University of California, Irvine
Department of Mechanical and Aerospace Engineering

MAE 112 - Homework 8
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1. Consider a crossed-field two-dimensional electromagnetic accelerator. The propellant flow begins with negligible velocity and is accelerated to 25,000 meters per second at the exit of the thruster. The exit plane pressure is 0.01 atmosphere while the ambient pressure is zero. Xenon gas with molecular weight of 54 and $\gamma = 5/3$ is the propellant. The gas temperature (T = 3,000 K) and the cross-sectional flow area are constant with downstream distance. The electric field is directed transverse to the flow with B = 1.8 webers per square meter. The conductivity $\sigma = 1800$ per ohm-meter. Neglect Hall current. Determine
a) the electric field E ion units of volts per meter at the exit;
b) the current density j_y at the exit in units of coulombs per meter squared per second;
c) the Lorentz force in units of newtons per cubic meter.

Solution:
Part(a): For constant temperature and area, the following equation can be derived where the magnetic field and electric field are related through:

$$\frac{B}{E} = \frac{1}{u}\frac{\gamma M^2 - 1}{\gamma M^2}$$

The sound speed at the exit can be computed by

$$c = \sqrt{\gamma RT} = \sqrt{\gamma \frac{R}{M} T} = \sqrt{\frac{5}{3}\frac{8.314 J/(K \cdot mol)}{0.1313 kg/mol} 3000K} = 562.68 m/s$$

The Mach number is

$$M_e = \frac{u_e}{c} = 44.43$$

By using the relation between magnetic field and electric field wee can get

$$E_e = \frac{B_e u_e \gamma M_e^2}{\gamma M_e^2 - 1} = \frac{1.8\frac{webers}{m^2} \times 25000\frac{m}{s} \times \frac{5}{3} \times 44.43^2}{\frac{5}{3} \times 44.43^2 - 1} = 45013.68 \frac{wb}{m \cdot s} = 45013.68 \frac{V}{m}$$

Part(b) The current density j_y is

$$j_y = \sigma(E - u_e B) = 1800\frac{1}{\Omega m} \times (45013.68\frac{V}{m} - 25000\frac{m}{s} \times 1.8\frac{Wb}{m^2}$$
$$Wb = Nms/C$$
$$\Omega = Vs/C$$
$$Nm = VC$$

The units of the first and second part of the equation can be converted by

$$\frac{V}{\Omega m^2} = \frac{V}{Vsm^2/C} = \frac{C}{m^2 s} \text{ and}$$
$$\frac{1}{\Omega m}\frac{m}{s}\frac{Wb}{m^2} = \frac{1}{\Omega m}\frac{m}{s}\frac{Nms/C}{m^2} = \frac{N}{\Omega m C} = \frac{\frac{VC}{m}}{\frac{Vs}{C} mC} = \frac{C}{m^2 s}$$

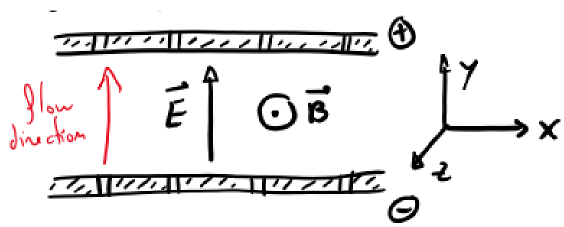
So the current density is

$$j_y = 24624 \frac{C}{m^2 s}$$

Part(c) The Lorentz force is

$$\vec{F} = \vec{j} \times \vec{B} = j_y B\hat{i} = 44323.2N$$

2. We have a crossed E and B field with the electric field vector pointing in the positive y-direction and the magnetic field vector pointing in the positive z-direction. Ohm's Law tells us that the current density has two components:
$j_y = \sigma(E - uB)/(1 + \Omega^2)$; and $j_x = -\sigma(E - uB)\Omega/(1 + \Omega^2)$. Futhermore, the velocity u = 0 and $\Omega = \omega_e/v$ where the gyro frequency $\omega_e = qB/m = 1.76 \times 10^{11}(weberssec)^{-1}B$; $v = 10^8$ collisions per second. $\sigma = 2000$ per ohm meter. Note that one weber = Newton-second per meter-coulomb; one ohm = volt-sec per coulomb; and one joule = Newton-meter = volt-coulomb. Design a Hall thruster that produces a Lorentz force in the positive y-direction of magnitude one kilo-Newton per cubic meter. i.e., choose the values of E and B in a consistent manner. Determine the consequential Lorentz force in the x-direction
Solution:
This problem relates to a Hall thruster. If we look at a sketch of the electrodes and the electric and magnetic fields, we have



As opposed to an electromagnetic thruster where we want to accelerate the propellant in the x-direction, here we want to accelerate the flow in the y-direction. Notice the electrodes are

perforated to allow particles to cross them. This is why u = 0 in this problem, being u the velocity component in x
Writing down the equations for the current density of the electric field we have

$$j_x = -\sigma(E - uB)\frac{\Omega}{1+\Omega^2} = \sigma E \frac{\Omega}{1+\Omega^2};\ j_y = \sigma(E - uB)\frac{1}{1+\Omega^2} = \sigma E \frac{1}{1+\Omega^2}$$

In order to have the Lorentz force dominate in y, j_x must dominate over j_y. That is $j_x/j_y >> 1$ or $\Omega >> 1$. The expression for Ω is

$$\Omega = \frac{\omega_e}{v} = \frac{1.76\times10^{11}B}{10^8} = 1.76 \times 10^3 B$$

If we pick $B = 2Wb/m^2$ as a reasonable value for the magnetic field, we get $\Omega = 3520$. With $\sigma = 2000$ per ohm-meter, the current density components become

$$j_x = -0.5682 \times EA/m^2 \text{ and } j_y = 1.614 \times 10^{-4}EA/m^2$$

As we can see, $j_x >> j_y$ in magnitude. The Lorentz force is evaluated as

$$\vec{F} = \vec{j} \times \vec{B} = j_y B\hat{i} - j_x B\hat{j} = F_x\hat{i} + F_y\hat{j}$$

Where the magnetic field vector $\vec{B} = (0, 0, B)$ and the current density vector is $\vec{j} = (j_x, j_y, 0)$.
We want

$$F_y = -j_x B = 1000\frac{N}{m^3} \to j_x = -500\frac{A}{m^2} = -0.5682 \times E \to E = 879.97\frac{V}{m}$$

And we verify that $F_X = j_y B = 0.284 N/m^3 << F_x$. With $j_y = (1.614 \times 10^{-4}) \times E = 0.142 A/m^2$

3. Consider a solid-core nuclear rocket where the propellant is helium gas. Suppose the gas temperature and the gas pressure for the flow entering the nozzle are 4000 K and 35 atmospheres, respectively. Assume that the Mach number at the nozzle entrance is sufficiently low so that we need not distinguish between static and stagnation values at that location. The helium mass flow rate is 100 kgm / second. Assume that $\gamma = 5/3$.
a) Assuming frozen isentropic flow of a perfect gas through the nozzle, determine the nozzle-throat area required.
b) Determine the thrust with an expansion through the nozzle to an exit pressure equal to 0.60 atmospheres with the ambient pressure of 0 atmospheres.
Solution:
Realize the nozzle is convergent-divergent. Thus, the throat must be choked(i.e., $M^* = 1$). With this condition, we can use mass conservation to find the throat area, A^*, that lets $\dot{m} = 100 kg/s$.
For helium gas we can use gas constant R = 2077 J/(kg K). For isentropic nozzle flow we can determine the throat area as

$$\dot{m} = \rho^* u^* A^* = \frac{P^*}{\sqrt{RT^*}}\sqrt{\gamma}A^* = \frac{P^o}{\sqrt{RT^o}}\Gamma(\gamma)A^* \to A^* = \frac{\dot{m}\sqrt{RT^o}}{P^o\Gamma(\gamma)} = 0.1119 m^2$$

With

$$\Gamma(\gamma) = \sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} = 0.7262$$

To determine thrust we need to solve for the exit conditions (or use the thrust equation that integrates everything together in terms of pressure ratios). Here, we pursue the first approach which is more of a step-by-step solution:

$$\begin{gathered}
\frac{P^o}{P_e} = \left(1 + \frac{\gamma-1}{2}M_e^2\right)^{\frac{\gamma}{\gamma-1}} \to M_e = 3.501 \\
\frac{T^o}{T_e} = 1 + \frac{\gamma-1}{2}M_e^2 \to T_e = 786.57K \\
u_e = \sqrt{2c_p(T^o - T_e)} = \sqrt{\frac{2\gamma R}{\gamma-1}(T^o - T_e)} = 5776.84 m/s \\
\frac{A_e}{A^*} = \frac{1}{M_e}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M_e^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \to A_e = 0.7605 m^2
\end{gathered}$$

Now that we have determined all the necessary variables, we find that the thrust of the nuclear rocket:

$$T = \dot{m}u_e + (p_e - p_a)A_e = \dot{m}u_e + p_e A_e = 584486.96 N$$

4. Consider a solid-core nuclear rocket where the propellant is hydrogen gas. Suppose the gas temperature and the gas pressure for the flow entering the nozzle are 3500 K and 40 atmospheres, respectively. The hydrogen mass flow rate is 2.0 kgm / second. Assume that $\gamma = 1.4$ for the mixture, which is dominated by the diatomic species, and c_p is constant for each species through the nozzle flow.

a) Determine the fraction of hydrogen mass in diatomic form(H_2) and the fraction in monatomic (H) form at the nozzle entrance.

b) Determine the total static enthalpy of the inflow to the nozzle; i.e., the sum of the sensible enthalpy and the chemical energy associated with the dissociated species.

c) Assuming frozen isentropic flow of a perfect gas through the nozzle, determine the nozzle-throat area required.

d) Determine the thrust for an exit pressure of 0.2 atmospheres and 0 atmospheres for ambient pressure.

e) Determine the temperature and the static sensible enthalpy at the exit. Also, determine the kinetic energy per unit mass at the exit.

Solution:

Part(a): Using CSU tool to find at equilibrium state ($T = 3500^o K, p = 30atm$) for a constant temperature and pressure reaction, the mass fractions are

	Initial State		Equilibrium State	
	mole fraction	mass fraction	mole fraction	mass fraction
H2	1.0000E+00	1.0000E+00	9.1140E-01	9.5364E-01
H	0.0000E+00	0.0000E+00	8.8603E-02	4.6355E-02

$$Y_{H_2} = 0.95364$$
$$Y_H = 0.046355$$

Part(b): Also from the CSU software we see the difference in enthalpy: The static specific enthalpy of the

	Initial State	Equilibrium State
Pressure (atm)	4.0000E+01	4.0000E+01
Temperature (K)	3.5000E+03	3.5000E+03
Volume (cm^3/g)	3.5616E+03	3.7265E+03
Enthalpy (erg/g)	5.3360E+11	6.3972E+11
Internal Energy (erg/g)	3.8924E+11	4.8868E+11
Entropy (erg/g K)	8.8263E+08	9.1678E+08

equilibrium state is 63972 J/g

Part(c):

First let us find the R and c_p for the mixture. Using $c_{p,H_2} = 14.55kJ/(kgK)$ and $c_{p,H} = 20.79kJ/(kgK)$

$$c_{p,mix} = Y_{H_2}c_{p,H_2} + Y_h c_{p,H} = 14839.18J/(kgK)$$
$$R = \frac{\gamma-1}{\gamma}c_{p,mix} = 4239.766J/(kgK)$$

The throat area can be found by

$$A^* = \frac{\dot{m}\sqrt{RT^o}}{P^o\Gamma(\gamma)} = 0.00278m^2$$

With

$$\Gamma(\gamma) = \sqrt{\gamma}(\frac{2}{\gamma+1})^{\frac{\gamma+1}{2(\gamma-1)}} = 0.6846$$
$$T^o = 3500K$$
$$P^o = 40atm = 4053000Pa$$

Part(d): With total pressure and exit pressure given, we could find exit Mach number by

$$M_e = \sqrt{\frac{2}{\gamma-1}((\frac{P^o}{P_e})^{\frac{\gamma-1}{\gamma}} - 1} = 4.66$$

Then we can find the exit temperature and exhaust velocity

$$T_e = T^o(1+\frac{\gamma-1}{2}M_e^2)^{-1} = 654.94K$$
$$u_e = \sqrt{2c_{p,mix}(T^o - T_e)} = 9188.94m/s$$

Thus, the thrust is

$$T = \dot{m}u_e + (p_e - p_a)A_e = 18434.218N$$

Part(e): Exit properties are already solved in part (d), let us take the value from part (d). Temperature at exit is

$$T_e = 654.94K$$

Kinetic energy per unit mass is

$$\frac{u_e^2}{2} = 4.2218 \times 10^7 J/kg$$

Sensible static enthalpy is

$$h = c_{p,mix}T_e = 9.7188 \times 10^6 J/kg$$