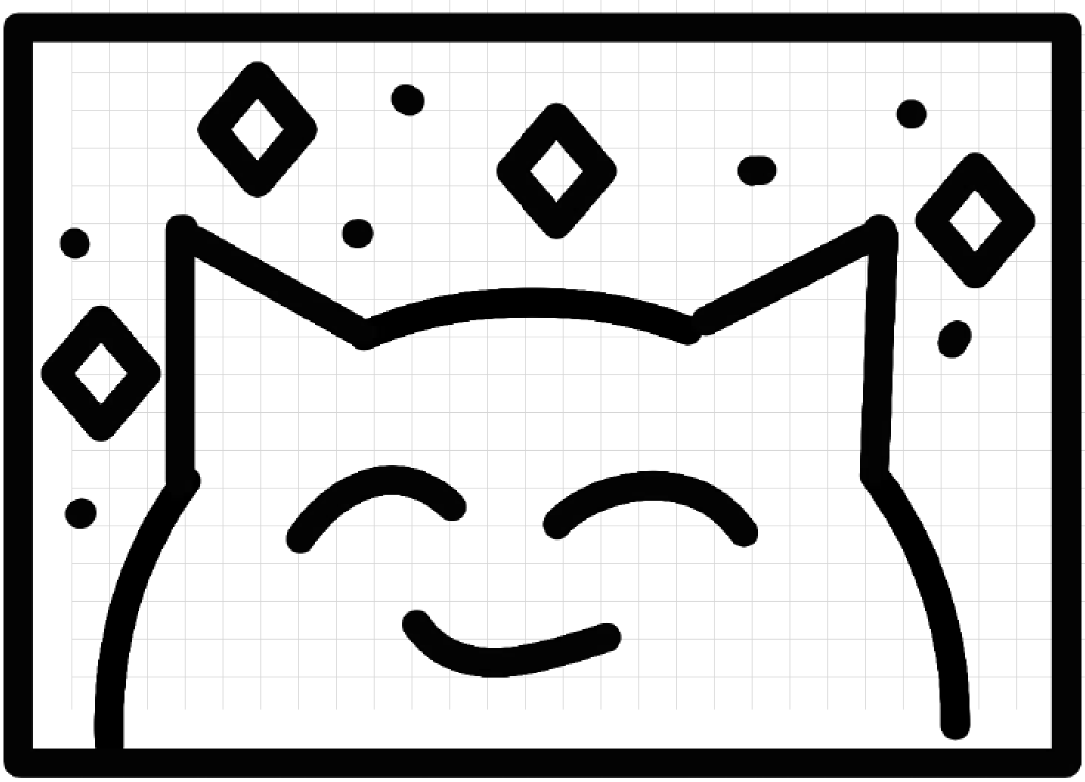
MAE 112

HW 7

TRIET



1) Helium has the ionization potential of $\varepsilon = 24.48$ electron volts. (1 ev = 1.602×10^{-19} joules.) The Saha equation, adjusted for degeneracy in the helium states, gives

$$\frac{n_{electron}n_{ion}}{n_{neutral}} = 4\frac{(2\pi m_e kT)^{3/2}}{h^3}e^{-\varepsilon/kT}$$

m_e, k, and h are respectively the electron mass, Boltzmann constant, and Planck constant. (use class notes or search for values.) We start with two moles of helium at standard atmospheric conditions. This initially cool helium gas with initially negligible ionization is heated at constant volume to 2900 K. At equilibrium, what is the free electron number density? What is the number density of positive ions?

Constants:

Electron mass: $m_e = 9.11\times10^{-31}$ [kg]

Boltzman constant: $k = 1.38062\times10^{-23}$ [J/k]

Planck constant: $h = 6.62607\times10^{-34}$ [Js]

Saha equation:

$$\frac{n_e n_i}{n_n} = 4\left(\frac{\sqrt{2\pi m_e k T}}{h}\right)^3 e^{-\varepsilon/kT}$$

$$= 4\left(\frac{\sqrt{2\times\pi\times9.11\times10^{-31}\times1.38662\times10^{-23}\times2900}}{6.62607\times10^{-34}}\right)^3 e^{\frac{-24.48\times1.602\times10^{-19}}{1.38662\times10^{-23}\times2900}}$$

$$\frac{n_e n_i}{n_n} = 4.3618\times10^{-16} \qquad (1)$$

Let say x is number of mole of ionized Helium:

We have:

$$n_i = n_e = x\frac{N_A}{V} \qquad (2)$$

$$n_n = (2-x)\frac{N_A}{V} \qquad (3)$$

$$(1),(2),(3) \Rightarrow \frac{n_e n_i}{n_n} = x^2 \frac{N_A^{\cancel{2}}}{V^{\cancel{2}}} \frac{\cancel{V}}{(2-x)\cancel{N_A}} = 4.3618 \times 10^{-16}$$

$$\frac{N_A}{V} \times \frac{x^2}{2-x} = 4.3618 \times 10^{-16} \qquad (*)$$

Calculate volume

$$PV = nRT \Rightarrow V = \frac{nRT}{P} = \frac{2 \times 8.314 \times 273}{101325}$$

$$V = 0.0448 \ [m^3]$$

$$(*) \Rightarrow \frac{6.022 \times 10^{23}}{0.0448} \cdot \frac{x^2}{2-x} = 4.3618 \times 10^{-16}$$

$$\frac{x^2}{2-x} = 3.245 \times 10^{-41}$$

$$x^2 = 3.245 \times 10^{-41} \times 2 - 3.245 \times 10^{-41} x$$

$$x^2 + 3.245 \times 10^{-41} x - 6.4898 \times 10^{-41} = 0$$

$$x = +8.056 \times 10^{-21} \ [mole]$$

Free electron number density:

$$n_e = x \frac{N_A}{V} = 8.056 \times 10^{-21} \times \frac{6.022 \times 10^{23}}{0.0448}$$

$$n_e = 108288.4643 \ \left[\frac{particle}{m^3}\right]$$

Positive ion number density:

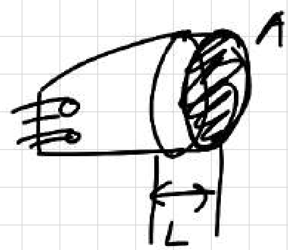
$$\boxed{n_i = n_e = 108288.4643 \ \left[\frac{particle}{m^3}\right]}$$

2) Consider an ion rocket with a voltage drop in potential between the accelerating electrodes placed 100 centimeters apart. Cesium of atomic weight 55 is the propellant. One volt equals one newton-meter per coulomb. The cross-sectional area is 40 cm^2. $\epsilon_0 =$ 8.85 x 10^{-12} $(coulomb)^2$/newton-meter. It is desired to obtain a specific impulse of 2500 seconds.

a) Determine the required voltage drop.
b) Determine the final jet (beam) velocity.
c) Determine the maximum thrust which is achievable here.
d) Determine the mass flow for operation with a current equal to 90% of the maximum beam current.

For Cs:

$$\frac{q}{m} = 7.25 \times 10^5 \left[\frac{C}{kg}\right]$$



ⓐ Exit velocity:

$$V_e = I_{sp}\, g = 2500 \times 9.81 = 24525 \,[m/s]$$

Energy Conservation:

$$KE = PE$$

$$\frac{1}{2} m (V_e^2 - 0^2) = qV$$

$$V = \frac{1}{2} \frac{m}{q} V_e^2 = \frac{1}{2} \times \frac{1}{7.25 \times 10^5} \times 24525^2$$

$$\boxed{V = 414.81 \,[V]}$$

ⓑ from ⓐ, $\boxed{V_e = 24525 \,[m/s]}$

ⓒ Max thrust:

$$T_{max} = \frac{8}{9} \epsilon_0 \left(\frac{V}{L}\right)^2 A$$

$$= \frac{8}{9} \times 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2} \left(\frac{414.81 \; V}{1 \quad m}\right)^2 0.004 \quad m^2$$

$$\boxed{T_{max} = 5.4144 \times 10^{-9} \ [N]}$$

ⓓ Thrust at 90% max current:

$$T_{max} = \sqrt{2\frac{m}{q}} \ j_{max} \sqrt{V} \ A$$

$$T = \sqrt{2\frac{m}{q}} \ 90\% \ j_{max} \sqrt{V} \ A$$

$$= 90\% \left(\sqrt{2\frac{m}{q}} \ j_{max} \sqrt{V} \ A \right)$$

$$= 90\% \ T_{max}$$

$$T = 90\% \times 5.4144 \times 10^{-9} = 4.873 \times 10^{-9} \ [N]$$

Mass flow rate at 90% max current:

$$\dot{m} = \frac{T}{V_e} = \frac{4.873 \times 10^{-9}}{24525} = \boxed{1.987 \times 10^{-13} \ [kg/s]}$$

3) Consider an ion rocket with a 600-volt drop in potential between the accelerating electrodes placed 60 centimeters apart. Xenon of atomic weight 54 is the propellant. The charge-to-mass ratio is 7.34×10^5 coulombs per kilogram. One volt equals one newton-meter per coulomb. The cross-sectional area is 100 cm^2. $\epsilon_0 = 8.85 \times 10^{-12}$ $(coulomb)^2$/newton-meter^2 Determine the

(a) exit velocity,
(b) specific impulse,
(c) maximum beam current,
(d) thrust at maximum beam current, and
(e) mass flow at maximum beam current.
(f) What are the thrust and the mass flow if the rocket operates on one-half of the maximum current?

(a) Energy conservation:

$$KE = PE$$

$$\frac{1}{2} m v_e^2 = qV$$

$$v_e = \sqrt{2 \frac{q}{m} V} = \sqrt{2 \times 7.34 \times 10^5 \times 600}$$

$$\boxed{v_e = 29678.275 \text{ [m/s]}}$$

(b) Specific impulse:

$$I_{sp} = \frac{v_e}{g} = \frac{29678.275}{9.81} = \boxed{3025.31 \text{ [s]}}$$

(c) Max current density:

$$j_{max} = \frac{4}{9} \varepsilon_0 \sqrt{2 \frac{q}{m}} \frac{V^{3/2}}{L^2}$$

$$= \frac{4}{9} \times 8.85 \times 10^{-12} \times \sqrt{2 \times 7.34 \times 10^5} \times \frac{600^{3/2}}{0.6^2}$$

$$j_{max} = 1.9456 \times 10^{-4} \text{ [A/m}^2\text{]}$$

Max beam current:

$$I_{max} = j_{max} A = 1.9456 \times 10^{-4} \times 0.01$$

$$\boxed{I_{max} = 1.9456 \times 10^{-6} \text{ [A]}}$$

(d) Thrust at I_{max}:

$$T_{I_{max}} = \sqrt{2\frac{m}{q}}\ j_{max} \sqrt{V}\ A$$

$$= \sqrt{2 \times \frac{1}{7.34\times10^{5}}}\ 1.9456\times10^{-4} \sqrt{600} \times 0.01$$

$$\boxed{T_{I_{max}} = 7.8667\times10^{-8}\ [N]}$$

(e) Mass flow rate at I_{max}:

$$\dot{m} = \frac{T}{V_e} = \frac{7.8667\times10^{-8}}{29678.275} = \boxed{2.6506\times10^{-12}\left[\frac{kg}{s}\right]}$$

(f) Current density: $j = \frac{I}{A} = \frac{0.5\ I_{max}}{A} = 0.5\ j_{max}$

$$j = 0.5\times 1.9456\times10^{-4}$$

$$j = 9.728\times10^{-5}\ [Amp/m^2]$$

Thrust at $0.5\ I_{max}$:

$$T = \sqrt{2\frac{m}{q}}\ j \sqrt{V}\ A$$

$$= \sqrt{2 \times \frac{1}{7.34\times10^{5}}}\ 9.728\times10^{-5} \sqrt{600} \times 0.01$$

$$\boxed{T = 3.9334\times10^{-8}\ [N]}$$

Mass flow rate at $0.5\ I_{max}$:

$$\dot{m} = \frac{T}{V_e} = \frac{3.9334\times10^{-8}}{29678.275} = \boxed{1.3253\times10^{-12}\left[\frac{kg}{s}\right]}$$