University of California, Irvine
Department of Mechanical and Aerospace Engineering

MAE 112 - Homework 7
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1. Helium has the ionization potential of $\varepsilon = 24.48$ electron volts (1 ev = $1.602 * 10^{-19}$ joules.) The Saha equation, adjusted for degeneracy in helium states, gives $\frac{n_{electron}n_{ion}}{n_{neutral}} = 4\frac{(2\pi m_e kT)^{3/2}}{h^3}e^{-\epsilon/kT}$ m_e, k, and h are respectively the electron mass, Boltzman constant, and Planck constant. (use class notes or search for values.) We start with two moles of helium at standard atmospheric conditions. This initially cool helium gas with initially negligible ionization is heated at constant volume to 2900 K. At equilibrium, what is the free electron number density? What is the number density of positive ions?

Solution:
Note the Saha equation provided here has a 4 in front of it differing from the one shown in the lecture slides. This is because some coefficients change depending on the species being ionized. The constants needed to solve the Saha equation are

$$m_e = 9.11 \times 10^{-31} kg$$

$$k = 1.38062 \times 10^{-23} J/K$$

$$h = 6.62607 \times 10^{-34} Js$$

Solving the Saha equation we get

$$\frac{n_e n_i}{n_n} = 4.36179 \times 10^{-16}$$

We are looking at the equilibrium condition for the first level of ionization. Thus, a neutral helium atom ionizes and forms 1 electron and 1 ion. If x moles of helium atoms ionize the components at equilibrium are

$$(2 - x)Neutrons \leftrightarrow (x)Electrons + (x)Ions$$

This is similar to the previous equilibrium relations we wrote for chemical equilibrium. The number density is defined as the number of particles per unit volume. It can be obtained from Avogadro's number $N_a = 6.022 \times 10^{23} particles/mole$ and molar density c.

$$n = N_A c = N_A \frac{moles}{volume}$$

$$n_e = n_i = \frac{N_A}{V} x \text{ and } n_n = \frac{N_A}{V}(2-x)$$

If we substitute back into the Saha equation

$$\frac{n_e n_i}{n_n} = \frac{N_A}{V}(\frac{x^2}{2-x}) = 4.36179 \times 10^{-16}$$

Before solving for x, we need to determine the volume occupied by the gas. The reaction happens at constant volume and we are given the initial conditions for helium: standard atmospheric conditions. Using the ideal gas law with $P_i = 101325Pa, T_i = 273K, R = 8.314J/molK, n = 2mol$

$$V = \frac{nRT}{P} = 0.0448m^3$$

We obtain a quadratic equation for x as

$$\frac{x^2}{2-x} = 3.2449 \times 10^{-41} \rightarrow x = 8.05593 \times 10^-21$$

$$n_e = n_i = \frac{N_A}{V} x = 1.08287 \times 10^5 particles/m^3$$

2. Consider an ion rocket with a voltage drop in potential between the acceleration electrodes placed 100 centimeters apart. Cesium of atomic weight 55 is the propellant. One volt equals one newton-meter per coulomb. The cross-sectional area is 40 cm^2 $\in_0 = 7.85 \times 10^{-12}$ $(coloumb)^2$/newton-meter. It is desired to obtain a specific impulse of 2500 seconds.
a) Determine the required voltage drop.
b) Determine the final jet (beam) velocity.
c) Determine the maximum thrust which is achievable here.
d) Determine the mass flow for operation with a current equal to 90% of the maximum beam current

Solution:

$$L = 1m, MW = 55g/m, \epsilon_0 = 7.85 \times 10^{-12} \frac{C^2}{Nm^2}, A = 40cm^2, I_{sp} = 2500s$$

From textbook table 14.1 for Cs,

$$\frac{q}{m} = 7.25 \times 10^5 \frac{C}{kg}$$

Part (a)
First we find exit velocity,

$$u_e = I_{sp}g = 24500m/s$$

For an Ion rocket we have

$$\frac{1}{2}mu_e^2 = qV_a$$

$$V_a = \frac{u_e^2}{2\frac{q}{m}} = 413.97Volts$$

Part (b)
Final jet (beam) velocity was found before in part a

$$u_e = 24500m/s$$

Part (c)
Maximum thrust is given by

$$\frac{T_{max}}{A} = \frac{8}{9}\varepsilon_0(\frac{V_a}{L})^2 = 1.196 \times 10^{-6}N/m^2$$

$$T_{max} = 4.783 \times 10^{-9} N$$

Part (d)
If the beam current is 90% of the maximum value, then the thrust is also 90% of the maximum thrust.

$$T = 0.9T_{max} = 4.305 \times 10^{-9} N \rightarrow \dot{m} = \frac{T}{u_e} = 1.757 \times 10^{-13} kg/s$$

3. Consider an ion rocket with a 600-volt drop in potential between the acceleration electrodes placed 60 centimeters apart. Xenon of atomic weight 54 is the propellant. The charge-to-mass ratio is 7.34×10^5 coulombs per kilogram. One volt equals one newton-meter per coulomb. The cross-sectional area is $100 cm^2$. $\in_0 = 8.85 \times 10^{-12}$ $(coloumb)^2$/newton-meter. Determine the
a) exit velocity,
b) specific impulse,
c) maximum beam current,
d) thrust at maximum beam current, and
e) mass flow at maximum beam current.
f) What are the thrust and the mass flow if the rocket operates on one-half of the maximum current?

Solution:
Part (a)
Keep in mind for the solution that the magnitudes of thrust and mass flows will be very small while exhaust velocity magnitudes will be very large for ion rockets.

$$u_e = \sqrt{2 \frac{q}{m} V_a} = 29678.275 m/s$$

Notice we achieve much higher exhaust velocities than chemical rockets/jet engines. The same happens for specific impulse. Part (b)

$$I_{sp} = \frac{u_e}{g} = 3028.395 s$$

Part (c)
Maximum beam current

$$j_{max} = \frac{4}{9}\varepsilon_0\sqrt{2\frac{q}{m}}\frac{V_a^{3/2}}{L^2} = 1.946 \times 10^{-4} Amperes/m^2$$

Part (d) The thrust at maximum beam current can also be written as

$$T = \sqrt{2\frac{m}{q}} j_{max} V_a^{1/2} A = 7.868 \times 10^{-8} N$$

Part (e) The mass flow at beam current maximum

$$\dot{m} = T/u_e = 2.651 \times 10^{-12} kg/s$$

Part (f) If the rocket operates on one half of the maximum current we will recalculate parts d and e now using $j_{half} = 9.73 \times 10^{-5}$

$$T = \sqrt{2\frac{m}{q}} j_{half} V_a^{1/2} A = 3.934 \times 10^{-8} N$$

$$\dot{m} = T/u_e = 1.325 \times 10^{-12} kg/s$$

4. Design an arc jet that produces 40 lbf of thrust. Give maximum chamber temperature and pressure. Describe propellant choice. Give mass flow of propellant, throat area, exit velocity, and exit pressure.

Solution:
Step 1: Choose the propellant. One example could be helium because it has low molecular weight and its dissociation is not very strong during high temperature.
Step 2: Choose the total temperature and total pressure for the arc jet.
Step 3: Design the nozzle. We can assume that the rocket is operating in outer space such that the flow will be expanded to vacuum (0 pressure) but the exit to throat area ratio can not go to infinity due to size limitations so we must choose a reasonable ratio such as $\frac{A_e}{A^*} = 30$. Then we can solve for the mach number, exhaust pressure, and exhaust temperature using the isentropic flow tables. Then we can use the following equations to find the exit velocity and then mass flow rate which is essentially the fuel mass flow rate. Note that $P_a = 0$ and $A_e = \frac{\dot{m}}{\rho_e u_e}$

$$u_e = \sqrt{2c_p(T_o - T_e)}$$

$$T_{design} = \dot{m}u_e + (P_e - P_a)A_e$$

$$T_{design} = \dot{m}u_e + P_e\frac{\dot{m}}{\rho_e u_e} = \dot{m}u_e + \frac{\dot{m}RT_e}{u_e}$$