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Propagation and quenching in a reactive Burgers–Boussinesq system

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Abstract
We investigate the qualitative behaviour of solutions of a Burgers–Boussinesq system—a reaction–diffusion equation coupled via gravity to a Burgers equation—by a combination of numerical, asymptotic and mathematical techniques. Numerical simulations suggest that when the gravity ρ is small the solutions decompose into a travelling wave and an accelerated shock wave moving in opposite directions. There exists $\rho_{\mathrm{cr}1}$ so that, when $\rho > \rho_{\mathrm{cr}1}$, this structure changes drastically, and the solutions become more complicated. The solutions are composed of three elementary pieces: a wave fan, a combustion travelling wave and an accelerating shock, the whole structure travelling in the same direction. There exists $\rho_{\mathrm{cr}2}$ so that when $\rho > \rho_{\mathrm{cr}2}$, the wave fan catches up with the accelerating shock wave and the solution is quenched, no matter how large the support of the initial temperature. We prove that the three building blocks (wave fans, combustion travelling waves and shocks) exist and we construct asymptotic solutions made up of these three elementary pieces. We finally prove, in a mathematically rigorous way, a quenching result irrespective of the size of the region where the temperature was above ignition—a major difference from what happens in advection–reaction–diffusion equations where an incompressible flow is imposed.

1. Introduction

In his pioneering papers [15,16], Kanel made the following discovery: consider an initial value problem

$$T_t = T_{xx} + f(T),$$

on the real line, $x \in \mathbb{R}$, with the initial data $T(0, x) = \chi_{[-L,L]}(x)$ which is the characteristic function of an interval $[-L, L]$. The nonlinearity $f(T)$ is Lipschitz and of the ignition type: there exists $\theta \in (0, 1)$ such that

$$\begin{aligned} f(T) &\equiv 0 \quad \text{on } [0, \theta] \cup \{1\}, \\ f(T) &> 0 \quad \text{on } (\theta, 1), \end{aligned} \tag{1.1}$$

the range of T being the interval $(0, 1)$. Kanel has shown that there exists L_0 so that if the initial 'hot spot' size L satisfies $L < L_0$ then there exists a time $t_0 > 0$ so that $0 < T(t_0, x) \leqslant \theta$ for all $x \in \mathbb{R}$—hence, the reaction ceases at this time and the solution decays to zero as $t \to +\infty$. We say that the solution *quenches* in that case. On the other hand, there exists L_1 so that if $L > L_1$ then the solution develops two travelling fronts, one going to the left, another to the right and $T(t, x) \to 1$ as $t \to +\infty$, uniformly on compact sets. Very recently Zlatos has shown that $L_0 = L_1$ [24].

The mathematical modelling of issues concerning flame propagation or quenching has attracted renewed attention recently, mainly regarding the effect that fluid flow has on the behaviour discovered by Kanel: see, for instance, the direct simulations and formal asymptotic analysis in [12–14, 21]. A number of mathematically rigorous results generalizing Kanel's results to ignition-type reactions in the presence of a fluid flow is also available. Here is a typical example of a result of this kind: suppose that $T(t, x, y)$ solves an advection–reaction–diffusion equation

$$\begin{aligned} &T_t + u(x, y) \cdot \nabla T - \Delta T = f(T), \\ &\partial_\nu T = 0, \qquad \text{for } (x, y) \in \mathbb{R} \times \partial\Omega, \\ &T(0, x, y) = T_0(x, y), \end{aligned} \tag{1.2}$$

in a cylinder $\Sigma = \{(x, y) \in \mathbb{R} \times \Omega\}$ where $\Omega \subset \mathbb{R}^n$ is bounded and where f is a smooth ignition-type source term as in (1.1). Assume for simplicity that the initial datum $T_0(x, y) = \chi_{[-L,L]}(x)$ depends only on the variable x, as in Kanel's problem. Then there exists a constant $L_0(u, f) > 0$ such that if $L < L_0(u, f)$, then $T(t, x, y)$ becomes uniformly smaller than θ in finite time. This is an example of finite time quenching. There also exists a constant $L_1(u, f) \geqslant L_0(u, f)$ such that if $L > L_1$, then $T(t, x, y) \to 1$ as $t \to +\infty$, uniformly on compact sets in $(x, y) \in \Sigma$. It is not known whether $L_0 = L_1$ when $u \not\equiv 0$. The main interest in these problems is in estimating the dependence of the quenching length L_0 on the amplitude and geometry of the flow $u(x, y)$. In particular, precise results are known in advection–reaction–diffusion equations when a strong incompressible flow is imposed: see, for instance, [5, 17] for quenching by a strong shear flow and [11] for quenching by a strong cellular flow. In both cases the critical size L_0 of an initially 'hot' region that can be quenched by the flow grows with the flow amplitude A, albeit at a rate depending on the flow geometry—$L_0 \sim CA$ for generic shear flows and $L_0 \sim CA^{1/4}$ in cellular flows. The increase in L_0 is due to improved mixing by the incompressible flow.

The goal of this paper is to investigate what happens when the fluid flow is no longer imposed, but rather obeys a hydrodynamic equation. What we wish to understand in this study is the following: what are the quenching rules when reaction–diffusion is coupled to hydrodynamics? In particular, is quenching still a matter of the size of the zone where the temperature exceeds the ignition temperature? Such an investigation was initiated for reaction–diffusion equations coupled to incompressible hydrodynamics in the Boussinesq approximation in the spirit of [1, 2, 4, 6, 7, 9, 10, 18–20, 22, 23]. This was done in [8] for a bounded domain with Dirichlet boundary conditions. The cases of Neumann boundary conditions and of unbounded domains are still under investigation.

In this paper we are interested in the effects of a compressible flow on the quenching phenomenon, a subject that seems not to have been as yet addressed in the mathematical literature. Studying this problem for the full compressible reactive Navier–Stokes problem is beyond our reach at the moment. We therefore investigate a drastically simplified model of a Boussinesq system, where the Navier–Stokes equation is replaced by the one-dimensional Burgers equation coupled to a temperature equation by a gravity force:

$$\begin{aligned} T_t - T_{xx} + uT_x &= f(T), \\ u_t - \nu u_{xx} + uu_x &= \rho T. \end{aligned} \tag{1.3}$$

Here $T(t, x)$ is the temperature and $u(t, x)$ the velocity of the fluid. The reaction term $f(T)$ satisfies the assumptions (1.1), and moreover we require that

$$f'(1) < 0. \tag{1.4}$$

The coefficient $\rho > 0$ represents the gravity and the coefficient $\nu \geqslant 0$ the kinematic viscosity.

System (1.3) has another physical interpretation for a one-dimensional system of discrete excitable particles. The particles are mobile and inertial, can mix by diffusion and can exchange momentum. A particle converts to an excited state if there is a high enough concentration of excited particles in its vicinity. Excited (and only excited) particles feel the presence of a force; all other properties of excited and non-excited particles are identical. Under some conditions the initially small excited region grows with time, as the excited particles spread around by diffusion and excite their neighbours. In addition, the driving force accelerates the excited particles and speeds up the process. However, if the force is too strong, the particles quickly spread around over a large area, their concentration drops below the threshold limit and transition of new particles to the excited state terminates. Even though the particles excited earlier are still present in the system, we call this event extinction or quenching.

The continuum representation of the problem is system (1.3); $T(x)$ is the fraction of excited particles ($0 \leqslant T \leqslant 1$) and $u(x)$ is the locally averaged velocity. The system of Burgers and advection–reaction–diffusion equations describes the transport of momentum and the transport of excited species, ρ is the driving force and $f(T)$ is the reaction term which accounts for the transition of particles from the non-excited to the excited state.

The outcome of this study is that the qualitative behaviour of the reactive system under investigation is markedly different from that of a reactive system in a passive incompressible flow. In particular, if the parameter ρ is sufficiently large, then quenching may occur irrespective of the size of the set where $T(0, x) \geqslant \theta$. We note that, as in the case of an imposed flow, the temperature goes to zero as $t \to +\infty$ as soon as it drops below θ everywhere, provided that the flow is decaying at infinity.

The paper is organized as follows. In section 2, we carry out a numerical investigation of system (1.3). The subsequent mathematical analysis is based on these numerical computations: it would be very difficult for us to find the correct qualitative behaviour of the solutions without them. A feature of the numerical simulations is that they are not very sensitive to the viscosity

ν; therefore we set $\nu = 0$ in the rest of the paper. Namely, we concentrate on the system

$$\begin{aligned} &T_t - T_{xx} + uT_x = f(T), \\ &u_t + uu_x = \rho T. \end{aligned} \tag{1.5}$$

Briefly, the numerical simulations show the following picture for solutions with a sufficiently large set where initially $T(0, x) = 1$: there exists a critical value $\rho_{\mathrm{cr1}} > 0$ so that for $\rho \in (0, \rho_{\mathrm{cr}_1})$ such solutions develop a left-going travelling wave which moves with a constant speed. On the right boundary they have a shock wave accelerating in time to the right. When $\rho > \rho_{\mathrm{cr1}}$ the gravity does not permit a left-going travelling wave to develop. Instead, the solution is made up of three elementary building blocks pieced together: a wave fan in the back, followed by a travelling wave and finally an accelerated shock. This whole structure propagates to the right. Finally, there exists a second critical threshold ρ_{cr2} so that for $\rho > \rho_{\mathrm{cr2}}$ the wave fan catches up with the shock, no matter how large the support of $T(0, x)$ and the reaction stops from this time onwards: it is quenched. This seems to be the main difference between active compressible and passive incompressible flows—when a compressible flow is sufficiently strong, all solutions are quenched, regardless of their initial size. It would be very interesting to investigate this phenomenon in more realistic reactive flow models. The numerical results are presented in section 2.

The next three sections are devoted to the mathematical study of the elementary solutions of system (1.5). In section 3 we establish the existence of wave fans, that is, self-similar solutions of (1.5) with $f(T) \equiv 0$. Suppressing the diffusivity in the temperature equation yields explicit wave fans; these can be seen in section 2. Establishing their existence when the temperature diffusivity is positive turns out to be a surprisingly difficult task: the whole programme is carried out in full detail in section 3.

In section 4 we prove the existence and study the qualitative properties of the combustion travelling waves. We expect similar results to hold for $\nu \neq 0$ as well, although the equations are different.

In section 5, we construct asymptotic solutions to the full system (1.5). We point out that what we prove here is that the ‘solutions’ that we have constructed only satisfy the system up to an error that is $O(t^{-1})$, and not that they are true solutions to the system. However, they are constructed by matched asymptotic expansions, and we believe that it is possible to construct true solutions to (1.5) on the basis of these asymptotic solutions. The latter investigation is not, however, in the scope of this paper and will be carried out elsewhere. The constructed solutions fully account for what we saw in the numerics of section 2.

In section 6 we prove a quenching result. For ρ sufficiently large there is numerical evidence from section 2 that the formal solution constructed in section 5 will quench. We prove rigorously that taking the asymptotic solution as the initial data, the temperature will drop below the ignition temperature in a finite time that we are able to estimate.

2. Numerical simulations

In this section we numerically investigate system (1.3) with $\nu = 1$; in other words, we consider

$$\begin{aligned} &T_t - T_{xx} + uT_x = f(T), \\ &u_t - u_{xx} + uu_x = \rho T. \end{aligned} \tag{2.1}$$

To be specific, we choose the following piecewise linear reaction rate:

$$\begin{aligned} &f(T) = \frac{\theta(1-T)}{(1-\theta)^2}, \qquad \theta < T < 1; \\ &f(T) = 0, \qquad \text{otherwise}, \end{aligned} \tag{2.2}$$

where θ is the ignition temperature. We set $\theta = 1/2$ in most simulations below. Our analysis does not depend on this particular choice of reaction—it is important only that the ignition cut-off exists and the reaction vanishes for $T > 1$ and $T < 0$. In the absence of advection ($\rho = 0$, $u = 0$) the reaction–diffusion equation for temperature with reaction rate (2.2) has a travelling wave solution moving with unit speed,

$$\begin{aligned} T^*(x,t) &= 1 - (1-\theta)\mathrm{e}^{\frac{1-\theta}{\theta}(x-t)}, \qquad x < t; \\ T^*(x,t) &= \theta \mathrm{e}^{-(x-t)}, \qquad x > t. \end{aligned} \tag{2.3}$$

Here the location $x = t$ corresponds to the point where temperature equals the ignition threshold value θ.

As an initial condition for T we use a smooth localized distribution of width h and steepness k,

$$T(x,0) = \tfrac{1}{2}\tanh[k(x+h/2)] - \tfrac{1}{2}\tanh[k(x-h/2)]. \tag{2.4}$$

We use mostly $k = 0.75$, which corresponds to a profile with a more gradual slope compared with the travelling wave (2.3), but that overall better matches the theoretical solution. In a few cases we use a steeper interface with $k = 1.5$. The initial velocity is zero in the majority of our simulations with the exception of the case where we study quenching by a prescribed initial velocity.

We solve system (2.1) with reaction rate (2.2) numerically, using fourth-order central differences discretization and third-order Adams–Bashforth integration in time. The simulations are done at resolution $\Delta x_0 = 1/16$ with an adaptive patch of a highly resolved mesh ($\Delta x_1 = \Delta x_0/64$) in the vicinity of the shock.

2.1. *Quenching without gravity*

We first consider the simple case $\rho = 0$. If initially $u = 0$, then the problem is reduced to a single scalar reaction–diffusion equation. If $u(0,x) \neq 0$ then the equation for $T(t,x)$ is driven by the flow $u(t,x)$ that satisfies the viscous Burgers equation. Much of the material of this section is well known and has received extensive mathematical treatment. It is, however, instructive to put it here: it will serve as a comparison basis for more sophisticated effects appearing when the gravity is present.

2.1.1. Zero velocity—Kanel's length. In the absence of gravity and initial velocity, initial data (2.4) evolve in agreement with the theoretical predictions [16, 24]. Namely, initially sufficiently large hot regions, $h > h^*$, develop two outward propagating travelling waves of the form (2.3), while initially small hot regions, $h < h^*$, are extinguished.

The critical length of the initially hot region h^*—Kanel's length—can be determined numerically. For the given θ, the critical width depends on the steepness of the interface. We show in figure 1 the initial profiles corresponding to Kanel's lengths $h^* = 1.50$, for $k = 0.75$, and $h^* = 1.00$, for $k = 1.50$ (in both cases $\theta = 0.5$). Note that for $k = 0.75$, the maximum in the profile only slightly exceeds the ignition temperature, $T_{\max} = 0.51$, while for steeper $k = 1.50$ the maximum is higher, $T_{\max} = 0.64$. The solution with critical width h^* is unstable. The solution with h just below critical decays with time, while the one just above critical grows and eventually develops into a pair of outward propagating fronts.

2.1.2. Stationary compression. When $\rho = 0$, the Burgers equation for $u(t,x)$ is uncoupled from the advection–reaction–diffusion equation for $T(t,x)$ and has a stationary solution,

$$u^*(x) = -U \tanh\frac{Ux}{2}, \tag{2.5}$$

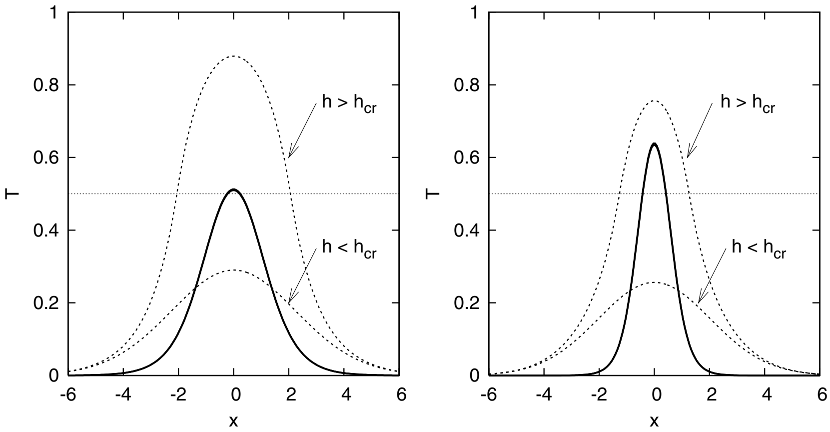

Figure 1. Quenching of a small hot spot by diffusion alone. Thick solid lines represent the initial profiles with $h = h^* = 1.50$ and $k = 0.75$ (left) and with $h = h^* = 1.00$ and $k = 1.50$ (right). Thin dashed lines correspond to the solutions at time $t = 2$ with the initial widths just below and just above critical ($\Delta h = \pm 0.01$). The dotted line is the threshold value $\theta = 0.5$.

where U is the absolute velocity at $x = \pm\infty$. The stationary solution represents compression, in the sense that $u^* < 0$ for $x > 0$ and $u^* > 0$ for $x < 0$. Although the velocity remains unaffected by temperature, it changes the temperature distribution and can facilitate quenching.

In this exercise, we study the quenching of initial data with different h by stationary velocity (2.5) with different intensities, U. Both $u(x)$ and $T(x)$ are aligned at $x = 0$, so that the compression is symmetric with respect to the centre of the hot spot.

We find that there exists a critical velocity, U_{cr}, that quenches any initial distribution of temperature, no matter how wide it is. The independence of the initial size is not surprising: if the initial distribution is wide, both fronts are located in the region where the velocity is nearly constant, $u(x) \approx \pm U$. The fronts are advected towards the centre with the speed $V \approx U - 1$ and eventually reach the centre. Near the centre the absolute velocity is lower, and the decrease in temperature due to compression might or might not be balanced by reaction. We found that if $U > U_{\mathrm{cr}} = 1.40$ the maximum of the temperature drops below the ignition threshold, that is, the hot spot completely extinguishes. For $1 < U < U_{\mathrm{cr}}$, the solution converges to a stationary profile $\tilde{T}(x)$. The shape of the stationary profile depends on the compression velocity; the profile is wider for lower U (see figure 2, left panel). When $U < 1$, the hot spot grows outwards.

The above discussion applies to initially wide profiles, $h \gg 1$, or more specifically, to profiles wider than $\tilde{T}(x)$. Narrower profiles converge to a narrower stationary solution. We performed a test where we kept the same $U = 1.3$ and $k = 0.75$ and varied h. For $h > 2.8$, all solutions converge to $\tilde{T}(x)$. For $1.66 > h > 2.7$, the solutions converge to different profiles (see figure 2, right panel). If $h < 1.65$, the solutions become extinct; recall that the Kanel size for this steepness at $U = 0$ is $h = 1.50$.

2.1.3. Non-stationary stretching. If compression facilitates quenching, stretching facilitates burning: it increases the area where T is above the reaction threshold. We consider the no-gravity case, $\rho = 0$, with the initial velocity profile $u(x, 0) = -u^*(x)$, where $u^*(x)$ is given by (2.5). We found that the critical size of the hot spot decreases with the stretching velocity U (see table below). Note that the stretching solution evolves even in the absence of gravity (figure 3).

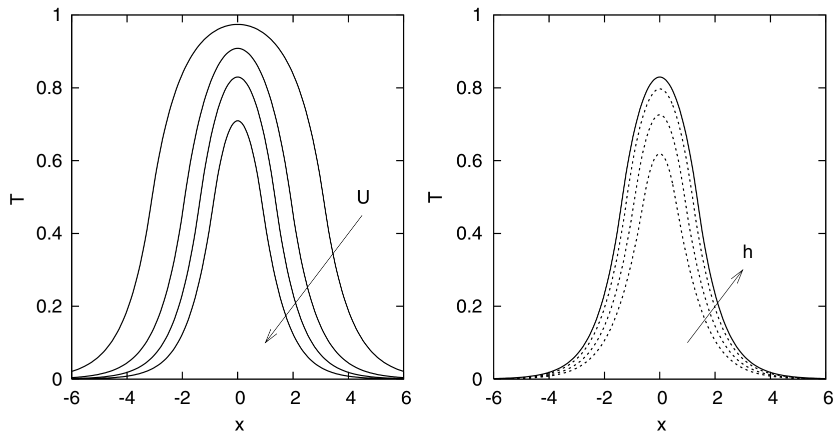

Figure 2. Stationary profiles of temperature in prescribed compressing velocity. In the left panel, the profiles are shown for compression $U = 1.1, 1.2, 1.3$, and 1.39; initial width of hot spot $h \gg 1$. In the right panel, the profiles are shown for different initial widths, $h > 2.8$ (——) and $h = 2.5$, 2.0 and 1.66 (- - - -), for $U = 1.3$.

U	$h_{\rm cr}$ ($k = 0.75$)	$h_{\rm cr}$ ($k = 1.50$)
0	1.50	1.00
1	1.48	0.94
2	1.48	0.87
4	1.47	0.80

Finally, we point out that the above statement, ‘compression facilitates quenching while stretching facilitates burning’, may sound counter-intuitive from the point of view of gas thermodynamics. We recall that in our model T has a physical meaning of the fraction of ‘hot particles’. A better model would include a reaction rate that depends on the density of the hot particles rather than their fraction. However implementing such a model involves introducing the concepts of density and pressure and an equation of state.

2.2. Quenching by the gravity force

Here we study the growth of an initially small hot spot with constant non-zero gravity in an initially quiescent fluid, $u(x, 0) = 0$. In the tested range, $1 \leqslant \rho \leqslant 8$, the gravity has no influence on the critical size of the initial hot spot. All solutions with h higher than Kanel's length grow, at least initially. For hot spots with the initial size above Kanel's length, the effect of the initial size is noticeable only at times $t \sim 1$. For $t \gg 1$, the difference in the initial size shows only as an offset in initial time. The growth of hot spots at later times depends only on the gravity ρ. And, depending on ρ, we observe three kinds of solutions.

When the gravity is small the left boundary of the hot spot moves to the left with a constant speed (see figure 4). The solution at large negative x resembles a travelling wave. The right boundary is extremely sharp and moves to the right accelerating. We will from now on refer to the right boundary of the hot region as ‘the shock’—because that is what it is when the viscous term u_{xx} is not present. As both boundaries move in opposite directions the hot spots at small gravities never quench.

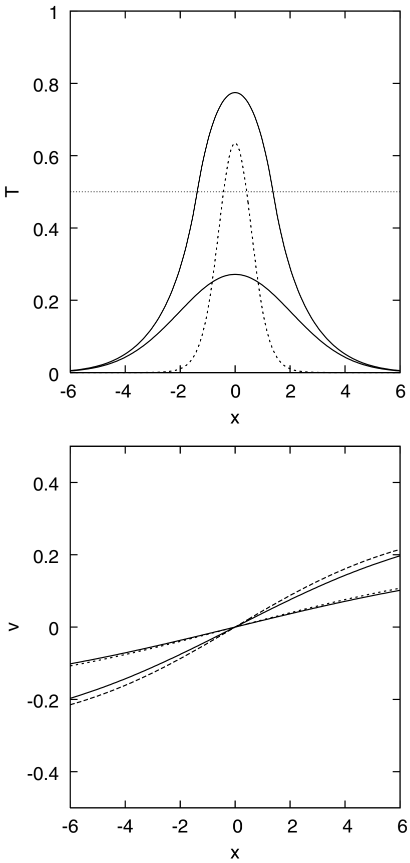

Figure 3. Left: the profiles of temperature and velocity at time $t = 2$ for stretching $U = 0.2$ and $U = 0.3$; the initial profile has the width $h = 1.00$ with $k = 1.50$. Right: the profiles of T and velocity at time $t = 1$ for stretching $U = 4.2$ and $U = 4.3$; the initial profile has width $h = 0.80$ with $k = 1.50$. The initial profiles are shown with dashed lines.

As the gravity is increased, the solution becomes more complicated (figure 5). The right boundary is still sharp, in the form of a shock, while the left boundary is stretched in the form of a long tail of partially burned fluid with the temperature below the ignition threshold. This part of the solution will be referred to as 'the ramp' or the 'wave fan'. Its analogue in the inviscid case is a rarefaction wave.

In the region between the ramp and the shock (or, for lower gravities, between two opposite moving fronts) the temperature mostly exceeds the ignition threshold. This is the only region where the reaction occurs; we refer to it as the 'combustion wave' or simply 'the wave'.

Even when both boundaries of the combustion wave move to the right, their dynamics are different and depend on the gravity. For moderate gravities, the shock moves faster than the right border of the ramp; such combustion waves do not quench. For high gravities, the ramp eventually catches up with the shock and the hot spot quenches. This kind of quenching can occur at times significantly exceeding the laminar front self-crossing time and after the hot region reaches sizes significantly exceeding Kanel's length. The object of the following two subsections is to get some intuition on how it happens as well as to obtain formally some orders of magnitude.

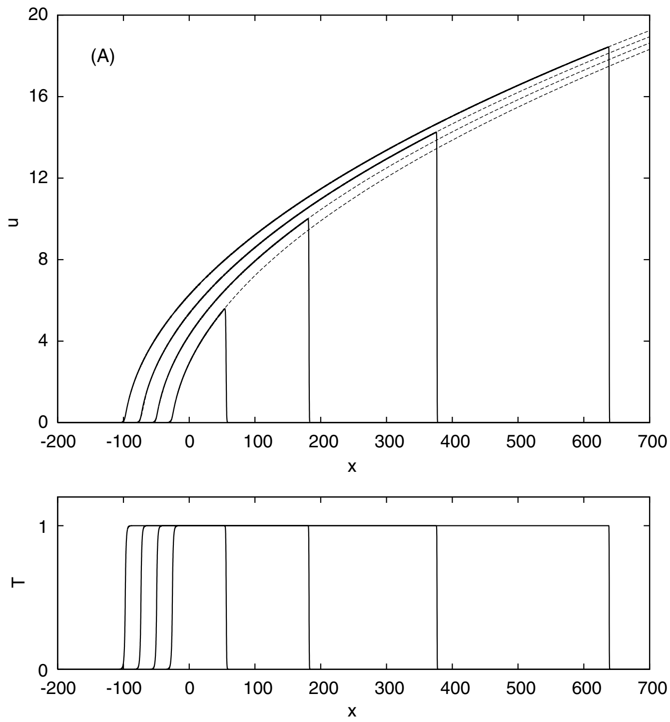

Figure 4. Profiles of velocity and concentration at times $t = 32, 64, 96, 128$ for $\rho = 1/4$ and $\theta = 0.5$. Profiles of velocity (——) are compared with velocity as suggested by equation (2.11) with $c = -0.75$ (- - - -).

2.2.1. Ramp–wave–shock structure of the velocity profile. The solution shown in figure 5 consists of three parts: a cold stationary fluid ahead of the shock, the combustion wave with T above the ignition temperature and the ramp where $T < \theta$. Initially the hot spot is located at $x = 0$. We denote by x_f the location of the shock and by x_b the location of the transition point between the ramp and the combustion wave. (The subscript 'f' refers to the 'front' and the subscript 'b' refers to the 'back' of the combustion wave.) Similarly, we denote the local velocities at corresponding points as $u_f \equiv u(x_f)$ and $u_b \equiv u(x_b)$ and the phase velocities as $v_f \equiv \dot{x}_f$ and $v_b \equiv \dot{x}_b$.

Below we construct approximate solutions at the ramp, the wave and the shock. Combining them together we find the speed of the shock, v_f, and the growth rate of the ramp, v_b. Comparing v_b and v_f in the next subsection, we estimate the criterion for quenching.

The ramp. In figure 5, both $T(x)$ and $u(x)$ appear to be linear in the ramp. In comparison with advection, dissipation effects are negligible on the scale of the ramp. Indeed, if L is the length of the ramp and U is the typical velocity in the ramp, then $T_{xx} \sim 1/L^2 \ll uT_x \sim U/L$ and $u_{xx} \sim U/L^2 \ll uu_x \sim U^2/L$ for large L and U. Neglecting dissipation and taking into account that $f(T) = 0$ in the ramp, we may find approximate solutions to (2.1) as

$$T(t, x) = \frac{2x}{\rho t^2}, \quad u(t, x) = \frac{2x}{t}, \qquad 0 < x < x_b. \tag{2.6}$$

In the ramp region, equations (2.6) agree with numerical simulations: see figure 6.

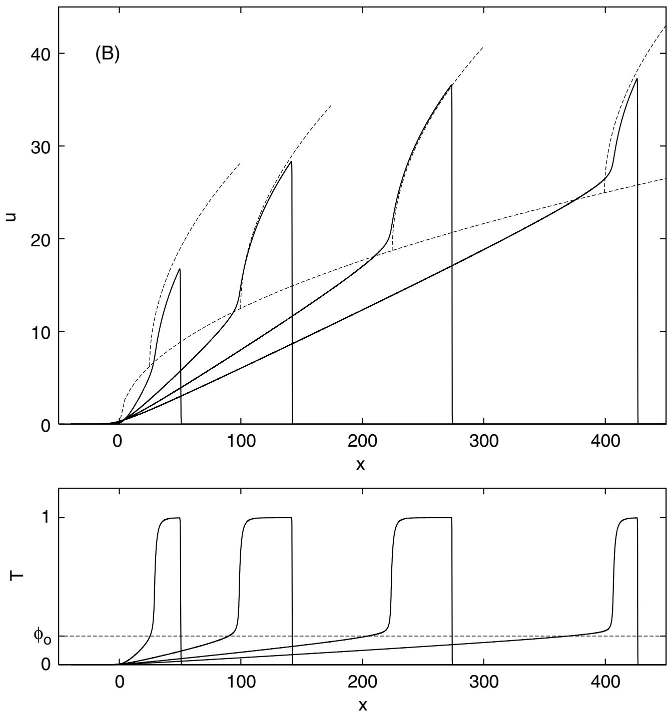

Figure 5. Profiles of velocity and concentration at times $t = 8, 16, 24, 32$ for $\rho = 4$ and $\theta = 0.5$. The dashed lines in the top panel show evolution $u_f(x_f)$ as suggested by equation (2.7) and velocity given by equation (2.10) at the times matching the simulation data. In the bottom panel, the dashed line shows $\phi_0 = 0.195$.

We return to figure 5. The transition between the ramp and the wave occurs at the same value of $T = \phi_0 < \theta$ which does not depend on time. Assuming that we know ϕ_0 we can estimate the location of the transition to the ramp x_b and corresponding velocities,

$$x_b = \tfrac{1}{2}\rho\phi_0 t^2, \qquad u_b = \rho\phi_0 t, \qquad v_b = \rho\phi_0 t. \tag{2.7}$$

In this case, $u_b = v_b$, but in general it does not have to be this way. This aspect will be investigated further in section 5.

The wave. Consider now the velocity in the wave part of the solution, $x_b < x < x_f$. In figure 5, the solutions at different times appear to have the same form $u_2(x)$, only shifted in time. We can assume that the origin is located at x_b. If we substitute velocity in the form $u(t, x) = u_2(x - x_b) + u_b$, where x_b and u_b are some functions of time, into (2.1), with $T = 1$—we of course do not have $T = 1$ everywhere, but this will at least give us some order of magnitude, we obtain

$$\left[(u_b - x_b')u_2' + u_b'\right] + u_2 u_2' = u_2'' + \rho. \tag{2.8}$$

We see from here that u_2 is time-independent only if u_b is linear in time and $x_b' = u_b$. And luckily x_b and u_b given by (2.7) satisfy this condition. The other possible combination is

$$x_b = ct, \qquad u_b = c, \qquad v_b = c, \tag{2.9}$$

which corresponds to a solution shifting with some constant velocity c.

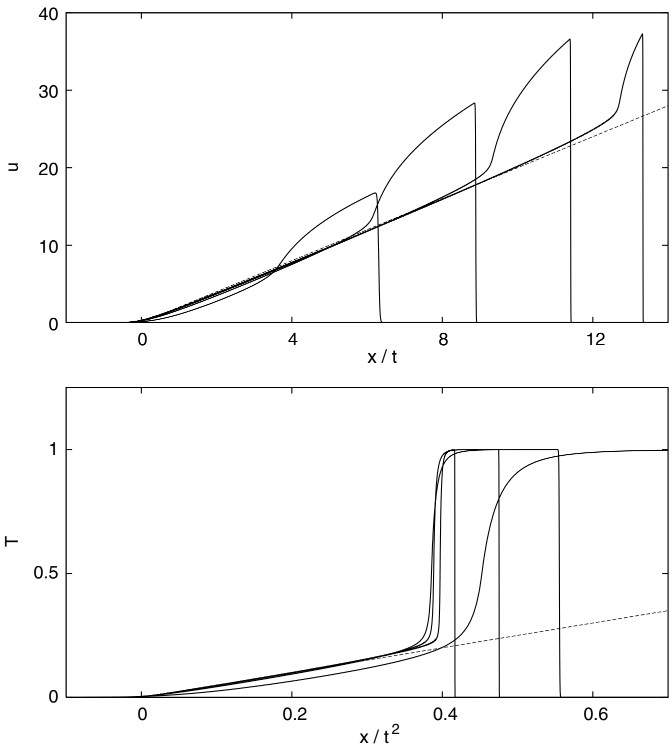

Figure 6. Rescaled profiles of temperature and velocity from figure 5 zoomed on the ramp region. The dashed lines correspond to equation (2.6).

When x_b and u_b are given by (2.7), the expression in square brackets in (2.8) is equal to $\rho\phi_0$ and (2.8) can be solved. Neglecting the dissipation terms (the same dimensional argument as for the ramp can be applied here), we obtain $u_2(x) = \sqrt{2\rho(1-\phi_0)x}$. The velocity profile in the wave is thus, approximately,

$$u(t,x) = \rho\phi_0 t + \sqrt{2\rho(1-\phi_0)(x - \tfrac{1}{2}\rho\phi_0 t^2)}, \quad x_b < x < x_f. \tag{2.10}$$

Similarly, when x_b and u_b are given by (2.9), the velocity in the wave is

$$u(t,x) = c + \sqrt{2\rho(x-ct)}, \qquad x_b < x < x_f. \tag{2.11}$$

In the numerical simulations we have examples of both types of solutions. For lower gravities, the left boundary of the combustion wave shifts to the left with constant speed; the numerical solution shown in figure 4 agrees with (2.11). For higher gravities, the left side of the combustion wave shifts to the right accelerating; in figure 5 the numerical solution is compared with (2.10). In both cases, the numerical data are fitted with one unknown parameter—the shift velocity c in the first case and temperature at the transition to the ramp ϕ_0 in the second case. Both parameters c and ϕ_0 are functions of gravity as seen in figure 7.

Figure 7 also shows numerical evidence of the existence of a first critical parameter, ρ_{cr1}, such that

- for $\rho < \rho_{cr1}$, the solution is of the constant shift type, with $c < 0$. When gravity approaches zero the left boundary of the combustion wave moves to the left with laminar speed, $|c| = 1$. For very small gravities $\rho \ll 1$, the speed is $c = \rho - 1$;

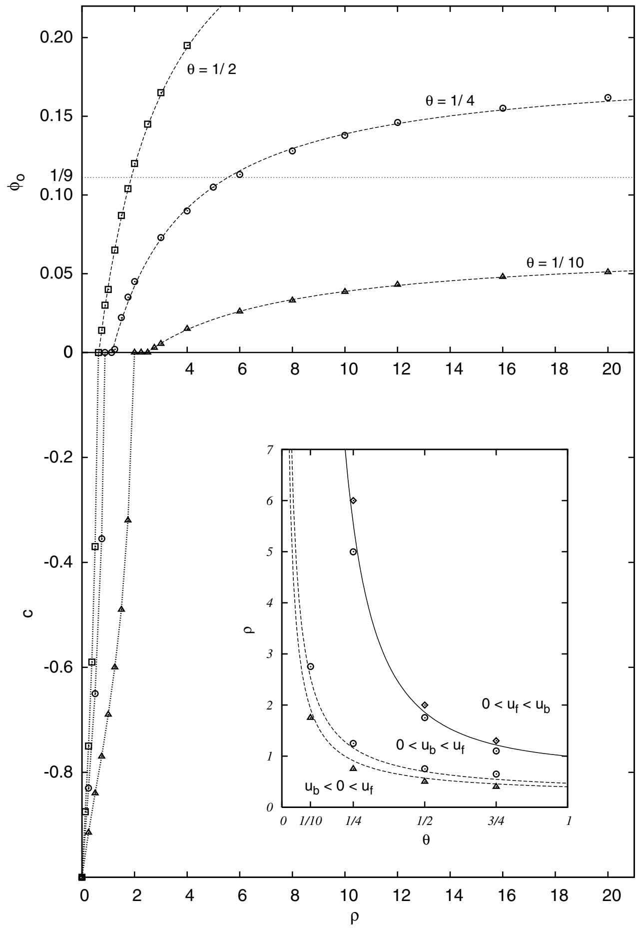

Figure 7. Parameters ϕ_0 and c as functions of ρ. Dashed lines correspond to the fits of the form $\phi_0(\rho) = a + b/(c - \rho)$. Symbols in the inset represent numerical solutions in different regimes while the lines are schematic borders between regimes.

- for gravities $\rho > \rho_{\mathrm{cr1}}$ the solution is of the accelerated shift type, with $\phi_0 > 0$. At $\rho = \rho_{\mathrm{cr1}}$ both parameters c and ϕ_0 are equal to zero, and the solution in the wave is stationary, $u(t, x) = \sqrt{2\rho x}$, $T(t, x) = 1$.

The shock. The front ahead of the combustion wave is driven by a Burgers shock, the mechanism for which is much stronger than the front propagation due to reaction. Moreover, at high shock speeds, the shock is extremely narrow; the reaction region is narrow as well, and the role of the reaction is reduced. In the vicinity of the shock we can neglect the reaction term in the temperature equation; then the solution is the classical Burgers shock of strength u_{f} propagating with speed $v_{\mathrm{f}} = u_{\mathrm{f}}/2$.

On the scale of the problem, the shock can be considered as a discontinuity located at x_{f} and moving with the speed $v_{\mathrm{f}} = \dot{x}_{\mathrm{f}}$. Then, according to (2.10), the location of the shock is given by the following differential equation:

$$\frac{\mathrm{d}x_{\mathrm{f}}}{\mathrm{d}t} = \frac{1}{2}u(x_{\mathrm{f}}, t) = \frac{1}{2}\left[\rho\phi_0 t + \sqrt{2\rho(1-\phi_0)(x_{\mathrm{f}} - \frac{1}{2}\rho\phi_0 t^2)}\right]. \qquad (2.12)$$

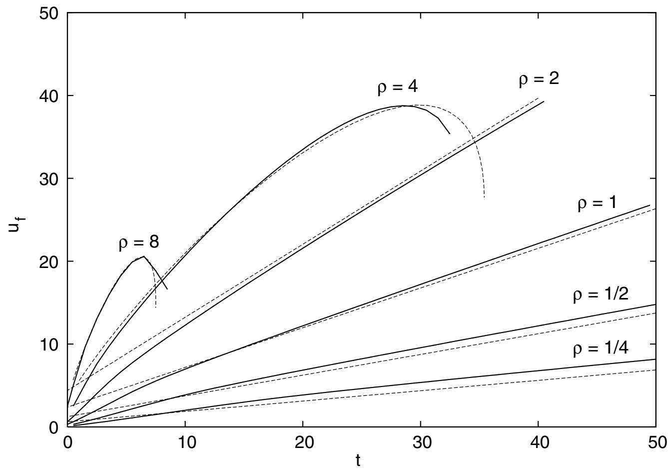

Figure 8. Velocity at the shock as a function of time for different gravities. The dashed lines show velocity at the shock $u_\mathrm{f} = \frac{1}{2}\rho t$ for $\rho = 1/4$ and $\rho = 1/2$, velocity $u_\mathrm{f} = \frac{1}{2}\rho t(1 - \phi_0)$ for $\rho = 1$ and $\rho = 2$, and velocity given by solving (2.12) for $\rho = 4$ and $\rho = 8$.

We compare the expected u_f with the results on numerical simulations in figure 8. The agreement between the analytical approximation for u_f and the numerical simulations, while not perfect, shows the correct qualitative behaviour of the analytical prediction. Notice that the decrease in velocity (slowing down of the shock) and abrupt termination of the curves (quenching) are observed only for high gravities. Thus, the numerical simulations suggest the existence of a critical value ρ_{cr2} such that, for $\rho < \rho_{\mathrm{cr2}}$, the shock solution exists for all time. We may also infer that

- for gravities $\rho > \rho_{\mathrm{cr2}}$ the temperature drops below the ignition threshold θ everywhere in space in a finite time. After this reaction ceases and the solution is quenched.

Notice that in the construction of our approximate solution we do not rely on the functional form of reaction rate $f(\phi)$. This is not surprising. The reaction rate is non-zero at only two narrow regions in the vicinity of x_f and x_b. As we discussed earlier, the reaction is negligible in the shock region because of the compression. The only region where the reaction is important is the transition between the ramp and the wave, the point whose behaviour is controlled by ϕ_0 and c. Although the reaction rate does not appear in the discussion, it is implicitly present in the model in the form of empirical parameters ϕ_0 and c.

2.2.2. Burning or quenching? In order to identify the regimes of burning and quenching we compare the speed of the transition point between the ramp and the wave, v_b, and the shock speed, v_f, in the limit $t \to \infty$. When $\rho < \rho_{\mathrm{cr1}}$, the end of the ramp moves with the speed $v_\mathrm{b} = c < 0$, and the velocity at the shock u_f can be approximated as $u_\mathrm{f} \approx \frac{1}{2}\rho t$. The shock speed, $v_\mathrm{f} = \frac{1}{2}u_\mathrm{f}$, is positive. The two sides of the combustion wave move in opposite directions and quenching never happens.

When $\rho > \rho_{\mathrm{cr1}}$, the ramp extends with the speed $v_\mathrm{b} = \rho t \phi_0$. The distance between the end of the ramp and the shock is then $x_\mathrm{f} - x_\mathrm{b} := y_\mathrm{f}$, and an equation for y_f is, from (2.12):

$$\dot{y}_\mathrm{f} = \tfrac{1}{2}\left(\sqrt{2\rho(1 - \phi_0)y_\mathrm{f}} - \rho\phi_0 t\right), \tag{2.13}$$

where ϕ_0 is the value of the temperature at the transition point. Note first that equation (2.13) has no global in time solution if $\phi_0 > 1/9$. Here is the reason: if we write $y_{\mathrm{f}} = \rho t^2 z$, then we have

$$2z + t\dot{z} = \tfrac{1}{2}[\sqrt{2(1-\phi_0)z} - \phi_0].$$

Changing the time variable $\tau = \ln t$ (for $t \geqslant 1$) this becomes

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}\tau} &= \frac{1}{2}q(\sqrt{z}), \\ q(s) &= \sqrt{2(1-\phi_0)}s - \phi_0 - 4s^2. \end{aligned} \tag{2.14}$$

An elementary study of $q(s)$ for $s > 0$ reveals that

- for $\phi_0 < 1/9$, there is $z_{\phi_0} > 0$ such that $q > 0$ on $[0, z_{\phi_0})$ and $q < 0$ on $(z_{\phi_0}, +\infty)$;
- for $\phi_0 > 1/9$ we have $q(s) < 0$ for $s > 0$.

In order for (2.14) to have a global solution we need z to be non-negative and $q(s)$ not to be uniformly negative for all $s \geqslant 0$. This is true only as long as $2(1-\phi_0) - 16\phi_0 > 0$ or $\phi_0 < 1/9$. Therefore, (2.14) has no global in time solutions and quenching occurs if $\phi_0(\rho) > 1/9$. The transition temperature ϕ_0 increases with the gravity ρ and for large ρ exceeds the critical value $1/9$. More precisely, as we show in theorem 4.1, ϕ_0 approaches the ignition threshold θ as ρ tends to infinity. Therefore, quenching happens for a sufficiently large ρ provided that $\theta > 1/9$.

Dependence on the ignition threshold θ. To illustrate the above quenching/propagation analysis, we show in figure 7 the dependence of $\phi_0(\rho)$ and $c(\rho)$ for three values of the ignition threshold θ in reaction rate (2.2).

As θ decreases, a stronger force (larger ρ) is needed to reach the quenching value $\phi_0 = 1/9$. Since $\phi_0 < \theta$, reactions with $\theta < 1/9$ cannot be quenched by any force. The dashed lines in figure 7 are functions of the form $\phi_0(\rho) = a + b/(c - \rho)$ fitted to the data. For the case of $\theta = 1/10$ the fit is bounded by asymptote $a = 0.0677 < 1/9$, which suggests that the quenching is impossible for this threshold value.

The value ρ_{cr1} with $\phi_0 = 0$ corresponds to the transition to a stationary left front; further reduction of ρ results in the opposite propagation of fronts, characterized by the speed of the left-going front $c(\rho)$ rather than by the value of ϕ_0. The speed $c < 0$ decreases with ρ down to $c = -1$ at $\rho = 0$, which corresponds to the speed of the undisturbed reaction–diffusion shock. It is interesting that the stationary left shock, when both $c(\rho) = 0$ and $\phi_0 = 0$, is observed for a range of ρ, rather than for a single value ρ_{cr1}.

To summarize, with the increase in forcing (see the inset in figure 7) the system with a particular reaction rate can exhibit the following regimes: (i) two shocks moving in opposite directions, $u_{\mathrm{b}} < 0 < u_{\mathrm{f}}$, no quenching; (ii) stationary left shock and right-propagating right shock, $u_{\mathrm{b}} = 0 < u_{\mathrm{f}}$, no quenching; (iii) both shocks move to right, first shock faster, $0 < u_{\mathrm{b}} < u_{\mathrm{f}}$, no quenching; (iv) both shocks move to right, second shock faster, $0 < u_{\mathrm{f}} < u_{\mathrm{b}}$, quenching.

In the rest of the paper we construct the building blocks of the observed numerical solutions—the ramp (the wave fan), the combustion wave and the shock—and show that they may be used to build an asymptotic solution. Moreover, we show that for sufficiently large gravity, a solution that has reached the wave fan–combustion wave–shock structure will ultimately quench.

3. Wave fans

In this section we search for non-reactive solutions of (1.5), that is, solutions of

$$\begin{aligned} T_t - T_{xx} + uT_x &= 0, \\ u_t + uu_x &= \rho T. \end{aligned} \tag{3.1}$$

If we additionally suppress the temperature diffusion, the sought-after solution would be an analogue of a rarefaction wave. The correct self-similarity scaling is

$$T(t,x) = \frac{1}{\rho t^{3/2}}\phi\left(\frac{x}{\sqrt{t}}\right), \qquad u(t,x) = \frac{1}{t^{1/2}}\psi\left(\frac{x}{\sqrt{t}}\right). \tag{3.2}$$

Setting the self-similar variable $\eta = x/\sqrt{t}$, we obtain the following system satisfied by $\phi(\eta)$ and $\psi(\eta)$:

$$\begin{aligned} -\phi'' + \left(\psi - \frac{\eta}{2}\right)\phi' - \frac{3\phi}{2} &= 0, \\ \left(\psi - \frac{\eta}{2}\right)\psi' - \frac{\psi}{2} &= \phi. \end{aligned} \tag{3.3}$$

We require ϕ to be zero at $(-\infty)$—remember that, without the heat diffusion, the rarefaction wave is equal to zero for large negative η. In addition, as we will need to match it with a travelling wave at large positive η, the function ϕ should have a bounded derivative on the whole domain. Also, we want ϕ and ψ to be positive. We will see that positivity plus a strong decay at infinity implies integrability plus a global Lipschitz bound.

The equation for the function ψ may be rewritten as

$$\left(\frac{\psi^2 - \eta\psi}{2}\right)' = \phi,$$

which implies the following quadratic equation for ψ:

$$\psi^2 - \eta\psi = 2\int_{-\infty}^{\eta} \phi(\eta')\,\mathrm{d}\eta'. \tag{3.4}$$

Choosing the positive root in the above equation we get

$$\psi(\eta) = \frac{\eta}{2} + \frac{1}{2}\sqrt{\eta^2 + 8\int_{-\infty}^{\eta} \phi(\eta')\,\mathrm{d}\eta'}. \tag{3.5}$$

Using this expression in the first equation in (3.3) leads to the following problem for the function $\phi(\eta)$ which we are now going to investigate:

$$\begin{aligned} &-\phi'' + \frac{1}{2}\left[\eta^2 + 8\int_{-\infty}^{\eta} \phi(\xi)\,\mathrm{d}\xi\right]^{\frac{1}{2}}\phi' - \frac{3\phi}{2} = 0, \\ &\phi > 0, \quad \phi \in L^1(\mathbb{R}_-), \quad \phi' \in L^\infty(\mathbb{R}). \end{aligned} \tag{3.6}$$

First, a definition: we say that ϕ *has a Gaussian decay at* $-\infty$ with exponent λ if $\eta \mapsto \mathrm{e}^{\lambda\eta^2}\phi(\eta)$ has at most polynomial growth as $\eta \to -\infty$. We have the following result for (3.6).

Theorem 3.1. *Equation (3.6) has at least one positive solution $\phi(\eta)$, having Gaussian decay at $-\infty$ with any exponent $\lambda \in (0, 1/4)$. For any such solution, there exists a number $b \in \mathbb{R}$ such that we have in addition, as $\eta \to +\infty$:*

$$\phi(\eta) = 2\eta + b(1 + o(1))\eta^{1/3}, \qquad \phi'(\eta) = 2 + O(\eta^{-2/3}). \tag{3.7}$$

We note here that the Gaussian decay has to be imposed. At this stage we do not know whether there are waves that have no Gaussian decay. On the other hand, there is no need to impose any bound on the growth of ϕ at $+\infty$: the Gaussian decay plus positivity implies a global Lipschitz bound, with a linear growth that will allow matching with another elementary solution.

Let us now explain why imposing a Gaussian decay is relevant. Equation (3.6), linearized around the rest state $\phi_0 = 0$ at $\eta = -\infty$, becomes

$$-\phi'' - \frac{\eta}{2}\phi' - \frac{3\phi}{2} = 0. \tag{3.8}$$

This equation has two integrable solutions:

$$\begin{aligned} h(\eta) &= \left(\frac{\eta^2}{4} - \frac{1}{2}\right)\mathrm{e}^{-\eta^2/4} = \frac{\mathrm{d}^2}{\mathrm{d}\eta^2}(\mathrm{e}^{-\eta^2/4}), \\ k(\eta) &= -\frac{1}{5}\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\left(\mathrm{e}^{-\eta^2/4}\int_0^\eta \mathrm{e}^{\zeta^2/4}\mathrm{d}\zeta\right) \sim -\eta^{-3}. \end{aligned} \tag{3.9}$$

A solution ϕ of (3.6), having no Gaussian decay, would satisfy, for some $A > 0$:

$$\phi(\eta) \sim Ak(\eta) \qquad \text{as } \eta \to -\infty. \tag{3.10}$$

Using (3.4) with the fact that $\psi \geqslant 0$ in mind, we see that then

$$\psi(\eta) \sim \frac{\eta \pm \sqrt{\eta^2 + A/2\eta^2}}{2} \sim \frac{A}{8|\eta|^3} \qquad \text{as } \eta \to -\infty.$$

But then, let us come back to the functions $T(t,x)$ and $u(t,x)$ defined by (3.2)—we have, by (3.10):

$$T(t,x) \sim \frac{1}{\rho|x|^3}, \qquad u(t,x) \sim \frac{t}{\rho|x|^3}. \tag{3.11}$$

However, this would mean that for large t, the flow grows in any compact region in x, which contradicts the numerics in section 2. On the other hand, solutions which have a Gaussian decay of ϕ as $\eta \to -\infty$ and grow linearly in η as $\eta \to +\infty$ do not have this problem. It is interesting to note that the Gaussian decay is impossible without the presence of diffusion in the temperature equation.

The proof of theorem 3.1 is via a shooting argument: we first construct a solution ϕ_δ to (3.3) on a half-line of the form $(-\infty, -\eta_1)$ for some large η_1, having a given value $\delta > 0$ at the endpoint: $\phi_\delta(-\eta_1) = \delta$. The real number δ is then adjusted to get $\phi_\delta(\eta) = O(\eta)$ as $\eta \to +\infty$.

Step 1: solution on a left half-line. Take any $\eta_1 \geqslant \sqrt{6}$. We claim that, given $\delta > 0$, the Dirichlet problem

$$-\phi_\delta'' + \frac{1}{2}\left[\eta^2 + 8\int_{-\infty}^{\eta} \phi_\delta(\xi)\,\mathrm{d}\xi\right]^{\frac{1}{2}}\phi_\delta' - \frac{3\phi_\delta}{2} = 0 \text{ on } (-\infty, -\eta_1), \qquad \phi_\delta(-\eta_1) = \delta \tag{3.12}$$

with an additional constraint that ϕ_δ has a Gaussian decay as $\eta \to -\infty$ has a positive solution. Indeed, let $h(\eta)$ be defined by (3.9) and define the operator

$$L_0 = -\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} - \frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}\eta} - \frac{3}{2}.$$

We have $L_0 h = 0$. Moreover, on the half-line $\mathbb{R}_-$, the function h is positive and increasing on $(-\infty, -\sqrt{6})$, decreasing on $(-\sqrt{6}, 0)$ and negative on $(-\sqrt{2}, 0)$. Now, take $r > 0$ and $\eta_1 \geqslant \sqrt{6}$. Let ϕ_r be the solution of the boundary value problem

$$-\phi_r'' + \frac{1}{2}\left[\eta^2 + 8\int_{-r}^{\eta} |\phi_r(\xi)|\,\mathrm{d}\xi\right]^{\frac{1}{2}} \phi_r' - \frac{3\phi_r}{2} = 0 \qquad \text{on } (-r, -\eta_1)$$

$$\phi_r(-r) = \frac{\delta h(-r)}{h(-\eta_1)}, \qquad \phi_r(-\eta_1) = \delta. \tag{3.13}$$

Let us show that (3.13) has a non-negative solution that satisfies

$$0 \leqslant \phi_r(\eta) \leqslant \bar{h}(\eta) = \frac{\delta h(\eta)}{h(-\eta_1)}. \tag{3.14}$$

Given a function $q(\eta) \in C([-r, -\eta_1])$ define the nonlinear mapping $\phi = \mathcal{M}q$, where ϕ is the unique solution of the linear boundary value problem

$$\begin{aligned} &-\phi'' + \frac{1}{2}\left[\eta^2 + 8\int_{-r}^{\eta} |q(\xi)|\,\mathrm{d}\xi\right]^{\frac{1}{2}} \phi' - \frac{3\phi}{2} = 0 \qquad \text{on } (-r, -\eta_1), \\ &\phi(-r) = \frac{\delta h(-r)}{h(-\eta_1)}, \qquad \phi(-\eta_1) = \delta. \end{aligned} \tag{3.15}$$

In order to see that (3.15) indeed has a unique solution we write $\phi = \bar{h} + w$ and observe that the existence of a strictly positive solution of $L_0 h = 0$ implies that the operator L_0, defined on the space $W_r = \{w \in C((-r, -\eta_1)) :\ w(-r) = w(-\eta_1) = 0\}$ with the standard domain $D(L_0)$ is invertible with a compact inverse. Upon defining $\zeta(\eta) = \mathrm{e}^{\eta^2/8}\phi(\eta)$ and

$$a(\eta, q) = \frac{1}{2}\left[\eta^2 + 8\int_{-a}^{\eta} |q(\xi)|\,\mathrm{d}\xi\right]^{\frac{1}{2}}$$

we obtain

$$-\zeta'' + \left(a(\eta, q) - \frac{\eta}{2}\right)\zeta' + \left(\frac{\eta^2}{16} + \frac{|\eta|}{4}a(\eta, q) - \frac{7}{4}\right)\zeta = 0. \tag{3.16}$$

Note that, for $|\eta| \geqslant 6$ we have

$$\frac{\eta^2}{16} + \frac{|\eta|}{4}a(\eta, q) - \frac{7}{4} \geqslant \frac{\eta^2}{8}. \tag{3.17}$$

Then the maximum principle implies that ζ cannot attain a negative minimum inside the interval $(-r, -\eta_1)$ and hence $\zeta \geqslant 0$, which, in turn, implies that $\phi \geqslant 0$.

Let us now show that in addition ϕ satisfies $\phi \leqslant \bar{h}(\eta)$. The function $\bar{h}(\eta)$ is monotonically increasing and is thus a supersolution to (3.15) in the sense that

$$-\bar{h}'' + \frac{1}{2}\left[\eta^2 + 8\int_{-r}^{\eta} |q(\xi)|\,\mathrm{d}\xi\right]^{\frac{1}{2}} \bar{h}' - \frac{3\bar{h}}{2} \geqslant 0 \qquad \text{on } (-r, -\eta_1),$$

$$\bar{h}(-r) = \frac{\delta h(-r)}{h(-\eta_1)}, \quad \bar{h}(-\eta_1) = \delta. \tag{3.18}$$

Given any $M \geqslant 1$ the difference $w_M = M\bar{h} - \phi$ satisfies

$$\begin{aligned} &-w_M'' + \frac{1}{2}\left[\eta^2 + 8\int_{-r}^{\eta} |q(\xi)|\,\mathrm{d}\xi\right]^{\frac{1}{2}} w_M' - \frac{3w_M}{2} \geqslant 0 \qquad \text{on } (-r, -\eta_1), \\ &w_M(-r) \geqslant 0, \qquad w_M(-\eta_1) \geqslant 0. \end{aligned} \tag{3.19}$$

Another consequence of the maximum principle is that w_M cannot attain an interior minimum in $(-r, -\eta_1)$ at a point where $w_M = 0$. This, combined with the fact that $w_M > 0$ for a sufficiently large M, and decreasing M until we do not have $w_M > 0$, yields that

$$\bar{M} := \inf\{M > 0 : \; w_M(\eta) \geqslant 0 \text{ for all } \eta \in (-r, -\eta_1)\} = 1.$$

Therefore, we have $0 \leqslant \phi(\eta) \leqslant \bar{h}(\eta)$ for all functions $q(x) \in C([-r, -\eta_1])$. As a consequence, the nonlinear operator $\mathcal{M}$ sends the closed set $E = \{\phi \in C([-r, -\eta_1]) : \; 0 \leqslant \phi(\eta) \leqslant \bar{h}(\eta)\}$ to itself. The elliptic regularity theory implies that the mapping $\mathcal{M}$ is compact. The Schauder fixed point theorem implies that it has a fixed point in E which is a solution of (3.13). In addition, the limit has to satisfy (3.14). Now, using (3.14) we may pass to the limit $r \to +\infty$ and obtain a solution of (3.12) with a Gaussian decay.

An unpleasant fact in the construction of ϕ_δ is that—as is usual with applications of the Schauder theorem—it yields no information on the uniqueness of the solution ϕ_δ or on the continuity of ϕ_δ with respect to δ. This inconvenience will be fixed later in the course of the proof of the proposition, by adjusting the shooting point η_1.

Step 2: some estimates for ϕ_δ. Still assume that $\eta_1 \geqslant 6$ is fixed. Let us consider the functions $h(\eta)$ and $k(\eta)$ defined by (3.9) and introduce the following quantities:

$$u(\eta) = \int_{-\infty}^{\eta} \phi_\delta(\xi)\, \mathrm{d}\xi, \qquad a(\eta) = \tfrac{1}{2}\sqrt{\eta^2 + 8u(\eta)}, \tag{3.20}$$

The starting point of this step is the following lemma.

Lemma 3.2. *For every $\eta_1 \geqslant 6$, any solution ϕ of (3.12) with a Gaussian decay satisfies*

$$\phi(\eta) \leqslant \delta \frac{h(\eta)}{h(-\eta_1)}. \tag{3.21}$$

Proof. Upon defining $\zeta(\eta) = \mathrm{e}^{\eta^2/8}\phi(\eta)$, as in step 1, we obtain—recall that $a(\eta)$ is defined by (3.20):

$$\tilde{L}\zeta := -\zeta'' + \left(a(\eta) - \frac{\eta}{2}\right)\zeta' + \left(\frac{\eta^2}{16} + \frac{|\eta|}{4}a(\eta) - \frac{7}{4}\right)\zeta = 0. \tag{3.22}$$

Note again that, for $|\eta| \geqslant 6$, we have

$$\frac{\eta^2}{16} + \frac{|\eta|}{4}a(\eta) - \frac{7}{4} \geqslant \frac{\eta^2}{8}. \tag{3.23}$$

For every $\varepsilon > 0$, the function

$$\bar{\zeta}_\varepsilon(\eta) = \left(\delta \frac{h(\eta)}{h(-\eta_1)} + \varepsilon k(\eta)\right)\mathrm{e}^{\eta^2/8}$$

satisfies $\tilde{L}\bar{\zeta}_\varepsilon \geqslant 0$. Moreover, we have, because of the Gaussian decay of ϕ: $\bar{\zeta}_\varepsilon(\eta) - \zeta(\eta) > 0$ for a sufficiently large negative η. Therefore, if it gets negative inside $(-\infty, -\eta_1)$ it has to reach a minimum—a situation precluded by (3.23). Hence, we have $\bar{\zeta}_\varepsilon(\eta) - \zeta(\eta) > 0$ for all $\eta < -\eta_1$ and all $\varepsilon > 0$—thus, (3.21) holds. □

Let us now examine what happens to ϕ_δ as δ becomes large. It follows from the upper bound in (3.14) that

$$\int_{-\infty}^{\eta} \phi_\delta \leqslant \sqrt{6}\delta/2 < 2\delta$$

and $\phi_\delta'(-\eta_1) > 0$. Therefore, as $\phi_\delta > 0$ and

$$\phi_\delta'(\eta) = \phi'(-\eta_1)\exp\left(A(-\eta_1) - A(\eta)\right) + \frac{3}{2}\int_\eta^{-\eta_1} \exp\left(A(\xi) - A(\eta)\right)\phi_\delta(\xi)\,\mathrm{d}\xi \geqslant 0,$$

where

$$A'(\eta) = -\frac{1}{2}\left[\eta^2 + 8\int_{-\infty}^{\eta}\phi(\xi)\,\mathrm{d}\xi\right]^{\frac{1}{2}},$$

the function ϕ_δ is increasing. Hence, we have

$$-\phi_\delta'' + \tfrac{1}{2}(4\sqrt{\delta} - \eta)\phi_\delta' - \tfrac{3}{2}\phi_\delta \geqslant -\phi_\delta'' + \tfrac{1}{2}\sqrt{\eta^2 + 16\delta}\phi_\delta' - \tfrac{3}{2}\phi_\delta \geqslant 0.$$

Then, just as in lemma 3.2, we may get a lower bound $\phi_\delta \geqslant \underline{\phi}_\delta$, where

$$\begin{aligned} &-\underline{\phi}_\delta'' + \frac{1}{2}(4\sqrt{\delta} - \eta)\underline{\phi}_\delta' - \frac{3\underline{\phi}_\delta}{2} = 0, \\ &\underline{\phi}_\delta(-\eta_1) = \delta. \end{aligned} \tag{3.24}$$

The function $\underline{\phi}_\delta$ is given explicitly by

$$\underline{\phi}_\delta(\eta) = \frac{\delta h(\eta - 4\sqrt{\delta})}{h(-\eta_1 - 4\sqrt{\delta})}.$$

Hence, for $\eta \in [-\eta_1 - 1, -\eta_1]$ we have for a sufficiently large $\delta > 0$:

$$\int_{-\infty}^{-\eta}\underline{\phi}_\delta(\xi)\,\mathrm{d}\xi \sim \frac{C\delta(\eta + 4\sqrt{\delta})}{(\eta_1 + 4\sqrt{\delta})^2} \sim C\sqrt{\delta}, \qquad \text{as } \delta \to +\infty,$$

where $C > 0$ is independent of δ. Using this information, we integrate (3.12) on $(-\infty, -\eta_1]$ and obtain

$$\begin{aligned} \phi_\delta'(-\eta_1) &= \frac{1}{2}\int_{-\infty}^{-\eta_1}\left(\eta^2 + 8\int_{-\infty}^{\eta}\phi_\delta(\xi)\,\mathrm{d}\xi\right)^{\frac{1}{2}}\phi_\delta'(\eta)\,\mathrm{d}\eta - \frac{3}{2}\int_{-\infty}^{-\eta_1}\phi_\delta(\eta)\,\mathrm{d}\eta \\ &\geqslant C_1\delta^{5/4} - C_2\delta \sim C_1\delta^{5/4} \qquad \text{as } \delta \to +\infty. \end{aligned}$$

The following estimates are therefore true for large δ:

$$C_1\sqrt{\delta} \leqslant \int_{-\infty}^{-\eta_1}\phi_\delta \leqslant C_2\delta, \qquad \phi_\delta(-\eta_1) = \delta, \qquad \phi_\delta'(-\eta_1) \geqslant C_3\delta^{5/4}. \tag{3.25}$$

Step 3: extension of ϕ_δ and its behaviour for $\eta > -\eta_1$. Estimates (3.25) and equation (3.20) enable us to extend ϕ_δ past $-\eta_1$, on a maximal interval $[-\eta_1, \eta_\infty^\delta)$—with, possibly, $\eta_\infty^\delta = +\infty$. Let us define the sets

$$\begin{aligned} X_-^{\eta_1} &= \{\delta > 0 : \ \exists\eta_2 > -\eta_1 \text{ such that } \phi_\delta(\eta_2) = 0\}, \\ X_+^{\eta_1} &= \{\delta > 0 : \ \phi_\delta > 0 \text{ and } \limsup_{\eta \to \eta_\infty^\delta}\phi_\delta'(\eta) = +\infty\}. \end{aligned} \tag{3.26}$$

It is clear that ϕ_δ also depends on η_1, but this dependence is not going to be indicated by a sub- or a subscript to keep the notation readable.

We begin the analysis for $\delta \notin X_-^{\eta_1}$ with the following lemma.

Lemma 3.3. *Assume that $\delta \notin X_-^{\eta_1}$, then $\phi_\delta' > 0$.*

Proof. Let us recall that if $\phi_\delta \geqslant 0$ (which is the case for $\delta \notin X_-^{\eta_1}$) then the inequality

$$\phi''_\delta - a(\eta)\phi'_\delta \leqslant 0,$$

holds, with the function $a(\eta)$ defined in equation (3.20). It follows that

$$\left(\exp\left(-\int a(\eta)\,\mathrm{d}\eta\right)\phi'_\delta\right)' \leqslant 0. \tag{3.27}$$

Therefore, for any η_2 and η_3 larger than η_1, with $\eta_3 > \eta_2$, we have

$$\exp\left(-\int_{\eta_2}^{\eta_3} a(\xi)\,\mathrm{d}\xi\right)\phi'_\delta(\eta_3) \leqslant \phi'_\delta(\eta_2).$$

It follows that if there exists a point η_2 where $\phi'_\delta(\eta_2) < 0$ then $\phi'_\delta(\eta) \leqslant \phi'_\delta(\eta_2) < 0$ for all $\eta \geqslant \eta_2$. Hence $\phi_\delta(\eta)$ has to vanish at some point. This contradicts the assumption that $\delta \notin X_-^{\eta_1}$ and finishes the proof of lemma 3.3. □

The main result of this step is the characterization of $X_+^{\eta_1}$.

Lemma 3.4. *Let $\delta \in X_+^{\eta_1}$, then $\eta_\infty^\delta < +\infty$.*

Proof. Consider $\delta \in X_+^{\eta_1}$; it is convenient to work with the logarithmic derivative of ϕ_δ:

$$\xi_\delta = \frac{\phi'_\delta}{\phi_\delta}. \tag{3.28}$$

Let us drop the subscript δ for the moment. The equation for ξ is

$$\xi' = a(\eta)\xi - \xi^2 - \tfrac{3}{2}. \tag{3.29}$$

The term $(-\xi^2)$ would, in principle, prevent a blow-up; it is the role of the—seemingly linear—term $a(\eta)\xi$ to force it. Assume, therefore, that $\eta_\infty = +\infty$, and let us try to reach a contradiction.

Case 1. Assume that there exists a sequence $(\eta_n)_n$ going to $+\infty$ such that $\lim_{n\to+\infty}\xi(\eta_n) = +\infty$. We claim then that $\xi'(\eta) > 0$ for all sufficiently large $\eta > 0$. Indeed, there exists $\eta_0 > 0$ such that $\xi'(\eta_0) > 0$. If $h := \xi'$ we have, by lemma 3.3,

$$h' + (-a(\eta) + 2\xi)h = \frac{1}{2}\frac{\eta + 4\phi}{a(\eta)}\xi > 0, \qquad h(\eta_0) > 0. \tag{3.30}$$

This implies $h(\eta) > 0$ for $\eta \geqslant \eta_0$.

To prove that we have a blow-up, we use an elementary numerical analysis procedure: pick $\eta_0 > 0$ so that $\xi' > 0$ on $(\eta_0, +\infty)$ and $\phi_\delta(\eta_0) \geqslant 10$. The value of $\xi(\eta_0)$ may be taken arbitrarily large, because $\xi(\eta_n) \to +\infty$. Let $\lambda_n = 1/(n+1)^2$ and set $\zeta_n = \eta_0 + \sum_{k\leqslant n}\lambda_k$. First, we have

$$\xi(\zeta_{n+1}) - \xi(\zeta_n) = \int_{\zeta_n}^{\zeta_{n+1}} \left[a(\zeta)\xi(\zeta) - \xi^2(\zeta)\right]\mathrm{d}\zeta - \frac{3\lambda_n}{2}.$$

The function $a(\zeta)$ for $\zeta \in (\zeta_n, \zeta_{n+1})$ may be bounded as

$$a(\zeta) = \frac{1}{2}\left(\zeta^2 + 8\int_{-\infty}^{\zeta}\phi_\delta(\xi)\,\mathrm{d}\xi\right)^{1/2} \geqslant \frac{1}{2}\left(8\int_{\zeta_n}^{\zeta}\phi_\delta(\xi)\,\mathrm{d}\xi\right)^{1/2}$$
$$\geqslant \left(2\int_{\zeta_n}^{\zeta}\phi_\delta(\eta_0)\exp\left(\int_{\eta_0}^{\zeta'}\xi(x)\,\mathrm{d}x\right)\mathrm{d}\zeta'\right)^{1/2}.$$

Therefore, using positivity and monotonicity of ψ we have

$$\xi(\zeta_{n+1})-\xi(\zeta_n)\geqslant\int_{\zeta_n}^{\zeta_{n+1}}\left(\sqrt{2\phi_\delta(\eta_0)}\left[\int_{\zeta_n}^{\zeta}\exp\left(\int_{\eta_0}^{\zeta'}\xi(x)\,\mathrm{d}x\right)\mathrm{d}\zeta'\right]^{1/2}\xi(\zeta)-\xi^2(\zeta)\right)\mathrm{d}\zeta-\frac{3\lambda_n}{2}$$

$$\geqslant\int_{\zeta_n}^{\zeta_{n+1}}\sqrt{2\phi_\delta(\eta_0)}\left[\int_{\zeta_n}^{\zeta}\exp\left(\int_{\zeta_{n-1}}^{\zeta_n}\xi(\zeta_{n-1})\,\mathrm{d}x\right)\mathrm{d}\zeta'\right]^{1/2}\xi(\zeta_n)\,\mathrm{d}\zeta$$

$$-\lambda_n\xi^2(\zeta_{n+1})-\frac{3\lambda_n}{2}$$

$$=\sqrt{2\phi_\delta(\eta_0)}\xi(\zeta_n)\mathrm{e}^{(\zeta_n-\zeta_{n-1})\xi(\zeta_{n-1})/2}\int_{\zeta_n}^{\zeta_{n+1}}\sqrt{\zeta-\zeta_n}\mathrm{d}\zeta-\lambda_n\xi^2(\zeta_{n+1})-\frac{3\lambda_n}{2}$$

$$\geqslant\lambda_n^{3/2}\xi(\zeta_n)\mathrm{e}^{\lambda_n\psi(\zeta_{n-1})/8}-\lambda_n\xi(\zeta_{n+1})^2-\frac{3\lambda_n}{2}.$$

As $\lambda_n\leqslant 1/2$ and we may take $\xi(\eta_0)\geqslant 10$ (and thus $\xi(\zeta_n)\geqslant 10$ for all n) it follows that

$$\xi(\zeta_{n+1})^2\geqslant C\lambda_n^{3/2}\xi(\zeta_n)\mathrm{e}^{\lambda_n\xi(\zeta_{n-1})/8}$$

or

$$\xi(\zeta_{n+1})\geqslant\frac{\mathrm{e}^{\xi(\zeta_{n-1})/(16(n+1)^2)}}{(n+1)^{3/2}}.$$

Now, choose $r>0$ so that $\mathrm{e}^{r(n-1)^4/[16(n+1)^2]}/(n+1)^{3/2}\geqslant r(n+1)^4$ for all $n\in\mathbb{N}$. An easy induction shows that, if $\xi(\eta_0)$ is large enough, we have $\xi(\zeta_n)\geqslant rn^4$. This contradicts the assertion $\eta_\infty=+\infty$.

Case 2. Assume that ξ is bounded. Then (3.29) may be integrated from $+\infty$ to yield

$$\xi(\eta)=\int_\eta^{+\infty}\left(\frac{3}{2}+\psi^2\right)\mathrm{e}^{-\int_\eta^\zeta a(\zeta')\,\mathrm{d}\zeta'}\,\mathrm{d}\zeta,$$

which, as $\xi\leqslant C$, implies $\xi\leqslant Ca(\eta)^{-1}$, where C does not depend on η. This implies in turn

$$0\leqslant\frac{\phi_\delta'}{\phi_\delta}\leqslant\frac{C}{\sqrt{\int_0^\eta\phi_\delta(\zeta)\,\mathrm{d}\zeta}},\tag{3.31}$$

which, after integration, yields

$$0\leqslant\phi_\delta(\eta)\leqslant C\left(1+\sqrt{\int_0^\eta\phi_\delta(\zeta)\,\mathrm{d}\zeta}\right).\tag{3.32}$$

Integrating (3.32) we obtain $\int_0^\eta\phi_\delta(\zeta)\,\mathrm{d}\zeta\leqslant C(1+\eta^2)$, which by (3.32) again translates into the bound $\phi_\delta(\eta)\leqslant C(1+\eta)$. But now, we may start again from the inequality $\xi\leqslant C/a(\eta)$, use the definition of ξ and the just obtained information: it follows that

$$\phi_\delta'(\eta)\leqslant\frac{C\phi}{a(\eta)}\leqslant\frac{C(1+\eta)}{\eta}\leqslant C.$$

This contradicts the fact that $\delta\in X_+^{\eta_1}$. We conclude that $\eta_\infty^\delta<+\infty$ for all $\delta\in X_+^{\eta_1}$. □

One important consequence of lemma 3.4 is the following.

Corollary 3.5. *There exists* $\delta_0>0$ *such that* $[\delta_0,+\infty)\subset X_+^{\eta_1}$.

Proof. Let us recall that the logarithmic derivative $\xi = \phi'_\delta/\phi_\delta$ satisfies equation (3.29). Moreover, if $\phi_\delta(\eta') \geqslant 0$ for all $\eta' < \eta$ then $a(\eta) \geqslant a_0 = a(-\eta_1)$. It follows that under this assumption and if $\xi(\eta) > 0$ we have

$$\xi' \geqslant a_0\xi - \xi^2 - \tfrac{3}{2}. \tag{3.33}$$

In addition, for large enough $\delta > 0$ we have, by estimate (3.25) of step 2:

$$\xi(-\eta_1) = \frac{\phi'_\delta(-\eta_1)}{\phi_\delta(-\eta_1)} \geqslant C\delta^{1/4}.$$

As the smallest root q_0 of the right side of (3.33) is smaller than $\xi(-\eta_1)$ and ϕ_δ may not become negative before as does the function ξ, it follows from the above that $\xi(\eta) > q_0$ for all $\eta > -\eta_1$. As a consequence, we have $\phi'_\delta > q_0\phi_\delta$ and thus ϕ_δ blows up at infinity (or at a finite distance) together with its derivative and so $\delta \in X_+^{\eta_1}$. □

Step 4: choice of the shooting point. Take, for definiteness, $\eta_1 = 7$. Corollary 3.5 implies the existence of $\delta_0 > 0$ such that if ϕ is a solution of (3.6) with Gaussian decay at $-\infty$, then $\phi(-7) \leqslant \delta_0$. By lemma 3.2 we have $\phi(\eta) \leqslant \delta_0 h(\eta)/h(-7)$, a quantity that decays to 0 as $\eta \to -\infty$. Pick any $\lambda_0 \in (1/8, 1/4)$, which will remain fixed until the end of the proof of theorem 3.1. By elementary elliptic regularity we may find a constant $\eta_0 > 7$ such that if ϕ is a solution of (3.6) with Gaussian decay at $-\infty$, then

$$\forall \eta \leqslant -\eta_0, \qquad 0 \leqslant \phi(\eta), \qquad \phi'(\eta) \leqslant \mathrm{e}^{-\lambda_0\eta^2}. \tag{3.34}$$

For $\eta_1 > \eta_0$, let us go back to problem (3.12). We may now prove the uniqueness and continuity with respect to δ that were lacking.

Lemma 3.6. *If $\eta_1 > 0$ is large enough and $\delta \in [0, \mathrm{e}^{-\lambda_0\eta_1^2}]$, problem (3.12) has exactly one solution that we still call ϕ_δ.*

Proof. Let ϕ_1 and ϕ_2 be two such solutions; define $\zeta(\eta) = \mathrm{e}^{\eta^2/8}(\phi_1(\eta) - \phi_2(\eta))$ and

$$a_i(\eta) = \frac{1}{2}\sqrt{\eta^2 + 8\int_{-\infty}^{\eta}\phi_i}. \tag{3.35}$$

The equation for ζ is

$$-\zeta'' + \left(a_1(\eta) - \frac{\eta}{2}\right)\zeta' + \left(\frac{\eta^2}{16} + \frac{|\eta|}{4}a(\eta) - \frac{7}{4}\right)\zeta = \mathrm{e}^{\eta^2/8}(a_2 - a_1)(\eta)\phi'_2(\eta). \tag{3.36}$$

Note again that, for $\eta \geqslant 6$, we have

$$\frac{\eta^2}{16} + \frac{|\eta|}{4}a_1(\eta) - \frac{7}{4} \geqslant \frac{\eta^2}{8}. \tag{3.37}$$

By definition (3.35) of a_i, we have

$$4\mathrm{e}^{\eta^2/8}|a_2(\eta) - a_1(\eta)| = \mathrm{e}^{\eta^2/8}\frac{\left|\int_{-\infty}^{\eta}(\phi_2 - \phi_1)\,\mathrm{d}x\right|}{a_1(\eta) + a_2(\eta)} \leqslant \frac{\mathrm{e}^{\eta^2/8}\int_{-\infty}^{\eta}\mathrm{e}^{-x^2/8}\,\mathrm{d}x}{a_1(\eta) + a_2(\eta)}\|\zeta\|_\infty \leqslant 4\frac{\|\zeta\|_\infty}{\eta_1^3}.$$

Combining the above inequality with estimate (3.34) and inequality (3.37), we obtain from (3.36)

$$\|\zeta\|_\infty \leqslant \frac{8\mathrm{e}^{-\lambda_0\eta_1^2}}{\eta_1^5}\|\zeta\|_\infty.$$

This implies $\zeta \equiv 0$ as soon as η_1 is large enough. □

Step 5: existence of the wave fan. We have to prove two things: first, the existence of a solution to (3.12) with Gaussian decay; second, the asymptotic behaviour of the constructed solution. Let us first worry about the existence: for this we fix any η_1 large enough such that

- lemma 3.6 holds and
- $\mathrm{e}^{-\lambda_0\eta_1^2} \in X_+^{\eta_1}$.

The second condition given above is realizable because if $\delta_0 \in X_+^{\eta_1}$ then as ϕ_δ satisfies the Gaussian decay bound $\phi_\delta(\eta) \leqslant \delta\bar{h}(\eta)/\bar{h}(-\eta_1)$, then when we increase η_1 the 'critical' δ_0 from corollary 3.5 is approximately multiplied by the factor $\mathrm{e}^{-\lambda_0\eta_1^2}$.

We now redefine the sets $X_\pm^{\eta_1}$ by restricting the values of δ to the interval $[0, \mathrm{e}^{-\lambda_0\eta_1^2}]$. For small $\delta > 0$, the function ϕ_δ is close on compact intervals to $\delta h(\eta)/h(-\eta_1)$. Hence, it vanishes at some point $\eta < 0$ close to $(-\sqrt{2})$—the negative point where $h(\eta)$ vanishes. This says that the set $X_-^{\eta_1}$ is non-empty. Moreover, the functions ϕ_δ may not attain a local minimum equal to zero. Therefore, the continuity of $\delta \mapsto \phi_\delta'(\eta)$ on compact sets implies that $X_-^{\eta_1}$ is open. On the other hand, we know that $X_+^{\eta_1}$ is non-empty. By the arguments in the proof of case 1 in lemma 3.4 and the continuity of $\delta \mapsto \phi_\delta(\eta)$ on compact intervals, it is also open. Consequently, there exists $\delta \in [0, \mathrm{e}^{-\lambda_0\eta_1^2}]\backslash(X_+^{\eta_1} \cup X_+^{\eta_1})$. This δ generates our desired solution $\phi(\eta)$ of (3.12).

Step 6: behaviour of ϕ at $+\infty$.—the first term in expansion (3.7) . If $u(\eta)$ and $a(\eta)$ are defined by expressions (3.20), then the equation for u is

$$-u''' + a(\eta)u'' - \tfrac{3}{2}u' = 0,$$

and there exists $C > 0$ so that

$$u(\eta) \leqslant C\eta^2 \qquad \text{for } \eta > 0 \tag{3.38}$$

as $\phi' \in L^\infty(\mathbb{R})$. This implies by integration from η to $+\infty$, with $\eta > 0$:

$$\begin{aligned} u''(\eta) &= \frac{3}{2}\int_\eta^{+\infty} \exp\left(-\int_\eta^\xi a(\zeta)\,\mathrm{d}\zeta\right) u'(\xi)\,\mathrm{d}\xi \\ &= \frac{3u'(\eta)}{2a(\eta)} + \frac{3}{2}\int_\eta^{+\infty} \exp\left(-\int_\eta^\xi a(\zeta)\,\mathrm{d}\zeta\right)\left(\frac{u'(\xi)}{a(\xi)}\right)' \mathrm{d}\xi := \frac{3u'(\eta)}{2a(\eta)} - f(\eta), \end{aligned} \tag{3.39}$$

the last line being obtained by integration by parts. The uniform bound for $\phi' = u''$ and positivity of u imply that $C_1\eta \leqslant a(\eta) \leqslant C_2\eta$, and $f(\eta)$ satisfies for $\eta > 0$

$$\begin{aligned} |f(\eta)| &\leqslant C\int_\eta^{+\infty} \exp\left\{-C\int_\eta^\xi \zeta\,\mathrm{d}\zeta\right\}\frac{\mathrm{d}\xi}{\xi} = \int_\eta^{+\infty} \exp\left\{-C(\xi^2-\eta^2)\right\}\frac{\mathrm{d}\xi}{\xi} \\ &= C\mathrm{e}^{C\eta^2}\int_\eta^{+\infty} \exp(-C\xi^2)\frac{\xi\,\mathrm{d}\xi}{\eta^2} \leqslant \frac{C}{\eta^2}, \end{aligned}$$

so that $f(\eta) = O(\eta^{-2})$ as $\eta \to +\infty$.

Let us show that

$$u(\eta) = \eta^2 + o(\eta^2), \tag{3.40}$$

where C is a constant depending on u. Note that the function $\phi(\eta)$ is increasing since it satisfies

$$-\phi'' + a(\eta)\phi' = \frac{3\phi}{2},$$

and integrating this equation from η to $+\infty$ we obtain

$$\phi'(\eta) = \frac{3}{2}\int_\eta^{+\infty} \exp\left(-\int_\eta^\xi a(\zeta)\,\mathrm{d}\zeta\right)\phi(\xi)\,\mathrm{d}\xi \geqslant 0.$$

It follows that

$$\lim_{\eta\to+\infty} u(\eta) = +\infty. \tag{3.41}$$

In order to improve this estimate to (3.40) we start with the inequality

$$u''(\eta) \geqslant \frac{3u'(\eta)}{\sqrt{8u(\eta)}} - \frac{C}{\eta^2},$$

which follows from (3.39) and holds for $\eta > 0$, and integrate it from 1 to η:

$$u'(\eta) \geqslant \frac{3\sqrt{u(\eta)}}{\sqrt{2}} - C.$$

Using (3.41) we conclude that $u(\eta) \geqslant C\eta^2$ with $C > 0$ and in particular

$$l = \liminf_{\eta\to+\infty} \frac{u(\eta)}{\eta^2} > 0.$$

Then for any $\delta > 0$ we can find $\eta(\delta)$ so that $u(\eta) \geqslant (l-\delta)\eta^2$ for all $\eta > \eta(\delta)$. Going back to (3.39) we observe that for $\eta > \eta(\delta)$ we have

$$u''(\eta) \geqslant \frac{3u'(\eta)}{\sqrt{\eta^2 + 8u}} - f(\eta) \geqslant \frac{3u'(\eta)}{\sqrt{\frac{\eta^2}{u}u + 8u}} - \frac{C}{\eta^2} \geqslant \frac{3u'(\eta)}{\sqrt{\frac{u}{l-\delta} + 8u}} - \frac{C}{\eta^2}.$$

Integrating this inequality between $\eta(\delta)$ and η we obtain for η sufficiently large

$$u'(\eta) \geqslant \frac{6\sqrt{u(\eta)}}{\sqrt{8 + \frac{1}{l-\delta}}} - C(\delta) \geqslant \frac{(6-\delta)\sqrt{u(\eta)}}{\sqrt{8 + \frac{1}{l-\delta}}}. \tag{3.42}$$

We used (3.41) in the last step given above. Therefore, we have

$$u(\eta) \geqslant \left[\frac{(6-\delta)}{2\sqrt{8 + \frac{1}{l-\delta}}}\eta - C(\delta)\right]^2,$$

as $\eta \to +\infty$. It follows that for any $\delta > 0$ we have

$$l \geqslant \frac{(6-\delta)^2}{4\left(8 + \frac{1}{l-\delta}\right)}.$$

Passing to the limit $\delta \to 0$ we see that

$$l \geqslant \frac{9}{\left(8 + \frac{1}{l}\right)},$$

and hence $l \geqslant 1$.

On the other hand, it follows from (3.38) that

$$L = \limsup_{\eta\to+\infty} \frac{u(\eta)}{\eta^2} < +\infty.$$

Then (3.39) implies for any $\delta > 0$ and $\eta > \eta(\delta)$ that

$$u''(\eta) = \frac{3u'(\eta)}{\sqrt{\eta^2 + 8u}} - f(\eta) \leqslant \frac{3u'(\eta)}{\sqrt{\frac{u}{L+\delta} + 8u}} + \frac{C}{\eta^2}.$$

Integrating this inequality between 1 and η we obtain

$$u'(\eta) \leqslant \frac{6\sqrt{u(\eta)}}{\sqrt{\frac{1}{L+\delta}+8}} + C(\delta) \leqslant \frac{(6+\delta)\sqrt{u(\eta)}}{\sqrt{\frac{1}{L+\delta}+8}} \tag{3.43}$$

for $\eta > \eta(\delta)$. Therefore, we have

$$u(\eta) \leqslant \left[\frac{(6+\delta)\eta}{2\sqrt{\frac{1}{L+\delta}+8}} + C(\delta)\right]^2.$$

In the limit $\eta \to +\infty$ we obtain

$$L \leqslant \frac{(6+\delta)^2}{4\left(\frac{1}{L+\delta}+8\right)},$$

which in the limit $\delta \to 0$ becomes $L \leqslant 1$. As $1 \geqslant L \geqslant l \geqslant 1$, we conclude that $L = l$ and

$$\lim_{\eta\to+\infty} \frac{u(\eta)}{\eta^2} = 1,$$

so that (3.40) indeed holds. Moreover, as $\phi = u'$, it follows now from (3.42) and (3.43) that

$$\lim_{\eta\to+\infty} \frac{\phi(\eta)}{\eta} = 2.$$

Step 7: the second term in expansion (3.7). Going back to $u(\eta)$ defined by expression (3.20) and $f(\eta)$ defined by (3.39), we set

$$u(\eta) = \eta^2 + v(\eta). \tag{3.44}$$

The equation for v is

$$v'' - \frac{v'}{\eta} + \frac{8v}{9\eta^2} = -\frac{4vv'}{9\eta^3} - f(\eta) + \frac{v'+2\eta}{\eta} g\left(\frac{v}{\eta^2}\right), \tag{3.45}$$

where $g(\eta)$ is a smooth function such that $g(0) = g'(0) = 0$. An asymptotic equation for (3.44) is

$$v'' - \frac{v'}{\eta} + \frac{8v}{9\eta^2} = 0, \tag{3.46}$$

which has two solutions: $\eta^{4/3}$ and $\eta^{2/3}$. We expect the function v to be asymptotic to $\eta^{4/3}$ and $v' = \phi - 2\eta$ to be asymptotic to $4\eta^{1/3}/3$, which would essentially finish the proof: the behaviour of ϕ' would be obtained by looking at (3.44).

Lemma 3.7. *If $v = u(\eta) - \eta^2$ is sought for the solution of (3.45), consider the differential equation with unknown w:*

$$w'' - \left(1 + g\left(\frac{v}{\eta^2}\right)\right)\frac{w'}{\eta} + \frac{8}{9\eta^2}\left(1 + \frac{v'}{2\eta}\right) w = 0. \tag{3.47}$$

For every small enough $\varepsilon > 0$, there exists $\eta^\varepsilon \geqslant \eta_0$ and a solution of (3.47), denoted by $w^\varepsilon(\eta)$, such that the following inequalities hold on $[\eta^\varepsilon, +\infty)$:

$$\eta^{4/3-\varepsilon} \leqslant |w^\varepsilon(\eta)| \leqslant \eta^{4/3+\varepsilon}, \qquad |w'_\varepsilon(\eta)| \leqslant \eta^{1/3+\varepsilon}, \qquad \left|\frac{w'_\varepsilon \eta)}{w_\varepsilon(\eta)} - \frac{4}{3\eta}\right| \leqslant \frac{\varepsilon}{\eta}. \tag{3.48}$$

Proof. We drop for simplicity the subscripts and superscripts and use once again the (slightly adjusted) logarithmic derivative of w: $q(\eta) = w'(\eta)/w(\eta) - 4/(3\eta)$. We have, using (3.47):

$$q' + \frac{5 - 3g(v/\eta^2)}{3\eta} q = -q^2 + \frac{4}{3\eta^2} g\left(\frac{v}{\eta^2}\right) - \frac{4}{9\eta^2}\frac{v'}{\eta} := -q^2 + \frac{h_0(\eta)}{\eta^2}. \tag{3.49}$$

If h were identically equal to zero we could take $q = 0$ (recall that we are simply looking for one solution w of (3.47))—this is, of course, not the case. Let us pick $\varepsilon > 0$. From step 6 we know that there exists $\eta^\varepsilon > 0$ such that

$$\forall \eta \geqslant \eta^\varepsilon, \qquad 3\left|g\left(\frac{v}{\eta^2}\right)\right| + |h_0(\eta)| \leqslant \varepsilon^2. \tag{3.50}$$

Let us look for a solution of (3.47) which has $q(\eta_\varepsilon) = 0$, then there exists a constant $C > 0$ independent of ε such that

$$|q(\eta)| \leqslant C \int_{\eta^\varepsilon}^{\eta} \left(\frac{\eta'}{\eta}\right)^{5/3 - C\varepsilon^2} \left(q^2 + \frac{\varepsilon^2}{\eta'^2}\right) \mathrm{d}\eta'. \tag{3.51}$$

We conclude by a classical stability argument: let η_1 be the first point where the inequality $q(\eta) \leqslant \sqrt{\varepsilon}\eta^{-1}$ is violated; if $\eta_1 < +\infty$ equation (3.49) implies—for ε small enough: $q(\eta_1) \leqslant C\varepsilon/\eta_1$, a contradiction. Therefore $\eta_1 = +\infty$, proving the desired inequality for w. The estimate for w' is obtained in a similar way. □

A similar argument shows that the other fundamental solution of (3.47) satisfies the estimate $|\tilde{w}_\varepsilon(\eta)| \leqslant C_\varepsilon \eta^{2/3+\varepsilon}$—we leave the details for the reader.

End of the proof of theorem 3.1 . We apply lemma 3.7 twice, by suitably varying the right-hand side and the coefficients in equation (3.45). Let us first observe that if $h \in C(\mathbb{R}_+)$ and a small $\varepsilon > 0$ and $\eta_\varepsilon > 0$ are given so that lemma 3.7 holds, the problem

$$w'' - \left(1 + g\left(\frac{v}{\eta^2}\right)\right)\frac{w'}{\eta} + \frac{8}{9\eta^2}\left(1 + \frac{v'}{2\eta}\right) w = h(\eta), \qquad w(\eta_\varepsilon) \text{ and } w'(\eta_\varepsilon) \text{ given}, \tag{3.52}$$

has a unique solution of the form

$$w(\eta) = w_{\text{free}}(\eta) + w^\varepsilon(\eta) \int_{\eta_\varepsilon}^{\eta} \int_{\eta_\varepsilon}^{\eta'} \exp\left(\int_{\eta'}^{\eta''} \left(2\frac{w_\varepsilon'(\zeta)}{w_\varepsilon(\zeta)} - \frac{1 + g(v/\zeta^2)}{\zeta}\right) \mathrm{d}\zeta\right) \frac{h(\eta'')}{w_\varepsilon(\eta'')} \mathrm{d}\eta''\, \mathrm{d}\eta', \tag{3.53}$$

where $w_{\text{free}}(\eta)$ is the solution of (3.47) with the data $(w(\eta_\varepsilon), w'(\eta_\varepsilon))$ and w_ε is the particular solution found in lemma 3.7. However, if $h = f$ we have $w = v$ and $h(\eta) = O(\eta^{-2})$. From lemma 3.7 we have

$$2\frac{w_\varepsilon'(\zeta)}{w_\varepsilon(\zeta)} - \frac{1 + g(v/\zeta^2)}{\zeta} \geqslant \left(\frac{5}{3} - C\varepsilon\right)\frac{1}{\zeta},$$

with a constant $C > 0$ independent of ε. Lemma 3.7 and the remark following its proof imply that we have $|w_{\text{free}}(\eta)| \leqslant C_\varepsilon \eta^{4/3+\varepsilon}$. Now, it follows from (3.53) with $h = f$ that

$$|v(\eta)| \leqslant C_\varepsilon \eta^{4/3+\varepsilon} + \eta^{4/3+\varepsilon} \int_{\eta_\varepsilon}^{\eta} \int_{\eta_\varepsilon}^{\eta'} \left(\frac{\eta''}{\eta'}\right)^{5/3 - C\varepsilon} \frac{|f(\eta'')|}{\eta''^{4/3-\varepsilon}} \mathrm{d}\eta''\, \mathrm{d}\eta' \leqslant C_\varepsilon \eta^{4/3+C\varepsilon}. \tag{3.54}$$

In a similar way we may obtain

$$|v'(\eta)| \leqslant C_\varepsilon \eta^{1/3+C\varepsilon}. \tag{3.55}$$

We now set

$$h_1(\eta) = -\frac{4vv'}{9\eta^3} - f(\eta) + \frac{v' + 2\eta}{\eta} g\left(\frac{v}{\eta^2}\right).$$

We have just proved the existence of $C > 0$ such that, for every small $\varepsilon > 0$, there are two large constants η_ε and C_ε for which we have

$$\forall \eta \geqslant \eta_\varepsilon, \qquad |h_1(\eta)| \leqslant \frac{C_\varepsilon}{\eta^{4/3-C\varepsilon}}. \tag{3.56}$$

Fix now $\varepsilon > 0$ once and for all such that

$$\frac{5}{3} - C\varepsilon > 1, \qquad \text{i.e. } \varepsilon < \frac{2}{3C}. \tag{3.57}$$

We have

$$\begin{aligned} v(\eta) &= v_{\text{free}}(\eta) + \eta^{4/3} \int_{\eta_\varepsilon}^{\eta} \int_{\eta_\varepsilon}^{\eta'} \left(\frac{\eta''}{\eta'}\right)^{5/3} \frac{h_1(\eta'')}{\eta''^{4/3}} \, \mathrm{d}\eta'' \, \mathrm{d}\eta', \\ v'(\eta) &= v'_{\text{free}}(\eta) + \frac{4}{3}\eta^{1/3} \int_{\eta_\varepsilon}^{\eta} \int_{\eta_\varepsilon}^{\eta'} \left(\frac{\eta''}{\eta'}\right)^{5/3} \frac{h_1(\eta'')}{\eta''^{4/3}} \, \mathrm{d}\eta'' \, \mathrm{d}\eta' + \eta^{4/3} \int_{\eta_\varepsilon}^{\eta} \left(\frac{\eta'}{\eta}\right)^{5/3} \frac{h_1(\eta')}{\eta'^{4/3}} \, \mathrm{d}\eta', \end{aligned} \tag{3.58}$$

where $v_{\text{free}}(\eta)$ is the solution of (3.46), with $(v_{\text{free}}(\eta_\varepsilon), v'_{\text{free}}(\eta_\varepsilon)) = (v(\eta_\varepsilon), v'(\eta_\varepsilon))$; hence it is a linear combination of $\eta^{4/3}$ and $\eta^{2/3}$. As for the integral term, we observe that

$$\left(\frac{\eta''}{\eta'}\right)^{5/3} \frac{h_1(\eta'')}{\eta''^{4/3}} = O(\eta'^{-5/3} \eta''^{C\varepsilon - 1}).$$

Hence, because of (3.57), the integral

$$\int_{\eta_\varepsilon}^{+\infty} \int_{\eta_\varepsilon}^{\eta'} \left(\frac{\eta''}{\eta'}\right)^{5/3} \frac{h(\eta'')}{\eta''^{4/3}} \, \mathrm{d}\eta'' \, \mathrm{d}\eta'$$

is finite; call it I. We have therefore $v(\eta) - v_{\text{free}}(\eta) = (I + o(1))\eta^{4/3}$, and a similar identity may be proved for $v'(\eta) - v'_{\text{free}}(\eta)$ by examining the equation for $v'(\eta) = \phi(\eta) - 2\eta$ in (3.58). This ends the proof of theorem 3.1. □

4. Combustion waves

In this section we seek travelling wave profiles that will play the role of the inner waves observed in section 2. Recalling that $\theta > 0$ is the ignition temperature, let us pick $\phi_0 \in [0, \theta)$ and investigate the following differential system, with unknowns (c, ϕ, ψ):

$$\begin{aligned} &-\phi'' + (c + \psi)\phi' = f(\phi), \\ &(c + \psi)\psi' = \rho(\phi - \phi_0), \\ &\phi(-\infty) = \phi_0, \qquad \phi(+\infty) = 1, \\ &\psi(-\infty) = 0, \qquad \psi(+\infty) = +\infty. \end{aligned} \tag{4.1}$$

In theorem 4.1 we present the two cases $(c > 0, \phi_0 = 0)$ and $(c = 0, \phi_0 > 0)$. The first case, described in part (i) of the theorem, represents the left-going travelling waves that were observed numerically in section 2 when the gravity $\rho \in (0, \rho_{\text{cr}1})$ is sufficiently small. The second case corresponds to the numerically observed profiles that connect the wave fan on the left to a shock on the right, for $\rho \in (\rho_{\text{cr}1}, \rho_{\text{cr}2})$. The critical threshold $\rho_{\text{cr}2}$ appears in the numerical simulations because the initial data vanish far on the right: this leads to a shock and opens the way to quenching for large ρ. If the initial data for temperature have the value $T \to 1$ as $x \to +\infty$ at $t = 0$, the solutions would have the form of a wave fan followed by a travelling wave for all $\rho > \rho_{\text{cr}1}$. This is reflected in part (ii) of the following theorem.

Theorem 4.1.

(i) *Assume that* $\phi_0 = 0$. *Then there exists* $\rho_{\max} > 0$ *such that system (4.1) has no solution* (c, ϕ, ψ) *with* $c \geqslant 0$ *for all* $\rho \geqslant \rho_{\max}$. *If* ρ *is small enough, there exists* $c > 0$ *such that system (4.1) has a solution* (ϕ, ψ).

(ii) *Assume that* $c = 0$. *If* $\rho > 0$ *is large enough, there exists* $\phi_0 \in (0, \theta)$ *such that system (4.1) has a solution. If* $\phi_0(\rho)$ *is the smallest of all* $\phi_0 \in (0, \theta)$ *such that system (4.1) has a solution, then we have*

$$\lim_{\rho \to +\infty} \phi_0(\rho) = \theta. \tag{4.2}$$

The second statement in part (ii) is essential for the quenching phenomenon—if the wave fan does catch up with the shock, the temperature drops below the value ϕ_0 everywhere and it is important that this value be close to θ.

Note that if the smooth reaction term is replaced by the Dirac mass $\delta_{\phi=1}$, the proof of existence of a travelling wave is much simpler. Recall that this regime (see, for instance, [3]) is the limit of a sequence of reaction terms with high activation energies. Then, we have an explicit expression for a travelling wave

$$\phi(x) = \begin{cases} \phi_0 \rho^{-1/2} \phi_\lambda(x) & \text{on } \mathbb{R}_-, \\ 1 & \text{on } \mathbb{R}_+, \end{cases}$$

where $\lambda = (16/[(\theta - \phi_0)\sqrt{\rho}])^{1/3}$, the family $\phi_\lambda(x)$ is defined by (4.4), while ϕ_0 is adjusted to satisfy the derivative jump $[\phi'](0) = 1$. We will not pursue this direction. The price to pay for this very simple existence proof is indeed a more difficult study of the quenching—where we crucially use the fact that the reaction term is globally Lipschitz.

Before we start proving anything about (4.1), let us note that any of its solutions satisfy $\psi' > 0$; hence, it may be reduced to

$$\begin{aligned} &-\phi'' + \left[c^2 + 2\rho \int_{-\infty}^{x} (\phi(y) - \phi_0)\mathrm{d}y\right]^{\frac{1}{2}} \phi' = f(\phi), \\ &\phi(-\infty) = \phi_0, \qquad \phi(+\infty) = 1. \end{aligned} \tag{4.3}$$

As in the proof of proposition 3.1 in the previous section, we proceed in several steps.

Step 1: non-existence. Our primary concern here is what happens as $x \to -\infty$ in (4.3), for different values of ϕ_0. For this we need a Liouville type lemma.

Lemma 4.2. *Consider the family of functions* $(\phi_\lambda^-)_{\lambda \in \mathbb{R}}$*—the dependence on* ρ *is omitted, for simplicity:*

$$\forall x < \lambda, \qquad \phi_\lambda^-(x) = -\frac{16}{(x - \lambda)^3}. \tag{4.4}$$

The only increasing solutions ϕ *of the equation*

$$-\phi'' + \left(2\rho \int_{-\infty}^{x} \phi(y)\,\mathrm{d}y\right)^{\frac{1}{2}} \phi' = 0, \qquad \phi(-\infty) = 0 \tag{4.5}$$

have the form $\rho^{-1/2}\phi_\lambda^-$.

Proof. It suffices to set $\rho = 1$; the complete result then follows by scaling. Set

$$u(x) = \int_{-\infty}^{x} \phi(y)\,\mathrm{d}y, \qquad \phi(x) = \eta(u(x)), \tag{4.6}$$

where ϕ is a solution of (4.5). An equation for $\eta(u)$ is

$$-\frac{\mathrm{d}}{\mathrm{d}u}\left(\eta\frac{\mathrm{d}\eta}{\mathrm{d}u}\right)+\sqrt{2u}\frac{\mathrm{d}\eta}{\mathrm{d}u}=0,\quad \eta(0)=0,\quad \frac{\mathrm{d}\eta}{\mathrm{d}u}>0. \tag{4.7}$$

An explicit solution of (4.7), derived from ϕ_λ^-, is $\eta(u)=u^{3/2}/\sqrt{2}$. Inspired by this explicit solution we introduce the new unknown

$$p(u)=\eta(u)^{2/3}, \tag{4.8}$$

which in turn satisfies

$$-(p^2p')'+\sqrt{2u}\,p^{1/2}p'=0,\qquad p(0)=0,\quad p'>0. \tag{4.9}$$

Claim 1. The function p' is locally bounded on $\mathbb{R}_+$. To see this, observe that since ϕ is increasing, and $\phi'>0$, we have $\phi''\leqslant C\phi'$ for $x<0$, with $C=\left(2\int_{-\infty}^0\phi(y)\,\mathrm{d}y\right)^{1/2}$. This implies $\phi'\leqslant C\phi$ or, equivalently, $\sqrt{p}\,p'\leqslant C$. Equation (4.9) may then be integrated from 0 to yield

$$p^2p'=\int_0^u\sqrt{2vp}\,p'\,\mathrm{d}v. \tag{4.10}$$

Because $p'>0$ it follows that

$$p^2p'\leqslant\sqrt{2up}\int_0^u p'\,\mathrm{d}v=\sqrt{2u}\,p^{3/2}. \tag{4.11}$$

Hence, we have $p^{3/2}\leqslant Cu^{3/2}$ or $p\leqslant Cu$. We may use this information in (4.10) and infer that

$$p^2p'\geqslant C\int_0^u pp'\,\mathrm{d}v=\frac{3}{4\sqrt{2}}p^2,$$

so that $p'\geqslant C>0$ and $p(u)\geqslant Cu$. Now we may go back to (4.11) and conclude that $p'\leqslant C_2<+\infty$.

Claim 2. The derivative $p'(u)$ has a limit as $u\to 0$. Expand (4.9) to get

$$-\frac{pp''}{2}-p'^2+\sqrt{\frac{u}{2p}}\,p'=0,\qquad p(0)=0, \tag{4.12}$$

and set

$$\underline{l}=\liminf_{u\to0}p'(u),\qquad \bar{l}=\limsup_{u\to0}p'(u).$$

Assume that $\underline{l}<\bar{l}$; then there exist two sequences $\underline{u}_n$ and $\bar{u}_n$, going to 0 as $n\to+\infty$, such that (i) we have $\lim_{n\to+\infty}p'(\underline{u}_n)=\underline{l}$ and $\lim_{n\to+\infty}p'(\bar{u}_n)=\bar{l}$ and (ii) $\bar{u}_n$ and $\underline{u}_n$ are, respectively, a local maximum and a local minimum for p'. Equation (4.12) implies

$$p'(\underline{u}_n)=\sqrt{\frac{\underline{u}_n}{2p(\underline{u}_n)}},\qquad p'(\bar{u}_n)=\sqrt{\frac{\bar{u}_n}{2p(\bar{u}_n)}}.$$

By the mean value theorem, there is $(\underline{v}_n,\bar{v}_n)$ such that $0\leqslant\underline{v}_n\leqslant\underline{u}_n$, $0\leqslant\bar{v}_n\leqslant\bar{u}_n$, so that

$$p'(\underline{u}_n)=\sqrt{\frac{1}{2p'(\underline{v}_n)}},\qquad p'(\bar{u}_n)=\sqrt{\frac{1}{2p'(\bar{v}_n)}},$$

yielding in turn

$$\underline{l}\geqslant\sqrt{\frac{1}{2\bar{l}}},\qquad \bar{l}\leqslant\sqrt{\frac{1}{2\underline{l}}}$$

and finally $-\underline{l}\geqslant\bar{l}$, a contradiction. As a consequence, we have $\underline{l}=\bar{l}=p'(0)=2^{-1/3}$.

Claim 3. We have $p(u) = u/2^{1/3}$. First assume p' to have both a global minimum $\underline{u}_0$ and a global maximum $\overline{u}_0$ on an interval $(0, r)$. Equation (4.12) and the mean value theorem imply that

$$p'(\underline{u}_0) \geqslant \sqrt{\frac{1}{2p'(\overline{u}_0)}}, \qquad p'(\overline{u}_0) \leqslant \sqrt{\frac{1}{2p'(\underline{u}_0)}}$$

and thus $p'(\underline{u}_0) = p'(\overline{u}_0)$. If p' has neither a minimum nor a maximum on intervals of the form $(0, r)$, then p'' has a constant sign; assume $p'' \geqslant 0$. Equation (4.12) then implies

$$p'(u) \leqslant \sqrt{\frac{u}{2p(u)}} \leqslant \sqrt{\frac{1}{2p'(0)}} = p'(0).$$

Hence, in this case we have $p'(u) = p'(0)$. The case $p'' \leqslant 0$ is treated in the same fashion. The only cases that remain to be treated are (i) when p' has a global minimum but no global maximum and (ii) the converse case. Assume for instance that (i) holds. Then there exists $u_1 > 0$ such that $p'' \leqslant 0$ on $(0, u_1)$ and $p'' \geqslant 0$ on $\{u \geqslant u_1\}$. Once again, we apply at that point both equation (4.9) and the mean value theorem: there exists $u_2 \in (0, u_1)$ such that

$$p'(u_1) = \sqrt{\frac{1}{2p'(u_2)}} \geqslant 2^{-1/3} = p'(0).$$

Consequently, p' is constant on $[0, u_1]$, hence everywhere else. □

Proof of the non-existence part of theorem 4.1 . Let us first take $\phi_0 = 0$ and prove that when $\rho > 0$ is sufficiently large, equation (4.3) has no solution for any $c > 0$. If ϕ is a solution, note that we have, classically: $\phi' > 0$. This is seen by the standard integration of (4.3) from x to $+\infty$, taking $f(\phi)$ as a non-negative right side.

Let us multiply (4.3) by ϕ' and integrate on $\mathbb{R}$. This yields

$$\int_{-\infty}^{0} \left(c^2 + 2\rho \int_{-\infty}^{x} \phi(y)\, \mathrm{d}y\right)^{1/2} \phi'^2(x)\, \mathrm{d}x \leqslant \int_0^1 f(\phi)\, \mathrm{d}\phi := M. \tag{4.13}$$

We may always assume that $\phi(0) = \theta$. Therefore, as ϕ is increasing, $\phi(x) \leqslant \theta$ for $x \leqslant 0$ so that $f(\phi) = 0$ there and we have

$$-\phi'' + \left(c^2 + 2\rho \int_{-\infty}^{x} \phi(y)\, \mathrm{d}y\right)^{1/2} \phi' = 0, \qquad \text{for } x \leqslant 0 \text{ and } \phi(0) = \theta. \tag{4.14}$$

Hence

$$\left(c^2 + 2\rho \int_{-\infty}^{x} \phi(y)\, \mathrm{d}y\right)^{1/2} = \frac{\phi''}{\phi'}$$

and inequality (4.13) becomes

$$(\phi'(0))^2 \leqslant 2M. \tag{4.15}$$

We therefore have to estimate $\phi'(0)$ in terms of ρ. Let us first note that $\phi(x) \leqslant \theta \mathrm{e}^{cx}$, which, as $\phi(0) = \theta$, implies

$$\phi'(0) \geqslant c\theta, \qquad \text{hence, by (4.15) } c \leqslant \sqrt{\frac{2M}{\theta}}. \tag{4.16}$$

To proceed further let us make the following change in the unknown variable, as in the proof of lemma 4.2:

$$u(x) = \int_{-\infty}^{x} \phi(y)\, \mathrm{d}y, \qquad \phi(x) = \eta(u(x)), \tag{4.17}$$

where ϕ is a solution of (4.14). An equation for $\eta(u)$ is

$$-\frac{d}{du}\left(\eta\frac{d\eta}{du}\right)+(c^2+2\rho u)^{1/2}\frac{d\eta}{du}=0, \qquad \eta(0)=0, \quad \frac{d\eta}{du}>0. \tag{4.18}$$

Then we set

$$p(u)=\eta(u)^{2/3}, \tag{4.19}$$

which in turn satisfies

$$-(p^2p')'+\left((c^2+2\rho u)p\right)^{1/2}p'=0, \qquad p(0)=0, \quad p'>0. \tag{4.20}$$

As in the proof of claim 1 in lemma 4.2, we have

$$p^2p'=\int_0^v\sqrt{(c^2+2\rho w)p(w)}\,p'(w)\,dw, \tag{4.21}$$

and hence—by the same argument as in the proof of claim 1 in lemma 4.2, we get

$$p^2p'=\int_0^u\sqrt{(c^2+2\rho w)p(w)}\,p'(w)\,dw\leqslant\sqrt{(c^2+2\rho u)p(u)}\,p(u).$$

It follows that

$$p(u)^{3/2}\leqslant K\int_0^u\sqrt{c^2+2\rho w}\,dw\leqslant K\sqrt{\rho}\left[\left(\frac{c^2}{2\rho}+u\right)^{3/2}-\left(\frac{c^2}{2\rho}\right)^{3/2}\right]\leqslant\frac{K}{\rho}(c^2+2\rho u)^{3/2}. \tag{4.22}$$

As a consequence, we have

$$p(u)\leqslant\frac{K}{\rho^{2/3}}(c^2+2\rho u).$$

This, in turn, implies

$$p^2p'=\int_0^u\sqrt{(c^2+2\rho w)p(w)}\,p'(w)\,dw\geqslant K\rho^{1/3}\int_0^u p(w)p'(w)\,dw=K\rho^{1/3}p^2(u)$$

so that $p(u)\geqslant K\rho^{1/3}u$. The function $\phi(x)$ thus satisfies the inequality

$$\phi(x)\geqslant K\rho^{1/2}u^{3/2},$$

which may be rewritten as

$$u'(x)\geqslant K\rho^{1/2}u^{3/2}. \tag{4.23}$$

The second inequality in (4.22) implies also the following upper bound for $u'(x)$:

$$u'(x)\leqslant\frac{K}{\rho}\left[(c^2+2\rho u)^{3/2}-c^3\right]. \tag{4.24}$$

Recall that $\phi(0)=u'(0)=\theta$—hence, we have from (4.23)

$$u(0)\leqslant K\left(\frac{\theta}{\rho^{1/2}}\right)^{2/3}\leqslant\frac{K}{\rho^{1/3}}, \tag{4.25}$$

while from (4.24) we obtain

$$u(0)\geqslant\frac{K}{\rho^{1/3}}-\frac{c^2}{2\rho}\geqslant\frac{K}{2\rho^{1/3}} \tag{4.26}$$

for $\rho>\rho_0$. We used (4.16) in the last step given above. Another consequence of (4.23) is that for $x\leqslant 0$ we have

$$\frac{1}{\sqrt{u(x)}}-\frac{1}{\sqrt{u(0)}}\geqslant K\rho^{1/2}|x|,$$

so that

$$u(x) \leqslant \frac{u(0)}{(1+\sqrt{u(0)}K\rho^{1/2}|x|)^2}.$$

With the help of (4.25) and (4.26) this becomes

$$u(x) \leqslant \frac{K}{\rho^{1/3}}\left(1+\frac{K}{\rho^{1/6}}\rho^{1/2}|x|\right)^{-2} \leqslant \frac{K}{(\rho^{1/6}+\rho^{1/2}|x|)^2}.$$

As a consequence, we have $u(-\rho^{-m}) \leqslant K/\rho^{1-2m}$ for $m \in (0, 1/3)$ and thus we get from (4.24)

$$\phi(\rho^{-m}) \leqslant \frac{K}{\rho}\rho^{3m} = K\rho^{3m-1} \ll 1.$$

As $\phi(0) = \theta$, there exists a point $\xi \in (-\rho^{-m}, 0)$ so that $\phi'(\xi) \geqslant K\rho^m$. However, the function ϕ is convex, thus $\phi'(0) \geqslant K\rho^m$ which contradicts (4.15) if ρ is large enough. This contradiction shows that no solution may exist for a sufficiently large ρ.

Step 2: existence for small ρ. The reason why solutions may exist when ρ is small is quite easy to understand: when $\rho = 0$, (4.3) reduces to

$$-\phi'' + c\phi' = f(\phi), \qquad \phi(-\infty) = 0, \quad \phi(+\infty) = 1,$$

which, of course, has a unique solution $(c_0, \bar{\phi})$, up to translation, when the nonlinearity f is of the ignition type. The idea is that the standard non-degeneracy property of this solution allows its continuation for tiny values of ρ. However, there is a technical point that makes the programme not completely trivial: the term $\rho\int_{-\infty}^{x}\phi(y)\,\mathrm{d}y$ grows linearly as $x \to +\infty$, whereas we would like to treat it as a small perturbation. This forces us to work a little more with the *a priori* estimates. The main element that will make a perturbation argument work is the following lemma.

Lemma 4.3. *There exists $\rho_0 > 0$ such that, for all $\rho \leqslant \rho_0$, for every $h(x) \in \mathrm{BUC}(\mathbb{R}_+)$—the space of all bounded, uniformly continuous functions on $\mathbb{R}_+$—there exists a unique $u(x) \in \mathrm{BUC}(\mathbb{R}_+)$ satisfying the boundary value problem*

$$L_\rho u := -u'' + \left(c_0^2 + 2\rho\int_{-\infty}^{x}\bar{\phi}(y)\,\mathrm{d}y\right)^{\frac{1}{2}} u' - f'(\bar{\phi})u = h \quad (x > 0), \qquad u(0) = 0. \tag{4.27}$$

Moreover there exists $C > 0$ independent of ρ such that

$$\|u\|_\infty \leqslant C\|h\|_\infty, \tag{4.28}$$

and the map $\rho \mapsto (u(0), u'(0))$ is continuous on $[0, \rho_0]$. Finally, if in addition $h \in L^1(\mathbb{R}_+)$, we have

$$\|u\|_{W^{1,1}(\mathbb{R}_+)\cap W^{1,\infty}(\mathbb{R}_+)} \leqslant C\|h\|_{L^1(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+)}. \tag{4.29}$$

Proof. What really matters is estimate (4.28). Once this property is at hand, (4.27) may be approximated by the following sequence of problems:

$$L_n u := -u'' + \left(c_0^2 + 2\rho\int_{-\infty}^{\max(x,n)}\bar{\phi}(y)\,\mathrm{d}y\right)^{\frac{1}{2}} u' - f'(\bar{\phi})u = h \quad (x > 0), \qquad u(0) = 0. \tag{4.30}$$

The operator L_n is Fredholm and estimate (4.28) is—as will become clear in the proof of the lemma—still valid for the solutions of (4.30). One then concludes by a standard compactness argument. The continuity of the map $\rho \mapsto (u(0), u'(0))$ is also inferred from compactness. Estimate (4.29), which is the main result, will easily follow.

Let us therefore assume that we have constructed a solution $u(x)$ to (4.27), and let us estimate it. Let us rewrite (4.27) as

$$-u'' + c_0 u' - f'(\bar\phi)u = l[u] := h - \frac{2\rho \displaystyle\int_{-\infty}^{x} \bar\phi(y)\,\mathrm{d}y}{c_0 + \sqrt{c_0^2 + 2\rho \displaystyle\int_{-\infty}^{x} \bar\phi(y)\,\mathrm{d}y}} u' \; (x > 0), \qquad u(0) = 0.$$

This problem has an explicit solution

$$u(x) = \int_0^x \frac{\bar\phi'(x)}{\bar\phi'(y)} \int_y^{+\infty} \frac{\bar\phi'(z)}{\bar\phi'(y)} \mathrm{e}^{c_0(y-z)} l[u](z)\,\mathrm{d}y\,\mathrm{d}z.$$

Using integration by parts to eliminate the derivative of u in the function l and the exponential decay of $\bar\phi'(x)$, we deduce that there exists a constant $C > 0$ independent of ρ such that

$$|u(x)| \leqslant C(\|h\|_\infty + \sqrt{\rho x}\|u\|_\infty). \tag{4.31}$$

Consider now $x_0 > 0$ and $\delta > 0$ such that $-f'(\bar\phi) \geqslant \delta$ on $[x_0, +\infty)$. On that interval equation (4.27) and the maximum principle yield that $u(x)$ cannot attain a maximum at a point where its value is larger than $\|h\|_\infty/\delta$. On the other hand, if $u(x)$ is monotonic on an infinite half-interval we set $u_\varepsilon(x) = u(x) - \varepsilon\bar u(x)$, where $\bar u(x)$ is

$$\bar u(x) = \int_0^x \exp\left(\int_0^y a(z)\,\mathrm{d}z\right)\mathrm{d}y, \qquad a(x) = \left(c_0^2 + 2\rho \int_{-\infty}^x \bar\phi(y)\,\mathrm{d}y\right)^{\frac12}.$$

As the function $a(y)$ tends to $+\infty$ as $x \to +\infty$, it follows that $u_\varepsilon(+\infty) = -\infty$ and thus has to attain a local maximum. Applying the maximum principle to u_ε and passing to the limit $\varepsilon \to 0$ we conclude that

$$\forall x \geqslant x_0, \qquad |u(x)| \leqslant |u(x_0)| + \frac{\|h\|_\infty}{\delta}.$$

Combining this with (4.31) yields

$$\|u\|_{L^\infty([x_0,+\infty))} \leqslant C(\|h\|_\infty + \sqrt{\rho x_0}\|u\|_\infty) + \frac{\|h\|_\infty}{\delta}. \tag{4.32}$$

But then (4.31) implies that

$$\|u\|_{L^\infty([0,x_0])} \leqslant C(\|h\|_\infty + \sqrt{\rho x_0}\|u\|_\infty). \tag{4.33}$$

Adding up (4.32) and (4.33), then choose ρ so that $C\sqrt{\rho x_0} < 1$ yields (4.28). We note that the above argument is valid for the family of operators L_n given by (4.30): inequality (4.32) does not change as long as $x_0 \leqslant n$, and this inequality does not use any bound on the first order term of L_n.

Finally, let us assume in addition that $h \in L^1(\mathbb{R}_+)$. We set for convenience

$$a(x) = \sqrt{c_0^2 + 2\rho \int_{-\infty}^x \bar\phi(y)\,\mathrm{d}y};$$

then we have

$$|a'(x)| \leqslant \rho c_0^{-1}. \tag{4.34}$$

With this fact in mind, we multiply (4.27) by sgn $u(x)$ and integrate it over $(0,+\infty)$. We get, after integration by parts:

$$\int_{x_0}^{+\infty}(\delta - a'(x))|u(x)|\,\mathrm{d}x \leqslant \|h\|_{L^1} + \int_0^{x_0}|f'(\bar{\phi})u|\,\mathrm{d}x.$$

Here x_0 and δ are chosen as in the previous step of the proof. The upper bound (4.34) and the L^∞ bound for u that we have already obtained imply that for a sufficiently small ρ we have an L^1-bound for u: $\|u\|_{L^1} \leqslant C\|h\|_{L^1\cap L^\infty}$. In order to improve it to a $W^{1,1}$ bound we note that the L^1-estimate for u and the fact that $h \in L^1$ imply that

$$u'(x) = -\int_x^{+\infty} g(y)\exp\left(-\int_x^y a(z)\,\mathrm{d}z\right)\mathrm{d}y,$$

with a function $g \in L^1$. As $a(x)$ is uniformly bounded from below by a positive constant, it follows that $u' \in L^1 \cap L^\infty$. □

The construction of a solution (c,ϕ) to (4.3) can now be done: it is a classical derivative matching problem. First, let us add to (4.3) the normalization condition $\phi(0) = \theta$. We know that a solution $\phi(x)$ of (4.3) has to be increasing—therefore, $f(\phi) \equiv 0$ for $x \leqslant 0$. The equation for ϕ is thus

$$-\phi'' + \left(c^2 + 2\rho\int_{-\infty}^x \phi(y)\,\mathrm{d}y\right)^{\frac{1}{2}}\phi' = 0 \quad \text{for } x < 0; \qquad \phi(-\infty) = 0, \quad \phi(0) = \theta. \tag{4.35}$$

We pick any $\mu \in (0, c_0/5)$ and c such that $|c - c_0| < 2\mu$. The implicit function theorem in the space $\{u \in \mathrm{BUC}(\mathbb{R}), \mathrm{e}^{-\mu x}u \in \mathrm{BUC}(\mathbb{R}_-)\}$ yields, for $\rho > 0$ small enough, the existence of a unique solution $\phi^-_{c,\rho}$ to (4.35). Moreover we have

$$\frac{\mathrm{d}\phi^-_{c,\rho}}{\mathrm{d}x}(0) = c\theta + O(\rho). \tag{4.36}$$

The details are standard and are therefore omitted.

Let us turn to the problem on the right half-line:

$$\begin{aligned} &-\phi'' + \left[c^2 + 2\rho\int_{-\infty}^x (\phi(y) - \phi_0)\,\mathrm{d}y\right]^{\frac{1}{2}}\phi' = f(\phi), \\ &\phi(0) = \theta, \quad \phi(+\infty) = 1, \end{aligned} \tag{4.37}$$

with the additional constraint $\phi'(0) = \phi^-_{c,\rho}(0)$. We look for a solution (c,ϕ) to (4.37) in the form $(c_0 + d, \bar{\phi} + u)$. We also extend ϕ to $\phi^-_{c,\rho}$ on $\mathbb{R}_-$—this is only necessary to assign a value to the integrals between $-\infty$ and x in (4.37). Write the problem as

$$\begin{aligned} &L_\rho u = g[u], \\ &u(0) = 0, \qquad u \in W^{1,1}(\mathbb{R}_+), \end{aligned} \tag{4.38}$$

with the operator L_ρ defined in (4.27),

$$\begin{aligned} g[u] = K[\phi]u^2 &- \frac{(c+c_0)d + 2\rho\int_{-\infty}^x u(y)\,\mathrm{d}y}{\sqrt{c_0^2 + 2\rho\int_{-\infty}^x \bar{\phi}(y)\,\mathrm{d}y} + \sqrt{c^2 + 2\rho\int_{-\infty}^x \bar{\phi}(y)\,\mathrm{d}y}}(u' + \bar{\phi}') \\ &- \frac{2\rho\int_{-\infty}^x u(y)\,\mathrm{d}y}{\sqrt{c_0^2 + 2\rho\int_{-\infty}^x \bar{\phi}(y)\,\mathrm{d}y} + \sqrt{c^2 + 2\rho\int_{-\infty}^x \bar{\phi}(y)\,\mathrm{d}y}}\bar{\phi}' \end{aligned}$$

and $K[\phi]u^2 = f(\bar\phi+u) - f(\bar\phi) - f'(\bar\phi)u$. Lemma 4.3 asserts that L_ρ is invertible and that L_ρ^{-1} sends $L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ to $W^{1,1}(\mathbb{R}_+) \cap W^{1,\infty}(\mathbb{R}_+)$ and thus equation (4.38) is equivalent to

$$u = L_\rho^{-1}(g[u]). \tag{4.39}$$

The Banach fixed point theorem yields the existence of two positive numbers ρ_0 and δ_0 such that, for each $(d, \rho) \in [-\delta_0, \delta_0] \times [0, \rho_0]$, equation (4.39) has a unique solution $u^+_{c,\rho}$ of the size $|d| + \rho$—and therefore the resulting $\phi^+_{c,\rho} = \bar\phi + u^+_{c,\rho}$ is $|d| + \rho$-close to $\bar\phi$.

Now, problem (4.3) reduces to the following equation: given ρ in some subinterval of $[0, \rho_0]$ containing 0, find c close to c_0 so that the equation

$$\frac{\mathrm{d}\phi^+_{c,\rho}}{\mathrm{d}x}(x=0) = \frac{\mathrm{d}\phi^-_{c,\rho}}{\mathrm{d}x}(x=0). \tag{4.40}$$

Now, a well-known Melnikov-type computation gives

$$\left.\frac{\partial^2\phi^+_{c,\rho}}{\partial c\partial x}\right|_{x=0,\rho=0,c=c_0} = -\frac{1}{c_0\theta}\int_0^{+\infty} \mathrm{e}^{-c_0 x}\bar\phi'^2(y)\,\mathrm{d}y.$$

This, combined with (4.36), implies that (4.40) can be solved uniquely, provided that ρ is chosen small enough. This ends the small ρ construction part and the proof of part (i) of theorem 4.1.

Step 3: proof of part (ii) of theorem 4.1.—the large values of ρ. Recall that we are looking for a pair (ϕ_0, ϕ) satisfying (4.3) with $c = 0$. We may impose the normalization condition $\phi(0) = \theta$, and the solution ϕ is increasing on $\mathbb{R}$. This implies, by lemma 4.2, that

$$\phi = \phi_0 + \rho^{-1/2}\phi_\lambda \qquad \text{for } x \leqslant 0, \tag{4.41}$$

with $\lambda = (16/(\theta-\phi_0)\sqrt{\rho})^{1/3}$. In particular, we have

$$\int_{-\infty}^0 [\phi(y) - \phi_0]\,\mathrm{d}y = 2^{1/3}(\theta-\phi_0)^{2/3}\rho^{-1/6} := c_\rho(\phi_0), \quad \phi'(0^-) = \frac{3}{2^{4/3}}(\theta-\phi_0)^{4/3}\rho^{1/6}. \tag{4.42}$$

The strategy is, once again, a shooting argument: we are going to solve the following Cauchy problem:

$$\begin{aligned} &-\phi'' + \left(c_\rho(\phi_0) + 2\rho\int_0^x \phi\right)^{\frac{1}{2}}\phi' = f(\phi), \\ &\phi(0) = \theta, \quad \phi'(0^+) = \frac{3}{2^{4/3}}(\theta-\phi_0)^{4/3}\rho^{1/6}, \end{aligned} \tag{4.43}$$

and adjust ϕ_0 so that

$$\lim_{x\to+\infty}\phi(x) = 1. \tag{4.44}$$

For all $\rho > 0$ and $\phi_0 > 0$ the Cauchy problem (4.43) has a unique maximal solution ϕ_{ρ,ϕ_0} defined on an interval of the form $[0, x_{\max}(\rho,\phi_0))$. Exactly as in section 3, for each $\rho > 0$ we define the following subsets of $[0, \theta]$:

$$\begin{aligned} X_-^\rho &= \{\phi_0 \in [0,\theta] :\ \exists x_0 > 0 \text{ such that } \phi_{\rho,\phi_0} = \theta\}, \\ X_+^\rho &= \{\phi_0 \in [0,\theta] :\ \phi_{\rho,\phi_0} > 0 \text{ and } x_{\max}(\rho,\phi_0) < +\infty\}. \end{aligned} \tag{4.45}$$

Lemma 4.4. *For every* $\rho > 0$ *there exists* $\overline{\phi}_0(\rho) > 0$ *such that every* $\phi_0 \in (\overline{\phi}_0(\rho), \theta]$ *belongs to* X_-^ρ.

Proof. Given $\rho > 0$, the Cauchy problem

$$\begin{aligned} &-\phi'' = f(\phi), \\ &\phi(0) = \theta,\ \phi'(0^+) = \frac{3}{2^{4/3}}(\theta - \phi_0)^{4/3}\rho^{1/6} \end{aligned} \tag{4.46}$$

has—by an easy explicit computation—a unique solution $\underline{\phi}$ which is larger than θ exactly on a finite interval $(0, \underline{x}(\rho, \phi_0))$ provided that $\phi_0 > 0$ is close enough to θ. The difference between $\underline{\phi}$ and ϕ_{ρ,ϕ_0} is easily estimated via the Gronwall lemma for large ρ since both $c_\rho(\phi_0)$ and $\underline{x}(\rho, \phi_0)$ are small. □

Because a solution ϕ to (4.3) is increasing in x, we are not really interested in the values of f outside the interval $(0, 1)$. Therefore we may extend f by 0 outside $(0, 1)$.

Lemma 4.5. *For every $\phi_0 < \theta$ there exists $\rho_0 > 0$ so that we have $\phi_0 \in X_+^\rho$ for all $\rho > \rho_0$.*

Proof. Quite the same as in lemma 3.4. The logarithmic derivative of $\phi = \phi_{\rho,0}$, denoted by ζ, satisfies

$$\zeta' = a(x)\zeta - \zeta^2 - \frac{f(\phi)}{\phi}, \tag{4.47}$$

where we have suppressed the subscripts and where we have set

$$a(x) = \sqrt{c_\rho(\phi) + 2\rho \int_0^x \phi(y)\, \mathrm{d}y}. \tag{4.48}$$

This equation, due to the boundedness of $f(\phi)/\phi$, is essentially the same as (3.29). In particular, if $\zeta(0)$ is large enough we have $\zeta > 0$ on its existence interval and $x_{\max}(\rho, 0) < +\infty$. This implies that $\phi_0 \in X_+^\rho$ if $\zeta(0)$ is sufficiently large. However, as $\zeta(0)$ is proportional to $\rho^{1/6}$, this is the case for a sufficiently large ρ. □

End of the proof of theorem 4.1. Take $\rho > 0$ large enough so that lemma 4.5 holds. As opposed to the construction of the wave fan–rarefaction wave, where the fact that the sets $X_\pm$ were open was non-trivial, here it is very easy to infer from the continuity of the solution of the Cauchy problem (4.43), with respect to its initial values, that the sets $X_\pm^\rho$ are open and non-empty. Consequently, there exists at least one ϕ_0 not in $X_+^\rho \cup X_-^\rho$—we need to show now that the solution generated by ϕ_0 tends to one as $x \to +\infty$. The strong maximum principle implies that the corresponding solution ϕ cannot have a local minimum in $\mathbb{R}_+$ and hence it is increasing. Assume that ϕ goes over the value 1 and let x_0 be such that $\phi(x_0) = 1$. Setting $\overline{c}_\rho(\phi_0) = c_\rho(\phi_0) + 2\rho \int_0^{x_0} \phi(y)\, \mathrm{d}y$, the equation for ϕ is

$$\begin{aligned} &-\phi'' + \left(\overline{c}_\rho(\phi_0) + 2\rho \int_{x_0}^x \phi\right)^{\frac{1}{2}} \phi' = 0, \\ &\phi(x_0) = 1, \qquad \phi' > 0. \end{aligned} \tag{4.49}$$

Equation (4.49) implies that $\phi' > 0$ and ϕ is convex on $(x_0, +\infty)$. Hence, the derivative $\phi'(x)$ has a limit $l > 0$ as $x \to +\infty$. This implies, once again by (4.49):

$$\phi''(x) \sim l\sqrt{\rho l x},$$

and thus l cannot be finite. Therefore, repeating the argument in the proof of lemma 3.4 we conclude that ϕ becomes infinite at a finite distance, which is a contradiction. Therefore, we have $\phi < 1$; hence, (4.44) is true. Finally, lemma 4.5 implies that (4.2) holds. □

Asymptotic behaviour of the travelling wave at $+\infty$. We end this section by additional information on the behaviour of a travelling wave solution of (4.3). The last lemma of this section is an estimate of how the wave solution converges to its rest state at $+\infty$ which we will need in the construction of an asymptotic solution when we match the travelling wave to the back of the shock. Due to the linear growth of the advection term, it is not a standard version of the stable manifold theorem. We could, at not too high a cost, derive a precise asymptotic expansion. The following weaker version will be sufficient for our purpose.

Lemma 4.6. *Let $\phi(x)$ be a solution of (4.3) and set $\alpha_0 = (-f'(1))\sqrt{2/\rho}$. There exists $B(\rho) > 0$ so that for each $\varepsilon > 0$, there exist $C_\varepsilon(\rho), C'_\varepsilon(\rho) > 0$ such that*

$$1 - C_\varepsilon \exp(-(\alpha_0 - B\varepsilon)\sqrt{x}) \leqslant \phi(x) \leqslant 1 - C'_\varepsilon \exp(-(\alpha_0 + B\varepsilon)\sqrt{x}) \tag{4.50}$$

and

$$0 \leqslant \phi'(x) \leqslant B\mathrm{e}^{-\alpha_0\sqrt{x}/2}.$$

Proof. The function $q(x) = 1 - \phi(x)$ satisfies

$$\mathrm{NL}(q) := -q'' + a(x)q' + f(1-q) = 0, \qquad q(-\infty) = 1, \qquad q(+\infty) = 0, \tag{4.51}$$

where

$$a(x) = \left(c^2 + 2\rho\int_{-\infty}^{x} \phi(y)\,\mathrm{d}y\right)^{1/2}.$$

Let us choose $x_\varepsilon > 1/\varepsilon^2$ so that for all $x \geqslant x_\varepsilon$ we have

$$(-f'(1) - \varepsilon)q \leqslant f(1-q) \leqslant (-f'(1) + \varepsilon)q$$

and

$$\sqrt{2\rho x}(1-\varepsilon) \leqslant a(x) \leqslant \sqrt{2\rho x}(1+\varepsilon).$$

Let us find a supersolution $\bar{q}(x) \geqslant 0$ such that $\bar{q}(x_\varepsilon) = q(x_\varepsilon)$, $\bar{q}' \leqslant 0$, $\bar{q}(+\infty) = 0$ and $\mathrm{NL}[\bar{q}] \geqslant 0$. This will imply that $q(x) \leqslant \bar{q}(x)$ and thus provide the lower bound on $\phi(x)$ in (4.50). For the last condition to hold it is sufficient to require that

$$\bar{L}[\bar{q}] := -\bar{q}'' + \sqrt{2\rho x}(1+\varepsilon)\bar{q}' + (-f'(1) - \varepsilon)\bar{q} \geqslant 0.$$

For a function of the form $s(x) = q(x_\varepsilon)\exp(-\alpha\sqrt{x - x_\varepsilon})$ we have

$$\bar{L}[s] = s(x)\left(O\left(\frac{1}{x}\right) + (-f'(1) - \varepsilon) - \frac{\alpha\sqrt{\rho}}{\sqrt{2}}(1+\varepsilon)\right).$$

Hence, for such a function to be a supersolution it is sufficient to take

$$\alpha < \sqrt{\frac{2}{\rho}}\left(\frac{-f'(1) - \varepsilon}{1+\varepsilon}\right), -\varepsilon,$$

so $\alpha \leqslant \alpha_0 - B(\rho)\varepsilon$ with $B(\rho)$ sufficiently large will suffice. The upper bound on ϕ in (4.50) is proved similarly. The bound on the derivative ϕ' in lemma 4.6 is obtained by differentiating (4.51):

$$-(q')'' + a(x)(q')' + (-f'(1-q) + a'(x))q' = 0.$$

As $a'(x) = O(1/\sqrt{x})$ for large x, the exponential bound for $q' = -\phi'$ follows by the same construction of subsolutions and supersolutions. $\square$

5. Large-time evolution: asymptotic solutions

The numerical simulations of section 2 indicate that, if the support of the initial data for temperature $T(0, x)$—or, at least, the measure of the set where it is above ignition—is very large, the solution has the following structure: it has the form of a ramp on the left, followed by a combustion wave, which is itself terminated by a shock that brings back both temperature and velocity to their rest states. This structure appears after some transient behaviour that we will not study in this paper and remains valid almost all the time before quenching occurs. In this section we derive an asymptotic relation for the shock position and discuss the time interval on which the above picture is valid. What happens after this time is the subject of section 6.

It is clear from the numerical simulations that quenching will be provoked by the dissipation at the accelerated shock and that the shock location is really what will eventually tell us the dynamics of our solution. In order to get an equation for the shock motion we have to construct the whole asymptotic solution, gluing together asymptotically the ramp, combustion wave and the shock constructed in the previous sections. First, we should identify the small parameter that controls the asymptotics—this is the aim of section 5.1. In section 5.2, we will see how to glue the ramp to the combustion wave; the role of the ramp being played by the self-similar solution constructed in section 3, and the role of the combustion wave will be played by the travelling wave constructed in section 4. In section 5.3 we will place the shock, thus terminating the description of the asymptotic solution.

5.1. Devising a length and time scale

Let us recall that if $y_{\mathrm{f}}(t)$ is the position of the shock relative to the endpoint of the ramp, that is the transition point in the temperature profile between the ramp and the combustion wave, an asymptotic equation for $y_{\mathrm{f}}(t)$ is given by (2.13):

$$\dot{y}_{\mathrm{f}} = \frac{1}{2}(\sqrt{2\rho(1-\phi_0)y_{\mathrm{f}}} - \rho\phi_0 t). \tag{5.1}$$

Here ϕ_0 is the value of the temperature at the transition. We will assume that $\phi_0 > 1/9$—recall that this ensures that (5.1) has no global in time solution $y_{\mathrm{f}}(t)$, hence $y_{\mathrm{f}}(t)$ reaches zero in a finite time—this is the time when quenching occurs since the transition value ϕ_0 is below the ignition temperature θ. Let us now worry about how large should be the support of the temperature for the solution to maintain the 'ramp–wave–shock' structure for a long time—this length will be our large parameter with respect to which we shall expand our solution.

Let us set

$$Q(z) = 2\int_0^z \frac{\mathrm{d}z'}{f(\sqrt{z'})}$$

and choose a pair of large positive numbers: (t_0, x_0). We specify the initial datum of y_{f} as $y_{\mathrm{f}}(t_0) = x_0$. The reason why we wish to start the integration of (5.1) from a large time t_0 will become clear in section 5.3. In a few words: we want to make sure that the transition layer (the shock width) in which the temperature goes from 1 to 0 is very narrow. From (2.14) we have, for $t \geqslant t_0$, using expression (2.14) for $f(\sqrt{z})$:

$$y_{\mathrm{f}}(t) = \rho t^2 Q^{-1}\left(\ln\frac{t}{t_0} + Q\left(\frac{x_0}{\rho t_0^2}\right)\right) \sim_{x_0/(\rho t_0)^2\to+\infty} \rho t^2 Q^{-1}\left(\ln\frac{t}{\sqrt{\rho^{-1}x_0}}\right). \tag{5.2}$$

Thus, for $t \sim \sqrt{\rho^{-1}x_0}$ we have $y_{\mathrm{f}}(t) \sim 0$, meaning that the shock has been caught up by the ramp—thus, presumably, that quenching has occurred around that time. The parameter x_0 will from then be the large parameter; we call it ε^{-1}, with $\varepsilon > 0$. The time interval over which we want to construct an approximate solution to (1.5) runs from t_0 to approximately $(\rho\varepsilon)^{-1/2}$. Recall that we want t_0 also very large; call it δ^{-1} and δ will be another small parameter. Our sole requirement for the moment is $x_0/\rho t_0^2 \gg 1$; hence $\delta \gg \sqrt{\varepsilon}$.

Before we proceed to the actual construction of the asymptotic solution, let us set the following definitions and notation: from now on, *and until the end of this section*, let us give the following names to the wave fan and travelling wave solutions to (1.5).

- A self-similar solution of (3.1) will be denoted by $t^{-3/2}(\phi_-, \psi_-)(x/\sqrt{t})$. Hence, the pair (ϕ_-, ψ_-) is a solution of (3.2).
- We assume that the ignition threshold $\theta > 1/9$ and ρ is sufficiently large. Then, according to theorem 4.1, a travelling wave solution of (1.5) with $c = 0$ and a temperature that converges to a value $\phi_0 \in (0, \theta)$ as $x \to -\infty$ exists and will be denoted by $(\phi_+, \psi_+)(x)$. Recall that (ϕ_+, ψ_+) is a solution of
$$\begin{aligned} &-\phi_+'' + \psi_+\phi' = f(\phi_+),\\ &\phi_+(-\infty) = \phi_0, \qquad \phi_+(+\infty) = 1,\\ &\psi_+(x) = \sqrt{2\rho\int_{-\infty}^{x}(\phi(y) - \phi_0)\,\mathrm{d}y}. \end{aligned} \tag{5.3}$$
Theorem 4.1 also implies that for sufficiently large ρ we have $\phi_0 > 1/9$, since θ satisfies this strict inequality.
- We denote by $(T^{\mathrm{app}}(t, x), u^{\mathrm{app}}(t, x))$ the approximate solution that we wish to construct. It will therefore NOT be an exact solution to (1.5); it will satisfy (1.5) up to a small error—typically, of the order $O(\sqrt{\varepsilon} + \sqrt{\delta})$ over a time interval $O(\sqrt{\rho^{-1}\varepsilon})$.

We now have all the ingredients: a set of elementary pieces (wave fan, travelling wave, shock) and two small parameters.

5.2. *Gluing a wave fan to a combustion wave*

Recall that we are not interested at this stage in the transients leading to the development of our composite wave. Hence, we let a pure wave fan of the non-reactive equation (3.1) evolve until we are satisfied with the size of its support and of its derivatives. Then we translate the profile so that the temperature at $x = 0$ is equal (perhaps up to some δ-correction) to ϕ_0. This will provide our initial datum to the left. To glue it to a combustion wave at later times, we proceed as follows: we resume the evolution of the wave fan and consider the place where it reaches the value ϕ_0-modulo, once again, a δ-correction. If the δ-correction is chosen carefully enough, it will be possible to translate the wave profile, then slightly modify the temperature, in such a way that the temperature component of the modified combustion wave matches exactly the wave fan and its slope. The velocity will then be set according to the equation for ψ_+ in (5.3). This will provide a δ-approximate solution to (1.5), over a large time interval.

The reference frame. Let us therefore consider a solution (ϕ_-, ψ_-) of (3.2). Choose $\tau_- > 0$ such that

$$\tau_- \geqslant \delta^{-1} \quad \text{and} \quad \forall\tau \geqslant \tau_-, \qquad \forall x, \quad \frac{1}{\rho\tau^{3/2}}\phi_-'\left(\frac{x}{\sqrt{\tau}}\right) \leqslant \min(\delta^2, \tau^{-2}). \tag{5.4}$$

This is possible since $\phi_-' \in L^\infty(\mathbb{R})$. For $\tau \geqslant \tau_-$, set

$$\tilde{T}_-(\tau, x) := (\rho\tau^{3/2})^{-1}\phi_-(x/\sqrt{\tau}), \qquad \tilde{u}_-(t, x) = \tau^{-1/2}\psi_-(x/\sqrt{\tau}). \tag{5.5}$$

We consider the places where $\tilde{T}_-$ reaches values close to ϕ_0. For that we obviously must have $x/\sqrt{\tau}$ very large. This implies $T_-(\tau, x) \sim 2x/\rho\tau^2$—hence, the gluing point $x \sim \rho\phi_0\tau^2/2$. We could try to directly consider the moving point $\tilde{x}_b(\tau) = \rho\phi_0\tau^2/2$. This is not, however, the most convenient choice. Instead, let us set

$$\tilde{x}_b(\tau) = \frac{\rho\phi_0\tau^2}{2} + \alpha\tau.$$

The subscript 'b' above stands for 'back'. Let us choose α according to the strategy that we have proposed above. The function ϕ_- comes with a function ψ_- accounting for the velocity, given by (3.5):

$$\psi_-(\eta) = \frac{\eta}{2} + \frac{1}{2}\sqrt{\eta^2 + 8\int_{-\infty}^{\eta} \phi_-(\eta')\,\mathrm{d}\eta'}. \tag{5.6}$$

By expansion (3.7) of theorem 3.1, we have, with $\eta = \tilde{x}_b/\sqrt{\tau}$:

$$\begin{aligned}\tilde{u}_-(\tau, \tilde{x}_b(\tau)) &= \frac{1}{2\sqrt{\tau}}\left[\eta + \sqrt{\eta^2 + 8\left(\eta^2 + \frac{3}{4}a\eta^{4/3} + o(\eta^{4/3})\right)}\right] = \frac{1}{2\sqrt{\tau}}[4\eta + a\eta^{1/3} + o(\eta^{1/3})] \\ &= \frac{1}{2\sqrt{\tau}}\left[\frac{4}{\sqrt{\tau}}\left(\frac{\rho\phi_0\tau^2}{2} + \alpha\tau\right) + \frac{a}{\tau^{1/6}}\left(\frac{\rho\phi_0\tau^2}{2} + \alpha\tau\right)^{1/3} + o\left(\tau^{1/2}\right)\right].\end{aligned}$$

We conclude that

$$\begin{aligned}&\tilde{u}_-(\tau, \tilde{x}_b(\tau)) = \rho\phi_0\tau + 2\alpha + \frac{a}{2}\left(\frac{\rho\phi_0}{2}\right)^{1/3} + o(1), \\ &\partial_\tau\tilde{u}_-(\tau, \tilde{x}_b(\tau)) = \rho\phi_0 + O\left(\frac{1}{\tau}\right).\end{aligned} \tag{5.7}$$

We choose α to ensure that

$$\tilde{u}_-(\tau, \tilde{x}_b(\tau)) = \dot{\tilde{x}}_b(\tau) + o(1), \tag{5.8}$$

hence $\alpha = -a(\rho\phi_0/16)^{1/3}$.

Let us now set the initial time to be $t = 1$; then for all $t \geqslant 1$ we choose

$$x_b(t) = \tilde{x}_b(\tau_- + t) = \frac{\rho\phi_0(\tau_- + t)^2}{2} - a\left(\frac{\rho\phi_0}{16}\right)^{1/3}(\tau_- + t). \tag{5.9}$$

Then, we change the reference frame by setting $x = x_b(t) + x'$ and drop the prime in order to alleviate the notation: set $x' := x$. *This will be our reference frame until the end of this section.* System (1.5) becomes, in this new reference frame:

$$\begin{aligned}&T_t - T_{xx} + (u - \dot{x}_b)T_x - f(T) = 0, \\ &u_t + (u - \dot{x}_b)u_x - \rho T = 0.\end{aligned} \tag{5.10}$$

The asymptotic solution on the left. Let μ be a smooth non-negative function, equal to 1 on the interval $[2\phi_0/3, 1]$ and equal to 0 on $[0, \phi_0/2]$. Our choice for $(T^{\mathrm{app}}, u^{\mathrm{app}})$ for $x < 0$ (in the new moving frame) is

$$T^-(t, x) = \tilde{T}_-(\tau_- + t, x_b(t) + x) + \gamma(t)\mu(\tilde{T}_-(\tau_- + t, x_b(t) + x)), \tag{5.11}$$

$$u^-(t, x) = \tilde{u}_-(\tau_- + t, x_b(t) + x) + \dot{x}_b(\tau_- + t). \tag{5.12}$$

The function γ is a correction of the order $o(1)$, to be chosen in a more precise fashion below, and $\tilde{T}_-$ is defined in (5.5). The multiplicative correction $\mu(\tilde{T})$ is non-zero only in the region where we have $\tilde{T}_-(\tau_- + t, x_{\rm b}(t) + x) \in (\phi_0/2, \phi_0]$. We have chosen to multiply the already small term $\gamma(t)$ by the cut-off μ in order to keep the correction of the same order as the main term T^- for large negative x where T^- decays as a Gaussian.

Let us define

$$\mathrm{NL}[T,u] = (\mathrm{NL}_1, \mathrm{NL}_2) := (T_t - T_{xx} + (u - \dot{x}_{\rm b})T_x - f(T),\ u_t + (u - \dot{x}_{\rm b})u_x - \rho T).$$

Then, as $\phi_0 < \theta$, we have $f(T_-) = 0$ for $x < 0$ and δ sufficiently small, so that

$$\begin{aligned}\mathrm{NL}_1(T^-, u^-) &= T_t^- - T_{xx}^- + (u^- - \dot{x}_{\rm b})T_x^- \\ &= \tilde{T}_t^- - \tilde{T}_{xx}^- + \tilde{u}^-\tilde{T}_x^- + \gamma\mu'(\tilde{T})[\tilde{T}_t^- - \tilde{T}_{xx}^- + \tilde{u}^-\tilde{T}_x^-] - \gamma\mu''(\tilde{T}^-)(\tilde{T}_x^-)^2 + \mu(\tilde{T})\dot{\gamma} \\ &= -\gamma\mu''(\tilde{T}^-)(\tilde{T}_x^-)^2 + \mu(\tilde{T})\dot{\gamma}\end{aligned}$$

and

$$\begin{aligned}\mathrm{NL}_2(T^-, u^-) &= u_t^- + (u^- - \dot{x}_{\rm b})u_x^- - \rho T^- = \tilde{u}_t^- + \tilde{u}^-\tilde{u}_x^- - \rho\tilde{T}^- - \rho\gamma(t)\mu(\tilde{T}^-) \\ &= -\rho\tilde{T}^- - \rho\gamma(t)\mu(\tilde{T}^-).\end{aligned}$$

We have therefore

$$\forall x \leqslant 0,\ \forall t \geqslant 1:\ \mathrm{NL}[T,u](x) = (\dot{\gamma}(t)\mu(T) - \gamma\mu''(\tilde{T}^-)(\tilde{T}_x^-)^2, -\rho\gamma(t))\mu(\tilde{T}^-).$$

Provided that γ is $o(1)$ as announced, the pair (T_-, u_-) is then an asymptotic solution on the left.

The asymptotic solution on the right. Let us now consider what happens for $x \in \mathbb{R}_+$. We seek our solution in the form $(T^{\rm app}, u^{\rm app})(t,x) = (T_+, u_+)(t,x) = (T_+, u_-(t,0) + v_+(t,x))$, where (T_+, u_+) satisfies (5.10) up to a small error and in addition satisfies the matching conditions

$$\begin{aligned}&T_+(t,0) = T_-(t,0) = \phi_0 + \gamma(t) + O((\delta^{-1}+t)^{-1}),\\ &\partial_x T_+(t,0) = \partial_x T_-(t,0) = 2(\delta^{-1}+t)^{-2} + o((\delta^{-1}+t)^{-2}),\\ &v_+(t,0) = 0.\end{aligned} \tag{5.13}$$

Recall that, because of (5.7) and (5.8), the system for (T_+, v_+) that we wish to satisfy approximately is—we drop the subscripts for convenience:

$$\begin{aligned}&T_t - T_{xx} + (v + o(1))T_x = f(T),\\ &v_t + (v + o(1))v_x = \rho(T - \phi_0 + o(1)).\end{aligned}$$

Hence, it is enough to find a pair (T_+, v_+) satisfying

$$\begin{aligned}&\overline{\mathrm{NL}}_1[T,v] := T_t - T_{xx} + vT_x - f(T),\\ &\overline{\mathrm{NL}}_2[T,v] := v_t + vv_x - \rho(T - \phi_0),\end{aligned} \tag{5.14}$$

up to an $o(1)$ error. Now, if (ϕ_+, ψ_+) is a solution of (5.3) normalized so that $\phi_+(0) = \theta$, we look for T_+ in the form

$$T_+(t,x) = \phi_+(x - x_+(t)), \tag{5.15}$$

the shift $x_+(t)$ being adjusted to satisfy the boundary condition for $\partial_x T_+(t,0)$ in (5.13). Using representation (4.41) this equation reduces to

$$\phi_+'(-x_+(t)) = 2(\delta^{-1}+t)^{-2} + o((\delta^{-1}+t)^{-2}) = \frac{48}{\sqrt{\rho}(x_+(t) + (16\theta\sqrt{\rho})^{-1/3})^4},$$

which defines a unique $x_+(t)$ satisfying

$$x_+(t) \sim \left(\frac{24}{\sqrt{\rho}}(t+\delta^{-1})\right)^{1/2}. \tag{5.16}$$

Let us recall from section 4 that we have

$$\text{if } \phi_+ \leqslant \theta,\ \phi_+ - \phi_0 = \frac{2}{3^{3/4}}\rho^{-1/8}(\phi_+')^{3/4} := h(\phi_+'). \tag{5.17}$$

Now, the function $\gamma(t)$ is chosen to satisfy the first equation in (5.13), namely:

$$\gamma(t) = h\left(\frac{48}{\sqrt{\rho}(x_+(t) + (16\theta\sqrt{\rho})^{-1/3})^4}\right) + o(1) = o(1).$$

This fully determines $T_+(t,x)$. Now, $v_+(t,x)$ is just computed as

$$v_+(t,x) = \sqrt{2\int_0^x (T_+(t,y)-\phi_0)\,\mathrm{d}y}. \tag{5.18}$$

We have $\overline{\mathrm{NL}}[T_+, v_+] = o(1)$ and thus $\mathrm{NL}[T_+, u_+] = o(1)$. This ends the construction of the right solution (T_+, u_+).

5.3. Terminating the combustion wave with a shock

The numerical simulations of section 2 show that the combustion wave terminates on the right by a hydrodynamic shock, that is, a moving point $y_{\mathrm{f}}(t)$ across which the unknown u jumps from a large value $u_+(t, y_{\mathrm{f}}(t))$ to approximately 0. The temperature profile follows exactly the velocity profile and undergoes a transition from (approximately) 1 to 0 inside a δ-wide layer. Let us recall that at time $t_0 = \delta^{-1}$ the shock is located at a position $x_0 = \varepsilon^{-1}$ with the restriction $\delta \gg \sqrt{\varepsilon}$. Note that this condition also ensures that the shock is far removed initially from the wave fan to the combustion wave transition which has initial width $x_+ = O(\delta^{-1/2}) \ll \varepsilon^{-1}$—consequently the shock does not interact with the wave fan at $t = 1$.

Let us now find the shock location at later times. From (5.18) we have

$$v_+(t, y_{\mathrm{f}}(t)) = \sqrt{2\int_0^{y_{\mathrm{f}}(t)} (T_+(t,y)-\phi_0)\,\mathrm{d}y} = \sqrt{2(1-\phi_0)y_{\mathrm{f}}(t)} + O(1);$$

the quantity $O(1)$ referring to the time t and the small parameter ε and coming from the integrability of $1 - T$ ensured by lemma 4.6. Assuming that 0 is a good approximation of $u(t,x)$ to the right of $y_{\mathrm{f}}(t)$, the Rankine–Hugoniot condition for the equation for u yields

$$\begin{aligned}\dot{y}_{\mathrm{f}}(t) &= \frac{1}{2}\left(\sqrt{2\int_0^{y_{\mathrm{f}}(t)} (T_+(t,y)-\phi_0)\,\mathrm{d}y} - \dot{x}_{\mathrm{b}}(t)\right)\\ &= \frac{1}{2}\left[\sqrt{2(1-\phi_0)y_{\mathrm{f}}(t)} - \rho\phi_0(\delta^{-1}+t)\right] + O(1).\end{aligned} \tag{5.19}$$

This is almost the same equation as (5.1) were it not for the $O(1)$ term. This is, however, not such a problem: a time-shift $t + \delta^{-1} \to t$, change in the unknown $y_{\mathrm{f}}(t) = \rho t^2 z(t)$ and in the independent variable $t = \mathrm{e}^{\tau}$, yields the equation

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \tfrac{1}{2}q(\sqrt{z}) + O(\mathrm{e}^{-2\tau}),$$

which has exactly the same dynamics as (2.14).

The above analysis completes, in principle, the asymptotic analysis, because it describes the dynamics of the two transition layers: the first connects the wave fan to the combustion wave and the second is the hydrodynamic shock. However, because we wish to pursue our analysis in section 6 beyond the time of validity of the fan–wave–shock picture, we also have to say something about the temperature profile. The crucial zone is around the shock: in this area we look for an expression for $T^{\mathrm{app}}(t,x)$ in the form

$$T^{\mathrm{app}}(t,x) = T_s(t, (\delta^{-1}+t)(x-y_{\mathrm{f}}(t))).$$

Define $v(t,x)$ as $v_+(t,x)$ to the left of the shock, that is, for $x < y_{\mathrm{f}}(t)$, and as $-\dot{x}_{\mathrm{b}}(t)$ to the right of the shock, for $x > y_{\mathrm{f}}(t)$. The equation to be satisfied by $T_s(t,y)$ is

$$\begin{aligned}-\partial_{yy}T_s &+ \frac{v(t, y_{\mathrm{f}}(t)+(\delta^{-1}+t)^{-1}y)-\dot{y}_{\mathrm{f}}}{\delta^{-1}+t}\partial_y T_s \\ &= \frac{f(T_s)-\partial_t T_s - y(\delta^{-1}+t)^{-1}\partial_y T_s}{(\delta^{-1}+t)^2}.\end{aligned} \tag{5.20}$$

An approximate equation for T_s is, therefore,

$$-\partial_{yy}T_s + c(t,y)\partial_y T_s = 0, \tag{5.21}$$

with the function $c(t,y)$ which is an odd function of y, discontinuous at $y=0$, and which takes values $c_-(t)$ for $y<0$ and $c_+(t) = -c_-(t)$ for $y>0$. To the left of the shock, that is, for $y<0$, we have, using expression (5.2) for $y_{\mathrm{f}}(t)$:

$$c(t,y) = \frac{\dot{x}_{\mathrm{b}} + \sqrt{2(1-\phi_0)y_{\mathrm{f}}}}{2(\delta^{-1}+t)} = \frac{1}{2}\left(2\rho(1-\phi_0)Q^{-1}\left[\ln\frac{\delta^{-1}+t}{\sqrt{\rho^{-1}x_0}}\right]\right)^{1/2} + \frac{\rho\phi_0}{2} := c_\delta(t).$$

An important feature of $c_\delta(t)$ is that we have

$$\dot{c}_\delta(t) = O(\delta^{-1}+t)^{-1}. \tag{5.22}$$

If we additionally impose the conditions $T_s(t,-\infty)=1$ and $T_s(t,+\infty)=0$, an expression for $T_s(t,y)$ is

$$T_s(t,y) = 1 - \frac{c_\delta(t)}{2}\int_{-\infty}^{y} \mathrm{e}^{-c_\delta(t)|z|}\,\mathrm{d}z. \tag{5.23}$$

The final expression for the approximate temperature in the reference frame of the location $x_{\mathrm{b}}(t)$ of the wave fan is therefore taken as

$$T^{\mathrm{app}}(t,x) = \begin{cases} T_-(t,x) & \text{for } x<0, \\ \inf\ (T_+(t,x), T_s(t,(\delta^{-1}+t)(x-y_{\mathrm{f}}(t)))) & \text{if } 0<x<y_{\mathrm{f}}(t), \\ T_s(t,(\delta^{-1}+t)(x-y_{\mathrm{f}}(t))) & \text{if } x>y_{\mathrm{f}}(t) \end{cases} \tag{5.24}$$

and for an approximate velocity as

$$u^{\mathrm{app}}(t,x) = \begin{cases} u_-(t,x) & \text{for } x<0, \\ u_-(t,0)+v_+(t,x) & \text{if } 0<x<y_{\mathrm{f}}(t), \\ 0 & \text{if } x>y_{\mathrm{f}}(t). \end{cases} \tag{5.25}$$

The function $T_-(t,x)$ in (5.24) is given by (5.11) and $T_+(t,x)$ by (5.15), while in (5.25) the function $u_-(t,x)$ is given by (5.12) and $v_+(t,x)$ by (5.18).

Since $\partial_x T_+ > 0$ and $\partial_x T_s < 0$, there is only one point where both coincide, and this is a point of discontinuity for T_x. However, it occurs at a point $x_d(t)$ where T is very close to 1.

Hence, we know from lemma 4.6 that the jump in T_x at $x_d(t)$ is exponentially small: there exist two positive constants k_1 and k_2 such that

$$k_1 \mathrm{e}^{-k_1\sqrt{x_d(t)}} \leqslant \mathrm{e}^{(t+\delta^{-1})c_\delta(t)(x_d(t)-y_\mathrm{f}(t))} \leqslant k_2 \mathrm{e}^{-k_2\sqrt{x_d(t)}}.$$

This implies that $y_\mathrm{f} - x_d = O(\sqrt{y_\mathrm{f}})$; thus the jump in T_{xx} produces a negligible Dirac mass—which one may even regularize by modifying T_+ and T_s by suitable cut-offs near $x_d(t)$ where the solution is very close to a constant. Therefore, we have

$$T_t^{\mathrm{app}} - T_{xx}^{\mathrm{app}} + v^{\mathrm{app}} T_x^{\mathrm{app}} - f(T^{\mathrm{app}}) = O((t+\delta^{-1})^{-1}),$$

except in an $O((t+\delta^{-1})^{-1})$ layer around $y_\mathrm{f}(t)$, where T is close to neither 1 nor 0 and where therefore $f(T)$ is not close to 0. We have set $v^{\mathrm{app}} = u^{\mathrm{app}} - u_-(t, 0)$ here. In the same fashion, we have

$$v_t^{\mathrm{app}} + v^{\mathrm{app}} v_x^{\mathrm{app}} - \rho(T^{\mathrm{app}} - \phi_0) = O((t+\delta^{-1})^{-1}),$$

once again except in the same layer where T^{app} is close to neither 1 nor 0. However we may write for all $t \in [0, \sqrt{\rho^{-1}\varepsilon}]$:

$$\|\overline{\mathrm{NL}}(T^{\mathrm{app}}(t, .), v^{\mathrm{app}}(t, .))\|_{L^1([y_\mathrm{f}(t)-1, y_\mathrm{f}(t)+1])} = O((t+\delta^{-1})^{-1}). \tag{5.26}$$

Thus we still get an approximate solution albeit not in the pointwise sense. This analysis is valid as long as the transition layers between the wave fan and the combustion wave and the wave and the rest state are well separated. In the next section we will consider what happens to the 'wave fan–combustion wave–shock' solution when the wave starts catching up with the shock.

6. The final quenching

The analysis of the previous sections shows that after a long time the solution consists of a wave fan on the left, followed by a combustion wave, which in turn ends with a shock. Here we consider such a profile as initial data and show that it can quench in a certain regime even when the data are large. We make the following assumptions on the initial data.

Assumptions on T_0. We assume that $T_0(0) = \phi_0$—this is the value 'in the back of the combustion wave', where transition from the ramp on the left to the wave on the right occurs. To the left of $x = 0$ the initial data for T_0 look like a ramp, that is, we assume that

$$0 \leqslant T_0'(x) \leqslant \frac{1}{\beta^2} \qquad \text{for all } x \leqslant 0, \tag{6.1}$$

with some $\beta \gg \rho$. The parameter β plays the role of the time it took the original solution to reach the profile that we are now taking as the initial data. The function T_0 looks like a combustion wave, connecting the values $T = \phi_0$ on the left and $T = 1$ on the right, between the points $x = 0$ and $x_0 = O(\beta^{\delta_0})$ with $0 \leqslant \delta_0 < 1/2$, where the shock is located, and falls off over a distance l_f after x_0:

$$T_0' > 0 \quad \text{on } (0, x_0); \qquad T_0' < 0 \quad \text{on } (x_0, +\infty); \qquad T_0(x_0) = 1 - O(\mathrm{e}^{-\rho^\alpha}),$$

$$T_0(x_0 + l_\mathrm{f}) \leqslant \frac{1}{\beta} \tag{6.2}$$

with some $\alpha > 0$. We assume that $l_\mathrm{f} \leqslant C\beta^{\gamma_\mathrm{f}}$ with $\gamma_\mathrm{f} < \delta_0 < 1/2$.

Assumptions on u_0. We assume that to the left of $x = 0$ the flow profile looks like a ramp and we have $u_0'(x) \sim O(\rho/\beta)$ for $x < 0$, while $u_0(0) = \beta$. The function $u_0(x)$ grows as in the combustion wave between $x = 0$ and $y_0 = x_0 + C_0\beta^{-1}$, where $u_0(x)$ has its shock, so that

$$|u_0'(x)| \leqslant C < +\infty, \quad u_0' > 0 \text{ on } (0, y_0); \qquad u_0 = 0 \text{ on } (y_0, +\infty), \tag{6.3}$$

and, moreover

$$u_{\max} = u_0(y_0) = \beta + O(\sqrt{\rho y_0}). \tag{6.4}$$

Recall that u and T satisfy

$$\begin{aligned} T_t - T_{xx} + uT_x &= f(T), \\ u_t + uu_x &= \rho T. \end{aligned} \tag{6.5}$$

Let us solve the Cauchy problem for (6.5) with the initial data (T_0, u_0) satisfying the above assumptions. The main result of this section is the following theorem.

Theorem 6.1. *Under the above assumptions on T_0 and u_0, let (T, u) be the solution of (6.5) with the Cauchy data (T_0, u_0). There exists $\bar{t} > 0$ such that $\|T(t,\cdot)\|_\infty \leqslant \theta$ for $t \geqslant \bar{t}$.*

In particular, assumptions on T_0 and u_0 in theorem 6.1 are satisfied if we take the approximate solution $(T^{\mathrm{app}}, u^{\mathrm{app}})$ constructed in section 5 at the time $t_0 = \varepsilon^{-1/2} - \varepsilon^{-1/3}$ and set $\beta \sim t_0$.

The strategy is the following: first, we prove that $u(t, .)$ is well approximated by a time shift of the solution of the pure Burgers equations, at least for a time much larger than $1/\beta$. This property of u is then exploited in the structure of the equation for T, which is proved to be quenched in a time of order

$$\bar{t} = K\beta^{\delta_0 - 1},$$

except in a zone of very small size. Quenching by diffusion is finally proved in this very small zone.

Before starting the construction we change the reference frame: we set $x' = x - (\rho\phi_0 t^2/2)$ so that equations become

$$\begin{aligned} T_t - \rho\phi_0 t T_{x'} + uT_{x'} &= T_{x'x'} + f(T), \\ u_t - \rho\phi_0 t u_{x'} + uu_{x'} &= \rho T. \end{aligned}$$

Next, we set

$$v(t, x') = u(t, x') - \rho\phi_0 t \tag{6.6}$$

and drop the primes to get the following equations in the new frame:

$$\begin{aligned} T_t + vT_x &= T_{xx} + f(T), \\ v_t + vv_x &= \rho[T - \phi_0]. \end{aligned}$$

The functions v and T in the new variables have the same initial values $u_0(x)$ and $T_0(x)$.

Step 1: an explicit approximation for $v(t, x)$. The maximum principle for entropy solutions of the inviscid Burgers equations implies that $\underline{v}(t, x) \leqslant v(t, x) \leqslant \overline{v}(t, x)$ with the functions $\underline{v}$ and $\overline{v}$ that satisfy

$$\underline{v}_t + \tfrac{1}{2}(\underline{v}^2)_x = -\rho, \qquad \overline{v}_t + \tfrac{1}{2}(\overline{v}^2)_x = \rho, \tag{6.7}$$

with the initial data $\underline{v}(0,x) = \overline{v}(0,x) = u_0(x)$. Let us also introduce $w(t,x)$ which is the entropy solution of the unforced Burgers equation:

$$w_t + \tfrac{1}{2}(w^2)_x = 0, \qquad w(0,x) = u_0(x). \tag{6.8}$$

Observe that we have

$$\overline{v}(t,x) = \rho t + w\left(t, x - \frac{\rho t^2}{2}\right), \qquad \underline{v}(t,x) = -\rho t + w\left(t, x + \frac{\rho t^2}{2}\right).$$

Therefore, the function $v(t,x)$ is bounded above and below as follows:

$$-\rho t + w\left(t, x + \frac{\rho t^2}{2}\right) \leqslant v(t,x) \leqslant \rho t + w\left(t, x - \frac{\rho t^2}{2}\right),$$

and for small times the problem is essentially reduced to understanding the behaviour of $w(t,x)$.

As $u_0(x)$ is smooth and increasing on the interval $(-\infty, y_0)$ and is equal to zero for $x > y_0$ the function $w(t,x)$ remains smooth on an interval $(-\infty, y_{\mathrm{f}}(t))$ and is equal to zero for $x > y_{\mathrm{f}}(t)$, where $y_{\mathrm{f}}(t)$ is the shock location for the function w at the time $t \geqslant 0$. The Rankine–Hugoniot condition implies that

$$\dot{y}_{\mathrm{f}} = \tfrac{1}{2} w(t, y_{\mathrm{f}}^-(t)).$$

We define the characteristics on the left of the shock:

$$\dot{X}(t;x) = w(t, X(t;x)), \qquad X(0) = x.$$

The map $X(t;x)$ is well defined and increasing both in t and x as long as $x < y_0$ and until the characteristic hits the shock. In addition we have $w(t, X(t;x)) = u_0(x)$ and therefore

$$X(t;x) = x + t u_0(x).$$

For $x < y_{\mathrm{f}}(t)$ we may define the inverse map $y(t;x) = X^{-1}(t,x)$ so that $x = y + t u_0(y)$. Now we can almost explicitly compute the shock location: set $y_s(t) = X^{-1}(t, y_{\mathrm{f}}(t))$, then

$$\dot{y}_{\mathrm{f}} = \tfrac{1}{2} u_0(y_s(t)), \qquad y_{\mathrm{f}}(0) = y_0, \tag{6.9}$$

$$y_{\mathrm{f}}(t) = y_s(t) + t u_0(y_s(t)). \tag{6.10}$$

Differentiating (6.10) in time we obtain

$$\dot{y}_s = -\frac{u_0(y_s)}{2(1 + t u_0'(y_s))}, \qquad y_s(0) = y_0. \tag{6.11}$$

Note that $y_s(t) < y_0$ for all $t > 0$ and $u_0'(x) > 0$ for $x < y_0$ so that the solution of (6.11) exists for all $t > 0$. Moreover, as $|u_0'| \leqslant C$ and $0 \leqslant t \leqslant \bar{t} = K\beta^{\delta_0 - 1}$ we have

$$-\frac{u_0(y_s)}{2} \leqslant \dot{y}_s \leqslant -\frac{u_0(y_s)}{2(1 + C\beta^{\delta_0 - 1})},$$

so that for $0 \leqslant t \leqslant \bar{t}$ we have $y_s(t) \geqslant y_0 - C u_{\max} \beta^{\delta_0 - 1}$. It follows that (at the expense of increasing the constant C in the last inequality given below)

$$u_0(y_0) \geqslant u_0(y_s(t)) \geqslant u_0\left(y_0 - C u_{\max}\beta^{\delta_0-1}\right) \geqslant u_0\left(y_0 - C(\beta + C\sqrt{\rho y_0})\beta^{\delta_0-1}\right)$$
$$\geqslant u_0\left(y_0 - C\beta^{\delta_0}\right).$$

Now, as $|u_0'(x)| \leqslant C\rho/\beta$ for $x < 0$ we conclude that (again at the expense of increasing the constant)

$$\beta + C\sqrt{\rho\beta^{\delta_0}} \geqslant u_0(y_0) \geqslant u_0(y_s(t)) \geqslant \beta - \frac{C\rho}{\beta^{1-\delta_0}} \qquad \text{for } 0 \leqslant t \leqslant \bar{t}.$$

As a consequence, we obtain that

$$\frac{\beta}{2} - \frac{C\rho}{\beta^{1-\delta_0}} \leqslant \dot{y}_{\mathrm{f}}(t) \leqslant \frac{\beta}{2} + C\sqrt{\rho\beta^{\delta_0}} \qquad \text{for } 0 \leqslant t \leqslant \bar{t}. \tag{6.12}$$

To summarize, we have shown that $v(t,x) \leqslant \rho t$ for $x \geqslant y_{\mathrm{f}}(t) + \rho t^2/2$ and $v(t,x) \geqslant w(t,x) - \rho t$ for $x \leqslant y_{\mathrm{f}}(t) - \rho t^2/2$ with $\dot{y}_{\mathrm{f}}$ satisfying (6.12).

Furthermore, for all $x < y_{\mathrm{f}}(t)$ we have $w(t,x) = u_0(y(t,x))$ and $0 \leqslant x - y(t,x) \leqslant u_{\max} t$. Therefore, $w(t,x)$ is large for all $-\sqrt{\beta} \leqslant x \leqslant y_{\mathrm{f}}(t)$ and $0 \leqslant t \leqslant \bar{t}$:

$$w(t,x) \geqslant u_0(-\sqrt{\beta} - Cu_{\max}\beta^{\delta_0 - 1}) \geqslant u_0(-\sqrt{\beta} - C\beta^{\delta_0}) \geqslant \beta - \frac{C\rho}{\beta}\sqrt{\beta} \geqslant \beta - \frac{C\rho}{\sqrt{\beta}}.$$

We have proved the following lemma.

Lemma 6.2. *There exists a 'shock location' function $y_{\mathrm{f}}(t)$ satisfying (6.12) with $y_{\mathrm{f}}(0) = y_0$ such that for any $K > 0$ there exists $\beta_0 > 0$ and $C > 0$ so that for all $0 \leqslant t \leqslant \bar{t} = K\beta^{\delta_0 - 1}$ we have for $\beta > \beta_0$ (i) $v(t,x) \leqslant \rho t$ for $x \geqslant y_{\mathrm{f}}(t) + \rho t^2/2$ and (ii) $v(t,x) \geqslant \beta - C\rho\beta^{-1/2} - \rho t$ for $-\sqrt{\beta} \leqslant x \leqslant y_{\mathrm{f}}(t) - \rho t^2/2$.*

Step 2: A uniform bound for temperature on the left. Now, in order to establish quenching we consider the coordinate system that moves with the speed $\dot{y}_{\mathrm{f}}(t)$: set $x'' = x - y_{\mathrm{f}}(t)$. The temperature equation takes the form (we drop the primes)

$$T_t + [v - \dot{y}_{\mathrm{f}}(t)]T_x = T_{xx} + f(T), \qquad T(0,x) = T_0(x). \tag{6.13}$$

The first step is to bound the temperature 'far on the left'.

Lemma 6.3. *In the new coordinate system for any $K > 0$ there exists $C(K) > 0$ so that we have $T(t, -\sqrt{\beta}) \leqslant \phi_0 + C(K)/\sqrt{\beta}$ for $0 \leqslant t \leqslant \bar{t} = K\beta^{\delta_0 - 1}$.*

Proof. First note that $f(T) \leqslant \Lambda T$ and thus $T(t,x) \leqslant \Phi(t,x)\mathrm{e}^{\Lambda t}$ with the function Φ that satisfies

$$\Phi_t + [v - \dot{y}_{\mathrm{f}}]\Phi_x - \Phi_{xx} = 0, \qquad \Phi(0,x) = T_0(x). \tag{6.14}$$

We can write $T(t,x) = E\{T_0(X(t;x))\}$ where the process $X(t;x)$ solves

$$\mathrm{d}Z = (\dot{y}_{\mathrm{f}} - v)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W.$$

However, as $|v| \leqslant u_{\max} \leqslant \beta + C\sqrt{\rho y_0}$ and $|\dot{y}_{\mathrm{f}}(t)| \leqslant \beta$, it follows that the probability that $Z(t)$ starting at $x = -\sqrt{\beta}$ exits the (very long) interval $(-3\sqrt{\beta}/2, -\sqrt{\beta}/2)$ before the (very short) time $K\beta^{\delta_0 - 1}$ with a sufficiently large $K > 0$ is smaller than the probability that $\max_{0 \leqslant t \leqslant K\beta^{\delta_0 - 1}} W(t) \geqslant K\sqrt{\beta}$, which is exponentially small in β as $\delta_0 \in (0, 1/2)$. The claim of lemma 6.3 now follows since $T_0(x) \leqslant \phi_0 + C/\beta^{3/2}$ for all $x \in (-3\sqrt{\beta}/2, -\sqrt{\beta}/2)$. □

Step 3: quenching in the middle and on the right. Lemmas 6.2 and 6.3 are really the piece of information that will lead us to quenching. Clearly, they do not use all of the information provided by the construction of our subsolutions and supersolutions for v; however, they will be sufficient—and it is not obvious that they could have been obtained in a simpler way.

By lemma 6.3 and the maximum principle, we have $T(t,x) \leqslant \phi_0 + C(K)/\sqrt{\beta} \leqslant \theta$ for the points $x \in (-\infty, -\beta^{1/2})$, as long as $t \leqslant \bar{t} = K\beta^{\delta_0 - 1}$. We want to prove that T falls under θ in the interval $(-\beta^{1/2}, +\infty)$ at some time $\tau \leqslant \bar{t}$. We will split this interval into three sub-intervals: $(-\beta^{1/2}, -N/\beta)$, $(-N/\beta, N/\beta)$ and $(N/\beta, +\infty)$ with a sufficiently large N.

1. The interval $(N/\beta, +\infty)$. We use the fact that the advection term in equation (6.13) is less than $-\beta/4$ for $x > 1/\beta$ and $0 \leqslant t \leqslant \bar{t}$ by part (i) in lemma 6.2 since $\beta^{2\delta_0-2} \ll 1/\beta$ because $\delta_0 \in (0, 1/2)$. Let

$$A(t,x) = \frac{1}{\beta} + \exp(-\mu(x + \beta(t-\bar{t})/16 - z_0))$$

with $z_0 = 1/\beta$ be the exponentially decaying solution of the problem

$$A_t - A_{xx} - \frac{3\beta}{16}A_x = \Lambda\left[A - \frac{1}{\beta}\right], \qquad -\mu^2 + \frac{\beta}{8}\mu = \Lambda,$$

with the constant Λ chosen so that $\Lambda(s - 1/\beta) \geqslant f(s)$ for $s \geqslant 1/\beta$—this is possible as $1/\beta \leqslant \theta/2$ for a sufficiently large β. The constant μ is chosen so that

$$\mu = \frac{\beta}{16} + \frac{1}{2}\sqrt{\frac{\beta^2}{64} - 4\Lambda} \geqslant \frac{\beta}{16}. \tag{6.15}$$

Note that, since $A_x(t,x) \leqslant 0$ and $v \leqslant C\rho\beta^{\delta_0-1}$ for $x > z_0$ and $0 \leqslant t \leqslant \bar{t}$, we have for $x \geqslant z_0$

$$A_t - A_{xx} + [v - \dot{y}_{\mathrm{f}}]A_x \geqslant A_t - A_{xx} - \frac{\beta}{8}A_x = \Lambda(A - \beta^{-1}) \geqslant f(A). \tag{6.16}$$

Moreover, at the endpoint $x = z_0$ we have

$$T(t,z_0) \leqslant 1 \leqslant \exp(-\mu\beta(t-\bar{t})/16) \leqslant A(t,z_0) \tag{6.17}$$

for all $0 \leqslant t \leqslant \bar{t}$. In order to apply the maximum principle we compare the initial data $T_0(x)$ and $A(0,x)$. First, for $x \geqslant x_0 - y_0 + l_{\mathrm{f}}$ we have $T_0(x) \leqslant 1/\beta \leqslant A(0,x)$. Now, at $x = x_0 - y_0 + l_{\mathrm{f}}$ we have

$$A(0, x_0 - y_0 + l_{\mathrm{f}}) = \frac{1}{\beta} + \exp(-\mu(x_0 - y_0 + l_{\mathrm{f}} - \beta\bar{t}/16 - z_0)) \geqslant 1,$$

provided that

$$x_0 - y_0 + l_{\mathrm{f}} - \beta\bar{t}/16 - z_0 \leqslant 0. \tag{6.18}$$

As $|x_0 - y_0| \leqslant C/\beta$, $z_0 = 1/\beta$ and $l_{\mathrm{f}} \leqslant C\beta^{\gamma_{\mathrm{f}}}$, inequality (6.18) indeed holds since $\gamma_{\mathrm{f}} < \delta_0$. The function $A(0,x)$ decreases in x—therefore, we also have

$$T_0(x) \leqslant 1 \leqslant A(0,x) \qquad \text{for all } x \leqslant x_0 - y_0 + l_{\mathrm{f}}.$$

We conclude that $T_0(x) \leqslant A(0,x)$ for all $x \geqslant z_0$. The maximum principle together with inequalities (6.16) and (6.17) implies that $T(t,x) \leqslant A(t,x)$ for all $x \geqslant z_0$ and all $0 \leqslant t \leqslant \bar{t}$. Therefore, at the points $x \geqslant z_0 + (N-1)/\beta$ we have at the time $\bar{t}$

$$T(\bar{t},x) \leqslant A(\bar{t},x) \leqslant A(\bar{t}, z_0 + (N-1)/\beta) = \frac{1}{\beta} + \exp(-\mu(N-1)/\beta) \leqslant \frac{\theta + \phi_0}{2}, \tag{6.19}$$

for a sufficiently large $N > 0$ since $\mu \geqslant \beta/16$.

2. The interval $(-\sqrt{\beta}, -N/\beta)$. We set $z_1 = -1/\beta$ and construct a supersolution for temperature in the interval $(-\sqrt{\beta}, z_1)$ using part (ii) of lemma 6.2 in a similar fashion. Take μ as in (6.15) and set

$$B(t,x) = \mathrm{e}^{\mu(x - \beta(t-\bar{t})/16 - z_1)} + \phi_0 + \frac{C}{\sqrt{\beta}},$$

so that

$$B_t - B_{xx} + \frac{3\beta}{16}B_x = \Lambda\left(B - \phi_0 - \frac{C}{\sqrt{\beta}}\right)$$

with $\Lambda > 0$ chosen so that $\Lambda(s - \phi_0 - C/\sqrt{\beta}) \geqslant f(s)$ for $s \geqslant \phi_0$—such a Λ exists for a sufficiently large $\beta > 0$ because $\phi_0 < \theta$. Then, as $B(t, x)$ is increasing in x for all $t \geqslant 0$ and $v(t, x) \geqslant 3\beta/4$ for $x \leqslant z_1$ and $0 \leqslant t \leqslant \bar{t}$, we have

$$B_t - B_{xx} + [v - \dot{y}_{\mathrm{f}}]B_x \geqslant B_t - B_{xx} + \frac{3\beta}{16}B_x = \Lambda\left(B - \phi_0 - \frac{C}{\sqrt{\beta}}\right) \geqslant f(B)$$

$$\text{for } -\sqrt{\beta} \leqslant x \leqslant z_1$$

for all $0 \leqslant t \leqslant \bar{t}$. At the two endpoints: $x_1 = -\sqrt{\beta}$ and $x_2 = z_1$, we have $T(t, x_j) \leqslant B(t, x_j)$, $j = 1, 2$ for all $0 \leqslant t \leqslant \tau$ simply because $T(t, x_1) \leqslant \phi_0 + C/\sqrt{\beta} \leqslant B(t, x_1)$ according to lemma 6.3 and $T(t, x_2) \leqslant 1 \leqslant B(\bar{t}, x_2) \leqslant B(t, x_2)$ since B is decreasing in t. Moreover, at the time $t = 0$ we have

$$B(-y_0, 0) = (1 - \phi_0)\mathrm{e}^{\mu(-y_0+1/\beta+\beta\bar{t}/16)} + \phi_0 + \frac{C}{\sqrt{\beta}} > 1,$$

as soon as K is sufficiently large, since $y_0 = C\beta^{\delta_0}$ and $\bar{t} = K\beta^{\delta_0-1}$. As the function $B(0, x)$ is increasing in x it follows that we have $T(0, x) \leqslant 1 < B(0, x)$ for all $x \geqslant -y_0$. However, for $-\sqrt{\beta} \leqslant x \leqslant -y_0$ we have $T(0, x) \leqslant \phi_0 < B(0, x)$—we conclude that $T(0, x) \leqslant B(0, x)$ for all $x \geqslant -\sqrt{\beta}$. Therefore, $B(t, x)$ is a supersolution for $T(t, x)$ and $T(t, x) \leqslant B(t, x)$ for all $0 \leqslant t \leqslant \tau$ and all $x \in (-\sqrt{\beta}, z_1)$. However, at the time $t = \bar{t}$ we then have for all $x \leqslant -N/\beta$

$$T(\bar{t}, x) \leqslant B(\bar{t}, x) \leqslant B(\bar{t}, -N/\beta) = (1 - \phi_0)\mathrm{e}^{\mu(-N/\beta-\beta\bar{t}/16-z_1+\beta\bar{t}/16)} + \phi_0 + \frac{C}{\sqrt{\beta}}$$

$$\leqslant (1 - \phi_0)\mathrm{e}^{\mu(-(N-1)/\beta)} + \phi_0 + \frac{C}{\sqrt{\rho}} \leqslant \frac{\theta + \phi_0}{2}$$

since $\mu \geqslant \beta/16$ and $\phi_0 < \theta$.

We conclude from the above that at the time $\bar{t}$ the function $T(t, x)$ is below the value θ everywhere except in the interval $x \in (-N/\beta, N/\beta)$. A slight generalization of that argument shows that (after increasing N) the same statement can be proved for all $t \in (\bar{t}/2, \bar{t})$.

3. The interval $(-N/\beta, N/\beta)$**.** This is now just quenching by diffusion. It follows from the previous calculations that in the slightly larger interval $(-2N/\beta, +2N/\beta)$ itself and for any time $t \in (\bar{t}/2, \bar{t})$ the function T may be bounded from above as

$$T(t, x) \leqslant \left[\frac{\phi_0 + \theta}{2} + \Phi(t, x)\right]\mathrm{e}^{\Lambda t}.$$

The function $\Phi(t, x)$ is the solution of the Dirichlet problem

$$\Phi_t + [v - \dot{y}_{\mathrm{f}}]\Phi_x = \Phi_{xx}, \qquad \Phi(t, -2N/\beta) = \Phi(t, +2N/\beta) = 0$$

with the Cauchy data

$$\Phi(\bar{t}/2, x) = \begin{cases} 1, & \text{for } -N/\beta \leqslant x \leqslant N/\beta \\ 0, & \text{for } -2N/\beta \leqslant x < -N/\beta \quad \text{and} \quad N/\beta < x \leqslant y_0 + 2N/\beta. \end{cases}$$

As $|v| + |\dot{y}_{\mathrm{f}}| \leqslant C\beta$ we have the inequality

$$\Phi_t - \Phi_{xx} \leqslant C\beta|\Phi_x|.$$

Now let $\bar{\Phi}$ solve

$$\bar{\Phi}_t - \bar{\Phi}_{xx} = C\beta|\bar{\Phi}_x|, \qquad \bar{\Phi}(t, -2N/\beta) = \bar{\Phi}(t, +2N/\beta) = 0$$

with $\bar{\Phi}(\bar{t}/2, x) = \Phi(\bar{t}/2, x)$. The maximum principle implies that $\bar{\Phi}(t, x) \geqslant \Phi(t, x)$. However, the function $\bar{\Phi}$ is symmetric about y_0 and solves the half-interval problem

$$\bar{\Phi}_t - \bar{\Phi}_{xx} = C\beta\bar{\Phi}_x, \qquad -2N/\beta < x < 0, \qquad \bar{\Phi}(t, -2N/\beta) = \bar{\Phi}_x(t, y_0) = 0.$$

Consider the principal eigenfunction $\xi(x)$ of this problem with the eigenvalue $\lambda(\beta)$:

$$-\xi_{xx} = C\beta\xi_x - \lambda(\beta)\xi, \qquad -2N/\beta < x < y_0, \qquad \xi(-2N/\beta) = \xi_x(y_0) = 0.$$

After rescaling: $x = z/\beta$, this becomes

$$-\beta^2\xi_{zz} = C\beta^2\xi_z - \lambda(\beta)\xi, \qquad -2 < z < 0, \qquad \xi(-2) = \xi_z(0) = 0,$$

and thus $\lambda(\beta) = -\lambda_0\beta^2$ with $\lambda_0 > 0$. It follows that $\bar{\Phi}(t,x) \leqslant C_0 e^{-\lambda_0\beta^2 t}$ and in particular $\bar{\Phi}(t,x) \leqslant \theta$ for all $-2N/\beta \leqslant x \leqslant 2N/\beta$ at the time $\bar{t} = C\delta_0^{\delta_0 - 1}$. The proof of theorem 6.1 is now complete. □

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