Solving Linear Systems of Equations

HHL Quantum Algorithm

(A. Harrow, A. Hassidim, and S. Lloyd *(Phys. Rev. Lett. 15, 150502 (2009))*

Variational Quantum Linear Solver

(Carlos Bravo-Pietro et. al. *(arXiv:1909.05820v2 [quant-ph]))*

Variational Quantum Algorithm for solving Poisson Equation

(H Liu et. al., *Phys. Rev. A 104, 022418 (2021))*

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- LSE (Linear System of Equations) represented in matrix form:

Problem: for matrix $\boldsymbol{A} \in \mathbb{C}^{N\times N}$ and vector $\vec{b} \in \mathbb{C}^{N}$ find vector $\vec{x} \in \mathbb{C}^{N}$ that solves the equation: $\boldsymbol{A}\vec{x} = \vec{b}$

- HHL algorithm requires that matrix ***A*** is *s*-sparse, i.e. it has at most *s* non-zero coefficients per row or column and has a low condition number ($k = \lambda_{max}/\lambda_{min}$). It also assumes that the user is not interested in the full solution but only in the outcome when a function $F(\vec{x})$ is applied.
- HHL is often used in qML algorithms, e.g. qSVM (quantum Support Vector Machine)
- HHL takes advantage of the QPE (Quantum Phase Estimation) algorithm, which rotates a set of qubits by the eigenphases of an applied unitary operator ***U***.

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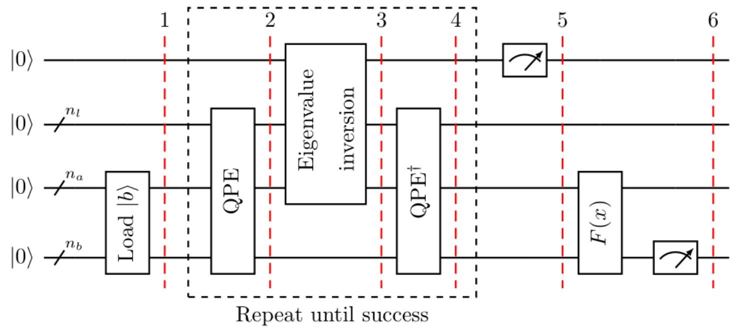
HHL Schematics:

Size of quantum registers:

- n_l to store eigenvalues of $\boldsymbol{A}$
- n_b containing the solution vector
- n_a auxiliary *(= n_b)*

(Note: the problem size is $N = 2^{n_b}$*)*



1. Load $|b\rangle$: so $|0\rangle_{n_b} \rightarrow |b\rangle_{n_b}$

2. Apply QPE with $U = e^{iAt} := \sum_{j=0}^{N-1} e^{i\lambda_j t} |u_j\rangle\langle u_j|$ so the register state is changed to: $\sum_{j=0}^{N-1} b_j |\lambda_j\rangle_{n_l} |u_j\rangle_{n_b}$

3. Add a scratch qubit and apply a rotation conditioned on $|\lambda_j\rangle$: $\sum_{j=0}^{N-1} b_j |\lambda_j\rangle_{n_l} |u_j\rangle_{n_b} \left(\sqrt{1 - \frac{C^2}{\lambda_j^2}} |0\rangle + \frac{C}{\lambda_j} |1\rangle \right)$

4. Apply inverse QPE (rotates $|\lambda_j\rangle_{n_l} \rightarrow |0\rangle_{n_l}$)

5. Measure scratch qubit and repeat steps 2-4 until the outcome is $|1\rangle$

6. Apply the observable M that is supposed to be measured: $F(\vec{x}) = \langle \vec{x} | M | \vec{x} \rangle$

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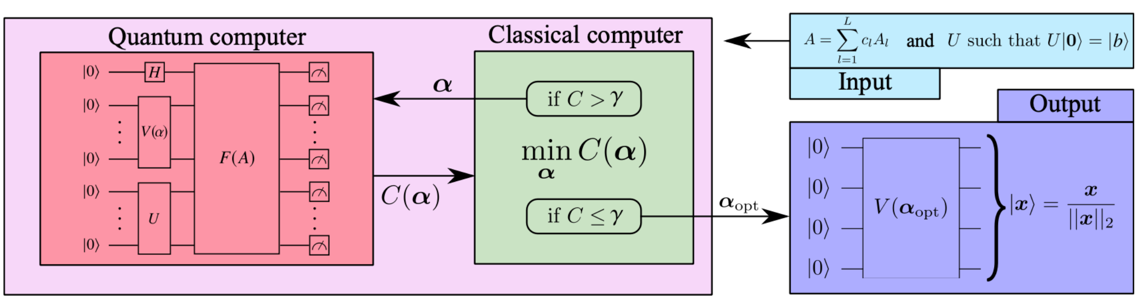
VQLS: hybrid quantum-classical algorithm



Concept: prepare a state $|x\rangle$ such that $A|x\rangle$ is proportional to $|b\rangle$; to this end define a gate sequence $V(\boldsymbol{\alpha})$ that prepares a possible solution: $|x(\boldsymbol{\alpha})\rangle = V(\boldsymbol{\alpha})|0\rangle$.

VQLS input preparation: a) construct $\boldsymbol{A}$ as sum of unitary operators $\boldsymbol{A}_l$ with complex coeffs. c_l
b) provide unitary operator that transforms $|0\rangle_{n_b} \rightarrow |b\rangle$

Parameters $\boldsymbol{\alpha}$ are provided to the quantum circuit, which outputs a cost function $C(\boldsymbol{\alpha})$.

If $C(\alpha) > \gamma$, the algorithm is run again with updated parameters: otherwise the algorithm terminates, and the ansatz is calculated with the last (optimal) parameters, this gives the state vector $|x\rangle$ (= normalized form of *x*).

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VQA for Poisson Equation

Similar approach like VQLS !

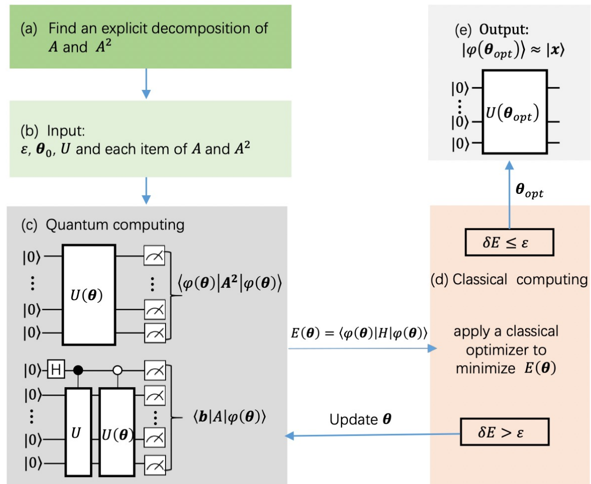
Potential energy is minimized (cost function):

$$E(\boldsymbol{\theta}) = \langle \psi(\boldsymbol{\theta}) \,|\, A\,(I - |f\rangle\langle f|)\,A \,|\, \psi(\boldsymbol{\theta}) \rangle$$

Both, A and A^2, are decomposed!

Matrix A must be modified to account for boundary conditions (periodic, Dirichlet, Neumann);

e.g. 1-dim. Poisson Eq., discretized by finite element method with N nodes in one direction:



$$A_{\text{periodic}} := \begin{bmatrix} 2 & -1 & 0 & & \dots & & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & & & 0 \\ 0 & -1 & 2 & -1 & \dots & & & 0 \\ \vdots & & & \ddots & & & & \vdots \\ 0 & & & \dots & 0 & -1 & 2 & -1 \\ -1 & 0 & & \dots & & 0 & -1 & 2 \end{bmatrix} \quad A_{\text{Dirichlet}} := \begin{bmatrix} 2 & -1 & 0 & & \dots & & & 0 \\ -1 & 2 & -1 & 0 & \dots & & & 0 \\ 0 & -1 & 2 & -1 & \dots & & & 0 \\ \vdots & & & \ddots & & & & \vdots \\ 0 & & & \dots & 0 & -1 & 2 & -1 \\ 0 & & & \dots & & 0 & -1 & 2 \end{bmatrix} \quad A_{\text{Neumann}} := \begin{bmatrix} 1 & -1 & 0 & & \dots & & & 0 \\ -1 & 2 & -1 & 0 & \dots & & & 0 \\ 0 & -1 & 2 & -1 & \dots & & & 0 \\ \vdots & & & \ddots & & & & \vdots \\ 0 & & & \dots & 0 & -1 & 2 & -1 \\ 0 & & & \dots & & 0 & -1 & 1 \end{bmatrix}$$

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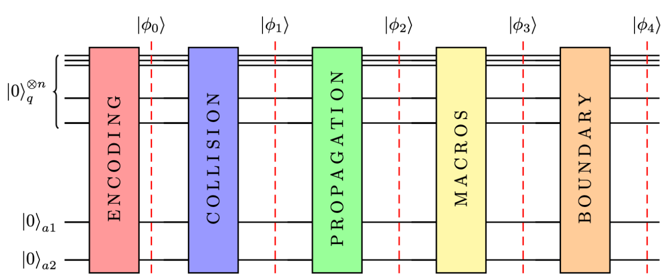
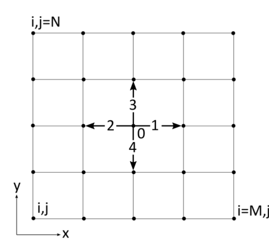
Better quantum approach to Navier-Stokes?

Budinski, *arXiv: 2103.03804v2 (2022)*

- using stream function-vorticity formulation for fluid flow equations;
- Lattice Boltzmann Method used for solving numerically for single time steps;



Conceptually promising approach using Hamiltonian Simulation!



… seems to be working on D2Q5 lattice.

Encoding of vorticity $\omega(\vec{x},0)$, stream function $\psi(\vec{x},0)$ and source term $S(\vec{x},0)$ appears difficult (approximate decomposition into unitaries);

Propagation step via quantum walk;

Measurement of macroscopic quantities from $|\phi_4(t)\rangle$.

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