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ON PARTICLE TRAJECTORIES IN LINEAR WATER WAVES

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Abstract. We determine the phase portrait of a Hamiltonian system of equations describing the motion of the particles in linear water waves. The particles experience in each period a forward drift which is minimal on the flat bed.

1. Introduction

The motion of water particles under the waves which advance across the water is a very old problem. Watching the sea it is oft possible to trace a wave as it propagates on the water's surface, but what one observes traveling across the sea is not the water but a wave pattern. The displacement of the water particles induces a much more rapid motion on the free surface wave, a fact supported by field evidence [17]. Due to the mathematical intractability of the governing equations for water waves, the classical approach [16, 19, 22] towards explaining this aspect of water waves consists in analysing the particle motion after linearisation of the governing equations. Within the linear water wave theory, the ordinary differential equations system describing the motion of the particles is nevertheless nonlinear and explicit solution are not available. In the first approximation of this nonlinear system, all particle paths are closed cf. [11, 14, 16, 17, 21, 22]. Support for this conclusion is given by the only known explicit solution with a non-flat free surface for the governing equations in water of infinite depth [12] solution for which all particle paths are circles of diameters decreasing with the distance from the free surface (see [3] and the discussion in [4, 13]). The conclusion seems to be supported by experimental evidence: photographs [11, 21, 22] or movie films [2] of small buoyant particles in laboratory wave tanks (see also the comments in [15]). However, an analysis of the average flow of energy within linear water wave theory (see [10, 14]) indicates that, due to the passage of a periodic surface wave, the water particles in the fluid experience on average a net displacement in the direction in which the waves are propagating, termed Stokes drift (Stokes [23] noticed this feature for infinitely deep water and Ursell [25] examined it in water of finite depth).

In this paper we show that for linear water wave no particle trajectory is closed, unless the free surface is flat. Each trajectory involves over a period a backward/forward movement of the particle, and the path is an elliptical

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arc (which degenerates on the flat bed) but with a forward drift. This result was first obtained in [9]. We obtain the same result but our analysis is more precise and uses only elementary analysis methods. As in [9], we shown that within linear water wave theory, the particle paths are almost closed and the more we approach the free surface, the more pronounced the deviation from a closed orbit becomes, in agreement with Stokes' observation (see [23]). The features encountered in the linear framework are confirmed to hold also for the governing equations by a different approach, see [5, 7].

The work is structured as follows: we recall first the governing equations for water waves and give the exact solutions of the linearised problem. Our major contribution, in Section 3, is a precise analysis of the phase portrait of a Hamiltonian system associated to the particle motion. This is the key point in determining the trajectories of the water particles.

2. Preliminaries

In this section we recall the governing equations for water waves and we present their linearization (for a more detailed discussion we refer to [14]).

2.1. The governing equations.

We consider a two-dimensional inviscid incompressible fluid in a constant gravitational field. To describe the waves propagating on the water surface we consider a cross section of the flow that is perpendicular to the crest line with Cartesian coordinates (x, y), the y-axis pointing vertically upwards and the x-axis being the direction of wave propagation, while the origin lies on the flat bed. Let $(u(t, x, y), v(t, x, y))$ be the velocity field of the flow over the flat bed $y = 0$ and let $y = h_0 + \eta(t, x)$ be the water's free surface. Here $h_0 > 0$ is the mean water level.

Constant density (homogeneity) is a physically reasonable assumption for gravity waves [14], and it implies the equation of mass conservation

$$u_x + u_y = 0. \tag{2.1}$$

Under the assumption of inviscid flow the equation of motion is EulerŠs equation

$$\begin{cases} u_t + uu_x + vu_y &= -P_x \\ v_t + uv_x + vv_y &= -P_y - g, \end{cases} \tag{2.2}$$

where $P(t, x, y)$ denotes the pressure and g is the gravitational constant of acceleration. The free surface decouples the motion of the water from that of the air, a fact that is expressed in [14] by the dynamic boundary condition

$$P = P_0 \qquad \text{on} \quad y = h_0 + \eta(t, x), \tag{2.3}$$

if we neglect surface tension, where P_0 is the (constant) atmospheric pressure. Since the same particles always form the free surface, we also have the kinematic boundary condition

$$v = \eta_t + u\eta_x \qquad \text{on} \quad y = h_0 + \eta(t, x). \tag{2.4}$$

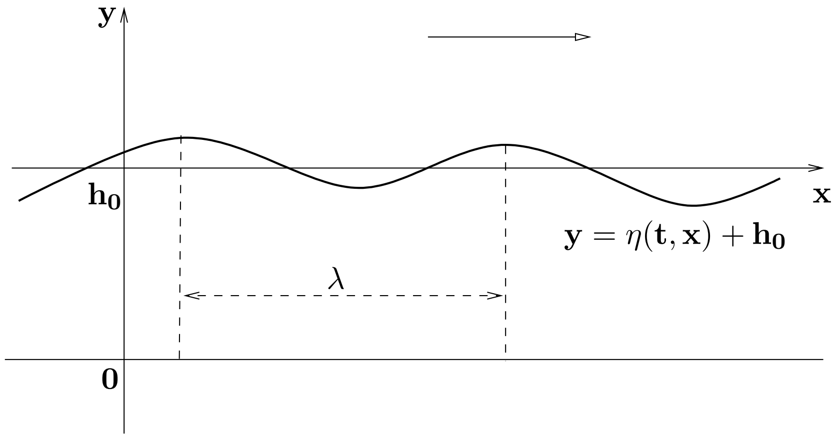

FIGURE 1. A periodic water wave propagating over a flow bed

On the flat bed we have the kinematic boundary condition

$$v = 0 \qquad \text{on} \quad y = 0, \tag{2.5}$$

expressing the fact that the flow is tangent to the horizontal bed (or, equivalently, that water cannot penetrate the rigid bed). Summarising, the governing equations for water waves are encompassed by the nonlinear free-boundary problem

$$\left\{\begin{array}{rcll} u_x + u_y & = & 0, & \\ u_t + uu_x + vu_y & = & -P_x, & \\ v_t + uv_x + vv_y & = & -P_y - g, & \\ P & = & P_0 & \text{on} \quad y = h_0 + \eta(t,x), \\ v & = & \eta_t + u\eta_x & \text{on} \quad y = h_0 + \eta(t,x), \\ v & = & 0 & \text{on} \quad y = 0. \end{array}\right. \tag{2.6}$$

In our discussion we suppose that at some distant point in the past a disturbance of the flat surface of still water was created and we analyze the subsequent motion of the water. The balance between the restoring gravity force and the inertia of the system governs the evolution of the mass of water and our primary objective is to understand the behaviour of the water particles below the free surface.

An important category of flows are those of zero vorticity (irrotational flows), characterized by the additional assumption

$$u_y = v_x. \tag{2.7}$$

The vorticity of a flow, $\omega = u_y - v_x$, measures the local spin or rotation of a fluid element so that in irrotational flows the local whirl is completely absent. Relation (2.7) ensures the existence of a velocity potential $\phi(t,x,y)$ defined up to a constant via

$$\phi_x = u, \qquad \phi_y = v.$$

In view of (2.1), ϕ is a harmonic function, i.e. $(\partial_x^2 + \partial_y^2)\phi = 0$ so that the methods of complex analysis become available for the study of irrotational flows.

Concerning the physical relevance of irrotational water flows, field evidence indicates that for waves entering a region of still water the assumption of irrotational flow is realistic [17]. Moreover, as a consequence of Kelvin's circulation theorem [1], a water flow that is irrotational initially has to be irrotational at all later times. It is thus reasonable to consider that water motions starting from rest will remain irrotational at later times. Nonzero vorticity is the hallmark of the interaction between waves and non-uniform currents cf. [8] and [9], for example tidal currents cf. [24].

2.2. Linear water waves. The problem (2.6) is nondimensionalised using a typical wavelength λ and a typical amplitude of the wave a. We define the set of nondimensional variables [14]

$$x \mapsto \lambda x, \quad y \mapsto h_0 y, \quad t \mapsto \frac{\lambda}{\sqrt{gh_0}} t,$$

$$u \mapsto u\sqrt{gh_0}, \quad v \mapsto v\frac{h_0\sqrt{gh_0}}{\lambda}, \quad \eta \mapsto a\eta$$

where, to avoid new notations, we have used the same symbols for the nondimensional variables $x, y, , t, u, v, \eta$ on the right-hand side. Setting the constant water density $\rho = 1$, the pressure in the new nondimensional variables is

$$P = P_0 + gh_0(1-y) + gh_0 p,$$

with the nondimensional pressure variable p measuring the deviation from the hydrostatic pressure distribution. We obtain the following boundary value problem in nondimensional variables

$$\left\{\begin{array}{rcll} u_t + uu_x + vu_y &=& -p_x, & \\ \delta^2(v_t + uv_x + vv_y &=& -p_y, & \\ u_x + v_y &=& 0, & \\ p &=& \varepsilon\eta & \text{on} \quad y = 1 + \varepsilon\eta, \\ v &=& \varepsilon(\eta_t + u\eta_x) & \text{on} \quad y = 1 + \varepsilon\eta, \\ v &=& 0 & \text{on} \quad y = 0, \end{array}\right. \tag{2.8}$$

where $\varepsilon = a/h_0$ is the amplitude parameter and $\delta = h_0/\lambda$ is the shallowness parameter. We observe from the fourth and the fifth equations in (2.8) that, on $y = 1 + \varepsilon\eta$, both v and p are proportional to ε. This is consistent with the fact that as $\varepsilon \to 0$ we must have $v \to 0$ and $p \to 0$, and it leads to the following scaling of the non-dimensional variables

$$p \mapsto \varepsilon p, \quad (u, v) \mapsto \varepsilon(u, v),$$

where we avoided again the introduction of a new notation. The problem (2.8) becomes

$$\begin{cases} u_t + \varepsilon(uu_x + vu_y) &= -p_x, \\ \delta^2[v_t + \varepsilon(uv_x + vv_y)] &= -p_y, \\ u_x + v_y &= 0, \\ p &= \eta \qquad \text{on} \quad y = 1 + \varepsilon\eta, \\ v &= \eta_t + \varepsilon u\eta_x \qquad \text{on} \quad y = 1 + \varepsilon\eta, \\ v &= 0 \qquad \text{on} \quad y = 0, \end{cases} \tag{2.9}$$

The linearized problem is now obtained by letting $\varepsilon \to 0$ in (2.9). We obtain

$$\begin{cases} u_t &= -p_x, \\ \delta^2 v_t &= -p_y, \\ u_x + v_y &= 0, \\ p &= \eta \qquad \text{on} \quad y = 1, \\ v &= \eta_t \qquad \text{on} \quad y = 1, \\ v &= 0 \qquad \text{on} \quad y = 0. \end{cases} \tag{2.10}$$

Looking for solutions of (2.10) representing waves traveling at speed $c_0 > 0$, we impose that all functions u, v, p and η have an (x,t)-dependence in the form of $x - c_0 t$. Furthermore, in seeking spatially periodic functions of period one, we are led to the fundamental Fourier mode *Ansatz*

$$\eta(x, t) = \cos[2\pi(x - c_0 t)].$$

For this specific η the problem (2.10) has the solution

$$u(t,x,y) = 2\pi c_0\delta\frac{\cosh(2\pi\delta y)}{\sinh(2\pi\delta)}\cos[2\pi(x-c_0t)],$$

$$v(t,x,y) = 2\pi c_0\frac{\sinh(2\pi\delta y)}{\sinh(2\pi\delta)}\sin[2\pi(x-c_0t)],$$

$$p(t,x,y) = \frac{\cosh(2\pi\delta y)}{\cosh(2\pi\delta)}\cos[2\pi(x-c_0t)],$$

provided $c_0^2 = \tanh(2\pi\delta)/(2\pi\delta)$. Returning to the original physical variables, we perform the change of variables

$$x \mapsto \frac{x}{\lambda}, \quad y \mapsto \frac{y}{h_0}, \quad t \mapsto \frac{\sqrt{gh_0}}{\lambda}t,$$

$$u \mapsto \frac{u}{\sqrt{gh_0}}, \quad v \mapsto v\frac{\lambda}{h_0\sqrt{gh_0}}, \quad \eta \mapsto \frac{\eta}{a},$$

(the variables on the right-hand side being the physical variables) to obtain the linear wave solution

$$\left\{\begin{array}{rcl} \eta(t,x) &=& \varepsilon h_0\cos(kx-\omega t),\\ u(t,x,y) &=& \varepsilon\omega h_0\dfrac{\cosh(ky)}{\sinh(kh_0)}\cos(kx-\omega t),\\ v(t,x,y) &=& \varepsilon\omega h_0\dfrac{\sinh(ky)}{\sinh(kh_0)}\sin(kx-\omega t),\\ P(t,x,y) &=& P_0+g(h_0-y)+\varepsilon gh_0\dfrac{\cosh(ky)}{\cosh(kh_0)}\cos(kx-\omega t), \end{array}\right. \tag{2.11}$$

of amplitude $\varepsilon h_0 > 0$ and wavelength $\lambda > 0$, propagating over the flat bed $y = 0$ and with mean water level $h_0 > 0$. Here

$$k = \frac{2\pi}{\lambda}, \quad \omega = \sqrt{gk\tanh(kh_0)},$$

are the wavenumber, respectively the frequency, and the dispersion relation

$$c = \frac{\omega}{k} = \sqrt{\frac{\tanh(kh_0)}{k}}$$

determines the speed c of the linear wave. The period of this wave is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{gk\tanh(kh_0)}}.$$

3. Particle trajectories

If $(x(t), y(t))$ is the particle below the linear wave (2.11) then

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v,$$

so that the motion of the particle is described by the system

$$\begin{cases} \dfrac{dx}{dt} &= M\cosh(ky)\cos[k(x-ct)], \\ \dfrac{dy}{dt} &= M\sinh(ky)\sin[k(x-ct)], \end{cases} \tag{3.1}$$

with initial data (x_0, y_0). We denoted

$$M := \frac{\varepsilon\omega h_0}{\sinh(kh_0)}. \tag{3.2}$$

The right-hand side of the differential system (3.1) is smooth so that the existence of a unique local smooth solution is ensured [20]. Also, since y is bounded, the right-hand side of (3.1) is bounded and therefore this unique solution is defined globally, i.e. for all $t = 0$, cf. [20]. Without actually solving (3.1) we would like to display the principal features of the solution.

To study the exact solutions to (3.1) it is convenient to re-write the system in a moving frame with scaled independent variables. The transformation

$$X = k(x-ct), \quad Y = ky, \tag{3.3}$$

maps (3.1) into

$$\begin{cases} \dfrac{dX}{dt} &= kM\cosh(Y)\cos(X), \\ \dfrac{dY}{dt} &= kM\sinh(Y)\sin(X). \end{cases} \tag{3.4}$$

Notice that in view of (3.2)

$$kM = k\frac{\varepsilon\omega h_0}{\sinh(kh_0)} = \varepsilon\omega\frac{kh_0}{\sinh(kh_0)} < kc = \omega, \tag{3.5}$$

since $s < \sinh(s)$ for $s > 0$ while $\varepsilon < 1$ within the framework of linear theory.

In order to determine the phase portrait, we take into consideration the Hamiltonian structure of (3.4). Setting

$$H(X,Y) = kM\sinh(Y)\cos(X) - kcY, \quad (X,Y) \in \mathbb{R}^2, \tag{3.6}$$

problem (3.4) re-writes

$$\begin{cases} \dfrac{dX}{dt} &= \partial_Y H, \\ \dfrac{dY}{dt} &= -\partial_X H. \end{cases} \tag{3.7}$$

Our goal is to determine the phase portrait corresponding to (3.6). In [9] the authors studied the phase portrait of (3.6) approximately, by using Morse's lemma. A precise analysis allows us in here to determine the phase portrait of the Hamiltonian system of equations describing the motion of the particles in linear water waves by using only elementary methods.

Since H is constant along the phase curves of (3.6), our task is equivalent to determining the level curves $H^{-1}(\{\alpha\})$ of H for all $\alpha \in \mathbb{R}$. Physically relevant is only the case when $Y \geq 0$. Moreover, since H is $2\pi-$periodic in the X variable, $\partial_Y H$ is even in both variables X, Y, and $\partial_X H$ is odd with respects to the same variables, the phase portrait of (3.7) is determined by the level sets of the restriction $H : [0, \pi] \times [0, \infty) \to \mathbb{R}$.

Obviously, $H^{-1}(\{0\}) \supset [0, \pi] \times \{0\}$. Since the level sets of H are disjoint, we are left to determine to which level set of H the remaining points (X, Y), $Y \neq 0$, belong. For convenience, we set

$$[0, \pi] \times (0, \infty) := \Omega$$

Let

$$P := (X_*, Y_*) := (0, \cosh^{-1}(c/M))$$

the solution of

$$\nabla H = 0 \quad \text{in} \quad \Omega,$$

which is the unique stationary solution of (3.6) within Ω. Then $P \in H^{-1}(\{\alpha_*\})$, for some $\alpha_* < 0$. Indeed, $H(P) < 0$ is equivalent to

$$\begin{aligned} H(0, Y_*) &< 0 & \Leftrightarrow \quad kM\sinh(y_*) - kcY_* &< 0 \\ \Leftrightarrow \quad \sinh(Y_*) &< \frac{c}{M} Y_* & \Leftrightarrow \quad \sinh(Y_*) &< \cosh(Y_*) Y_*. \end{aligned}$$

The function

$$h : [0, \infty) \to (-\infty, 0], \quad h(Y) = \sinh(Y) - \cosh(Y)Y \tag{3.8}$$

is bijective and decreasing, since $h'(Y) = -\sinh(Y)Y < 0$, $h(0) = 0$ and $\lim_{Y \to \infty} h(Y) = -\infty$. Hence, since $Y_* > 0$, it follows that $P \in H^{-1}(\{\alpha_*\})$ for

$$\alpha_* := kM(\sinh(Y_*) - \cosh(Y_*)Y_*). \tag{3.9}$$

We make now a crucial observation. Let $\alpha \in \mathbb{R}$ be given with $H(X, Y) = \alpha$ for some $(X, Y) \in \Omega$. In view of $kM\sinh(Y)\cos(Y) - kcY = \alpha$, it must hold that

$$X = f_\alpha(Y) := \arccos\left(\frac{\alpha + kcY}{kM\sinh(Y)}\right),$$

with $f_\alpha : D(f_\alpha) \subset \mathbb{R} \to [-1, 1]$. The definition domain $D(f_\alpha)$ of f_α consists of all the points $Y > 0$ which satisfy the inequalities

$$-1 \le \frac{\alpha + kcY}{kM\sinh(Y)} =: g_\alpha(Y) \le 1. \tag{3.10}$$

Hence $H^{-1}(\{\alpha\}) \cap \Omega = G_{f_\alpha}$, with G_{f_α} denoting the graph of f_α.

We are left to determine for which $Y > 0$ the relation (3.10) is valid. A simple calculation shows that

$$g'_\alpha(Y) = \frac{kc\,h(Y) - \alpha\cosh(Y)}{kM\sinh^2(Y)} \quad \text{for all } Y > 0, \tag{3.11}$$

where h is the function defined by the relation (3.8). The main result of this paper is the following theorem:

Theorem 3.1. *Let $\alpha \in \mathbb{R}$ be given. There exists a function $f_\alpha : D(f_\alpha) \subset (0,\infty) \to \mathbb{R}$ such that*

$$H^{-1}(\{\alpha\}) = G_{f_\alpha} \text{ for } \alpha \neq 0, \quad \text{and} \quad H^{-1}(\{0\}) = G_{f_0} \cup ([0,\pi] \times \{0\}).$$

More precisely,

(i) *If $\alpha \geq 0$, then $D(f_\alpha) = [Y(\alpha), \infty)$ for some $Y(\alpha) > 0$. The function $f_\alpha : D(f_\alpha) \to [0, \pi/2)$ is bijective and increasing.*

(ii) *For $\alpha < 0$, there exists a unique $\widetilde{Y}(\alpha) > 0$ such that*

(a) *If $\alpha < \alpha_*$, then $D(f_\alpha) = [\widetilde{Y}(\alpha), \infty)$,*

$$f_\alpha(\widetilde{Y}(\alpha)) = \pi, \quad \text{and} \quad \lim_{Y \to \infty} f_\alpha(Y) = \pi/2.$$

Moreover, the minimum of f_α is less $\pi/2$.

(b) *If $\alpha > \alpha_*$, there exist positive real numbers $Y_i(\alpha)$, $i = 1,2$, with $\widetilde{Y}(\alpha) < Y_1(\alpha) < Y_2(\alpha)$ such that $D(f_\alpha) = [\widetilde{Y}(\alpha), Y_1(\alpha)] \cup [Y_2(\alpha), \infty)$. The restriction $f_\alpha : [\widetilde{Y}(\alpha), y_1(\alpha)] \to [0,\pi]$ is bijective with $f(\widetilde{Y}(\alpha)) = \pi$.*

Additionally, we have

(1) $Y : [0,\infty) \to [Y_\bullet, \infty)$, *where $Y_* < Y_\bullet$, is bijective and increasing.*

(2) $\widetilde{Y} : (-\infty, 0) \to (0,\infty)$ *is bijective and decreasing.*

(3) $Y_1 : (\alpha_*, 0) \to (0, Y_*)$ *is bijective and decreasing.*

(4) $Y_2 : (\alpha_*, 0) \to (Y_*, Y_\bullet)$ *is bijective and increasing.*

All the functions mentioned above are continuous and smooth in the interior of their definition domain.

Remark 3.2. Having proved Theorem 3.1, the phase portrait is complete (see Figure 2). The motion along the phase curves is determined by the fact that

$$-\partial_X H(X,Y)) = kM \sinh(Y) \sin(X) > 0$$

provided $X \in (0, \pi/2)$. The phase portrait also discloses that the stationary solution (X_*, Y_*) is a saddle node for (3.6). Solutions starting in $(X_0, Y_0) \in G_{f_{\alpha_*}}$ are attracted by (X_*, Y_*) provided $Y_0 < Y_*$. On the oder hand, if $Y_0 > Y_*$ they tend to infinity.

Proof of Theorem 3.1. Consider first the case $\alpha \geq 0$. In view of $h \leq 0$, we obtain that g_α is strictly decreasing. L'Hospital's rule shows that

$$\lim_{Y \to \infty} g_\alpha(Y) = 0, \qquad \lim_{Y \to 0} g_\alpha(Y) = \begin{cases} \infty & , \quad \alpha > 0, \\ c/M & , \quad \alpha = 0, \end{cases}$$

so that we find a unique $Y(\alpha) > 0$ such that $g_\alpha(Y(\alpha)) = 1$. Let $Y_\bullet := Y(0)$.

We show first that the function $Y : [0,\infty) \to [Y_\bullet, \infty)$ is a strictly increasing bijection. Indeed, $g : (0,\infty) \times (0,\infty)$ be the function defined by $g(\alpha, Y) := g_\alpha(Y)$. In view of

$$\partial_Y g(\alpha, Y(\alpha)) = \frac{kc\,h(Y(\alpha)) - \alpha \cosh(Y)}{kM \sinh^2(Y(\alpha))} < 0,$$

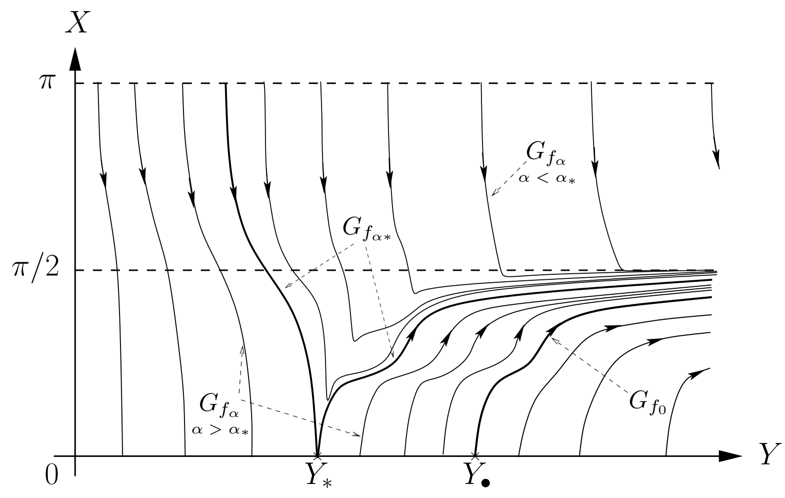

FIGURE 2. Phase portrait in the moving frame

we infer from the implicit function theorem that the map Y is smooth in $(0,\infty)$. Differentiating the relation $g(\alpha, Y(\alpha)) = 1$ with respect to the variable α we get

$$\partial_\alpha g(\alpha, Y(\alpha)) + \partial_Y g(\alpha, Y(\alpha))Y'(\alpha) = 0,$$

and, in view of the previous relation and $\partial_\alpha g(\alpha, Y) = 1/(kM\sinh(Y)) > 0$, we get that $Y'(\alpha) > 0$ for all $\alpha > 0$. Particularly, we obtain that $Y \in C([0,\infty))$, and the desired assertion is immediate.

Consequently, $g_\alpha(Y) \in [-1,1]$ if and only if $Y \in [Y(\alpha),\infty)$, and

$$H^{-1}(\{\alpha\}) \cap \Omega := \{(f_\alpha(Y), Y) : Y \in [Y(\alpha),\infty)\}$$

for all $\alpha \geq 0$. The function f_α is strictly increasing and

$$f_\alpha(Y(\alpha)) = 0, \quad f'_\alpha(Y(\alpha)) = \infty, \quad \text{and} \quad \lim_{Y\to\infty} f_\alpha(Y) = \pi/2.$$

We determine now a relation between $Y_\bullet$ and Y_*. Recall that $Y_\bullet$ is the solution of the equation $MY_\bullet - kM\sinh(Y_\bullet) = 0$, whereas Y_* solves $kc - kM\cosh(Y_*) = 0$. Hence, y_* is exactly the point where

$$[0,\infty) \ni Y \mapsto -H(0,y) = kcY - kM\sinh(Y)$$

changes monotony. Since $H(0,0) = 0$ and $-H(0,Y_*) = -\alpha_* > 0$, cf. (3.9), it must hold that $Y_\bullet > Y_*$.

Assume that $g_\alpha : [Y(\alpha),\infty) \to [0,1]$ is convex. Then

$$g_\alpha\left(\frac{Y_1+Y_2}{2}\right) \leq \frac{g_\alpha(Y_1)+g_\alpha(Y_2)}{2},$$

for all $Y_1, Y_2 \geq Y(\alpha)$. Since arccos is a decreasing function we get

$$\arccos\left(g_\alpha\left(\frac{Y_1+Y_2}{2}\right)\right) \geq \arccos\left(\frac{g_\alpha(Y_1)+g_\alpha(Y_2)}{2}\right),$$

and since $\arccos|_{[0,1]}$ is concave, we conclude that

$$\arccos\left(g_\alpha\left(\frac{Y_1+Y_2}{2}\right)\right) \geq \frac{\arccos(g_\alpha(Y_1))+\arccos(g_\alpha(Y_2))}{2},$$

so that $f_\alpha = \arccos \circ g_\alpha$ is concave.

Summarising, in order to prove that f_α is concave we are left to show that $g''_\alpha \geq 0$. Taking into consideration the relation $\cosh^2(Y) - \sinh^2(Y) = 1$, we have

$$\begin{aligned} &kM\sinh^3(Y)g''_\alpha(Y) \\ &\quad = \alpha + \alpha\cosh^2(Y) + kcY + kc\cosh(Y)(\cosh(Y)Y - 2\sinh(Y)) \geq 0 \end{aligned}$$

provided $Y \geq 2$. Consequently, f_α is a concave function on $[\max\{2, Y(\alpha)\}, \infty)$ and the graph of f_α approaches asymptotically the line $X = \pi/2$ as $Y \to \infty$. This proves assertion (i) of the Theorem 3.1.

We are left to determine to which level set $H^{-1}(\{\alpha\})$, $\alpha < 0$, the points in the remaining domain

$$\Omega \setminus (\{(X,Y) \,:\, Y_\bullet \leq Y < \infty \text{ and } f_0(Y) < X \leq \pi\} \cup [0,\pi] \times (0, Y_\bullet))$$

belong.

Thus, let $\alpha < 0$. Notice that in this case

$$\lim_{Y\to 0} g_\alpha(Y) = -\infty \quad \text{and} \quad \lim_{Y\to\infty} g_\alpha(Y) = 0.$$

We claim that for each $\alpha \leq 0$ there exists a unique $\overline{Y}(\alpha) \geq 0$ such that $g'_\alpha(\overline{Y}(\alpha)) = 0$, that is

$$Mh(\overline{Y}(\alpha)) - \alpha\cosh(\overline{Y}(\alpha)) = 0. \tag{3.12}$$

To prove this assertion, we define the smooth mapping $f : [0,\infty) \to \mathbb{R}$ by $f(Y) = kc\,h(Y) - \alpha\cosh(Y)$. Recall that f is exactly the numerator of in the expression (3.11). If $\alpha = 0$, put simply $\overline{Y}(0) = 0$. For negative α we compute

$$f'(Y) = -(kcY + \alpha)\sinh(Y),$$

thus $f'(Y) = 0$ if and only if $Y \in \{0, -\alpha/(kc)\}$. Moreover, f' is positive in the interval defined by these numbers, and negative on $(-\alpha/(kc), \infty)$. The maximum of f, which is achieved at $-\alpha/(kc)$ is positive, since

$$\begin{aligned} f\left(-\frac{\alpha}{kc}\right) &= kc\left(\sinh\left(-\frac{\alpha}{kc}\right) + \frac{\alpha}{kc}\cosh\left(-\frac{\alpha}{kc}\right)\right) - \alpha\cosh\left(-\frac{\alpha}{kc}\right) \\ &= -kc\left(\sinh\left(-\frac{\alpha}{kc}\right)\right) > 0. \end{aligned}$$

Since $f(0) = -\alpha > 0$ and $\lim_{Y\to\infty} f(Y) = -\infty$, we find a unique $\overline{Y}(\alpha) > 0$ such that $f(\overline{Y}(\alpha)) = 0$, and we are done. Let us stress that $\overline{Y}(\alpha)$ is exactly the point where g_α attains its positive maximum.

Using the implicit function as we did before, yields that $\overline{Y} : (-\infty, 0] \to [0, \infty)$ is a continuous bijection, which is smooth in the interior of its definition domain. In view of (3.12) and

$$\frac{h(Y)}{\sinh(Y)\cosh(Y)} \to_{Y\to l} \begin{cases} 0 & , \quad l = 0, \\ 0, & , \quad l = \infty, \end{cases}$$

we obtain that

$$g_\alpha(\overline{Y}(\alpha)) = \frac{c}{M}\left(\frac{h(\overline{Y}(\alpha))}{\sinh(\overline{Y}(\alpha))\cosh(\overline{Y}(\alpha))} + \frac{\overline{Y}(\alpha)}{\sinh(\overline{Y}(\alpha))}\right)$$

$$\to_{\alpha\to l} \begin{cases} c/M & , \quad l = 0, \\ 0 & , \quad l = -\infty, \end{cases}$$

Since

$$\frac{d}{d\alpha}\left(\frac{\alpha + kc\overline{Y}(\alpha)}{kM\sinh(\overline{Y}(\alpha))}\right) = \frac{1}{kM\sinh(\overline{Y}(\alpha))} > 0$$

we deduce that

$$(-\infty, 0] \ni \alpha \mapsto g_\alpha(\overline{Y}(\alpha)) \in (0, c/M]$$

is a continuous bijection. Hence, there exists a unique $\alpha_\odot$ with $g_{\alpha_\odot}(\overline{Y}(\alpha_\odot)) = 1$, which means that

$$\alpha_\odot + kc\overline{Y}(\alpha_\odot) - kM\sinh(\overline{Y}(\alpha_\odot)) = 0. \tag{3.13}$$

We show now that, in fact, $\overline{Y}(\alpha_\odot) = Y_*$. Indeed, we multiply the relation (3.13) by $\cosh(\overline{Y}(\alpha_\odot))$ and, by adding it to (3.12) evaluated at $\alpha = \alpha_\odot$, we get

$$kc - kM\cosh(\overline{Y}(\alpha_\odot)) = 0.$$

Consequently, $\overline{Y}(\alpha_\odot) = Y_*$.

Lastly, given $\alpha < 0$, let $\widetilde{Y}(\alpha)$ denote the unique solution of the equation $g_\alpha(Y) = -1$ within $(0, \infty)$. The function $[(-\infty, 0) \ni \alpha \mapsto \widetilde{Y}(\alpha) \in (0, \infty)]$ is, in virtue of $\widetilde{Y}(\alpha) < \overline{Y}(\alpha)$, decreasing. Letting $L := \lim_{\alpha\to 0}\widetilde{Y}(\alpha)$, we obtain from $g_\alpha(\widetilde{Y}(\alpha)) = -1$ when $\alpha \to 0$ that $L = 0$. A similar argument yields that $\lim_{\alpha\to 0}\widetilde{Y}(\alpha) = \infty$, so that $\widetilde{Y}$ is also onto.

We show now that $\alpha_\odot = \alpha_*$, where α_* is the constant defined by (3.9). In view of $g_{\alpha_\odot}(\overline{Y}(\alpha_\odot)) = 1$, the maximum of $g_{\alpha_\odot}$ is 1, whence $[\widetilde{Y}(\alpha_\odot), \infty)$ is the maximal subset of $(0, \infty)$ for which (3.10) hold true when $\alpha = \alpha_\odot$. The function $f_{\alpha_\odot} : [\widetilde{Y}(\alpha_\odot), \infty) \to [-\pi, \pi]$ is a parametrisation of the level set $H^{-1}(\{\alpha_\odot\})$, and since $f_{\alpha_\odot}(\overline{Y}(\alpha_\odot)) = 0$, we deduce that $P = (0, Y_*) = (0, \overline{Y}(\alpha_\odot)) \in H^{-1}(\{\alpha_\odot\})$, which in turn leads to $\alpha_* = \alpha_\odot$.

Our analysis reveals that for $\alpha > \alpha_*$, there exist precisely two points $Y_1(\alpha) < Y_2(\alpha)$ such that $g_\alpha(Y_i(\alpha)) = 1$ for $i = 1, 2$ and all $\alpha > \alpha_*$. It is not difficult to see that $Y_1 : (\alpha_*, 0) \to (0, Y_*)$ is a smooth and decreasing bijection. Moreover, $Y_2 : (\alpha_*, 0) \to (Y_*, Y_\bullet)$ is a smooth and increasing bijection. We show only that $\lim_{\alpha\to 0} Y_2(\alpha) = Y_\bullet$. The other assertions follow similarly. Clearly, $\lim_{\alpha\to 0} Y_2(\alpha) := Y_0$ is finite and positive. Letting $\alpha \to 0$ in the relation $g_\alpha(Y_2(\alpha)) = 1$, we obtain that

$$kMY_0 - kM\sinh(Y_0) = 0,$$

thus $Y_0 = Y_\bullet$.

Hence, if $\alpha > \alpha_*$, we have that $f_\alpha : [\widetilde{Y}(\alpha), Y_1(\alpha)] \cup [Y_2(\alpha), \infty) \to [-\pi, \pi]$, $f_\alpha(Y_i(\alpha)) = 0$, $i = 1, 2$, $f_\alpha(\widetilde{Y}(\alpha)) = \pi$, $\lim_{Y\to\infty} f_\alpha(Y) = \pi/2$, and

$$f'_\alpha(\widetilde{Y}(\alpha)) = f'_\alpha(Y_1(\alpha)) = -f'_\alpha(Y_2(\alpha)) = -\infty.$$

Moreover, f_α is decreasing on $[\widetilde{Y}(\alpha), Y_1(\alpha)]$ and increasing on $[Y_2(\alpha), \infty)$.

If $\alpha < \alpha_*$, then $[\widetilde{Y}(\alpha), \infty)$ are all the solutions of (3.10) within $(0, \infty)$, thus $H^{-1}(\{\alpha\})$ may be parametrised by the mapping $f_\alpha : [\widetilde{Y}(\alpha), \infty) \to (0, \pi]$. The minimum of f_α is less than $\pi/2$ and $\lim_{Y\to\infty} f_\alpha(Y) = \pi/2$. This completes the proof of Theorem 3.1. □

Knowing the phase curves $(X(t), Y(t))$, the particle trajectories $(x(t), y(t))$ in the linear wave (2.11) are given, via (3.3), by

$$x(t) = \frac{X(t)}{k} + ct \quad y(t) = \frac{Y(t)}{k}. \tag{3.14}$$

The phase-plane analysis done in Theorem 3.1 reveals that a necessary condition for (3.4) to be physically realistic solution is that the wave profile is located underneath Y_*, meaning that

$$kh_0(1+\varepsilon) \le Y_* = \cosh^{-1}\left(\frac{\sinh(kh_0)}{\varepsilon h_0 k}\right). \tag{3.15}$$

This restriction ensures that the particles do not leave the fluid domain. Whence, a sufficient condition for (2.11) to be a realistic model of water waves of average depth $h_0 > 0$ and wave number k, is that the small amplitude $\varepsilon > 0$ satisfies

$$\varepsilon \le \max\left\{1, \frac{\sinh(kh_0)}{kh_0\cosh(2kh_0)}\right\},$$

which gives a quantitative meaning to the assumption "$\varepsilon < 1$ is small."

We describe now the trajectory of the water particle (x_0, y_0) in the fluid. Assume that the particle is located initially at $X(0) = x_0/k = \pi$ and $Y(0) = y_0/k \ge 0$. The time $\theta = \theta(Y_0)$ needed for $(X(0), Y(0))$ to reach the line $X = -\pi$ fulfills (cf. [9]) $\theta > 2\pi/(ck)$. In view of (3.1) and (3.3) we know that $dx/dt > 0$ as long as $X(t) \in (-\pi/2, \pi/2)$, and $dx/dt < 0$ for $X(t) \in (-\pi, -\pi/2) \cup (\pi/2, \pi)$, while $dy/dt \ge 0$ when $X(t) \in (0, \pi)$ and $dy/dt \le 0$ for $X(t) \in (-\pi, 0)$ (equality holds iff $y_0 = 0$). The particle attains therefore the maximum height when $X(t) = 0$, i.e. $t = \theta/2$, and $y(t)$ decreases afterwards to y_0. Via (3.14),

$$x(\theta) - x(0) = \frac{X(\theta) - X(0)}{k} + c\theta = -\frac{2\pi}{k} + c\theta > 0, \tag{3.16}$$

so that in time θ the particle traces a loop that fails to close up. There is a small forward drift explicitly expressed by (3.16) in terms of θ and $2\pi/(ck)$,

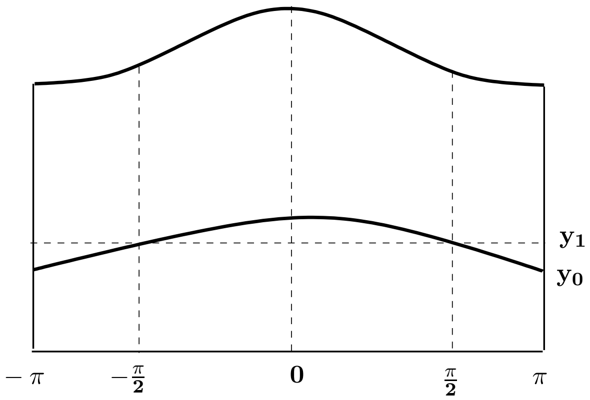

Figure 3. Particle paths in the moving frame

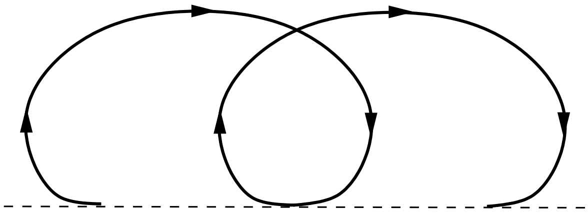

Figure 4. Particle trajectory above the flat bed

which is minimal on the flat bed, cf. [9].

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