

COMPLEX VARIABLE

LECTURE 9

Surveying the algebraic and geometric structure of the complex number system through

OBJECTIVE:

- Complex number
- Graphical representation
- Fundamental operations
- Conjugates
- Absolute value/modulus
- Power of imaginary unit
- Polar form and argument

Complex Numbers:

$z = a + ib$; where a and b are any real number and i is the imaginary unit.

- $i = \sqrt{-1}$ and $i^2 = -1$.
- If $a = 0$, the number $z = ib$ is called **purely imaginary**,
- If $b = 0$, the number $z = a$ is called **real**.
- Real part of z is: **$\text{Re}\{z\} = a$**
- Imaginary part of z is: **$\text{Im}\{z\} = b$** .

Example: For $z = 2 - 4i$, $\text{Re}\{z\} = 2$ and $\text{Im}\{z\} = -4$.

Example: For $z = \frac{-1+2i}{3}$, $\text{Re}\{z\} = -\frac{1}{3}$ and $\text{Im}\{z\} = \frac{2}{3}$.

Conjugate:

- Conjugate of a complex number $z = a + ib$ is $\bar{z} = a - ib$.
- The geometric interpretation of a complex conjugate is the reflection along the real axis.

Example: If $z = 2 + 3i$ then conjugate of z

$$\bar{z} = 2 - 3i.$$

Example: If $z = -2 - i$ then conjugate of z

$$\bar{z} = -2 + i.$$

Example: If $z = -\frac{i}{3}$ then conjugate of z

$$\bar{z} = \frac{i}{3}.$$

Example: If $z = 5$ then conjugate of z

$$\bar{z} = 5.$$

Absolute value/Modulus:

- The distance from the origin to any complex number is the **absolute value** or **modulus**.
- Absolute value of a complex number $z = a + ib$ denoted by $\text{mod } z$ or $|z|$

$$\text{mod } z = |z| = \sqrt{a^2 + b^2}$$

Example: If $z = 4 - 3i$ then

$$\text{mod } z = |z| = \sqrt{(4)^2 + (-3)^2} = 5.$$

Some properties of conjugate:

$$1. \bar{\bar{z}} = z$$

$$2. \overline{z + w} = \bar{z} + \bar{w}$$

$$3. \overline{zw} = \bar{z}\bar{w}$$

$$4. \overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}, w \neq 0.$$

Some properties of modulus:

$$1. |z_1 \cdot z_2| = |z_1||z_2|$$

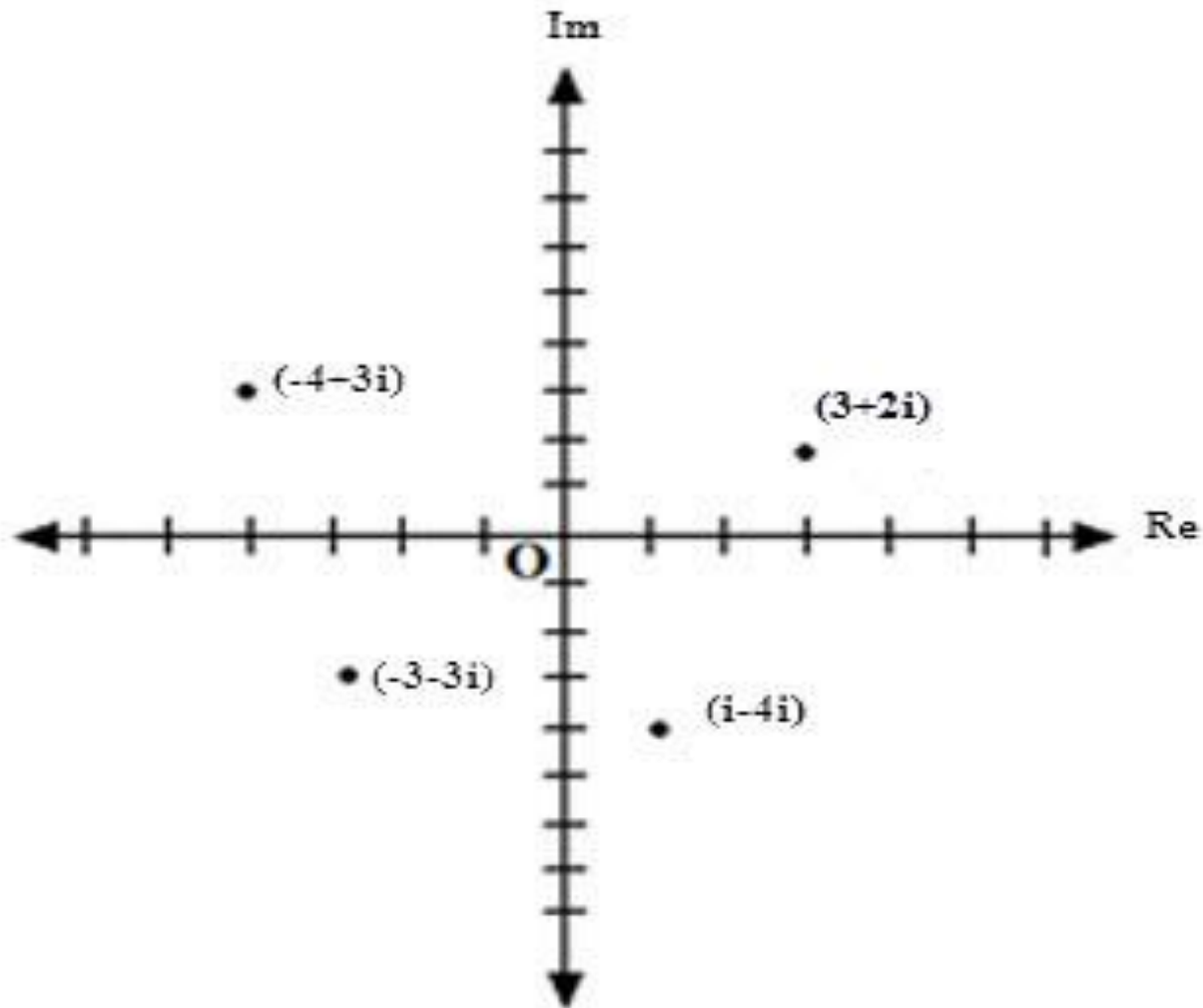
$$2. |z|^2 = z \cdot \bar{z}$$

$$3. \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Graphical Representation of Complex Number/ Argand Diagram:

- Mathematician Argand represented a complex number in a diagram known as **Argand diagram**.
- A complex number $z = a + ib$ can be represented as an ordered pair of real number (a, b) .
- A complex number can be represented by points in a xy plane which is called **complex plane/ Argand diagram**.
- The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis**.

Complex Numbers in complex plane:



Fundamental operations with complex number:

➤ Addition and Subtraction:

The **sum** and **difference** of complex numbers is defined by adding or subtracting the corresponding components where $a, b \in \mathbb{R}$ i.e.:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Example: Let $z_1 = (3 + i)$ and $z_2 = (1 - 7i)$

$$\therefore z_1 + z_2 = (3 + 1) + (1 - 7)i = 4 - 6i$$

$$z_1 - z_2 = (3 - 1) + (1 + 7)i = 2 + 8i.$$

➤ Product: **Product** of two complex numbers is also a complex number.

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + bi(c + di) \\ &= ac + adi + bci + bdi^2 = (ac - bd) + i(ad + bc).\end{aligned}$$

Example: Let $z_1 = (3 + i)$ and $z_2 = (1 - 7i)$.

$$\therefore z_1 * z_2 = (3 + i)(1 - 7i) = 3 - 21i + i - 7i^2 = 3 - 20i + 7 = 10 - 20i.$$

➤ Division:

For the division of two complex numbers, multiply both the numerator and denominator by the conjugate of the denominator.

$$\frac{(a+bi)}{(c+di)} = \frac{(a+bi)}{(c+di)} \cdot \frac{(c-di)}{(c-di)} = \frac{(ac+bd)}{(c^2+d^2)} + i \frac{(bc-ad)}{(c^2+d^2)}.$$

Problem: Express $\frac{-3+i}{7-3i}$ in the form $a + ib$.

$$\begin{aligned}\text{Solution: } \frac{-3+i}{7-3i} &= \frac{-3+i}{7-3i} * \frac{7+3i}{7+3i} = \frac{-21+7i-9i+3i^2}{7^2-(3i)^2} = \frac{-21-2i-3}{49+9} \\ &= \frac{-24-2i}{58} = -\frac{12}{29} - i \frac{1}{29}.\end{aligned}$$

Problem: Find $\text{Re}\{z\}$ and $\text{Im}\{z\}$ where $z = \frac{3-2i}{1-2i}$.

$$\text{Solution: Here, } z = \frac{3-2i}{1-2i} = \frac{3-2i}{1-2i} * \frac{1+2i}{1+2i} = \frac{3+6i-2i+4}{1+4} = \frac{7+4i}{5} = \frac{7}{5} + \frac{4}{5}i.$$

$$\therefore \text{Re}\{z\} = \frac{7}{5} \text{ and } \text{Im}\{z\} = \frac{4}{5}.$$

Power of imaginary unit i :

Power of imaginary unit i :

$$i^0 = 1, \quad i^1 = i, \quad i^2 = -1, \quad i^3 = i^2 \cdot i = (-1) \cdot i = -i,$$

$$i^4 = i^3 \cdot i = (-i) \cdot i = 1, \quad i^5 = i^4 \cdot i = i; \quad i^6 = i^5 \cdot i = -1; \quad i^7 = i^6 \cdot i = -i.$$

By induction, for any positive integer n :

$$i^{4n} = 1; \quad i^{4n+1} = i; \quad i^{4n+2} = -1; \quad i^{4n+3} = -i.$$

If n is a negative integer, then

$$i^{-n} = (i^{-1})^n = \left(\frac{1}{i}\right)^n = \left(\frac{i}{i \cdot i}\right)^n = (-i)^n.$$

Problem: Evaluate $i^{105} + i^{23} + i^{20} - i^{34}$.

Solution:

$$\begin{aligned} & i^{105} + i^{23} + i^{20} - i^{34} \\ &= i^{4 \cdot 26 + 1} + i^{4 \cdot 5 + 3} + i^{4 \cdot 5} - i^{4 \cdot 8 + 2} \\ &= i - i + 1 + 1 = 2. \end{aligned}$$

Problem: If $z_1 = 1 - i$, $z_2 = -2 - 3i$ and $z_3 = 2i$, then evaluate

$$|3z_1^2 + z_1\bar{z}_2 - 5z_3|.$$

Solution: $|3z_1^2 + z_1\bar{z}_2 - 5z_3| = |3(1 - i)^2 + (1 - i)(-2 + 3i) - 5 * 2i|$

$$= |3(1 - 2i - 1) + (-2 + 3i + 2i + 3) - 10i|$$

$$= |-6i + 1 + 5i - 10i| = |1 - 11i| = \sqrt{(1)^2 + (-11)^2} = \sqrt{122}.$$

Problem: Evaluate $\left|\frac{(1-i)^2}{1+i}\right|$.

Solution: $\left|\frac{(1-i)^2}{1+i}\right| = \left|\frac{1-2i-1}{1+i}\right| = \left|\frac{-2i}{1+i}\right| = \frac{\sqrt{(-2)^2}}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$

Problem: Evaluate $\operatorname{Re}\left\{\frac{(1-i)^2}{1+i}\right\}$.

Solution: $\operatorname{Re}\left\{\frac{(1-i)^2}{1+i}\right\} = \operatorname{Re}\left\{\frac{-2i}{1+i} \cdot \frac{1-i}{1+i}\right\} = \operatorname{Re}\left\{\frac{-2i+2i^2}{1^2-i^2}\right\}$

$$= \operatorname{Re}\left\{\frac{-2-2i}{2}\right\} = \operatorname{Re}\{-1 - i\} = -1.$$

Polar form of Complex Number and Argument:

- If P a point in the complex plane corresponding to the complex number (a, b) or, $a + ib$ then,

$$a = r \cos \theta \text{ and } b = r \sin \theta.$$

Where $r = |z| = \sqrt{a^2 + b^2}$ is the distance of (a, b) from the origin,

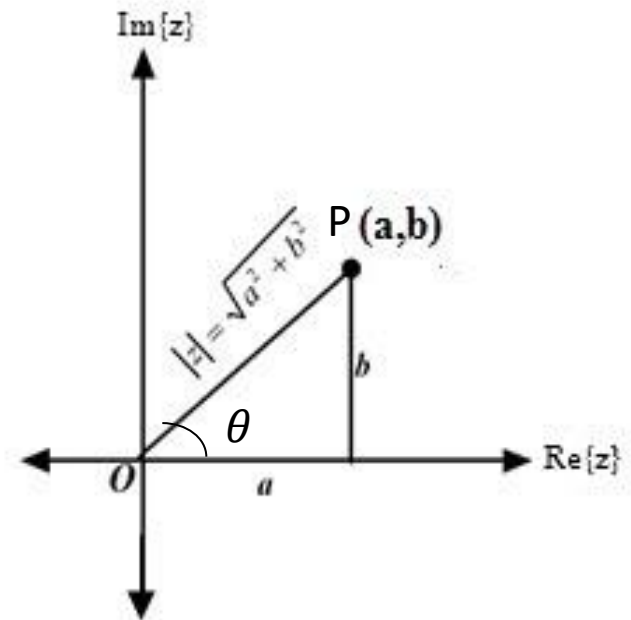
And θ is called the **amplitude or argument** of z which is the angle with real axis denoted by $\arg\{z\}$.

Hence, we can write z in polar form as:

$$z = r \cos \theta + i r \sin \theta$$

$$= r(\cos \theta + i \sin \theta)$$

$$\therefore z = r e^{i\theta} \text{ [Euler Formulae].}$$



Principal argument: The **principal value** of the argument (sometimes called the **principal argument**) is the unique value of the argument that is in the range $0 \leq \theta < 2\pi$ or, $-\pi < \theta \leq \pi$ and is denoted by $\text{Arg } z$.

$$\arg z = \text{Arg } z + 2n\pi, \quad (n = 0, \pm 1, \pm 2, \dots)$$

Some important properties of argument:

$$\triangleright \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$$\triangleright \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

$$\triangleright \arg(z^n) = n \arg(z).$$

Transformation of complex numbers:

Polar to rectangular: $z = re^{i\theta} \rightarrow z = a + ib$

$$a = r\cos\theta \text{ and } b = r\sin\theta$$

Rectangular to Polar: $z = a + ib \rightarrow z = re^{i\theta}$

$$r = \sqrt{a^2 + b^2} \text{ and } \tan\theta = \frac{b}{a}$$

$$\theta = \begin{cases} \tan^{-1}\left(\frac{b}{a}\right); & \text{if } a > 0, b \geq 0 \text{ (1st quadrant and on +ve real axis)} \\ \tan^{-1}\left(\frac{b}{a}\right) + \pi; & \text{if } a < 0 \text{ (2nd, 3rd quadrant and on -ve real axis)} \\ \tan^{-1}\left(\frac{b}{a}\right) + 2\pi; & \text{if } a > 0, b < 0 \text{ (4th quadrant)} \\ \frac{\pi}{2}; & \text{if } a = 0, b > 0 \text{ (on +ve imaginary axis)} \\ \frac{3\pi}{2}; & \text{if } a = 0, b < 0 \text{ (on -ve imaginary axis)} \\ \text{Undefined}; & \text{if } a = 0, b = 0 \end{cases}.$$

Problem: Find the rectangular form of $z = \sqrt{2}e^{i\frac{\pi}{4}}$.

Solution: Here $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

$$a = r \cos \theta = \sqrt{2} \cos \frac{\pi}{4} = 1$$

$$b = r \sin \theta = \sqrt{2} \sin \frac{\pi}{4} = 1$$

Hence $z = a + ib = 1 + i$.

Problem: Find the rectangular form of $z = 2e^{i\frac{5\pi}{6}}$.

Solution: Here $r = 2$ and $\theta = \frac{5\pi}{6}$

We know that, $a = r \cos \theta = 2 \cos\left(\frac{5\pi}{6}\right) = -2 \frac{\sqrt{3}}{2} = -\sqrt{3}$

$$\text{And } b = r \sin \theta = 2 \sin\left(\frac{5\pi}{6}\right) = 2 \frac{1}{2} = 1$$

Hence $z = a + ib = -\sqrt{3} + i$.

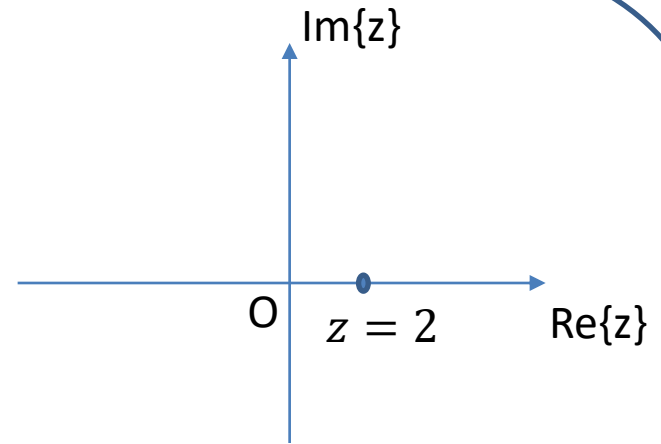
Problem: Find the polar form of $z = 2$.

Solution: Here $a = 2$ and $b = 0$

We know that, $r = \sqrt{a^2 + b^2} = \sqrt{2^2} = 2$

And $\tan \theta = \frac{b}{a} = \frac{0}{2} = 0$, $\theta = 0$

Hence, $z = re^{i\theta} = 2$.



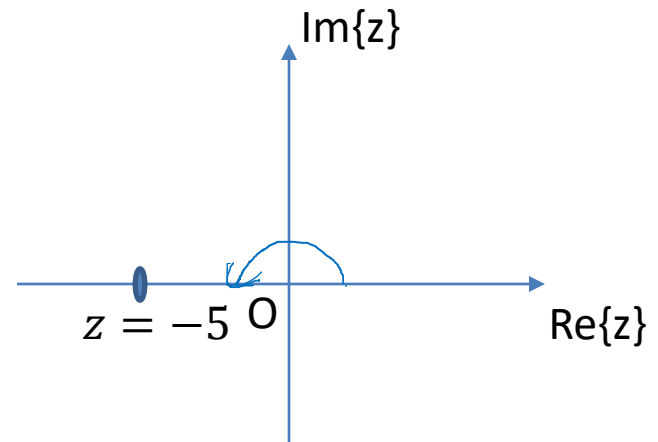
Problem: Find the polar form of $z = -5$.

Solution: Here $a = -5$ and $b = 0$

We know that, $r = \sqrt{a^2 + b^2} = \sqrt{(-5)^2} = 5$

And $\tan \theta = 0$, $\theta = \pi$

Hence, $z = re^{i\theta} = 5e^{i\pi}$.



Problem: Find the polar form of $z = 3i$.

Solution: Here $a = 0$ and $b = 3$

We know that, $r = \sqrt{a^2 + b^2} = \sqrt{3^2} = 3$

And $\theta = \frac{\pi}{2}$

Hence, $z = re^{i\theta} = 3e^{i\frac{\pi}{2}}$.

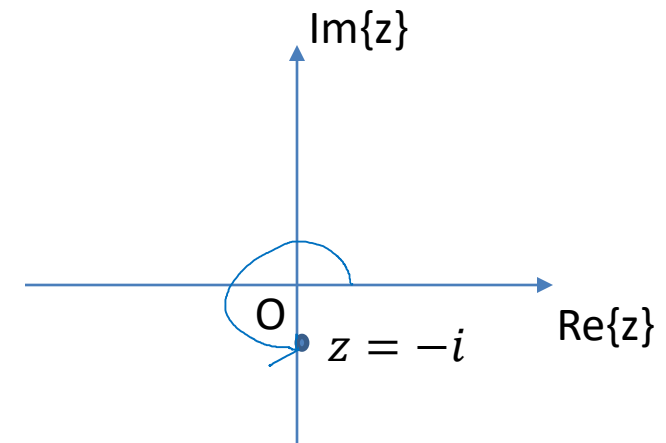
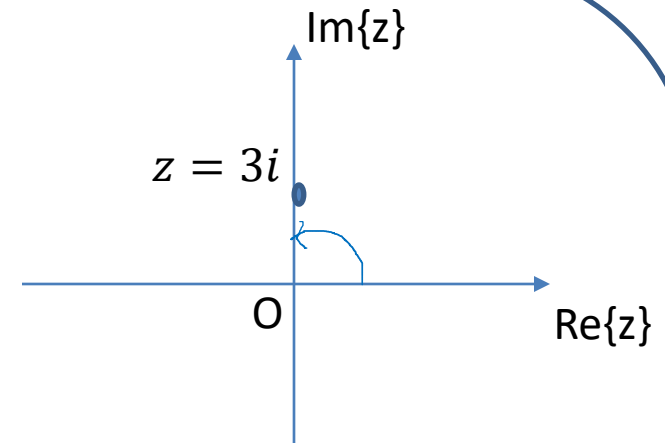
Problem: Find the polar form of $z = -i$.

Solution: Here $a = 0$ and $b = -1$

We know that, $r = \sqrt{a^2 + b^2} = \sqrt{(-1)^2} = 1$

And $\theta = \frac{3\pi}{2}$

Hence, $z = re^{i\theta} = e^{i\frac{3\pi}{2}}$.



Problem: Find the polar form of $z = \sqrt{3} + 3i$.

Solution: Here $a = \sqrt{3}$ and $b = 3$

We know that, $r = \sqrt{a^2 + b^2}$

$$= \sqrt{(\sqrt{3})^2 + 3^2} = 2\sqrt{3}$$

$$\text{And } \tan \theta = \frac{b}{a} = \frac{3}{\sqrt{3}} = \sqrt{3} = \tan \frac{\pi}{3}$$

$$\therefore \theta = \frac{\pi}{3}$$

$$\text{Hence, } z = re^{i\theta} = 2\sqrt{3}e^{i\frac{\pi}{3}}.$$

Problem: Find the polar form of $z = -1 + i$.

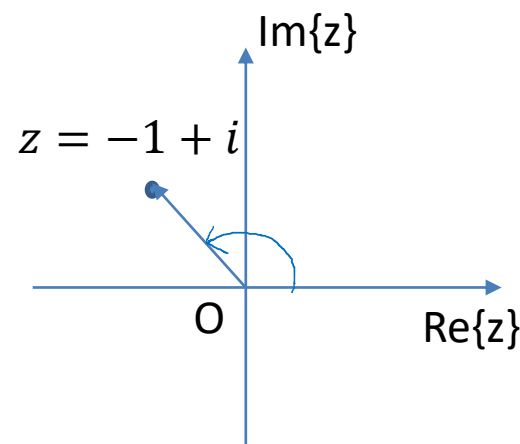
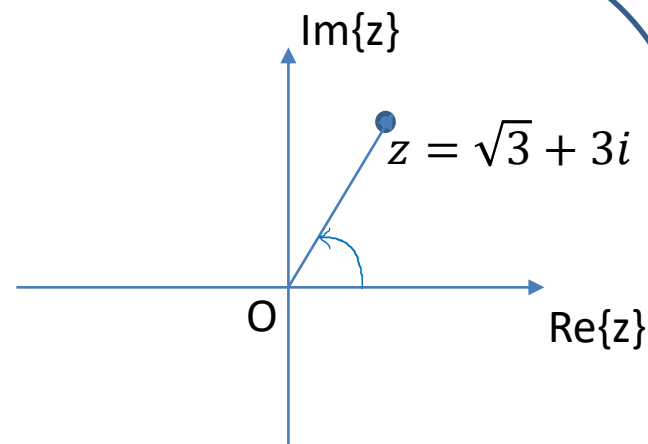
Solution: Here $a = -1$ and $b = 1$

$$\text{Now, } r = \sqrt{a^2 + b^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

$$\text{And } \tan \theta = -1 = -\tan \frac{\pi}{4} = \tan(\pi - \frac{\pi}{4})$$

$$\theta = \frac{3\pi}{4}$$

$$\text{Hence, } z = re^{i\theta} = \sqrt{2}e^{i\frac{3\pi}{4}}.$$



Problem: Find the polar form of $z = -2\sqrt{3} - 6i$.

Solution: Here $a = -2\sqrt{3}$ and $b = -6$

We know that, $r = \sqrt{a^2 + b^2}$

$$= \sqrt{(-2\sqrt{3})^2 + (-6)^2} = 4\sqrt{3}$$

$$\text{And } \tan \theta = \left(\frac{-6}{-2\sqrt{3}} \right) = \sqrt{3} = \tan \frac{\pi}{3} = \tan(\pi + \frac{\pi}{3})$$

$$\theta = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$$

Hence, $z = re^{i\theta} = 4\sqrt{3}e^{i\frac{4\pi}{3}}$.

Problem: Find the polar form of $z = \sqrt{3} - i$.

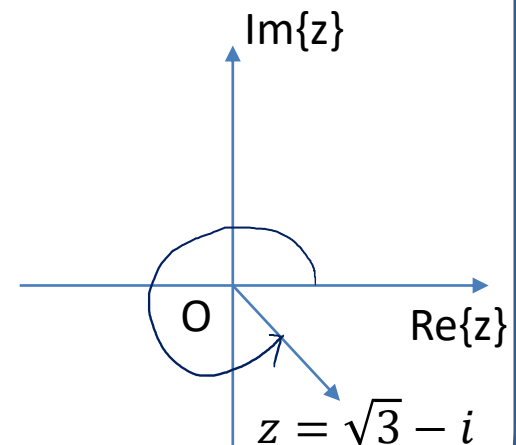
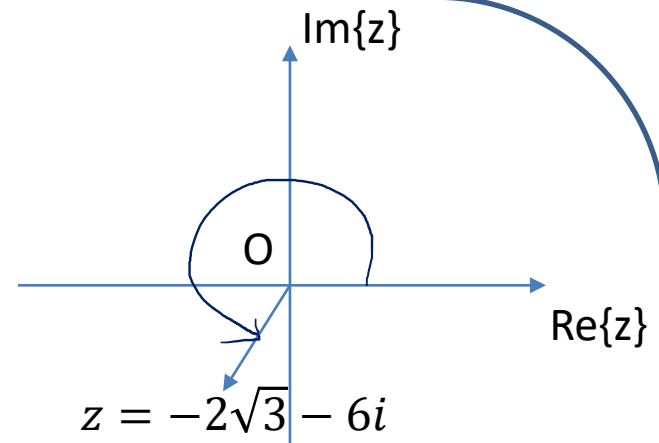
Solution: Here $a = \sqrt{3}$ and $b = -1$

$$r = \sqrt{a^2 + b^2} = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$$

$$\tan \theta = \left(\frac{-1}{\sqrt{3}} \right) = -\frac{1}{\sqrt{3}} = -\tan \frac{\pi}{6} = \tan(2\pi - \frac{\pi}{6})$$

$$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

Hence, $z = re^{i\theta} = 2e^{i\frac{11\pi}{6}}$.



Problem: Find the polar form of $z = \left(\frac{1-i}{1+i}\right)^{18}$.

Solution: Let $z_1 = 1 - i$ and $z_2 = 1 + i$.

For z_1 : $a = 1$ and $b = -1$

$$\therefore r = \sqrt{(1)^2 + (-1)^2} = \sqrt{2} \text{ and}$$

$$\tan \theta = -1 = -\tan \frac{\pi}{4} = \tan \left(2\pi - \frac{\pi}{4}\right) \therefore \theta = \frac{7\pi}{4}$$

$$\text{Hence, } z_1 = \sqrt{2}e^{i\frac{7\pi}{4}}.$$

Similarly,

For z_2 : $a = 1$ and $b = 1$

$$\therefore r = \sqrt{(1)^2 + (1)^2} = \sqrt{2} \text{ and}$$

$$\tan \theta = 1 = \tan \frac{\pi}{4} \therefore \theta = \frac{\pi}{4}$$

$$\text{Hence, } z_2 = \sqrt{2}e^{i\frac{\pi}{4}}.$$

$$\text{So, } z = \left(\frac{1-i}{1+i}\right)^{18} = \left(\frac{\sqrt{2}e^{i\frac{7\pi}{4}}}{\sqrt{2}e^{i\frac{\pi}{4}}}\right)^{18} = \left(e^{i\frac{3\pi}{2}}\right)^{18} = e^{i27\pi} = e^{i(24+3)\pi}$$

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Finding principal argument ($\text{Arg } z$) from general argument ($\arg z$):

➤ If $\arg\{z\}$ is in $0 \leq \theta < 2\pi$ then $\arg z = \text{Arg } z$.

Example: For $z = 3e^{i\frac{\pi}{6}}$, $\arg z = \frac{\pi}{6} = \text{Arg } z$.

➤ If $\arg z \geq 2\pi$ then subtract maximum no. of $2n\pi$

Example: For $z = 15e^{i\frac{15\pi}{2}}$, $\arg z = \frac{15\pi}{2} = 7.5\pi$

So, $\text{Arg } z = 7.5\pi - 6\pi = 1.5\pi = \frac{3\pi}{2}$.

➤ If $\arg z < 0$ then add minimum no. of $2n\pi$

Example: For $z = 2e^{-i5\pi}$, $\arg z = -5\pi$

So, $\text{Arg } z = -5\pi + 6\pi = \pi$.

Problem: Find the principal argument of $z = (2 + 2i)^4$.

Solution: Let $z_1 = 2 + 2i$; $a = 2, b = 2$;

$$\therefore r = \sqrt{4 + 4} = 2\sqrt{2} \text{ and } \tan \theta = \frac{2}{2} = 1 = \tan \frac{\pi}{4}$$

$$\text{Hence, } z = (2 + 2i)^4 = \left(2\sqrt{2}e^{i\frac{\pi}{4}}\right)^4 = (2\sqrt{2})^4 e^{i\pi}$$

$$\therefore \arg z = \pi; \quad \therefore \text{Principal argument: } Arg z = \pi.$$

Problem: Find the principal argument of $z = \left(\frac{2\sqrt{3}+2i}{1-\sqrt{3}i}\right)^6$.

Solution: Let $z_1 = 2\sqrt{3} + 2i$ and $z_2 = 1 - \sqrt{3}i$.

For z_1 : $a = 2\sqrt{3}$ and $b = 2$

$$\therefore r = \sqrt{(2\sqrt{3})^2 + (2)^2} = 4 \text{ and } \tan \theta = \frac{1}{\sqrt{3}} = \tan \frac{\pi}{6} \therefore z_1 = 4e^{i\frac{\pi}{6}}.$$

Similarly, For z_2 : $a = 1$ and $b = -\sqrt{3}$

$$\therefore r = \sqrt{(1)^2 + (-\sqrt{3})^2} = 2 \text{ and } \tan \theta = -\sqrt{3} = -\tan \frac{\pi}{3} = \tan \left(2\pi - \frac{\pi}{3}\right)$$

$$\therefore \theta = \frac{5\pi}{3} \therefore z_2 = 2e^{i\frac{5\pi}{3}}$$

$$\text{So, } z = \left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^6 = \left(\frac{4e^{i\frac{\pi}{6}}}{2e^{i\frac{5\pi}{3}}}\right)^6 = 2^6 \frac{e^{i\pi}}{e^{i10\pi}} = 2^6 e^{i\pi - i10\pi} = 2^6 e^{-i9\pi}.$$

$$\therefore \arg z = -9\pi; \quad \therefore \text{Principal argument: } Arg z = -9\pi + 10\pi = \pi.$$

Problem: Find the principal argument of $z = \left(\frac{1-i}{1+i}\right)^{18}$.

Solution: Let $z_1 = 1 - i$ and $z_2 = 1 + i$.

For z_1 : $a = 1$ and $b = -1$

$$\therefore r = \sqrt{(1)^2 + (-1)^2} = \sqrt{2} \text{ and}$$

$$\tan \theta = -1 = -\tan \frac{\pi}{4} = \tan \left(2\pi - \frac{\pi}{4}\right) \therefore \theta = \frac{7\pi}{4}$$

$$\text{Hence, } z_1 = \sqrt{2}e^{i\frac{7\pi}{4}}.$$

For z_2 : $a = 1$ and $b = 1$

$$\therefore r = \sqrt{(1)^2 + (1)^2} = \sqrt{2} \text{ and } \tan \theta = 1 = \tan \frac{\pi}{4} \therefore \theta = \frac{\pi}{4}$$

$$\text{Hence, } z_2 = \sqrt{2}e^{i\frac{\pi}{4}}.$$

$$\text{So, } z = \left(\frac{1-i}{1+i}\right)^{18} = \left(\frac{\sqrt{2}e^{i\frac{7\pi}{4}}}{\sqrt{2}e^{i\frac{\pi}{4}}}\right)^{18} = \left(e^{i\frac{3\pi}{2}}\right)^{18} = e^{i27\pi}$$

$$\therefore \arg z = 27\pi; \quad \therefore \text{Principal argument: } \text{Arg } z = 27\pi - 26\pi = \pi.$$

Exercise:

1. Express $\frac{(1+i)^2}{1-i}$ in terms of $a + ib$.
2. Evaluate each of the followings:
(a) $\operatorname{Re} \left\{ \frac{1+\sqrt{3}i}{1-i} \right\}$, (b) $\left| \frac{z}{\bar{z}} \right|$, (c) $\operatorname{Im} \left\{ \frac{z}{\bar{z}} \right\}$.
3. Convert the following numbers into polar form:
(a) $z = -1 + i$, (b) $z = -3 - \sqrt{3}i$, (c) $z = \frac{(1-i)^2}{1+i}$.
4. Convert the following numbers into rectangular form:
 $z = \sqrt{3} e^{i\frac{\pi}{3}}$ and $z = 2e^{i\frac{\pi}{4}}$.
5. Find the principle argument of the followings:
(a) $z = (-1 - i)^4$, (b) $z = (-2 + 2\sqrt{3}i)^3$, $z = \frac{(1+i)^3}{(1-i)}$.

MULTIPLE CHOICE QUESTION

1. The standard form of the complex number $\frac{(1+i)^2}{1-i}$ is
(a) $-1-i$ (b) $-1+i$ (c) $1+i$ (d) None
2. The polar form of the complex number $-1+i$ is
(a) $\sqrt{2}e^{i\frac{5\pi}{4}}$ (b) $-\sqrt{2}e^{i\frac{5\pi}{4}}$ (c) $\sqrt{2}e^{i\frac{3\pi}{4}}$ (d) None
3. The standard rectangular form of the complex number $z = \sqrt{3}e^{i\frac{\pi}{2}}$ is
(a) $-\sqrt{3}+i$ (b) $-\sqrt{3}+\sqrt{3}i$ (c) $\sqrt{3}i$ (d) None
4. The principle argument of the complex number $(-1+i)^4$ is
(a) π (b) 2π (c) 3π (d) None