HOMEWORK I

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(1) a) Since $P_n(t) = Pr(X(t) = n)$, we have $\sum_{n=0}^{\infty} P_n(t) \leq 1$.

Let S_k be the k-th sojourn time, then S_k has Exponential (λ_k) distribution.

$(\Rightarrow)$ $\sum_{n=0}^{\infty} P_n(t) = 1$ implies $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty$.

Suppose by contradiction that $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} < \infty$, then $\mathbb{E}\left[\sum_{n=0}^{\infty} S_n\right] = \cancel{\sum \frac{1}{\lambda_n}} \sum_{n=0}^{\infty} \frac{1}{\lambda_n} < \infty$

or the expected time for the population to explode is finite. Thus $0 < Pr(X(t) = +\infty) = 1 - \sum_{n=0}^{\infty} P_n(t)$ or $\sum_{n=0}^{\infty} P_n(t) < 1$ (contradiction!)

$(\Leftarrow)$ $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty$ implies $\sum_{n=0}^{\infty} P_n(t) = 1$.

We have $\mathbb{E}\left[\exp\left(-\sum_{n=0}^{\infty} S_n\right)\right] = \prod_{n=0}^{\infty} \mathbb{E}\left[e^{-S_n}\right]$ (by independence)

$$= \prod_{n=0}^{\infty} \frac{\lambda_n}{1+\lambda_n} \leq \prod_{n=0}^{\infty} \frac{1}{e^{1/\lambda_n}} = \frac{1}{\exp\left(\sum_{n=0}^{\infty} \frac{1}{\lambda_n}\right)} \to 0$$

as $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty$.

Therefore $P\left(\sum_{n=0}^{\infty} S_n = \infty\right) = P\left(\exp\left(-\sum_{n=0}^{\infty} S_n\right) = 0\right) = 1$

which implies $Pr(X(t) = \infty) = 0$, so $\sum_{n=0}^{\infty} P_n(t) = 1$.

b) "Explosive process": choose λ_n s.t. $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} < \infty \Rightarrow$ any geometric series works. $(\lambda_n = n^{\alpha}, \alpha > 1)$. $\blacksquare$

② a) Use Forward Kolmogorov equation with $\lambda_n = \lambda n + \nu$, $\mu_n = \mu n$:

$$P'_{i,j}(t) = [\lambda(j-1)+\nu] P_{i,j-1}(t) - [(\lambda+\mu)j+\nu] P_{i,j}(t)$$
$$(1) \qquad + \mu(j+1) P_{i,j+1}(t), \quad j \geq 1.$$

We deduce

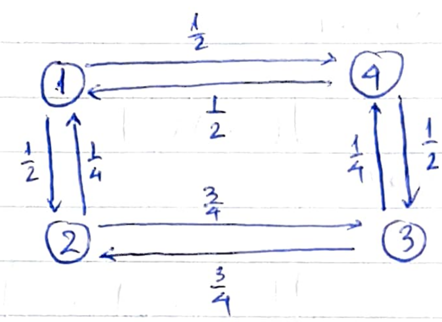
$$\frac{d\bar{n}}{dt} = \frac{d}{dt} \mathbb{E}[X(t)]$$
$$= \frac{d}{dt} \sum_{j=1}^{\infty} j P_{i,j}(t)$$
$$= \sum_{j=1}^{\infty} j \frac{d}{dt} P_{i,j}(t)$$
$$\overset{\text{by }(1)}{=} \sum_{j=1}^{\infty} j \Big([\lambda(j-1)+\nu] P_{i,j-1}(t) - [(\lambda+\mu)j+\nu] P_{i,j}(t) + \mu(j+1) P_{i,j+1}(t) \Big)$$
$$= \sum_{j=1}^{\infty} \left[\lambda(j-1)j P_{i,j-1}(t) - (\lambda+\mu)j^2 P_{i,j}(t) + \mu j(j+1) P_{i,j+1}(t) \right] + \sum_{j=1}^{\infty} \nu j \left[P_{i,j-1}(t) - P_{i,j}(t) \right]$$
$$= (\lambda - \mu)\bar{n} + \nu.$$

b) Example: The logistic model (in computational challenge 2) with immigration:

$$\lambda_n = rn\left(1 - \frac{n}{2K}\right) + \nu$$
$$\mu_n = \frac{rn^2}{2K}$$
$$r = 0.015, \quad K = 2, \quad \nu = 0.01.$$

~~Correspond~~ Corresponding deterministic equation $\frac{dn}{dt} = rn\left(1-\frac{n}{K}\right)+v$. ∎

(3) a)



b) We have $p(1\to 2)>0$, $p(2\to 3)>0$, $p(3\to 4)>0$, $p(4\to 1)>0$, so all the states communicate with each other, therefore the chain is irreducible.

Moreover, the chain has finite states, so it is positive recurrent.

For any state $i \in \{1,2,3,4\}$, we have $P^2_{ii} > 0$, $P_{ii} = 0$, so the chain is periodic with period $= 2$.

c) The stationary distribution π satisfies $\begin{cases} \pi = \pi P \\ \sum_{i=1}^{4} \pi_i = 1. \end{cases}$

Solving the system yields $\pi = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right)$ ∎

(4) Let S_n be the sojourn time of n-th protein, then S_n has Poisson(λ)

T_n be the time ~~promoter~~ that the promoter is occupied by n-th protein, then T_n are i.i.d r.v.s with mean μ.

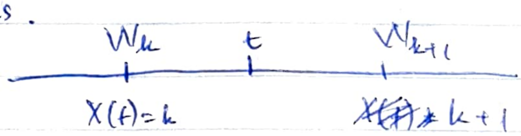
Then

$$\frac{\text{Unoccupied Time}}{\text{Total Time}} = \lim_{N\to\infty} \frac{\sum_{n=0}^{N} S_n}{\sum_{n=0}^{N}(S_n+T_n)} = \lim_{N\to\infty} \frac{\frac{1}{N}\sum_{n=0}^{N} S_n}{\frac{1}{N}\sum_{n=0}^{N}(S_n+T_n)}$$

$$= \frac{E[S_n]}{E[S_n]+E[T_n]} \quad \text{(by Law of Large Numbers)}$$

$$= \frac{1}{1+\lambda\mu}$$ ∎

(5) Let $X(t)$ be the Poisson process modelling the sequence of action potentials.



Then, we conclude that

$$T(t) = \min_{X(t) \le k \le X(t)+1} |t - W_k|.$$

Fixed ~~s ∈ R~~, $s \in \mathbb{R}$,

$$\Pr(T(t) > s) = \Pr\left(\min_{X(t) \le k \le X(t)+1} |t - W_k| > s \right)$$

$$= \Pr\left(|t - W_k| > s,\ |t - W_k| > s \mid X(t) = k \right)$$

$$= \Pr\left(t - W_k > s,\ W_{k+1} - t > s \mid X(t) = k \right)$$

$$= \Pr\left(W_k < t - s,\ W_{k+1} > t + s \mid X(t) = k \right)$$

$$= \Pr\left(X(t+s) - X(t-s) = 0,\ X(t-s) = k \mid X(t) = k \right)$$

$$= \Pr\left(X(t+s) - X(t) = 0 \mid X(t) = k \right) \Pr\left(X(t) - X(t-s) = 0 \mid X(t) = k \right)$$

$$= e^{-\lambda s} \cdot e^{-\lambda s}$$

$$= e^{-2\lambda s}$$

Therefore

$$F_{T(t)}(s) = 1 - \Pr(T(t) > s) = 1 - e^{-2\lambda s},$$

so

$$f_{T(t)}(s) = 2\lambda e^{-2\lambda s}.$$

Hence, $T(t)$ has Exponential (2λ) and $\mathbb{E}[T(t)] = \frac{1}{2\lambda}$ ∎