MATH-2415, Ordinary and Partial Differential Equations **Name:**
Summer 2023
Problem Set 3
Due June 18, 2023 by midnight

Directions: You can either

(I) Show all your work on the pages of the assignment itself, or

(II) Use separate paper to work out the problems. The paper must be plain white paper. No notebook paper, no lined paper. Engineering paper is acceptable.

For either selection, **clearly show all work that leads to your final answer**. Don't make me hunt for your steps! And DON'T turn in **any** scratched out work! Your final product must be clear and legible, or it will be returned, ungraded. **Please submit your work to Blackboard as a single pdf file**.

1. When a vertical beam of light passes through a transparent substance, the rate at which its intensity I decreases is proportional to $I(t)$, where t represents the thickness of the medium (in feet). In clear seawater the intensity 3 feet below the surface is 25% of the initial intensity I_0 of the incident beam. What is the intensity of the beam 15 feet below the surface?

$\frac{dI}{dt} = -kI \rightarrow \frac{dI}{I} = -k\,dt \quad \int \frac{1}{I}\,dI = -k\int dt \quad \ln I = -kt + c_1 \quad e^{\ln I}$

$\hookrightarrow = e^{-kt+c_1} \rightarrow = ce^{-kt}$

$\hookrightarrow I_0 = ce^{0}c \rightarrow I = I_0 e^{-kt} \rightarrow .25 I_0 = I_0 e^{-3k} \; 0.25$

$\hookrightarrow = e^{-3k} \quad \ln(.25) = -3k \quad k = -\frac{\ln(.25)}{3} \rightarrow I = I_0 e^{-.462t}$

$\hookrightarrow I = I_0 e^{-.462 \cdot 15} \rightarrow \boxed{I = 0.00098 I_0 \text{ W/ft}}$

2. A large tank initially holds 300 gallons of a brine solution (i.e. salt water). A brine solution with a concentration of 2 lb/gal is pumped into the tank at a rate of 3 gal/min. The solution in the tank is thoroughly mixed, and pumped out of the tank at a rate of 3 gal/min. If 50 lbs of salt is dissolved in the initial 300 gallons, how much salt is in the tank at time t? How much salt is in the tank after a very long time? [Note: this is similar to a problem we did in class, but not exactly the same]

$$\frac{dA}{dt} = R_{in} - R_{out} \qquad R_{in} = 6A(t)$$

$$R_{out} = \frac{A(t)}{100} \rightarrow \boxed{\frac{dA}{dt} = 6t - \frac{t}{100}}$$

Amount of salt approaches ∞ over time

3. In class we solved the following problem:

The rate at which a substance evaporates is proportional to its surface area. If a spherical object has a radius of 0.75 cm just after it was manufactured and a radius of 0.30 cm after 6 months due to evaporation,

a) How long will it take for the radius to be 0.15 cm?
b) How long will it take for the volume of the object to be one third of its initial volume?

To solve this problem, we rewrote the differential equation

$$\frac{dV}{dt} = kA$$

in terms of the radius of the sphere r. Here you will solve this problem a different way. Instead of rewriting the ODE in terms of r, you will rewrite the ODE in terms of V by expressing the surface area A in terms of V. Then solve this equation for $V(t)$ and answer the questions in parts a) and b).

$V = \frac{4}{3}\pi r^3 = \frac{4}{3} r A = \frac{4}{3}\sqrt{\frac{A}{\pi}}\, A$

$A = \pi r^2$

$r = \sqrt{\frac{A}{\pi}}$

$A = \pi r^2 = \frac{\sqrt[3]{\pi}\left(\sqrt[3]{6V}\right)^2}{4}$

$V = \frac{4}{3}\pi r^3 \rightarrow \sqrt[3]{\frac{3V}{4\pi}} = r$

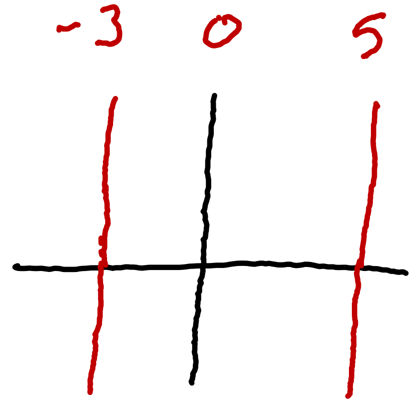
$V(t) = \int \frac{\sqrt[3]{\pi}\left(\sqrt[3]{6V}\right)^2}{4}\, dV \rightarrow V(t) =$

4. What is the longest interval on which each of the following initial value problems is guaranteed to exist?

a) $(x^2 - 2x - 15)y' - \frac{y}{x} = 4x^2 \qquad y(-5) = 3$

b) $(x^2 - 2x - 15)y' - \frac{y}{x} = 4x^2 \qquad y(-1) = 2$

c) $(x^2 - 2x - 15)y' - \frac{y}{x} = 4x^2 \qquad y(10) = -1$



$$y' - \frac{y}{x(x^2-2x-15)} = \frac{4x^2}{x^2-2x-15}$$

$$y' - \underbrace{\frac{y}{x(x-5)(x+3)}}_{p(x)} = \underbrace{\frac{4x^2}{(x-5)(x+3)}}_{g(x)}$$

a) $y(-5) = 3 \rightarrow -\infty < x < -3$

b) $y(-1) = 2 \rightarrow -3 < x < 0$

c) $y(10) = -1 \rightarrow 5 < x < \infty$

5. What is the general solution of the following Bernoulli equation?

$$\frac{dy}{dx} + y = xy^{2/3}$$

If the initial condition is $y(0) = 0$, what is the particular solution?

$p(x) = 1 \qquad g(x) = x \qquad n = \frac{2}{3}$

$u = y^{1-\frac{2}{3}} = y^{\frac{1}{3}} = \sqrt[3]{y}$

$y = u^3$

$\mu(x) = e^{\int p(x)dx} = e^{\int dx} = e^x$

$u(x) = \frac{1}{e^x}\left[\int xe^x\,dx + c\right]$

$\hookrightarrow u(x) = \frac{1}{e^x}\left[(x-1)e^x + c\right] \rightarrow = (x-1) + \frac{c}{e^x}$

$\boxed{y(x) = u(x)^3 = \left(x + \frac{c}{e^x} + 1\right)^3}$

$y(0) = \left(0 + \frac{c}{e^0} + 1\right)^3 \qquad c = -1$

$\boxed{y(x) = \left(x - \frac{1}{e^x} + 1\right)^3}$ particular solution

6. What is the general solution of the following Bernoulli equation?

$$3xy^2\frac{dy}{dx} + 3y^3 = 1$$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{3xy^2} \qquad n=-2 \qquad p(x) = \frac{1}{x} \qquad g(x) = \frac{1}{3x}$$

$$u = y^{1-n} = y^{1+2} = y^3 \qquad y = \sqrt[3]{u}$$

$$\mu(x) = e^{\int p(x)dx} = e^{\int \frac{dx}{x}} = e^{\ln|x|} = \pm x$$

$$\hookrightarrow u(x) = \frac{1}{\mu(x)}\left[\int \mu(x) g(x)\, dx + C\right]$$

$$\hookrightarrow u(x) = \frac{1}{x}\left[\int \frac{1}{3}\, dx + C\right] = \frac{1}{x}\left[\frac{x}{3} + C\right] = \frac{1}{3} + \frac{C}{x}$$

$$\boxed{y(x) = \sqrt[3]{\frac{1}{3} + \frac{C}{x^3}}}$$

7. Determine if any of the equations are exact. If so, find the general solution.

a) $y(2x^2y^2+1)y' + x(y^4+1) = 0$ Exact

b) $2x^2y' + 3x + 4xy = 0$ Exact

c) $(\cos^2 x + y\sin 2x)y' + y^2 = 0$ Not Exact

I don't understand this so I kinda guessed

8. Determine if the following equation is exact. If so, solve the initial value problem.

$$x\frac{dy}{dx} + 3x + y = 0 \qquad y(1) = 1$$

$\frac{dy}{dx} + \frac{y}{x} = 3$ $\qquad p(x) = \frac{1}{x} \qquad g(x) = 3$

$\mu(x) = e^{\int \frac{dx}{x}} = x$

$u(x) = \frac{1}{x}\left[\int 3x\,dx + c\right] \rightarrow = \frac{1}{x}\left[\frac{3}{2}x^2 + C\right]$

$\hookrightarrow y(x) = \frac{3}{2}x + \frac{c}{x}$ $\qquad y(1) = \frac{3}{2}(1) + \frac{c}{1} = \frac{3}{2} + C$

$C = -\frac{1}{2}$

$$\boxed{y(x) = \frac{3}{2}x + \frac{\left(-\frac{1}{2}\right)}{x}}$$