MATH-2415, Ordinary and Partial Differential Equations **Name:**
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Final Exam
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Clearly show all work that leads to your final answer.

1. Solve the following IVPs:

a) $xy' + y = 2x\cos x \qquad y(\pi) = 0$

b) $y' = \dfrac{2t}{\sin y} \qquad y(1) = \pi$

$$y' + \frac{1}{x}y = 2\cos x$$

$$u(x) = e^{\int \frac{1}{x}dx} = x$$

$u = x \quad v = \sin x$

$du = dx \quad dv = \cos x\, dx$

$$y(x) = \frac{1}{x}\left[2\int x\cos x\, dx\right]$$

$$\frac{1}{x}\left[2\int u\, dv\right]$$

$$\frac{1}{x}\left[2x\sin x - 2\int \sin x\, dx\right] \to \frac{1}{x}\left[2x\sin x + 2\cos x + C\right]$$

$$2\sin x + \frac{2}{x}\cos x + \frac{C}{x}$$

$$y(\pi) = 2\sin(\pi) + \frac{2}{\pi}\cos(\pi) + \frac{C}{\pi} = 0 \qquad C = 2$$

$$y(x) = 2\sin(x) + \frac{2}{x}\cos(x) + \frac{2}{x}$$

2. Is the following equation exact? If so, find the general solution.

$$\underbrace{\left(2xe^{3y} + e^{x}\right)}_{M} + \underbrace{\left(3x^2 e^{3y} - y^2\right)}_{N}\frac{dy}{dx} = 0$$

$\frac{\partial M}{\partial y} = 6xe^{3y}$ $\quad \frac{\partial N}{\partial x} = 6xe^{3y}$ $\quad$ Exact

$K(x,y) = \int M(x,y)\,dx = \int 2xe^{3y} + e^{x}\,dx = x^2 e^{3y} + e^{x} + h(y)$

$\frac{\partial K}{\partial y} = 3x^2 e^{3y} + \frac{\partial h}{\partial y} = N(x,y)$ $\quad \frac{\partial h}{\partial y} = -y^2$ $\quad h(y) = \int -y^2$

$= -\frac{y^3}{3}$

$k(x,y) = x^2 e^{3y} + e^{x} - \frac{y^3}{3} + C_1$

3. a) Solve the second-order ODE:

$$y'' - 6y' + 12y = 0$$

b) If the right-hand side equaled $7x^2e^x + 157\sin x$ instead of 0, what form of particular solution would you need for the method of undetermined coefficients (you do not need to solve this nonhomogeneous equation)

4. Solve the following heat conduction equation with the prescribed initial condition and homogeneous boundary conditions:

$$u_{xx} = u_t$$

$$u(0,t) = 0 \qquad u(20,t) = 0$$

$$u(x,0) = \begin{cases} 10 - x & 0 \le x \le 10 \\ 0 & 10 \le x \le 20 \end{cases}$$

5. Consider the following system of linear first-order ODEs:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 16 & 0 \end{pmatrix} \mathbf{x}$$

a) Solve the system and describe the behavior of the solutions as $t \to \infty$ for different choices of c_1 and c_2.

b) Convert the system of first-order ODE's into a single second-order ODE and solve it. Verify that the solution is consistent with the solutions for $x_1(t)$ and $x_2(t)$ from part a).