LOG DISTANCE

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(Survit Sra, MIT 2013) Prove that

$$d_L(a,b) = \sqrt{\log \frac{a+b}{2\sqrt{ab}}}$$

is a distance function on $\mathbb{R}^+$.

Proof. Obviously, $d_L(a,b) = 0 \iff a = b$ by AM-GM inequality. Also, $d_L(a,b) = d_L(b,a)$ is clear. It remains to show that

$$d_L(a,b) \le d_L(a,c) + d_L(b,c) \tag{1}$$

for all $a, b, c > 0$. First, let us rewrite

$$\begin{aligned} d_L(a,b) &= \sqrt{\log \frac{1}{2}\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right)} \\ &= \sqrt{\log f\left(\sqrt{\frac{a}{b}}\right)} \\ &= g\left(\frac{a}{b}\right) = g\left(\frac{b}{a}\right) \end{aligned}$$

where

$$f(x) = \frac{1}{2}\left(x + \frac{1}{x}\right)$$

and

$$g(x) = \sqrt{\log f(\sqrt{x})}.$$

Observe that the functions $f(x)$ is increasing on the interval $(1, +\infty)$ and the functions $\log x$ and $\sqrt{x}$ are increasing on $(0, +\infty)$. Since g is a composition $\sqrt{x} \circ \log \circ f \circ \sqrt{x}$ of functions increasing on $(1, +\infty)$, g must be increasing on $(1, +\infty)$ as well. In other words, if $1 < x < y$ then $0 < g(x) < g(y)$.

To prove (1), we can assume $a < b$ without loss of generality. (If $a = b$ then the inequality is obvious since the left hand side is 0 and right hand side is obviously ≥ 0.)

There are several possibilities for c:

- $c = a$ or $c = b$: (1) is obvious for then one of the term on the RHS equals to the LHS.
- $c < a$ so c is the smallest number amongst the a, b, c; i.e. $c < a < b$: then $\frac{a}{c}, \frac{b}{c}, \frac{b}{a} > 1$ and

$$(1) \iff g\left(\frac{b}{a}\right) \le g\left(\frac{a}{c}\right) + g\left(\frac{b}{c}\right)$$

which is true because then

$$1 < \frac{b}{a} < \frac{b}{c}$$

so we even have a stronger inequality

$$g\left(\frac{b}{a}\right) < g\left(\frac{b}{c}\right).$$

- $c > b$ so c is the largest number amongst the a, b, c; in other words, $a < b < c$: Then

$$(1) \iff g\left(\frac{b}{a}\right) \le g\left(\frac{c}{a}\right) + g\left(\frac{c}{b}\right)$$

which is similarly true because

$$1 < \frac{b}{a} < \frac{c}{a}.$$

- Finally, $a < c < b$ or c is the middle number amongst the a, b, c: Then

$$(1) \iff g\left(\frac{b}{a}\right) \le g\left(\frac{c}{a}\right) + g\left(\frac{b}{c}\right)$$

Unfortunately, we cannot yet make any conclusion. But observe that it suffices to show that

$$g(X^2Y^2) \le g(X^2) + g(Y^2) \tag{2}$$

for all real numbers $X, Y > 1$. For then, we can plug in $X = \sqrt{\frac{c}{a}}$ and $Y = \sqrt{\frac{b}{c}}$ whence $XY = \sqrt{\frac{b}{a}}$ to get (1). Conversely, if the original inequality (1) is true for all $a, b, c > 0$ then it must be true for $a = \frac{1}{X^2}, b = Y^2, c = 1$ which gives us (2). By doing this, we **eliminate one variable** AND also the **inner square-root** as

$$g(X^2) = \sqrt{\log f(\sqrt{X^2})} = \sqrt{\log f(X)}.$$

Let $G(x) = g(x^2)$.

For a fixed value of $Y > 1$, consider the function

$$h(X) = g(X^2) + g(Y^2) - g(X^2Y^2) = G(X) + G(Y) - G(XY)$$

as a function of X only. We claim that this function is increasing on $[1, \infty)$ and so

$$h(X) \ge h(1) = g(1) + g(Y) - g(Y) = g(1) = 0$$

which proves our inequality (2).

To prove our claim, we show that $h'(X) > 0$ on $(1, \infty)$. One has

$$h'(X) = G'(X) - YG'(XY)$$

and

$$G'(x) = \frac{x^2 - 1}{2x(x^2 + 1)G(x)}$$

so

$$\begin{aligned} h'(X) > 0 &\iff G'(X) > YG'(XY) \\ &\iff \frac{X^2-1}{2X(X^2+1)G(X)} > Y\frac{(XY)^2-1}{2XY((XY)^2+1)G(XY)} \\ &\iff \frac{X^2-1}{(X^2+1)G(X)} > \frac{(XY)^2-1}{((XY)^2+1)G(XY)} \\ &\iff \frac{(X^2+1)G(X)}{X^2-1} < \frac{((XY)^2+1)G(XY)}{(XY)^2-1} \\ &\iff k(X) < k(XY) \end{aligned}$$

where the function

$$k(x) = \frac{(x^2+1)G(x)}{x^2-1} = \left(1 + \frac{2}{x^2-1}\right) G(x).$$

The final inequality is true because $X < XY$ (from assumption $Y > 1$) and $k(x)$ is an increasing function on $(1, +\infty)$ since

$$k'(x) = \frac{-4x}{(x^2-1)^2}\, G(x) + \left(1 + \frac{2}{x^2-1}\right) G'(x)$$

and so

$$\begin{aligned} k'(x) > 0 &\iff -4xG(x) + (x^4-1)G'(x) > 0 \\ &\iff (x^4-1)\frac{x^2-1}{2x(x^2+1)G(x)} > 4xG(x) \\ &\iff \frac{(x^2-1)^2}{x^2} > 8G(x)^2 = 8\log f(x) \\ &\iff K(x) > 0 \end{aligned}$$

where the function

$$K(x) = \frac{(x^2-1)^2}{x^2} - 8\log f(x).$$

One has

$$K'(x) = \frac{2(x^2-1)^3}{x^3(x^2+1)}$$

which is evidently positive on $(1, +\infty)$. Hence, $K(x)$ is increasing on $(1, +\infty)$ and so $K(x) \geq K(1) = 0$ with equality only achieved when $x = 1$.

□