University of California, Irvine
Department of Mechanical and Aerospace Engineering

MAE 112 - Homework 4
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Prepared by: Wes Hellwig

Note, this homework solution uses area-mach tables (TA.1), normal shock tables (TA.2), and the beta-theta oblique shock chart ($\beta - \theta$ chart). These charts were introduced in MAE 130C. The tables are faster than solving the mathematical relations but both will give the similar answers.

1. Compare a normal shock with an oblique shock. Suppose the inflowing velocity of the air had a Mach number of 2.0 at a temperature of 250 K and an ambient pressure of 0.70 atm.
(a) With the normal shock, determine the pressure, stagnation pressure, temperature, velocity, and Mach number behind (downstream of) the shock.
(b) Suppose we aim for a downstream stagnation pressure that is 15% higher than the value found in part (a). What is the angle of oblique shock here to the incoming velocity vector? Use the charts from Chapter 3, making the best interpolations you can.
(c) Determine the downstream values for the temperature, Mach number, velocity component normal to the oblique shock, and velocity component parallel to the oblique shock.

Solution:
(a) Start by obtaining the total pressure before the shock.

$$p_{01} = p_1 \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\frac{\gamma}{\gamma-1}} \tag{1}$$

From TA.2 with $M_1 = 2$, $p_2/p_1 = 4.5$, $T_2/T_1 = 1.687$, $p_{02}/p_{01} = 0.7209$, and $M_2 = 0.5774$ which gives $p_2 = 3.15\ atm$, $T_2 = 421.75\ K$, $p_{02} = 3.948\ atm$.

$$u_2 = M_2\sqrt{\gamma R T_2} = 237.67\ m/s \tag{2}$$

(b) New stagnation pressure is $p_{02} = 1.15(3.948) = 4.5402\ atm$, thus, $p_{02}/p_{01} = 0.829$. Looking at TA.2 we see that an inflow Mach number of approximately 1.76 produces this stagnation pressure ratio. When using TA.2 with an oblique shock however, the Mach number listed in the table is the shock-normal Mach number i.e., $M_{1n} = 1.76$. We know $M_1 = 2$ from the problem statement so $M_{1n} = M_1 sin(\beta)$.

$$\beta = sin^{-1}\left(\frac{M_{1n}}{M_1}\right) = 61.64^\circ \tag{3}$$

Looking for this value in the $\beta - \theta$ chart with $M_1 = 2$ gives $\theta \approx 23^\circ$ which would be the wedge angle.

(c) From normal shock chart, $T_2/T_1 = 1.502$ so $T_2 = 375.5\ K$.

$$M_2 = \frac{M_{2n}}{sin(\beta - \theta)} = 1.002 \tag{4}$$

$$u_2 = M_2\sqrt{\gamma R T_2} = 389.20\ m/s \tag{5}$$

$$u_{2n} = M_{2n}\sqrt{\gamma R T_2} = 243.04\ m/s \tag{6}$$

Then, using Pythagorean theorem,

$$u_{2t} = \sqrt{u^2 - u_{2n}^2} = 303.98\ m/s \tag{7}$$

2. Consider a Kantrowitz-Donaldson diffuser designed for a flight Mach number of 1.75. The entrance area equals 1.5 ft^2 and the ambient air temperature and pressure are 500°R and 0.7 atmosphere. The flow is isentropic everywhere except across the normal shockwave. Determine:
(a) the minimum throat cross-sectional area such that a normal shock may be stabilized at the entrance,
(b) the maximum mass flow
(c) the maximum stagnation pressure possible at the end of the diffuser (with subsonic flow only in the divergent portion).
In each of these optimizations, consider the flight Mach number fixed at the design value while the final pressure (at the end of the diffuser) is allowed to adjust.

Solution:
(a) To have a shock wave stabilized at the entrance, that means our incoming flow for diffuser will be subsonic (after shockwave). For a subsonic flow in the convergent part of the diffuser, it will accelerate until Mach number reaches 1. Then a divergent nozzle is required to either accelerate it or decelerate it (depending on the backpressure at the exit). Thus, the smallest area we can have for the throat is when $M^* = 1$. When flow reaches Mach number of 1, let us call this area as A_design^*. If the throat area $A^* > A_design^*$, the flow won't reach Mach number of 1 at the throat (which is ok because the flow will then be decelerated in divergent part after the throat). But if the throat area $A^* < A_design^*$, we know that the flow should be already at sonic ($M = 1$) before it reaches the throat, let's consider what will happen if we assume the flow can keep going after passing this point. Since the area is still decreasing, it is still a convergent channel. We know the flow can not go to supersonic because a convergent-divergent channel is required and here the channel is still converging. Can the flow go to subsonic then? If the flow becomes subsonic, that means the Mach number drops, which is impossible because the Mach number of subsonic flow should increase in a convergent channel! So, our assumption was not physically possible, i,e. the flow can't keep going downstream if it reaches Mach number of 1 before the throat. The good thing is that information will be propagated to upstream, so that our inlet condition will change in order to satisfy this small throat area, that means the shockwave will change its position and it is no longer stabilized at the entrance.

$M_1 = 1.75 \rightarrow$ TA.2 we get $M_2 = 0.6281 \rightarrow$ TA.1 gives $A_1/A^* = 1.157$. Since we know the inlet area (A_1), $A^* = 1.3\ ft^2$.

(b) Maximum mass flow occurs when the full inlet area captures air, i.e., shock is either swallowed by the intake (sitting at the throat for example), or sitting right at the inlet BUT NOT OUTSIDE THE INLET.

$$\dot{m}_{max} = \rho_1 u_1 A_1 = p_1 M_1 A_1 \sqrt{\frac{\gamma}{RT_1}} = 159.8\ lbm/s \tag{8}$$

(c) As we have discussed in the lecture, to achieve the highest stagnation we will have to minimize the strength of the shockwave inside the diffuser, the lowest strength is achieved when shock sits right at the throat. (Supersonic incoming flow is decelerated through the convergent section and the Mach number is lowest at the throat, thus, the shockwave has minimum strength)

At this condition, the throat is no longer sonic but supersonic. Let it be state 3 here where $A_3 = 1.3\ ft^2$. We can directly solve the area relation between any two locations in the nozzle.

$$\frac{A_1}{A_3} = \frac{M_3}{M_1}\left(\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_3^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{9}$$

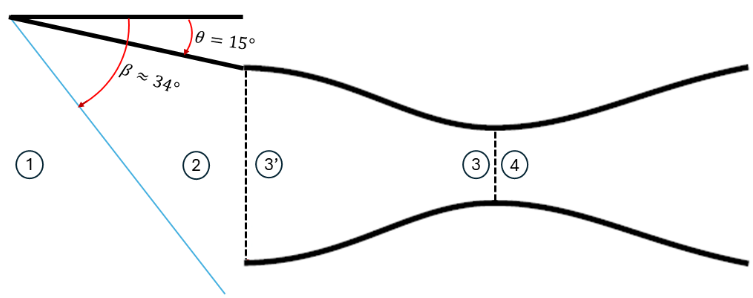
This gives $M_3 = 1.532 \rightarrow$ TA.2 gives $p_{04}/p_{03} = 0.9199$.

$$p_{03} = p_{01} = p_1\left(1+\frac{\gamma-1}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}} = 3.727\ atm \tag{10}$$

So, $p_{04} = 3.428\ atm$.

3. Consider a ramjet in flight at a Mach number of 2.75 with ambient conditions at 298 K and 0.9 atmosphere of pressure. The air capture area is 0.70 square meters. The inlet design involves first a wedge that deflects the stream by an angle of 15 degrees followed by a Kantrowitz-Donaldson (K-D) diffuser. Operation is at design conditions except for part (h). (a) What is the mass flow through the ramjet?
(b) What is the stagnation temperature for that flow through the inlet/diffuser?
(c) What are the stagnation-pressure values ahead of and immediately behind the first shock?
(d) What is the flow Mach number immediately behind the first shock? What is the flow Mach number at the entrance to the K-D diffuser?
(e) What is the Mach number at the diffuser throat?
(f) What is the final stagnation pressure?
(g) Determine the value of the polytropic efficiency for this inlet design.
(h) Determine the polytropic efficiency value for a shock at the entrance of the K- D diffuser.

Solution:



(a)

$$\dot{m} = \rho_1 u_1 A_1 = p_1 M_1 A_1 \sqrt{\frac{\gamma}{RT_1}} = 710.24\ kg/s \tag{11}$$

(b)

$$T_{01} = T_1 \left(1 + \frac{\gamma - 1}{2} M_1^2\right) = 748.725\ K \tag{12}$$

(c) With a wedge angle of $\theta = 15°$ and $M_1 = 2.75$, the $\beta - \theta$ chart gives $\beta \approx 34°$. Thus, $M_{1n} = 1.538 \rightarrow$ TA.2 gives $p_{02}/p_{01} = 0.917$.

$$p_{01} = p_1 \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\frac{\gamma}{\gamma-1}} = 22.626\ atm \tag{13}$$

So, $p_{02} = 20.748\ atm$.

(d) $M_{1n} = 1.538 \rightarrow$ TA.2 gives $M_{2n} = 0.69$

$$M_2 = \frac{M_{2n}}{sin(\beta - \theta)} = 2.12 \tag{14}$$

(e) To find the Mach number in front of the shock wave sitting at the throat, we first need to find out what the throat to inlet area ratio is. We know that K-D diffusers can stabilize shockwaves at two locations. Either the shockwave is stabilized at the entrance and the throat Mach number is 1, or, the shockwave sits at the throat and the Mach number right before this shock is greater than 1. The design condition has the Mach number sitting at the throat because the shock strength will be minimized in this case, which preserves total pressure. A designer wants to preserve as much total pressure as possible to achieve the highest efficiency/thrust.

To find the inlet to throat area ratio, consider the shock sitting at the inlet. $M_2 \rightarrow$ gives $M_{3'} = 0.56 \rightarrow$ TA.1 gives $A_{3'}/A^* = 1.24$. Now, we could also obtain the diffuser inlet area, $A_{3'}$, although this is not necessary to solve part (e). We will obtain this through mass conservation. $M_{1n} = 1.538 \rightarrow$ TA.2 gives $p_2/p_1 = 2.58$ and $T_2/T_1 = 1.343$, therefore, $p_2 = 2.322\ atm$ and $T_2 = 400.21\ K$.

$$A_2 = A_{3'} = \frac{\dot{m}}{p_2 M_2}\sqrt{\frac{RT_2}{\gamma}} = 0.408\ m^2 \tag{15}$$

$$A_{throat} = A^* = \frac{A^*}{A_{3'}} A_{3'} = 0.329\ m^2 \tag{16}$$

Now that we know the area ratio, we will consider the shock sitting at the throat. This means there is no longer a shock at the inlet. Using the area relation to relate any two areas between isentropic flow we can get the Mach number into the throat shock. Note, $A_2 = A_{3'}$ and $A_3 = A^* = A_4$.

$$\frac{A_2}{A_3} = \frac{M_3}{M_2}\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_3^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} = 1.24 \tag{17}$$

Solving this relation gives $M_3 = 1.86$.

(f) $M_3 = 1.86 \rightarrow$ TA.2 gives $p_{04}/p_{03} = 0.7857$. Since there is no shock at the inlet, the total pressure at station 2 is the same as station 3. Thus, $p_{04} = 16.302\ atm$.

(g) For a polytropic compression, efficiency is related to temperature and pressure by the following relation.

$$\frac{T_{final}}{T_{initial}} = \left(\frac{p_{final}}{p_{initial}}\right)^{\frac{1}{e}\frac{\gamma-1}{\gamma}} \tag{18}$$

So, we need the static temperature and pressure at station 4. These can be obtained either through the static ratios from the shock table or through isentropic relations evaluated at station 4. I do the later since we already have p_{04} and remembering that total temperature is conserved through a shock, thus $T_{04} = T_{01}$. $M_3 = 1.86 \rightarrow$ TA.2 gives $M_4 = 0.6036$.

$$p_4 = p_{04}\left(1+\frac{\gamma-1}{2}M_4^2\right)^{-\frac{\gamma-1}{\gamma}} = 12.74\ atm \tag{19}$$

$$T_4 = T_{04}\left(1+\frac{\gamma-1}{2}M_4^2\right)^{-1} = 697.873\ K \tag{20}$$

Rearranging Eq. (18) gives an expression for the efficiency.

$$e = \frac{\gamma-1}{\gamma}\frac{ln\left(\frac{p_4}{p_1}\right)}{ln\left(\frac{T_4}{T_1}\right)} = 0.89 \tag{21}$$

(h) If the shock were at the diffuser inlet we would evaluate Eq. (21) between stations 1 and 3'. $M_2 = 2.12 \rightarrow$ TA.2 gives $p_{3'}/p_2 = 0.51$ so $p_{3'} = 11.842\ atm$.

$$T_{3'} = T_{01}\left(1+\frac{\gamma-1}{2}M_4^2\right)^{-1} = 704.536\ K \tag{22}$$

$$e = \frac{\gamma-1}{\gamma}\frac{ln\left(\frac{p_{3'}}{p_1}\right)}{ln\left(\frac{T_{3'}}{T_1}\right)} = 0.86 \tag{23}$$

4. Suppose a particular compressor has a compression ratio $P_3/P_2 = 25$; the incoming air temperature is 300 K and its pressure is 1.2 atm. 20 kgm per sec. of air flows through the compressor.
(a) If the adiabatic efficiency is 90%, what is the final temperature?
(b) What is the power required?
(c) What is the minimum number of stages (pairs of rotor and stator sections) required to protect against separation due to adverse pressure gradients?

Solution:
(a) Recall from lecture that, for a compressor, adiabatic efficiency is defined as:

$$\eta_{ad} = \frac{h_{2,is} - h_1}{h_2 - h_1} = \frac{\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}}{\frac{T_2}{T_1} - 1} \tag{24}$$

This equation can be rearranged to find the final temperature. Note, this equation is for any compressor between stations 1 and 2. The problem statement gives stations 2 and 3 so we use those instead of 1 and 2.

$$T_2 = T_1\left(1 + \frac{1}{\eta_{ad}}\left[\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]\right) = 802.83\ K \tag{25}$$

(b) Power required is defined as $P = \dot{m}(h_{02} - h_{01})$ which, for constant specific heat, reduces to $P = \dot{m}c_p(T_{02} - T_{01})$. Furthermore, the Mach number in the compressor will be very low so $T_0 \approx T$. $c_{p,air} = 1004.5 J/kg - K$

$$P = \dot{m}c_p(T_2 - T_1) = 10101818.2\ Watts \tag{26}$$

(c) The minimum number of stages is defined by a pressure coefficient across the stages which must be less than 0.6 to prevent flow separation and other unwanted effects. The derivation on slide 11 of the compressors PowerPoint shows that this pressure coefficient corresponds to a maximum stage pressure ratio of:

$$\frac{p_{afterstage}}{p_{beforestage}} \leq 1.6 \tag{27}$$

Therefore, the number of stages required is:

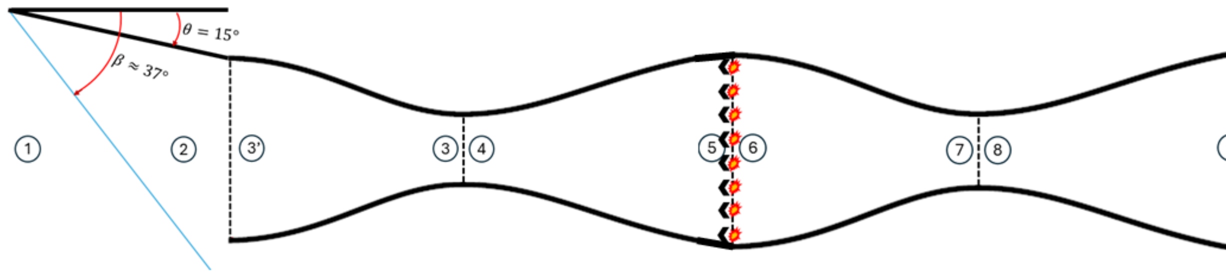
$$min\ \#\ stages = \frac{P_3/P_2}{1.6} = 15.625 \tag{28}$$

Since partial stages do not exist, the minimum number of stages must be rounded up to 16.

5. Do a preliminary design on a ramjet engine which produces 5000 lbf of thrust. Size constraints limit the intake cross-sectional capture area to 0.80 square feet. The engine is intended to cruise at a Mach number of 2.5. Assume ambient air conditions are one atmosphere of pressure and 500°R. Indicate your choices of inlet type, fuel, temperature at entrance to nozzle, and extent of expansion in nozzle. All choices must be rational and defensible, of course. Indicate mass flows of air and fuel, mixture ratio(s), thrust specific fuel consumption, exhaust velocity, stagnation pressure ratios across each component, throat area, nozzle exit area, and nozzle exit pressure.

Solution:

To begin, we will choose a K-D diffuser with a 15 degree wedge. The fuel will be octane C_8H_{18}. We will also design the nozzle so it is perfectly expanded. For ease, I am going to solve in SI units so the parameters are:
$Thrust = 22241.11\ N$
$A_1 = 0.0743\ m^2$



$M_1 = 2.5$
$T_1 = 277.778\ K$

The first half of this problem (freestream through oblique shock, and diffuser) will be the exact same steps as problem 3. Using isentropic relations we get the freestream total pressure and temperature and using the mass flow equation from problem 3 we get $\dot{m}_a$.

$$p_{01} = 17.086\ atm \qquad T_{01} = 625.0\ K \qquad \dot{m}_a = 78.87\ kg/s \tag{29}$$

Next, we get flight velocity $u_1 = M_1\sqrt{\gamma R T_1} = 835.2\ m/s$. Using the oblique shock chart we get $\beta \approx 37°$ which is used to get $M_{1n} = M_1 sin(\beta) = 1.5 \rightarrow$ TA.2 $p_2/p_1 = 2.458$, $T_2/T_1 = 1.32$, $p_{02}/p_{01} = 0.9298$, and $M_{2n} = 0.7011$. Thus, $M_2 = M_{2n}/sin(\beta - \theta) = 1.87$.

Next, we need to get the Mach number before the shock at the diffuser throat. This is done exactly the same as in problem 3. We assume the shock is at the diffuser inlet, then use the normal shock relations (TA.2) to cross the shock ($M_2 \rightarrow M_{3'}$), then $M_{3'} \rightarrow$ isentropic area mach relations (TA.1) to get the diffuser inlet-to-throat area ratio. $M_{3'} = 0.6016 \rightarrow$ TA.1 gives $A_{3'}/A^* = A_2/A^* = 1.18624$. Note in the following equation, $A^* = A_3$.

$$\frac{A_2}{A_3} = \frac{M_3}{M_2}\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_3^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} = 1.18624 \tag{30}$$

So, the diffuser throat Mach number before the throat shock is $M_3 = 1.636$. We can also use mass continuity to get $A_2 = A_{3'}$.

$$A_2 = A_{3'} = \frac{\dot{m}}{p_2 M_2}\sqrt{\frac{RT_2}{\gamma}} = 0.0.04643\ m^2 \tag{31}$$

$$A_3 = A_4 = A^* = \frac{A^*}{A_{3'}}A_{3'} = 0.04\ m^2 \tag{32}$$

Now we use $M_3 \rightarrow$ TA.2 to cross the shock in the diffuser throat which gives $p_4/p_3 = 2.9558$, $T_4/T_3 = 1.4132$, $p_{04}/p_{03} = 0.88146$ and $M_4 = 0.65784$. We are now at the combustor and need a lower Mach number around 0.1. So, choose $M_5 = M_6 = 0.1$ in the area relation to know how much the combustor area needs to expand from the diffuser throat.

$$\frac{A_4}{A_5} = \frac{M_5}{M_4}\left(\frac{1+\frac{\gamma-1}{2}M_4^2}{1+\frac{\gamma-1}{2}M_5^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} = 0.1938 \tag{33}$$

Now that the Mach number in the combustor is very low, the static and total pressure and temperature will be very close and we will assume they are the same, so, $T_{05} \approx T_{01} = 625.0\ K$. Note, in the equation below, $p_{03}/p_{02} = 1$ since $2 \rightarrow 3$ is isentropic without the shock.

$$p_{06} = p_{05} = p_{04} = \frac{p_{04}}{p_{03}}\frac{p_{03}}{p_{02}}\frac{p_{02}}{p_{01}}p_{01} = 14.00\ atm \tag{34}$$

Now we come to the chemistry. If you balance the stoichiometric equation between octane and air, you find that the mixture ratio $f = 0.3155$ which is way too high. The jet would run out of fuel almost immediately.

So, we will burn very lean (a lot of excess air). I choose $f = 0.02$. Recall the mass flow rate of air is $\dot{m}_a = 78.87\ kg/s$.

$$\dot{m}_f = f\dot{m}_a = 1.5774\ kg/s \tag{35}$$

We can then get the fuel flow rate in mole (rather than mass) basis to balance the chemical equation. We get the air flow rate similarly.

$$\dot{N}_f = \frac{\dot{m}_f}{MW_f} = 13.81\ mol/s \tag{36}$$

$$\dot{N}_a = \frac{\dot{m}_a}{MW_a} = 2723.4\ mol/s \tag{37}$$

Now we can balance the chemical equation

$$13.81\mathrm{C_8H_{18}} + 2723.4(\mathrm{O_2} + 3.76\mathrm{N_2}) \rightarrow 110.47\mathrm{CO_2} + 124.28\mathrm{H_2O} + 2550.80\mathrm{O_2} + 10240.00\mathrm{N_2} \tag{38}$$

and normalize it so there is 1 mole of fuel.

$$\mathrm{C_8H_{18}} + 197.21(\mathrm{O_2} + 3.76\mathrm{N_2}) \rightarrow 8\mathrm{CO_2} + 9\mathrm{H_2O} + 184.72\mathrm{O_2} + 741.54\mathrm{N_2} \tag{39}$$

Putting this reactant combination into the CSU software gives a flame temperature of $T_6 = T_{06} = 797.0\ K$. Now lets use the thrust equation to get the exit velocity. Remember, since we chose a perfectly expanded nozzle, the $(p_e - p_a)A_e$ term drops out of the thrust equation.

$$u_e = \frac{T + \dot{m}_a u_1}{\dot{m}_a + \dot{m}_f} = 1095.3\ m/s \tag{40}$$

Since the mixture is so lean, the molecular weight will be close to that of air, but, for correctness, I calculated the ratio of specific heats including all the species in their equilibrium composition output by the CSU software. Where the subscript stands for "combustion", $\gamma_c = 1.35$. With γ_c we can get the nozzle throat area ratio with respect to the combustion chamber area.

$$\frac{A_6}{A^*} = \frac{1}{M_6}\left[\frac{2}{\gamma_c + 1}\left(1 + \frac{\gamma_c - 1}{2}M_6^2\right)\right]^{\frac{\gamma_c+1}{2(\gamma_c-1)}} = 5.8536 \tag{41}$$

$$A_6 = A_5 = \frac{A_5}{A_4}\frac{A_4}{A_2}A_2 = 0.202\ m^2 \tag{42}$$

$$A^*_{nozzle} = \frac{A^*}{A_6}A_6 = 0.0345\ m^2 \tag{43}$$

The exit Mach number can be obtained with the isentropic pressure relation knowing $p_e = p_a$ as was chosen.

$$M_9 = \sqrt{\frac{2}{\gamma_c - 1}\left[\left(\frac{p_06}{p_e}\right)^{\frac{\gamma_c - 1}{\gamma_c}} - 1\right]} = 2.369 \tag{44}$$

Then using the area-mach relation once again (Eq. (41)), but now with M_9 we get the nozzle exit area to throat area ratio.

$$\frac{A_9}{A^*} = \frac{1}{M_9}\left[\frac{2}{\gamma_c + 1}\left(1 + \frac{\gamma_c - 1}{2}M_9^2\right)\right]^{\frac{\gamma_c+1}{2(\gamma_c-1)}} = 2.442 \tag{45}$$

$$A_e = A_9 = \frac{A_9}{A^*}A^*_{nozzle} = 0.0843\ m^2 \tag{46}$$

Lastly, the thrust specific fuel consumption is $TSFC = \dot{m}_f/T = 0.2553\ \frac{kg/hr}{N}$