Complex Variable, Laplace & Z-transformation

Lecture 03

LT of Unit Step Function

This Lecture Covers -

1. Definition of Unit Step Function.
2. Rectangular Pulse.
3. Laplace Transformation of Unit Step Function.
4. Examples & Exercises on Laplace Transformation of Unit Step Function.

Definition of Unit Step function

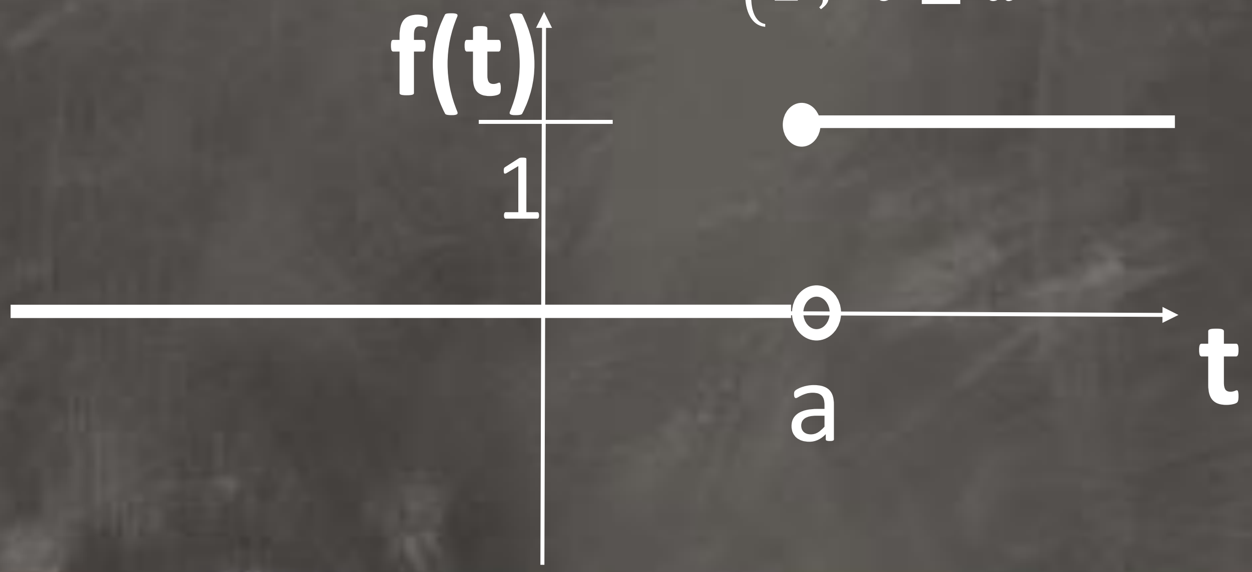
The Unit Step or Heaviside's function is defined as follows:

$$f(t) = u(t) = \begin{cases} 0\,; & t < 0 \\ 1\,; & t \geq 0 \end{cases}$$

Shifted Unit Step Function

The Unit Step or Heaviside's function is defined as follows:

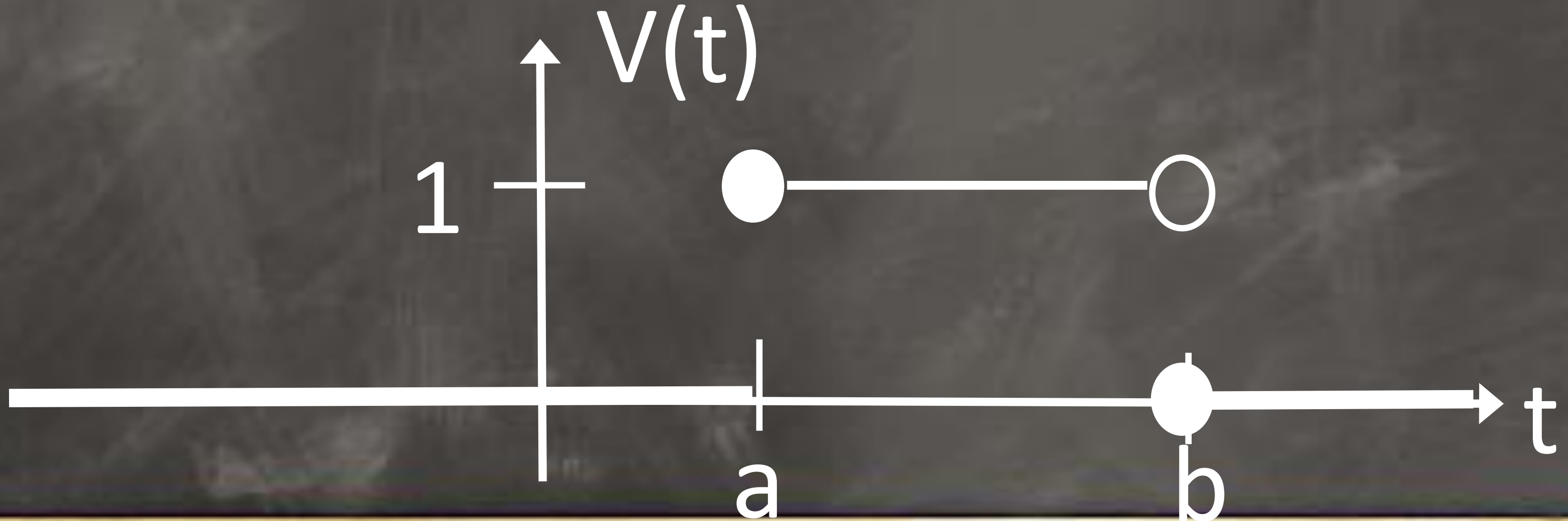
$$u_a(t) = u(t-a) = \begin{cases} 0\,; & t < a \\ 1\,; & t \geq a \end{cases}$$



Rectangular Pulse

A common situation in a circuit is for a voltage v(t), to be applied at a particular time (say $t = a$) and removed later at $t = b$ (say). We write such a situation using unit step function as:

$$v(t) = u(t-a) - u(t-b) = \begin{cases} 1 \, ; a \leq t < b \\ 0 \, ; \text{otherwise} \end{cases}$$



Laplace Transformation of Unit Step Function and Examples

Formulae:

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$\mathcal{L}\{f(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$$

Example 1:

$$\mathcal{L}\{t^2\,u(t-3)\}$$
$$= e^{-3s}\,\mathcal{L}\{f(t+3)\}$$
$$= e^{-3s}\,\mathcal{L}\{(t+3)^2\}$$
$$= e^{-3s}\,\mathcal{L}\{t^2+6t+9\}$$
$$= e^{-3s}\,[\,\mathcal{L}\{t^2\}+6\,\mathcal{L}\{t\}+\mathcal{L}\{9\}]$$
$$= e^{-3s}\,\left[\frac{2!}{s^3}+6\frac{1}{s^2}+9\frac{1}{s}\right].$$

Ans.

Laplace Transformation of Unit Step Function and Examples

$$\mathcal{L}\{f(t)\,u(t-a)\}$$
$$= e^{-as}\,\mathcal{L}\{f(t+a)\}$$

Example 1:

$$\mathcal{L}\{\sin t\;\,u(t)\}$$
$$= e^{-0\times s}\,\mathcal{L}\{f(t+0)\}$$
$$= \mathcal{L}\{\sin t\}$$
$$= \frac{1}{s^2+1}$$

Ans.

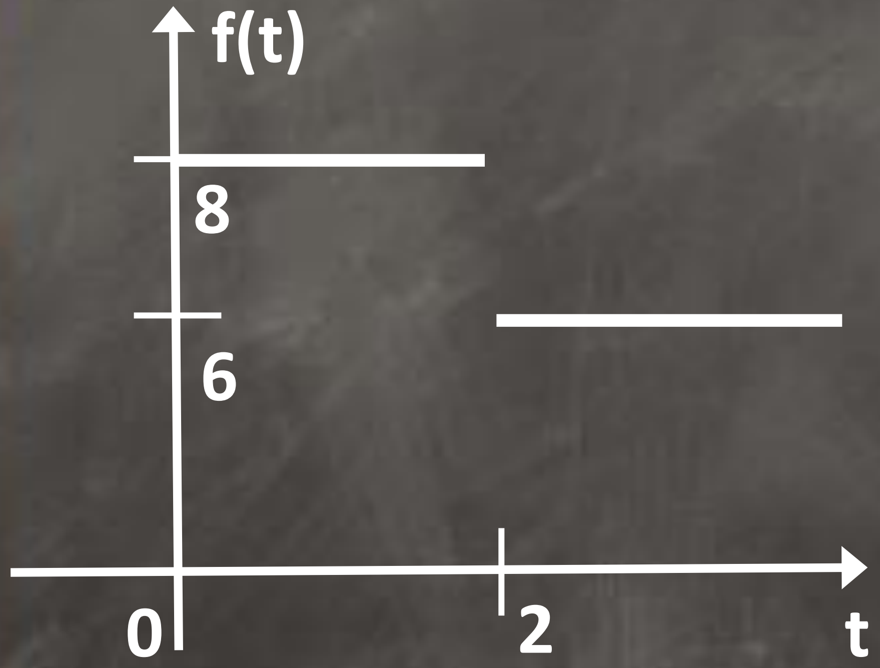
Example 2:

$$\mathcal{L}\{e^{-2t}\,u_\pi(t)\}$$
$$= \mathcal{L}\{e^{-2t}\,u(t-\pi)\}$$
$$= e^{-\pi s}\,\mathcal{L}\{f(t+\pi)\}$$
$$= e^{-\pi s}\,\mathcal{L}\left\{e^{-2(t+\pi)}\right\}$$
$$= e^{-\pi s}\,[\,\mathcal{L}\{e^{-2t}\,e^{-2\pi}\}]$$
$$= e^{-\pi s}\;\,e^{-2\pi}\,\mathcal{L}\{e^{-2t}\}$$
$$= e^{-\pi(s+2)}\,\frac{1}{s+2}.$$

Ans.

Laplace Transformation of Unit Step Function and Examples

Example 3. Given $f(t) = \begin{cases} 8\,; 0 < t < 2 \\ 6\,; \quad\quad t > 2 \end{cases}$, Sketch the function $f(t)$, also express $f(t)$ in terms of Unit step function and hence find it's Laplace transformation.



$$
\begin{aligned}
f(t) &= 8[u(t) - u(t-2)] + 6\,u(t-2) \\
&= 8\,u(t) - 8\,u(t-2) + 6\,u(t-2) \\
&= 8\,u(t) \;\; - 2\,u(t-2)
\end{aligned}
$$

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$
\begin{aligned}
F(s) &= 8\,\mathcal{L}\{u(t)\} - 2\,\mathcal{L}\{u(t-2)\} \\
&= 8\frac{e^{0\times s}}{s} - 2\frac{e^{-2s}}{s} \\
&= 8\frac{1}{s} - 2\frac{e^{-2s}}{s}.
\end{aligned}
$$

Exercise Set on Laplace Transformation of Unit Step function

Sketch the following function and find their Laplace Transformations:

1. $f(t) = t\,u(t-1)$, $\boxed{f(t) = \begin{cases} 0 & t < 1 \\ t & t \geq 1 \end{cases}}$

2. $f(t) = (t-1)\,u(t-3)$,

3. $f(t) = (t+2)^2\,u(t-1)$,

4. $f(t) = e^{-2t}\,u(t-3)$,

5. $f(t) = 4\cos t\;u(t-\pi)$.

Sketch the following function, also express $f(t)$ in terms of unit step function and find it's Laplace Transformation:

6. $f(t) = \begin{cases} t\,; 0 < t < 1 \\ 2\,; \quad t > 1 \end{cases}$ $\boxed{\begin{aligned} &f(t) \\ &= u(t) - u(t-1) \\ &+ 2u(t-1) \end{aligned}}$

7. $f(t) = \begin{cases} t^2\,;\ \ 0 \leq t < 1 \\ t - 3\,; t \geq 1 \end{cases}$

Learning Outcomes:

- ❑ In engineering applications:
  - ✓ we frequently encounter functions whose values change abruptly at specified values of time *t*.
  - ✓ One common example is when a voltage is switched on or off in an electrical circuit at a specified value of time *t.*
  - ✓ The switching process can be described mathematically by the function called the **Unit Step Function.**
- ❑ In this lecture we overviewed
  - ✓ The general concept of unit step function and also
  - ✓ Discussed the process of Laplace Transformation of unit step function.

Sample MCQ

1. If $f(t) = \begin{cases} 1-t & ; 0 < t < 1 \\ 0\ ; & t > 1 \end{cases}$ then what is $F(s)$?

(a) $\frac{2s+e^{-s}-1}{s^2}$ (b) $\frac{s+e^{-s}-1}{s^2}$ (c) $\frac{s-e^{-s}-1}{s^2}$ (d) $\frac{s^2+e^{-s}-1}{s^2}$

2. If $V(t) = \begin{cases} 0, & t < 3 \\ 2t+8, & 3 < t < 5 \\ 0, & t > 5 \end{cases}$. then which of the following is corresponding unit step function?

(a) $(2t+8) \cdot [u(t-3) - u(t-5)]$ (b) $[u(t-3) - u(t-5)]$

(c) $[u(t+3) - u(t-5)]$ (d) $[u(t+3) - u(t+5)]$

3 . $\mathcal{L}\{t^2\, u(t-3)\} = ?$

(a) $e^{-3s}\left[\frac{2}{s^3} + \frac{1}{s^2} + 9\frac{1}{s}\right]$ (b) $e^{-3s}\left[\frac{2}{s^3} + 6\frac{1}{s^2} + 9\frac{1}{s}\right]$ (c) $e^{-3s}\left[\frac{2}{s^3} + 6\frac{1}{s^2} + \frac{1}{s}\right]$ (d) $e^{-3s}\left[\frac{1}{s^3} + 6\frac{1}{s^2} + 9\frac{1}{s}\right]$