CIRCUIT THEOREMS

EEE301

INTRODUCTION

- The previous approaches to circuit analysis do not modify the circuit prior to analysis and present a disadvantage; for a large, complex circuit, tedious computation is involved i.e. no of nodes or meshes are proportional to the equations generated.
- To handle the complexity, engineers over the years have developed some theorems to simplify circuit analysis
- These theorems include; Thevenin’s and Norton’s theorems, Source transformation, Superposition ,Maximum Power Transfer, Millmans theorem etc.

LINEARITY PROPERTY

- Linearity is the property of an element describing a linear relationship between cause and effect.
- The property is a combination of both the homogeneity (scaling) property and the additivity property.
- **Homogeneity property** requires that if the input (also called the *excitation) is multiplied by a constant, then the output (also called the response) is multiplied by the same constant.*
- *Using Ohms law,*

$$v = iR \quad (1)$$

- If the current is increased by a constant *k, then voltage increases* correspondingly by *k, that is,*

$$kiR = kv \quad (2)$$

- **Additivity property** requires that the response to a sum of inputs is the sum of the responses to each input applied separately. Using the voltage-current relationship of a resistor,

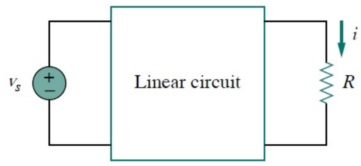
$$\left.\begin{aligned} v_1 &= i_1 R \\ v_2 &= i_2 R \end{aligned}\right\} \quad (3)$$

- *Applying $i_1 + i_2$ we get*

$$v = (i_1 + i_2)R = i_1 R + i_2 R = v_1 + v_2 \quad (4)$$

- A resistor is a linear element because the voltage-current relationship satisfies both the **homogeneity** and the **additivity** properties
- ***A linear circuit is one whose output is linearly related (or directly proportional) to its input***.

Linearity illustrated

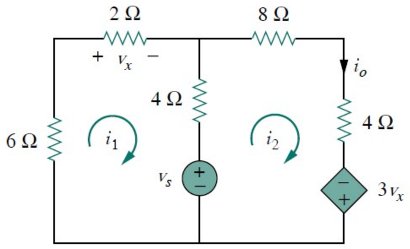


A linear circuit with input v_s *and* output *i*.

- The linear circuit has no independent sources inside it.
- It is excited by a voltage source *vs , which serves as the input.*
- *The* circuit is terminated by a load *R. We may take the current i through R as* the output.
- Suppose v_s *= 10 V gives i = 2 A. According to the linearity* principle, v_s *= 1 V will give i = 0.2 A. By the same token, i = 1 mA* must be due to v_s *=5 mV.*

Example

- Find i_o *when* v_s *= 12 V and* v_s *= 24 V.*



- Apply KVL to both loops

$$12i_1 - 4i_2 + v_s = 0 \quad (1)$$

$$-4i_1 + 16i_2 - 3v_x - v_s = 0 \quad (2)$$

- Observe, $v_x = 2i_1$
- In (2)

$$-10i_1 + 16i_2 - v_s = 0 \quad (3)$$

- Add Eq (1) & (3)

$$2i_1 + 12i_2 = 0 \quad \Longrightarrow \quad i_1 = -6i_2$$

- Substitute in (1)

$$-76i_2 + v_s = 0 \quad \Longrightarrow \quad i_2 = \frac{v_s}{76}$$

When $v_s = 12$ V,

$$i_o = i_2 = \frac{12}{76}\text{ A}$$

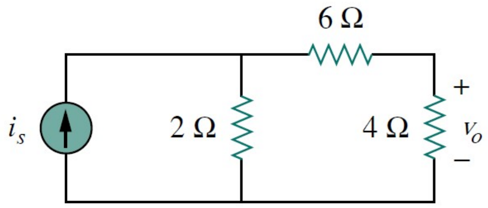
When $v_s = 24$ V,

$$i_o = i_2 = \frac{24}{76}\text{ A}$$

when V_s is doubled, i_o *doubles.*

Practice Problem

- Find v_o *when* i_s *= 15 and* i_s *= 30 A.*



SUPERPOSITION THEOREM

- If a circuit has two or more independent sources, one way to determine the value of a specific variable (voltage or current) is to use nodal or mesh analysis.
- A theory proposes we determine the contribution of each **independent source** to the **variable** and then add them up.
- Being a linear circuit, it is **homogenous** and **additive** .
- The approach is known as the *superposition.*

- The superposition principle states that the **voltage** across (or **current through**) an element **in a linear circuit** is the **algebraic sum** of the voltages across (or currents through) that element due to **EACH independent source** acting alone.
- *The principle of superposition helps us to analyze a linear circuit with more than one independent source by calculating the contribution of each independent source separately.*

HOW TO APPLY SUPERPOSITION

1. We consider one independent source at a time while all other independent sources are *turned off (killed, made inactive, deadened,* or *set equal to zero). This implies that we* replace every voltage source by 0 V (or a short circuit), and every current source by 0 A (or an open circuit). This way we obtain a simpler and more manageable circuit.
2. Dependent sources are left intact **because they are controlled by circuit variables**.

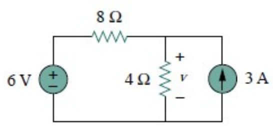
Steps to apply Superposition Principle:

1. **Systematically** turn off all independent sources except one source. Find the output (voltage or current) due to that active source using nodal or mesh analysis.
2. **Repeat** step 1 for each of the other independent sources.
3. Find the total contribution by **adding algebraically** all the contributions due to the independent sources.

- Note: when current i_1 *flows through resistor R, the power is* $p_1 = Ri_1^2$, and when current i_2 *flows through R, the power is* $p_2 = Ri_2^2$
- If current $i_1 + i_2$ *flows through R, the power absorbed* is $p_3 = R(i_1 + i_2)^2 = Ri_1^2 + Ri_2^2 + 2Ri_1i_2 = p_1 + p_2$.
- *Thus, the power relation is nonlinear.*
- If the power value is needed, the current through (or voltage across) the element must be calculated first using superposition.

Example

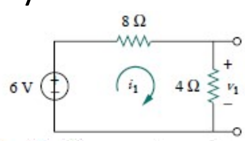
- Use the superposition theorem to find *v in the circuit*



- *We determine the contributions due to the 6-V voltage source and*the 3-A current source, respectively.

$$v = v_1 + v_2$$

- To obtain *v1, we set the current* source to zero, apply KVL



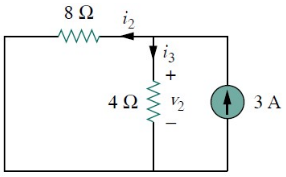
$$12i_1 - 6 = 0 \quad \Longrightarrow \quad i_1 = 0.5 \text{ A}$$

$$v_1 = 4i_1 = 2 \text{ V}$$

- Can use voltage division to get v_1

$$v_1 = \frac{4}{4+8}(6) = 2 \text{ V}$$

- To get v_2, *we set the voltage source to zero, apply* current division.

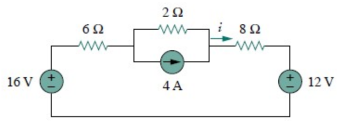


$$i_3 = \frac{8}{4+8}(3) = 2 \text{ A}$$

$$v_2 = 4i_3 = 8 \text{ V}$$

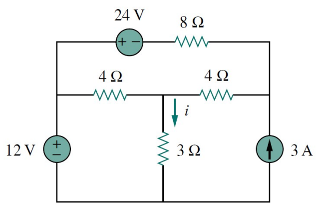
- Algebraic sum,
  $v = v_1 + v_2 = 2 + 8 = 10 \text{ V}$

- Find i



Example

- Use superposition theorem to find i.

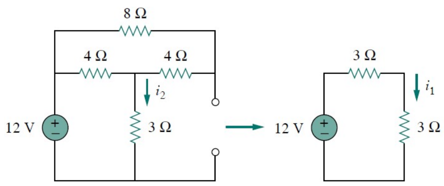


- Three sources, therefore

$$i = i_1 + i_2 + i_3$$

- Where i_1, i_2, *and* i_3 *are due to the 12-V, 24-V, and 3-A sources respectively*

- Consider the 12V source, others killed off!



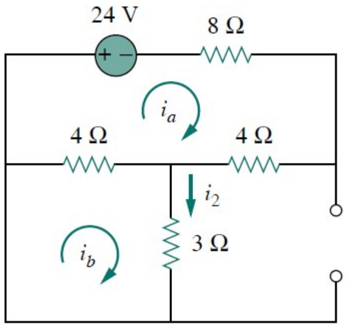
- The 12Ω *parallel with 4* Ω gives 12 × *4/16 = 3* Ω*.*

$$i_1 = \frac{12}{6} = 2 \text{ A}$$

- To get i_2, *consider the circuit*

Example

- Mesh analysis is suitable



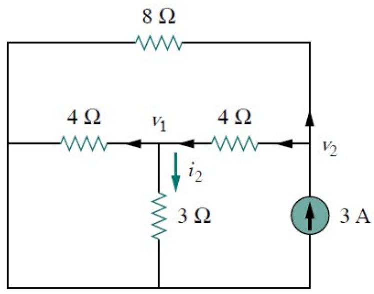
$$16i_a - 4i_b + 24 = 0 \quad \Longrightarrow \quad 4i_a - i_b = -6 \qquad (1)$$

$$7i_b - 4i_a = 0 \quad \Longrightarrow \quad i_a = \frac{7}{4}i_b \qquad (2)$$

- Substituting Eq. (2) into Eq. (1) gives

$$i_2 = i_b = -1$$

- To get i_3, consider the circuit



- Using nodal analysis,

Example

$$3 = \frac{v_2}{8} + \frac{v_2 - v_1}{4} \quad \Longrightarrow \quad 24 = 3v_2 - 2v_1 \qquad (3)$$

$$\frac{v_2 - v_1}{4} = \frac{v_1}{4} + \frac{v_1}{3} \quad \Longrightarrow \quad v_2 = \frac{10}{3}v_1 \qquad (4)$$

- Substituting Eq. (4) into Eq. (3) leads to *v_1 = 3 and*

$$i_3 = \frac{v_1}{3} = 1 \text{ A}$$

- Thus,

$$i = i_1 + i_2 + i_3 = 2 - 1 + 1 = 2 \text{ A}$$

SOURCE TRANSFORMATION

- ***Source transformation*** *is another tool for* simplifying circuits. Basic to these tools is the concept of *equivalence.*
- An equivalent circuit is one whose *v-i characteristics are* identical with the original circuit.
- It is possible in circuit analysis to substitute a voltage source in series with a resistor for a current source in parallel with a resistor, or vice versa, as shown below.

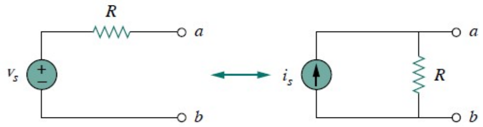


Fig 1. Transformation of independent sources.

- A source transformation is the process of replacing a voltage source v_s in series with a resistor *R by a current source* i_s *in parallel* with a resistor *R, or vice versa.*
- Source transformation is the process of replacing a voltage source $\boldsymbol{v_s}$ in series with a resistor *R by a current source* $\boldsymbol{i_s}$ *in parallel* with a resistor *R, or vice versa.*

SOURCE EQUIVALENCE

- If the sources in Fig 1 are turned off, the equivalent resistance at terminals *a-b in both circuits is R.* Also, when terminals a-b are short-circuited,
- The short-circuit current flowing from *a to b is*

$$i_{sc} = v_s/R$$

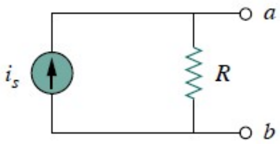
- In the circuit on the left-hand side and $i_{sc} = i_s$ *for the circuit on the right-hand* side.
- Thus, $v_s/R = i_s$ in order for the two circuits to be equivalent. Hence, source transformation requires that

$$v_s = i_s R \qquad \text{or} \qquad i_s = \frac{v_s}{R}$$

- Source transformation also applies to dependent sources, provided the dependent variable is cautiously treated.

Observations

1. Note that the arrow of the current source is directed toward the positive terminal of the voltage source.



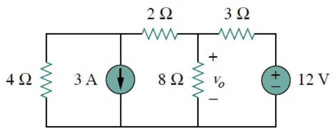
2. Note also that source transformation is not possible when *R = 0, which is the case with an ideal voltage source.*

$$i_s = \frac{v_s}{R}$$

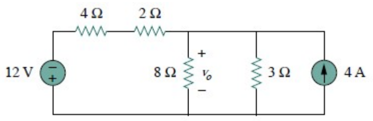
- However, for a practical, non-ideal voltage source, *R = 0.*
- Similarly, an ideal current source with *R =∞cannot be* replaced by a finite voltage source.

Example

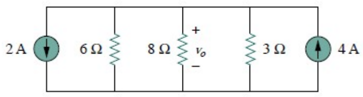
- Use source transformation to find v_o *in the circuit below*



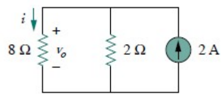
- *Transform current and Voltage sources to obtain:*



- *Combine 4Ω and 2Ω resistors and transform to 12V source to current source.*



- Combine the sources algebraically; -2A+ 4A, the equivalent parallel resistor 2Ω. (Read about Millman's Theorem!)



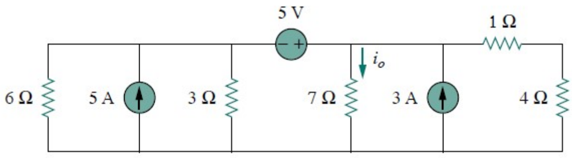
- We use current division

$$i = \frac{2}{2+8}(2) = 0.4\text{A}$$

$$v_o = 8i = 8(0.4) = 3.2 \text{ V}$$

$$v_o = (8 \parallel 2)(2 \text{ A}) = \frac{8 \times 2}{10}(2) = 3.2 \text{ V}$$

- Find i_o *in the circuit below using source transformation.*



THEVENIN'S THEOREM

- A households power outlet may be connected to different appliances constituting a variable load.
- Each time the variable element is changed, the entire circuit has to be analyzed all over again.
- To avoid this problem, Thevenin's theorem provides a technique by which the fixed part of the circuit is replaced by an equivalent circuit.

- According to Thevenin's theorem, the linear circuit in Fig 1.

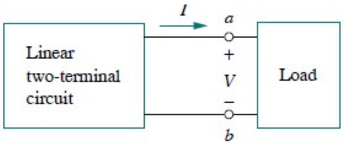


Fig 1. Original circuit

- Can be replaced by Fig 2.

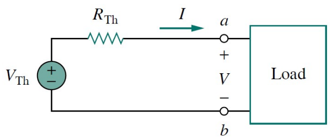


Fig 2 Thevenin equivalent circuit

Thevenins Theorem

- Thevenin's theorem states that **a linear two-terminal circuit** can be **replaced by an equivalent circuit** consisting of a **voltage source V_{Th}** *in __series__ with* a **resistor R_{Th}*, where V_{Th} is the open-circuit voltage*** *at the terminals* and ***R_{Th} is the input or equivalent resistance*** *at the terminals when* the **independent sources are turned off**.
- Remember, 2 circuits are equivalent if they have the same V-I characteristics at their terminals. If circuits in Figs 1 & 2 have terminals open circuited, no currents flows therfore the open circuit voltage acros terminals a-b must be equal to the voltage source V_{Th}

$$V_{\mathrm{Th}} = v_{oc}$$

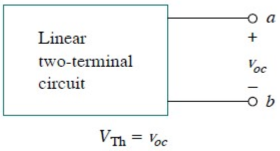


Fig 3 Finding V_{Th}.

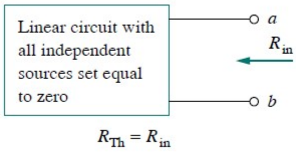
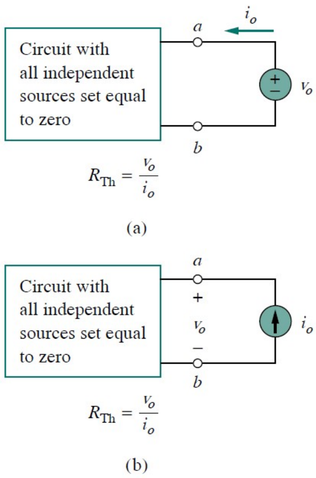


Fig 4 Finding R_{Th}.

- Again with the **load disconnected** and terminals *a-b open-circuited,* **we turn off all independent sources**. The input resistance (or equivalent resistance) of the dead circuit at the terminals *a-b in Fig.* $R_{\mathrm{Th}} = R_{\mathrm{in}}$ to R_{Th}

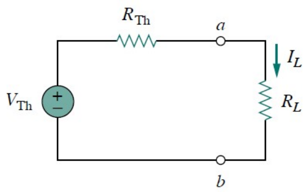
Finding R_{Th}

- CASE I: If the network has no dependent sources, we turn off all independent sources. *RTh is the input resistance of the network looking* between terminals *a-b, as shown in Fig. 4.*
- CASE II: If the network has dependent sources, we turn off all independent sources. As with **superposition**, dependent sources are not to be turned off because they are controlled by circuit variables. Two approaches illustrated by Fig 5 a & b.
- In either approach we may assume any value of v_o and i_o. For example, we may use v_o = 1 V or i_o = 1 A.
- It often occurs that R_{Th} takes a negative value. In this case, the negative resistance (v = −iR) implies that the circuit is supplying power



I_L, V_L

- If equivalent circuit is below,

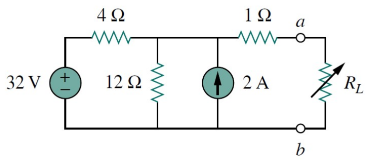


$$I_L = \frac{V_{Th}}{R_{Th} + R_L}$$

$$V_L = R_L I_L = \frac{R_L}{R_{Th} + R_L} V_{Th}$$

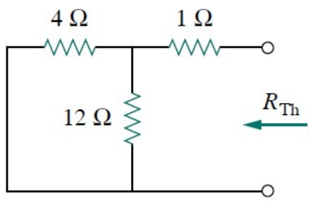
- Thevenin equivalent is a simple voltage divider, yielding V_L *by mere inspection.*

- Example: Find the Thevenin equivalent circuit of the circuit shown below, to the left of the terminals *a-b. Then find the current through* R_L = *6, 16,* and 36 .

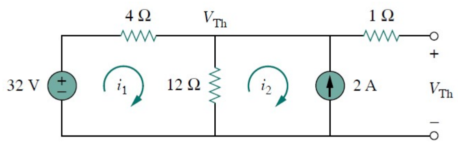


- Solution: We find R_{Th} *by* ***turning off*** *the 32-V voltage source (**replacing it with*** **a short circuit**) and the 2-A current source (**replacing it with an open circuit**).

EXAMPLE



$$R_{Th} = 4 \parallel 12 + 1 = \frac{4 \times 12}{16} + 1 = 4\ \Omega$$



- To find *VTh, consider the circuit above. Applying mesh* analysis to the two loops, we obtain

$$-32 + 4i_1 + 12(i_1 - i_2) = 0, \qquad i_2 = -2\text{ A}$$

- Solving for i_1, we get i_1 = 0.5 A. Thus

$$V_{Th} = 12(i_1 - i_2) = 12(0.5 + 2.0) = 30\text{ V}$$

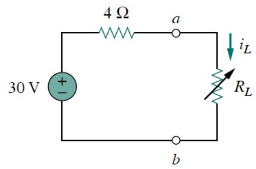
- Easier using nodal analysis at top node.

$$\frac{32 - V_{Th}}{4} + 2 = \frac{V_{Th}}{12}$$

$$96 - 3V_{Th} + 24 = V_{Th}$$

$$V_{Th} = 30\text{ V}$$

- The Thevenin Equivalent circuit is



EXAMPLE

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{30}{4 + R_L}$$

When $R_L = 6$,

$$I_L = \frac{30}{10} = 3\text{ A}$$

When $R_L = 16$,

$$I_L = \frac{30}{20} = 1.5\text{ A}$$

When $R_L = 36$,

$$I_L = \frac{30}{40} = 0.75\text{ A}$$

Try practice problem 4.8

NORTON'S THEOREM

- In 1926, about 43 years after Thevenin published his theorem, E. L. Norton, an American engineer at Bell Telephone Laboratories, proposed a similar theorem.
- Norton's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a current source I_N *in parallel with* a resistor R_N, *where* I_N *is the short-circuit current through the terminals* and R_N *is the input or equivalent resistance at the terminals when the* independent sources are turned off.
- In fact, from what we know about source transformation, the Thevenin and Norton resistances are equal:

$$R_N = R_{\mathrm{Th}} \quad (1)$$

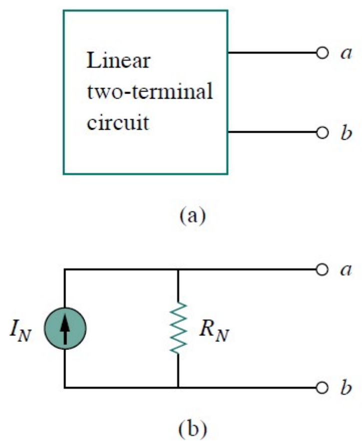
Norton's Theorem

- To find the Norton current I_N, *we determine the short-circuit current* flowing from terminal *a to b*

$$I_N = i_{sc} \quad (2)$$

- Observe the close relationship between Norton's and Thevenin's theorems: $R_N = R_{Th}$

$$I_N = \frac{V_{\mathrm{Th}}}{R_{\mathrm{Th}}} \quad (3)$$



(a) Original circuit, (b) Norton equivalent circuit.

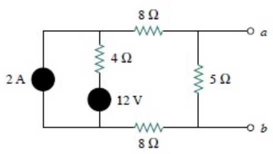
Norton Thevenin Nexus

- This is essentially source transformation. For this reason, **source transformation** is often called **Thevenin-Norton transformation**.
- Since V_{Th}, I_N, *and* R_{Th} *are related according to Eq. (3), to determine* the Thevenin or Norton equivalent circuit requires that we find:

- The open-circuit voltage v_{oc} *across terminals a and b.*
- The short-circuit current i_{sc} *at terminals a and b.*
- The equivalent or input resistance R_{in} *at terminals a and b when* all independent sources are turned off.

$$V_{Th} = v_{oc}$$
$$I_N = i_{sc}$$
$$R_{Th} = \frac{v_{oc}}{i_{sc}} = R_N$$

EXAMPLES



- We find R_N *in the same way we find* R_{Th} *in the Thevenin equivalent circuit.*
- Set the independent sources equal to zero. This leads to the circuit, from which we find R_N

$$R_N = 5 \parallel (8+4+8) = 5 \parallel 20 = \frac{20 \times 5}{25} = 4\ \Omega$$

- We ignore the 5- *resistor because it has been short-circuited. Applying* mesh analysis, we obtain

$$i_1 = 2\text{ A}, \qquad 20i_2 - 4i_1 - 12 = 0$$

- Thus,

$$i_2 = 1\text{ A} = i_{sc} = I_N$$

- Find the Norton equivalent circuit for the circuit in Fig below.