Lecture 1
Momentum Balance

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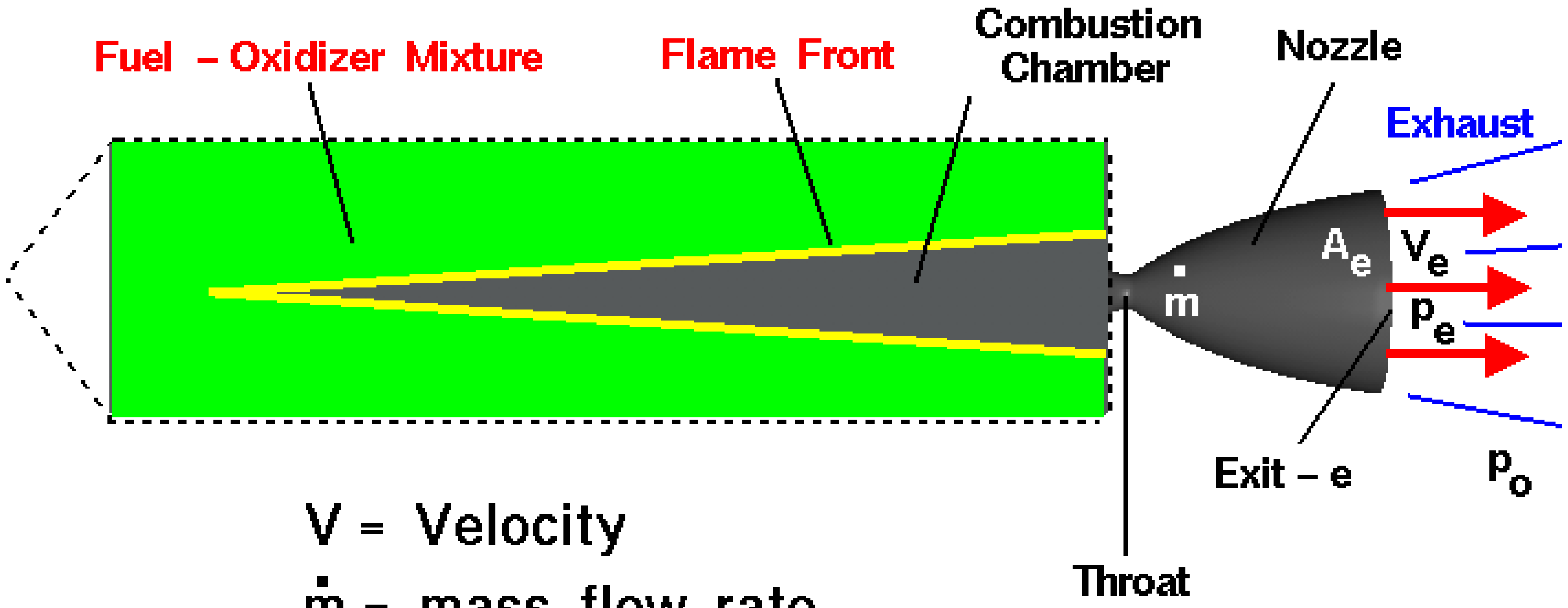
Solid Rocket Engine

Glenn Research Center



V = Velocity

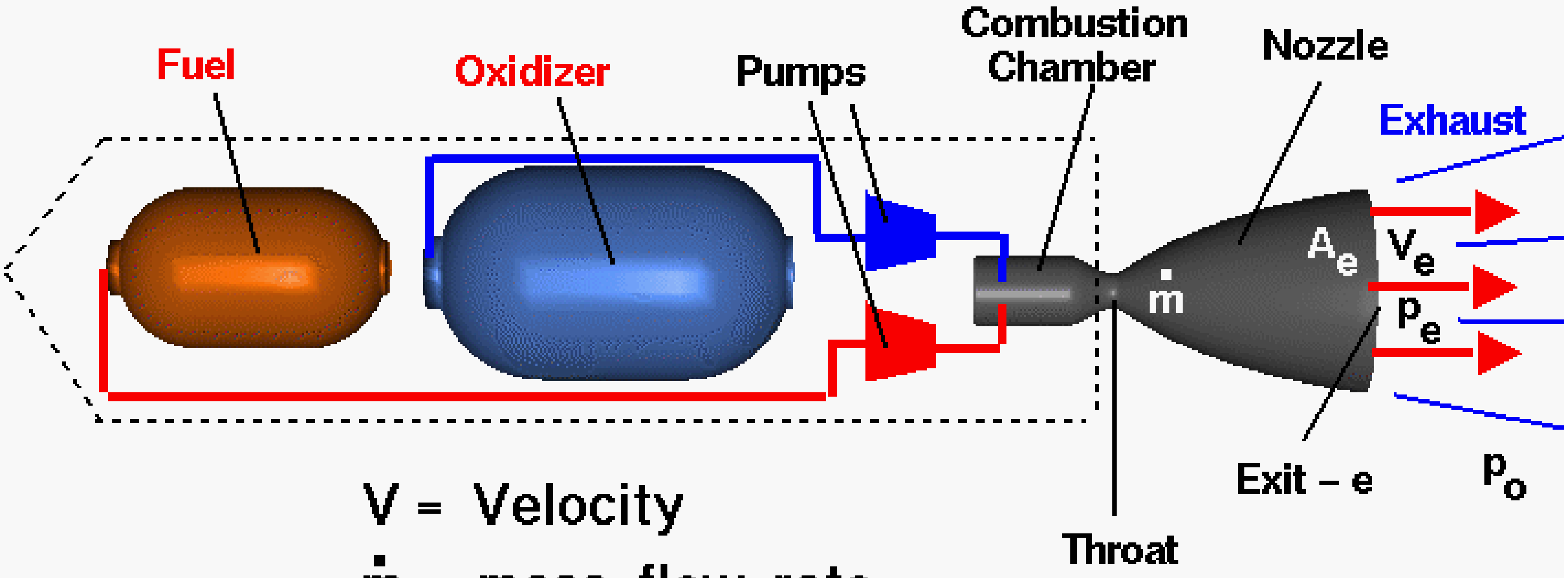
$\dot{m}$ = mass flow rate

p = pressure

Thrust = $F = \dot{m}\, V_e + (p_e - p_0)\, A_e$

Liquid Rocket Engine

Glenn Research Center



V = Velocity

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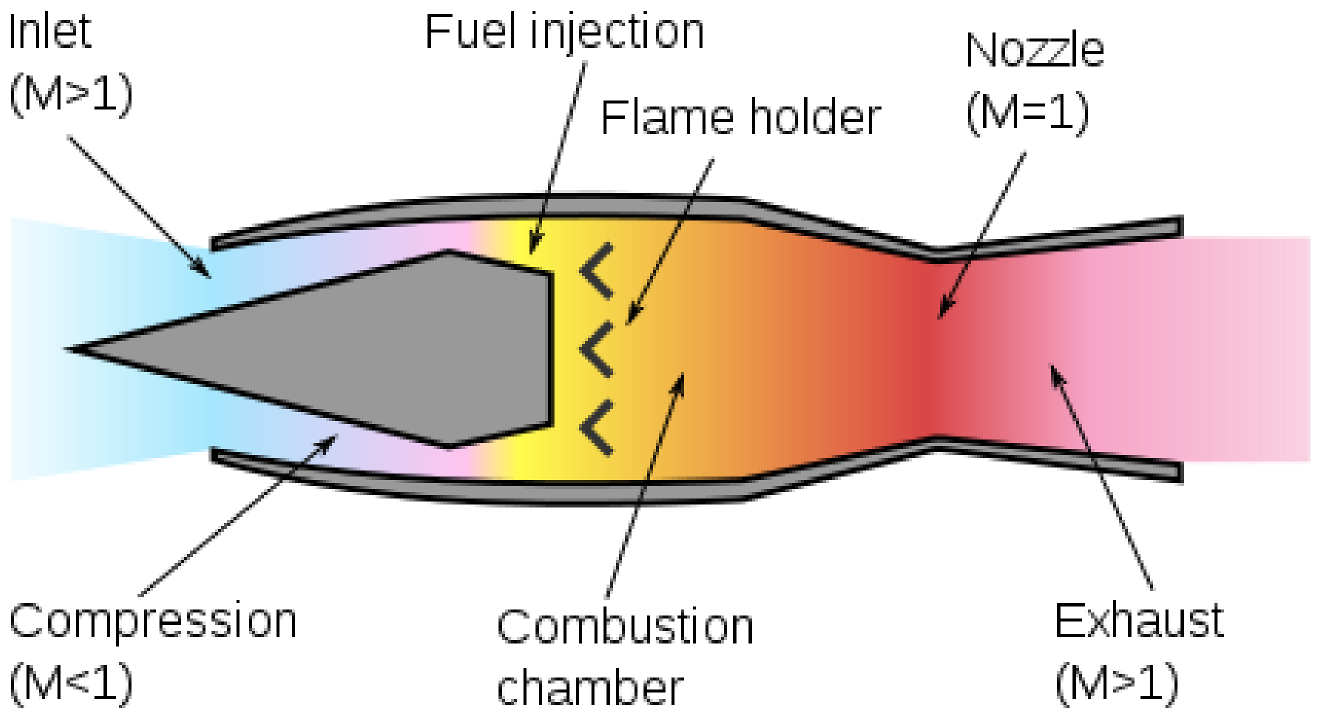
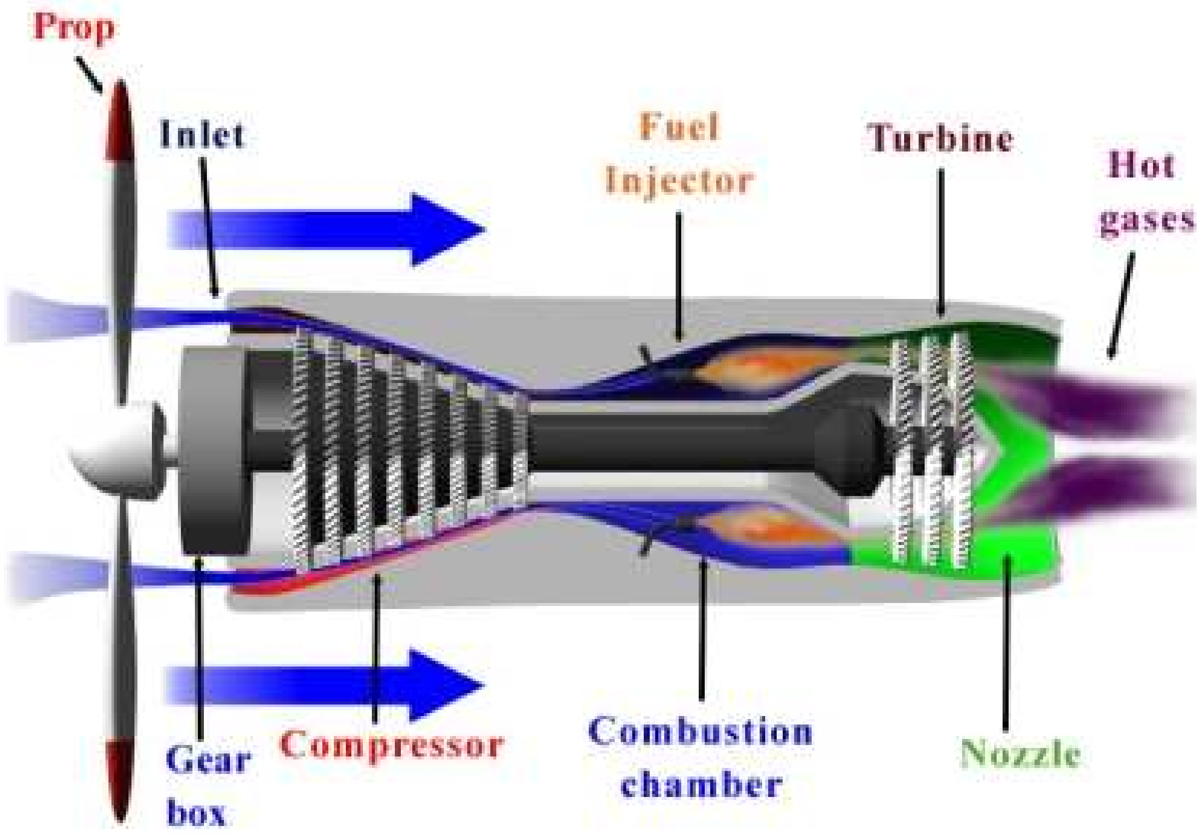
Ramjet



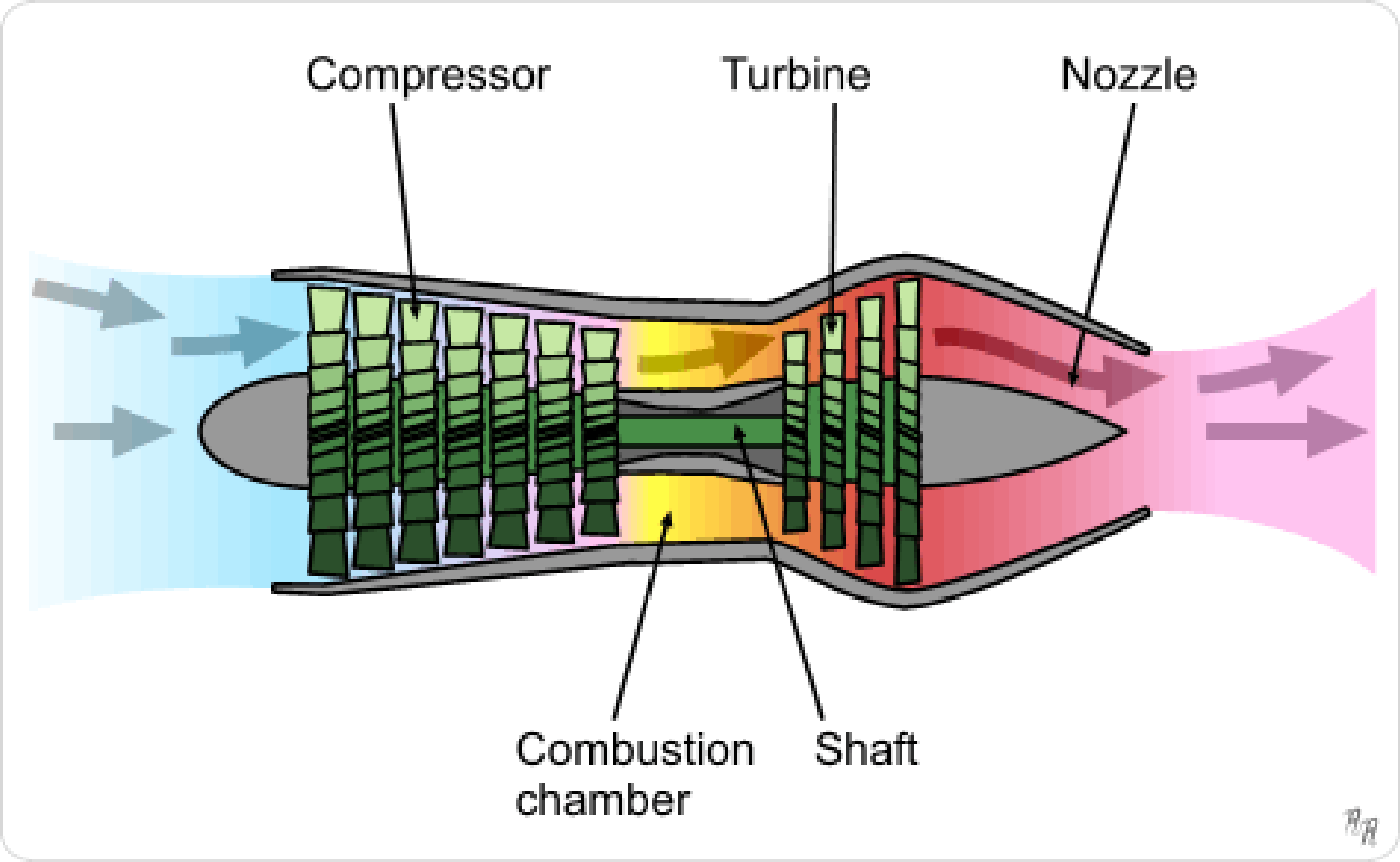
High flight Mach, no need of compressor and turbine, light, simple,
But need assist for taking off

Turboprop Engine

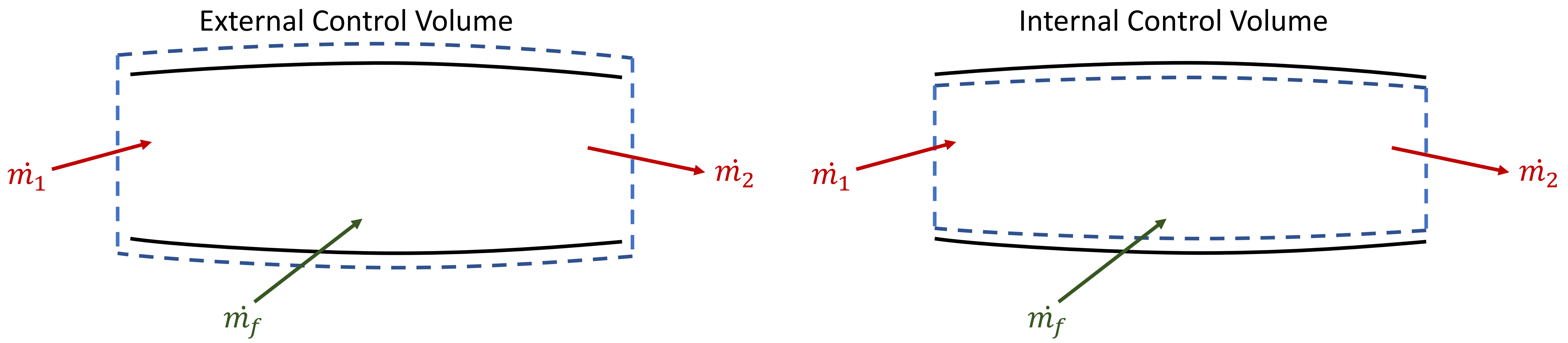


High power density, high propulsion efficiency (large mass flow rate of propellent)
Loss of efficiency, vibrations, noise at high speed.

Turbojet with Axial Compressor



Conservation of Mass



$\dot{m}_f$ = mass flow rate of fuel or liquid propellant

We have the same result in both cases. Steady-state result is: $\dot{m}_2 = \dot{m}_1 + \dot{m}_f$

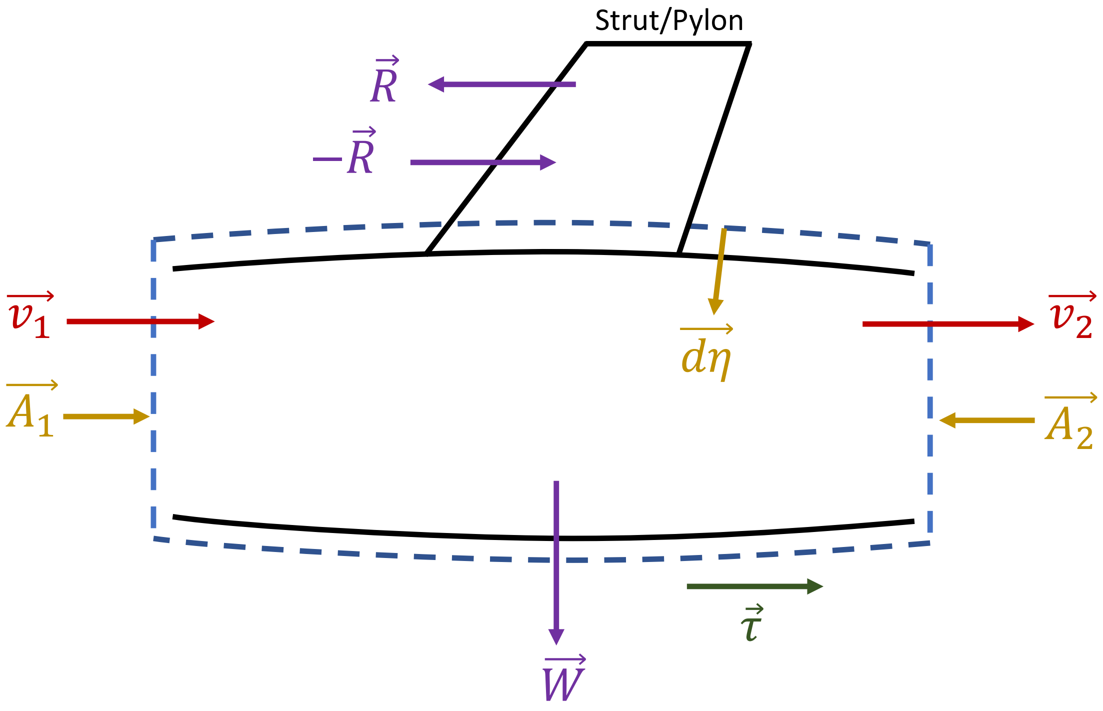
Note: In solid propellant case, propellant is already inside the engine, so a steady-state case does not exist because the mass continually decreases!

Conservation of Momentum

- Fix the frame of reference to the engine
- When the engine is accelerating, the frame (of reference) is non-Newtonian and reversed D'Alembert force exists
- ***Question:*** For the case where $\frac{dm}{dt} \neq 0$, does Newton's 2nd law say that $\sum \vec{F} = m\frac{d\vec{v}}{dt}$ or that $\sum \vec{F} = \frac{d}{dt}(m\vec{v})$?
- ***Answer:*** Noting that $\frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt}$, under a Galilean transformation, $\vec{v}$ varies and therefore the second term varies. Newton's law is invariant under a Galilean transformation so: $\sum \vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$ is the correct form!

Conservation of Momentum

Let's first consider the External Control Volume



- $\vec{R}$ - Reaction force on strut
- $\vec{W}$ - Gravity force
- $\vec{\tau}$ - Friction force due to flow
- $\overrightarrow{d\eta}$ - Infinitesimal area vector (vector points inward)
- $\vec{A}$ - Inlet/Exit area (vector points inward)
- $\vec{v}$ - Velocity of fluid

Consider steady-state: $\overrightarrow{I_1} = \dot{m}_1 \overrightarrow{v_1}$ $\quad \overrightarrow{I_2} = \dot{m}_2 \overrightarrow{v_2}$ $\quad d\eta = |d\vec{\eta}|$

Conservation of Momentum

Consider steady-state: $\overrightarrow{I_1} = \dot{m}_1\overrightarrow{v_1}$ $\overrightarrow{I_2} = \dot{m}_2\overrightarrow{v_2}$ $d\eta = |d\vec{\eta}|$

$$0 = \overrightarrow{I_1} - \overrightarrow{I_2} + P_1\overrightarrow{A_1} + P_2\overrightarrow{A_2} + \int_s P d\vec{\eta} + \int_s \vec{\tau} d\eta + \overrightarrow{W} - \vec{R} - m\vec{a}$$

($-m\vec{a}$ is reversed D'Alembert force)

Consider P_a to be pressure at infinity

$$\int_s d\vec{\eta} + \overrightarrow{A_1} + \overrightarrow{A_2} = 0$$

$$\int_s P_a d\vec{\eta} + P_a\overrightarrow{A_1} + P_a\overrightarrow{A_2} = 0$$

Combine by subtracting

$$\overrightarrow{I_1} - \overrightarrow{I_2} + (P_1 - P_a)\overrightarrow{A_1} + (P_2 - P_a)\overrightarrow{A_2} + \int_s (P - P_a) d\vec{\eta} + \int_s \vec{\tau} d\eta + \overrightarrow{W} - \vec{R} - m\vec{a} = 0$$

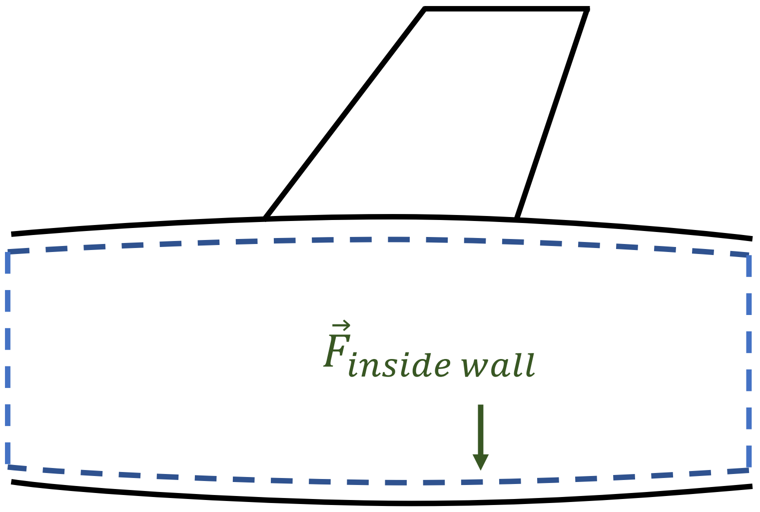
Conservation of Momentum

Define Thrust Vector: $\vec{T} \equiv \vec{I_1} - \vec{I_2} + (P_1 - P_a)\overrightarrow{A_1} + (P_2 - P_a)\overrightarrow{A_2}$

Define Aerodynamic Forces: $\overrightarrow{F_a} = \int_s (P - P_a) d\vec{\eta} + \int_s \vec{\tau} d\eta$

Note: $\overrightarrow{F_a} = \vec{L} + \vec{D}$ (Lift plus Drag)

Now consider the Internal Control Volume



The weight, strut force, and reversed D'Alembert force are all outside the control volume

$\vec{F}_{inside\ wall}$ is the force on the inside walls
(This results from pressure and friction forces on the walls)

Conservation of Momentum

Consider steady-state again: The forces sum to zero

$$\vec{T} = \overrightarrow{I_1} - \overrightarrow{I_2} + (P_1 - P_a)\overrightarrow{A_1} + (P_2 - P_a)\overrightarrow{A_2}$$

$$\vec{T} - \vec{F}_{inside\ wall} = 0 \longleftarrow$$ ($-\vec{F}_{inside\ wall}$ is the reaction to the force at the wall)

This is the net force on the inside walls of the engine

We define this as $\vec{T}$, the thrust vector. This is consistent with the definition above.
For convenience, $\vec{T}$ is defined using P_a as reference pressure so that it goes to zero in quiescent situation.

Balancing the forces of the walls of the engine:

$$\vec{F}_{inside\ wall} \longrightarrow \vec{T} + \overrightarrow{F_a} + \overrightarrow{W} - \vec{R} - m\vec{a} = 0$$

This is consistent with the external control volume case!

One-Dimensional Case

Assume:

- $P_1 = P_a$, $P_2 = P_e$
- $\overrightarrow{v_1} = V$ (Flight Velocity)
- $\overrightarrow{v_2} = U$ (Exhaust Velocity)
- $\overrightarrow{A_2} = A_e$
- $\vec{T} = T$
- $\dot{m}_1 = \dot{m}_{\mathrm{air}} = \dot{m}_a$
- $\dot{m}_2 = \dot{m}_{\mathrm{f}} + \dot{m}_a$

Then:

$$\vec{T} = (\dot{m}_{\mathrm{f}} + \dot{m}_a)U - \dot{m}_a V + (P_e - P_a)A_e$$

In the special case of the rocket where:

$$\dot{m}_a = 0$$

$$\dot{m}_{\mathrm{f}} = \dot{m}_{\mathrm{p}} \text{ (Flow rate of the propellants)}$$

Then, for the rocket:

$$\vec{T} = \dot{m}_{\mathrm{p}} U + (P_e - P_a)A_e$$

This is not an external force applied to the rocket – see definition as force applied to walls.

One-Dimensional Case

Let's look at the following: Consider Lift, Drag, Strut Force, and Gravity Force as ZERO

$$\vec{T} - m\vec{a} = 0 \qquad \text{or} \qquad \vec{T} - m\frac{d\vec{v}}{dt} = 0$$

In one dimension:

$$\dot{m}_{\mathrm{p}}U + (P_e - P_a)A_e = m\frac{d\vec{v}}{dt}$$

Interpretation: Impulse = Change in momentum

Apply this to a fixed-mass system!

M = Mass of Rocket

$$Impulse = (P_e - P_a)A_e\, dt$$

One-Dimensional Case

$$Momentum\ at\ time, t\ = MV$$

$$Momentum\ at\ time, t + dt\ = (M - dM)(V + dV) + (V - U)dM$$

$$= MV + MdV\ - \cancel{VdM}\ - dMdV + \cancel{VdM} - UdM$$

H.O.T.

$$Change\ in\ Momentum = MdV - UdM + H.O.T.$$

$$(P_e - P_a)A_e\, dt = MdV - UdM$$

$$(P_e - P_a)A_e + U\frac{dM}{dt} = M\frac{dV}{dt}$$

Now: $\frac{dM}{dt} = \dot{m}_{\mathrm{p}}$ $\leftarrow$ Note that $\frac{dM}{dt}$ is positive because $-\,dM$ was taken as change in mass.

One-Dimensional Case

Finally:

$$(P_e - P_a)A_e + \dot{m}_{\mathrm{p}}U = Ma$$

So conservation of momentum applied to a fixed mass system yields a result that agrees with, $\vec{F} = m\vec{a}$, applied to a system with varying mass!

Thermodynamic Units

Chemists work with moles while propulsion engineers work with mass units!

- $n =$ Number of moles
- $V =$ Volume
- $^{n}/_{V} = C$ is concentration
- $H =$ enthalpy/mole
- $nH =$ enthalpy (energy)
- $CH =$ enthalpy/Volume
- $h =$ enthalpy per unit mass, $h = {}^{H}/_{W}$
- $\rho =$ density, $^{m}/_{V}$
- $\rho h = CH =$ enthalpy/Volume
- $W =$ molecular mass (weight)
- $nW = m$ or $CW = \rho$
- $u = \text{velocity}, \quad \rho u = \frac{\text{mass}}{\text{area time}}, \quad Cu = \frac{\text{moles}}{\text{area time}}$
- $\rho u h = \frac{\text{enthalpy}}{\text{area time}} = CuH =$ enthalpy flux