# Complex Variable, Laplace & Z- transformation

## Lecture 05

Inverse Laplace Transform using partial fraction and inverse of Unit Step function

# This Lecture Covers-

1. Inverse Laplace transformation using partial fraction with type unrepeated, repeated and complex or irrational factors.

2. Inverse Laplace transformation associated with unit step function.

# Inverse Laplace Transformation Using Partial Fraction

## Example on type Unrepeated Factors

**Example: 01**

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 5s + 6}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s-2)}\right\}$$

Let, $\frac{1}{(s-3)(s-2)} \equiv \frac{A}{s-3} + \frac{B}{s-2}$

$\Rightarrow 1 = A(s-2) + B(s-3)$

$If\ s = 2, B = -1$ and $if\ s = 3, A = 1$

Now,

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s-2)}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{(s-3)} - \frac{1}{(s-2)}\right\}$$
$$= e^{3t} - e^{2t}.$$

To Practice-

1. $F(s) = \frac{s+1}{s\,(s-2)\,(s+3)}$ ,

2. $F(s) = \frac{6}{(s+2)\,(s-4)}$ ,

3. $F(s) = \frac{6s - 17}{s^2 - 5s + 6}$.

# Inverse Laplace Transformation Using Partial Fraction

## Example on type Repeated Factors

**Example: 01**

$$\mathcal{L}^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$$

Let, $\frac{4s+5}{(s-1)^2(s+2)} \equiv \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$

$4s+5 \equiv A(s-1)(s+2) + B(s+2) + c(s-1)^2$

Equating coefficients of $s^2$,

$$A + C = 0,$$

Equating coefficients of $s$,

$$A + B - 2C = 4$$

Equating constant term only,

$$-2A + 2B + C = 5$$

Solving the equations, we get

$$A = \frac{1}{3}, \quad B = 3 \text{ and } C = -\frac{1}{3}.$$

Now,

$$\mathcal{L}^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1/3}{s-1} + \frac{3}{(s-1)^2} - \frac{1/3}{s+2}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{1}{3}e^t + 3e^t\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{3}e^{-2t}$$

$$= \frac{1}{3}e^t + 3t\,e^t - \frac{1}{3}e^{-2t}.$$

**To Practice-**

1. $F(s) = \frac{s}{(s+1)^2}$ ,
2. $F(s) = \frac{7s^2+14s-9}{(s-1)^2(s-2)}$ .

# Inverse Laplace Transformation Using Partial Fraction

## Example on type of irreducible Factors

**Example: 01**

$$\mathcal{L}^{-1}\left\{\frac{3s^2+13s+26}{s(s^2+4s+13)}\right\}$$

Let, $\frac{3s^2+13s+26}{s(s^2+4s+13)} \equiv \frac{A}{s} + \frac{Bs+C}{s^2+4s+13}$

$$3s^2+13s+26$$
$$= A(s^2+4s+13) + (Bs+C)s$$

Equating coefficients of $s^2$,

$A + B = 3,$

Equating coefficients of $s$

$4A + C = 13$

Equating constant term,

$13A = 26$

Solving the equations, we get

$A = 2,\quad B = 1 \quad$ and $\quad C = 5$ .

Now,

$$\mathcal{L}^{-1}\left\{\frac{3s^2+13s+26}{s(s^2+4s+13)}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{2}{s} + \frac{s+5}{(s+2)^2+3^2}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{2}{s} + \frac{s+2}{(s+2)^2+3^2} + \frac{3}{(s+2)^2+3^2}\right\}$$
$$= 2 + e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + e^{-2t}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= 2 + e^{-2t}\cos 3t + e^{-2t}\sin 3t$$

## To Practice-

1. $F(s) = \frac{20}{(s^2+4s+1)\,(s+1)}$ ,

2. $F(s) = \frac{s}{(s^2+4)\,(s-1)}$ .

# Inverse Laplace Transformation Associated with Unit Step Function

## Formula

- $\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u(t-a),$
- $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a).$

**Example 1:**
Find and sketch $f(t)$,
where $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\}.$

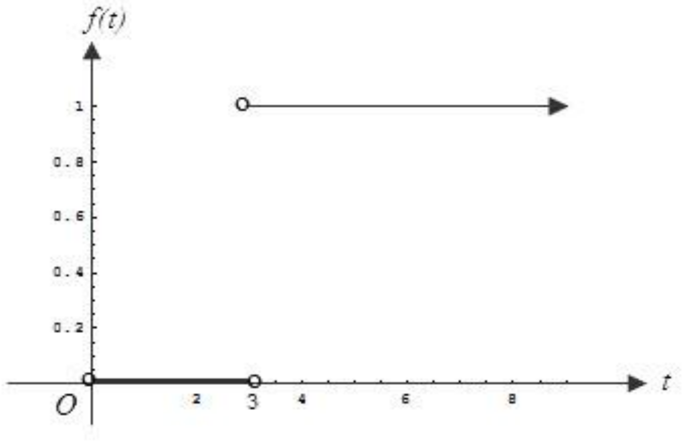
**Solution:** we know that

$$\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = u(t-a) = u_a(t)$$

**So,**

$$f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\}$$

$$= u(t-3) = u_3(t) = \begin{cases} 0, & t < 3 \\ 1, & t > 3 \end{cases}.$$



# Inverse Laplace Transformation Associated with Unit Step Function

**Example 2:**

Find and sketch $f(t)$, where $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\}$.

**Solution:**

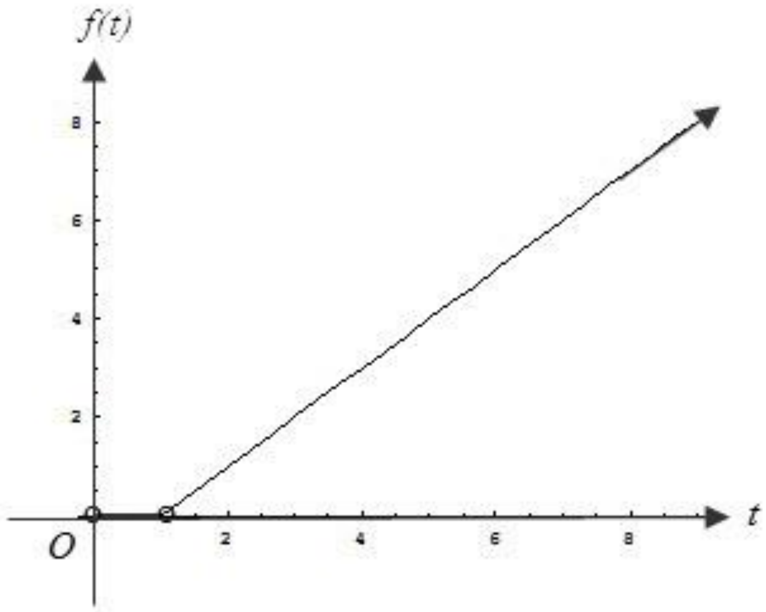
Let $\quad F(s) = \frac{1}{s^2} \quad$ and $\quad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t = f(t).$

We know that,

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t)$$

So, $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\} = f(t-1)u_1(t) = (t-1)u_1(t)$

$$= \begin{cases} 0, & t < 1 \\ t-1, & t > 1 \end{cases}.$$



# Inverse Laplace Transformation Associated with Unit Step Function

**Example 3:**

Find and sketch $f(t)$, where $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$.

**Solution:**

Let $F(s) = \frac{1}{s^2+1}$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = f(t)$.

We know that,

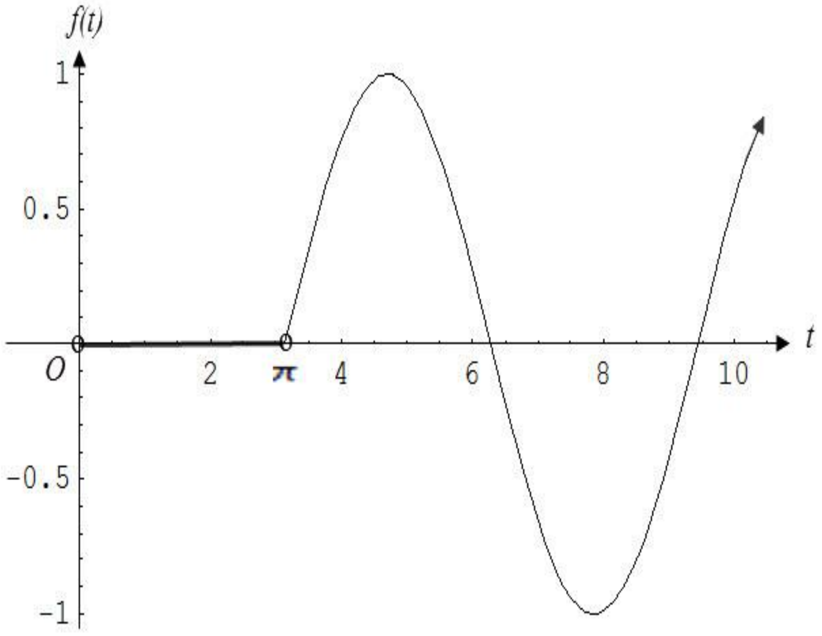
$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t)$$

So, $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = f(t-\pi)u_\pi(t)$

$$= \sin(t-\pi)\,u_\pi(t)$$

$$= \begin{cases} 0, & t < \pi \\ -\sin(\pi - t), & t > \pi \end{cases}$$

$$= \begin{cases} 0, & t < \pi \\ -\sin\, t, & t > \pi \end{cases}.$$



# Inverse Laplace Transformation Associated with Unit Step Function

**To Practice**

1. $F(s) = 3\left(\frac{e^{-5s}}{s}\right)$

2. $F(s) = 4\left(\frac{e^{-3s}}{s^2}\right)$

3. $F(s) = \frac{se^{-\pi s}}{s^2+25}$

4. $F(s) = \frac{2\left(e^{-3s}-3e^{-4s}\right)}{s}$

5. $F(s) = \frac{5\left(e^{-\pi s}+e^{-2\pi s}\right)}{s^2+25}$

## Learning Outcomes

After completing this lecture you will know how to evaluate inverse Laplace transformation using partial fraction and also inverse Laplace transformation associated with unit step function.

# Sample MCQ

1. $\mathcal{L}^{-1}\left\{\frac{6}{(s+2)\,(s-4)}\right\} = ?$

(a) $e^{t} - e^{-2t}$ (b) $e^{4t} - e^{-2t}$ (c) $e^{4t} - e^{2t}$ (d) $e^{3t} - e^{-2t}$

2. $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\} = ?$

(a) $e^{-t} - t\,e^{-t}$ (b) $e^{t} - t\,e^{-t}$ (c) $e^{-t} - e^{-t}$ (d) $e^{-t} - t\,e^{t}$

3. $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)\,(s-1)}\right\} = ?$

(a) $\frac{2}{5}\sin 2t - \frac{1}{5}\cos 2t + \frac{1}{5}\,e^{t}$ (b) $\frac{2}{3}\sin 2t - \frac{1}{5}\cos 2t + \frac{1}{5}\,e^{t}$

(c) $\sin 2t - \frac{1}{5}\cos 2t + \frac{1}{5}\,e^{t}$ (d) $\frac{2}{5}\sin 2t + \frac{1}{5}\cos 2t + \frac{1}{5}\,e^{t}$