HOMEWORK 4 - Thuyen Dang (1657399)

1. • The only one pure strategy Nash equilibrium is

$$s_1 + s_2 = 1{,}000{,}000 \qquad (*)$$

• Fix s_1, which is the sum of money son I wants. If son II wants $s_2 > 1000000 - s_1$, he'll get 0. On the other hand, if son II wants $s_2 < 1000000 - s_1$, he'll get this amount, which is less than the optimal value.

• Similarly, if son I choose a amount different to $1000000 - s_2$, he'll get less money.

• Therefore, $(*)$ is the only pure Nash equilibrium. ▨

2. a) • Payoff matrix of 1st round:

$$\begin{array}{cc} & \text{Player I} \\ & \begin{array}{cc} \text{GRIM} & \text{ALLD} \end{array} \\ \text{Player I} \begin{array}{c} \text{GRIM} \\ \text{ALLD} \end{array} & \begin{pmatrix} R, R & S, T \\ T, S & P, P \end{pmatrix} \end{array}$$

• Payoff matrix of n th round $(n \geq 2)$

$$\begin{array}{cc} & \text{Player II} \\ & \begin{array}{cc} \text{GRIM} & \text{ALLD} \end{array} \\ \text{Player I} \begin{array}{c} \text{GRIM} \\ \text{ALLD} \end{array} & \begin{pmatrix} R, R & P, P \\ P, P & P, P \end{pmatrix} \end{array}$$

- Therefore, the payoff matrix after m rounds is

$$\begin{array}{cc|cc} & & \text{Player II} & \\ & & \text{GRIM} & \text{ALLD} \\ \text{Player I} & \begin{array}{c}\text{GRIM}\\ \text{ALLD}\end{array} & \multicolumn{2}{c}{\begin{pmatrix} mR,\ mR & S+(m-1)P,\ T+(m-1)P \\ T+(m-1)P,\ S+(m-1)P & mP,\ mP \end{pmatrix}} \end{array}$$

b) Suppose $m > \dfrac{T-P}{R-P}$, then

$$\mathbb{E}(\text{GRIM}, \text{GRIM}) = mR > T + (m-1)P = \mathbb{E}(\text{ALLD}, \text{GRIM})$$

Therefore, GRIM is stable against invasion by ALLD.

c) - Payoff matrix of nth round $(1 \leq n \leq m-1)$.

$$\begin{array}{cc|cc} & & \text{Player II} & \\ & & \text{GRIM} & \text{GRIM}^* \\ \text{Player I} & \begin{array}{c}\text{GRIM}\\ \text{GRIM}^*\end{array} & \multicolumn{2}{c}{\begin{pmatrix} R,R & R,R \\ R,R & R,R \end{pmatrix}} \end{array}$$

- Payoff matrix of final round m.

$$\begin{array}{cc|cc} & & \text{Player II} & \\ & & \text{GRIM} & \text{GRIM}^* \\ \text{Player I} & \begin{array}{c}\text{GRIM}\\ \text{GRIM}^*\end{array} & \multicolumn{2}{c}{\begin{pmatrix} R,R & S,T \\ T,S & P,P \end{pmatrix}} \end{array}$$

- Therefore, the payoff matrix after m rounds is

$$\begin{array}{cc|cc} & & \text{Player II} & \\ & & \text{GRIM} & \text{GRIM}^* \\ \text{Player I} & \begin{array}{c}\text{GRIM}\\ \text{GRIM}^*\end{array} & \multicolumn{2}{c}{\begin{pmatrix} mR,\ mR & (m-1)R+S,\ (m-1)R+T \\ (m-1)R+T,\ (m-1)R+S & (m-1)R+P,\ (m-1)R+P \end{pmatrix}} \end{array}$$

- Since $T > R$ and $P > S$, we have $(m-1)R+T > mR$ and $(m-1)R+P > (m-1)R+S$, so GRIM^* dominates GRIM.

d) • If everyone uses GRIM*, then everyone is bound to get L on the last round, so

GRIM* vs GRIM* in m rounds

is equivalent to

GRIM vs GRIM in $(m-1)$ round.

• From result of c), the dominant strategy against GRIM* is

GRIM**: same as GRIM, but always defects on last 2 rounds.

e) Continuing the argument, we obtain ALLD. ∎

(3) a) Probability of 1st game occurs : 1

" 2nd " " : δ

" 3rd " " : δ^2

$\vdots$

" nth " " : δ^n

$\vdots$

$\Rightarrow$ The expected number of rounds play is

$$\langle m \rangle = 1 \cdot 1 + 1 \cdot \delta + 1 \cdot \delta^2 + \ldots + 1 \cdot \delta^n + \ldots$$

$$= 1 + \delta \sum_{n=0}^{\infty} \delta^n$$

$$= 1 + \frac{\delta}{1-\delta}.$$

b) From 2a), the expected payoff for :

- GRIM vs ALLD $= S + (\langle m \rangle - 1)P = S + \dfrac{\delta P}{1-\delta}$
- GRIM vs GRIM $= \langle m \rangle R = \left(1 + \dfrac{\delta}{1-\delta}\right)R$
- ALLD vs ALLD $= \langle m \rangle P = \left(1 + \dfrac{\delta}{1-\delta}\right)P.$

c) From 2b), we only need to show

$$1 + \frac{\delta}{1-\delta} > \frac{T-P}{R-P} \Leftrightarrow \frac{1}{1-\delta} > \frac{T-P}{R-P}$$

$$\Leftrightarrow 1 - \delta < \frac{R-P}{T-P} \Leftrightarrow \delta > \frac{T-R}{T-P}$$ (always true by assumption)

Therefore, GRIM is stable against ALLD $\blacksquare$

4 States : $\{cc, cd, dc, dd\}$. Stationary distribution $\begin{cases} \pi = (\pi_1, \pi_2, \pi_3, \pi_4) \\ \pi = \pi M \\ \sum \pi_i = 1 \end{cases}$

$\boxed{\text{Matrix I}}$ TFT vs GRIM

a) $$M_1 = \begin{pmatrix} (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \\ (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 & \varepsilon^2 & \varepsilon(1-\varepsilon) \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \end{pmatrix}$$

b) $$I - M_1 \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ Need to solve $\begin{cases} \pi(I - M_1) = 0 \\ \sum \pi_i = 1 \end{cases}$

or $$\begin{cases} \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \\ \pi_1 = 0 \\ \pi_2 = 0 \\ \pi_3 = 0 \\ -\pi_1 - \pi_2 - \pi_3 = 0 \end{cases} \Rightarrow \boxed{\pi = (0,0,0,1)}$$

$\boxed{\text{Matrix II}}$ TFT vs ALLC

a) $$M = \begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \\ (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \end{pmatrix}$$

b) $$I - M \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

As for $\boxed{\text{Matrix I}}$, $\pi = (0,0,0,1)$.

$\boxed{\text{Matrix III}}$ GRIM vs ALLC

a) $$M = \begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \end{pmatrix}$$

b) $$I - M \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \pi = (0, 0, 0, 1)$$

$\boxed{\text{Matrix IV}}$ TFT vs TFT

a) $$M = \begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 & \varepsilon^2 & \varepsilon(1-\varepsilon) \\ \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 & \varepsilon^2 & \varepsilon(1-\varepsilon) \\ \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 & \varepsilon^2 & \varepsilon(1-\varepsilon) \end{pmatrix}$$

b) $$I - M \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \pi = (0, 0, 0, 1)$$

Matrix V   GRIM vs GRIM

a) $$M = \begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \end{pmatrix}$$

b) $$I - M \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \pi = (0, 0, 0, 1)$$

Matrix VI   ALLC vs ALLC

a) $$M = \begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \end{pmatrix}$$

b) $$I - M \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \pi = (0, 0, 0, 1)$$

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c) Payoff for each strategy is P

d) $(0, 0, 0, 1)$ is the only Nash equilibrium, since as $\varepsilon \to 0$, we always get same payoff.