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On the streamlines and particle paths of gravitational water waves

Mats Ehrnström

Department of Mathematics, Lund University, PO Box 118, 221 00 Lund, Sweden

E-mail: mats.ehrnstrom@math.lu.se

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Abstract
We investigate steady symmetric gravity water waves in a finite depth. For non-positive vorticity it is shown that the particles display a mean forward drift, and for a class of waves we prove that the size of this drift is strictly increasing from bottom to surface. This includes the case of particles within irrotational waves. We also provide detailed information concerning the streamlines and the particle trajectories.

Mathematics Subject Classification: 35Q35, 76B15

1. Introduction

This paper is concerned with the streamlines and the particle trajectories of steady gravity water waves in a finite depth. Such water waves are one of the most common wave formations at sea. As a result of dispersion, wind-generated gravity waves eventually sort themselves out [22,24]. Larger waves move faster than smaller ones and swell is generated: approximately two-dimensional wave-trains of periodic and symmetric waves moving with constant speed across the sea. The exact mathematical theory for such waves is well established, in particular for irrotational flows [18,31]. Those model very well the situation when the waves propagate into a region of still water. There is, however, experimental evidence that for some situations such a model is inadequate [29]. One example is tidal flow, which is more correctly modelled by waves entering a rotational current of constant vorticity [30]. Therefore the importance of water motion with a non-vanishing curl—i.e. in the presence of vorticity—has recently drawn a lot of attention (see, e.g. [8, 12, 21, 35]). For us it is relevant that for arbitrary vorticity distributions there exist symmetric waves [9], and that any wave for which the surface profile is monotone from crest to trough is necessarily symmetric [6].

In a number of recent papers the exact behaviour of the fluid particles within such waves has been investigated [4, 5, 11, 15, 19, 20]. The background is as follows. For over a century it has been known that the very first approximation of steady irrotational gravity water waves display closed elliptic particle trajectories [28]. However, as was first noted in [11], a thorough study of the linearized system shows that the particle paths indeed have the shape of an oval, but are not closed. This is also true for other types of waves, as has been shown in [5, 20]. Since the linearized problem can be solved explicitly, details of the particle paths can be more easily studied. In particular, it can be seen that all particles traverse oval orbits. When vorticity is present things are not as transparent, not even for linear waves on a current of constant vorticity. In [15] it is, for example, shown that when the size of the vorticity is large, the particle paths of linear waves need not all be oval; some particles may move constantly forward along with the wave. Near the flat bed the particles, however, always behave like the classical first approximation: they move slightly forward in non-closed oval shapes. As discussed below this is all in relation to some reference speed, i.e. the generalized Stokes requirement (2.5).

For exact water waves the details are far more elusive since closed expressions are not available. The investigations [4, 19] show, however, that even for exact irrotational water waves the particles display a mean forward drift. The notion of *mean forward drift* means that each time a particle reaches its highest point it has moved some distance along with the wave. The papers [4, 19] also assert that for irrotational Stokes waves in a finite—as well as an infinite—depth all particles move in oval orbits. In this paper, we show that the mean forward drift is preserved for all negative vorticity distributions. For irrotational waves and small enough rotational waves, we are able to show that this forward drift is strictly increasing from bottom to top. A proof of this for linear waves without vorticity was given in [5]. In addition we establish some surprisingly nice properties of the velocity field and the particle paths, in particular for irrotational waves. We have not been able to confirm the oval orbital shapes for rotational waves.

The novelty of our approach lies in the fact that we establish precise pointwise information about the velocity field and its derivatives within the entire fluid domain. In this way we extend the propositions in [4, 19], providing further understanding even for the irrotational case. To be exact, the earlier investigations are based on the irrotational counterparts of this paper's lemmas 3.1 (iii) and 3.2 (i), together with an analysis of the zero level set of harmonic functions. The implications for particle paths are found via means similar to those used to prove theorem 4.1 (i). The resulting statement is that there are no closed trajectories, and the oval orbits with the forward drift are described. Apart from the fact that we investigate rotational flows, several other results are presented, in particular lemma 3.2 (ii) and the results of theorem 4.1.

The proof techniques rely heavily on sharp maximum principles, for which we refer the reader to the excellent sources [16, 26]. For steady rotational waves, to our knowledge, this is the first investigation of its kind apart from [15]. Some of the results here obtained can be extended to deep-water waves and solitary waves. However, it should be noted that there are important differences between these types of waves. Notably, the investigation [7] shows that the particle trajectories within irrotational solitary waves differ in fundamental ways from those in periodic waves, and in [13] it is proved that the class of vorticities allowed for in deep-water waves is much more restrictive than for a finite depth.

The paper is organized as follows. Section 2 gives the mathematical background, while the main results are proved in sections 3 and 4. A synthesis and an analysis of the particle paths are given in section 5, presented as two, hopefully illustrative, examples.

2. Mathematical formulation

Let $d > 0$ be the depth below the mean water level $y = 0$, so that the flat bottom can be described by $y = -d$. The *free surface* can be represented by a function

$$\eta \in C^3(\mathbb{R}, \mathbb{R}).$$

We require that $\eta(0) = \max_{x\in\mathbb{R}}\{\eta(x)\}$ be the vertical coordinate of the crest, unique within a period. Naturally $\min_{x\in\mathbb{R}}\{\eta(x)\} > -d$, so that the trough is above the flat bed $y = -d$. We shall be concerned with the non-trivial case when $\max \eta > \min \eta$. The wave is steady of period $L > 0$—without loss of generality we may take $L = 2\pi$—and we require the surface profile to be monotone between crests and troughs. It is therefore symmetric around the crest [6], and we have that

$$\eta(x + 2\pi) = \eta(x), \qquad \eta(x) = \eta(-x) \qquad \text{and} \qquad \eta'(x) < 0 \text{ for } x \in (0, \pi).$$

We let Ω_η denote the *fluid domain* and define it as the interior of its boundary

$$\partial\Omega_\eta \equiv \{y = -d\} \cup \{x, \eta(x))\}_{x\in\mathbb{R}}.$$

A solution to the water wave problem is then defined as a function $\psi \in C^2(\overline{\Omega}_\eta)$ such that

$$\begin{cases} \Delta\psi = -\gamma(\psi), & (x, y) \in \Omega_\eta \\ |\nabla\psi|^2 + 2gy = C, & y = \eta(x) \\ \psi = 0, & y = \eta(x) \\ \psi = -p_0, & y = -d, \end{cases} \tag{2.1}$$

that is even and 2π-periodic in the x-variable. In (2.1) p_0 is called the *relative mass flux*, the *vorticity function* $\gamma\colon [0, -p_0] \to \mathbb{R}$ is continuously differentiable, $g > 0$ is the gravitational constant and C is a constant related to the energy. The setting is that of gravitational water waves, meaning that the influence of capillarity is neglected in (2.1), and the water is assumed to be inviscid. The *stream function* ψ is defined (up to a constant) by

$$\psi_x = -v, \qquad \psi_y = u - c < 0, \tag{2.2}$$

where u, v are the horizontal and the vertical velocities, respectively, and $c > 0$ is the constant horizontal speed of propagation. The notion of relative mass flux introduced in [9] captures the physical fact that the amount of water passing any vertical line is constant throughout the fluid domain:

$$\int_{-d}^{\eta(x)} (u(x, y) - c)\, \mathrm{d}y = p_0, \qquad x \in \mathbb{R},$$

holds since $u - c = \psi_y$, and ψ is constant on the surface $y = \eta(x)$ as well as on the bottom $y = -d$.

Provided that $u - c = \psi_y < 0$, system (2.1) can be deduced from the Euler equations (see, e.g. [9, 34] for a more detailed discussion). This assumption is supported by physical measurements [24]: for a wave not near breaking or spilling, the speed of an individual fluid particle is far less than that of the wave itself. For irrotational waves it is however known that there exist so-called highest waves for which the crest is a stagnation point (see, e.g. [1]), i.e. $\nabla\psi = 0$. While the exact problem is still open for waves with vorticity [10, 32], there are indications that for some classes of vorticity there do exist steady waves with particle layers not satisfying $\psi_y < 0$ [15, 23]. In this paper we shall, however, consider only waves that are not near breaking or stagnation, so that $\psi_y < 0$ in $\overline{\Omega}_\eta$.

A hodograph transform converts the free boundary problem (9) into a problem with a fixed boundary. Let us express the height

$$h \equiv y + d$$

above the flat bed in terms of the new space variables

$$q \equiv x, \qquad p \equiv -\psi. \tag{2.3}$$

Note that $\psi_y < 0$ so that (2.3) is a local change of variables, with

$$h_q \equiv -\frac{\psi_x}{\psi_y} = \frac{v}{u-c}, \qquad h_p \equiv -\frac{1}{\psi_y} = \frac{1}{c-u}.$$

The above local coordinate transform is actually a global change of variables (see [9]) so that we can transform problem (2.1) into these variables to obtain

$$\begin{cases} (1+h_q^2)\,h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} + \gamma(-p)\,h_p^3 = 0 & \text{in} \quad p_0 < p < 0, \\ 1 + h_q^2 + (2gh - Q)\,h_p^2 = 0 & \text{on} \quad p = 0, \\ h = 0 & \text{on} \quad p = p_0, \end{cases} \tag{2.4}$$

with h even and of period 2π in the q variable. This is an elliptic equation—since $h_p > 0$—with a nonlinear boundary condition. Instead of studying (2.1) in the domain Ω_η, which depends on η, we investigate (2.4) in the fixed rectangle $R \equiv (-\pi, \pi) \times (p_0, 0)$, looking for functions $h \in C^2(\overline{R})$ that are 2π-periodic in q. Note that knowing $h(q, 0)$ is equivalent to knowing the free surface $y = \eta(x)$, as $h(q, 0) = \eta(q) + d$.

Let $(x, \sigma(x))$ denote the parametrization of a (general) streamline

$$\{(x, y) : \psi(x, y) = -p\}.$$

Note that since $\psi_y < 0$ the above definition is sensible, and we have

$$\sigma'(x) = -\frac{\psi_x(x, \sigma(x))}{\psi_y(x, \sigma(x))} = h_q(q, p).$$

To normalize the reference frame Stokes made a now commonly accepted proposal. In the case of irrotational flow he required that the horizontal velocity should have a vanishing mean over a period. Stokes' definition of the wave speed unfortunately cannot be directly translated to waves with vorticity, a consequence of the fact that if div $\nabla\psi \not\equiv 0$, then ψ_y has different means at different depths in view of the divergence theorem. In the setting of periodic waves with vorticity we propose the requirement

$$\int_{-\pi}^{\pi} u(x, -d)\,\mathrm{d}x = 0, \tag{2.5}$$

a 'Stokes' condition' at the bottom. This is consistent with deep-water waves (cf [13]), and is also the choice made in [30]. (In [25], however, the normalization is done at the surface of the underlying flow.) Calculations performed on linear water waves with constant vorticity indicate that this is the natural choice, since that and only that choice recovers the well-established bound $\sqrt{gh}$ for the wave speed [15]. We emphasize that (2.5) is only a convention for fixing the reference frame; except from the assertion of forward drift it does not change the results of this paper. Without such a reference it is however meaningless to discuss whether physical particle paths are closed or not.

3. Streamlines and the horizontal velocity

In this section we establish the main results for the steady reference frame.

Lemma 3.1.

(i) *Every streamline satisfies $\sigma' < 0$ for $x \in (0, \pi)$, and the maximal steepness of the streamlines is a strictly monotone function of depth.*

(ii) *For $\gamma' \geqslant 0$ and $\gamma \leqslant 0$, the maximal horizontal velocity, $\max_{x\in\mathbb{R}} u(x, \sigma(x))$, is strictly increasing from bottom to surface.*

(iii) *The vertical velocity is strictly positive for $x \in (0, \pi)$, and if $\gamma' \leqslant 0$, then $\max_{x\in\mathbb{R}} |v(x, \sigma(x))|$ is strictly increasing from bottom to surface.*

Lemma 3.2 (the horizontal velocity).

(i) *If $\gamma \leqslant 0$, then the horizontal velocity u is non-increasing from crest to trough, i.e.*

$$\mathrm{D}_x u(x, \eta(x)) \leqslant 0 \qquad \text{for } x \in (0, \pi). \tag{3.1}$$

(ii) *If $\gamma = 0$ and $|\eta'| \leqslant 1/\sqrt{3}$, then along any streamline $(x, \sigma(x))$ holds*

$$\mathrm{D}_x u(x, \sigma(x)) < 0 \qquad \text{for } x \in (0, \pi),$$

and the pointwise steepness of the streamlines is everywhere decreasing with depth.

(iii) *If $\gamma(0) \geqslant 0$ and $\gamma', \gamma'' \leqslant 0$ then*

$$\partial_x u(x, y) < 0 \qquad \text{in } (0, \pi)$$

for waves in a neighbourhood of the bifurcation point in [9].

(iv) *If the horizontal velocity attains its maximum at the surface, then it does so either at the crest or at the concave part of the surface where*

$$(c - u)\gamma \leqslant g = -\eta''(c - u)^2.$$

Corollary 3.3.
If (3.1) holds, then for $x \in (0, \pi)$ we have the uniform bound $\eta'' \geqslant -\frac{g}{C-2g\eta(0)}$, together with

$$\eta'(x) \geqslant -\frac{gx}{C - 2g\eta(0)} \qquad \text{and} \qquad \eta(x) \geqslant \eta(0) - \frac{gx^2}{2(C - 2g\eta(0))}.$$

Remark 3.4. Part (iii) of lemma 3.2 equivalently states that $\partial_y v > 0$ within the half-period $(0, \pi)$. Note also that the equality of part (iv) in lemma 3.2, as well as corollary 3.3, is based solely on the surface conditions of 2.1, and hence unrelated to, e.g. periodicity, symmetry and depth.

Proof of lemma 3.1.

(i) For details see [9, equation (5.18)]. The key idea is that h_q is annihilated by

$$(1 + h_q^2)\partial_p^2 - 2h_p h_q \partial_p \partial_q + h_p^2 \partial_q^2 + 2h_q h_{pp} \partial_q + [3\gamma(-p)h_p^2 - 2h_q h_{pq}]\partial_p. \tag{3.2}$$

The second statement follows from applying the strong maximum principle to subdomains $(0, \pi) \times (-d, p)$ of this half-period.

(ii) Consider $h_p = 1/(c - u) > 0$. Since h_p belongs to the kernel of the uniformly elliptic operator

$$(1 + h_q^2)\partial_p^2 - 2h_q h_p \partial_p \partial_q + h_p^2 \partial_q^2 - 2h_p h_{qp} \partial_q + h_p(2h_{qq} + 3\gamma h_p)\partial_p - \gamma' h_p^2, \tag{3.3}$$

the strong maximum principle implies that $\max u$ is never attained in the interior of any C^2-subdomain of the fluid. In view of that

$$u_y = \psi_{yy} = -\gamma(-p_0) \geqslant 0$$

at the flat bed, it is a consequence of the Hopf boundary point lemma that u does not attain its maximum on the bottom. The proposition then follows by periodicity.

(iii) Since $\sigma' = -\psi_x/\psi_y$ and $\psi_y < 0$, the positivity of v is immediate from (i). Note also that v vanishes on the flat bed. Since $v \in \mathrm{Ker}\{\Delta + \gamma'(\psi)\}$ we may apply the strong maximum principle to any subdomain

$$\{(x, y): 0 < x < \pi, -d < y < \sigma(x)\}.$$

Symmetry yields the assertion.

Proof of lemma 3.2.

(i) See [33, theorem 2.2]

(ii) We first note that

$$\partial_q h_p = \left(\frac{\partial x}{\partial q}\partial_x + \frac{\partial y}{\partial q}\partial_y\right)\frac{1}{c - u(x, \sigma(x))} = \frac{D_x u(x, \sigma(x))}{[c - u(x, \sigma(x))]^2}.$$

It thus suffices to show that $h_{qp} < 0$ everywhere in $q \in (0, \pi)$. In view of the above calculation, lemma 3.2 (i) shows that $h_{qp} \leqslant 0$ along the surface for $q \in [0, \pi]$. At the bottom we have $h_{qp} < 0$ for $q \in (0, \pi)$. This follows from lemma (i) and the Hopf boundary point lemma: since $\sigma' = h_q$ vanishes on the flat bed, and h_q satisfies the strong maximum principle according to (3.2), it must have a negative derivative in the direction inwards the fluid domain. Along the vertical sides where $q = 0$ and $q = \pi$, symmetry implies that $h_{qp} = 0$. Along the boundary containing a half-period we thus conclude that $h_{qp} \leqslant 0$. The aim is now to establish that the strong maximum principle holds for h_{qp} in $q \in [0, \pi]$, according to which $h_{qp} \geqslant 0$ at an interior point of the half-period would force $h_{qp} = \text{const}$ everywhere.

We begin by differentiating (3.2) with respect to p. That gives

$$\begin{aligned}
&\big[(1 + h_q^2)\partial_p^2 - 2h_q h_p \partial_q \partial_p + h_p^2 \partial_q^2 - 2h_p h_{qp}\partial_q + (3\gamma(-p)h_p^2 - 2h_q h_{qp})\partial_p \\
&\qquad + 2h_{pp}h_{qq} + 6\gamma(-p)h_p h_{pp} - 3\gamma'(-p)h_p^2 - 2h_{qp}^2\big]\, h_{qp} \\
&\qquad = -2h_p h_{pp} h_{qqq} - 2h_q h_{qq} h_{ppp}.
\end{aligned} \tag{3.4}$$

Now, rewrite (3.2) and (3.3) as

$$\begin{aligned}
h_{qqq} = \frac{1}{h_p^2}\big(&2h_p h_q (h_{qp})_q - 2h_q h_{pp} h_{qq} \\
&-(1 + h_q^2)(h_{qp})_p - (3\gamma(-p)h_p^2 - 2h_q h_{qp})(h_{qp})\big),
\end{aligned}$$

and

$$\begin{aligned}
h_{ppp} = \frac{1}{(1 + h_q^2)}\big(&2h_q h_p (h_{qp})_p - h_p^2 (h_{qp})_q \\
&+ 2h_p (h_{qp})^2 - h_p(2h_{qq} + 3\gamma(-p)h_p)h_{pp} + \gamma'(-p)h_p^3\big).
\end{aligned}$$

To proceed, substitute those expressions into (3.4), to eliminate the third order expressions h_{qqq} and h_{ppp}. We consider then the result as an operator acting on h_{qp}, and deal with the coefficients one at a time.

First, leave the highest derivatives untouched in (3.4), i.e. $(1+h_q^2)\partial_p^2 - 2h_q h_p \partial_q \partial_p + h_p^2 \partial_q^2$. The terms involving ∂_q are

$$\left(4h_q h_{pp} - 2h_q h_{qp} - \frac{2h_q h_p^2 h_{qq}}{1 + h_q^2}\right)\partial_q,$$

and those involving ∂_p are

$$\left(3\gamma(-p)h_p^2 - 2h_q h_{qp} + \frac{4h_q^2 h_p^2 h_{qq}}{h_p(1+h_q^2)} - \frac{2h_{pp}(1+h_q^2)}{h_p}\right)\partial_p.$$

Collecting the zero order terms we obtain

$$\begin{aligned}
&2h_{pp}h_{qq}h_{qp} + 6\gamma h_p h_{pp} h_{qp} - 3\gamma' h_p^2 h_{qp} - 2(h_{qp})^3 \\
&\quad + \frac{2h_{pp}}{h_p}\left(2h_q(h_{qp})^2 - 2h_q h_{pp} h_{qq} - 3\gamma h_p^2 h_{qp}\right) \\
&\quad + \frac{2h_q h_{qq}}{1+h_q^2}\left(2h_p(h_{qp})^2 - 2h_p h_{qq} h_{pp} - 3\gamma h_p^2 h_{pp} + \gamma' h_p^3\right) \\
&= \frac{1}{h_p\left(1+h_q^2\right)}\Big(h_p\left(1+h_q^2\right)\left[2\left(h_{pp}h_{qq} - (h_{qp})^2\right) - 3\gamma' h_p^2\right] h_{qp} \\
&\quad + 4h_q\left((h_{qp})^2 - h_{qq}h_{pp}\right)\left[\left(1+h_q^2\right)h_{pp} + h_p^2 h_{qq}\right] \\
&\quad + 2h_q h_p^3 h_{qq}\left(\gamma' h_p - 3\gamma h_{pp}\right)\Big) \\
&= \frac{1}{\left(1+h_q^2\right)}\Big(\left(1+h_q^2\right)\left[2\left(h_{pp}h_{qq} - (h_{qp})^2\right) - 3\gamma' h_p^2\right] h_{qp} \\
&\quad + 4h_q\left((h_{qp})^2 - h_{qq}h_{pp}\right)\left[2h_q h_{qp} - \gamma h_p^2\right] \\
&\quad + 2h_q h_p^2 h_{qq}\left(\gamma' h_p - 3\gamma h_{pp}\right)\Big),
\end{aligned}$$

where we have made repeated use of the elliptic equality in (2.4). Bringing everything together, a last simplification yields the identity

$$\begin{aligned}
&\Bigg[(1+h_q^2)\partial_p^2 - 2h_p h_q \partial_p \partial_q + h_p^2 \partial_q^2 + \left(4h_q h_{pp} - 2h_p h_{qp} - \frac{2h_q h_p^2 h_{qq}}{1+h_q^2}\right)\partial_q \\
&\quad + \left(3\gamma h_p^2 - 2h_q h_{qp} + \frac{4h_q^2 h_{qq} h_p}{1+h_q^2} - \frac{2h_{pp}(1+h_q^2)}{h_p}\right)\partial_p \\
&\quad + \frac{2(h_{qq}h_{pp} - h_{qp}^2)(1-3h_q^2) - 3\gamma' h_p^2}{1+h_q^2}\Bigg] h_{qp} \\
&= \frac{2h_q h_p^2}{1+h_q^2}\left(\gamma(h_{qq}h_{pp} + 2h_{qp}^2) - \gamma' h_p h_{qq}\right).
\end{aligned} \tag{3.5}$$

By multiplying the elliptic equation in (2.4) with h_{pp} and then completing the squares, we obtain for $\gamma = 0$ that

$$0 = (h_q h_{pp} - h_p h_{qp})^2 + h_{pp}^2 + h_p^2(h_{qq}h_{pp} - h_{qp}^2), \tag{3.6}$$

which forces $h_{qp}^2 \geqslant h_{qq}h_{pp}$ everywhere in the fluid. Finally, lemma 3.1 (i) and the assumption guarantees that $h_q^2 \leqslant 1/3$ everywhere. This means that the coefficient in front of h_{qp} in (3.5) is non-positive, so that h_{qp} satisfies the strong maximum principle.

(iii) We shall determine ψ_{xy} in a manner similar to that used in [14]. Since $\psi = 0$ along the surface, we have $\psi_x = -\eta'\psi_y$, and insertion into the Bernoulli surface condition of (2.1) yields that

$$\psi_y^2 = \frac{C - 2g\eta}{1+\eta'^2}. \tag{3.7}$$

By differentiation along the surface $(x, \eta(x))$, it follows that

$$\psi_{xy} + \eta'\psi_{yy} = -\partial_x\sqrt{(C-2g\eta)/(1+\eta'^2)}. \tag{3.8}$$

On the other hand, we may differentiate $\psi_x = \eta'\psi_y$ once more along the surface, obtaining

$$\psi_{xx} + \eta'^2\psi_{yy} + 2\eta'\psi_{xy} + \eta''\psi_y = 0. \tag{3.9}$$

A third equality is supplied by

$$\psi_{xx} + \psi_{yy} = -\gamma(0). \tag{3.10}$$

We now combine (3.8), (3.9) and (3.10) to isolate ψ_{xy}. The calculation—cf [14] for details—yields that at the surface ψ_{xy} can be determined as

$$\psi_{xy} = \frac{\eta'(2\eta''(C-2g\eta) + (1-\eta'^4)g + \gamma(0)\sqrt{C-2g\eta}(1+\eta'^2)^{3/2})}{(1+\eta'^2)^{5/2}\sqrt{C-2g\eta}}. \tag{3.11}$$

Since $C - 2g\eta$, g, $\gamma(0)$ are all positive, and $\eta' < 0$ in $(0, \pi)$, we have that for η' and η'' small enough, $\psi_{xy} \leqslant 0$ at the surface. Moreover, $\psi_{xy} = 0$ for $x = k\pi$, $k \in \mathbb{Z}$, by symmetry. And on the bottom holds $\psi_{xy} \leqslant 0$, according to lemma 3.1 (iii) and the boundary condition $\psi_x(x, -d) = 0$. Since ψ_{xy} obeys the maximum principle,

$$(\Delta + \gamma')\psi_{xy} = -\gamma''\psi_x\psi_y \geqslant 0,$$

a non-negative maximum thus cannot be attained in the interior of the half-period $0 < x < \pi$.

(iv) According to (3.7), differentiation along the surface gives

$$D_x\psi_y^2(x, \eta(x)) = \frac{2\eta'[(2g\eta - C)\eta'' - g(1+\eta'^2)]}{(1+\eta'^2)^2}. \tag{3.12}$$

Thus a maximum of ψ_y along the surface implies that either $\eta' = 0$ or $(2g\eta - C)\eta'' = g(1+\eta'^2)$. The equality $g = -\eta''(c-u)^2$ is obtained by substituting the second expression into the Bernoulli surface condition of (2.1). To obtain the inequality, first note that if the maximum is attained at the surface, then $\psi_{yy} \geqslant 0$ at that point. But (3.8), (3.9) and (3.10) can be used to show that (cf [14] for details)

$$\psi_{yy} = \frac{\eta''(C-2g\eta)(\eta'^2-1) + 2g\eta'^2(1+\eta'^2) - \gamma(0)\sqrt{C-2g\eta}(1+\eta'^2)^{3/2}}{(1+\eta'^2)^{5/2}\sqrt{C-2g\eta}}, \tag{3.13}$$

so we need only substitute $(2g\eta - C)\eta'' = g\left(1+\eta'^2\right)$ into that expression to get

$$\psi_{yy}\Big|_{\eta''=\frac{g(1+\eta'^2)}{(2g\eta-C)}} = \frac{g\sqrt{1+\eta'^2} - \gamma(0)\sqrt{C-2g\eta}}{\left(1+\eta'^2\right)\sqrt{C-2g\eta}} \geqslant 0. \tag{3.14}$$

Proof of corollary 3.3. Recall (3.12). It follows from the assumption that

$$\frac{2\eta'\eta''}{1+\eta'^2} \leqslant \frac{-2g\eta'}{C-2g\eta}, \quad \text{meaning} \quad \frac{\mathrm{d}}{\mathrm{d}x}\log\left(\frac{1+\eta'^2}{C-2g\eta}\right) \leqslant 0. \tag{3.15}$$

This can be integrated to

$$\frac{1+\eta'^2}{C-2g\eta} \leqslant \frac{1}{C-2g\eta(0)}. \tag{3.16}$$

Since $\eta' < 0$ in $(0, \pi)$ we may rearrange to obtain

$$\frac{-\eta'}{\sqrt{2g(\eta(0) - \eta(x))}} \leqslant \frac{1}{\sqrt{C - 2g\eta(0)}}. \tag{3.17}$$

The assertion concerning η is established by integrating (3.17). The bound on η' then follows from employing the lower bound on η to (3.16). Finally, the uniform bound on η'' is immediate from combining the left-hand side of (3.15) with (3.16).

4. The forward drift

Theorem 4.1.

(i) For $\gamma \leqslant 0$ there are no closed particle trajectories. In particular all fluid particles display a mean forward drift.
(ii) If $\gamma = 0$ with $|\eta'| \leqslant 1/\sqrt{3}$ then the mean forward drift is strictly increasing from bed to surface.
(iii) If $\gamma < 0$ then for all waves in a neighbourhood of the bifurcation point found in [9], the mean forward drift is strictly increasing from bed to surface.

Remark 4.2. In the proof of theorem 4.1 (i) we describe a relationship between the closedness of paths and a certain time, called τ. That idea comes from [11]. It is crucial in all recent investigations concerning particle trajectories in periodic water waves.

Lemma 4.3. *For $(x, \sigma(x))$ a non-trivial streamline, and $\gamma \leqslant 0$, the quantity*

$$\int_0^\pi |\psi_y(x, \sigma(x))|(1 + \sigma'^2(x))\, \mathrm{d}x$$

is non-decreasing as a function of depth, and

$$\int_0^\pi |\psi_y(x, \sigma(x))|\, \mathrm{d}x < c\pi. \tag{4.1}$$

Proof of lemma 4.3. For two streamlines $(x, \sigma_1(x))$ and $(x, \sigma_2(x))$ with $\sigma_1(x) < \sigma_2(x)$, let

$$\Sigma \equiv \{(x, y) : 0 < x < \pi,\ \sigma_1(x) < y < \sigma_2(x)\}.$$

According to the divergence theorem we have

$$\begin{aligned}
-\int_\Sigma \gamma\, \mathrm{d}A = \int_\Sigma \nabla \cdot \nabla\psi\, \mathrm{d}A &= \int_{\sigma_1} \nabla\psi \cdot \frac{\nabla\psi}{|\nabla\psi|}\, \mathrm{d}s - \int_{\sigma_2} \nabla\psi \cdot \frac{\nabla\psi}{|\nabla\psi|}\, \mathrm{d}s \\
&= \int_0^\pi \left(|\nabla\psi(x, \sigma_1(x))| \sqrt{1 + \sigma_1'^2(x)} - |\nabla\psi(x, \sigma_2(x))| \sqrt{1 + \sigma_2'^2(x)} \right) \mathrm{d}x \\
&= \int_0^\pi \left(\left|\psi_y(x, \sigma_1(x))\right| \left(1 + \sigma_1'^2(x)\right) - \left|\psi_y(x, \sigma_2(x))\right| \left(1 + \sigma_2'^2(x)\right) \right) \mathrm{d}x.
\end{aligned}$$

This implies that for any non-trivial streamline $(x, \sigma(x))$,

$$\int_0^\pi |\psi_y(x, \sigma(x))| \left(1 + \sigma'^2(x)\right) \mathrm{d}x = c\pi + \int_0^\pi \int_{-d}^{\sigma(x)} \gamma\, \mathrm{d}A,$$

in view of that $\int_0^\pi \psi_y(x, -d)\, \mathrm{d}x = -c\pi$ by the normalization (2.5). The lemma follows.

Proof of theorem 4.1.

(i) Let $(\mathbb{X}(t), \mathbb{Y}(t))$ be any physical trajectory, so that

$$(x(t), y(t)) = (\mathbb{X}(t) - ct, \mathbb{Y}(t))$$

is the corresponding path in the steady variables. We have $\dot{x}(t) = u - c \leqslant -\delta < 0$. Thus $(x(t), y(t))$ passes any $x \in \mathbb{R}$, and there is no loss of generality in choosing $x(0) = \pi$. We may also define τ through

$$x(\tau) := -\pi.$$

Since

$$\sigma' = \frac{v}{u-c} = \frac{\dot{y}}{\dot{x}},$$

the streamlines describe the flow of the particles *in the steady reference frame*. By symmetry we thus have $y(\tau) = y(0)$. It moreover follows from lemma 3.1 that $y(0)$ is the lowest point of the trajectory, attained below the trough, and in view of symmetry $v(\tau/2)$ is the highest, attained below the crest. In between $\dot{y}(t) \neq 0$. Hence

$$y(T) = y(0) \qquad \text{implies} \;\; x(T) - x(0) = 2\pi n,$$

for some $n \in \mathbb{Z}$. In particular $n = -1$ for $T = \tau$.
Returning to the physical variables, this means that any new time a particle $(\mathbb{X}(t), \mathbb{Y}(t))$ attains its lowest (or highest) position it has moved a distance of

$$\mathbb{X}(\tau) - \mathbb{X}(0) = c\tau - 2\pi$$

in the horizontal direction. We infer that a physical particle trajectory is closed if and only if $\tau = 2\pi/c$. We also see from this reasoning that if $\tau > 2\pi/c$, then the particle displays a mean forward drift, and contrariwise.
The good thing is that τ can be evaluated:

$$\tau/2 = t(0) - t(\pi) = -\int_0^\pi \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x$$
$$= -\int_0^\pi \frac{\mathrm{d}x}{\dot{x}(x, \sigma(x))} = \int_0^\pi \frac{\mathrm{d}x}{c - u(x, \sigma(x))} = \int_0^\pi \frac{\mathrm{d}x}{|\psi_y(x, \sigma(x))|}.$$

According to Hölders inequality

$$\pi^2 = \left(\int_0^\pi \mathrm{d}x\right)^2$$
$$\leqslant \int_0^\pi |\psi_y(x, \sigma(x))|\,\mathrm{d}x \int_0^\pi \frac{\mathrm{d}x}{|\psi_y(x, \sigma(x))|} \leqslant c\pi \int_0^\pi \frac{\mathrm{d}x}{|\psi_y(x, \sigma(x))|},$$

so that $\tau \geqslant 2\pi/c$. Note that lemma 4.3 guarantees that there is strict inequality above the flat bed. So the only way we could have equality is if there is equality in Hölder on the flat bed, implying $\psi_y(x, -d) = -1$. But if ψ_y is constant at the bottom, then so is h_p, and hence

$$h_{qp}(q, p_0) = 0.$$

On the other hand, h_q satisfies the strong maximum principle (cf (3.2)). Since $h_q = \sigma'$, it is strictly negative within the half-period $(0, \pi)$ (cf lemma 3.1), and it vanishes at the flat bed. The Hopf boundary point lemma then forces

$$h_{qp}(q, p_0) < 0, \qquad \text{for} \;\; 0 < q < \pi.$$

Thus the Hölder inequality must be strict, and we conclude that

$$\tau > \frac{2\pi}{c}.$$

(ii) For two streamlines $(x, \sigma_1(x))$ and $(x, \sigma_2(x))$ with $\sigma_1(x) < \sigma_2(x)$ we are interested in the difference

$$\int_0^\pi \left(\frac{1}{|\psi_y(x, \sigma_2(x))|} - \frac{1}{|\psi_y(x, \sigma_1(x))|} \right) \mathrm{d}x,$$

the positivity of which we want to prove. Let p_1 and p_2 correspond to σ_1 and σ_2, respectively. Then we need to prove exactly that

$$\int_0^\pi (h_p(q, p_2) - h_p(q, p_1))\,\mathrm{d}q = \int_\Sigma h_{pp}\,\mathrm{d}q\,\mathrm{d}p > 0,$$

for $\Sigma \equiv (0, \pi) \times (p_1, p_2)$. As follows from (2.4),

$$h_{pp} = \frac{2h_q h_p h_{qp}}{1 + h_q^2} - \frac{h_{qq} h_p^2}{1 + h_q^2}.$$

Recall that $\sigma' = h_q$ so that, according to lemma 3.1 (i), we have $h_q < 0$ in Σ. Furthermore, as explained in its proof, lemma 3.2 (ii) asserts that $h_{qp} < 0$ in Σ. In view of that $h_p > 0$, we see that the first term is positive everywhere in Σ. As for the second, integration by parts in the q-variable yields that

$$-\int_\Sigma \frac{h_{qq} h_p^2}{1 + h_q^2}\,\mathrm{d}q\,\mathrm{d}p = 2\int_\Sigma \arctan(h_q) h_p h_{qp}\,\mathrm{d}q\,\mathrm{d}p,$$

since h_q vanishes in vertical sides $q = 0$ and $q = \pi$. Summing up, it follows from the oddness of arctan that

$$\int_\Sigma h_{pp}\,\mathrm{d}q\,\mathrm{d}p > 0.$$

(iii) Consider the quotient

$$\left(\int_0^\pi \frac{\mathrm{d}x}{|\psi_y(x, \sigma_2(x))|} - \int_0^\pi \frac{\mathrm{d}x}{|\psi_y(x, \sigma_1(x))|} \right) \Big/ (\sigma_2(x) - \sigma_1(x)),$$

the sign of which determines the change of τ and thus of the mean drift. The Lebesgue dominated convergence theorem can be applied to consider the limit $\sigma_2 \to \sigma_1$, being

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} \int_0^\pi \frac{\mathrm{d}x}{|\psi_y(x, \sigma(x))|} = \int_0^\pi \frac{\psi_{yy}(x, \sigma(x))\,\mathrm{d}x}{\psi_y^2(x, \sigma(x))}. \tag{4.2}$$

Since $\psi_{yy} = -\gamma > 0$ at the bifurcation point, it follows by continuity that the expression in (4.2) is positive in a neighbourhood of the trivial flow from which the non-trivial waves bifurcate.

5. The particle trajectories

We are now ready to discuss what our results mean for the streamlines and particle trajectories. We shall do so with the aid of two examples.

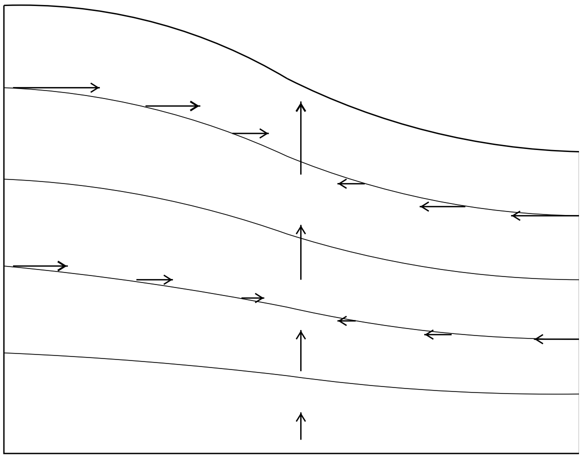

Figure 1. The streamlines and the velocity field for irrotational gravity waves of small amplitude. The steepness of the streamlines and the vertical velocity is pointwise increasing from bottom to surface. The horizontal velocity is everywhere decreasing along the streamlines, and it is increasing from bottom and up beneath the crest, whilst decreasing beneath the trough.

5.1. Irrotational waves

Irrotational waves display extraordinary regular features (see figure 1). From the bottom and upwards, the angles between the streamlines and the horizontal plane are pointwise increasing, and for small enough waves this is true also for the vertical velocity. The same for the maximal horizontal velocity, which for every streamline is attained below the crest, wherefrom it strictly decreases towards the trough. The surface is bounded below by a concave parabola, the curvature of which is determined by gravity and the maximal horizontal velocity.

In the language of particle paths, every particle traverses a non-closed oval orbit as the wave passes above. This forward drift is strictly increasing from the bottom to the surface. As the wave propagates above, the particle moves upwards starting from the time a trough passes until the next crest passes (see figure 2). At the top of its orbit the particle attains its maximal horizontal velocity. The movement then continues in a symmetric way, and the particle begins its descent with the horizontal speed strictly decreasing until it reaches its minimum value as the next trough passes.

5.2. Waves of negative vorticity

Some of the above features persist for waves of negative vorticity. The maximal steepness of the streamlines is strictly increasing from bed to surface, but we lack a proof of this property holding along any vertical line. At the surface the horizontal velocity is non-increasing from crest to trough, and so is it at the bottom; in between we do not know.

The surface is bounded below by the same parabola as the irrotational waves. The fluid particles show a mean forward drift. The forward drift is always strictly increasing from bed to surface in some neighbourhood of the bifurcation point from laminar flows (cf [9]). We have not been able to verify the oval shape of the trajectories throughout the fluid since there is no general control of the horizontal velocity in the interior of the fluid. At present the possibility of particles moving constantly forward cannot be ruled out.

Remark 5.1. We remark that for water of infinite depth there is an explicit solution due to Gerstner [17], and rediscovered by Rankine [27]. For that solution, with a particular non-zero

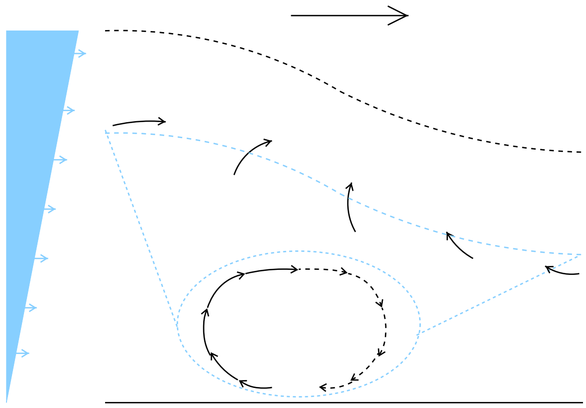

Figure 2. (Left) For irrotational waves and small waves of negative vorticity the forward drift is strictly increasing from bottom to surface. Centre: to an observer standing still as the wave passes the particles in irrotational waves traverse non-closed oval orbits, corresponding to what an observer travelling along with the wave understands as a streamline. Not only the forward drift but also the vertical size of these orbits is increasing from bottom to surface, and the horizontal velocity of each particle is strictly increasing from the bottom of the orbit to its top.

(This figure is in colour only in the electronic version)

vorticity, all paths are circular. We refer to the discussion in [3]. There is also an extension to three-dimensional edge waves in [2].

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