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Benjamin–Feir instability in nonlinear dispersive waves

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ABSTRACT

In this paper, the authors extended the derivation to the nonlinear Schrödinger equation in two-dimensions, modified by the effect of non-uniformity. The authors derived several classes of soliton solutions in $2+1$ dimensions. When the solution is assumed to depend on space and time only through a single argument of the function, they showed that the two-dimensional nonlinear Schrödinger equation is reduced either to the sine-Gordon for the hyperbolic case or sinh-Gordon equations for the elliptic case. Moreover, the authors extended this method to obtain analytical solutions to the nonlinear Schrödinger equation in two space dimensions plus time. This contains some interesting solutions such as the plane solitons, the N multiple solitons, the propagating breathers and quadratic solitons. The authors displayed graphically the obtained solutions by using the software Mathematica 5.

1. Introduction

The Benjamin–Feir instability (BFI) corresponds to the loss of stability of Stokes nonlinear travelling waves to sideband perturbations [1,2]. An alternative treatment which leads to the same results, based on a nonlinear evolution equation for the envelope of the Stokes wave was given by [3–5]. A further treatment of the BFI mechanism of the two-dimensional Stokes waves in deep water was given by Stuart and DiPrima [4]. They used the method of the nonlinear Schrödinger equation (NLSE) that allows the analysis of the sideband perturbations.

The theoretical prediction of the Benjamin–Feir sideband instability is a breakup of a nonlinear wave, spreading the energy over a number of small-amplitude waves eventually. These predictions were found to be in remarkable agreement with their experimental results. This indicates that the monochromatic gravity waves – produced in the experiment – transfers energy to a wavenumber adjacent to that of the carrier wave by the nonlinear interactions [1,2]. Benjamin–Feir observed what was apparently a disintegration of a mechanically produced finite-amplitude deep water wave. They interpreted that this disintegration (or breakup) is due to a sideband instability of the finite-amplitude wave. Based upon linear stability analysis, they described further the mechanism as a resonant coupling between the primary wave train with a pair of wave modes at sideband frequencies and wavenumbers fractionally different from both the fundamental frequency and wavenumber. With the identification of solitons, it has been conjectured that the final state of an unstable Stokes wave would be one or more solitons (which is further illustrated in this work). In consequence of coupling, the energy is transferred from the primary wave to the sidebands at a rate that can increase exponentially in either time or distance as the nonlinear interactions develop. The general conclusion is that finite-amplitude water waves are unstable [6,7].

Although the technique as well as the experiment of Benjamin–Feir is quite remarkable, the discovery of the BFI led to a question regarding the possible end-states of a nonlinear wave train undergoing modulational instability. This question was

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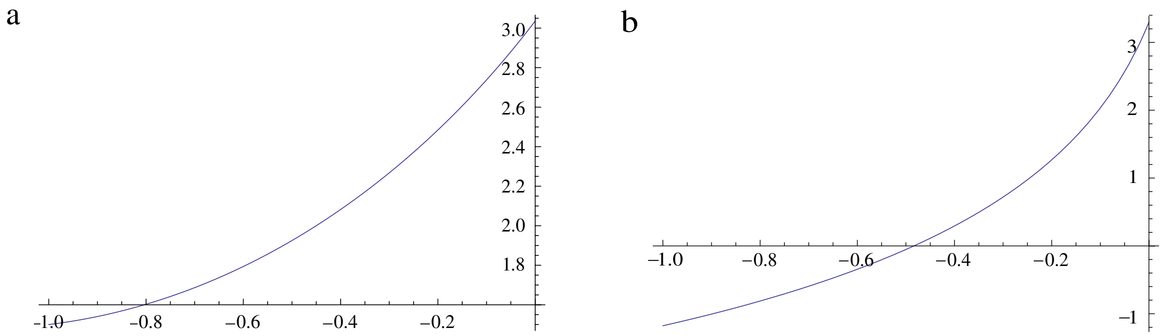

Fig. 1. (a) The numerical solution for the sinh-Gordon equation in the plane with $C = 1$ in the interval $[-1, 0]$, $\varphi[-1] = 1.5$ and $\varphi'[-1] = 0.3$. (b) The one soliton solution for the sinh-Gordon equation in the plane with $C = 1$ in the interval $[-1, 0]$ and $\zeta_0 = -0.08$.

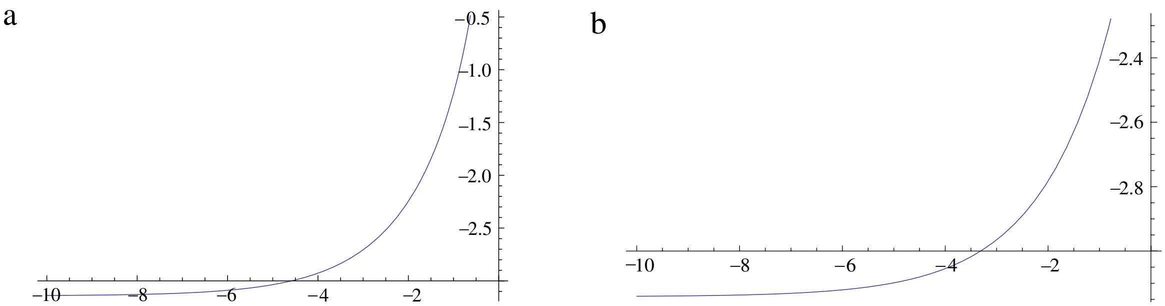

Fig. 2. (a), (b) The one soliton solution for the sinh-Gordon equation in the plane with $C = 1$ in the interval $[-10, 0]$, $\zeta_0 = -0.1; -1$.

answered by an even more remarkable experimental and computational work of Yuen and Lake [5] and Lake et al. [8]. They have shown that the sideband instability in the evolution of a nonlinear wave train does not lead to either a disintegration of the wave train (as observed by Benjamin–Feir) or a loss of coherence (as suggested by Benney) [9]. Instead, it has been confirmed by the experimental work of Lake et al. [8] that unstable modulations grow to a maximum limit and then subside. The energy is transferred from the primary wave train to the sidebands for a definite period of time, then it is recollected back into the primary wave mode. Their experimental findings have also been verified by numerical computations using the NLSE with unstable perturbations. Also, numerical computations indicated that the longtime evolution of an unstable wave train leads to a series of modulation-demodulation cycles in the absence of viscosity. This striking feature of the modulation-demodulation cycles involved in the evolution of an unstable wave train is known as the Fermi-Pasta–Ulam recurrence phenomenon. The works of Yuen and Lake [5] and Lake et al. [8] were accepted as a conclusive evidence of the Benjamin–Feir modulational instability, the development of solitons, and the Fermi-Pasta–Ulam recurrence phenomenon [6]. The critical feature of this instability is that the class of perturbations has a wavelength which is different from that of the basic travelling wave. A rigorous proof of the BFI for the Stokes' travelling wave, when the fluid depth is sufficiently large, has been given in [10,11]. For interfacial waves, without a basic velocity difference, a comprehensive analytical treatment of the BFI is given in [12], with numerical results for large amplitudes in [13]. Numerical results on the BFI of interfacial gravity waves with a velocity difference are given in [14].

In general, the effect of non-uniformity (both temporal and spatial) on the development of a modulated wave was studied by using a multiple scales perturbation technique in one-dimension [15] and we found the solution of the NLSE in one-dimension. Moreover, we extended the derivation to the NLSE in two-dimensions, which is modified by the effect of non-uniformity. We applied the function transformation method on the two-dimensional NLSE (which was transformed to either sine-Gordon or sinh-Gordon equations), which leads to a general equation which depends only on one function ζ and can be solved. We obtained the general solution of the equations in ζ which leads to a general soliton solution of the two-dimensional NLSE. The two-dimensional NLSE was transformed to a sinh-Gordon equation for the elliptic case and to a sine-Gordon equation for the hyperbolic case. It contains some interesting specific solutions such as *a* plane solitons, the *N* multiple solitons, the propagating breathers and the quadratic solitons which contained the circular shape, elliptical shape and hyperbolic shape solitons. As an illustration, we used Mathematica 5 to solve the original problem. We started from the governing partial differential equation (PDE) and by using Mathematica 5 we obtained soliton solutions. This solution is in good agreement with the analytical solutions. We displayed graphically the obtained solutions by using Mathematica 5 (Figs. 1–4).

This paper is organised as follows: There is an introduction in section one. In section two, the basic equations and the boundary conditions governing the problem together with the multiple scale method are given. The effect of non-uniformity

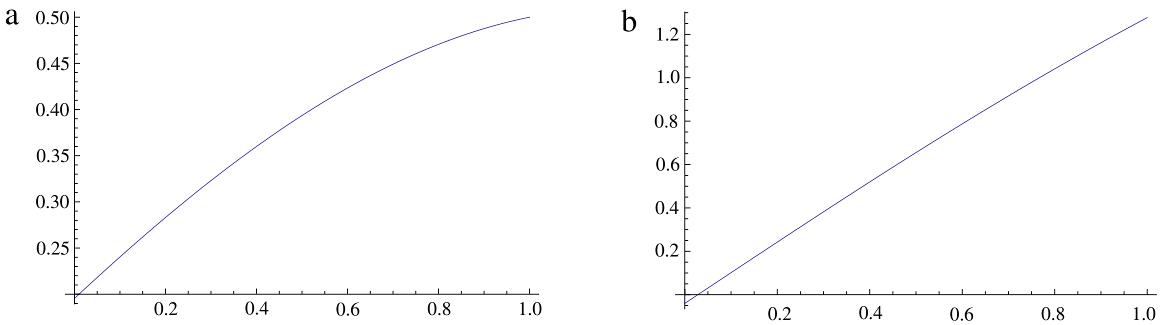

Fig. 3. (a) The numerical solution for the sine-Gordon equation in the plane with $C' = 1$ in the interval [0, 1], $\Psi[1] = 0.5$ and $\Psi'[1] = 0.1$. (b) The one soliton solution for the sine-Gordon equation in the plane with $C' = 1$ in the interval [0, 1], and $\zeta_0 = -0.02$.

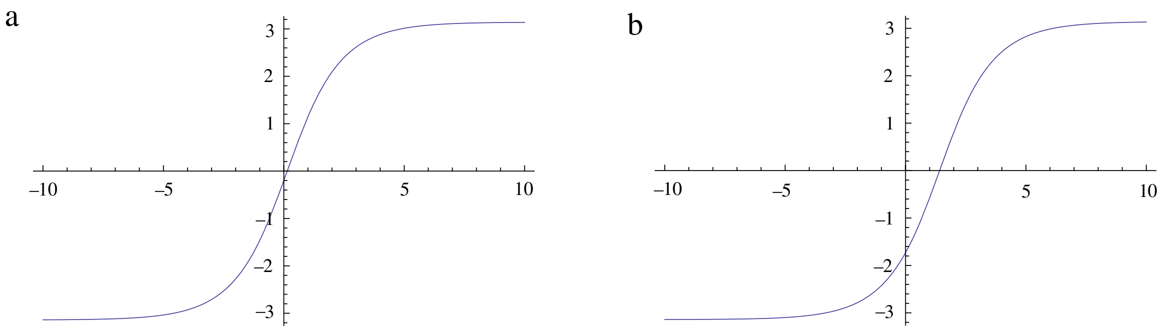

Fig. 4. (a), (b) The one soliton solution for the sine-Gordon equation in the plane with $C' = 1$ in the interval $[-10, 10]$, $\zeta_0 = -0.1; -1$.

on the development of a modulated wave is briefly illustrated by using a multiple scales perturbation technique in section three. We extend the derivation to the NLSE in two-dimensions, modified by the effect of non-uniformity and we discuss the soliton stability for the soliton solutions. In section four, we obtain the general soliton solutions to the standard NLSE in two-dimensions in both elliptic and hyperbolic cases. Finally, the paper ends with a conclusion in section five.

2. Problem formulation (non-uniformity case)

Van Duin [15] considered a fixed Cartesian coordinate system Oxz. The z-axis points vertically upwards, with $z = 0$ corresponding to the undisturbed free water surface. The x-axis is aligned with the propagation direction of a Stokes wavepacket. Since the fluid motion is irrotational, incompressible and deep with respect to the characteristic wavelength, a velocity potential $\phi(x, z, t)$ satisfying Laplace's equation

$$\phi_{xx} + \phi_{zz} = 0, \quad -\infty < z < \zeta, \tag{1}$$

where $\zeta(x, t)$ denotes the position of the undulating free surface. The boundary conditions at an infinite depth reads

$$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 \to 0 \quad \text{at } z \to -\infty. \tag{2}$$

At the air–water interface, $z = \zeta(x, t)$, we have the conditions

$$\frac{\partial \phi}{\partial z} = \frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x}, \tag{3}$$

$$2g\zeta + 2\frac{\partial \phi}{\partial t} + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 = 0. \tag{4}$$

He considered the motion of a Stokes wavepacket in the two-dimensional water waves of infinite depth. The evolution of a packet is described by two PDEs: the NLSE with a forcing term and a linear equation, which is of either elliptic or hyperbolic type depending on whether the group velocity of a Stokes wavepacket is less than or greater than the velocity of long gravity waves [16,17].

3. Two-dimensional NLS equation

We study the effect of non-uniformity (both temporal and spatial) on the development of a modulated wave in two-dimensions. Using a multiple scales perturbation technique, we derive the NLSE in two-dimensions, modified by the effect of non-uniformity. As a preliminary it will be useful to consider the equation

$$\frac{\partial^2 \Psi}{\partial t^2} - d^2\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}\right) + b^2\Psi + a\Psi^3 = 0. \tag{5}$$

This again turns out to be a representative model of weakly non-linear, dispersive waves in two-dimensions, when a small parameter ε is a measure of the amplitude of the wave, relevant slow variables are [18]

$$x_n = \varepsilon^n x, \qquad y_n = \varepsilon^n y, \qquad t_n = \varepsilon^n t. \tag{6}$$

The phase function θ is defined by

$$\frac{\partial \theta}{\partial x} = l(x_2, y_2, t_2), \qquad \frac{\partial \theta}{\partial y} = m(x_2, y_2, t_2), \qquad \frac{\partial \theta}{\partial t} = -\omega(x_2, y_2, t_2), \tag{7}$$

where l and m are the direction cosines of the wave vector along the x and y directions respectively. If θ is twice continuously differentiable, which gives the compatibility relationship, these are related according to the consistency relation

$$\frac{\partial l}{\partial t_2} + \frac{\partial \omega}{\partial x_2} = 0, \qquad \frac{\partial m}{\partial t_2} + \frac{\partial \omega}{\partial y_2} = 0. \tag{8}$$

The degree of non-uniformity, or modulation depth, is determined by the particular choice of the slow variables in Eq. (7). As a solution of Eq. (5), we take

$$\Psi = \varepsilon\left\{A\left(x_1, x_2, y_1, y_2, t_1, t_2\right) e^{i\theta} + c.c.\right\} + 0(\varepsilon^3), \tag{9}$$

then Eq. (5) becomes

$$\begin{aligned}
&-\varepsilon\omega^2 A e^{i\theta} - 2i\omega\varepsilon^2 \frac{\partial A}{\partial t_1} e^{i\theta} - \varepsilon^3\left(2i\omega\frac{\partial A}{\partial t_2} + 2iA\frac{\partial \omega}{\partial t_2} - \frac{\partial^2 A}{\partial t_1^2}\right) e^{i\theta} + +\varepsilon d^2 l^2 A e^{i\theta} \\
&\quad - 2ild^2\varepsilon^2 \frac{\partial A}{\partial x_1} e^{i\theta} - d^2\varepsilon^3\left(2il\frac{\partial A}{\partial x_2} + 2iA\frac{\partial l}{\partial x_2} - \frac{\partial^2 A}{\partial x_1^2}\right) e^{i\theta} + \varepsilon d^2 m^2 A e^{i\theta} \\
&\quad - 2imd^2\varepsilon^2 \frac{\partial A}{\partial y_1} e^{i\theta} - d^2\varepsilon^3\left(2im\frac{\partial A}{\partial y_2} + 2iA\frac{\partial m}{\partial y_2} + \frac{\partial^2 A}{\partial y_1^2}\right) e^{i\theta} + c.c. + b^2 A\varepsilon e^{i\theta} + a\varepsilon^3\left(Ae^{i\theta} + \overline{A}e^{-i\theta}\right)^3 = 0,
\end{aligned} \tag{10}$$

since $\Psi^3 = \varepsilon^3\left(A^3 e^{3i\theta} + 3A\left|A\right|^2 e^{i\theta} + 3A\left|A\right|^2 e^{-i\theta} + A^3 e^{-3i\theta}\right)$. Equating the coefficients of terms powers θ in Eq. (10), we get, for the first order, the term $\alpha\varepsilon$ should vanish, and gives the dispersion relation

$$\omega^2 = b^2 + l^2 d^2 + m^2 d^2. \tag{11}$$

For the second order, the term $\alpha\varepsilon^2$ should also vanish

$$\omega\frac{\partial A}{\partial t_1} + ld^2\frac{\partial A}{\partial x_1} + md^2\frac{\partial A}{\partial y_1} = 0,$$

then we have

$$\frac{\partial A}{\partial t_1} + v_l\frac{\partial A}{\partial x_1} + v_m\frac{\partial A}{\partial y_1} = 0, \tag{12}$$

where $v_l = \frac{ld^2}{\omega}$ and $v_m = \frac{md^2}{\omega}$ are the group velocities of the wave packet along the x and y axes respectively. For the third order, the term $\alpha\varepsilon^3$ also should vanish

$$2i\omega\left(\frac{\partial A}{\partial t_2} + \frac{ld^2}{\omega}\frac{\partial A}{\partial x_2} + \frac{md^2}{\omega}\frac{\partial A}{\partial y_2} + \frac{A}{2\omega}\frac{\partial \omega}{\partial t_2} + \frac{d^2 A}{2\omega}\frac{\partial l}{\partial x_2} + \frac{d^2 A}{2\omega}\frac{\partial m}{\partial y_2}\right) + d^2\frac{\partial^2 A}{\partial x_1^2} + d^2\frac{\partial^2 A}{\partial y_1^2} - \frac{\partial^2 A}{\partial t_1^2} = 3a\left|A\right|^2 A. \tag{13}$$

Eq. (13) leads to

$$\begin{aligned}
&2i\omega\left(\frac{\partial A}{\partial t_2} + v_l\frac{\partial A}{\partial x_2} + v_m\frac{\partial A}{\partial y_2} + \frac{A}{2}\frac{\partial v_l}{\partial x_2} + \frac{A}{2}\frac{\partial v_m}{\partial y_2}\right) \\
&\quad + \left(d^2 - v_l^2\right)\frac{\partial^2 A}{\partial x_1^2} - 2v_l v_m\frac{\partial^2 A}{\partial x_1\partial y_1} + \left(d^2 - v_m^2\right)\frac{\partial^2 A}{\partial y_1^2} = 3a\left|A\right|^2 A.
\end{aligned} \tag{14}$$

We study the stability of the so-called fundamental wave in two-dimensions, using A_0 instead of A. This depends on the slow variables x_2, y_2 and t_2 only, and it is described by the equation

$$2i\omega\left(\frac{\partial A_0}{\partial t_2}+v_l\frac{\partial A_0}{\partial x_2}+v_m\frac{\partial A_0}{\partial y_2}+\frac{A_0}{2}\frac{\partial v_l}{\partial x_2}+\frac{A_0}{2}\frac{\partial v_m}{\partial y_2}\right)=3a\left|A_0\right|^2A_0. \tag{15}$$

Then the equation

$$\frac{\partial}{\partial t_2}\left(|A_0|^2\right)+\frac{\partial}{\partial x_2}\left(v_l\,|A_0|^2\right)+\frac{\partial}{\partial y_2}\left(v_m\,|A_0|^2\right)=0, \tag{16}$$

derived from (16), shows that the energy of the fundamental wave propagates with the local group velocity. Depending on whether the group velocity increases or decreases in the direction of propagation, the associated wave will be called expansive or compressive wave. Then Eq. (15) leads to

$$2i\left(\frac{\partial A}{\partial t_2}+v_l\frac{\partial A}{\partial x_2}+v_m\frac{\partial A}{\partial y_2}\right)+P_1\frac{\partial^2A}{\partial x_1^2}+2P_2\frac{\partial^2A}{\partial x_1\partial y_1}+P_3\frac{\partial^2A}{\partial y_1^2}=Q\,|A|^2A-CA, \tag{17}$$

where

$$P_1=\frac{1}{\omega}\left(d^2-v_l^2\right),\qquad P_3=\frac{1}{\omega}\left(d^2-v_m^2\right),\qquad P_2=-\frac{1}{\omega}\left(v_mv_l\right),$$
$$C=1/2\left(\frac{\partial v_l}{\partial x_2}+\frac{\partial v_m}{\partial y_2}\right),\qquad Q=\frac{3a}{\omega}.$$

The PDE (17) is elliptic or hyperbolic depending upon the sign of $P=P_2^2-P_1P_3$,

$$P=\frac{d^4}{\omega^4}\left(l^2d^2+m^2d^2-\omega^2\right).$$

For the elliptic case, when P is negative, we introduce the transformations

$$\zeta_1=\frac{x_1}{P_1^{1/2}},\qquad \eta_1=\left(P_3-\frac{P_2^2}{P_1}\right)^{-1/2}\left(y_1-\frac{P_2}{P_1}x_1\right). \tag{18}$$

Then we obtain the elliptic equation

$$2i\frac{\partial A}{\partial\tau}+\frac{\partial^2A}{\partial\zeta_1^2}+\frac{\partial^2A}{\partial\eta_1^2}=Q\,|A|^2A-CA. \tag{19}$$

Eq. (19) is the standard elliptic NLSE in 2 + 1 dimensions. For the hyperbolic case, when P is positive. We introduce the transformations

$$\zeta_1=\frac{x_1}{P_1^{1/2}},\qquad \eta_1=\left(\frac{P_2^2}{P_1}-P_3\right)^{-1/2}\left(\frac{P_2}{P_1}x_1-y_1\right). \tag{20}$$

Proceeding as before, we obtain the hyperbolic equation

$$2i\frac{\partial A}{\partial\tau}+\frac{\partial^2A}{\partial\zeta_1^2}-\frac{\partial^2A}{\partial\eta_1^2}=Q\,|A|^2A-CA. \tag{21}$$

Eqs. (19) and (21) can be expressed in the form

$$2i\frac{\partial A}{\partial\tau}+\Delta_1\frac{\partial^2A}{\partial\zeta_1^2}+\Delta_2\frac{\partial^2A}{\partial\eta_1^2}=Q\,|A|^2A-CA, \tag{22}$$

with $\Delta_1=1$, and $\Delta_2=1$ when $P_2^2-P_1P_3<0$, and $\Delta_2=-1$ when $P_2^2-P_1P_3>0$. It is easy to show that two integrals of motion exist for Eq. (22)

$$I_1=\iint|A|^2\,d\zeta_1d\eta_1, \tag{23}$$

$$I_2=\iint\left(\Delta_1\left|\frac{\partial A}{\partial\zeta_1}\right|^2+\Delta_2\left|\frac{\partial A}{\partial\eta_1}\right|^2+\frac{Q}{2}\,|A|^4-\frac{C}{2}\,|A|^2\right)d\zeta_1d\eta_1. \tag{24}$$

Furthermore, it is possible by direct calculation to show that

$$\frac{\partial^2}{\partial\tau^2}\iint\left(\zeta_1^2+\eta_1^2\right)|A|^2\,d\zeta_1 d\eta_1$$
$$=8\iint\left(\Delta_1^2\left|\frac{\partial A}{\partial\zeta_1}\right|^2+\Delta_2^2\left|\frac{\partial A}{\partial\eta_1}\right|^2+\frac{Q}{4}\left(\Delta_2+1\right)|A|^4-\frac{C}{4}\left(\Delta_2+1\right)|A|^2\right)d\zeta_1 d\eta_1, \tag{25}$$

where the integrals are over the whole ζ_1 and η_1 plane. For the elliptic case $\Delta_2=1$ the right hand side of the Eq. (25) is just $8I_2$. Since I_2 is a constant of motion, Eq. (25) can be integrated twice to give

$$\iint\left(\zeta_1^2+\eta_1^2\right)|A|^2\,d\zeta_1 d\eta_1=4\tau^2 I_2+c_1\tau+c_2. \tag{26}$$

The sign of I_2 is now important since it is the $4\tau^2 I_2$ term which dominates as τ increases. If $Q<0$ and $C>0$, then it is possible to have $I_2<0$, for quite a broad class of initial data. If therefore $I_2<0$, then the right-hand side of Eq. (26) can change sign after a finite value of τ. Since the integral on the right-hand side of Eq. (26) has a positive definite integrand, this behaviour implies the existence of a singularity in A after a finite time and the solution ceases to exist [19,20].

4. General soliton solutions of the elliptic and hyperbolic NLS equations

Now we shall find solution classes to both elliptic and hyperbolic NLS equations. By applying the function transformation method, we come to find the solutions of Eq. (22) in the form

$$A=\theta\left(\tau,\zeta_1,\eta_1\right)\exp i\left(c_0\tau+c_1\zeta_1+c_2\eta_1\right), \tag{27}$$

where $\theta\left(\tau,\zeta_1,\eta_1\right)=\theta^*\left(\tau,\zeta_1,\eta_1\right)$ and c_0,c_1,c_2 are real constants. By inserting (27) into (22), we get

$$2i\frac{\partial\theta}{\partial\tau}+2ic_1\Delta_1\frac{\partial\theta}{\partial\zeta_1}+2ic_2\Delta_2\frac{\partial\theta}{\partial\eta_1}+\Delta_1\frac{\partial^2\theta}{\partial\zeta_1^2}+\Delta_2\frac{\partial^2\theta}{\partial\eta_1^2}=\left(-c+2c_0+\Delta_1 c_1^2+\Delta_2 c_2^2\right)\theta+Q\theta^3.$$

For the elliptic case, we have

$$2i\frac{\partial\theta}{\partial\tau}+2ic_1\frac{\partial\theta}{\partial\zeta_1}+2ic_2\frac{\partial\theta}{\partial\eta_1}+\frac{\partial^2\theta}{\partial\zeta_1^2}+\frac{\partial^2\theta}{\partial\eta_1^2}=C\theta+Q\theta^3, \tag{28}$$

where $2c_0+c_1^2+c_2^2-c=C$ (constant). Let us make a function transformation,

$$\theta=\sqrt{C/Q}\,\sinh\left(\phi/2\right), \tag{29}$$

then we have

$$2i\frac{\partial\phi}{\partial\tau}+2ic_1\frac{\partial\phi}{\partial\zeta_1}+2ic_2\frac{\partial\phi}{\partial\eta_1}+\frac{\partial^2\phi}{\partial\zeta_1^2}+\frac{\partial^2\phi}{\partial\eta_1^2}+\frac{1}{2}\tanh\left(\phi/2\right)\left[\left(\frac{\partial\phi}{\partial\zeta_1}\right)^2+\left(\frac{\partial\phi}{\partial\eta_1}\right)^2\right]=C\sinh\left(\phi\right). \tag{30}$$

Setting $\phi=\phi\left(\zeta\right)$ which is a function of another function ζ only we easily see that

$$\frac{\partial^2\phi}{\partial\zeta_1^2}=\left(\frac{\partial\zeta}{\partial\zeta_1}\right)^2\frac{d^2\phi}{d\zeta^2}+\frac{\partial^2\zeta}{\partial\zeta_1^2}\frac{d\phi}{d\zeta},\quad\text{and}\quad\frac{\partial^2\phi}{\partial\eta_1^2}=\left(\frac{\partial\zeta}{\partial\eta_1}\right)^2\frac{d^2\phi}{d\zeta^2}+\frac{\partial^2\zeta}{\partial\eta_1^2}\frac{d\phi}{d\zeta}. \tag{31}$$

By substituting (31) into (30), we see that

$$\left(2i\frac{\partial\zeta}{\partial\tau}+2ic_1\frac{\partial\zeta}{\partial\zeta_1}+2ic_2\frac{\partial\zeta}{\partial\eta_1}+\frac{1}{2}\frac{\partial^2\zeta}{\partial\zeta_1^2}+\frac{1}{2}\frac{\partial^2\zeta}{\partial\eta_1^2}\right)\frac{d\phi}{d\zeta}$$
$$+\left[\left(\frac{\partial\phi}{\partial\zeta_1}\right)^2+\left(\frac{\partial\phi}{\partial\eta_1}\right)^2\right]\left[\frac{d^2\phi}{d\zeta^2}+\frac{1}{2}\tanh\left(\phi/2\right)\left(\frac{d\phi}{d\zeta}\right)^2\right]=C\sinh\left(\phi\right). \tag{32}$$

Explicitly, some solutions of (32) obey the following system of equations:

$$i\frac{\partial\zeta}{\partial\tau}+ic_1\frac{\partial\zeta}{\partial\zeta_1}+ic_2\frac{\partial\zeta}{\partial\eta_1}=\frac{\partial^2\zeta}{\partial\zeta_1^2}+\frac{\partial^2\zeta}{\partial\eta_1^2}=0\quad\text{and}\quad\left(\frac{\partial\zeta}{\partial\zeta_1}\right)^2+\left(\frac{\partial\zeta}{\partial\eta_1}\right)^2=1, \tag{33}$$

$$\frac{d^2\phi}{d\zeta^2}+\frac{1}{2}\tanh\left(\phi/2\right)\left(\frac{d\phi}{d\zeta}\right)^2=C\sinh\left(\phi\right). \tag{34}$$

Eq. (34) is equivalent to a sinh-Gordon equation; its solution is a well-known soliton

$$\phi = 4\tanh^{-1}\left[\exp\left(\sqrt{\frac{C}{2}}\zeta + \zeta_0\right)\right] - \pi, \quad \zeta_0 = \text{constant}. \tag{35}$$

Substituting (35) and (29) into (27), we obtain the soliton solutions of the two-dimensional NLSE (22) in the form

$$\theta = \sqrt{C/Q}\sinh\left(2\tanh^{-1}\left[\exp\left(\sqrt{\frac{C}{2}}\zeta + \zeta_0\right)\right] - \frac{\pi}{2}\right),$$

$$A = \sqrt{C/Q}\sinh\left(2\tanh^{-1}\left[\exp\left(\sqrt{\frac{C}{2}}\zeta + \zeta_0\right)\right] - \frac{\pi}{2}\right)\exp i\left(c_0\tau + c_1\zeta_1 + c_2\eta_1\right). \tag{36}$$

Now we come to find a kind of general solutions of (33) in the form

$$\zeta = F\left(\xi_j\right) + d_0\tau + d_1\zeta_1 + d_2\eta_1, \qquad \xi_j = b_{j0}\tau + b_{j1}\zeta_1 + b_{j2}\eta_1 + \varepsilon_j, \tag{37}$$

where $d_0, d_1, d_2, b_{j0}, b_{j1}, b_{j2}, \varepsilon_j$ are constants, and $F\left(\xi_j\right)$ denotes an arbitrary function of ξ_j. Combining (33) and (37) we easily obtain

$$i\frac{\partial\zeta}{\partial\tau} + ic_1\frac{\partial\zeta}{\partial\zeta_1} + ic_2\frac{\partial\zeta}{\partial\zeta_2} = i\left(b_{j0} + c_1b_{j1} + c_2b_{j2}\right)\frac{\partial F}{\partial\xi_j} + i\left(d_0 + c_1d_1 + c_2d_2\right),$$

$$\left(\frac{\partial\zeta}{\partial\zeta_1}\right)^2 + \left(\frac{\partial\zeta}{\partial\eta_1}\right)^2 = \left[\left(b_{j1}\right)^2 + \left(b_{j2}\right)^2\right]\left(\frac{\partial F}{\partial\xi_j}\right)^2 + \left(2d_1b_{j1} + 2d_2b_{j2}\right)\frac{\partial F}{\partial\xi_j} + d_1^2 + d_2^2 = 1,$$

$$\frac{\partial^2\zeta}{\partial\zeta_1^2} = b_{j1}b_{k1}\frac{\partial^2F}{\partial\xi_j\partial\xi_k} = 0 \quad \text{and} \quad \frac{\partial^2\zeta}{\partial\eta_1^2} = b_{j2}b_{k2}\frac{\partial^2F}{\partial\xi_j\partial\xi_k} = 0,$$

where $F\left(\xi_j\right)$ to be arbitrary leads to the conditions

$$b_{j0} + c_1b_{j1} + c_2b_{j2} = 0, \qquad d_0 + c_1d_1 + c_2d_2 = 0, \qquad b_{j1}b_{k1} + b_{j2}b_{k2} = 0,$$

$$d_1d_1 + d_2d_2 = 1 \quad \text{and} \quad d_1b_{j1} + d_2b_{j2} = 0. \tag{38}$$

The general solutions (37) make (35) and (34), a form of general soliton solutions of the two-dimensional NLSE and sinh-Gordon equation.

For the hyperbolic case, we have

$$2i\frac{\partial\Phi}{\partial\tau} + 2ic_1\frac{\partial\Phi}{\partial\zeta_1} - 2ic_2\frac{\partial\Phi}{\partial\eta_1} + \frac{\partial^2\Phi}{\partial\zeta_1^2} - \frac{\partial^2\Phi}{\partial\eta_1^2} = Q\Phi^3 - C'\Phi, \tag{39}$$

where $2c_0 + c_1^2 - c_2^2 - c = -C'$ (constant). Let us make a function transformation

$$\Phi = \sqrt{\frac{C'}{Q}}\sin\left(\frac{\psi}{2}\right), \tag{40}$$

where Ψ is a function in τ, ζ_1 and η_1

$$2i\frac{\partial\Phi}{\partial\tau} + 2ic_1\frac{\partial\Phi}{\partial\zeta_1} - 2ic_2\frac{\partial\Phi}{\partial\eta_1} = \sqrt{\frac{C'}{Q}}\cos\left(\Psi/2\right)\left[i\frac{\partial\Psi}{\partial\tau} + ic_1\frac{\partial\Psi}{\partial\zeta_1} - ic_2\frac{\partial\Psi}{\partial\eta_1}\right],$$

$$\frac{\partial^2\Phi}{\partial\zeta_1^2} = \frac{1}{2}\sqrt{\frac{C'}{Q}}\left[\cos\left(\Psi/2\right)\frac{\partial^2\Psi}{\partial\zeta_1^2} - \frac{1}{2}\sin\left(\Psi/2\right)\left(\frac{\partial\Psi}{\partial\zeta_1}\right)^2\right],$$

$$\frac{\partial^2\Phi}{\partial\eta_1^2} = \frac{1}{2}\sqrt{\frac{C'}{Q}}\left[\cos\left(\Psi/2\right)\frac{\partial^2\Psi}{\partial\eta_1^2} - \frac{1}{2}\sin\left(\Psi/2\right)\left(\frac{\partial\Psi}{\partial\eta_1}\right)^2\right],$$

$$Q\Phi^3 - C'\Phi = -\frac{C'}{2}\sqrt{\frac{C'}{Q}}\cos\left(\Psi/2\right)\sin\Psi.$$

By substituting (40) into Eq. (39), we get

$$i\frac{\partial\Psi}{\partial\tau} + ic_1\frac{\partial\Psi}{\partial\zeta_1} - ic_2\frac{\partial\Psi}{\partial\eta_1} + \frac{1}{2}\left(\frac{\partial^2\Psi}{\partial\zeta_1^2} - \frac{\partial^2\Psi}{\partial\eta_1^2}\right) - \frac{1}{4}\tan\left(\Psi/2\right)\left(\left(\frac{\partial\Psi}{\partial\zeta_1}\right)^2 - \left(\frac{\partial\Psi}{\partial\eta_1}\right)^2\right) = -\frac{C'}{2}\sin\Psi. \tag{41}$$

Setting $\Psi = \Psi(\zeta)$ which is a function of another function ζ only and by substituting (31) into (41), we easily see that

$$\left(i\frac{\partial\zeta}{\partial\tau} + ic_1\frac{\partial\zeta}{\partial\zeta_1} - ic_2\frac{\partial\zeta}{\partial\eta_1} + \frac{1}{2}\frac{\partial^2\zeta}{\partial\zeta_1^2} - \frac{1}{2}\frac{\partial^2\zeta}{\partial\eta_1^2}\right)\frac{d\Psi}{d\zeta} + \left[\left(\frac{\partial\zeta}{\partial\zeta_1}\right)^2 - \left(\frac{\partial\zeta}{\partial\eta_1}\right)^2\right]\left[\frac{1}{2}\frac{d^2\Psi}{d\zeta^2} - \frac{1}{4}\tan(\Psi/2)\left(\frac{d\Psi}{d\zeta}\right)^2\right] = -\frac{C'}{2}\sin\Psi. \tag{42}$$

Explicitly, some solutions of (42), obey the following system of equations:

$$i\frac{\partial\zeta}{\partial\tau} + in_1\frac{\partial\zeta}{\partial\zeta_1} - in_2\frac{\partial\zeta}{\partial\eta_1} = \frac{\partial^2\zeta}{\partial\zeta_1^2} - \frac{\partial^2\zeta}{\partial\eta_1^2} = 0 \quad \text{and} \quad \left(\frac{\partial\zeta}{\partial\zeta_1}\right)^2 - \left(\frac{\partial\zeta}{\partial\eta_1}\right)^2 = 1, \tag{43}$$

$$\frac{d^2\Psi}{d\zeta^2} - \frac{1}{2}\tan(\Psi/2)\left(\frac{d\Psi}{d\zeta}\right)^2 = -C'\sin(\Psi). \tag{44}$$

Eq. (44) is equivalent to a sine-Gordon equation, its solution is a well-known soliton

$$\Psi = 4\tan^{-1}\exp\left(\sqrt{\frac{C'}{2}}\zeta + \zeta_0\right) - \pi, \ \zeta_0 = \text{constant}. \tag{45}$$

Applying (45) and (40) into (27), we obtain the soliton solutions of the two-dimensional NLSE (22) in the form

$$A = \sqrt{C'/Q}\tanh\left(\sqrt{\frac{C'}{2}}\zeta + \zeta_0\right)\exp i(c_0\tau + c_1\zeta_1 + c_2\eta_1), \tag{46}$$

where ζ denotes a solution of Eq. (43) which has several solution classes, it includes many interesting solitons of the two-dimensional NLSE. Now we come to find a kind of general solution of Eq. (45) in the form

$$\zeta = F(\xi_j) + m_0\tau + m_1\zeta_1 + m_2\eta_1, \qquad \xi_j = b_{j0}\tau + b_{j1}\zeta_1 + b_{j2}\eta_1 + \varepsilon_j, \tag{47}$$

where $m_0, m_1, m_2, b_{j0}, b_{j1}, b_{j2}, \varepsilon_j$ are constants, and $F(\xi_j)$ denotes an arbitrary function of ξ_j. Combining (47) and (43), we easily obtain

$$\begin{aligned}
&i\frac{\partial\zeta}{\partial\tau} + ic_1\frac{\partial\zeta}{\partial\zeta_1} - ic_2\frac{\partial\zeta}{\partial\zeta_2} = i(b_{j0} + c_1b_{j1} - c_2b_{j2})\frac{\partial F}{\partial\xi_j} + i(m_0 + c_1m_1 - c_2m_2),\\
&\left(\frac{\partial\zeta}{\partial\zeta_1}\right)^2 + \left(\frac{\partial\zeta}{\partial\eta_1}\right)^2 = \left[(b_{j1})^2 - (b_{j2})^2\right]\left(\frac{\partial F}{\partial\xi_j}\right)^2 + (2m_1b_{j1} - 2m_2b_{j2})\frac{\partial F}{\partial\xi_j} + m_1^2 - m_2^2 = 1,\\
&\frac{\partial^2\zeta}{\partial\zeta_1^2} = b_{j1}b_{k1}\frac{\partial^2 F}{\partial\xi_j\partial\xi_k} = 0, \qquad \frac{\partial^2\zeta}{\partial\eta_1^2} = b_{j2}b_{k2}\frac{\partial^2 F}{\partial\xi_j\partial\xi_k} = 0,
\end{aligned}$$

where $F(\xi_j)$ to be arbitrary leads to the conditions

$$\begin{aligned}
&b_{j0} + c_1b_{j1} - c_2b_{j2} = 0, \qquad m_0 + c_1m_1 - c_2m_2 = 0, \qquad b_{j1}b_{k1} - b_{j2}b_{k2} = 0,\\
&m_1m_1 + m_2m_2 = 1, \quad \text{and} \quad m_1b_{j1} - m_2b_{j2} = 0.
\end{aligned} \tag{48}$$

The general solutions (47) make (45) and (46), a form of general soliton solutions of the two-dimensional NLSE and sine-Gordon equation. They contain some interesting specific solutions, such as the plane solitons, the N multiple solitons, the propagating breathers and the quadratic solitons. We will discuss the elliptic and hyperbolic cases respectively as the following.

4.1. The plane solitons.

This is a simple case. The equations of the elliptic case and sinh-Gordon equation; equation of the hyperbolic case and sine-Gordon equation have the following solutions.

4.1.1. The elliptic case

$$\phi = 4\tanh^{-1}\left[\exp\left(\sqrt{\frac{C}{2}}(d_0\tau + d_1\zeta_1 + d_2\eta_1) + \zeta_0\right)\right] - \pi, \tag{49}$$

$$A = \sqrt{C/Q}\sinh\left(2\tanh^{-1}\left(\exp\left(\sqrt{\frac{C}{2}}(d_0\tau + d_1\zeta_1 + d_2\eta_1) + \zeta_0\right)\right) - \frac{\pi}{2}\right)(\exp(i(c_0\tau + c_1\zeta_1 + c_2\eta_1))). \tag{50}$$

4.1.2. The hyperbolic case

$$\Psi = 4\tan^{-1}\left[\exp\left(\sqrt{\frac{C'}{2}}\left(m_0\tau + m_1\zeta_1 + m_2\eta_1\right) + \zeta_0\right)\right] - \pi, \tag{51}$$

$$A = \sqrt{C'/Q}\tanh\left(\sqrt{\frac{C'}{2}}\left(m_0\tau + m_1\zeta_1 + m_2\eta_1\right) + \zeta_0\right)\left(\exp\left(i\left(c_0\tau + c_1\zeta_1 + c_2\eta_1\right)\right)\right). \tag{52}$$

In this case, Eqs. (49)–(52) denotes hyperplanar soliton solutions of the two-dimensional NLSE, sinh-Gordon and sine-Gordon equations.

4.2. The N multiple soliton solutions

Let us select $F(\xi_j)$ in the form

$$F\left(\xi_j\right) = \ln \Sigma_{j=1}^{N} \exp\left(b_{j0}\tau + b_{j1}\zeta_1 + b_{j2}\eta_1 + \varepsilon_j\right), \tag{53}$$

4.2.1. The elliptic case

Application of (37) and (53) leads to

$$\zeta = \ln\left(\Sigma_{j=1}^{N} \exp\left[a_{j0}\tau + a_{j1}\zeta_1 + a_{j2}\eta_1 + \varepsilon_j\right]\right), \tag{54}$$

where $a_{j\alpha} = d_\alpha + b_{j\alpha}$, $\alpha = 0, 1, 2$, the conditions in (38) are simplified to

$$a_{j0} + c_1 a_{j1} + c_2 a_{j2} = 0, \quad \text{and} \quad a_{j1}a_{k1} + a_{j2}a_{k2} = 1.$$

Taking j equal to $1, 2, \ldots, N$ respectively, the N multiple wave solutions are:

$$\phi = 4\tanh^{-1}\exp\left[\sqrt{\frac{C}{2}}\left(\ln\left(\Sigma_{j=1}^{N}\exp\left[a_{j0}\tau + a_{j1}\zeta_1 + a_{j2}\eta_1 + \varepsilon_j\right]\right)\right) + \zeta_0\right] - \pi, \tag{55}$$

$$\begin{aligned} A &= \sqrt{C/S}\left(\exp i\left(c_0\tau + c_1\zeta_1 + c_2\eta_1\right)\right) \\ &\quad \times \sinh\left(2\tanh^{-1}\exp\left(\sqrt{\frac{C}{2}}\left(\ln\left(\Sigma_{j=1}^{N}\exp\left[a_{j0}\tau + a_{j1}\zeta_1 + a_{j2}\eta_1 + \varepsilon_j\right]\right)\right) + \zeta_0\right) - \frac{\pi}{2}\right). \end{aligned} \tag{56}$$

4.2.2. The hyperbolic case

Application of (47) and (53) leads to

$$\zeta = \ln\left(\Sigma_{j=1}^{N}\exp\left[D_{j0}\tau + D_{j1}\zeta_1 + D_{j2}\eta_1 + \varepsilon_j\right]\right), \tag{57}$$

where $D_{j\alpha} = m_\alpha + b_{j\alpha}$, $\alpha = 0, 1, 2$, the conditions in (48) are simplified to

$$D_{j0} + n_1 D_{j1} - n_2 D_{j2} = 0 \quad \text{and} \quad D_{j1}D_{k1} - D_{j2}D_{k2} = 1. \tag{58}$$

Then the N multiple wave solutions are

$$\Psi = 4\tan^{-1}\exp\left(\sqrt{\frac{C'}{2}}\left(\ln\left(\Sigma_{j=1}^{N}\exp\left[D_{j0}\tau + D_{j1}\zeta_1 + D_{j2}\eta_1 + \varepsilon_j\right]\right)\right) + \zeta_0\right) - \pi, \tag{59}$$

$$A = \sqrt{\frac{C'}{Q}}\tanh\left(\sqrt{\frac{C'}{2}}\left(\ln\left(\Sigma_{j=1}^{N}\exp\left[D_{j0}\tau + D_{j1}\zeta_1 + D_{j2}\eta_1 + \varepsilon_j\right]\right)\right) + \zeta_0\right)\left(\exp i\left(n_0\tau + n_1\zeta_1 + n_2\eta_1\right)\right). \tag{60}$$

Eqs. (55)–(56) and (59)–(60) denote the N multiple soliton solutions of the two-dimensional NLSE, sinh-Gordon and sine-Gordon equations, respectively.

4.3. The propagating breathers

4.3.1. The elliptic case

Considering $N = 2$ in (53) we get

$$\zeta = \ln\left(\exp\left[a_{10}\tau + a_{11}\zeta_1 + a_{12}\eta_1 + \varepsilon_1\right]\right) + \exp\left[a_{20}\tau + a_{21}\zeta_1 + a_{22}\eta_1 + \varepsilon_2\right], \tag{61}$$

which corresponds to a 2-soliton solutions of the two-dimensional NLSE, we set

$$a_{10} = -a_{20}, \qquad a_{11} = -a_{21}, \qquad a_{12} = a_{22}, \qquad \varepsilon_1 = \ln s, \qquad \varepsilon_2 = \ln s + i\pi. \tag{62}$$

Then (61) becomes

$$\zeta = \ln\left[2A \exp\left(a_{12}\eta_1\right) \sinh\left(a_{10}\tau + a_{11}\zeta_1\right)\right],$$

and (38) gives the conditions

$$a_{11}^2 + a_{12}^2 = 1, \qquad a_{10} + 2c_1 a_{11} = 0, \qquad c_2 a_{12} = 0.$$

If a_{10}, a_{11} and A are imaginary numbers such as

$$a_{10} = ia_0, \qquad a_{11} = ia_1, \qquad 2A = -iB,$$

then (61) takes the form

$$\zeta = \ln\left[B \exp\left(a_{12}\eta_1\right) \sin\left(a_0\tau + a_1\zeta_1\right)\right].$$

Then the solutions of Eqs. (34) and (20) become

$$\phi = 4\tanh^{-1}\exp\left(\sqrt{\frac{C}{2}}\ln\left[B\exp\left(a_{12}\eta_1\right)\sin\left(a_0\tau + a_1\zeta_1\right)\right] + \zeta_0\right) - \pi, \tag{63}$$

$$A = \sqrt{C/Q}\,\sinh\left(\tanh^{-1}\exp\left(\sqrt{\frac{C}{2}}\ln\left[B\exp\left(a_{12}\eta_1\right)\sin\left(a_0\tau + a_1\zeta_1\right)\right] + \zeta_0\right) - \frac{\pi}{2}\right) \times \left(\exp i\left(c_0\tau + c_1\zeta_1 + c_2\eta_1\right)\right). \tag{64}$$

4.3.2. The hyperbolic case

Considering $N = 2$ in (53), we get

$$\zeta = \ln\left(\exp\left[D_{10}\tau + D_{11}\zeta_1 + D_{12}\eta_1 + \varepsilon_1\right]\right) + \exp\left[D_{20}\tau + D_{21}\zeta_1 + D_{22}\eta_1 + \varepsilon_2\right], \tag{65}$$

which corresponds to a 2-soliton solutions of the two-dimensional NLSE (hyperbolic equation), and

$$\zeta = \ln\left[\lambda\exp\left(D_{12}\eta_1\right)\sin\left(D_0\tau + D_1\zeta_1\right)\right], \tag{66}$$

where $D_{10} = iD_0,\ D_{11} = iD_1,\ 2s = -i\lambda$. Substituting from Eq. (66) into Eqs. (45) and (46), the solutions become

$$\Psi = 4\tan^{-1}\exp\left(\sqrt{\frac{C'}{2}}\ln\left[\lambda\exp\left(D_{12}\eta_1\right)\sin\left(D_0\tau + D_1\zeta_1\right)\right] + \zeta_0\right) - \pi, \tag{67}$$

$$A = \sqrt{C'/Q}\,\tanh\left(\sqrt{\frac{C'}{2}}\ln\left[\lambda\exp\left(D_{12}\eta_1\right)\sin\left(D_0\tau + D_1\zeta_1\right)\right] + \zeta_0\right)\left(\exp\left(i\left(n_0\tau + n_1\zeta_1 + n_2\eta_1\right)\right)\right). \tag{68}$$

Then (63)–(64) and (67)–(68) are multidimensional breathers which propagate in the ζ_1 direction and have the same value on the right line $a_{12}\eta =$ constant and $D_{12}\eta =$ constant.

4.4. The quadratic solitons

4.4.1. The elliptic case

We take the solution (53) in the form

$$\zeta = b_{j0}b_{j0}\tau^2 + b_{j1}b_{j1}\zeta_1^2 + b_{j2}b_{j2}\eta_1^2 + 2b_{j1}b_{j2}\zeta_1\eta_1 + \left(2\varepsilon_j b_{j1} + 2b_{j0}b_{j1}\tau + d_1\right)\zeta_1 + \left(2\varepsilon_j b_{j2} + 2b_{j0}b_{j2}\tau + d_2\right)\eta_1 + \left(2\varepsilon_j b_{j0} + d_0\right)\tau + \varepsilon_j\varepsilon_j, \quad j = 1, 2, \ldots, N. \tag{69}$$

It describes some general quadratic surfaces at any definite time. These quadratic surfaces include all specific ones such as:

The circle.

By choosing the constants of (69) as

$$b_{ji}b_{jk} = \delta_{ik} \quad j = 1, 2, \ldots, N, \tag{70}$$

then (69) becomes

$$\zeta = \tau^2 + \zeta_1^2 + \eta_1^2 + \left(2\varepsilon_j b_{j1} + d_1\right)\zeta_1 + \left(2\varepsilon_j b_{j2} + d_2\right)\eta_1 + \left(2\varepsilon_j b_{j0} + d_0\right)\tau + \varepsilon_j\varepsilon_j. \tag{71}$$

We obtain a circle with radius

$$R = \left[\left(\varepsilon_j b_{ji} + \frac{d_i}{2}\right)\left(\varepsilon_k b_{ki} + \frac{d_i}{2}\right) - \tau^2 - \left(2\varepsilon_j b_{j0} + d_0\right)\tau - \varepsilon_j\varepsilon_j\right]^{1/2}, \tag{72}$$

which satisfies $R^2 > 0$, and with center

$$G\left(-2\varepsilon_j b_{j1} - 2b_{j0}b_{j1}\tau - d_1, -2\varepsilon_j b_{j2} - 2b_{j0}b_{j2}\tau - d_2\right), \tag{73}$$

which moves along a space line. Thus (71), (35) and (36) gives some circular shape solitons of the two-dimensional NLSE and sinh-Gordon equation.

The ellipses. Under the conditions

$$b_{j1}b_{j1} > 0, \qquad b_{j2}b_{j2} > 0, \qquad b_{ji}b_{jk} = 0, \quad i \neq k, \; j = 1, 2, \ldots, N, \tag{74}$$

then (69) becomes

$$\zeta = b_{j0}b_{j0}\tau^2 + b_{j1}b_{j1}\zeta_1^2 + b_{j2}b_{j2}\eta_1^2 + \left(2\varepsilon_j b_{j1} + d_1\right)\zeta_1 + \left(2\varepsilon_j b_{j2} + d_2\right)\eta_1 + \left(2\varepsilon_j b_{j0} + d_0\right)\tau + \varepsilon_j\varepsilon_j. \tag{75}$$

Eq. (75) are ellipses while (35) and (36) are the elliptical shape solitons.

The hyperbolas. In the conditions (69), let the constants satisfy

$$b_{j1}b_{j1} > 0, \qquad b_{j2}b_{j2} < 0, \qquad b_{ji}b_{jk} = 0 \quad i \neq K, \; j = 1, 2, \ldots, N. \tag{76}$$

We obtain the hyperbolas by inserting the condition (76) in Eqs. (35) and (36), i.e. this gives the corresponding hyperbolic shape solitons of the two-dimensional NLSE and sinh-Gordon equation.

4.4.2. The hyperbolic case

We take the solution (53) in the form

$$\begin{aligned}\zeta = {} & b_{j0}b_{j0}\tau^2 + b_{j1}b_{j1}\zeta_1^2 + b_{j2}b_{j2}\eta_1^2 + 2b_{j1}b_{j2}\zeta_1\eta_1 + \left(2\varepsilon_j b_{j1} + 2b_{j0}b_{j1}\tau + m_1\right)\zeta_1 \\ & + \left(2\varepsilon_j b_{j2} + 2b_{j0}b_{j2}\tau + m_2\right)\eta_1 + \left(2\varepsilon_j b_{j0} + m_0\right)\tau + \varepsilon_j\varepsilon_j. \end{aligned} \tag{77}$$

It describes some general quadratic surfaces at any definite time. These quadratic surfaces include all specific ones such as:

The circle.

The solution (77) becomes

$$\zeta = \tau^2 + \zeta_1^2 + \eta_1^2 + \left(2\varepsilon_j b_{j1} + m_1\right)\zeta_1 + \left(2\varepsilon_j b_{j2} + m_2\right)\eta_1 + \left(2\varepsilon_j b_{j0} + m_0\right)\tau + \varepsilon_j\varepsilon_j, \tag{78}$$

we obtain a circle with radius

$$R = \left[\left(\varepsilon_j b_{ji} + \frac{m_i}{2}\right)\left(\varepsilon_k b_{ki} + \frac{m_i}{2}\right) - \tau^2 - \left(2\varepsilon_j b_{j0} + m_0\right)\tau - \varepsilon_j\varepsilon_j\right]^{1/2}, \tag{79}$$

which satisfies $R^2 > 0$, and with center

$$G\left(-2\varepsilon_j b_{j1} - 2b_{j0}b_{j1}\tau - m_1, -2\varepsilon_j b_{j2} - 2b_{j0}b_{j2}\tau - m_2\right), \tag{80}$$

which moves along a space line. Thus (78), (45) and (46) give some circular shape solitons of the two-dimensional NLSE and sine-Gordon equation.

The ellipses.

$$\zeta = b_{j0}b_{j0}\tau^2 + b_{j1}b_{j1}\zeta_1^2 + b_{j2}b_{j2}\eta_1^2 + \left(2\varepsilon_j b_{j1} + m_1\right)\zeta_1 + \left(2\varepsilon_j b_{j2} + m_2\right)\eta_1 + \left(2\varepsilon_j b_{j0} + m_0\right)\tau + \varepsilon_j\varepsilon_j. \tag{81}$$

Eq. (81) are ellipses while (45) and (46) are the elliptical shape solitons.

The hyperbolas. From the conditions (77), we obtain some hyperbolas which (45) and (46) give the corresponding hyperbolic shape solitons of the two-dimensional NLSE and sine-Gordon equation.

5. Conclusion

The effect of non-uniformity on the development of a modulated wave was studied by using the multiple scales perturbation technique in one-dimension and we found the solution of the NLSE in one-dimension. We extended the derivation to the NLSE in two-dimensions, modified by the effect of non-uniformity. Applying the function transformation method, the two-dimensional NLSE was transformed to either sine-Gordon or sinh-Gordon equations, which depend only on one function ζ and can be solved. The general solution of the equations in ζ leads to a general soliton solution of the two-dimensional NLSE. Further the two-dimensional NLSE was transformed to either a sinh-Gordon equation for the elliptic case or to a sine-Gordon equation for the hyperbolic case. It contains some interesting specific solutions such as plane solitons, the N multiple solitons, the propagating breathers and the quadratic solitons which contain the circular shape, elliptical shape and hyperbolic shape solitons. As an illustration, we used the ready made package Mathematica 5 to solve the original problem. We started from the governing PDEs and by using Mathematica 5 we obtained soliton solutions. This solution was in good agreement with the analytical one. We displayed graphically the obtained solutions by using Mathematica 5 (Figs. 1–4).

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