MATH-2415, Ordinary and Partial Differential Equations **Name:**
Instructor: Michael Fellinger
Spring 2022
Midterm 2
Due March 24, 2022 by 8:00pm

Directions:

You can
(I) Print this sheet and show all work on the sheet itself, or

(II) Use separate paper to work out the problems. The paper must be plain white paper. No notebook paper, no lined paper. Engineering paper is acceptable.

For either selection, clearly show all work that leads to your final answer.

You can scan your work and save the pages as a single pdf. Or you can take pictures of your work, add the pictures to Word or Powerpoint and export the pages to a single pdf. You will submit the pdf to me via email.

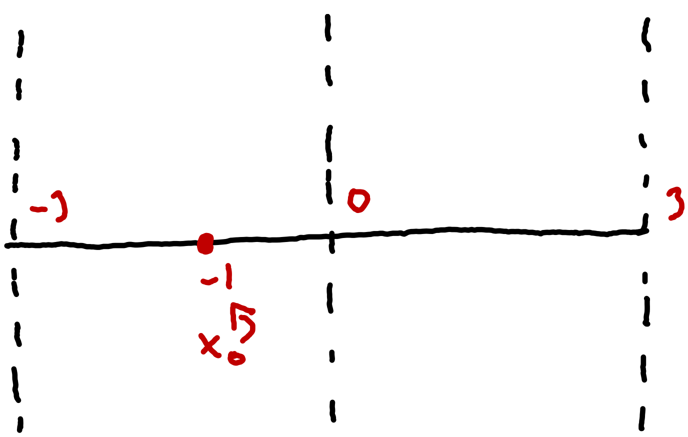
1. Determine the longest interval on which a unique solution exists for the following ODE (note that you do not need to solve the equation to answer this question):

x_0 x_0

$$y'' + \frac{4}{x}y' + 18x^2 y = \frac{5}{x^2-9}, \qquad y(-1) = -2, \qquad y'(-1) = 0$$

$p(x) = \frac{4}{x}$ $q(x) = 18x^2$ $g(x) = \frac{5}{x^2-9}$

Disc. @ $x=0$, $x=\pm 3$



$\boxed{-3 < x < 0}$

2. Solve the second-order differential equations with constant coefficients:

a) $y'' + 2y' - y = 0$

b) $9y'' - 6y' + y = 0$

c) $y'' - 4y' + 13y = 0$

a) $y = e^{rx} \rightarrow y' = re^{rx} \rightarrow y'' = r^2 e^{rx}$

$\rightarrow y'' + 2y' - y = (r^2 + 2r - 1)e^{rx} = 0$

$\rightarrow r^2 + 2r - 1 = 0 \rightarrow r = -1 \pm \sqrt{2}$

$\rightarrow \boxed{y(x) = C_1 e^{(-1+\sqrt{2})x} + C_2 e^{(-1-\sqrt{2})x}}$

b) $9r^2 - 6r + 1 = 0 \rightarrow r = \frac{1}{3}$ ← repetered

$$\boxed{y(x) = c_1 e^{x/3} + c_2 x e^{x/3}}$$

c) $r^2 - 4r + 13 = 0 \rightarrow r = 2 \pm 3i$ ← Fake

$$\boxed{y(x) = c_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x}$$

3. Consider the second-order homogeneous equation: $2x^2y'' + 3xy' - y = 0 \quad (x > 0)$

a) Verify that $y_1 = x^{1/2}$ and $y_2 = x^{-1}$ are solutions of the ODE.

b) Find the Wronskian, $W[y_1, y_2]$

c) Do y_1 and y_2 form a fundamental set of solutions for the given ODE? If so, state the general solution.

a) $y_1 = x^{\frac{1}{2}} \rightarrow y_1' = \frac{1}{2}x^{-\frac{1}{2}} \rightarrow y_1'' = -\frac{1}{4}x^{-\frac{3}{2}}$

$2x^2\left(-\frac{1}{4}x^{-\frac{3}{2}}\right) + 3x\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - x^{\frac{1}{2}} = -\frac{1}{2}x^{\frac{1}{2}} + \frac{3}{2}x^{-\frac{1}{2}} - x^{\frac{1}{2}} = 0$

$y_2 = x^{-1} \rightarrow y_2' = -x^{-2} \rightarrow y_2'' = 2x^{-3}$

$2x^2(2x^{-3}) + 3x(-x^{-2}) - x^{-1} = 4x^{-1} - 3x^{-1} - x^{-1} = 0$

b) $W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^{\frac{1}{2}} & x^{-1} \\ \frac{1}{2}x^{-\frac{1}{2}} & -x^{-2} \end{vmatrix} = -x^{\frac{1}{2}}x^{-2} - \frac{1}{2}x^{-\frac{1}{2}}x^{-1}$

$= -x^{-\frac{3}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \boxed{-\frac{3}{2}x^{-\frac{3}{2}}}$

c) Since $W[y_1, y_2] \neq 0$

these solutions are linearly independent and form a fundamental set

$$\boxed{y(x) = C_1x^{\frac{1}{2}} + C_2x^{-1}}$$

4. Consider the second-order differential equation: $y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{3x}$

Determine the form of the particular solution needed to use the *method of undetermined coefficients* for the nonhomogeneous equation **(You do not need to solve the ODE!)**

$g_1(x) = 3x^2 \qquad g_2(x) = -5\sin 2x \qquad g_3(x) = 7xe^{3x}$

$y_{p_1}(x) = Ax^2 + Bx + C$

$y_{p_2}(x) = D\sin 2x + E\cos 2x$

$y_{p_3}(x) = (Fx + G)e^{3x}$

$$\boxed{y_p(x) = Ax^2 + Bx + C + D\sin 2x + E\cos 2x + (Fx + G)e^{3x}}$$

5. Solve the homogeneous second-order initial value problem:

$$y'' + 2y' + 5y = 0, \qquad y(0) = 3, \qquad y'(0) = 4$$

$$r^2 + 2r + 5 = 0 \rightarrow r = -1 \pm 2i$$

$$y(x) = C_1 e^{-x} \cos 2x + C_2 e^{-x} \sin 2x$$

$$y(0) = C_1 = 3 \rightarrow y(x) = 3e^{-x} \cos 2x + C_2 e^{-x} \sin 2x$$

$$y'(x) = -(x+6) e^{-x} \sin 2x + (2C_2 - 3) e^{-x} \cos 2x$$

$$y'(0) = 2C_2 - 3 = 4 \qquad C_2 = \frac{7}{2}$$

$$\boxed{y(x) = 3e^{-x} \cos 2x + \frac{7}{2} e^{-x} \sin 2x}$$

6. Solve the nonhomogeneous second-order ODE using the method of undetermined coefficients:

$$y'' - 3y' - 10y = x - 3e^{5x}$$

y_c: $r^2 - 3r - 10 = 0$ $\quad r = -2, 5$

$y_c(x) = C_1 e^{-2x} + C_2 e^{5x}$

y_p: $y_p = Ax + B + Cxe^{5x}$

$y_p' = A + Ce^{5x} + 5Cxe^{5x}$

$y_p'' = 10Ce^{5x} + 25Cxe^{5x}$

$y_p'' - 3y_p' - 10y_p = 10Ce^{5x} + 25Cxe^{5x}$

$- 3A - 3Ce^{5x} - 15Cxe^{5x}$

$- 10Ax - 10B - 10Cxe^{5x}$

$= 7Ce^{5x} - 3a - 10b - 10ax = x - 3e^{5x} + 0$

$$\left.\begin{array}{l} -10A = 1 \\ 7C = -3 \\ -3A - 10B = 0 \end{array}\right\} \quad \begin{array}{l} A = \frac{1}{10} \\ B = \frac{3}{100} \\ C = -\frac{3}{7} \end{array}$$

$$\boxed{y(x) = C_1 e^{-2x} + C_2 e^{5x} - \frac{1}{10}x + \frac{3}{100} - \frac{3}{7}xe^{5x}}$$