Lecture 8. (5F)

29.11.2023

System of Equations:

$$\begin{cases} y' = f_1(t, y, z) \\ z' = f_2(t, y, z) \\ y(t_0) = y_0 \text{ and } z(t_0) = z_0 \end{cases} \quad \text{— (1)}$$

* Runge-Kutta 2nd order method:

$$y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$$

$$z_1 = z_0 + \frac{1}{2}[m_1 + m_2]$$

Where,

$$k_1 = h f_1(t_0, y_0, z_0)$$

$$k_2 = h f_1(t_0 + h, y_0 + k_1, z_0 + m_1)$$

$$m_1 = h f_2(t_0, y_0, z_0)$$

$$m_2 = h f_2(t_0 + h, y_0 + k_1, z_0 + m_1).$$

Exercise

2. Given the initial value problem

$$\frac{dy}{dx} = x + y^2 - z, \quad \frac{dz}{dx} = x^2 - 3y + z^2$$

with $y(1) = 2$ and $z(1) = 2.5$.

(a) Estimate $y(1.2)$ and $z(1.2)$ using the R-K 2nd order method with step size $h = 0.1$.

Solution: Let,

$$f_1(x, y, z) = x + y^2 - z$$

$$f_2(x, y, z) = x^2 - 3y + z^2$$

$$x_0 = 1, \quad y_0 = 2, \quad z_0 = 2.5, \text{ and } h = 0.1$$

RK-2 method:

$$y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$$

$$z_1 = z_0 + \frac{1}{2}[m_1 + m_2]$$

Now,

$$k_1 = h f_1(x_0, y_0, z_0)$$

$\therefore k_1 = 0.1\, f_1(1, 2, 2.5)$

$= 0.1 \times 2.5$

$= 0.25$

$m_1 = 0.1\, f_2(1, 2, 2.5)$

$= 0.1 \times 1.25$

$= 0.125$

$k_2 = h\, f_1(x_0 + h,\ y_0 + k_1,\ z_0 + m_1)$

$= 0.1\, f_1(1.1, 2.25, 2.625)$

$= 0.3538$

$m_2 = h\, f_2(1.1, 2.25, 2.625)$

$= 0.1\, f_2(1.1, 2.25, 2.625)$

$= 0.1351$

$y(1.1) = y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$

$$y_1 = 2 + \frac{1}{2}[0.25 + 0.3538]$$
$$= 2.3019$$

$$z(1.1) = z_1 = z_0 + \frac{1}{2}[m_1 + m_2]$$
$$= 2.5 + \frac{1}{2}[0.125 + 0.1351]$$
$$= 2.6301$$

Now, we will start with

$x_1 = 1.1$, $y_1 = 2.3019$, $z_1 = 2.6301$

and $h = 0.1$

$$K_1 = h f_1(x_1, y_1, z_1)$$
$$= 0.1\, f_1(1.1, 2.3019, 2.6301)$$
$$= 0.3768$$

$$m_1 = h f_2(1.1, 2.3019, 2.6301)$$
$$= 0.1\, f_2(1.1, 2.3019, 2.6301)$$
$$= 0.1222$$

$$k_2 = h f_1(x_1 + h,\ y_1 + k_1,\ z_1 + m_1)$$
$$= 0.1 f_1(1.2,\ 2.6788,\ 2.7523)$$

$$= 0.1 \times 5.6236$$
$$= 0.5623$$

$$m_2 = 0.1 f_2(1.2,\ 2.6788,\ 2.7523)$$

$$= 0.0979$$

$$y(1.2) = y_2 = y_1 + \frac{1}{2}[k_1 + k_2]$$
$$= 2.3019 + \frac{1}{2}[0.3768 + 0.5623]$$
$$= 2.7716$$

$$z(1.2) = z_2 = z_1 + \frac{1}{2}[m_1 + m_2]$$
$$= 2.6301 + \frac{1}{2}[0.1222 + 0.0979]$$
$$= 2.7402.$$

IV Higher order Differential Equations:

$$y'' = f(x, y, y')$$

with $y(x_0) = c_0$ and $y'(x_0) = c_1$

we introduce,

$$y' = z$$

We can rewrite the above initial value problem as,

$$y' = z$$

$$z' = f(x, y, z)$$

with $y(x_0) = c_0$ and $z(x_0) = c_1$

5. Consider the boundary value problem

$$y'' - 2y' + 2y = x$$

$$y(0) = 1, \quad y(1) = 2.$$

(a) Derive a recurrence relation for the above differential equation using three point central difference formula for derivative with $h = 1/3$.

Soln: Using central difference formulas, we have,

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} - 2\,\frac{y_{n+1} - y_{n-1}}{2h} + 2y_n = x_n$$

or,

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{\frac{1}{9}} - \frac{y_{n+1} - y_{n-1}}{1/3} + 2y_n = x_n$$

or, $9y_{n+1} - 18y_n + 9y_{n-1}$

$- 3y_{n+1} + 3y_{n-1} + 2y_n = x_n$

or, $6y_{n+1} - 16y_n + 6y_{n-1} = x_n$

Ⓑ Using the above finite difference formula (recurrence relation) to solve the above boundary value problem.

with $h = 1/3$, the nodal points are, $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = 2/3$ and $x_3 = 1$.

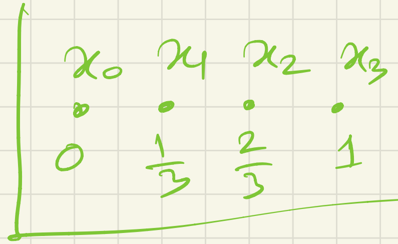
For, $n = 1$, $x_1 = \frac{1}{3}$, $y_0 = 1$,

$6y_2 - 16y_1 + 6y_0 = x_1$

or, $6y_2 - 16y_1 + 6 = 1/3$

or, $6y_2 - 16y_1 = \frac{1}{3} - 6$

∴ $6y_2 - 16y_1 = -\frac{17}{3}$ ——— (i)



For $n = 2$, $y_3 = y(x_3) = y(1) = 2$

$$6y_3 - 16y_2 + 6y_1 = x_2$$

or, $6\times 2 - 16y_2 + 6y_1 = \frac{2}{3}$

or, $-16y_2 + 6y_1 = \frac{2}{3} - 12$

or, $-16y_2 + 6y_1 = \frac{2-36}{3}$

∴ $-16y_2 + 6y_1 = -\frac{34}{3}$ ——— (ii)

Solving (i) & (ii),

$$y_2 = 0.978, \quad y_1 = 0.7212$$

ⓒ Express the above initial value problem as a system of first order differential equations.

$$y'' - 2y' + 2y = x$$

$$y(0) = 1, \quad y(1) = 2$$

Soln: Let, $y_1 = y$ ——— (I)

and $y_2 = y'$ ——— (II)

Then we have,

$$y_2' = y''$$

Therefor, $y_2 = y'$ [from 1st eqn]

$$= y_1'$$

Our system of equations,

$$y_1' = y_2$$

$$y_2' - 2y_2 + 2y_1 = x$$

$y(0) = 1$
$y(1) = 2$

$$\begin{cases} y_1' = y_2 \\ y_2' = 2y_2 - 2y_1 + x \end{cases}$$

(d) Use MATLAB function
"sol = bvp4c(odefun, bcfun, solinit)"
to solve the BVP.

(i) Estimate the values of y at $x = 1/3$ and $2/3$

(ii) plot the solution curve $y(x)$.

[Next class]