Lecture 3. math 5(F) 13.11.23

Numerical Integration

$$\boxed{\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ (\Delta x_r)_{max}\to 0}} \sum_{r=1}^{n} f(c_r)\,\Delta x_r}$$

where, $\Delta x_r = x_r - x_{r-1}$

and $x_{r-1} < c_r < x_r$, $r = 1, 2, 3, \ldots n$

* re-write the formula as:

$$\boxed{\int_a^b f(x)\,dx = \sum_{r=1}^{n} \underbrace{\omega_r}_{\text{Weighting faction}} f(x_r) + \underbrace{E}_{\text{Error}}}$$

* Gaussian Quadrature Rule:

* If you have two unknown, we will use $f(x) = 1$ and $f(x) = x$.

* If there are 3 unknowns $f(x) = 1$, $f(x) = x$ and $f(x) = x^2$ and so on.

Exercise 8

6 (b) Derive the quadrature rule:

$$\int_0^h f(x)\,dx \approx a f\left(\frac{h}{3}\right) + b f(h)$$

in each case, find its degree of precision.

Sol: Since there two unknowns, we use $f(x)=1$ and $f(x)=x$.

$f(x)=1$, $f\left(\frac{h}{3}\right)=1$

$f(h)=1$

$$\int_0^h f(x)\,dx = a f\left(\frac{h}{3}\right) + b f(h)$$

$$\Rightarrow \int_0^h 1\,dx = a\cdot 1 + b\cdot 1$$

$$\Rightarrow [x]_0^h = a+b$$

$$\therefore \quad h = a + b \quad \text{———} \quad (1)$$

Again,

$$f(x) = x,$$

$$f\left(\frac{h}{3}\right) = \frac{h}{3}$$

$$f(h) = h$$

$$\int_0^h f(x)\,dx = a f\left(\frac{h}{3}\right) + b f(h)$$

$$\text{or, } \int_0^h x\,dx = a \cdot \frac{h}{3} + b \cdot h$$

$$\text{or, } \left[\frac{x^2}{2}\right]_0^h = \frac{ah}{3} + bh$$

$$\text{or, } \frac{h^2}{2} = \frac{ah}{3} + bh$$

$$\therefore \quad \frac{h}{2} = \frac{a}{3} + b \quad \text{or, } b = \frac{h}{2} - \frac{a}{3} \quad \text{—} \quad (ii)$$

From I $\Rightarrow$

$$h = a + \frac{h}{2} - \frac{a}{3}$$

$$\Rightarrow \quad h - \frac{h}{2} = a - \frac{a}{3}$$

$$\Rightarrow \quad \frac{h}{2} = \frac{2a}{3}$$

$$\therefore \quad a = \frac{3h}{4}$$

From II $\Rightarrow$

$$b = \frac{h}{2} - \frac{a}{3}$$

$$= \frac{h}{2} - \frac{h}{4}$$

$$= \frac{2h - h}{4}$$

$$= \frac{h}{4}$$

Therefore, the quadrature rule

$$\int_0^h f(x)\,dx \approx \frac{3h}{4} f\left(\frac{h}{3}\right) + \frac{h}{4} f(h).$$

*6 (remaining part) Use the above quadrature rule to estimate the following integrals to 3 d.p.

(i) $$\int_{0.2}^{0.8} \cos\left(\sqrt{1+x^2}\right) dx$$

We have, the quadrature rule

$$\int_0^h f(x)\,dx \simeq \frac{3h}{4} f\left(\frac{h}{3}\right) + \frac{h}{4} f(h)$$

or, $$\int_{x_0}^{x_1} f(x)\,dx = \frac{3h}{4} f\left(x_0 + \frac{h}{3}\right) + \frac{h}{4} f(x_0 + h)$$

Now, $x_0 = 0.2$

$x_1 = 0.8$

$h = 0.8 - 0.2 = 0.6$

Here, $f(x) = \cos\sqrt{1+x^2}$

$$\int_{0.2}^{0.8} \cos\left(\sqrt{1+x^2}\right) dx = \frac{3\times 0.6}{4} f\left(0.2+\frac{0.6}{3}\right)$$

$$+ \frac{0.6}{4} f(0.2+0.6)$$

$$= 0.45\, f(0.4) + 0.15\, f(0.8)$$

$$= 0.45 \times 0.474 + 0.15 \times 0.286$$

$$= 0.256$$

4) Newton-Cotes Quadrature Rule

* Trapezoidal rule:

For $n=1 \longrightarrow$ one interval

$$\int_{x_0}^{x_1} f(x)\,dx \simeq a f(x_0) + b f(x_1)$$

where, $h = x_1 - x_0$.

$\longrightarrow$ Using $f(x)=1$ &

$f(x)=x$

You can find a & b.

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}\left[f(x_0)+f(x_1)\right]$$

$\longrightarrow$ Do it

Exercise (8)

1. The table shows the power P supplied to the driving wheels of a car as a function of the speed v. If the mass of the car is m = 2000 kg, determine the time Δt it takes for the car to accelerate from 1 m/s to 6 m/s. Use trapezoidal rule for integration.

Hint: $\Delta t = m \int_1^6 \left(\frac{v}{P}\right) dv$

V	0	1	1.8	2.4	3.5	4.4	5.1	6
P	0	4.7	12.2	19.0	31.8	40.1	43.8	43.2

Let, $f(x) = \frac{v}{P}$ and $dv = dx$; [For formulation]

$$\int_1^6 f(x)\,dx = \int_1^{1.8} f(x)\,dx + \int_{1.8}^{2.4} f(x)\,dx + \int_{2.4}^{3.5} f(x)\,dx$$

$$+ \int_{3.5}^{4.4} f(x)\,dx + \int_{4.4}^{5.1} f(x)\,dx + \int_{5.1}^{6} f(x)\,dx$$

$$\int_1^6 \left(\frac{v}{p}\right) dv = \frac{\overset{h}{1.8-1}}{2}\left[f(1)+f(1.8)\right] + \frac{2.4-1.8}{2}\left[f(1.8)+f(2.4)\right]$$

$$+ \frac{3.5-2.4}{2}\left[f(2.4)+f(3.5)\right] + \frac{4.4-3.5}{2}\left[f(3.5)+f(4.4)\right]$$

$$+ \frac{5.1-4.4}{2}\left[f(4.4)+f(5.1)\right] + \frac{6-5.1}{2}\left[f(5.1)+f(6)\right]$$

$$= \frac{0.8}{2}\left[\frac{1}{4.7}+\frac{1.8}{12.2}\right] + \frac{0.6}{2}\left[\frac{1.8}{12.2}+\frac{2.4}{19}\right]$$

$$+ \frac{1.1}{2}\left[\frac{2.4}{19}+\frac{3.5}{31.8}\right] + \frac{0.9}{2}\left[\frac{3.5}{31.8}+\frac{4.4}{40.1}\right]$$

$$+ \frac{0.7}{2}\left[\frac{4.4}{40.1}+\frac{5.1}{43.8}\right] + \frac{0.9}{2}\left[\frac{5.1}{43.8}+\frac{6}{43.2}\right]$$

Note: Since $f(x) = \frac{v}{p}$

$$= 0.4\left[0.213+0.147\right] + 0.3\left[0.147+0.126\right]$$

$$+ 0.55\left[0.126+0.110\right] + 0.45\left[0.110+0.109\right]$$

$$+ 0.35\left[0.109+0.116\right] + 0.45\left[0.116+0.138\right]$$

$= 0.144 + 0.0819 + 0.1298 + 0.09855$
$+ 0.07875 + 0.1143$

$= 0.6473$

Now,

$$\Delta t = m \int_1^6 \frac{v}{p}\, dv$$

$$= 2000 \times 0.6473$$

$$= 1294.6 \text{ s}$$

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$$\int_{0.4}^{1} f(x)\,dx$$

x	0.4	0.5	0.7	1.0
$f(x)$	1.083	1.133	1.287	1.649

$$\int_{0.4}^{1} f(x)\,dx = \int_{0.4}^{0.5} f(x)\,dx + \int_{0.5}^{0.7} f(x)\,dx + \int_{0.7}^{1.0} f(x)\,dx$$

$$= \frac{0.5-0.4}{2}\left[f(0.4)+f(0.5)\right] + \frac{0.7-0.5}{2}\left[f(0.5)+f(0.7)\right] + \frac{1.0-0.7}{2}\left[f(0.7)+f(1)\right]$$

$$= \frac{0.1}{2} [1.133 + 1.083)$$

$$+ \frac{0.2}{2} (1.133 + 1.287)$$

$$+ \frac{0.3}{2} (1.287 + 1.649)$$

$$= \frac{0.1}{2} (2.216) + \frac{0.2}{2} (2.420)$$

$$+ \frac{0.3}{2} (2.936)$$

$$= 0.793$$