A point's Temp increase quick

Chapter-6

~~Grand~~ Gradient, Divergence and curl

Gradient: $\nabla T = \frac{\hat{u}_1}{h_1}\frac{\partial T}{\partial u_1} + \frac{\hat{u}_2}{h_2}\frac{\partial T}{\partial u_2} + \frac{\hat{u}_3}{h_3}\frac{\partial T}{\partial u_3}$

(i) **cartesian:**

$u_1 = x$, $u_2 = y$, $u_3 = z$ and

$h_1 = h_2 = h_3 = 1$

$$\therefore \nabla T = \frac{\hat{x}_1}{1}\frac{\partial T}{\partial x} + \frac{\hat{y}}{1}\frac{\partial T}{\partial y} + \frac{\hat{z}}{1}\frac{\partial T}{\partial z}.$$

$$\boxed{\nabla T = \hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}.}$$

(ii) **Cylindrical:**

$u_1 = r$, $u_2 = \varphi$, $u_3 = z$ and

~~$h_1 = h_2 = h_3 =$~~

$h_1 = h_3 = 1$, $h_2 = r$.

$$\therefore \nabla T = \frac{\hat{r}}{1}\frac{\partial T}{\partial r} + \frac{\hat{\varphi}}{r}\frac{\partial T}{\partial \varphi} + \frac{\hat{z}}{1}\frac{\partial T}{\partial z}.$$

$$\boxed{\therefore \nabla T = \hat{r}\frac{\partial T}{\partial r} + \frac{\hat{\varphi}}{r}\frac{\partial T}{\partial \varphi} + \hat{z}\frac{\partial T}{\partial z}}$$

(iii) Spherical:

$u_1 = R,\ u_2 = \theta,\ u_3 = \varphi$ and

$h_1 = 1,\ h_2 = R,\ h_3 = R\sin\theta$.

$$\nabla T = \frac{\hat{R}}{1}\frac{\partial T}{\partial R} + \frac{\hat{\theta}}{R}\frac{\partial T}{\partial \theta} + \frac{\hat{\varphi}}{R\sin\theta}\frac{\partial T}{\partial \varphi}$$

$$\boxed{\therefore \nabla T = \hat{R}\frac{\partial T}{\partial R} + \frac{\hat{\theta}}{R}\frac{\partial T}{\partial \theta} + \frac{\hat{\varphi}}{R\sin\theta}\frac{\partial T}{\partial \varphi}}$$

ap-1|
EX-1.1: Find the gradient of the scalar functions ⓐ $T(x,y,z) = \frac{xyz}{(x^2+y^2+z^2)}$ at point $(1,1,1)$

Solution:

Give that $T(x,y,z) = \frac{xyz}{x^2+y^2+z^2}$

we know that

$$\nabla T = \hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}$$

$$= \hat{x}\frac{\partial}{\partial x}\left[\frac{xyz}{x^2+y^2+z^2}\right] + \hat{y}\frac{\partial}{\partial y}\left[\frac{xyz}{x^2+y^2+z^2}\right]$$

$$+ \hat{z}\frac{\partial}{\partial z}\left[\frac{xyz}{x^2+y^2+z^2}\right].$$

$$= \hat{x}\left[\frac{(x^2+y^2+z^2) - (xyz)\cdot 2x}{(x^2+y^2+z^2)^2}\right]$$

$$+ \hat{y}\left[\frac{(x^2+y^2+z^2)\cdot xz - (xyz)\cdot 2y}{(x^2+y^2+z^2)^2}\right]$$

$$+ \hat{z}\left[\frac{(x^2+y^2+z^2)\, xy - (xyz)\cdot 2z}{(x^2+y^2+z^2)^2}\right]$$

$\boxed{\text{:y}}$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Now at point $(1,1,1)$

$$\nabla T = \hat{x}\left[\frac{(1^2+1^2+1^2)\cdot 1\cdot 1 - (1\cdot1\cdot1)\cdot 2\cdot 1}{(1^2+1^2+1^2)^2}\right] + \hat{y}\left[\frac{(1^2+1^2+1^2)\cdot 1\cdot 1 - (1\cdot1\cdot1)\, 2\cdot 1}{(1^2+1^2+1^2)^2}\right]$$

$$+ \hat{z}\left[\frac{(1^2+1^2+1^2)\cdot 1\cdot 1 - (1\cdot1\cdot1)\cdot 2\cdot 1}{(1^2+1^2+1^2)^2}\right]$$

$$= \hat{x}\,\frac{1}{9} + \hat{y}\,\frac{1}{9} + \hat{z}\,\frac{1}{9}.$$

$$\therefore \nabla T = \frac{1}{9}(\hat{x}+\hat{y}+\hat{z})$$

Ex: 1.1

Find the gradient of the scalar functions

(b) $T(r, \varphi, z) = \frac{z \cos \varphi}{1+r^2}$ at point $(1, \pi, 2)$

Solution:

Given that

$$T(r, \varphi, z) = \frac{z \cos \varphi}{1+r^2}$$

we know that

$$\nabla T = \hat{r} \frac{\partial T}{\partial r} + \hat{\varphi} \frac{\partial T}{\partial \varphi} + \hat{z} \frac{\partial T}{\partial z}$$

$$= \hat{r} \frac{\partial}{\partial r}\left(\frac{z \cos \varphi}{1+r^2}\right) + \hat{\varphi} \frac{\partial}{\partial \varphi}\left(\frac{z \cos \varphi}{1+r^2}\right) + \hat{z} \frac{\partial}{\partial z}\left(\frac{z \cos \varphi}{1+r^2}\right)$$

Ex: 2.4

Find the gradient of the scalar functions

(i) $T(R,\theta,\varphi) = R\cos\theta\sin\varphi$ at point $\left(2, \frac{\pi}{2}, \frac{\pi}{4}\right)$

Solution:

Given that

~~T = R cos~~

$$T(R,\theta,\varphi) = R\cos\theta\sin\varphi$$

we know that

$$\nabla T = \hat{R}\frac{\partial T}{\partial R} + \frac{\hat{\theta}}{R}\frac{\partial T}{\partial \theta} + \frac{\hat{\varphi}}{R\sin\theta}\frac{\partial T}{\partial \varphi}$$

$$= \hat{R}\frac{\partial}{\partial R}(R\cos\theta\sin\varphi) + \frac{\hat{\theta}}{R}\frac{\partial}{\partial\theta}(R\cos\theta\sin\varphi) + \frac{\hat{\varphi}}{R\sin\theta}\frac{\partial}{\partial\varphi}(R\cos\theta\sin\varphi)$$

$$= \hat{R}(\cos\theta\sin\varphi) + \frac{\hat{\theta}}{R}(-R\sin\theta\sin\varphi) + \frac{\hat{\varphi}}{R\sin\varphi}(R\cos\theta\cos\varphi)$$

$$\therefore \nabla T = \hat{R}(\cos\theta\sin\varphi) - \hat{\theta}(\sin\theta\sin\varphi) + \hat{\varphi}(\cot\theta\cos\varphi)$$

$$\left[\because \frac{\cos\theta}{\sin\theta} = \cot\theta\right]$$

At point $\left(2, \frac{\pi}{2}, \frac{\pi}{4}\right)$.

$$\nabla T = \hat{R}\left(\cos\frac{\pi}{2}\sin\frac{\pi}{4}\right) - \hat{\theta}\left(\sin\frac{\pi}{2}\sin\frac{\pi}{4}\right) + \hat{\varphi}\left(\cot\frac{\pi}{2}\cos\frac{\pi}{4}\right)$$

$$= \hat{R}\left(0\times\frac{1}{\sqrt{2}}\right) - \hat{\theta}\left(1\times\frac{1}{\sqrt{2}}\right) + \hat{\varphi}\left(0\times\frac{1}{\sqrt{2}}\right)$$

$$\therefore \nabla T = -\frac{1}{\sqrt{2}}\hat{\phi}$$

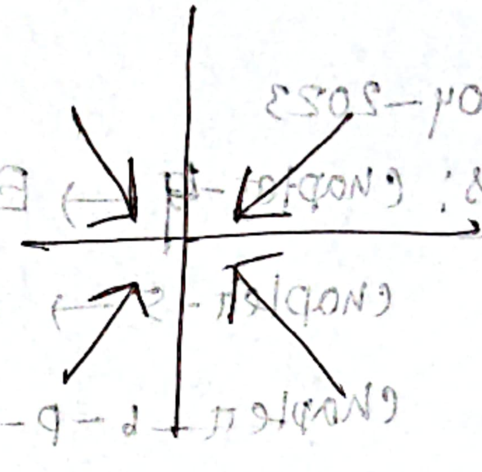
Quiz-2

Date : 09-04-2023

Divergence and curl

$\frac{1}{\delta v}$ — $= \nabla$

$\nabla \cdot A > 0$



$\nabla \cdot A < 0$

$\nabla \cdot A = 0 \rightarrow$ solenoidal

$\nabla \cdot A \neq 0 \rightarrow$ incompressible.

$\nabla T \qquad \nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$] 3D form

$\nabla \cdot A$ · Divergence dot product এর মত.

Curl

$\nabla \times A = 0 \longrightarrow$ irrotational.

$\nabla \times A \neq 0 \longrightarrow$ rotational.

M-4 (class Lecture)

P-7 Example-3.1

Determine divergence and curl. Also check each of the flowing vector fields are solenoidal, conservative or both.

(a) $\bar{A} = \hat{x}\,x^2 + \hat{y}\,2xy$

(b) $\bar{A} = \hat{r}\,\dfrac{\sin\varphi}{r^2} + \hat{\varphi}\,\dfrac{\cos\varphi}{r^2}$

(c) $\bar{A} = \hat{R}\,(Re^{-R})$

Solution:

Given that $A = \hat{x}x^2 + \hat{y}\,2xy$

Here. $A_1 = x^2$, $A_2 = 2xy$

Thus $\operatorname{div} A = \nabla\cdot A = \dfrac{\partial}{\partial x}(x^2) + \dfrac{\partial}{\partial y}(2xy) + \dfrac{\partial}{\partial z}(0)$

$= 2x + 2x + 0$

$\therefore \nabla\cdot A = 4x \neq 0$

Hence vector field A is not solenoidal.

Solution:

$u_1 = x$, $u_2 = y$ $u_3 = z$, $h_1 = h_2 = h_3 = 1$

$$\operatorname{curl} A = \nabla\times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{d}{dx} & \dfrac{d}{dy} & \dfrac{d}{dz} \\ x^2 & 2xy & 0 \end{vmatrix} = \hat{x}\left[\frac{d}{dy}(0) - \frac{d}{dz}(2xy)\right] - \hat{y}\left[\frac{d}{dx}(0) - \frac{d}{dz}(x^2)\right]$$

$+\hat{z}\left[\frac{d}{dx}(2xy)-\frac{d}{dy}(xy)\right]$

$= 0-0+\hat{z}(2y-0)$

$\therefore \nabla\times A = \hat{z}\, 2y \neq 0$

Thus vector field A is not conservative.

ⓑ solution:

Given that

$\hat{r}\cdot\frac{\sin\varphi}{r^2}+\hat{\varphi}\,\frac{\cos\varphi}{r^2}$

Here,

$A_1=\frac{\sin\varphi}{r^2} \qquad A_2=\frac{\cos\varphi}{r^2}$

$\operatorname{div} A = \nabla\cdot A = \frac{\partial}{\partial r}\frac{\sin\varphi}{r^2}+\frac{d}{dy}\left(\frac{\cos\varphi}{r^2}\right)$

(c) Solution:

$\vec{A} = \hat{R}\, R e^{-R}$

$A_1 = R e^{-R}$ $\quad A_2 = A_3 = 0$

(i) $\nabla \cdot A = \frac{1}{R^2 \sin\theta} \left[\frac{d}{dR} \left(R^2 \sin\theta \cdot R e^{-R} \right) + \frac{\partial}{\partial\theta}(0) + \frac{d}{d\phi}(0) \right]$

$= \frac{1}{R^2 \sin\theta} \sin\theta \left[\frac{d}{dR} \left(R^3 e^{-R} \right) \right]$

$= \frac{1}{R^2} \left[R^3 \frac{d}{dR} e^{-R} - e^{-R} \frac{d}{dR} (R^3) \right]$

$= \frac{1}{R^2} \left[-R^3 e^{-R} + 3R^2 e^{-R} \right]$

$= \cancel{\frac{1}{R^2}} \cdot \cancel{R^2} \cdot e^{-R}(3-R)$

$\nabla \cdot A = e^{-R}(3-R) \neq 0$, not solenoidal.

And curl $A = \nabla \times A = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \hat{R} & \hat{\theta} R & \hat{\phi} R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ R e^{-R} & 0 & 0 \end{vmatrix}$

$\therefore \nabla \times A = 0$

Thus A is conservative.

18 9 218

curl

$\nabla \times A = 0$ $\boxed{\text{irrotational}}$

$\nabla \times A \neq 0$ $\boxed{\text{Rotational}}$

Divergence

$$\text{div } A = \nabla \cdot A = \frac{1}{h_1 \cdot h_2 \cdot h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(h_1 A_2 h_3)\right.$$

$\boxed{\nabla = \text{Nelda}}$

P-81

Ex-3.2

Test whether $\bar{A} = \hat{x}(y^2\cos x + z^3) + \hat{y}(2y\sin x - y) + \hat{z}(3xz^2+2)$ is a conservative field?

If conservative find the scalar potential T such that $\bar{A} = \nabla T$. Find workdone in the field from $(0,1,-1)$ to $(\frac{\pi}{2},-1,2)$

Solution: For conservative field, curl $A = \nabla \times A = 0$.

Let $T(x,y,z)$ be a scalar potential of A, i.e.

$$\boxed{\bar{A} = \nabla T} \Rightarrow T = \int A.dl$$

position vector $dl = \hat{x}dx + \hat{y}dy + \hat{z}dz$

$$\Rightarrow T = \int \left[\hat{x}(y^2\cos x + z^2) + \hat{y}(2y\sin x - y) + \hat{z}(3xz^2+2)\right] \cdot (\hat{x}dx + \hat{y}dy + \hat{z}dz)$$

$$= \int \left[(y^2\cos x + z^2)dx + (2y\sin x - y)dy + (3xz^2+2)dz\right]$$

$$= \int \left[y^2\cos x + z^2 dx + 2y\sin x dy - y dy + 3xz^2 dz + 2dz\right]$$

$$= y^2\sin x + xz^3 + y^2\sin x - \frac{1}{2}y^2 + xz^3 + 2z + c$$

$$T = 2x z^3 - \frac{1}{2}y^2 + 2z + 2y^2\sin x + c$$

Workdone $= \int_{(0,1,-1)}^{(\frac{\pi}{2},-1,2)} \vec{A}\cdot d\vec{l}$

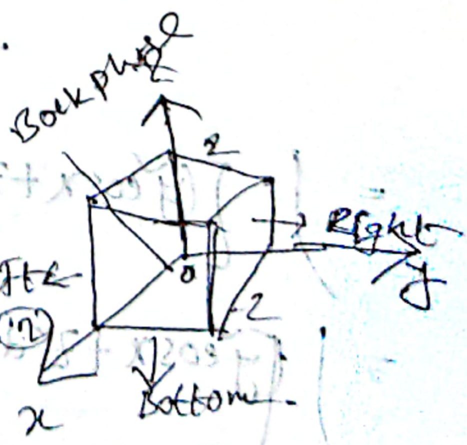
$= \left[2xz^3 - \frac{1}{2}y^4 + 2z + 2y^2\sin x\right]_{(0,1,-1)}^{(\frac{\pi}{2},-1,2)}$

$= 8\pi + 8$ (Ans)

$\boxed{\text{Laplacian}}$

Short question → Gauss divergence theorem $\boxed{P-14}$

(A) $\vec{A} = \hat{x}xz - \hat{y}yz - \hat{z}xy$



Solution: Surface integral

(1) Front face:

$x = 2$, $d\vec{s} = \hat{x}\,dy\,dz$, $\vec{A} = \hat{x}2z - \hat{y}yz - \hat{z}2y$

$\therefore \int_{FR} \vec{A}\cdot d\vec{s} = \int_0^2\int_0^2 (\hat{x}2z - \hat{y}yz - \hat{z}2y)\cdot(\hat{x}\,dy\,dz)$

$$= \int_0^2 \int_0^2 (2z\, dy\, dz) = \int_0^2 2z \left[\int_0^2 dy\right] dz$$

$$= \int_0^2 2z [y]_0^2 dz = \int_0^2 2z \cdot 2dz$$

$$= 4\left[\frac{1}{2}z^2\right]_0^2 = 8$$

(2) Back face

$x = 0$, $ds = -\hat{x}\, dy\, dz$,

$\bar{A} = -\hat{y}\, yz^2$

$$\int_{BF} \bar{A}\cdot ds = \int_0^2 \int_0^2 (-\hat{y}\, yz^2)\cdot(-\hat{x}\, dy\, dz)$$

$$= 0.$$

Thus surface integral

$$\oint_s \bar{A}\cdot ds = 8 + 0 - \frac{32}{3} + 0 - 4 + 4 = -\frac{8}{3}$$

$$\left[\int_v \nabla\cdot A\, dv = \oint_s \bar{A}\cdot ds\right]$$

$\operatorname{div} A = \nabla\cdot A = \frac{\partial}{\partial x}(A_1) + \frac{\partial}{\partial y}(A_2) + \frac{\partial}{\partial z}(A_3)$

Now $\nabla\cdot A = z - z^2$

$$\int_v \nabla\cdot A\, dv = \int_0^2\int_0^2\int_0^2 (z - z^2)\, dx\, dy\, dz$$

$$= \int_0^2 \int_0^2 (z - z^2)[x]_0^2\, dy\, dz$$

$$= 2\int_0^2 (z - z^2)[y]_0^2\, dz$$

$$\cancel{= 2\int_0^2 [y]^2}$$

$$= 2.2 \int_0^2 (z - z^2)\, dz = 4\left[\frac{1}{2}z^2 - \frac{1}{3}z^3\right]_0$$