Complex variable, Laplace & Z- transformation

Lecture 06

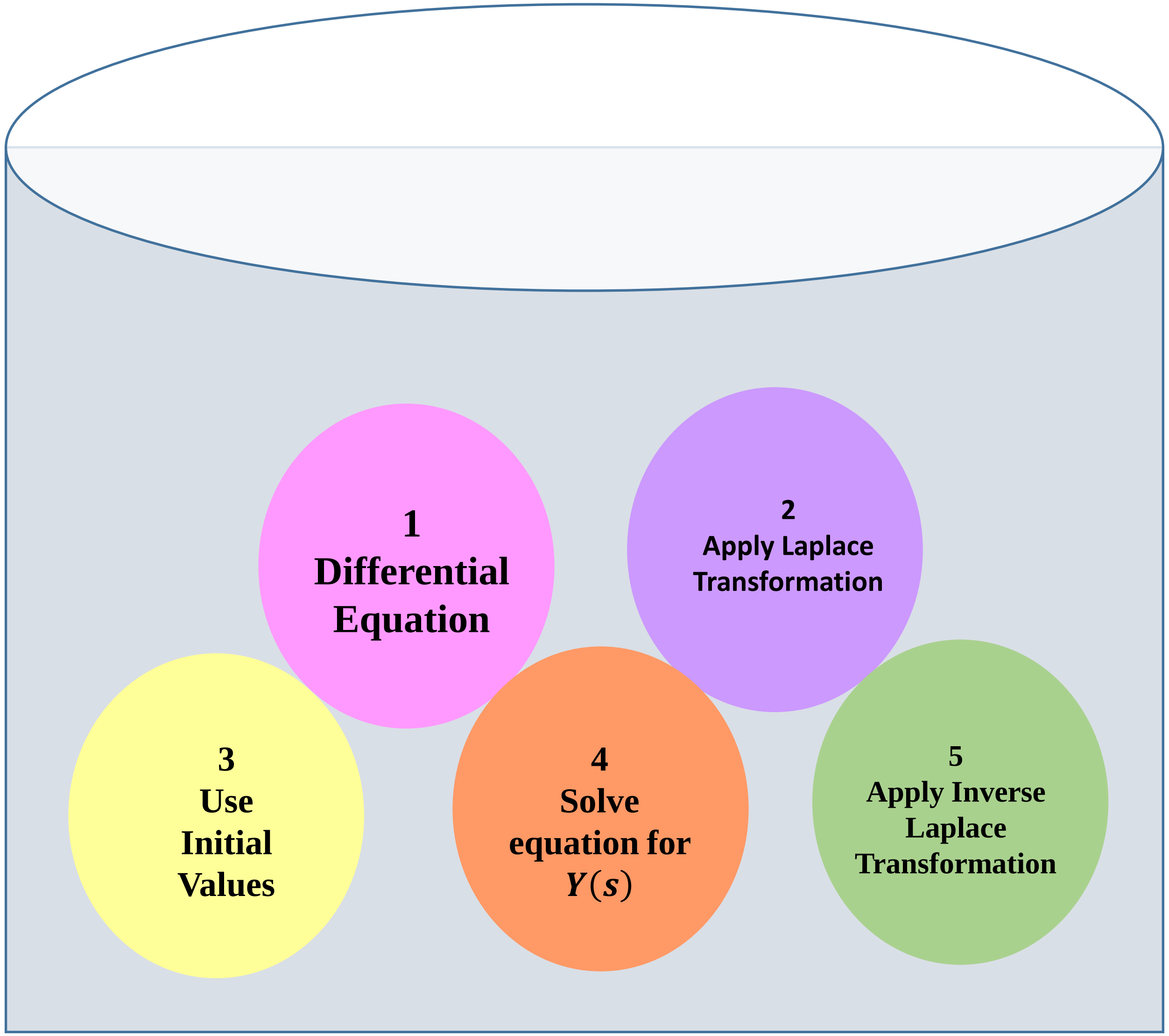
This Lecture Covers-

1. Process of Solving Differential Equations using Laplace transformation.

2. Some Important formulae.

3. Example and exercises of solving differential equations using Laplace Transformation.

Process of Solving Differential Equations using Laplace Transformation



Important Formulae

1. $\mathcal{L}\{\dot{f}(t)\} = \mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0).$

2. $\mathcal{L}\{\ddot{f}(t)\} = \mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s) - sf(0) - \dot{f}(0)$ where $f(0)$, and $\dot{f}(0)$are the initial values of f and$\dot{f}$ for $\mathrm{t} = 0$

3. $\mathcal{L}\{\dddot{f}(t)\} = \mathcal{L}\left\{\frac{d^3f(t)}{dt^3}\right\} = s^3F(s) - s^2f(0) - s\dot{f}(0) - \ddot{f}(0).$

The general case for the Laplace transform of an n^{th} derivative is

$$\mathcal{L}\{f^n(t)\} = \mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

Examples

$$\mathcal{L}\{y(t)\} = Y(s)$$
$$\mathcal{L}\{\dot{y}(t)\} = sY(s) - y(0)$$

Solve the differential equation $\dot{y}(t) - y(t) = 3$ for $y(0) = 1$;

Solution: Given,

$$\dot{y}(t) - y(t) = 3$$

$$\Rightarrow \mathcal{L}\{\dot{y}(t)\} - \mathcal{L}\{y(t)\} = \mathcal{L}\{3\}$$

$$\Rightarrow s\,Y(s) - y(0) - Y(s) = \frac{3}{s}$$

$$\Rightarrow (s-1)\,Y(s) - 1 = \frac{3}{s}$$

$$\Rightarrow (s-1)\,Y(s) = \frac{3}{s} + 1$$

$$\Rightarrow Y(s) = \frac{(s+3)}{s\,(s-1)}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = -\mathcal{L}^{-1}\left\{\frac{3}{s}\right\} + 4\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$\Rightarrow y(t) = -3 + 4\,e^{t}$$

Ans.

Now,

$$\frac{(s+3)}{s\,(s-1)} = \frac{A}{s} + \frac{B}{s-1}$$

$$A = \left[\frac{s+3}{s-1}\right]_{s=0} = \frac{(0+3)}{0-1} = -3$$

$$B = \left[\frac{s+3}{s}\right]_{s=1} = \frac{(1+3)}{1} = 4$$

So, $\frac{(s+3)}{s\,(s-1)} = -\frac{3}{s} + \frac{4}{s-1}$

Exercise

Apply Laplace transform to solve the following ordinary differential equations and hence justify your answer, where $\dot{y} \equiv \frac{dy(t)}{dt}$ and $\ddot{y} \equiv \frac{d^2y(t)}{dt^2}$:

1. $\dot{y}(t) = 3;\quad y(0) = 2.$
2. $\dot{y}(t) = 4t;\quad y(0) = 1.$
3. $\dot{y}(t) = 2t - 1;\quad y(0) = 3.$
4. $\dot{y}(t) = t^2;\quad y(0) = 4.$
5. $\dot{y}(t) = e^{2t};\quad y(0) = 2.$
6. $\dot{y}(t) + y(t) = 2\,;\ y(0) = 0.$
7. $\ddot{y}(t) = 5\,;\quad y(0) = 1, \dot{y}(0) = 2.$
8. $\ddot{y}(t) - 2\,\dot{y}(t) = \cos t\,;\quad y(0) = 0, \dot{y}(0) = 1.$
9. $\ddot{y}(t) + 3\,\dot{y}(t) - y(t) = e^t;\quad y(0) = \dot{y}(0) = 0.$
10. $\ddot{y}(t) - 7\dot{y}(t) + 12y(t) = 0, y(0) = 2, \dot{y}(0) = 1.$
11. $\ddot{y}(t) + y(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}, y(0) = 0, \dot{y}(0) = 0.$

Learning Outcomes

After completing this lecture student will learn solving differential equation using Laplace transformation.

Sample MCQ

For $\dot{y}(t) = 3t\ ;\quad y(0) = 2$ answer the following questions:

1. What is the Laplace transformation of given differential equation?

(a) $sY(s) - y(0) = \frac{3}{s^2}$ (b) $sY(s) - 2$ (c) Only a (d) Both a and b

2. For the given DE $Y(s)$ =

(a) $\frac{1}{s^3} + \frac{2}{s}$ (a) $\frac{3}{s^3} + \frac{1}{s}$ (a) $\frac{3}{s^3} + \frac{2}{s}$ (a) $\frac{3}{s^3} - \frac{2}{s}$

3. What is the Inverse Laplace transformation of $Y(s)$ for given differential equation?

(a) $t^2 + 2$ (b) $\frac{3}{2}\,t^2 - 2$ (c) $\frac{3}{2}\,t^2 + 2$ (d) $-\frac{3}{2}\,t^2 + 2$