$$(1+e^x)\frac{dy}{dx}+e^x y=0 \qquad y(0)=6$$

Standard Form: $\frac{dy}{dx}+\frac{e^x}{1+e^x}y=0$ $\qquad p(x)=\frac{e^x}{1+e^x}$ $\quad g(x)=0$

Subs: $u=1+e^x$
$du=e^x dx$

Integrating Factor:

$$\mu(x)=e^{\int p(x)dx}=e^{\int \frac{e^x}{1+e^x}dx}=e^{\int \frac{du}{u}}=e^{\ln|u|}=|u|=|1+e^x|=1+e^x$$

General Solution: $y(x)=\frac{1}{\mu(x)}\left[\int \mu(x)\,\cancel{g(x)}^{\,0}\,dx+C\right]=\frac{C}{\mu(x)}=\frac{C}{1+e^x}$

IC: $y(0)=\frac{C}{1+e^0}=\frac{C}{2}=6$ given IC $\longrightarrow$ $C=12$

Particular Solution:

$$\boxed{y(x)=\frac{12}{1+e^x}}$$

Example: Find the general solution of

$$\cos x \frac{dy}{dx} + (\sin x)y = 1$$

and state an interval over which the general solution is defined.

Standard Form: $\frac{dy}{dx} + \underbrace{\frac{\sin x}{\cos x}}_{\tan x} y = \underbrace{\frac{1}{\cos x}}_{\sec x}$

Note: tan and sec are discontinuous at $x = \pm n\frac{\pi}{2}$ where n is odd

$\rightarrow \frac{dy}{dx} + (\tan x)y = \sec x$

So an interval where solution is defined is $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$p(x) = \tan x = \frac{\sin x}{\cos x}$ $\quad g(x) = \sec x = \frac{1}{\cos x}$

$\mu(x) = e^{\int p(x)\,dx}$ $\quad \mu(x) = e^{\int \frac{\sin x}{\cos x} dx}$

Subs: $u = \cos x$

$-du = \sin x\,dx$

$\hookrightarrow = e^{-\int \frac{du}{u}} = e^{-\ln|u|} = e^{\ln|u|^{-1}} = |u|^{-1} = |\cos x|^{-1} = (\cos x)^{-1} = \sec x$

no abs since cos x is positive over chosen range

IC $y(x) = \frac{1}{\mu(x)}\left[\int \mu(x) g(x)\,dx + C\right] = \frac{1}{\sec x}\left[\int \sec^2 x\,dx + C\right]$

re lookup in table of integrals

$\hookrightarrow = \frac{1}{\sec x}\left[\tan x + C\right] = \cos x\left[\frac{\sin x}{\cos x} + C\right] = \boxed{\sin x + C\cos x \text{ on } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

Section 2.2: Separable Differential Equations

A general first-order ODE can be written as

$$\frac{dy}{dx} = f(x, y)$$

If the equation is linear, we can use the method of integrating factors to solve it.

If the equation is nonlinear, there is no general method for solution. In *some* cases, we can solve the equation by direct integration. An ODE belonging to this subclass is called **separable.**

To begin, we rewrite the above equation in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

If $M(x, y) = M(x)$ and $N(x, y) = N(y)$, then we will have

$$M(x) + N(y)\frac{dy}{dx} = 0$$

We can write this equation in **differential form** to get one term that depends only on x and one term that depends only on y. Start by multiplying both sides of the equation by the differential dx:

$$M(x)dx + N(y)\frac{dy}{dx}dx = 0,$$

Now use the definition of the differential dy, where $dy = \frac{dy}{dx}dx$, to eliminate the derivative from the equation:

$$M(x)dx + N(y)dy = 0$$

Upon rearranging, we have

$$N(y)dy = -M(x)dx$$

We can then solve the equation by integrating the left-hand side with respect to y and the right-hand side with respect to x. If we can solve the resulting equation to get $y = \phi(x)$, we will have an *explicit solution.* If we can't do this, we will have an *implicit solution.*

We have already used this method when we solved the differential equation describing the motion of an object falling through the atmosphere near the surface of the earth. Let's do a few more examples using this method.

Example: Solve $(1+x)dy - ydx = 0$

$(1+x)dy = ydx$

$dy = \frac{y}{1+x}dx$

$\frac{1}{y}dy = \frac{1}{1+x}dx$

subs.

$u = 1+x$

$du = dx$

$\int \frac{1}{y}dy = \int \frac{1}{1+x}dx$

$\ln|y| = \ln|u| + C$

$\ln|y| = \ln|1+x| + C$ raise to e

$|y| = |1+x| \cdot C$

$\boxed{y = C(1+x)}$ This only works on linear equations

Example: Solve the following initial value problem (IVP)

$$\frac{dy}{dx} = -\frac{x}{y}, \qquad y(4) = 3$$

Differential Form:

$y\,dy = -x\,dx$

General Solution of a family of circles

Integrate:

$\int y\,dy = -\int x\,dx \rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + c \rightarrow x^2 + y^2 = \underbrace{2C}_{R^2} \rightarrow \boxed{x^2 + y^2 = R^2}$

IC: $y(4) = 3$

$4^2 + 3^2 = R^2 \rightarrow 25 = R^2 \rightarrow \boxed{R = 5}$

In summary, if the differential equation can be put into the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

the equation is called **separable**. We can then rewrite it in differential form,

$$N(y)dy = -M(x)dx$$

Integrate both sides of the equation to obtain a general solution involving a single arbitrary integration constant c. If we also have an initial condition, we can determine c to obtain a particular solution.

Let's look at this procedure in more detail now. If we let H_1 and H_2 denote the antiderivatives of M and N, we have

$$H_1'(x) = M(x) \qquad H_2'(y) = N(y)$$

In terms of these antiderivatives, the DE is

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0$$

Since y is a function of x, using the chain rule the second term is

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y)$$

Now, we rewrite the DE as

$$\frac{d}{dx}(H_1(x) + H_2(y)) = 0$$

If we integrate both sides with respect to x, we get

$$H_1(x) + H_2(y) = c$$

The solution to the DE in this form is called an *implicit solution*. An *explicit solution* is any function $y = \phi(x)$ that satisfies the DE.

In practice, we usually obtain the implicit solution by integrating each term in the differential form. *If* we can easily solve the implicit solution for y as a function of x, we can get the explicit form. If not, leave your answer in the implicit form.

- **Example:** Use separation of variables to solve the following equation

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2}$$

- **Example:** Use separation of variables to solve the initial value problem and determine the interval in which the solution exists

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)} \qquad y(0) = -1$$

The set of implicit solutions are the equations of the integral curves of the DE (these are the trajectories we sketched from direction fields in a previous lecture).

To find an integral curve passing through a given point (x_0, y_0), substitute x_0 and y_0 into the implicit solution to determine c.

Here are some integral curves for the previous two examples:

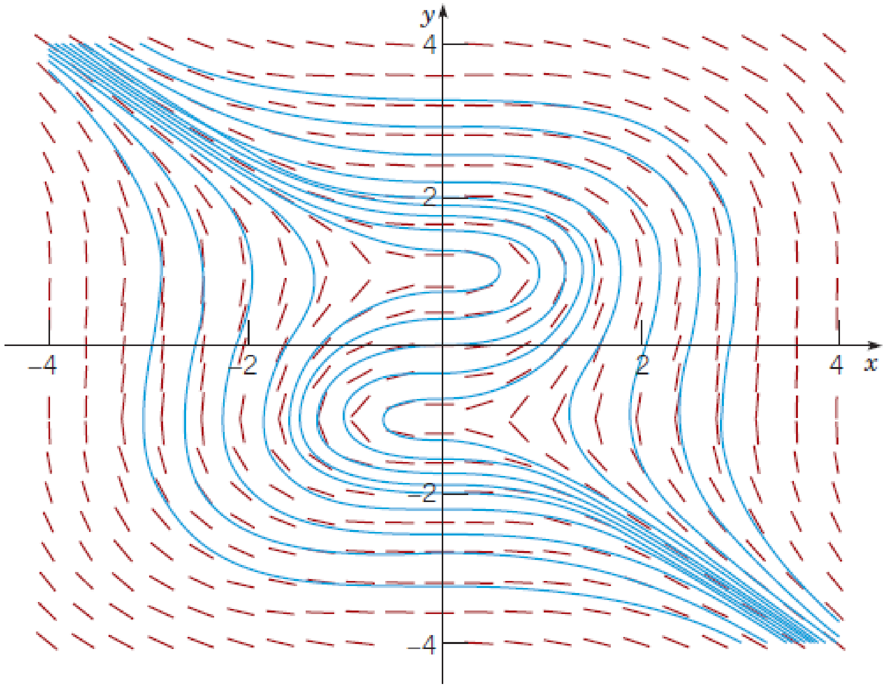

FIGURE 2.2.1 Direction field and integral curves of $y' = x^2/(1-y^2)$.

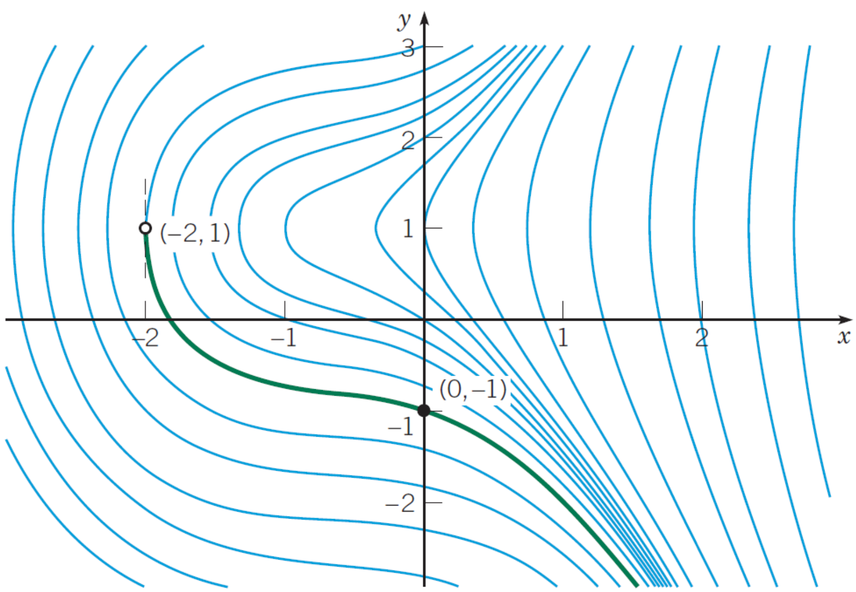

FIGURE 2.2.2 Integral curves of $y' = (3x^2+4x+2)/2(y-1)$; the solution satisfying $y(0) = -1$ is shown in green and is valid for $x > -2$.

Another thing to be on the lookout for is constant solutions, $y = y_0$. Sometimes these solutions satisfy an ODE, but we don't find them when we apply the methods we have studied so far.

- **Example:** Solve $xy^4 dx + (y^2 + 2)e^{-3x} dy = 0$

- **Example:** Does the following DE have a constant solution?

$$y' = \frac{(y-4)e^x}{y^2}$$

Section 2.3: Modeling with First-Order Differential Equations

In Lecture 01, we saw that differential equations are useful mathematical models of processes involving relations between the rates at which things happen (e.g., we constructed an ODE to model the relation between the forces on a falling object and the rate at which its velocity changes).

Formulating these types of models can be difficult, but we identified some general procedures that are useful for model construction:

1. Identify the variables in the problem (both independent and dependent) and assign symbols to identify them (usually letters)

2. Choose the units for each quantity in the problem

3. Identify the underlying principles governing the behavior of the system (e.g., we used Newton's 2^{nd} law to determine the equation of motion of the falling object)

4. Express the principle in step 3 in terms of the variables chosen in step 1

5. Check that your equation is dimensionally consistent

6. More complex problems may require formulating a system of several differential equations

Note that mathematical models are almost always an approximate description of the behavior of interest. Be aware of the limitations of your model, and that the model may need to be revised to account for more factors affecting the behavior under investigation

Once we have a model, we may be able to solve the equations exactly. Often this is too difficult, so we may have to make further mathematical approximations to obtain a solution (e.g., linearizing a non-linear problem). The simplified problem should be examined carefully to make sure it still captures the essential features of the problem.

You should also check that the results you obtain make sense (are they physically reasonable?)

1. Compare to experiment if possible

2. Is the behavior consistent with observations in certain regimes (e.g., in the long-time limit)?

3. If we know what the results should be for certain values of parameters, is our model consistent with these results?

If the answer is no, we will need to examine the model and/or the simplifications we made.

This section of your text gives several interesting examples of problems that can be modelled using first-order ODEs (mixing, compound interest, chemicals in a pond, escape velocity).

Let's look at a few additional examples.

- **Example:** Bacterial growth

If the growth rate of bacteria is proportional to the number of bacteria, how long will it take for the initial number of bacteria to double?

- **Example:** Half-life of a radioactive substance

The half-life of a radioactive substance is the time it takes for one-half of the atoms in the initial amount of the substance to transform into another substance. The rate of disintegration of the substance is proportional to the remaining amount of the substance. Find the half-life of a radioactive isotope if 0.043% of the isotope has disintegrated after 20 years.

- **Example:** Water draining from a tank

Torricelli's law from hydrodynamics can be used to show that the equation describing the height h of water that is draining from a tank in the shape of a right circular cylinder is

$$\frac{dh}{dt} = -\frac{A_{\mathrm{h}}}{A_{\mathrm{w}}}\sqrt{2gh}$$

where $g = 32$ ft/s^2, A_{w} is the cross-sectional area of the water, and A_{h} is the cross-sectional area of the hole. What is $h(t)$ if $h(0) = 20$ ft, $A_{\mathrm{h}} = 0.25$ ft^2, and $A_{\mathrm{w}} = 50$ ft^2? How long does it take for the tank to empty?

- **Example:** Determine the equations of motion (position and velocity as functions of time) for a particle with constant acceleration moving in one dimension.

- **Example:** A large tank initially holds 300 gallons of a brine solution (i.e., salt water). A brine solution with a concentration of 2 lb/gal is pumped into the tank at a rate of 3 gal/min. The solution in the tank is thoroughly mixed, and pumped out of the tank at a rate of 2 gal/min. If 50 lbs of salt is dissolved in the initial 300 gallons, how much salt is in the tank at time t?