# Lecture 5
# Nozzle Flow

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# Pressure Ratio

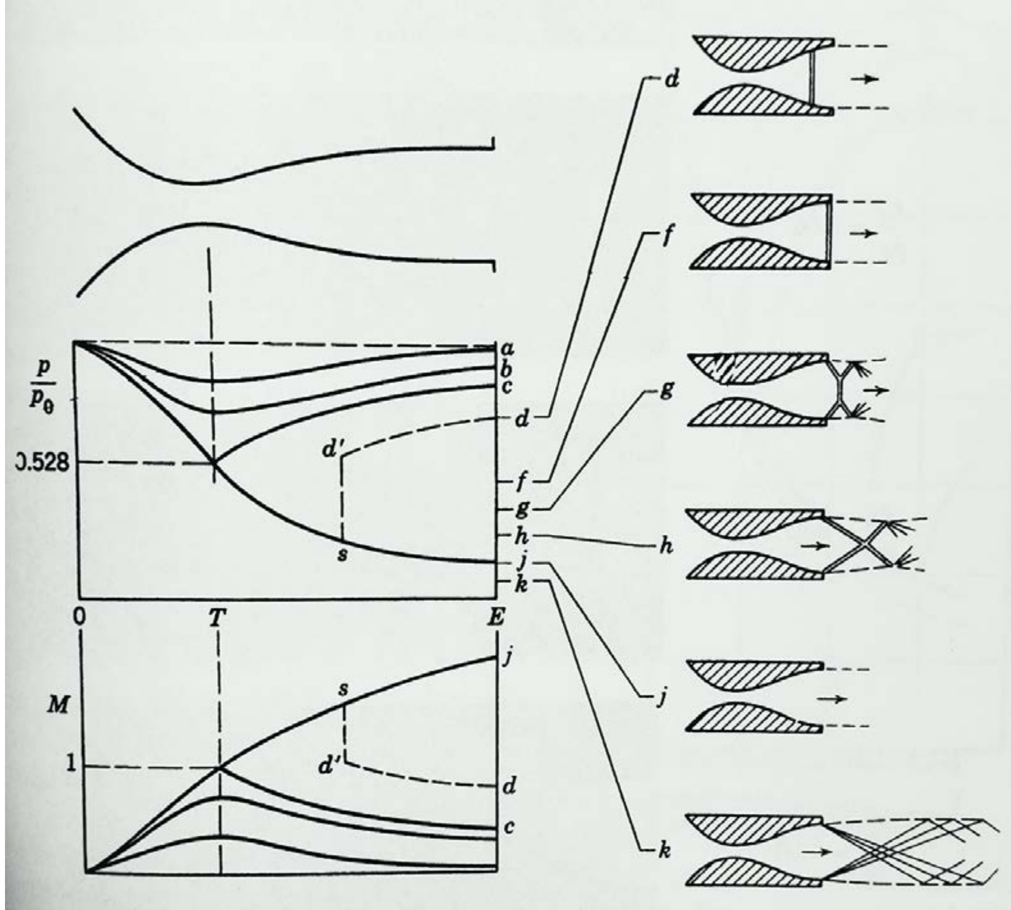
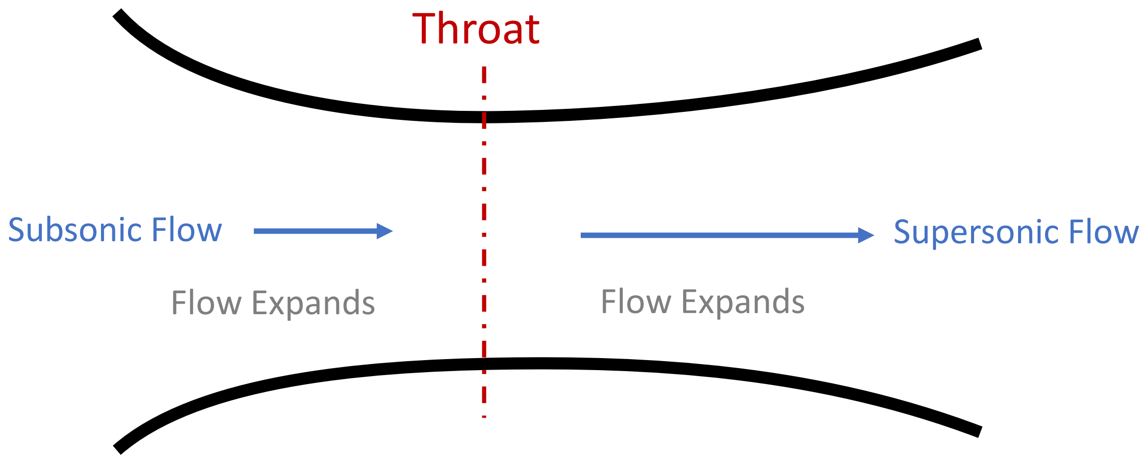


Fig. 5·3 Effect of pressure ratio on flow in a Laval nozzle.

Effects of back pressure on nozzle flow – Liepmann & Roshko (1993)

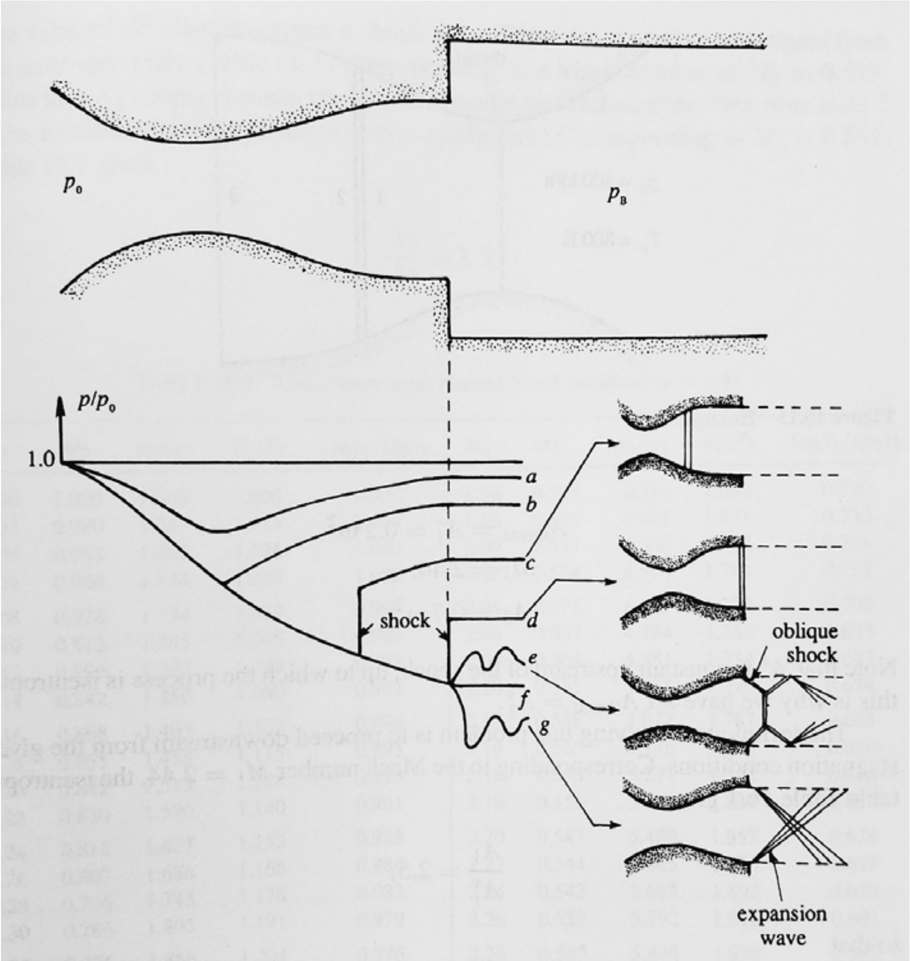



The exit pressure determines the flow!

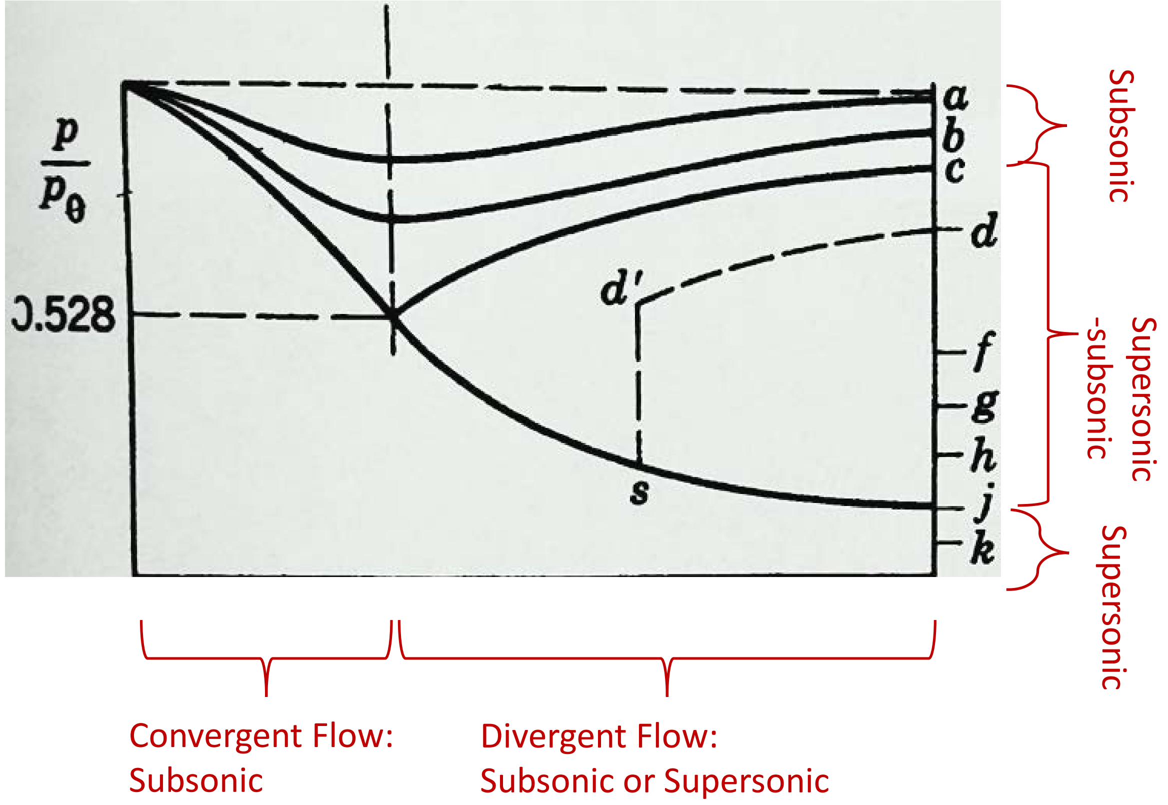
The length and thereby the exit cross-sectional area can be varied at fixed ambient pressure:

- At optimal length and exit area, $P_e = P_a$
- At shorter length and smaller exit area, $P_e > P_a$ (Underexpanded)
    - *Additional expansion is required*
- At longer length and smaller area, $P_e < P_a$ (Overexpanded)
    - *Compression shock occurs*

# Pressure Ratio




Effects of back pressure on nozzle flow – Kundu & Cohen (2008)



# Momentum Equation

$$T = (\dot{m}_{\mathrm{f}} + \dot{m}_a)U - \dot{m}_a V + (P_e - P_a)A_e$$

$P_1 = P_a$ here

***Define:*** $T' = (\dot{m}_{\mathrm{f}} + \dot{m}_a)U + (P_e - P_a)A_e$

This is the part of $T$ that is affected by the nozzle design. Note that for a rocket, $T' = T$.

Look at the effect of change in nozzle length, $dx$

$\dot{m} = \dot{m}_{\mathrm{f}} + \dot{m}_a$ is fixed

$$dT' = \dot{m}dU + (P_e - P_a)dA_e + A_e dP_e$$

$U, A_e, P_e$ change with changes in nozzle length.

***From the momentum (Euler) equation:*** $\rho u du = -dp$ ***or*** $\frac{\dot{m}}{A} du = -dp$

***So:*** $\frac{\dot{m}}{A_e} dU = -dP_e$ ***Then:*** $\dot{m}dU = -A_e dP_e$

# Momentum Equation

***Then:*** $dT' = -\cancel{A_e dP_e} + (P_e - P_a)dA_e + \cancel{A_e dP_e}$

$$dT' = (P_e - P_a)dA_e$$

$dT' = 0$ when $P_e = P_a$ so an extrema is reached at this point.
It is the maximum effective exhaust.

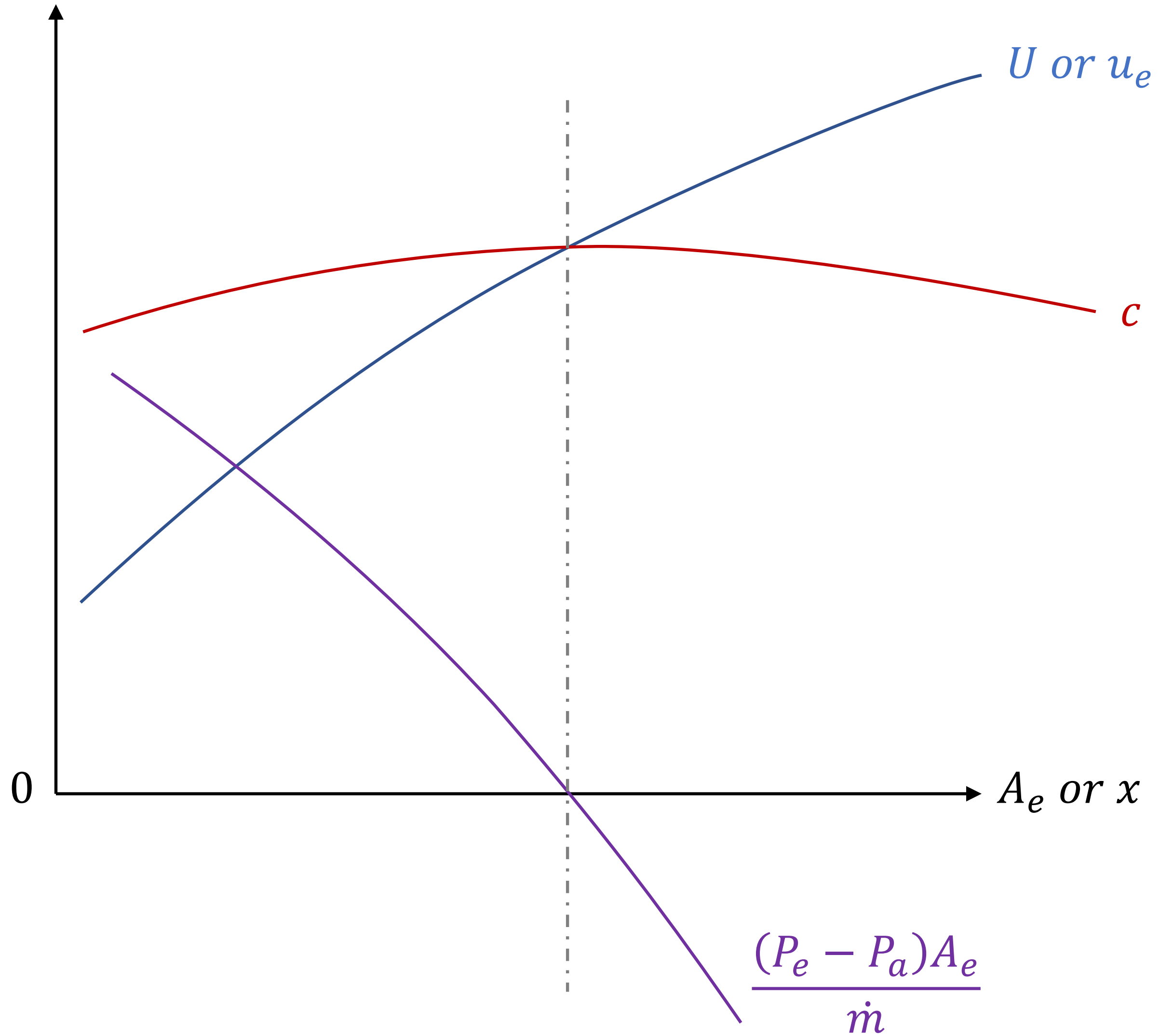
***Define the effective exhaust velocity, c:***

$$\boxed{c \equiv \frac{T'}{\dot{m}} = U + \frac{(P_e - P_a)A_e}{\dot{m}}}$$

# Momentum Equation

$$c \equiv \frac{T'}{\dot{m}} = U + \frac{(P_e - P_a)A_e}{\dot{m}}$$

$c$ therefore $T' = c\dot{m}$ reaches a maximum when $P_e = P_a$!



# Momentum & Energy

***Consider:*** isentropic, isoenergetic flow through a nozzle where, $h^\circ = c_p T^\circ = stagnation\ enthalpy$, is constant through the nozzle.

$$c_p T^\circ = c_p T + {u^2}/{2}$$

or

$$u = \sqrt{2c_p(T^\circ - T)} = \sqrt{2c_p T^\circ\left(1 - \frac{T}{T^\circ}\right)}$$

- $c_p = c_v + R$
- $\gamma = \frac{c_p}{c_v}$
- $\frac{c_p}{c_p} = \frac{1}{\gamma} + \frac{R}{c_p}$
- $c_p = R\frac{\gamma}{\gamma - 1}$

$$U = u_e = \sqrt{\frac{2\gamma R T^\circ}{\gamma - 1}}\left(1 - \frac{T_e}{T^\circ}\right)^{1/2}$$

$$P = \rho R T$$

$$\frac{T}{T^\circ} = \left(\frac{P}{P^\circ}\right)^{\frac{\gamma-1}{\gamma}}, \left(\frac{\rho}{\rho^\circ}\right)^{\gamma} = \frac{P}{P^\circ}$$

$$\frac{T_e}{T^\circ} = \left(\frac{P_e}{P^\circ}\right)^{\frac{\gamma-1}{\gamma}}$$

$$\boxed{U = u_e = \sqrt{\frac{2\gamma R T^\circ}{\gamma - 1}}\left(1 - \left(\frac{P_e}{P^\circ}\right)^{\frac{\gamma-1}{\gamma}}\right)^{1/2}}$$

# Mass Flow Through Nozzle

$$\dot{m} = \rho_t u_t A^* = \rho_t a_t A^*$$

The $t$ subscript refers to the throat of the nozzle and $A^*$ is the throat area.

$$P = \rho R T \qquad P^\circ = \rho^\circ R T^\circ \qquad \frac{P}{P^\circ} = \left(\frac{\rho}{\rho^\circ}\right)^\gamma$$

$$\frac{a_t^2}{2} = \frac{u_t^2}{2} = c_p(T^\circ - T_t)$$

$$\frac{\gamma R T_t}{2} = c_p(T^\circ - T_t) \qquad \frac{T^\circ}{T_t} = 1 + \frac{\gamma R}{2c_p} = 1 + \frac{\gamma - 1}{2} = \frac{\gamma + 1}{2}$$

$$\frac{\rho_t}{\rho^\circ} = \left(\frac{T_t}{T^\circ}\right)^{\frac{1}{\gamma-1}} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}$$

# Mass Flow Through Nozzle

$$a_t = \sqrt{\gamma R T_t} = \sqrt{\gamma R T^\circ}\left(\frac{T_t}{T^\circ}\right)^{1/2} = \boxed{a_t = \sqrt{\gamma R T^\circ}\left(\frac{2}{\gamma+1}\right)^{1/2}}$$

$$\dot{m} = \rho^\circ \sqrt{\gamma R T^\circ} A^* \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}+\frac{1}{\gamma-1}}$$

$$\dot{m} = \frac{P^\circ A^*}{\sqrt{R T^\circ}} \underbrace{\sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}_{\Gamma(\gamma)} = \boxed{\dot{m} = \frac{P^\circ A^*}{c^*}}$$

Where the characteristic velocity is:

$$c^* = \frac{\sqrt{R T^\circ}}{\Gamma}$$

Desire high temp - low molecular weight

$$T' = \dot{m} U + (P_e - P_a) A_e = \dot{m} c$$

$$T' = P^\circ A^* \left\{ \frac{2\gamma^2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{(\gamma-1)}}\left[1-\left(\frac{P_e}{P^\circ}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{1/2}$$

if $P_e = P_a$, i.e., optimal situation

# Thrust Coefficient

$$C_F \equiv \frac{T'}{P^\circ A^*} = \frac{\dot{m}c}{P^\circ A^*} = \frac{c}{c^*} \qquad \boxed{c = C_F c^*}$$

Note that this coefficient is non-dimensional and is useful in comparing large nozzles to small nozzles

$$C_F = \left\{\frac{2\gamma^2}{\gamma - 1}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1 - \left(\frac{P_e}{P^\circ}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{1/2} + \left(\frac{P_e}{P^\circ} - \frac{P_a}{P^\circ}\right)\frac{A_e}{A^*}$$

We can show that $A_e/A^*$ is function of $P_e/P^\circ$ if the flow is isentropic and choked at the throat.

$$\rho_t u_t A^* = \rho_e u_e A_e$$

$$\frac{A_e}{A^*} = \left(\frac{\rho_t}{\rho_e}\right)\left(\frac{a_t}{u_e}\right) = \left(\frac{\rho_t}{\rho^\circ}\right)\left(\frac{\rho^\circ}{\rho_e}\right)\frac{\sqrt{\gamma R T^\circ}\left(\frac{2}{\gamma + 1}\right)^{1/2}}{\sqrt{\frac{2\gamma R T^\circ}{\gamma - 1}}\left(1 - \left(\frac{P_e}{P^\circ}\right)^{\frac{\gamma-1}{\gamma}}\right)^{1/2}}$$

# Thrust Coefficient

$$\frac{A_e}{A^*} = \left(\frac{2}{\gamma+1}\right)^{\left(\frac{1}{2}+\frac{1}{\gamma-1}\right)} \sqrt{\frac{\gamma-1}{2}\left(\frac{\rho^\circ}{\rho_e}\right)\left(1-\left(\frac{P_e}{P^\circ}\right)^{\frac{\gamma-1}{\gamma}}\right)}^{\,-1/2}$$

$$\boxed{\frac{A_e}{A^*} = \left(\frac{2}{\gamma+1}\right)^{\left(\frac{1}{2}+\frac{1}{\gamma-1}\right)} \sqrt{\frac{\gamma-1}{2}\left(\frac{P^\circ}{P_e}\right)^{\frac{1}{\gamma}}\left(1-\left(\frac{P_e}{P^\circ}\right)^{\frac{\gamma-1}{\gamma}}\right)}^{\,-1/2}}$$

***So:*** $A_e/A^*$ can be determined by $P_e/P^\circ$

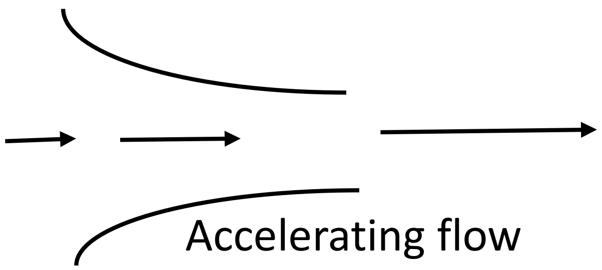
***Then:*** $C_F$ can be determined as a function of $P_e/P^\circ$

The pressure ratio clearly determines the area ratio or nozzle size for isentropic flow!

If the nozzle only converged to the throat without a divergent section: $A_e = A^*$

$$\frac{P_e}{P^\circ} = \frac{P_t}{P^\circ} = \left(\frac{\rho_t}{\rho^\circ}\right)^{\gamma} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\left[1-\left(\frac{P_e}{P^\circ}\right)^{\frac{\gamma-1}{\gamma}}\right]^{1/2} = \left[1-\frac{2}{\gamma+1}\right]^{1/2} = \left(\frac{\gamma-1}{\gamma+1}\right)^{1/2}$$



Accelerating flow through convergent nozzle

# Thrust Coefficient

$$C_{F_{convergent}} = \left\{ \frac{2\gamma^2}{\gamma + 1} \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma+1}{\gamma-1}} \right\}^{1/2} + \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma-1}} - \frac{P_a}{P^\circ}$$

1. For each ambient pressure, $P_a$, there is an optimal nozzle size
2. The optimum value of size and thrust coefficient increases as ambient pressure decreases

For practical considerations, we might compromise to avoid long, heavy nozzles as high-altitude operation!