University of California, Irvine
Department of Mechanical and Aerospace Engineering

MAE 112 - Homework 3
Fall 2024

Prepared by: Wes Hellwig

1. A rocket nozzle has initial pressure and temperature of fifty atmospheres and 5000°R with $\gamma = 1.25$; $c_p = 0.30 Btu/lbm^\circ R$; and $A^* = 1.5 ft^2$. The flow is slightly over-expanded to a Mach number $M_e = 3.5$ at the nozzle exit with the ambient pressure at 0.50 atmosphere. Assume 95% for nozzle polytropic efficiency. Calculate: (a) the characteristic velocity c^*; (b) the mass flow $\dot{m}$; (c) nozzle exit pressure and cross-sectional area (beware of tables and graphs constructed for air flow); (d) nozzle exit velocity U; and (e) effective exhaust velocity c.

Solution:

(a) Characteristic velocity is defined as $c^* \equiv \sqrt{RT_0}/\Gamma$ (as derived in lecture, see Nozzle Flow slideset slide 9). Given that the nozzle is imperfect (the efficiency does not equal 1) $\Gamma = \Gamma(\gamma, e)$ where e is the polytropic efficiency. Recall from thermo that a polytropic process follows $pV^n = C$ where n is the polytropic index. When $n = \gamma$, the process is also isentropic. Pressure and temperature are related by Eq. (1) (including the efficiency factor e); when $e = 1$, the equation produces the typical pressure-temperature relation.

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{(\gamma-1)e}} \tag{1}$$

Following the derivation in class, $\Gamma(\gamma, e)$ can be found as:

$$\Gamma(\gamma, e) = \sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{(2-e)\gamma+e}{2e(\gamma-1)}} \tag{2}$$

We also need the gas constant which we can get from specific heat and γ. First, lets use dimensional analysis to get c_p converted to more useful units.

$$c_p = 0.3 Btu/lbm^\circ R\left(\frac{778\ ft * lbf}{1\ Btu}\right)\left(\frac{32.174\ lbm * ft/s^2}{1\ lbf}\right) = 7509.4116\frac{ft^2}{s^{2\circ}R} \tag{3}$$

$$R = c_p\frac{\gamma-1}{\gamma} = 1501.88\frac{ft^2}{s^{2\circ}R} \tag{4}$$

Therefore, $c^* = 4295.32$ ft/s.

(b)

$$\dot{m} = \frac{p_0 A^*}{c^*} = \frac{50atm(2116.22(lbf/ft^2)/atm)(32.174(lbm * ft/s^2)/lbf)(1.5ft^2)}{4295.32ft/s} = 1188.86lbm/s \tag{5}$$

(c) Nozzle exit pressure and cross-sectional area can be found with a the total-to-static relations involving the efficiency factor. If the efficiency was $e = 1$, these equations would be the isentropic equations from MAE 130C. Combining Eq. (1) with the isentropic equations (Eqs. (6-7)) gives the useful relation (Eq. (8)).

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2 \tag{6}$$

$$\frac{p_0}{p} = \left[1 + \frac{\gamma-1}{2}M^2\right]^{\frac{\gamma}{\gamma-1}} \tag{7}$$

$$\frac{p_0}{p} = \left[1 + \frac{\gamma - 1}{2}M^2\right]^{\frac{\gamma}{(\gamma-1)e}} \tag{8}$$

Similarly, the area ratio factoring in efficiency can be found as:

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{\frac{(2-e)\gamma+e}{2e(\gamma-1)}} \tag{9}$$

Solving Eqs. (8-9) for p and A and plugging in $M = M_e = 3.5$ we get $p_e = 0.3768atm$ and $A_e = 20.395ft^2$.

(d) The exit temperature is obtained with Eq. (6), $T_e = 1975.3°R$ which is used to find the exit velocity.

$$u_e = \sqrt{2c_p(T_0 - T_e)} = 4765.89\ ft/s \tag{10}$$

(e) Lastly, effective exhaust velocity is

$$c \equiv \frac{T}{\dot{m}} = \frac{\dot{m}u_e + (p_e - p_a)A_e}{\dot{m}} = 4621.987\ ft/s \tag{11}$$

2. Consider a nozzle with initial upstream entry pressure and temperature of thirty atmospheres and $4000°R$. The value of $\gamma = 1.2$ and the value of $c_p = 0.30Btu/1bm°R$. The throat area is $0.75ft^2$. The flow is perfectly expanded to the ambient pressure of 0.70 atmospheres. Calculate: (a) the mass flow, (b) the exhaust velocity, (c) the exit area, and (d) the thrust coefficient.

Solution:
(a) Since efficiency is not mentioned here, we assume it is isentropic. Thus, $\Gamma = \Gamma(\gamma)$ and the isentropic relations apply without alternation.

$$\Gamma(\gamma) = \sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} = 0.6485 \tag{12}$$

$$\dot{m} = \frac{p_0 A^*}{\sqrt{RT_0}}\Gamma(\gamma) = 444.04\ lbm/s \tag{13}$$

(b) The gas constant is found in the same manner as problem 1 part (a). Exit temperature is found with Eq. (14) to be $T_e = 2138.24°R$.

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} \tag{14}$$

$$u_e = \sqrt{2c_p(T_0 - T_e)} = 5287.85\ ft/s \tag{15}$$

(c) Exit area is found via the area-Mach relation (Eq. (17)). First, we need the exit Mach number.

$$M_e = \frac{u_e}{a_e} = \frac{u_e}{\sqrt{\gamma R T_e}} = 2.95 \tag{16}$$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \tag{17}$$

Plugging the exit Mach number into Eq. (17) and multiplying by A^* we find $A_e = 4.71\ ft^2$.

(d) Since the nozzle is perfectly expanded $T = \dot{m}u_e = 2348016.9\ lbm * ft/s^2$ i.e., the pressure term drops out of the thrust equation.

$$C_F = \frac{T}{p_0 A^*} = 1.53 \tag{18}$$

3. Consider a rocket engine that uses liquid oxygen and liquid ethanol C_2H_5OH fuel aka ethyl alcohol. The oxygen mass-flow rate is 2.0 times greater than the fuel mass-flow rate. Ethanol is stored at 298K while the oxygen is stored at 80K just slightly below its boiling point. Oxygen has a heat of vaporization of 6.81kJ/mole while the value for ethanol is 38.6 kJ/mole. The heat of formation of liquid ethanol is -277.0 kJ/mole. The specific heat at constant pressure for gaseous oxygen is 30.77 joules/mole K. The liquids are sprayed into the combustion chamber.

- (a) How much energy per mole is required to vaporize and heat a mole of oxygen to the temperature of 298K?
- (b) What is the expected ideal flame temperature? The ideal flame temperature aka theoretical flame temperature is the value with no dissociation. Assume that the products are H_2O, CO_2, and CO. You should make this calculation without using the online computer code. A key step is to determine what fraction of the carbon will appear in CO_2 and what fraction will appear in CO.

Solution:
(a) The energy required to vaporize and heat one mole of oxygen from 80K to 298K is found via Eq. (19). Notice I am choosing to define the result as negative because it requires heat. You can define it positive as well but remember which because it will be important in part (b).

$$Q_{O_2} = -h_{vape,O_2} + \int 298K80Kc_{p,O_2}dT = -13517.86\ J/mol \tag{19}$$

(b) To find the ideal adiabatic flame temperature we first need the balanced chemical reaction. Assume 1 mole of fuel which has a mass of 46 g and thus $\dot{m}_{fuel} = 46\ g/s$. Thus, we know our oxidizer flow rate is $\dot{m}_{O_2} = 2.0\dot{m}_{fuel} = 92\ g/s$. Dividing by the molecular weight of O_2, i.e. $MW_{O_2} = 32\ g/mol$, gives us the coefficient in front of O_2 in the chemical reaction, 2.875. The products are balanced as we have done so before.

$$\mathrm{C_2H_5OH + 2.875O_2 \rightarrow 1.75CO_2 + 3H_2O + 0.25CO} \tag{20}$$

Now, we can do the total; enthalpy balance that we have done before with one minor change. We need to account for the heat of vaporization from part (a). This will appear in the LHS, similar to the hf,m term. Since we already calculated both the energy required to vaporize and raise the temperature to 298, these two things can be combined under the value obtained for part (a).

$$\sum_{Reactants} n_m \left(h_{f,m} + \int_{T_{ref}}^{T_{initial}} c_{p,m}\, dT \right) = \sum_{Products} n_m \left(h_{f,m} + \int_{T_{ref}}^{T_{adiabatic}} c_{p,m}\, dT \right) \tag{21}$$

The LHS of the equation becomes:

$$\sum_{Reactants} n_m \left(h_{f,m} + \int_{T_{ref}}^{T_{initial}} c_{p,m}\, dT \right) = n_{C_2H_5OH}(h_{f,C_2H_5OH}) + n_{O_2}(-h_{vape,O_2} + c_{p,O_2}(80K - 298K)) \tag{22}$$

The quantity $-h_{vape,O_2} + c_{p,O_2}(80K - 298K)$ is what was calculated in part (a) so these terms will be replaced by Q_{O_2} going forward. The heat of vaporization of the fuel was not needed because it is already included in the heat of formation of fuel at the reference temperature. Because the fuel is stored at the reference temperature, the temperature integral of fuel specific heat dropped out.

$$h_{f,C_2H_5OH} + n_{O_2}Q_{O_2} = n_{CO_2}(h_{f,CO_2} + c_{p,CO_2}\Delta T) + n_{H_2O}(h_{f,H_2O} + c_{p,H_2O}\Delta T) + n_{CO}(h_{f,CO} + c_{p,CO}\Delta T) \tag{23}$$

This equation can then be solved for T_{ad} where $\Delta T = T_{ad} - T_{ref}$.

$$T_{ad} = T_{ref} + \frac{h_{f,C_2H_5OH} + n_{O_2}Q_{O_2} - n_{CO_2}h_{f,CO_2} - n_{H_2O}h_{f,H_2O} - n_{CO}h_{f,CO}}{n_{CO_2}c_{p,CO_2} + n_{H_2O}c_{p,H_2O} + n_{CO}c_{p,CO}} \tag{24}$$

Using the values in the table below, $T_{ad} = 6681.6K$. This is very high because the specific heats were taken to be constant near the reference temperature. In reality, the ability of a subtance to store heat increases with temperature, thus at higher temperatures, more energy is stored in the molecules which detracts from the maximum temperature. If specific heat values at 1500K or so were chosen (as was done in HW2) a lower maximum temperature would have been obtained.

	$h_{f,m}$ (kJ/kmol)	$c_{p,m}$ (kJ/kmol-K)
C_2H_5OH	-277,000	n/a
O_2	0	30.77
CO_2	-393,500	37.14
H_2O	-241,800	34.74
CO	-110,500	28.56

4. Suppose we have a rocket combustor that has hot products produced at the following conditions:

- $T = 4600°R$
- $p = 75\ atm$
- $\gamma = 1.25$
- $MW_{mix} = 27\ g/mol$

(a) Design a nozzle that will produce 75,000 pounds of thrust with an ambient pressure of one atmosphere. In particular, determine the following quantities: mass flow rate, exit pressure, exit or exhaust velocity, effective exhaust velocity, thrust coefficient, throat cross- sectional area, and exit cross-sectional area.
(b) Design a nozzle that produces 100,000 pounds of thrust with an ambient pressure at vacuum conditions. Limit the nozzle exit cross-sectional area to no more than thirty times the throat cross-sectional area. Determine the same quantities as described in Part (a).

Solution:
(a) First, it is necessary to know the exit pressure. To achieve maximum efficiency, assume perfect expansion, thus, $p_e = p_a$. The exit Mach number is then determined by Eq. (25) to be $M_e = 3.3123$.

$$p_0 = p\left(1 + \frac{\gamma - 1}{\gamma}M^2\right)^{\frac{\gamma}{\gamma-1}} \tag{25}$$

Using the exit Mach number, exit temperature and exhaust velocity can be found by Eqs. (26-27). To get exhaust velocity we need $c_p = R\gamma/(\gamma - 1) = 1539.63\ J/kg \cdot K$. The specific gas constant R was obtained by dividing the universal gas constant by the molecular weight.

$$T_0 = T\left(1 + \frac{\gamma - 1}{\gamma}M^2\right) \qquad T_e = 1939.75°R = 1077.64K \tag{26}$$

$$u_e = \sqrt{2c_p(T_0 - T_e)} = 2133.28m/s \tag{27}$$

Next, mass flow rate is found using the thrust equation $T = \dot{m}u_e + (p_e - p_a)A_e$. Remember, since it is perfectly expanded, the pressure term disappears.

$$\dot{m} = \frac{T}{u_e} = 156.4\ kg/s \tag{28}$$

To find A^* we will use conservation of mass. Additionally, Eqs. (25-26) are used again to find pressure and temperature in the throat, $p^* = 4217113.245\ Pa$ and $T^* = 2271.6\ K$, respectively.

$$\dot{m} = \rho^* u^* A^* \qquad p^* = \rho^* RT^* \qquad \dot{m} = \frac{p^*}{RT^*}u^* A^* \tag{29}$$

Since the throat must be sonic i.e., $M^* = 1$, we can find the throat velocity u^* with the Mach number.

$$M = \frac{u}{\sqrt{\gamma RT}} \qquad u^* = \sqrt{\gamma RT} \tag{30}$$

Substituting this relation into the third equation in Eq. (29), we obtain an equation in which A^* is the only unknown.

$$\dot{m} = p^* A^* \sqrt{\frac{\gamma}{RT^*}} \qquad A^* = 0.0277\ m^2 \tag{31}$$

The exit area can either be obtained by conservation of mass or by direct application of the area-Mach relation. Using area-Mach:

$$\frac{A_e}{A^*} = \frac{1}{M_e}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_e^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} = 8.664 \qquad \Rightarrow A_e = \frac{A_e}{A^*}A^* = 0.24\ m^2 \tag{32}$$

Effective exhaust velocity is $c = T/\dot{m} = 2133.28\ m/s$ and thrust coefficient is:

$$C_F = \frac{\dot{m}c}{p_0 A^*} = 1.585 \tag{33}$$

(b) Here we have a vacuum nozzle which is impossible to have perfectly expanded (the exit area would have to be infinite). So, with the limited area ratio of 30, we can find the exit Mach number for this limiting case using the area-Mach relation, $M_e = 4.3015$. Using Eqs. (25-27) we get $p_e = 19043.5\ Pa$, $T_e = 1388.52°R = 771.4\ K$, and $u_e = 2343.9\ m/s$. THe exit area is found by combining the ideal gas law with the mass flow rate equation and then substituting for mass flow rate in the thrust equation which produces:

$$T = p_e A_e \left(\frac{u_e^2}{RT_e} + 1\right) \tag{34}$$

which can be solved for $A_e = 0.9681\ m^2$. Using our chosen area ratio of 30 the throat area is found to be $A^* = 0.03227\ m^2$. Effective exhaust velocity and thrust coefficient are found with the equations listed for part (a). We obtain $c = 2445.42\ m/s$ and $C_F = 1.814$.

5. Consider a jet engine flying at a Mach number of 1.4. A normal shock sits at the entrance of the divergent diffuser. The diffuser entrance cross-sectional area is 2.5 ft^2 . The ambient conditions are 500°R for temperature and 0.8 atmosphere for pressure.
(a) What is the stagnation pressure immediately in front (upstream) of the shock? What is the stagnation pressure immediately behind (downstream) the shock? What is the Mach number immediately behind the shock? What is the mass flow through the diffuser?
(b) What is the minimum cross-sectional area required at the downstream end of the diffuser in order to assure that the Mach number of the flow there does not exceed 0.10?

Solution:
(a) To find the stagnation pressure in front of the shock we use Eq. (35) and find it to be $p_0 1 = 2.546\ atm$. Across the shock we use the normal shock relations which are tabulated in "Modern Compressible Flow" by John D. Anderson (the MAE 130C textbook). For a Mach number of 1.4, the total pressure ratio is 0.9582 so $p_0 2 = 2.44\ atm$. From the same line of the tabulated data the exit Mach number is $M_2 = 0.7397$.

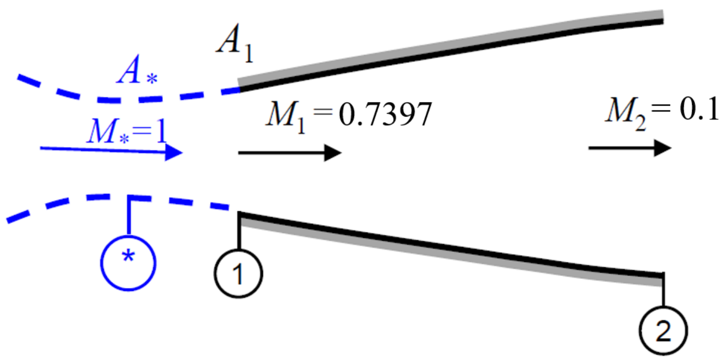
$$p_0 = p\left(1 + \frac{\gamma-1}{\gamma}M^2\right)^{\frac{\gamma}{\gamma-1}} \tag{35}$$

Mass flow can be calculated before or after the shock. We will do it before the shock. From the Mach number the velocity can be determined by first the speed of sound is needed. $a_1 = \sqrt{\gamma RT_1} = 334.08\ m/s$ therefore $u_1 = M_1 a_1 = 467.72\ m/s$.

$$\dot{m} = \frac{p_1}{RT_1}u_1 A_1 = 110.45\ kg/s \tag{36}$$

(b) This question implies that we must relate different sections of the diffuser to one another. We only know the area of the inlet and need to use isentropic relations to determine the area of the exit such that the Mach number is ≤ 0.1. This can be done with an imaginary sonic throat (hopefully you are familiar with this from 130C).

The idea behind an imaginary sonic throat is to relate different areas of a convergent or divergent duct to the same sonic throat. In other words, imagine that the flow immediately after the shock did not come from a shock but instead came from a longer diffuser as shown in the picture below. In this image the dashed



blue section is imaginary while the rest is the actual diffuser. Now, using the area-Mach relation, the various areas can be related. From the area-Mach chart, the two ratios below are found by looking at rows where the Mach number is 0.7397≈0.74 and 0.1. If you want to be precise, you can interpolate between rows.

$$A_2 \geq \frac{A_2}{A^*}\frac{A^*}{A_1}A_1 = 13.62\ ft^2 \tag{37}$$