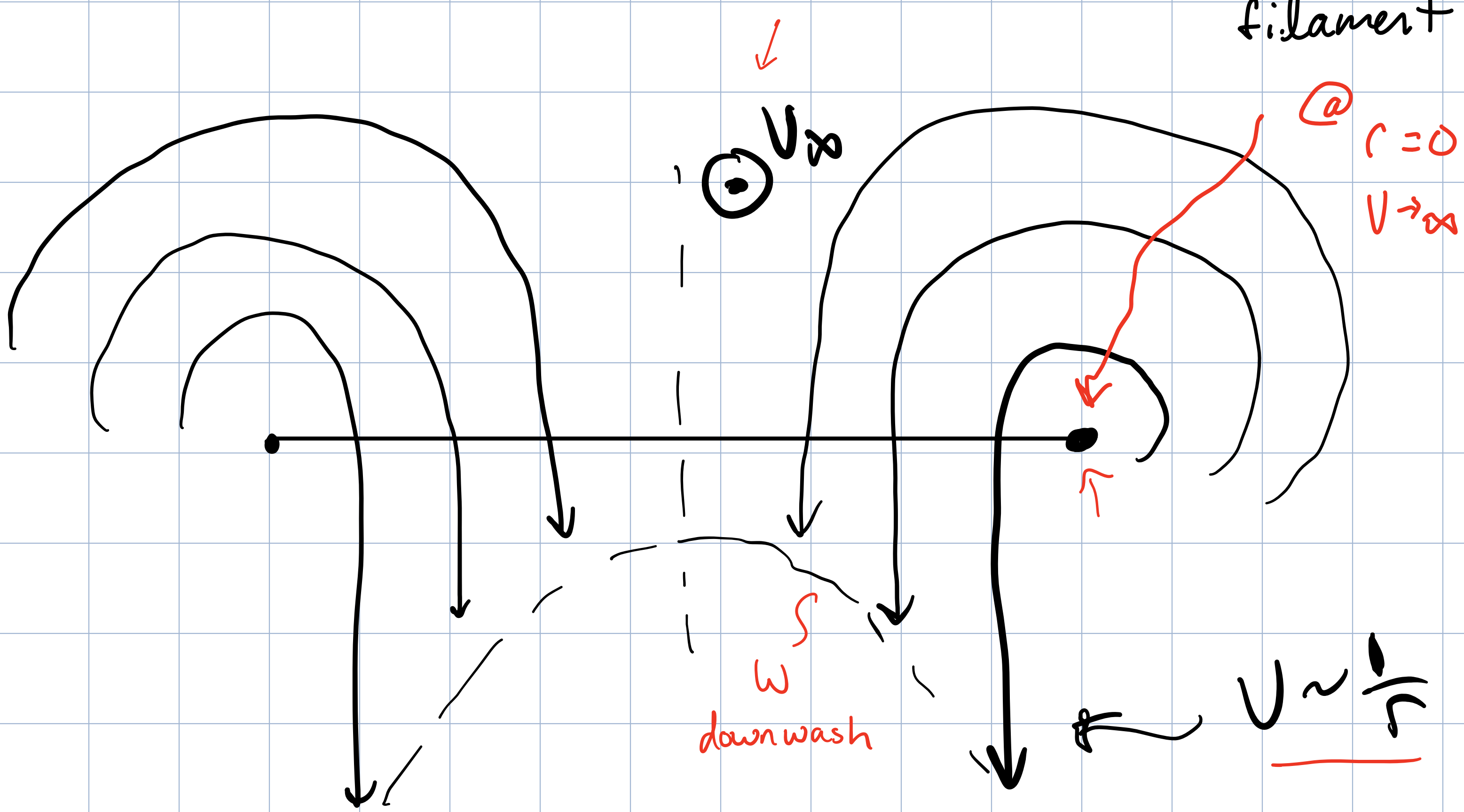
MAE 158 Fall 2024
Lecture 5

Announcements:

Today's Objectives: Induced Drag (Ch. 9) + compressible Flow (Ch. 7)

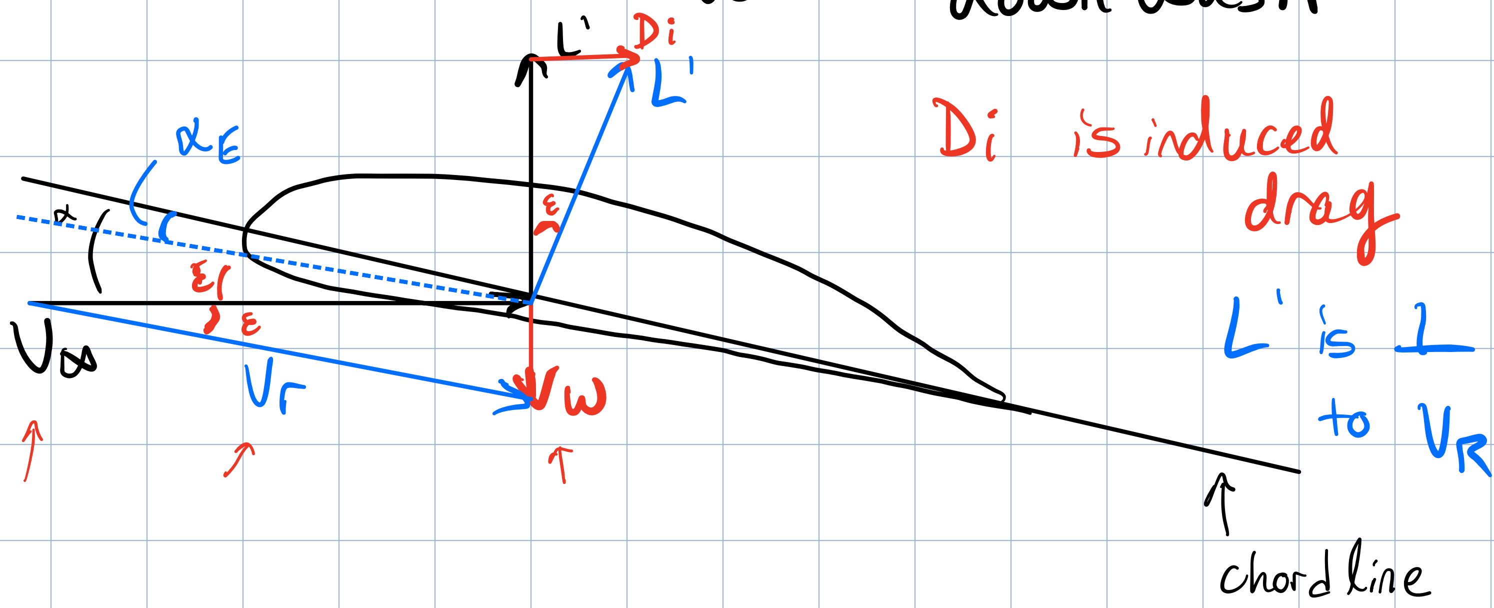
Last time:

a single horseshoe vortex filament



velocity distribution behind wing

↳ w → down wash



w Rotate $V_\infty \to V_r$

L' rotated & D_i produced

$\alpha \equiv$ geometric angle of attack → angle between V_∞ & chord line

$D_i = L' \tan \varepsilon$ (induced drag)

$= L' \varepsilon$

$= L' \dfrac{w}{V_\infty}$

typically, ε & w are small

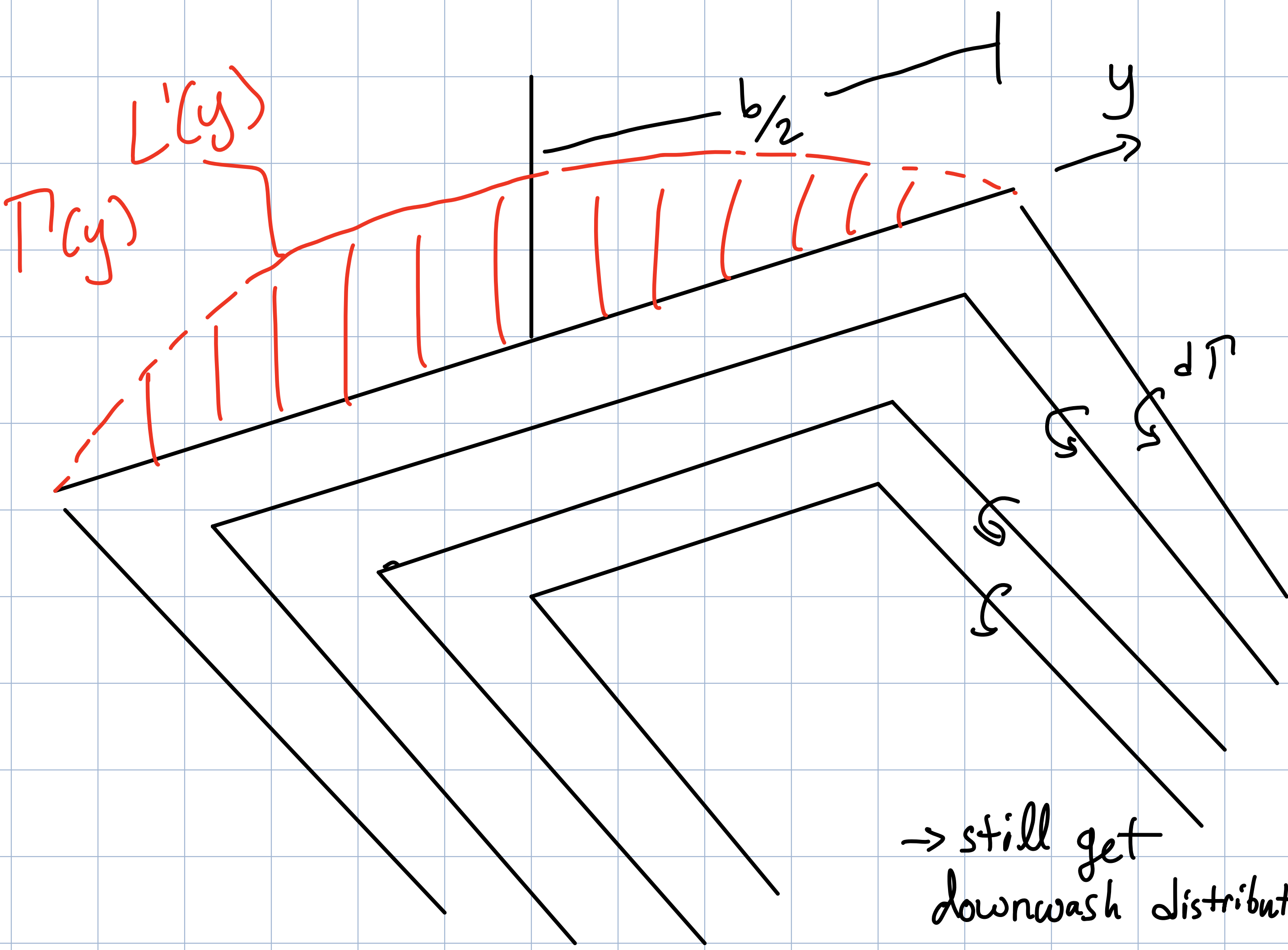
$$\varepsilon \approx \frac{w}{V_\infty}$$

$\alpha_E \equiv$ effective angle of attack

$$\alpha_E = \alpha - \varepsilon$$
$$= \alpha - w/V_\infty$$

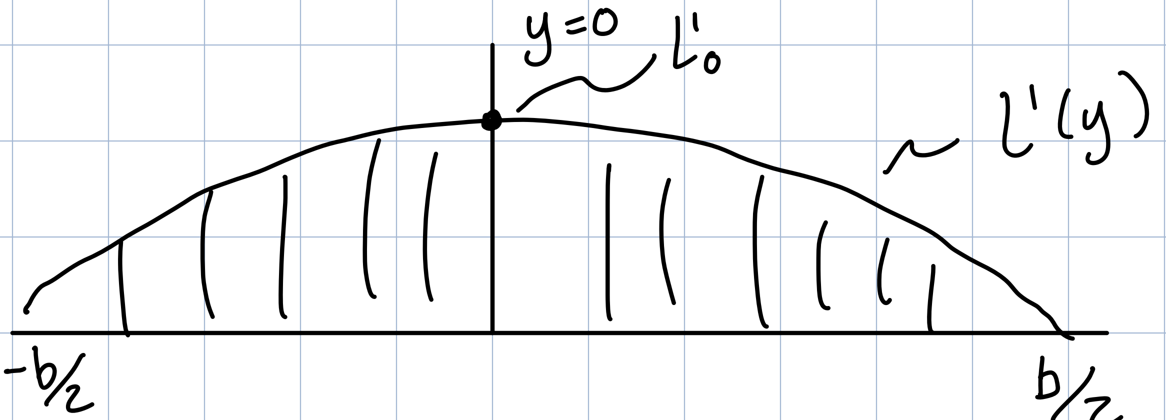
- What is w, the downwash?

to get w, model the wing as a series of horseshoe vortices, strength $d\Gamma$



→ still get downwash distribution

if the lift distribution is elliptical then the induced drag will be minimum for the total lift produced



$$L'(y) = L'_0 \sqrt{1-(2y/b)^2}$$
$$\Gamma(y) = \Gamma_0 \sqrt{1-(2y/b)^2}$$

Note, if $L'(0) = L'_0 = \rho V \Gamma_0$

then $L = \int_{-b/2}^{b/2} L'(y)\,dy$

$$= \int_{-b/2}^{b/2} \rho V \Gamma_0 \sqrt{1-(2y/b)^2}\,dy$$

$$= \frac{\pi}{4} \rho \underset{\substack{\uparrow \\ V_\infty}}{V} \Gamma_0 b$$ → total lift

Solve for w (downwash) of elliptical lift distribution

$$D_i = L' \, w/V_\infty$$

use Biot-Savart Law

$$dw = -\frac{d\Gamma}{4\pi} \frac{1}{y_0 - y}$$

y_0 is a location where we want to determine downwash that is produced by a filament @ y

$$w(y_0) = -\frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{\frac{d\Gamma}{dy}}{y_0 - y} \, dy$$

$$= -\frac{\Gamma_0}{\pi b^2} \int_{-b/2}^{b/2} \frac{y}{\sqrt{1-(2y/b)^2}} \frac{dy}{y_0 - y}$$

$$= -\frac{\Gamma_0}{2b}$$ $\rightsquigarrow$ const.

elliptical lift distribution:

- w is const. along the span
- ε is const. along the span

$$D_i' = L\, w/V_\infty = L\varepsilon$$

$$C_{D_i} = C_L\, w/V_\infty$$

Recall that $\varepsilon = w/V_\infty = \dfrac{\Gamma_0}{2b V_\infty}$

$$L = \Gamma_0 \frac{\pi}{4} \rho V_\infty b$$

then $\varepsilon = \dfrac{L}{\pi \frac{1}{2}\rho V_\infty^2 b^2}$

Note $C_L = \dfrac{L}{\frac{1}{2}\rho V_\infty^2 S}$

then $\varepsilon = \dfrac{C_L}{\pi R}$

$$\mathcal{R} = \frac{b^2}{S_{REF}} \equiv \text{Aspect Ratio}$$

Wing

Lift

$$\boxed{C_{D_i} = C_L \cdot \varepsilon = \frac{C_L^2}{\pi \mathcal{R}}}$$

geometrical factor of wing

elliptical lift distribution

$$D_i = \tfrac{1}{2}\rho V_\infty^2 \cdot S_{REF}\, C_{D_i}$$

S_{REF}

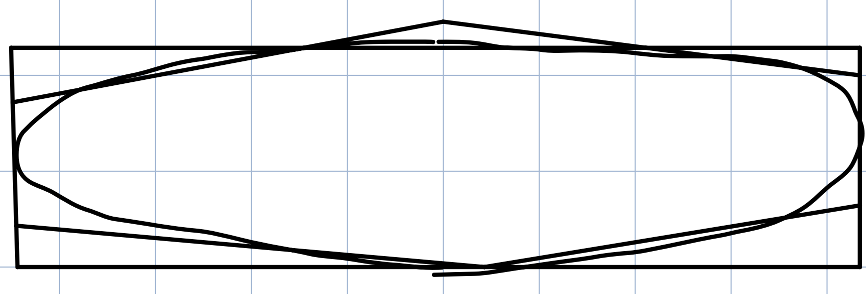
$L = \tfrac{1}{2}\rho U^2 S C_L$

$$= \frac{1}{2}\rho V_\infty^2 S \left(\frac{C_L^2}{\pi \mathcal{R}}\right)$$

$$= \frac{1}{\pi q} \cdot \left(L^2/b^2\right)$$

$\tfrac{1}{2}\rho U^2$

if distribution of Lift is Not elliptical ....



$$C_{Di} = \frac{C_L^2}{\pi \mathcal{R} e}$$

$e \equiv$ span efficiency factor

shevelle → "u"

if elliptical → $e = 1$
if not elliptical → $e < 1$

for C_{Di} → must use C_L of the wing! not, C_l of the airfoil

if you only have C_l of the airfoil, must convert to C_L in 3D → know what $\mathcal{R}$ will be geometric

Recal: $\alpha_E = \alpha - \varepsilon$

effective AoA

$\varepsilon = \frac{C_L}{\pi AR}$

consider lift curve slope:

2D: $C_l = a_0 \cdot \alpha + \text{const}$

$a_0 = 2\pi\eta$ (α is in Rad)

in 3D: $C_L = a_0 \cdot \alpha_E$

$$= a_0 \left(\alpha - \frac{C_L}{\pi AR}\right)$$

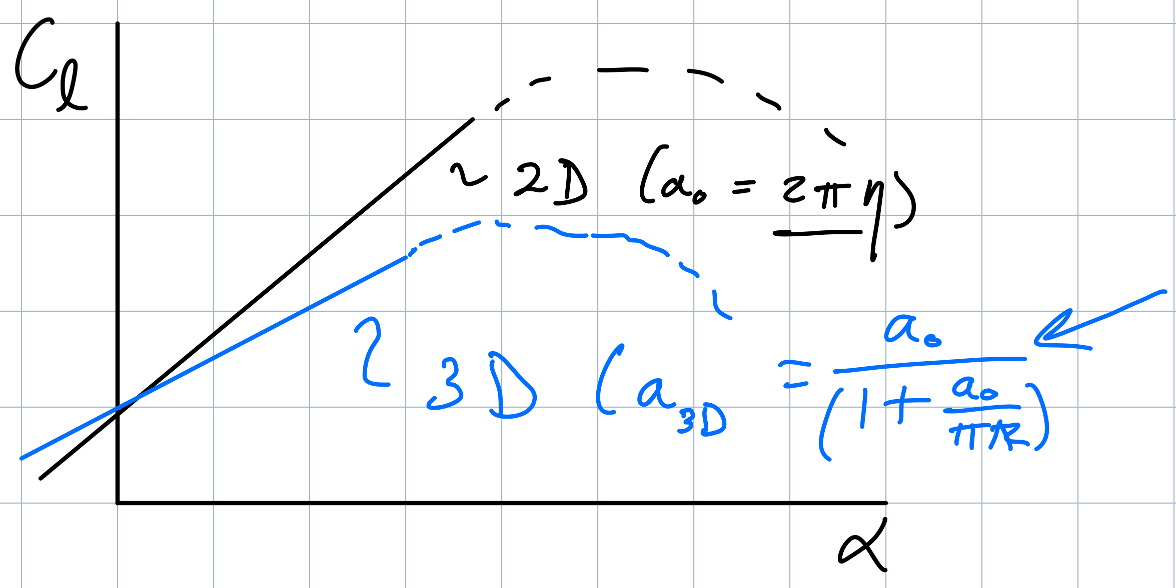
$a_0 = 2\pi\eta$

thus,

$$C_L = \frac{a_0}{\left(1 + \frac{a_0}{\pi AR}\right)} \cdot \alpha$$

geometric α

$$a_{3D} = \frac{dC_L}{d\alpha} = \frac{a_0}{\left(1 + \frac{a_0}{\pi AR}\right)}$$



Must use C_L when $\rightarrow D_i$

$$D_{total} = D_p + D_i + D_c$$

D_p: skin fric + pressure drag

D_i: $C_{D_i} = \dfrac{C_L^2}{\pi AR e}$

D_c: compressibility

compressible flow

$\rho_1 \neq \rho_2 \rightarrow$ pressure changes $\rightarrow$ change ρ

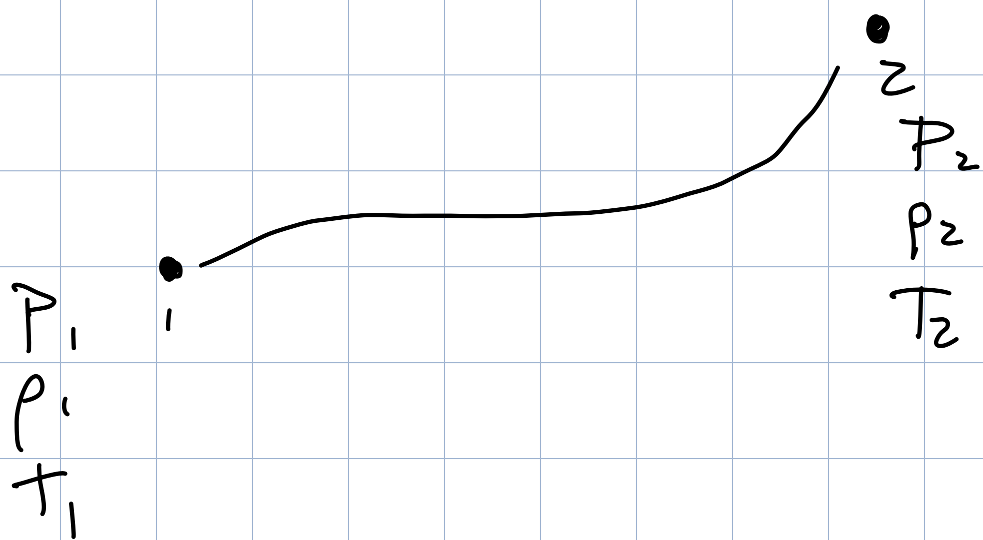
$M > 0.3 \rightarrow$ condition for

compressible Flow

$$\text{Mach } \# = \frac{V}{a} \qquad a \rightarrow \text{speed of sound}$$

· Relations for compressible Flow between ρ, V, P, M

· considering a process from 1 to 2



a few concepts for different processes to consider

– adiabatic: No heat is added or taken

- Reversible: No friction is dissipated

- Isentropic: adiabatic & Reversible

most aerodynamic flows, if outside of BL, Reversible, & adiabatic

thus; can apply isentropic gas laws

$$\left(\frac{P_2}{P_1}\right) = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

γ = ratio of specific heats

$\gamma = 1.4$ if air

equation of state $\rho = \frac{P}{RT}$

then $$\frac{P_2}{P_1} = \left(\frac{P_2/RT_2}{P_1/RT_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

also: $$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{1/\gamma-1}$$

→ established Relation between ρ, P, T

how about T, V ?

energy equation

$$C_p \cdot T + \frac{V^2}{2} = \text{const} =: C_p \cdot T_t$$

↑ specific heat capacity

$$\equiv 1004.7 \ \text{J/kgK} \quad \text{for air}$$

$$= 6006 \ \frac{\text{lb-ft}}{\text{slug R}} \quad \text{for air}$$

how about P, V ? $\rho \neq \text{const}$

Euler equation $dp = -\rho V dV$

denote $(\,)_T \equiv$ stagnation property where $V = 0$

Sub in isentropic Relation $\rho = \rho_T \left(\frac{p}{p_t}\right)^{1/\gamma}$

thus $dp = -\left(p/p_t\right)^{1/\gamma} \rho_T \, V \, dV$

integrate $0 = \int_{p_t}^{p} dp \left(p/p_t\right)^{-1/\gamma} \frac{1}{\rho_t} + \int_0^V V \, dV$

$\hookrightarrow \frac{\gamma}{\gamma - 1}\left(\frac{p_t}{\rho_t}\right)\left(\frac{p}{p_t}\right)^{\frac{\gamma-1}{\gamma}} + \frac{V^2}{2} = \frac{\gamma}{\gamma - 1}\frac{p_t}{\rho_t}$

Compressible Bernoulli equation

get Relations between M & stagnation, static properties

$M = V/a \qquad a = \sqrt{\gamma R T}$

-or- $a = \sqrt{\gamma \frac{p}{\rho}}$

standard sea level air $\quad T = 288\,K$

$R = 287\ Nm/kgK$

$$a = \sqrt{(1.4)(287\ \tfrac{Nm}{kgK})(288K)}$$

$= 340\ m/s$ @ sea level

$= 1116\ ft/s$

$$V = a \cdot M = \sqrt{\gamma R T} \cdot M$$

go back to energy equation $\quad c_p T_1 + \frac{V_1^2}{2} = c_p T_2 + \frac{V_2^2}{2}$

Relate stagnation to static properties

$(V=0) \rightsquigarrow (\ )_T$

$$c_p = \frac{\gamma R}{\gamma - 1} \qquad a^2 = \gamma R T_1$$

$$\frac{T_t}{T_1} = 1 + \frac{V_1^2}{2c_pT_1}$$

$$= 1 + \frac{\gamma-2}{2}M_1^2$$

plug in isentropic flow Relations

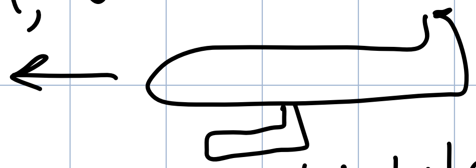
$$\frac{P_t}{P_0} = \left(1+\frac{\gamma-1}{2}M_1^2\right)^{\gamma/\gamma-1}$$

$$\frac{\rho_t}{\rho_1} = \left(1+\frac{\gamma-1}{2}M_1^2\right)^{1/\gamma-1}$$

$$M_1 = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{P_t}{P_1}\right)^{\frac{\gamma-1}{\gamma}}-1\right]}$$

↳ get M_1 knowing local Stagnation (total) pressure & Static pressure

M, V



$\hookrightarrow P_t$

$\longrightarrow$ what is M_1?