ANOTHER DISTANCE FUNCTION

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Let $f(t) = t\log t$. For $a, b > 0$, define

$$d(a,b) = \sqrt{\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)}$$

then $d(a,b)$ is a metric on $\mathbb{R}^+$.

Proof. First, d is well-defined because $f(t)$ is a convex function i.e. $f''(t) = \frac{1}{t} > 0$ for $t > 0$.

The main thing to prove is the triangle inequality. Observe that

$$\begin{aligned}
d(a,b) &= \sqrt{\frac{a\log a + b\log b}{2} - \left(\frac{a+b}{2}\right)\log\frac{a+b}{2}} \\
&= \sqrt{\frac{1}{2}\left(a\log a + b\log b - (a+b)\log\frac{a+b}{2}\right)} \\
&= \sqrt{\frac{1}{2}\left(a\left[\log a - \log\frac{a+b}{2}\right] + b\left[\log b - \log\frac{a+b}{2}\right]\right)} \\
&= \sqrt{\frac{1}{2}\left(a\log\frac{2a}{a+b} + b\log\frac{2b}{a+b}\right)} \\
&= \sqrt{\frac{a+b}{2}\left(\frac{a}{a+b}\log\frac{2a}{a+b} + \frac{b}{a+b}\log\frac{2b}{a+b}\right)} \\
&= \sqrt{\frac{a+b}{2}}\, K\left(\frac{a}{a+b}\right)
\end{aligned}$$

where the function $(0,1) \to \mathbb{R}_{\geq 0}$ is given by

$$K(t) := \sqrt{t\log(2t) + (1-t)\log(2(1-t))} = \sqrt{\frac{f(2t)+f(2-2t)}{2}}.$$

(Check that $K(t)$ is well-defined and $K(t)$ is decreasing on $[0, 1/2]$ and increasing on $[1/2, 1]$.)

Now the triangle inequality

$$\begin{aligned}
& d(a,b) \leq d(a,c) + d(b,c) \\
\iff & \sqrt{\frac{a+b}{2}}\, K\left(\frac{a}{a+b}\right) \leq \sqrt{\frac{a+c}{2}}\, K\left(\frac{c}{a+c}\right) + \sqrt{\frac{c+b}{2}}\, K\left(\frac{c}{c+b}\right) \\
\iff & K\left(\frac{a}{a+b}\right) \leq \sqrt{\frac{a+c}{a+b}}\, K\left(\frac{c}{a+c}\right) + \sqrt{\frac{c+b}{a+b}}\, K\left(\frac{c}{c+b}\right)
\end{aligned}$$

Notice that the rational functions $\frac{a}{a+b}, \frac{a+c}{a+b}, \ldots$ appearing on the last inequality are all homogeneous of degree 0. So the triangle inequality is equivalent to

$$K\left(\frac{a}{a+b}\right) \leq \sqrt{\frac{a+1}{a+b}}\, K\left(\frac{1}{a+1}\right) + \sqrt{\frac{1+b}{a+b}}\, K\left(\frac{1}{1+b}\right) \tag{1}$$

for all $a, b > 0$ obtained by normalizing $c = 1$. (If (1) is true, applying it with a/c and b/c in place of a and b yields the original inequality.)

We do one more normalization:

$$\begin{aligned}
(1) \iff & \sqrt{a+b}\, K\left(\frac{a}{a+b}\right) \leq \sqrt{1+a}\, K\left(\frac{1}{a+1}\right) + \sqrt{1+b}\, K\left(\frac{1}{1+b}\right) \\
\iff & \sqrt{a\left(1+\frac{b}{a}\right)}\, K\left(\frac{1}{1+\frac{b}{a}}\right) \leq \sqrt{1+a}\, K\left(\frac{1}{a+1}\right) + \sqrt{1+b}\, K\left(\frac{1}{1+b}\right) \\
\iff & \sqrt{a}\, k(b/a) \leq k(a) + k(b)
\end{aligned}$$

where the function $k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is given by

$$\begin{aligned}
k(s) &:= \sqrt{1+s}\, K\left(\frac{1}{1+s}\right) \\
&= \sqrt{1+s}\sqrt{\frac{1}{1+s}\log\frac{2}{1+s} + \frac{s}{1+s}\log\frac{2s}{1+s}} \\
&= \sqrt{\log\frac{2}{1+s} + s\log\frac{2s}{1+s}}.
\end{aligned}$$

As

$$k'(s) = \frac{\log\frac{2s}{1+s}}{2k(s)}$$

the sign of $k'(s)$ is the same as the sign of $\log\frac{2s}{1+s}$ and so the function $k(s)$ is increasing on $[1, +\infty)$ and decreasing on $(0, 1)$.

Note that $\sqrt{a}\; k(b/a) = \sqrt{b}\; k(a/b)$ so without loss of generality, we can assume $a < b$. Changing variable $x = b/a > 1$ and so $b = ax$, one further has

$$\begin{aligned}
(1) \iff & \sqrt{a}\, k(x) \leq k(a) + k(ax) \\
\iff & \frac{k(x)}{\sqrt{x}} \leq \frac{k(a)}{\sqrt{ax}} + \frac{k(ax)}{\sqrt{ax}}
\end{aligned}$$

for all $a > 0$ and $x > 1$. From the second form, the inequality is true if $a \geq 1$ because then $x < ax$ and similar to the function $k(s)$, the function $s \mapsto \frac{k(s)}{\sqrt{s}}$ is increasing on $[1, +\infty)$ and decreasing on $(0, 1]$ so we even have $\frac{k(x)}{\sqrt{x}} \leq \frac{k(ax)}{\sqrt{ax}}$. Note that we shouldn't check $\frac{k(s)}{\sqrt{s}}$ is increasing: check $\frac{k(s)^2}{s}$ instead.

Now let $a \in (0, 1)$ be arbitrary. Observe that:

$$\begin{aligned}
\sqrt{s}\, k\left(\frac{1}{s}\right) &= \sqrt{s}\,\sqrt{\log\frac{2}{1+1/s} + \frac{1}{s}\log\frac{2/s}{1+1/s}} \\
&= \sqrt{s\log\frac{2s}{s+1} + \log\frac{2}{s+1}} \\
&= k(s)
\end{aligned}$$

so if $1 < x \le \frac{1}{a}$, we also have a stronger inequality

$$\sqrt{a}\, k(x) \le \sqrt{a}\, k\left(\frac{1}{a}\right) = k(a)$$

due to $k(s)$ is increasing on $[1, +\infty)$ so (1) is also true.

Finally, it remains to consider the case $x > \frac{1}{a} > 1$. Consider the following function on $(\frac{1}{a}, +\infty)$:

$$h(x) = k(ax) + k(a) - \sqrt{a}\, k(x).$$

We want to show that $h(x)$ is increasing by showing that $h'(x) > 0$ so $h(x) \ge h(\frac{1}{a}) = 0$:

$$\begin{aligned} h'(x) > 0 &\iff ak'(ax) - \sqrt{a}\, k'(x) > 0 \\ &\iff \sqrt{ax}\, \frac{\log \frac{2ax}{1+ax}}{2k(ax)} > \sqrt{x}\, \frac{\log \frac{2x}{1+x}}{2k(x)} \\ &\iff g(ax) > g(x) \end{aligned}$$

where the function $g(s)$ is defined on $(1, +\infty)$ by

$$g(s) := \sqrt{s}\, \frac{\log \frac{2s}{1+s}}{2k(s)}.$$

The final inequality $g(ax) > g(x)$ is true because $g(s)$ is a decreasing function on $(1, +\infty)$ and $1 < ax < x$.

$\square$