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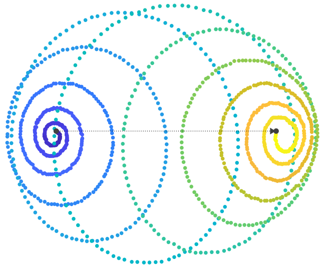
Stokes drift: theory and experiments



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An important facet of water wave dynamics is the fact that Stokes' 1847 approximate theory of water waves predicts mean Lagrangian velocities even when mean Eulerian velocities are zero. This motion, known as Stokes drift, is important to a wide variety of oceanic processes. Reflecting the difficulty of avoiding effects associated with the boundaries in wave tanks, the first convincing experimental evidence confirming this behaviour has only recently been given in van den Bremer *et al.* (*J. Fluid Mech.*, vol. 879, 2019, pp. 168–186). This is an important result given prior evidence that the exact rotational waves first studied by Gerstner in 1802 may exist. Nonetheless, despite more than 200 years of work on the theory of water waves, much remains to be discovered.

Key words: surface gravity waves

1. Introduction

While the theory of water waves has a long history (Craik 2004), much of what is commonly used today is based on the seminal work of Sir Gabriel Stokes, first published in 1847 (Stokes 1847). In his analysis of irrotational water waves, Stokes formalized the approximation of the fully nonlinear boundary value problem by implicitly defining a solution that takes the form of a power series in wave steepness. Through this analysis he made a striking finding: while the time-averaged (mean) Eulerian velocity at any point in the fluid might be zero, there is a net motion of particles, i.e., a non-zero Lagrangian mean velocity, now known as the Stokes drift (e.g. Bühler 2014).

Stokes drift, which emerges as a property of the averaging of wave effects on the mean (wave averaged) flow, plays an important role in oceanic flows. For example, wave averaging the Navier–Stokes equations produces a body force acting on the mean velocity field that is proportional to the cross product of the mean vorticity and the Stokes drift (Craik & Leibovich 1976). It is this 'extra' force that supports the instability that can produce Langmuir cells. Transport of materials in the ocean in the presence of waves is due to the Lagrangian mean velocity, the sum of the Stokes drift and the Eulerian mean velocity (e.g. Monismith & Fong 2004).

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2. Overview

Given that the theory behind the Stokes drift is 172 years old, it is striking that the recent paper by van den Bremer *et al.* (2019) is the first definitive, quantitative and unambiguous demonstration that the mean Lagrangian velocity in the absence of other flows can be accurately calculated from the lowest-order irrotational wave velocity field.

Why has it taken 172 years for experimental confirmation of Stokes' theory to be made? It turns out that the experiment is surprisingly hard to do properly. The principal challenge is that the finite length of the tank, i.e., the existence of endwalls, forces a condition of no net flow through any cross-section, at least in steady state (Swan 1990). In the absence of vorticity and of viscous effects, the Eulerian return flow required to balance the Stokes drift of a spatially uniform wave field should be independent of depth (van den Bremer *et al.* 2019). However, in the laboratory, as time proceeds, the mean Eulerian flow changes as the effects of sidewall, surface and bottom boundary layers begin to influence the flow in the interior of the tank (Longuet-Higgins 1953). Moreover, how the waves are absorbed at the far end of the tank can also influence the flow in the interior (Swan 1990; Swan & Sleath 1990). Thus, rather than using a constant wave field, it is preferable to use wave groups, as van den Bremer *et al.* (2019) did, and thus to suitably modify the calculation of the Eulerian return flow, a flow that is driven by radiation stress gradients also associated with the mean set down of the group (Longuet-Higgins & Stewart 1962).

Focusing on the case where the group is long relative to the depth, van den Bremer *et al.* (2019) express the leading-order Stokes drift and Eulerian return flow in terms of group properties, enabling them to convert measured free surface time series into predictions of the Lagrangian mean flow, expressed in terms of net displacements due to the passage of the group. As an aside, one interesting result of considering groups rather than steady waves is that the Stokes drift velocity is divergent. It also has a non-zero vertical component proportional to the rate of change with time of the group envelope amplitude, behaviour not found for steady wave fields, and which is considered contentious by McWilliams, Restrepo & Lane (2004).

Lagrangian particle tracks measured by van den Bremer *et al.* (2019) show that displacements in both horizontal and vertical directions as the group passes through the measurement section can be predicted using Stokes' theory. However, the most significant aspect of this work is the degree to which theory and particle tracking derived net motions match, despite the challenges of separating the wheat (net displacements due to wave passage that ranged from a few millimetres to several centimetres) from the chaff (seiching in the tank, eddies and vertical motions of the particles due to their small buoyancy). Thus, 172 years after Stokes' seminal work, clear confirmation of his result for net motions under irrotational water waves has at last been given!

3. Future

The results in van den Bremer *et al.* (2019) may not be the last word on this topic. In 1802 Gerstner used the Lagrangian momentum equations to derive an exact solution to the inviscid water wave problem valid up to and including the deformed free surface (Gerstner 1802). These waves, known as Gerstner waves, closely resemble finite-amplitude Stokes waves, with the principal (and likely only observable) difference being that particle orbits are closed so that there is a rotational Eulerian mean flow that exactly cancels the lowest-order Stokes drift (Kinsman 1965).

The problem with Gerstner's surface waves is that they cannot be generated by conservative forces (Lamb 1932), and so should not be expected to arise in nature. However, there is evidence that the Stokes drift cancellation as described by Gerstner has been observed in the lab (Monismith *et al.* 2007). The most striking example of this behaviour is the observation of mean Lagrangian flows under waves in the ocean reported by Smith (2006): as wave groups passed by his instruments, the mean Lagrangian velocity beneath them did not measurably change, exactly the behaviour that would be expected for Gerstner waves. However, while shallow water velocity measurements reported by Lentz *et al.* (2008) also showed Stokes drift cancellation, this behaviour can be attributed to the vortex force associated with the product of the planetary vorticity and the Stokes drift, a result first predicted by Ursell (1950) for steady waves in an infinite ocean.

Thus, a central question that remains is: under what circumstances, if any, can Gerstner waves arise? For example, it is possible that since Gerstner waves involve vorticity, surface waves on underlying flows with some vorticity distributions might be Gerstner-like. However, unlike irrotational waves for which a variety of solutions to initial value problems involving wave generation are known (e.g. Stoker 1992), no such solutions have been found for Gerstner waves. In this regard, the recent work of Abrashkin (2019) in which he appears to have derived a solution for unsteady Gerstner waves forced by spatially variable pressure forces clearly deserves further attention.

In conclusion, the experiments of van den Bremer *et al.* (2019) show that the Stokes drift, mean motions predicted nearly two centuries ago, can take the form predicted by irrotational wave theory as Stokes first showed. Nonetheless, the fact that wave behaviour outside of what Stokes predicted may also exist also shows that, despite nearly two centuries of study since Stokes' foundational paper, interesting problems in water wave mechanics remain to be solved both through new theory and innovative experiments like those of van den Bremer *et al.*

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