



COMPLEX VARIABLE

Lecture 10

Complex roots

OBJECTIVE

- Obtaining complex roots of *polynomial equations* using De Moivre's theorem
- Discussion and sketch of complex inequality.

De`Moivre's Theorem:

If $z = r(\cos\theta + i\sin\theta)$ and n is positive integer, then

$$z^n = r^n(\cos n\theta + i \sin n\theta) = (re^{i\theta})^n = r^n e^{in\theta}$$

- To find the n th power of a complex number, take the n th power of the absolute value or length and multiply the argument by n .

Problem: Find all values of z for which $z^3 + 2 - i2\sqrt{3} = 0$ and also locate these values in the complex plane.

Solution: Given, $z^3 + 2 - i2\sqrt{3} = 0$.

Here the numbers of roots are 3.

$$z^3 + 2 - i2\sqrt{3} = 0$$

$$\Rightarrow z = (-2 + i2\sqrt{3})^{\frac{1}{3}}$$

$$\Rightarrow z = (4 e^{i\frac{2\pi}{3}})^{\frac{1}{3}}$$

$$\Rightarrow z = (2^2 e^{i(\frac{2\pi}{3} + 2n\pi)})^{\frac{1}{3}} [\because \theta = \theta + 2n\pi]$$

$$\Rightarrow z_n = 2^{\frac{2}{3}} e^{i(\frac{2\pi + 6n\pi}{9})}; n = 0, 1, 2$$

$$\text{when } n = 0, z_0 = 2^{\frac{2}{3}} e^{i(\frac{2\pi}{9})}$$

$$\text{when } n = 1, z_1 = 2^{\frac{2}{3}} e^{i(\frac{8\pi}{9})}$$

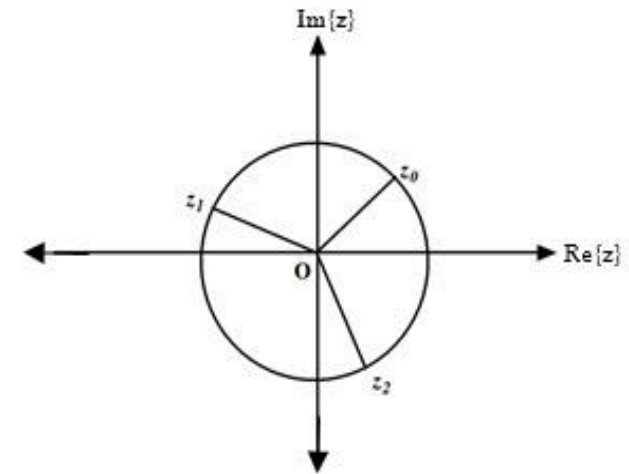
$$\text{when } n = 2, z_2 = 2^{\frac{2}{3}} e^{i(\frac{14\pi}{9})}$$

$$r = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$$

$$\tan \theta = \left(\frac{2\sqrt{3}}{-2} \right) = -\sqrt{3}$$

$$= -\tan \frac{\pi}{3} = \tan(\pi - \frac{\pi}{3})$$

$$\therefore \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$



The distance of each root from the origin is same as $2^{\frac{2}{3}}$ and the angular distance $\frac{2\pi}{3}$ of two consecutive roots are same.

Problem: Find all values of z for which $z^4 - 81 = 0$ and also locate these values in the complex plane.

Solution: Given, $z^4 - 81 = 0$.

Here the numbers of roots are 4.

$$z^4 - 81 = 0$$

$$\Rightarrow z = (81)^{\frac{1}{4}}$$

$$\Rightarrow z = (81 e^{i0})^{\frac{1}{4}}$$

$$\Rightarrow z = (3^4 e^{i(0+2n\pi)})^{\frac{1}{4}} [\because \theta = \theta + 2n\pi]$$

$$\Rightarrow z_n = 3 e^{i\left(\frac{2n\pi}{4}\right)}; n = 0, 1, 2, 3$$

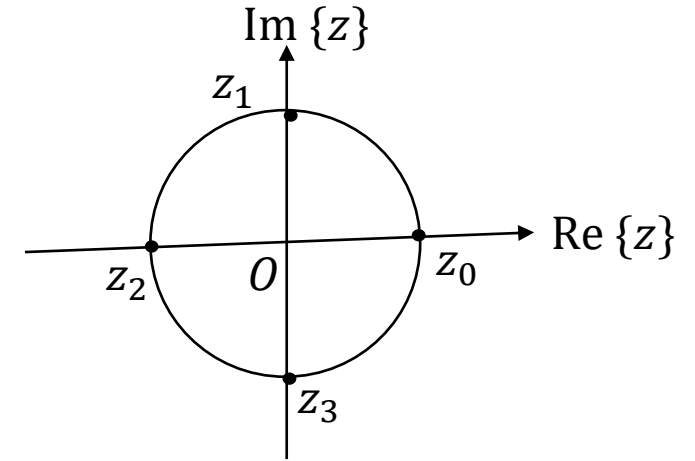
$$\text{when } n = 0, z_0 = 3 e^{i0} = 3$$

$$\text{when } n = 1, z_1 = 3 e^{i\frac{\pi}{2}}$$

$$\text{when } n = 2, z_2 = 3 e^{i\pi}$$

$$\text{when } n = 3, z_3 = 3 e^{i\frac{3\pi}{2}}$$

$$\begin{aligned} r &= \sqrt{(81)^2} = 81 \\ \tan \theta &= 0 \\ \therefore \theta &= 0 \end{aligned}$$



The distance of each root from the origin is same as 3 and the angular distance $\frac{\pi}{2}$ of two consecutive roots are same.

Problem: Describe and graph the locus represented by $1 < |z + i| \leq 2$.

Solution: Given, $1 < |z + i| \leq 2$.

$$\Rightarrow 1 < |x + iy + i| \leq 2$$

$$\Rightarrow 1 < |x + i(y + 1)| \leq 2$$

$$\Rightarrow 1 < \sqrt{x^2 + (y + 1)^2} \leq 2$$

$$\Rightarrow 1 < (x - 0)^2 + (y - (-1))^2 \leq 2^2$$

\therefore Given inequality represents the region between two concentric circles of radii 1 and 2 with center $(0, -1)$.

Problem: Describe and graph the locus represented by $|z + 2 - 3i| > 3$.

Solution: Given, $|z + 2 - 3i| > 3$.

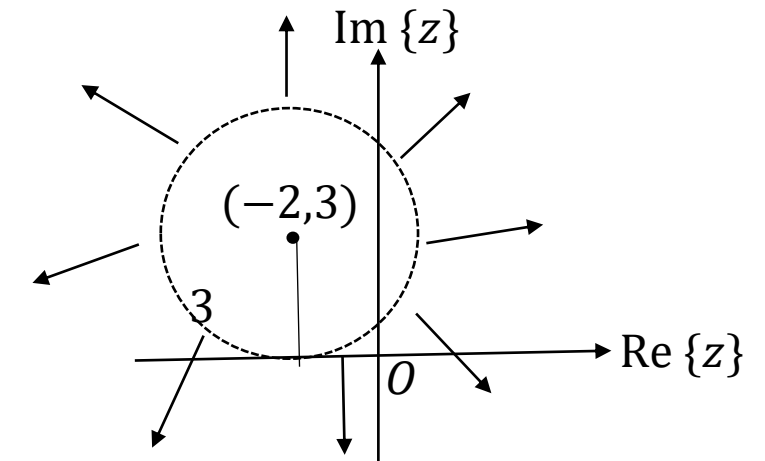
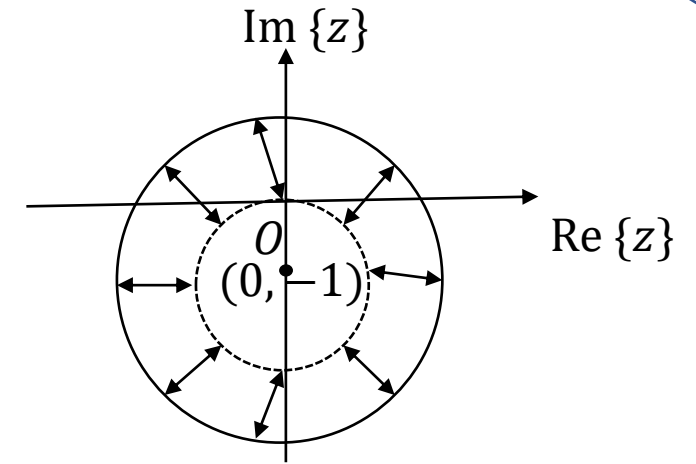
$$\Rightarrow |x + iy + 2 - 3i| > 3$$

$$\Rightarrow |(x + 2) + i(y - 3)| > 3$$

$$\Rightarrow \sqrt{(x + 2)^2 + (y - 3)^2} > 3$$

$$\Rightarrow (x + 2)^2 + (y - 3)^2 > 3^2$$

\therefore Given inequality represents the region outside the circle of radius 3 with center $(-2, 3)$.



Problem: Describe and graph the locus represented by $\text{Re}\{z^2\} \leq 4$

Solution: Given, $\text{Re}\{z^2\} \leq 4$

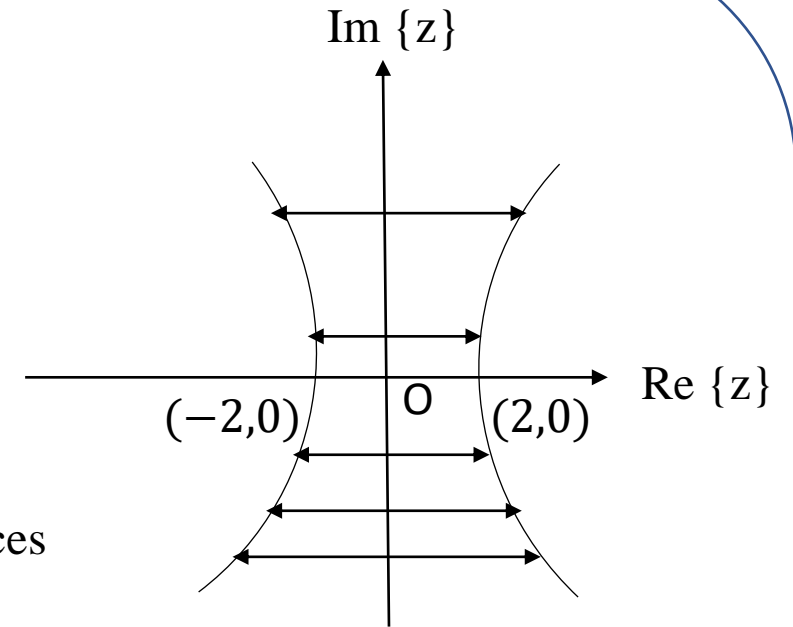
$$\Rightarrow \text{Re}\{(x + iy)^2\} \leq 4$$

$$\Rightarrow \text{Re}\{x^2 + i 2xy - y^2\} \leq 4$$

$$\Rightarrow x^2 - y^2 \leq 2^2$$

$$\Rightarrow \frac{x^2}{2^2} - \frac{y^2}{2^2} \leq 1$$

\therefore Given inequality represents the region inside the hyperbola with vertices $(\pm 2, 0)$ and center $(0, 0)$.



Problem: Describe and graph the locus represented by $\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{4}$.

Solution: Given, $\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{4}$.

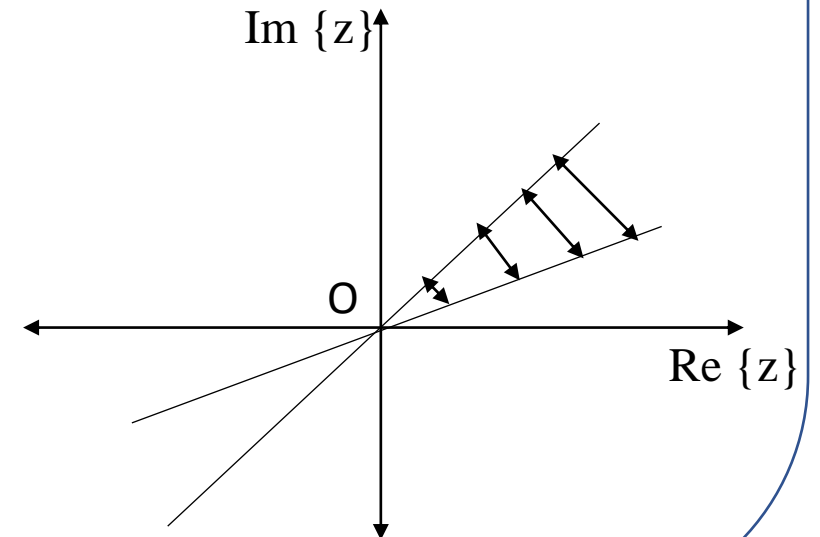
$$\Rightarrow \frac{\pi}{6} \leq \tan^{-1}\left(\frac{y}{x}\right) \leq \frac{\pi}{4}$$

$$\Rightarrow \tan \frac{\pi}{6} \leq \frac{y}{x} \leq \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{1}{\sqrt{3}} \leq \frac{y}{x} \leq 1$$

$$\Rightarrow \frac{1}{\sqrt{3}}x \leq y \leq x$$

\therefore Given inequality represents the region between the lines $y = \frac{1}{\sqrt{3}}x$ and $y = x$ in 1st quadrant.



Exercise

1. Find all values of z for the following equations and also locate these values in the complex plane:

(a) $z^2 + 9 = 0$.

(b) $z^3 - \sqrt{3} - i = 0$.

(c) $z^3 = -i$.

(d) $z^4 - 1 = 0$.

2. Describe and graph the locus represented by each of the followings:

(a) $|z + 2i| > 4$.

(b) $1 < |z - 2 + i| \leq 3$.

(c) $\text{Im}\{z^2\} = 9$.

(d) $|z - 1| \leq 1$.

(e) $\text{Re}\{z^2\} < 4$.

(f) $\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{3}$.

MULTIPLE CHOICE QUESTION

1. The complex equation $z^3 - i + 1 = 0$ has how many roots?
(a) 2 (b) 3 (c) 1 (d) None
2. The angular distance of each individual root from the next root is
(a) same (b) different (c) both (d) None
3. The roots of the complex equation $z^2 - i + 1 = 0$ are
(a) $\sqrt{2}e^{i\frac{3\pi}{4}}, \sqrt{2}e^{i\frac{11\pi}{4}}$ (b) $\sqrt{2}e^{i\frac{\pi}{4}}, \sqrt{2}e^{i\frac{3\pi}{4}}$ (c) $\sqrt{2}e^{i\frac{3\pi}{4}}, \sqrt{2}e^{i\frac{7\pi}{4}}$ (d) None