MAE 112 PROPULSION – FALL 2022

SOLUTIONS FOR MIDTERM EXAM

PROBLEM #1（30 pts）

Data:

Normal decane burns adiabatically at a constant pressure of ten atmospheres. The mixture is lean with 100% excess air in terms of moles. The air and the fuel enter at 700°R in gaseous form. Consider the products to be CO2, CO, H2O, H2, O2, and N2.

Find:

(a) (15 pts) Write all of the required equations with known quantities and parameters substituted into the equation. Identify the unknowns.
(b) (10 pts) In an ideal case where no CO or H2 occurs in the products, what would be the mole fractions for CO2 and H2O in the products?
(c) (5 pts) Which of the two temperatures from 1a and 1b is larger? Why?

Solution:

Part(a)
First, write the chemical reaction for the ideal stoichiometric case with no dissociation. Then we can obtain reactant composition

$$C_{10}H_{22} + 15.5\left(O_2 + 3.76N_2\right) \rightarrow 10CO_2 + 11H_2O + 58.28N_2$$

Our actual reaction will have 100% excess air

$$C_{10}H_{22} + 31\left(O_2 + 3.76N_2\right) \rightarrow aCO_2 + bCO + cH_2O + dH_2 + eO_2 + fN_2$$

To solve for the 6 coefficient we need 6 equations. 4 come from mass balance and 2 come from Chemical equilibrium. The mass balance gives

$$\begin{aligned} C &\rightarrow \quad a + b = 10 \\ O &\rightarrow \quad 2a + b + c + 2e = 62 \\ H &\rightarrow \quad 2c + 2d = 22 \\ N &\rightarrow \quad 2f = 233.12 \end{aligned}$$

For chemical equilibrium we have two dissociations

$$CO_2 \leftrightarrow CO + \tfrac{1}{2}O_2$$
$$H_2O \leftrightarrow H_2 + \tfrac{1}{2}O_2$$

which give the following equations

$$K_1 = \frac{X_{CO} X_{O_2}^{1/2}}{X_{CO_2}} = p^{-1/2} K_{p_1}\left(T_2\right)$$
$$K_2 = \frac{X_{H_2} X_{O_2}^{1/2}}{X_{H_2O}} = p^{-1/2} K_{p_2}\left(T_2\right)$$

with T_2 stands for flame temperature or final temperature.
Each mole fraction is related to the total number of moles at equilibrium $N_p = a + b + c + d + e + f$. For instance:

$$X_{CO_2} = \frac{a}{N_p} = \frac{a}{a+b+c+d+e+f}$$

To obtain T_2, the adiabatic energy equation of the chemical reaction is

$$Q = \left(H_{p2} - H_{pf}\right) - \left(H_{R1} - H_{Rf}\right) + H_{RPf} = 0$$

where $\left(H_{p2} - H_{pf}\right)$ is the enthalpy change of product mixture from reference state f to final state 2, $\left(H_{R1} - H_{Rf}\right)$ is the enthalpy change of reactant mixture from initial state 1 to reference state f, H_{RPf} is the heat of reaction at the reference state f

$$H_{RPf} = H_{Pf} - H_{Rf} = \sum_j \left(n_j Q_{fj}\right)_p - \sum_i \left(n_i Q_{fi}\right)_R$$

The enthalpy change for reactants and products from the initial state to reference state are

$$H_{P2} - H_{Pf} = N_P \int_{T_f}^{T_2} c_{p,P}(T) dT \qquad (1)$$

with T_2 being the flame temperature, and $c_{p,P} = \sum_j X_j c_{pj}$.

$$H_{R1} - H_{Rf} = N_R \int_{T_f}^{T_1} c_{p,R}(T) dT \qquad (2)$$

With T_1 being the initial temperature.

Part(b)

For ideal case, no dissociation occurs and since we have excess air, the fuel burns completely

$$C_{10}H_{22} + 31\left(O_2 + 3.76N_2\right) \rightarrow aCO_2 + bH_2O + cO_2 + dN_2$$

The equations to fins equilibrium compositions are only mass balance:

$$C \rightarrow \quad a = 10$$
$$O \rightarrow \quad 2a + b + 2c = 62$$
$$H \rightarrow \quad 2b = 22$$
$$N \rightarrow \quad 2d = 233.12$$

Solve the system of equations to get $a = 10,\ b = 11,\ c = 15.5,\ d = 116.56$, So total number of moles of products is

$$N_P = a + b + c + d = 153.06$$
$$X_{CO_2} = \frac{a}{N_P} = 0.0653$$
$$X_{H_2O} = \frac{a}{N_P} = 0.0719$$

Part(c)

Both ideal and non-ideal flame temperature are very close to each other because the dissociation in the non-ideal case is not very strong. However, ideal flame temperature should be slightly higher (in fact it is the highest by definition) since no energy has been taken to dissociate a product.

PROBLEM #2 (30 pts)

Data: Convergent-divergent nozzle with stagnation pressure, $p_0 = 25$ atm, and stagnation temperature, $T_0 = 2000$ K. Fluid properties of the gas are ratio of specific heats, $\gamma = \frac{c_p}{c_v} = 1.25$, and specific heat at constant pressure, $c_p = 0.30$ btu/(lbm °R). The flow is perfectly expanded to ambient pressure $p_e = p_a = 1$ atm. The throat area is $A^* = 0.01$ m^2.
Find: exhaust velocity, u_e (10 pts); mass flow, $\dot{m}$ (10 pts); and thrust coefficient, C_F (10 pts).

Solution:

To find the exhaust velocity, we need to know how the flow expands across the nozzle. Taking the isoenergetic condition and knowing that for a perfect gas $h = c_p T$, we have:

$$h^o = h + \frac{u^2}{2} \rightarrow c_p T_0 = c_p T + \frac{u^2}{2}$$

Taking the original equation and using the properties of perfect gases, the exhaust velocity can be found as

$$u_e = \sqrt{\frac{2\gamma R T_0}{\gamma - 1}} \left(1 - \left(\frac{p_e}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right)^{1/2}$$

The stagnant value for pressure is the initial pressure. That is, the goal of a nozzle is to accelerate the flow from nearly stationary flow (stagnant conditions $u \approx 0$).
Remember,

$$c_p = R \frac{\gamma}{\gamma - 1}$$

Thus, $c_p = 1256$ J/(kg K) and $R = 251.2$ J/(kg K). Then,

$$u_e = \sqrt{4000 c_p} \left(1 - \left(\frac{1}{25}\right)^{\frac{\gamma-1}{\gamma}}\right)^{1/2} = 1544.3 \text{ m/s}$$

To obtain mass flow, we could do it at any given section of the nozzle. However, it is very useful and simple to do it at the throat where we know (for a convergent-divergent nozzle with perfect expansion) that it must be sonic conditions. Thus, we can use

$$\dot{m} = \frac{p^o A^*}{\sqrt{R T^o}} \sqrt{\gamma} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} = \frac{p^o A^*}{\sqrt{R T^o}} \Gamma(\gamma) = 23.52 \text{ kg/s}$$

From lecture slides, the thrust coefficient is

$$C_F = \left\{ \frac{2\gamma^2}{\gamma - 1} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}} \left[1 - \left(\frac{p_e}{p^o}\right)^{\frac{\gamma-1}{\gamma}}\right] \right\}^{\frac{1}{2}} + \left(\frac{p_e}{p^o} - \frac{p_a}{p^o}\right) \frac{A_e}{A^*} = 1.434$$

PROBLEM #3 (40 pts)

<u>Data:</u> Consider a Kantrowitz-Donaldson diffuser designed for a flight Mach number of 1.41. Entrance is $0.5\,ft^2$ and the ambient air temperature and pressure are 550°R and 0.5 atmosphere. The flow is isentropic everywhere except across the normal shockwave.

<u>Find:</u>

(a) (15 pts) the minimum cross-sectional area of the throat such that a normal shock may be stabilized at the entrance,

(b) (10 pts) the maximum mass flow

(c) (15 pts) the maximum stagnation pressure possible at the end of the diffuser (with subsonic flow only in the divergent portion).

<u>Solution:</u>

Part(a)

To have a shock wave stabilized at the entrance, that means our incoming flow for diffuser will be subsonic (after shockwave). For a subsonic flow in the convergent part of the diffuser, it will accelerate until Mach number reaches 1. Then a divergent nozzle is required to either accelerate it or decelerate it (depending on the backpressure at the exit). Thus, the smallest area we can have for the throat is when $M^* = 1$. When flow reaches Mach number of 1, let us call this area as A^*_{design}. If the throat area $A^* > A^*_{design}$, the flow won't reach Mach number of 1 at the throat (which is ok because the flow will then be decelerated in divergent part after the throat). But if the throat area $A^* < A^*_{design}$, we know that the flow should be already at sonic (M=1) before it reaches the throat, let's consider what will happen if we assume the flow can keep going after passing this point. Since the area is still decreasing, it is still a convergent channel. We know the flow can not go to supersonic because a convergent-divergent channel is required and here the channel is still converging. Can the flow go to subsonic then? If the flow becomes subsonic, that means the Mach number drops, which is impossible because the Mach number of subsonic flow should increase in a convergent channel! So, our assumption was not physically possible, i,e. the flow can't keep going downstream if it reaches Mach number of 1 before the throat. The good thing is that information will be propagated to upstream, so that our inlet condition will change in order to satisfy this small throat area, that means the shockwave will change its position and it is no longer stabilized at the entrance.

Now we just need to solve for the A^* when shock wave is right at the entrance, this area then is our smallest area to keep the shock wave stable at the entrance

Before the normal shock (let's call it state 1), we have $M_1 = 1.41$, the real inlet mach number should be the Mach number after shock wave (let's call it state 2)

$$M_2 = \sqrt{\frac{M_1^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_1^2 - 1}} = 0.7338$$

The area ratio can be obtained by

$$\frac{A_2}{A^*} = \frac{1}{M_2}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}{M_2}^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} = 1.0719$$

So smallest throat area is

$$A^* = \frac{A_2}{1.0719} = 0.4665\ ft^2$$

Part(b)

If the Mach number is fixed at the design value. The maximum mass flow rate we can get for KD diffuser is when shock sit right at the entrance or gets swallowed inside the diffuser, in which the air intake area is the same as the entrance area. (if shock is detached, which means we need more diffusion process in front of the entrance, the air intake area is actually less than the entrance area (less mass flow rate for the same flight condition))

Before the shock wave, we have

$$\begin{aligned}\dot{m}_{max} &= \rho_1 u_1 A_i = \frac{p_1}{RT_1} M_1 \sqrt{\gamma R T_1} A_i \\ &= \frac{0.5\times 2116.2\frac{lbf}{ft^2}}{1717\frac{ft\cdot lbf}{slug\cdot R}\times 550R} \times 1.41 \times \sqrt{1.4 \times 53.33\frac{ft\cdot lbf}{lb\cdot R} \times 550R} \times 0.5ft^2 \\ &= 0.036049\frac{lb}{ft^3} \times 1.41 \times 1149.43 ft/s \times 0.5ft^2 = 29.21\ \frac{lb}{s}\end{aligned}$$

The maximum mass flow rate is solved here, the maximum mass flow rate per unit area happens at the smallest area (throat).

$$\frac{\dot{m}}{A^*} = 58.42\frac{lb}{ft^2\cdot s}$$

Part(c)

As we have discussed in the lecture, to achieve the highest stagnation we will have to minimize the strength of the shockwave inside the diffuser, the lowest strength is achieved when shock sits right at the throat. (Supersonic incoming flow is decelerated through the convergent section and the Mach number is lowest at the throat, thus, the shockwave has minimum strength)

At this condition, the throat is no longer sonic but supersonic. Let it be state 3 here where $A_3 = 0.4665ft^2$. We can directly solve the area relation between any two location in nozzle

$$\frac{A_1}{A_3} = \frac{M_3}{M_1}\left(\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_3^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

We know $A_1 = 0.5ft^2$, $A_3 = 0.4665ft^2$ and $M_1 = 1.41$, so we can solve for M_3 here.

So $M_3 = 1.2601$, this is the Mach number before the shock. With this we could find the stagnation pressure drop across the shock wave that sits in the throat of the KD diffuser, let us call the state right after the shock wave state 4

$$\frac{p_{04}}{p_{03}} = \frac{\left[\left(\frac{\gamma+1}{2}M_3^2\right)/\left(1+\frac{\gamma-1}{2}M_3^2\right)\right]^{\gamma/(\gamma-1)}}{\left(\frac{2\gamma}{\gamma+1}M_3^2-\frac{\gamma-1}{\gamma+1}\right)^{1/(\gamma-1)}} = 0.9857$$

The stagnation pressure p_{03} equals to the stagnation pressure at the entrance because the diffusion is isentropic before the shock wave.

$$p_{03} = p_{01} = p_1\left(1 + \frac{\gamma-1}{2}M_1^{\ 2}\right)^{\gamma/(\gamma-1)} = 1.6234\ atm$$

So we have

$$p_{04} = p_{03} \times 0.9857 = 1.6002\ atm$$

Comments:

Note that one important feature of the KD nozzle is that it is able to have a stabilized shock wave at the entrance when there is perturbation in the diffuser. But to optimize our design, we want to make the diffuser smaller so that we call this as our optimized design (i,e. shock can be stable at the entrance and throat is smallest or choked with Mach number being 1). Of course you could have a larger throat area so that the throat Mach number is still subsonic when the shock sit at the entrance of the diffuser.