Lecture 7 (5F)

27.11.2023

Quiz 2: Numerical Integration (Lecture Note 8).

Friday (Time: will be informed)

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Chapter:

Ordinary Differential Equations

Ordinary differential equation can be written in the form:

$$\frac{dy}{dx} = f(x, y) \qquad (1).$$

$$y(x_0) = y_0$$

* Solution procedure:

$\boxed{I}$ Taylor series solution

The solution of the equation is a function of x.

The Taylor series expansion of $y(x)$ about x_0 is

$$y(x_0+h) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

where $y_0^{(n)}$ is the value of

$$\frac{d^n y}{dx^n} \text{ at } x = x_0.$$

IV Euler Method:

$$\frac{dy}{dx} = f(x,y) \qquad \text{—— (1)}$$

$$y(x_0) = y_0$$

$$y_{n+1} = y_n + h f(x_n, y_n),$$

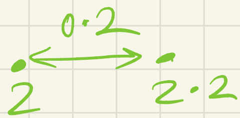
$$n = 0, 1, 2, \cdots$$

Exercise

1. Given initial value problem

$$y' = 2x^2 - y + 3y^2 \quad \text{with } y(2) = 0.5$$

(a) Estimate the values of $y(2.2)$ using Euler's method with step size $h = 0.2$.



Soln: Here, $f(x, y) = 2x^2 - y + 3y^2$

and $x_0 = 2$, $\quad y_0 = 0.5$

Taking $h = 0.2$,

$$y_1 = y(x_0 + h) = y_0 + hf(x_0, y_0)$$

or, $y(2+0.2) = 0.5 + 0.2\, f(2, 0.5)$

or, $y(2.2) = 0.5 + 0.2\left[2(2)^2 - 0.5 + 3(0.5)^2\right]$

$$= 0.5 + 0.2(8.25)$$

$$= 2.15.$$

III Runge-kutta method:

* Second order Runge-kutta method (RK-2 method).

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

we have,

$$y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$$

where,

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h, y_0 + k_1).$$

Exercise

1. (b) $y' = 2x^2 - y + 3y^2$

$y(2) = 0.5$

Estimate the values of $y(2.6)$ using Runge-Kutta method of order 2.

Sol: Here, $f(x,y) = 2x^2 - y + 3y^2$

$$x_0 = 2, \quad y_0 = 0.5$$

Taking the step-size, $h = 0.6$

Now,
$$k_1 = hf(x_0, y_0)$$
$$= 0.6\, f(2, 0.5)$$
$$= 0.6 \times \left[2(2)^2 - 0.5 + 3(0.5)^2\right]$$
$$= 4.95$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$
$$= 0.6\, f(2.6, 5.45)$$

$$= 0.6\left[2(2.6)^2 - 5.45 + 3(5.45)^2\right]$$

$$= 58.3065$$

$$\therefore y_1 = y(2.6) = y_0 + \frac{1}{2}\left[k_1 + k_2\right]$$

$$= 0.5 + \frac{1}{2}\left[4.95 + 58.3065\right]$$

$$= 32.12$$

Fourth order Runge-Kutta method (RK-4 method)

$$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

where,

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$K_4 = hf(x_0+h, y_0+K_3)$$

Exercise

1. (c) $y' = 2x^2 - y + 3y^2$ with $y(2)=0.5$

Estimate a value of $y(2.8)$ using Runge-Kutta method of order 4.

Soln: $f(x,y) = 2x^2 - y + 3y^2$

$x_0 = 2, \quad y_0 = 0.5$

Let, the step size, $h = 0.8$

$$K_1 = hf(x_0, y_0)$$
$$= 0.8 f(2, 0.5)$$
$$= 6.6$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$
$$= 0.8 f(2+0.4, 0.5+3.3)$$

$$= 0.8 f(2.4, 3.8)$$

$$= 40.832$$

$$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$= 0.8 f(2 + 0.4, 0.5 + 20.416)$$

$$= 0.8 f(2.4, 20.916)$$

$$= 1042.4329$$

$$K_4 = h f(x_0 + h, y_0 + K_3)$$

$$= 0.8 f(2.8, 0.5 + 1042.4329)$$

$$= 0.8 f(2.8, 1042.9329)$$

$$= 2609679.879$$

$$y_1 = y(x_0 + h) = y_0 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$y(2.8) = 0.5 + \frac{1}{6}\left[6.6 + 2\times 40.832 + 2\times 1042.4329 + 2609679.879\right]$$

$$= 435309.3348$$

Ans

4. System of Equations

$$\frac{dy}{dx} = f_1(\cancel{x}, y, z)$$

$$\frac{dz}{dx} = f_2(x, y, z)$$

Initial conditions:

$$y(x_0) = y_0 \quad , \quad z(x_0) = z_0$$

* Solution procedure:

- Euler's Method for the above system

$$y_1 = y_0 + h f_1(x_0, y_0, z_0)$$

$$z_1 = z_0 + h f_2(x_0, y_0, z_0)$$

[4] Given the system

$$\frac{dy}{dx} = y^2 + xz + x^2$$

$$\frac{dz}{dx} = y + 2x + yz$$

with $y(1) = -1$ and $z(1) = 1$.

Estimate $y(1.1)$ and $z(1.1)$ using Euler's method.

Soln: Here, $f_1(x, y, z) = y^2 + xz + x^2$

$$f_2(x, y, z) = y + 2x + yz$$

$$x_0 = 1, \quad y_0 = -1, \quad z_0 = 1.$$

Let, $h = 0.1$

$$y_1 = y(x_0 + h) = y_0 + h f_1(x_0, y_0, z_0)$$

$$\text{or, } y(1.1) = -1 + 0.1 f_1(1, -1, 1)$$

$$= -1 + 0.1 \times 3$$

$$= -0.7$$

$$z_1 = z(x_0 + h) = z_0 + h f_2(x_0, y_0, z_0)$$

$$\text{or, } z(1.1) = 1 + 0.1 f_2(1, -1, 1)$$

$$= 1 + 0.1 \times 2$$

$$= 1.2$$