The code for this homework is published at:

https://github.com/trienthuyendang/M6397-StochasticProcesses/tree/master/Homework

Question 1. a) In this model, Y_1 acts as prey, while Y_2 acts as predator. The prey Y_1 reproduce with a constant growth rate c_1 and the parents remaining in the population (first reaction). The prey Y_1 serves as "food" supply for Y_2, while predator Y_2 reproduces with rate c_2 only when the food is available (second reaction). Moreover, Y_2 has constant death rate c_3 (final reaction).

The corresponding ODE is

$$\frac{dY_1}{dt} = c_1Y_1 - c_2Y_1Y_2, \tag{1a}$$

$$\frac{dY_2}{dt} = c_2Y_1Y_2 - c_3Y_2. \tag{1b}$$

Suppose $c_i \geq 0$ for $i = 1, 2, 3$. One the one hand, *species Y_1 can exists in isolation.* Indeed, without Y_2 we have $c_2 = 0$, so (1a) becomes $\frac{dY_1}{dt} = c_1Y_1$, or Y_1 will be constant if $c_1 = 0$ or experience explosion in population if $c_1 > 0$. On the other hand, in the absence of Y_1, (1b) becomes $\frac{dY_2}{dt} = -c_3Y_2$. This implies that Y_2 cannot reproduce without Y_1. The population of Y_2 is constant if $c_3 = 0$ and becomes extinct eventually if $c_3 > 0$.

b) We simulate by Gillespie algorithm and obtain Figure 1. Table 1 presents corresponding parameters for the algorithm.

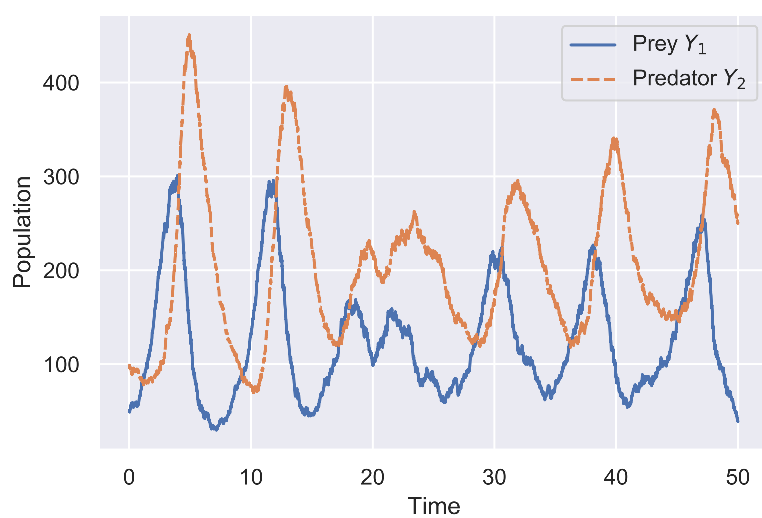

Figure 1 – Dynamics of prey Y_1 and predator Y_2 with $c_1 = 1, c_2 = 0.005, c_2 = 0.6, Y_1(0) = 50$, $Y_2(0) = 100$.

Reaction	Propensity Function	Change of (Y_1, Y_2)
$Y_1 \xrightarrow{c_1} 2Y_1$	c_1Y_1	(1,0)
$Y_1 + Y_2 \xrightarrow{c_2} 2Y_2$	$c_2Y_1Y_2$	(-1,1)
$Y_2 \xrightarrow{c_3} \varnothing$	c_3Y_3	(0,-1)

Table 1 – Parameters for Gillespie algorithm

c) To make it more likely for species Y_1 to go extinct, the parameters should satisfy $\frac{dY_1}{dt} \geq 0$ (prey population decreases) and $\frac{dY_2}{dt} \geq 0$ (predator population increases). Take into account with (1), we obtain $Y_2 \geq \frac{c_1}{c_2}$ and $Y_1 \geq \frac{c_3}{c_2}$. Before extinction, Y_1, Y_2 are greater than 1, so both inequalities satisfies if $\max\left\{\frac{c_1}{c_2}, \frac{c_3}{c_2}\right\} \leq 1$. This can be achieved by setting $c_2 \geq 1$. The bigger c_2, the faster Y_1 go extinct. The argument in a) shows that if species Y_1 goes extinct, species Y_2 dies out eventually. The simulation is shown in Figure 2.

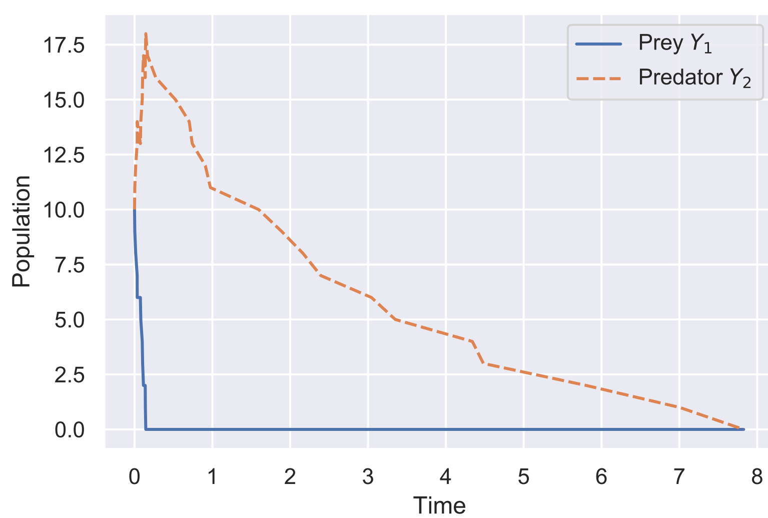

Figure 2 – Dynamics of prey Y_1 and predator Y_2 with $c_1 = 1, c_2 = 1, c_2 = 0.6, Y_1(0) = Y_2(0) = 10$.

□

Question 2. a) The fixed points are obtained by solving

$$k - \alpha_1[X] - k_a[X][Y] = 0, \text{ and}$$
$$k - \alpha_2[Y] - k_a[X][Y] = 0.$$

Subtracting side by side, we get $\alpha_1[X] = \alpha_2[Y]$, plugging back to the first equation, we get a quadratic equation which is solve in Python (see code). The fixed points of two cases are the same. They are

$$(-3162.7776997, -316.27777) \quad \text{and} \quad (3161.7776997, 316.17777).$$

In our context, only the positive one makes sense.

b) Table 2 presents parameters for Gillespie algorithm. Note that A, B, C are replaced by $\varnothing$ since we don't care about those reactants. The system volumes of those reactants are assumed to be 1.[1] Run the Gillespie algorithm, we obtain Figure 3. We now find the

Reaction	Propensity Function	Change of (X, Y)
$\varnothing \xrightarrow{k} X$	k	(1,0)
$\varnothing \xrightarrow{k} Y$	k	(0,1)
$X + Y \xrightarrow{k_a} \varnothing$	$k_a XY$	(-1,-1)
$X \xrightarrow{\alpha_1} \varnothing$	$\alpha_1 X$	(-1,0)
$Y \xrightarrow{\alpha_2} \varnothing$	$\alpha_2 X$	(0,-1)

Table 2 – Parameters for Gillespie algorithm

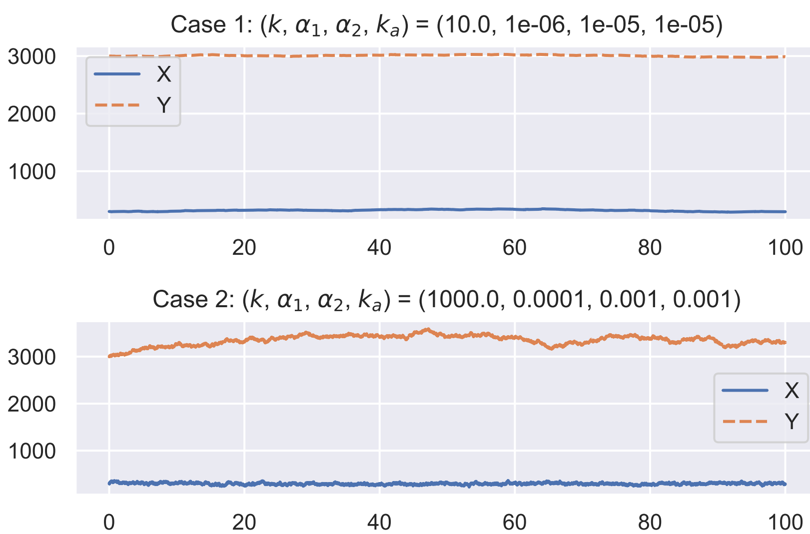

Figure 3 – Dynamics of reactants X and Y.

stationary distribution. We have

$$f_{x,y} \xrightarrow{k} f_{x+1,y}$$
$$f_{x,y} \xrightarrow{k} f_{x,y+1}$$
$$f_{x,y} \xrightarrow{\alpha_1 + k_a xy} f_{x-1,y}$$
$$f_{x,y} \xrightarrow{\alpha_2 + k_a xy} f_{x,y-1}$$

[1] The assumption is necessary to compute propensity function, see Radek Erban and Jonathan Chapman, *Stochastic Modelling of Reaction-Diffusion Processes*.

This yields the master equation

$$\frac{df_x}{dt}(t) = k\left[f_{x-1}(t) - f_x(t)\right] + \left[\alpha_1 + k_a(x+1)y\right] f_{x+1}(t) - \left[\alpha_1 + k_a xy\right] f_x(t), \tag{2a}$$

$$\frac{dg_y}{dt}(t) = k\left[g_{y-1}(t) - g_y(t)\right] + \left[\alpha_2 + k_a x(y+1)\right] g_{y+1}(t) - \left[\alpha_2 + k_a xy\right] g_y(t). \tag{2b}$$

The detail balance equations give us recursive relations for stationary distribution $f(x) := \lim_{t\to\infty} f_x(t),\ g(y) := \lim_{t\to\infty} g_y(t)$:

$$f(x) = \frac{k}{\alpha_1 + k_a xy} f(x-1),$$

$$g(y) = \frac{k}{\alpha_2 + k_a xy} g(y-1).$$

c) We observe that

$$\frac{\text{Parameter in Case 2}}{\text{Parameter in Case 1}} = 100,$$

which means in a same period of time, there are much more reactions happen in Case 2 than in Case 1 (about 100 times!). This explain why the graph in Case 2 oscillates more vigorous than in Case 1.

□

Question 3. a) We first write down the reactions from the original ODE in Table 3. Those reactions give us the transition probability of the continuous time Markov chain.

Reaction	Propensity Function	Change of (r,p)
$\varnothing \xrightarrow{k_l+\phi(p)} r$	$k_l + \phi(p)$	(1,0)
$r \xrightarrow{\gamma_r} \varnothing$	$\gamma_r r$	(-1,0)
$r \xrightarrow{k_p} r + p$	$k_p r$	(0,1)
$p \xrightarrow{\gamma_p} \varnothing$	$\gamma_p p$	(0,-1)

Table 3 – Parameters for Gillespie algorithm

$$\Pr(r(t+h) = n+1, p(t+h) = m \mid r(t) = n, p(t) = m) = (k_l + \phi(m))h + O(h),$$
$$\Pr(r(t+h) = n-1, p(t+h) = m \mid r(t) = n, p(t) = m) = \gamma_r nh + O(h),$$
$$\Pr(r(t+h) = n, p(t+h) = m+1 \mid r(t) = n, p(t) = m) = k_p nh + O(h),$$
$$\Pr(r(t+h) = n, p(t+h) = m+1 \mid r(t) = n, p(t) = m) = \gamma_p mh + O(h).$$

b) Setting $k_l = 0$, $\gamma_p = \gamma_r = k_p = k_0 = 1$, $K = 0.5$, the system becomes

$$\frac{dr}{dt} = \phi(p) - r, \tag{3a}$$

$$\frac{dp}{dt} = r - p, \tag{3b}$$

where $\phi(p) = \frac{\left(\frac{p}{K}^n\right)}{1+\left(\frac{p}{K}\right)^n}$. The fixed point(s) are solutions of $p = r, r = \phi(r)$.

- $n = 1$:
  Then $r = \phi(r) = \frac{r}{K+r}$, so either $r = 0$ or $r = \frac{1}{2}$. We obtain 2 fixed points $(r, p) = (0, 0)$ or $(r, p) = \left(\frac{1}{2}, \frac{1}{2}\right)$. To check the stability of the fixed points, one can either compute the Jacobian or draw the phase plane. Here we use Python to draw the phase portrait, see Figure 4. The stream plot shows that $(0, 0)$ is a saddle point, while $\left(\frac{1}{2}, \frac{1}{2}\right)$ is a stable

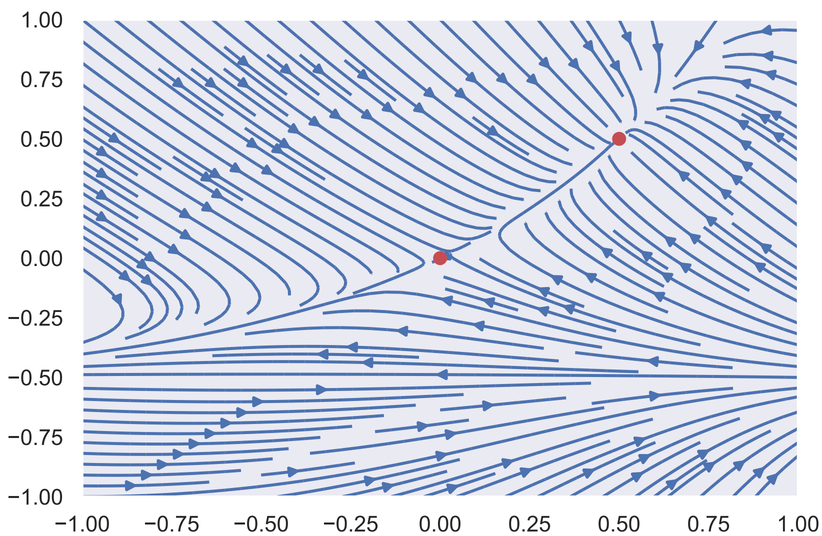

Figure 4 – Phase portrait and fixed points when $n = 1$.

  fixed point.
- $n = 10$: Then $r = \phi(r) = \frac{r^{10}}{K^{10}+r^{10}}$. So either $r = 0$, which gives the first fixed point $F_1 = (0, 0)$ or $r^{10} - r^9 + K^{10} = 0$. Solving this polynomial in Python for real roots, we then obtain two other fixed points: $F_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $F_3 \approx (0.9990147351, 0.9990147351)$. The stream plot in Figure 5 shows that F_1 and F_3 are stable, while F_2 is saddle.

c) We plot the stationary distribution of protein in Figure 6. The ratio of the standard deviation to the mean of the distribution for strong regulation is 0.5360628035416339. The ratio of the standard deviation to the mean of the distribution for strong regulation is 0.5007762996948194.

□

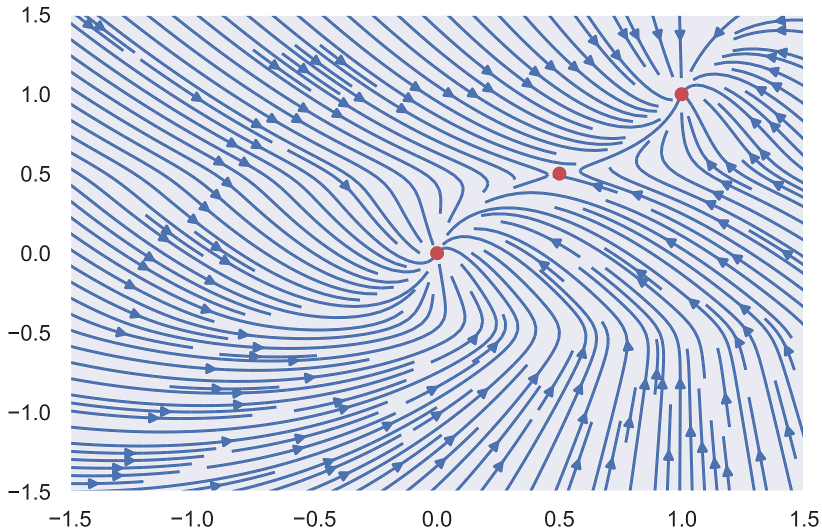

Figure 5 – Phase portrait and fixed points when $n = 10$.

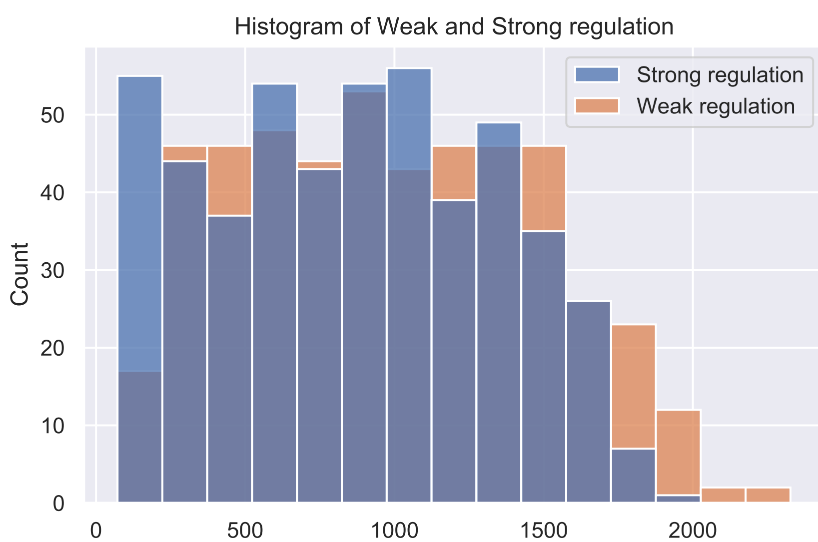

Figure 6 – Stationary distribution of protein.

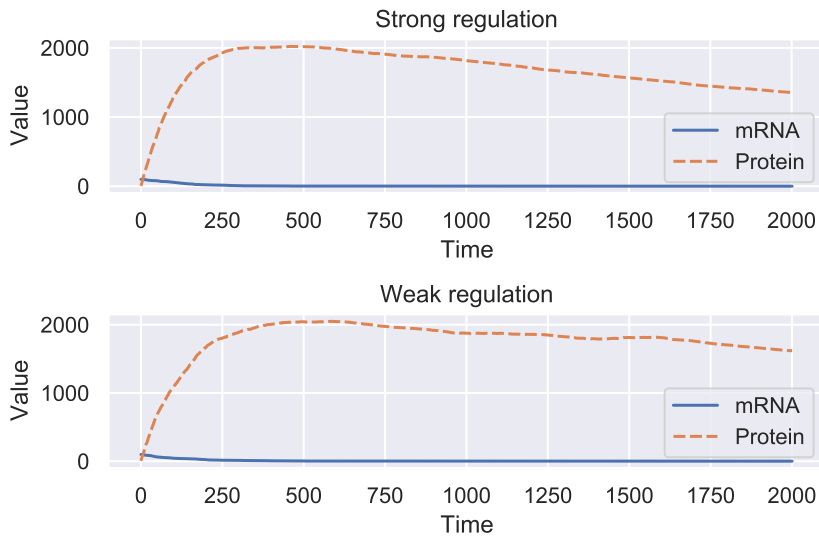

Figure 7 – Trajectory when terminal time $T = 2000$.

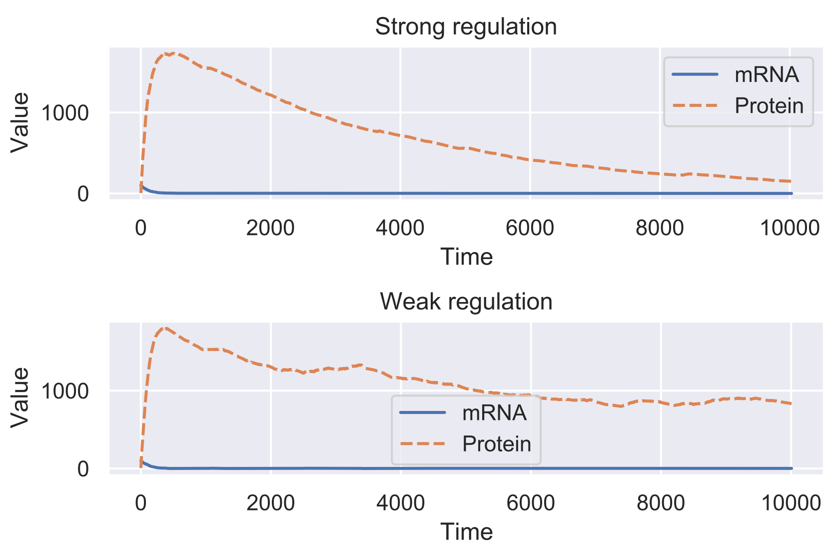

Figure 8 – Trajectory when terminal time $T = 10000$.