Final Term: Lecture 01 — Date: 6/11/23

Section F

Numerical Differentiation

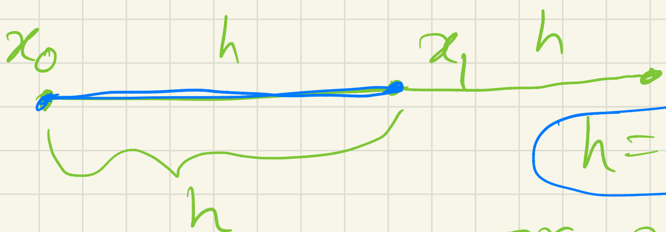
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if h is finite,

$$f'(x) \simeq \frac{f(x+h) - f(x)}{h}$$

$$f'(x_i) \simeq \frac{f(x_i+h) - f(x_i)}{h}$$

$$\simeq f'(x_i, h)$$



$$\boxed{h = x_1 - x_0}$$

or, $x_1 = x_0 + h$

$x_2 = x_1 + h$

$= x_0 + 2h$

$x_n = x_0 + nh$

Taylor's series:

$$f(x+h) = f(x) + h\underline{f'(x)} + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots \quad (1)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \cdots \quad (2)$$

$(1) \Rightarrow$

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2!}f''(x) + O(h^2)$$

$h \rightsquigarrow 0$

$$\boxed{f'(x) \approx \frac{f(x+h) - f(x)}{h}} \quad (3)$$

Forward difference formula.

$(2) \Rightarrow$

$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h)$$

$h \to 0$

$$\boxed{f'(x) = \frac{f(x) - f(x-h)}{h}} \quad \text{--- (4)}$$

Backward difference formula:

(3) + (4) $\Rightarrow$

$$2f'(x) = \frac{f(x+h) - f(x)}{h} + \frac{f(x) - f(x-h)}{h}$$

$$= \frac{f(x+h) - f(x-h)}{h}$$

$$\boxed{\therefore \; f'(x) = \frac{f(x+h) - f(x-h)}{2h}}$$

Central difference formula.

First derivatives (Formula):

- $$f'(x_0) \approx \frac{f_1 - f_0}{h} = \frac{f(x_0+h) - f(x_0)}{h};$$

$O(h)$ = order of error = 1

→ 2-points forward difference

- $$f'(x_0) \approx \frac{f_0 - f_{-1}}{h}$$

$$= \frac{f(x_0) - f(x_0-h)}{h}; \quad O(h)$$

→ 2 points backward difference

- $$f'(x_0) \approx \frac{f_1 - f_{-1}}{2h}$$

$$\boxed{= \frac{f(x_0+h) - f(x_0-h)}{2h}}; \quad O(h^2)$$

→ 3 points central difference

$$f'(x_0) \approx \frac{-3f_0 + 4f_1 - f_2}{2h}, \quad O(h^2)$$

↳ 3-points forward difference

$$f'(x_0) \approx \frac{3f_0 - 4f_{-1} + f_{-2}}{2h}, \quad O(h^2)$$

↳ 3 points backward difference

Second Derivatives:

- $$f''(x_0) \approx \frac{f_0 - 2f_1 + f_2}{h^2}; \quad O(h)$$

  ↳ 3 points forward difference

- $$f''(x_0) \approx \frac{f_0 - 2f_{-1} + f_{-2}}{h^2}; \quad O(h)$$

  ↳ 3 point backward difference

$$f''(x_0) \approx \frac{f_{-1} - 2f_0 + f_1}{h^2} \; ; \; O(h^2)$$

↳ 3-point central difference

$$= \frac{f(x_0 - h) - 2f(x_0) + f(x_0 + h)}{h^2}$$

Exercise 7

1. The distance S of a runner from a fixed point is measured (in m) at intervals of half a second. The obtained data are

Time t (s)	0	0.5	1	1.5	2
Distance S (m)	0	3.65	6.8	9.9	12.15

① Use two points difference formulas to approximate the runner's speed at times $t = 0$ s and $t = 2$ s.

we have 2-points forward difference formula

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

at time $t = 0$ s, $h = 0.5 - 0 = 0.5$

$$V(0) = S'(0) = \frac{S(0+0.5) - S(0)}{0.5}$$

$$= \frac{S(0.5) - S(0)}{0.5} = \frac{3.65 - 0}{0.5}$$

$= 7.3 \ ms^{-1}$

Again, we have the 2 points backward difference formula:

$$f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h}$$

at time, $t = 2$ s, $h = 2 - 1.5 = 0.5$

$$V(2) = S'(2) = \frac{S(2) - S(2 - 0.5)}{0.5}$$

$$= \frac{S(2) - S(1.5)}{0.5}$$

$$= \frac{12.15 - 9.9}{0.5}$$

$$= 4.5 \ ms^{-1}$$

(b) Use three points central difference formula to approximate the runner's speed at times $t = 0.5$ s and $t = 1.25$ s.

1.25

t	0	0.5	1	1.5	2
s	0	3.65	6.8	9.9	12.15

The three points central difference formula for first derivative,

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

at time $t = 0.5$ s, $h = \frac{1-0}{2} = 0.5$

$$v(0.5) = s'(0.5) = \frac{s(0.5+0.5) - s(0.5-0.5)}{2 \times 0.5}$$

$$= \frac{s(1) - s(0)}{1}$$

$$= \frac{6.8 - 0}{1} = 6.8\ ms^{-1}$$

Again, at $t = 1.25\,s$,

$$h = \frac{1.5 - 1}{2} = \frac{0.5}{2} = 0.25$$

$$V(1.25) = S'(1.25) = \frac{S(1.25 + 0.25) - S(1.25 - 0.25)}{2 \times 0.25}$$

$$= \frac{S(1.50) - S(1)}{0.5}$$

$$= \frac{9.9 - 6.8}{0.5}$$

$$= 6.2\ ms^{-1}$$

ⓒ Use three points central difference formula to approximate the runner's acceleration at time $t = 1s$.

t	0	0.5	1	1.5	2
s	0	3.65	6.8	9.9	12.15

we have, 3 points central difference formula for 2nd derivate:

$$f''(x_0) = \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2}$$

at time $t = 1s$, $h = 0.5$

$$a(1) = s''(1) = \frac{s(1-0.5) - 2s(1) + s(1+0.5)}{(0.5)^2}$$

$$= \frac{s(0.5) - 2s(1) + s(1.5)}{(0.5)^2}$$

$$= \frac{3.65 - 2 \times 6.8 + 9.9}{(0.5)^2} = -0.2 \ ms^{-2}$$

(d) Write down MATLAB Code to estimate the speed and acceleration at time $t = 1.5\,s$ using three points central difference formulas.

```
>> clear
>> x = [0  0.5  1  1.5  2];
>> y = [0  3.65  6.8  9.9  12.25];
>> h = x(4) - x(3);
>> D1 = (y(5) - y(3))/(2*h)
>> D2 = (y(3) - 2*y(4) + y(5))/(h^2)
```

Rough

$t \rightarrow$	$0 \nearrow x(1)$	$0.5 \nearrow x(2)$	$1 \nearrow x(3)$	$1.5 \nearrow x(4)$	$2 \nearrow x(5)$
$S \rightarrow$	$0 \downarrow y(1)$	$3.65 \downarrow y(2)$	$6.8 \downarrow y(3)$	$9.9 \downarrow y(4)$	$12.15 \downarrow y(5)$

$$S'' = \frac{S(x_0-h) - 2S(x_0) + S(x_0+h)}{h^2}$$

$$= \frac{S(1.5-0.5) - 2S(1.5) + S(1.5+0.5)}{h^2}$$

$$= \frac{S(1) - 2S(1.5) + S(2)}{h}$$

$$= \boxed{\frac{y(3) - 2y(4) + y(5)}{h}}$$