Lecture 10. (5F) 06.12.23

3. Given the system

$$\frac{dy}{dx} = y^2 + xz + x^2 \quad \text{and} \qquad y'$$

$$\frac{dz}{dx} = y + 2x + yz \qquad z'$$

with $y(1) = -1$ and $z(1) = 1$.

(b) Use central difference formula to derive a recurrence relation and estimate the values of $y(1.2)$ and $z(1.2)$.

Soln: By 3-point central difference formula:

$$y_n' = \frac{y_{n+1} - y_{n-1}}{2h}$$

Now, $$\left(\frac{dy}{dx}\right)_n = \left(y^2 + xz + x^2\right)_n$$

or, $$\frac{y_{n+1}-y_{n-1}}{2h} = y_n^2 + x_n z_n + x_n^2$$

or, $$y_{n+1} - y_{n-1} = 2h\left(y_n^2 + x_n z_n + x_n^2\right)$$

$$\therefore\ y_{n+1} = y_{n-1} + 2h\left(y_n^2 + x_n z_n + x_n^2\right) \quad \text{——— (I)}$$

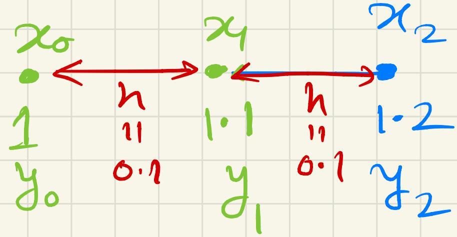
Again,

$$\left(\frac{dz}{dx}\right)_n = \left(y + 2x + yz\right)_n$$

or, $$\frac{z_{n+1} - z_{n-1}}{2h} = y_n + 2x_n + y_n z_n$$

$$\therefore\ z_{n+1} = z_{n-1} + 2h\left(y_n + 2x_n + y_n z_n\right) \quad \text{——— (II)}$$

we have, $x_0 = 1$, $y_0 = -1$ and $z_0 = 1$



Let, $h = 0.2$

Using $n = 1$, the recurrence relation ① and ⑪ :

$$y_2 = y_0 + 2h\left(y_1^2 + x_1 z_1 + x_1^2\right)$$

$$z_2 = z_0 + 2h\left(y_1 + 2x_1 + y_1 z_1\right)$$

By Euler's method, Let, $f_1(x,y,z) = y^2 + xz + x^2$
$f_2(x,y,z) = y + 2x + yz$

$$\begin{aligned} y_1 &= y_0 + h f_1(x_0, y_0, z_0) \\ &= -1 + 0.1 f_1(1,-1,1) \\ &= -1 + 0.1(1+1+1) \\ &= -1 + 0.3 \\ &= -0.7 \end{aligned}$$

$$z_1 = z_0 + h f_2(x_0, y_0, z_0)$$
$$= 1 + 0.1\, f_2(1, -1, 1)$$
$$= 1 + 0.1\,(-1 + 2\cdot 1 - 1)$$
$$= 1$$

Now,

$$y_2 = y_0 + 2h\,(y_1^2 + x_1 z_1 + x_1^2)$$
$$= -1 + 2\times 0.1\,\big((-0.7)^2 + (1.1)\cdot 1 + (1.1)^2\big)$$

$$\therefore y(1.2) = -0.44$$ Ans

$$z_2 = z_0 + 2h\,(y_1 + 2x_1 + y_1 z_1)$$
$$= 1 + 2\times 0.1\,\big(-0.7 + 2(1.1) + (-0.7)\cdot 1\big)$$

$$z(1.2) = 1.16$$ Ans

3 (e) Use MATLAB function
"[x1, y1] = ode45(F, [x0, xn], [y0, z0])
(i) to estimate the values of of y
in $1 \le x \le 1.5$ using $h = 0.1$.
(ii) to plot the solution curve
in [1, 1.5].

y → y(1)
z → y(2)

Soln:

```
>> clear all
>> F = @(x,y) [y(1)^2 + x*y(2)
               +x^2 ; y(1)+2*x+y(1)*y(2)]
>> [x1, y1] = ode(F, [1:0.1:1.5], [-1, 1])
>> solution = (x1, y1)
>> plot (x1, y1)
```

4. The equation for a LCR circuit with applied voltage E is

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E$$

where, $L=1\,H$, $R=10\,\Omega$, $C=5F$ and $E=20\,V$ given that $q=1$ and $\frac{dq}{dt}=2$ at $t=1$.

(a) Use 3-point central difference formula for derivatives to a derive a recurrence relation for the above IVP.

Soln:

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E$$

or, $$q''+10q'+0.2q=20$$

with $q=1$ and $q'=2$ at $t=1$

or, $q(1)=1$ and $q'(1)=2$.

so that,

$$q'' + 10q' + 0.2q = 20$$

with $q(1) = 1$ and $q'(1) = 2$.

By using 3-point central difference formula:

$$\frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} + 10 \cdot \frac{q_{n+1} - q_{n-1}}{2h} + 0.2 q_n = 20$$

or, $$q_{n+1} - 2q_n + q_{n-1} + 5h(q_{n+1} - q_{n-1}) + 0.2h^2 q_n = 20h^2$$

or, $$q_{n+1}(1+5h) + (0.2h^2 - 2)q_n + (1-5h)q_{n-1} = 20h^2$$

$$\therefore \; q_{n+1} = \frac{1}{1+5h}\left[20h^2 - (0.2h^2-2)q_n - (1-5h)q_{n-1}\right].$$

(c) Express the above IVP as a system of first order differential equations.

Soln: $q'' + 10q' + 0.2q = 20$

$t_0 = 1, \; q_0 = 1, \; q_0' = 2$

Let, $x = q$

and, $y = q' = x'$

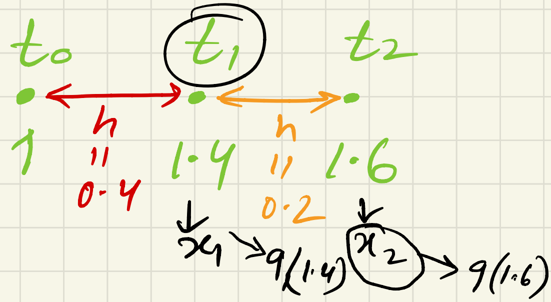
or, $y' = q'' = 20 - 10q' - 0.2q$

$= 20 - 10y - 0.2x$

i.e.,
$$\begin{cases} x' = y \\ y' = 20 - 10y - 0.2x \end{cases}$$

with $x_0 = 1, \; y_0 = 2, \; t_0 = 1.$

(b) Use (a) and (c) to estimate the values of q at $t = 1.4$ and 1.6.



Soln: By (c),

$$x' = y$$

$$y' = 20 - 10y - 0.2x$$

$$t_0 = 1, \quad x_0 = 1, \quad y_0 = 2$$

Here, $f_1(t, x, y) = y$

$$f_2(t, x, y) = 20 - 10y - 0.2x$$

we have, $h_1 = 0.4$ and $h_2 = 0.2$,

$t_1 = 1.4$ and $t_2 = 1.6$.

By Euler's method,

$$x(1.4) = x_1 = x_0 + h_1 f_1(t_0, x_0, y_0)$$

$$\therefore x(1.4) = x_1 = 1 + 0.4\, f_1(1, 1, 2)$$

$$= 1 + 0.4 \times 2$$

$$= 1 + 0.8$$

$$= 1.8$$

$$\boxed{x = q}$$

$$\therefore q(1.4) = 1.8$$

From (a), the recurrence relation,

$$q_{n+2} = \frac{1}{1+5h}\Big[20h^2 - (0.2h^2 - 2)q_n - (1-5h)\, q_{n-1}\Big]$$

For, $n = 1$,

$t_1 \quad t_2$

$1.4 \xleftrightarrow{h_2} 1.6$

$h_2 = 0.2$

$$q_2 = \frac{1}{1+5h_2}\Big[20h_2^2 - (0.2h_2^2 - 2)q_1 - (1-5h_2)\, q_0\Big]$$

$$= \frac{1}{1+5\times 0.2}\Big[20\,(0.2)^2 - (0.2(0.2)^2 - 2)\times 1.8 - (1-5\times 0.2)\times 1\Big]$$

$= 2.1928$

$\therefore q(1.6) = 2.1928.$

(d) Estimate a value of $q(1.2)$ using Runge-kutta method of order 2. (Homework)

Hint: Use the system of first order equation

$$x' = y$$

$$y' = 20 - 10y - 0.2x$$

$$x(1) = 1, \quad y(1) = 2.$$

$$x = q$$
$$y = q'$$

$$\boxed{x(1.2) = q(1.2)}$$

e. Use MATLAB function "[x1, y1] = ode45(F, [x0, xn], y0)"

(i) to estimate the values of q in $1 \le t \le 2$ using $h = 0.2$.

(ii) to plot the solution curve in [1, 2].

$t \to t$
$x \to x(1)$
$y \to x(2)$

Soln:

```
>> clear all
>> F = @(t,x) [x(2); 20-10*x(2) - 0.2*x(1)];
>> [x1, y1] = ode45(F, [1:0.2:2], [1,2])
>> solution = (x1, y1)
>> plot (x1, y1)
```