Quantifying Predicates

Having predicates by themselves is no real improvement on propositional logic. We need a way to be able to talk about multiple propositions (even an infinite number of them) at the same time. For this we have quantifiers

There are two common quantifiers: $\forall$ and $\exists$.

The Universal Quantifier $\forall$

We use the universal quantifier when we want to assert that a predicate is true for every element in a given set. So

$$(\forall x \in S)[P(x)]$$

says that $P(x)$ is true for every x in the set S. Notice that it is now a proposition, either true or false.

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If the set is obvious (or the set of everything) then the set can be omitted and at times we can include extra information in the quantifier. E.g., in $(\forall x > 0)[e^{\ln x} = x]$ the fact that $x \in \mathbb{R}$ is implied and the inequality in the quantifier is a short way of saying

$$(\forall x \in \mathbb{R})[(x > 0) \Rightarrow (e^{\ln x} = x)].$$

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$$(\forall x)[P(x) \Rightarrow M(x)].$$

Proof by Induction

Proof by induction or mathematical induction is a method of proving a predicate is true for all of an infinite sequence of numbers, usually the natural numbers. It uses the "argument" below

Base Case

Inductive step

$$\begin{array}{l} P(0) \\ (\forall k \in \mathbb{N})[P(k) \Rightarrow P(k+1)] \\ \hline \therefore (\forall n \in \mathbb{N})[P(n)] \end{array}$$

$$\begin{array}{ll} & P(0) \\ (k=0) & P(0) \Rightarrow P(1) \\ \hline & \therefore P(1) \\ (k=1) & P(1) \Rightarrow P(2) \\ \hline & \therefore P(2) \end{array}$$

to prove that something is true for all natural numbers.

(Equivalently and more commonly, we could prove that that $P(k-1) \Rightarrow P(k)$ for almost all natural numbers (i.e. not zero). Also zero doesn't have to be the first number that P is true for.)

A First Example of Proof by Induction

Lemma

For all natural numbers n, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Proof.

<u>Base Case:</u> $(n=1)$ ✓

$$\sum_{k=1}^{1} k = 1 \stackrel{?}{=} \frac{2}{2} = \frac{1(1+1)}{2}$$

<u>Inductive Step:</u> $\left(\sum_{k=1}^{L} k = \frac{L(L+1)}{2} \Rightarrow \sum_{k=1}^{L+1} k = \frac{(L+1)(L+2)}{2}\right)$

Assume $\sum_{k=1}^{L} k = \frac{L(L+1)}{2}$

$$\sum_{k=1}^{L+1} k = \sum_{k=1}^{L} k + (L+1) = \frac{L(L+1)}{2} + (L+1) = \frac{L^2+L}{2} + \frac{2L+2}{2} = \frac{L^2+3L+2}{2} = \frac{(L+1)(L+2)}{2}$$

$$= 1+2+3+\dots+(L-1)+L+(L+1)$$

□

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Now if T is a set such that $|\mathscr{P}(T)| = 2^{|T|}$ and S is set made by adding a new element to T (i.e. $S = T \cup \{a\}$, where $a \notin T$)

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Strong Induction

This type of induction is sometimes called weak induction because there's another type called strong induction shown by the following argument:

$$\begin{array}{l} P(0) \\ (\forall n \in \mathbb{N} - \{0\})\big[(\forall m < n)[P(m)] \Rightarrow P(n)\big] \\ \hline \therefore (\forall n \in \mathbb{N})[P(n)] \end{array}$$

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We'll return to strong induction when we get to analyzing recursive algorithms.

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For example, $(\exists x \in \mathbb{R})[x^2 = 0]$ and $(\exists x \in \mathbb{R})[x^2 = 1]$ are both true, but $(\exists x \in \mathbb{R})[x^2 = -1]$ is false.

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Or, if we make $D(x)$ mean "x is a dog" we could translate the claim into

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Or even

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})\big[(x \geq 0) \Rightarrow [y \cdot y = x]\big],$$

since when $x < 0$ any y will do.

Important Concepts

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- A statement in which all variables are bound is considered a sentence and is, in essence, a proposition.

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Main take-away: the word "any" can be particularly problematic, especially in the phrase "for any".

Goldbach's conjecture states that every even integer greater than two is the sum of two primes. Let's rewrite that using quantifiers, letting $\mathbb{Z}$ be the integers, and P the set of primes.

$$(\forall n \in \mathbb{Z})\left[(2n > 2) \Rightarrow (\exists p, q \in P)\left[p + q = 2n\right]\right]$$

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- $P(x)$, "x is a person"
- $T(t)$, "t is a time"
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$$(\forall x)\{P(x) \Rightarrow (\exists y)(\exists t)[P(y) \wedge T(t) \wedge L(x,y,t)]\}$$

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$$\text{or } (\forall x)\,[P(x) \Rightarrow (\exists y)(\exists t)[P(y) \wedge T(t) \wedge L(x, y, t)]]$$