The code for this homework is published at:

https://github.com/trienthuyendang/M6397-StochasticProcesses/tree/master/Homework

Question 1. a) We follow the argument in Section 8.4 of the book of Gerstner et al.

On the one hand, the transition law P^{trans}, which represents the probability density of finding a membrane potential u at time $t+\Delta t$ if at time t membrane potential is u', is given by

$$\begin{aligned} P^{\text{trans}}(u, t+\Delta t \mid u', t) = \left[1 - \Delta t \sum_k \nu_k(t)\right] \delta\left(u - u' \,\mathrm{e}^{-\Delta t/\tau_m}\right) \\ + \Delta t \sum_k \nu_k(t) \delta\left(u - u' \,\mathrm{e}^{-\Delta t/\tau_m} - w_k\right). \end{aligned} \tag{1}$$

On the other hand, we know that the membrane potential is given by a differential equation (i.e., (8.20) in the book) with Poisson distribution input spikes. So Picard's theorem and continuation of solutions implies that the evolution of the membrane potential is a Markov process (the memoryless property comes from the fact that we can continue the solution trajectory by regarding the old end time/value as the new initial time/value). So the probability that membrane potential is u at time $t+\Delta t$ is given by Chapman-Kolmogorov equation,

$$p(u, t+\Delta t) = \int_{\mathbb{R}} P^{\text{trans}}(u, t+\Delta t \mid u', t) p(u', t) \,\mathrm{d}u'. \tag{2}$$

Substituting (1) into (2), we obtain,

$$\begin{aligned} p(u, t+\Delta t) &= \left[1 - \Delta t \sum_k \nu_k(t)\right] \int_{\mathbb{R}} \delta\left(u - u' \,\mathrm{e}^{-\Delta t/\tau_m}\right) p(u', t) \,\mathrm{d}u' \\ &\quad + \Delta t \sum_k \nu_k(t) \int_{\mathbb{R}} \delta\left(u - u' \,\mathrm{e}^{-\Delta t/\tau_m} - w_k\right) p(u', t) \,\mathrm{d}u' \\ &= \left[1 - \Delta t \sum_k \nu_k(t)\right] \mathrm{e}^{\Delta t/\tau_m} p\left(\mathrm{e}^{\Delta t/\tau_m} u, t\right) \\ &\quad + \Delta t \sum_k \nu_k(t) \,\mathrm{e}^{\Delta t/\tau_m} p\left(\mathrm{e}^{\Delta t/\tau_m}(u - w_k), t\right). \end{aligned} \tag{3}$$

Here we use two changes of variables: $t_1 = u - u' \,\mathrm{e}^{-\Delta t/\tau_m}$ for the first integral and $t_2 = u - u' \,\mathrm{e}^{-\Delta t/\tau_m} - w_k$ for the second integral, then apply $\int_{\mathbb{R}} \delta(s) f(s) \,\mathrm{d}s = f(0)$ and $\delta(\alpha s) = |\alpha|^{-1} \delta(s)$.

Since $|\Delta t| \ll 1$, we have the Taylor expansions

$$\begin{aligned} \mathrm{e}^{\Delta t/\tau_m} &= 1 + \frac{\Delta t}{\tau_m} + O(\Delta t^2), \\ p\left(\mathrm{e}^{\Delta t/\tau_m} u, t\right) &= p(u, t) + \frac{\Delta t}{\tau_m} u \frac{\partial}{\partial u} p(u, t) + O(\Delta t^2), \end{aligned}$$

$$p\left(\mathrm{e}^{\Delta t/\tau_m}(u-w_k),t\right) = p(u-w_k,t) + \frac{\Delta t}{\tau_m}(u-w_k)\frac{\partial}{\partial u}p(u-w_k,t) + O(\Delta t^2).$$

Thus (3) becomes

$$\begin{aligned}
p(u,t+\Delta t) &= \mathrm{e}^{\Delta t/\tau_m}\, p\left(\mathrm{e}^{\Delta t/\tau_m}\, u,t\right) \\
&\qquad + \Delta t\sum_k \nu_k(t)\,\mathrm{e}^{\Delta t/\tau_m}\left[p\left(\mathrm{e}^{\Delta t/\tau_m}(u-w_k),t\right) - p\left(\mathrm{e}^{\Delta t/\tau_m}\, u,t\right)\right] \\
&= \left(1+\frac{\Delta t}{\tau_m}\right)\left(p(u,t) + \frac{\Delta t}{\tau_m}u\frac{\partial}{\partial u}p(u,t)\right) \\
&\qquad + \Delta t\sum_k \nu_k(t)\left[p(u-w_k,t) - p(u,t)\right] + O(\Delta t^2) \\
&= \left(1+\frac{\Delta t}{\tau_m}\right)p(u,t) + \frac{\Delta t}{\tau_m}u\frac{\partial}{\partial u}p(u,t) \\
&\qquad + \Delta t\sum_k \nu_k(t)\left[p(u-w_k,t) - p(u,t)\right] + O(\Delta t^2),
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{p(u,t+\Delta t)-p(u,t)}{\Delta t} &= \frac{1}{\tau_m}p(u,t) + \frac{1}{\tau_m}u\frac{\partial}{\partial u}p(u,t) \\
&\qquad + \sum_k \nu_k(t)\left[p(u-w_k,t) - p(u,t)\right] + O(\Delta t)
\end{aligned}$$

Letting $\Delta t \to 0$, we conclude that

$$\frac{\partial}{\partial t}p(u,t) = \frac{1}{\tau_m}p(u,t) + \frac{1}{\tau_m}u\frac{\partial}{\partial u}p(u,t) + \sum_k \nu_k(t)\left[p(u-w_k,t) - p(u,t)\right] \tag{4}$$

If we further assume that $|w_k| \ll 1$, then by the asymptotic expansion

$$p(u-w_k,t) = p(u,t) - w_k\frac{\partial}{\partial u}p(u,t) + \frac{1}{2}w_k^2\frac{\partial}{\partial u^2}p(u,t) + O(w_k^3),$$

we can write (4) as (ignoring $O(w^3)$),

$$\begin{aligned}
\frac{\partial}{\partial t}p(u,t) &= \left[\frac{1}{\tau_m} + \left(\frac{1}{\tau_m}u - \sum_k \nu_k(t)w_k\right)\frac{\partial}{\partial u}\right]p(u,t) + \frac{1}{2}\sum_k \nu_k(t)w_k^2\frac{\partial}{\partial u^2}p(u,t) \\
&= \frac{\partial}{\partial u}\left[\left(\frac{1}{\tau_m}u - \sum_k \nu_k(t)w_k\right)p(u,t)\right] + \frac{1}{2}\sum_k \nu_k(t)w_k^2\frac{\partial}{\partial u^2}p(u,t)
\end{aligned}$$

Multiplying both sides by τ_m, we obtain equation (8.41) in the book.

b) If

$$p(u,t) = \frac{1}{\sqrt{2\pi\left\langle \Delta u^2(t)\right\rangle}}\exp\left[-\frac{(u(t\mid\hat{t}) - u_0(t))^2}{2\left\langle\Delta u^2(t)\right\rangle}\right], \tag{5}$$

then

$$\begin{aligned}
\frac{\partial}{\partial u} p(u,t) &= \frac{\left(u_0(t) - u(t \mid \hat{t})\right)}{\left\langle \Delta u^2(t) \right\rangle^{\frac{3}{2}} \sqrt{2\pi}} \exp\left[-\frac{\left(u(t \mid \hat{t}) - u_0(t)\right)^2}{2\left\langle \Delta u^2(t) \right\rangle} \right] \\
\frac{\partial^2}{\partial u^2} p(u,t) &= \frac{\left(u_0(t) - u(t \mid \hat{t})\right)^2 - \left\langle \Delta u^2(t) \right\rangle}{\left\langle \Delta u^2(t) \right\rangle^{\frac{5}{2}} \sqrt{2\pi}} \exp\left[-\frac{\left(u(t \mid \hat{t}) - u_0(t)\right)^2}{2\left\langle \Delta u^2(t) \right\rangle} \right] \\
\frac{\partial}{\partial t} p(u,t) &= \frac{\left(u_0(t) - u(t \mid \hat{t})\right)\left(u_0'(t) - u'(t \mid \hat{t})\right)}{\left\langle \Delta u^2(t) \right\rangle^{\frac{3}{2}} \sqrt{2\pi}} \exp\left[-\frac{\left(u(t \mid \hat{t}) - u_0(t)\right)^2}{2\left\langle \Delta u^2(t) \right\rangle} \right]
\end{aligned}$$

Plugging into (8.41), we see that p defined as in (5) satisfies this equation.

□

Question 2. a) The expected time for a LIF neuron to fire satisfies the equation

$$\begin{aligned}
&A(x)\partial_x T(x) + \frac{1}{2} B(x) \partial_x^2 T(x) = -1, \\
&T(0) = T(\theta) = 0.
\end{aligned} \tag{6}$$

Using the formula in slide 7 lecture 16, we get $A(x) = RI_0 - x$, $B(x) = \sigma^2$.

b) We use the codes of Group 4 - Challenge 6 to plot the histogram of 1000 trials:

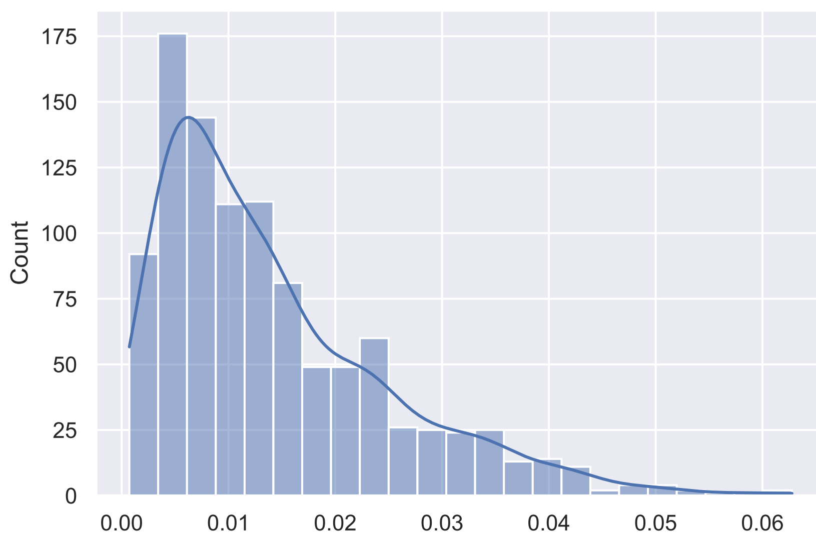

Figure 1 – Histogram of expected time for a LIF neuron to fire.

□

Question 3. a) If the times between spikes are distributed according to a gamma distribution $\Gamma(\alpha, \beta)$, then the coefficient of variation of the inter-spike interval is

$$c_v = \frac{\sigma}{\mu} = \frac{\sqrt{\frac{\alpha}{\beta^2}}}{\frac{\alpha}{\beta}} = \frac{1}{\sqrt{\alpha}}.$$

b) If inter-event time follows the Erlang distribution with fixed λ and increasing k, then the coefficient of variation of the inter-spike interval is

$$c_v = \frac{\sigma}{\mu} = \frac{\sqrt{\frac{k}{\lambda^2}}}{\frac{k}{\lambda}} = \frac{1}{\sqrt{k}}.$$

The Erlang distribution is a special case of Gamma distribution, i.e., if we let the shape parameter $\alpha = k$, where the latter is a positive integer, then $Gamma(k, \lambda) \sim Erlang(k, \lambda)$. In this case, we get the same coefficient of variation.

$\square$

Question 4. Let $n = Nx,\ u = 2\mathcal{U}\frac{1}{N},\ v = 2\mathcal{V}\frac{1}{N}$, we have

$$\begin{aligned}
T(n+1 \mid n) &= (1-u)x(1-x) + v(1-x)^2 \\
&= \left(1 - 2\mathcal{U}\frac{1}{N}\right) x(1-x) + 2\mathcal{V}\frac{1}{N}(1-x)^2, \\
T(n-1 \mid n) &= (1-v)(1-x)x + ux^2 \\
&= \left(1 - 2\mathcal{V}\frac{1}{N}\right) x(1-x) + 2\mathcal{U}\frac{1}{N}x^2, \\
T(n \mid n+1) &= (1-v)\left(1 - x - \frac{1}{N}\right)\left(x + \frac{1}{N}\right) + u\left(x + \frac{1}{N}\right)^2 \\
&= \left(1 - 2\mathcal{V}\frac{1}{N}\right)\left[x(1-x) + (1-2x)\frac{1}{N} - \frac{1}{N^2}\right] + 2\mathcal{U}\frac{1}{N}\left(x^2 + 2x\frac{1}{N} + \frac{1}{N^2}\right) \\
&= x(1-x) + \left[(1-2x) - 2\mathcal{V}x(1-x) + 2\mathcal{U}x^2\right]\frac{1}{N} \\
&\qquad + \left[-2\mathcal{V}(1-2x) - 1 + 4\mathcal{U}x\right]\frac{1}{N^2} + O\left(\frac{1}{N^3}\right), \\
T(n \mid n-1) &= (1-u)\left(x - \frac{1}{N}\right)\left(1 - x + \frac{1}{N}\right) + v\left(1 - x + \frac{1}{N}\right)^2 \\
&= \left(1 - 2\mathcal{U}\frac{1}{N}\right)\left[x(1-x) + (2x-1)\frac{1}{N} - \frac{1}{N^2}\right] + 2\mathcal{V}\frac{1}{N}\left((1-x)^2 + 2(1-x)\frac{1}{N} + \frac{1}{N^2}\right) \\
&= x(1-x) + \left[(2x-1) - 2\mathcal{U}x(1-x) + 2\mathcal{V}(1-x)^2\right]\frac{1}{N} \\
&\qquad + \left[2\mathcal{U}(1-2x) - 1 + 4\mathcal{V}(1-x)\right]\frac{1}{N^2} + O\left(\frac{1}{N^3}\right).
\end{aligned}$$

To avoid abusing notation, let

$$Q(x,t) \coloneqq P(Nx,t) = P(n,t),$$

then

$$\begin{aligned}
\frac{\mathrm{d}P(n,t)}{\mathrm{d}t} &= \frac{\partial}{\partial t}Q(x,t), \\
P(n+1,t) &= Q\left(x + \frac{1}{N}, t\right)
\end{aligned}$$

$$
\begin{aligned}
&= Q(x,t) + \frac{1}{N}\frac{\partial}{\partial x}Q(x,t) + \frac{1}{2N^2}\frac{\partial^2}{\partial x^2}Q(x,t) + O\left(\frac{1}{N^3}\right) \\
P(n-1,t) &= Q\left(x - \frac{1}{N}, t\right) \\
&= Q(x,t) - \frac{1}{N}\frac{\partial}{\partial x}Q(x,t) + \frac{1}{2N^2}\frac{\partial^2}{\partial x^2}Q(x,t) + O\left(\frac{1}{N^3}\right)
\end{aligned}
$$

We now compute each term in the right hand side of the master equation:

$$
\begin{aligned}
&P(n,t)\left[T(n+1 \mid n) + T(n-1 \mid n)\right] \\
&\quad = Q(x,t)\left[(1-u)x(1-x) + v(1-x)^2 + (1-v)(1-x)x + ux^2\right] \\
&\quad = \left[2x(1-x) + (2x-1)(ux - v(1-x))\right]Q(x,t) \\
&\quad = \left[2x(1-x) + (2x-1)\left(2\mathcal{U}\frac{1}{N}x - 2\mathcal{V}\frac{1}{N}(1-x)\right)\right]Q(x,t), \\
&P(n+1,t)T(n \mid n+1) \\
&\quad = \left(Q(x,t) + \frac{1}{N}\frac{\partial}{\partial x}Q(x,t) + \frac{1}{2N^2}\frac{\partial^2}{\partial x^2}Q(x,t)\right) \\
&\qquad \times \left[(1-v)\left(1 - x - \frac{1}{N}\right)\left(x + \frac{1}{N}\right) + u\left(x + \frac{1}{N}\right)^2\right] + O\left(\frac{1}{N^3}\right) \\
&\quad = \left(Q(x,t) + \frac{1}{N}\frac{\partial}{\partial x}Q(x,t) + \frac{1}{2N^2}\frac{\partial^2}{\partial x^2}Q(x,t)\right) \\
&\qquad \times \left\{x(1-x) + \left[(2x-1) - 2\mathcal{U}x(1-x) + 2\mathcal{V}(1-x)^2\right]\frac{1}{N}\right. \\
&\qquad\quad \left. + \left[2\mathcal{U}(1-2x) - 1 + 4\mathcal{V}(1-x)\right]\frac{1}{N^2}\right\} + O\left(\frac{1}{N^3}\right), \\
&P(n-1,t)T(n \mid n-1) \\
&\quad = \left(Q(x,t) - \frac{1}{N}\frac{\partial}{\partial x}Q(x,t) + \frac{1}{2N^2}\frac{\partial^2}{\partial x^2}Q(x,t)\right) \\
&\qquad \times \left[(1-v)\left(1 - x - \frac{1}{N}\right)\left(x + \frac{1}{N}\right) + u\left(x + \frac{1}{N}\right)^2\right] + O\left(\frac{1}{N^3}\right) \\
&\quad = \left(Q(x,t) - \frac{1}{N}\frac{\partial}{\partial x}Q(x,t) + \frac{1}{2N^2}\frac{\partial^2}{\partial x^2}Q(x,t)\right) \\
&\qquad \times \left\{x(1-x) + \left[(2x-1) - 2\mathcal{U}x(1-x) + 2\mathcal{V}(1-x)^2\right]\frac{1}{N}\right. \\
&\qquad\quad \left. + \left[2\mathcal{U}(1-2x) - 1 + 4\mathcal{V}(1-x)\right]\frac{1}{N^2}\right\} + O\left(\frac{1}{N^3}\right).
\end{aligned}
$$

Substituting into the master equation and simplifying, we obtain

$$
\frac{\partial}{\partial t}Q(x,t) = \frac{1}{N^2}\frac{\partial^2}{\partial x^2}\left[x(1-x)Q(x,t)\right] + \frac{2}{N^2}\frac{\partial}{\partial x}\left\{\left[\mathcal{U}x - \mathcal{V}(1-x)\right]Q(x,t)\right\} + O\left(\frac{1}{N^3}\right).
$$

Let $\tau = \frac{2t}{N^2}$ be the rescaled time unit, then $\frac{\partial}{\partial t} = \frac{2}{N^2}\frac{\partial}{\partial \tau}$, so the above equation becomes

$$\frac{\partial}{\partial \tau}Q(x,\tau) = \frac{1}{2}\frac{\partial}{\partial x^2}\left[x(1-x)Q(x,\tau)\right] + \frac{\partial}{\partial x}\left\{\left[\mathcal{U}x - \mathcal{V}(1-x)\right]Q(x,\tau)\right\} + O\left(\frac{1}{N}\right).$$

Passing to limit $N \to \infty$, we obtain the Fokker-Planck equation,

$$\frac{\partial}{\partial \tau}Q(x,\tau) = \frac{1}{2}\frac{\partial}{\partial x^2}\left[x(1-x)Q(x,\tau)\right] + \frac{\partial}{\partial x}\left\{\left[\mathcal{U}x - \mathcal{V}(1-x)\right]Q(x,\tau)\right\}.$$

□