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On weakly nonlinear modulation of waves on deep water

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We propose a new approach for modeling weakly nonlinear waves, based on enhancing truncated amplitude equations with exact linear dispersion. Our example is based on the nonlinear Schrödinger (NLS) equation for deep-water waves. The enhanced NLS equation reproduces exactly the conditions for nonlinear four-wave resonance (the “figure 8” of Phillips) even for bandwidths greater than unity. Sideband instability for uniform Stokes waves is limited to finite bandwidths only, and agrees well with exact results of McLean; therefore, sideband instability cannot produce energy leakage to high-wave-number modes for the enhanced equation, as reported previously for the NLS equation. The new equation is extractable from the Zakharov integral equation, and can be regarded as an intermediate between the latter and the NLS equation. Being solvable numerically at no additional cost in comparison with the NLS equation, the new model is physically and numerically attractive for investigation of wave evolution. © 2000 American Institute of Physics. [S1070-6631(00)50010-5]

I. INTRODUCTION

Direct numerical integration of the sea surface remains a formidable computational task that is beyond our reach. Asymptotic techniques have been developed to approximately describe weakly nonlinear, and sometimes also weakly dispersive, surface waves.

For weakly nonlinear waves, the Zakharov integral equation^{1–4} accounts for resonant wave interactions with no constraint on the bandwidth of modulation. It provides an effective tool to compute the dynamics of the waves, however, its numerical simulation is still quite expensive.

For weakly nonlinear and narrow-banded waves, the nonlinear Schrödinger (NLS) equation is an attractive alternative due to its analytical simplicity and computational efficiency. The NLS equation has enjoyed much attention due to its relevance to many modulation processes and its exact integrability.⁵ However, numerical simulation of the NLS equation has revealed that a wave field with an initially narrow-banded spectrum can suffer energy leakage to high wave number modes, whereby the model breaks down due to violation of its own bandwidth constraint.^{6,7} After this discovery the NLS equation became somewhat unfashionable for the study of ocean surface waves.

The energy leakage is due to the truncation to narrow

bandwidth. Due to the form of the linear dispersion relation for ocean waves, the region of unstable wave vector perturbations for a uniform Stokes wave locally resembles a hyperbola in the wave vector plane for narrow bandwidths. The NLS equation simply extends the hyperbola to infinitely broad bandwidths, resulting in an unbounded region of instability.

The NLS equation has been enhanced slightly for broader bandwidth and larger steepness.^{8,9} This work has been successful both in the sense of describing more physical effects (e.g., asymmetric evolution of wave packets and sidebands), and improving the resolution in bandwidth. Although these equations still suffer from sideband instability for high-wave-number modes, numerical simulation^{10,11} shows evidence that energy leakage is not a problem in practice provided the spectrum is discretized only within the domain of validity of the equation.

Typically, the broadening of the bandwidth is done by multiple scales asymptotic expansions, adding an ever increasing number of linear dispersive terms in the style of a power series expansion. This approach eventually becomes unattractive due to the lengthy expressions and the poor convergence properties outside some spectral “radius of convergence.” As a more efficient approach one could introduce Padé approximants to the linear dispersion relation, however, we prefer not to approximate the linear dispersion relation at all.

Our idea is, starting out with any truncated amplitude equation, to enhance the equation with a pseudodifferential operator that captures the full linear dispersive behavior. For

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example, for deep-water waves, combining the exact linear operator with the simplest cubic nonlinearity, we propose the following model:

$$\frac{\partial B}{\partial t} + \frac{1}{4\pi^2} \int_{-\infty}^{\infty} i(\omega(\mathbf{k}_0 + \boldsymbol{\lambda}) - \omega_0) e^{i\boldsymbol{\lambda} \cdot (\mathbf{x} - \mathbf{y})} B(\mathbf{y}, t) d\mathbf{y} d\boldsymbol{\lambda} + \frac{i\omega_0 k_0^2}{2} |B|^2 B = 0. \quad (1)$$

Here $B(\mathbf{x}, t)$ is the complex envelope function for modulation of the surface displacement, relative to a characteristic wave vector \mathbf{k}_0 and frequency ω_0 .

Equation (1) is a special case of the Zakharov integral equation for the limiting case that only the nonlinear part is truncated subject to a bandwidth constraint. Therefore, Eq. (1) may be regarded as intermediate between the Zakharov integral equation (keeping its full linear part) and the NLS equation (keeping its nonlinearity and overall simplicity).

The analytical advantage of the new equation can be assessed already by considering the sideband instability of Stokes waves and the conditions for neutrally stable four-wave resonance. We show that the “figure 8” pattern of Phillips¹² for exact four-wave resonance is exactly reproduced even for interactions with bandwidth greater than unity. The region of unstable wave vector perturbations for a Stokes wave agrees well with the exact results of McLean.¹³ In particular, the branches of the instability region that extend to infinitely broad bandwidth are completely eliminated. This is thus no longer an avenue that can lead to energy leakage.

The NLS equation and the new equation are well-suited for numerical integration with operator splitting methods. A common numerical scheme is to integrate the linear dispersive part analytically in Fourier space and integrate the nonlinear part numerically in physical space with a split-step method.^{14,15} In both cases the linear part is integrated with the same speed regardless of whether it is truncated or exact. Therefore, numerical integration of Eq. (1) can be performed at no additional cost compared with the NLS equation.

A final advantage of the new approach is that we believe it can provide good and efficient models for description of realistic ocean waves. Ocean waves are on average weakly nonlinear with moderately narrow bandwidth; a steepness of 0.12 and a bandwidth of 0.4 are typical for relatively rough sea conditions. The broader bandwidth NLS equation of Trulsen and Dysthe⁹ was specifically designed for this purpose. The new approach provides a model that has improved bandwidth resolution, is exact in the linear limit, and has improved stability properties.

The heuristic approach of combining the exact linear operator with the simplest types of nonlinearity was used by various researchers for problems such as (i) description of peaking and breaking of shallow-water waves;¹⁶ (ii) Swift–Hohenberg equation in convection;¹⁷ (iii) the Frankel equation in the theory of flame propagation;¹⁸ and (iv) viscous shear flows.¹⁹ However, we are not aware that this method was used before for description of wave modulation processes.

II. LINEAR EVOLUTION EQUATION

For simplicity, we consider the case of infinitely deep water; generalization to deep water with a flat bottom requires minor changes in the subsequent analysis. We start with the equations for the surface displacement $\zeta(\mathbf{x}, t)$ and velocity potential $\phi(\mathbf{x}, z, t)$ of an inviscid and incompressible fluid:

$$\frac{\partial \zeta}{\partial t} + \nabla \phi \cdot \nabla \zeta = \frac{\partial \phi}{\partial z} \quad \text{at } z = \zeta, \quad (2)$$

$$\frac{\partial \phi}{\partial t} + g\zeta + \frac{1}{2}(\nabla \phi)^2 = 0 \quad \text{at } z = \zeta, \quad (3)$$

$$\nabla^2 \phi = 0 \quad \text{for } -\infty < z < \zeta, \quad (4)$$

$$\frac{\partial \phi}{\partial z} \rightarrow 0 \quad \text{at } z \rightarrow -\infty. \quad (5)$$

Here g is the acceleration of gravity, $\mathbf{x} = (x, y)$ is a horizontal position vector, z is the vertical position, t is time, and $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$.

The full linearized solution of (2)–(5) can be obtained by Fourier transform. The surface displacement can be expressed as

$$\zeta(\mathbf{x}, t) = \frac{1}{2} \int_{-\infty}^{\infty} b(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega(\mathbf{k})t)} d\mathbf{k} + \text{c.c.} \quad (6)$$

Here $\mathbf{k} = (k_x, k_y)$ is a horizontal wave vector and the frequency $\omega(\mathbf{k})$ is given by the linear dispersion relation

$$\omega(\mathbf{k}) = (g|\mathbf{k}|)^{1/2}. \quad (7)$$

We assume that the surface displacement $\zeta(\mathbf{x}, t)$ may be represented as a modulation of a wave train with the characteristic wave vector $\mathbf{k}_0 = (k_0, 0)$ and frequency $\omega_0 = \omega(\mathbf{k}_0)$:

$$\zeta(\mathbf{x}, t) = \frac{1}{2} B(\mathbf{x}, t) e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t)} + \text{c.c.} \quad (8)$$

Equating (6) and (8), we obtain

$$B(\mathbf{x}, t) e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t)} = \int_{-\infty}^{\infty} b(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega(\mathbf{k})t)} d\mathbf{k}, \quad (9)$$

$$\begin{aligned} B(\mathbf{x}, t) &= \int_{-\infty}^{\infty} b(\mathbf{k}_0 + \boldsymbol{\lambda}) e^{i[\boldsymbol{\lambda} \cdot \mathbf{x} - (\omega(\mathbf{k}_0 + \boldsymbol{\lambda}) - \omega_0)t]} d\boldsymbol{\lambda} \\ &= \int_{-\infty}^{\infty} \hat{B}(\boldsymbol{\lambda}, t) e^{i\boldsymbol{\lambda} \cdot \mathbf{x}} d\boldsymbol{\lambda}. \end{aligned} \quad (10)$$

We use $\boldsymbol{\lambda} = (\lambda, \mu)$ to denote the modulation wave vector relative to \mathbf{k}_0 .

The Fourier transform $\hat{B}(\boldsymbol{\lambda}, t)$ satisfies the equation

$$\frac{\partial \hat{B}}{\partial t} + i[\omega(\mathbf{k}_0 + \boldsymbol{\lambda}) - \omega_0] \hat{B} = 0. \quad (11)$$

The spectral evolution equation (11) can be formally written in physical space as

$$\frac{\partial B}{\partial t} + L(\partial_x, \partial_y) B = 0, \quad (12)$$

$$L(\partial_x, \partial_y) = i\{[(1 - i\partial_x)^2 - \partial_y^2]^{1/4} - 1\}, \quad (13)$$

or equivalently

$$\frac{\partial B}{\partial t} + \frac{1}{4\pi^2} \int_{-\infty}^{\infty} i[\omega(\mathbf{k}_0 + \boldsymbol{\lambda}) - \omega_0] e^{i\boldsymbol{\lambda} \cdot (\mathbf{x} - \mathbf{y})} B(\mathbf{y}, t) d\mathbf{y} d\boldsymbol{\lambda} = 0. \quad (14)$$

Linear evolution equations at all orders can now be obtained by expanding (12) in powers of the derivatives, or so-called gradient expansions. After normalization with the central wave number k_0 and frequency ω_0 for simplicity, expansion to fifth-order derivatives yields

$$\begin{aligned} \frac{\partial B}{\partial t} + \frac{1}{2} \frac{\partial B}{\partial x} + \frac{i}{8} \frac{\partial^2 B}{\partial x^2} - \frac{i}{4} \frac{\partial^2 B}{\partial y^2} - \frac{1}{16} \frac{\partial^3 B}{\partial x^3} + \frac{3}{8} \frac{\partial^3 B}{\partial x \partial y^2} \\ - \frac{5i}{128} \frac{\partial^4 B}{\partial x^4} + \frac{15i}{32} \frac{\partial^4 B}{\partial x^2 \partial y^2} - \frac{3i}{32} \frac{\partial^4 B}{\partial y^4} + \frac{7}{256} \frac{\partial^5 B}{\partial x^5} \\ - \frac{35}{64} \frac{\partial^5 B}{\partial x^3 \partial y^2} + \frac{21}{64} \frac{\partial^5 B}{\partial x \partial y^4} = 0. \end{aligned} \quad (15)$$

Hence we recover the linear part of the classical cubic NLS equation up to second order, the linear part of the modified NLS equation of Dysthe⁸ up to third order, and the linear part of the broader bandwidth modified NLS equation of Trulsen and Dysthe⁹ up to fifth order.

At each successive order, the truncated polynomial approximation (15) becomes better within some radius of convergence in the modulation wave vector plane, but becomes worse outside due to the poor convergence properties of power series expansions. However, there is in principle no need to approximate (12) by a long-wavelength expansion. In the following we shall use the exact linear dispersive equation (12) instead of its conventional power-series approximation (15).

III. WEAKLY NONLINEAR EVOLUTION EQUATION: CONNECTION TO THE NLS AND ZAKHAROV INTEGRAL EQUATIONS

For the NLS equation one may assume that the velocity potential and surface displacement can be expanded in harmonic expansions

$$\phi = \bar{\phi} + \frac{1}{2}(A e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t) + k_0 z} + A_2 e^{2i(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t) + 2k_0 z} + \dots + \text{c.c.}), \quad (16)$$

$$\zeta = \bar{\zeta} + \frac{1}{2}(B e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t)} + B_2 e^{2i(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t)} + \dots + \text{c.c.}). \quad (17)$$

After substituting (16) and (17) into the original problem (2)–(5), one may obtain the NLS equation or modifications of the NLS equation with increasing order of accuracy, e.g., these of Zakharov,¹ Dysthe,⁸ and Trulsen and Dysthe.⁹ Their linear parts can be recovered by truncation of (12) at appropriate orders. These linear terms are balanced by appropriate nonlinear terms.

Here we introduce the entire linear part without truncation, and combine it with the leading order nonlinear terms. Using only the simplest cubic nonlinear term borrowed from the NLS equation, we obtain

$$\frac{\partial B}{\partial t} + L(\partial_x, \partial_y)B + \frac{i\omega_0 k_0^2}{2} |B|^2 B = 0, \quad (18)$$

where the operator L is given by (13). Using the additional nonlinear terms of the modified NLS equation of Dysthe⁸ we get

$$\begin{aligned} \frac{\partial B}{\partial t} + L(\partial_x, \partial_y)B + \frac{i\omega_0 k_0^2}{2} |B|^2 B + \frac{3\omega_0 k_0}{2} |B|^2 \frac{\partial B}{\partial x} \\ + \frac{\omega_0 k_0}{4} B^2 \frac{\partial B^*}{\partial x} + ik_0 \frac{\partial \bar{\phi}}{\partial x} B = 0, \end{aligned} \quad (19)$$

$$\frac{\partial \bar{\phi}}{\partial z} = \frac{\omega_0}{2} \frac{\partial}{\partial x} |B|^2 \quad \text{at } z=0, \quad (20)$$

$$\nabla^2 \bar{\phi} = 0 \quad \text{for } -\infty < z < 0, \quad (21)$$

$$\frac{\partial \bar{\phi}}{\partial z} = 0 \quad \text{at } z \rightarrow -\infty. \quad (22)$$

Equations (18) and (19)–(22) constitute our main result.

Equation (18) could be also obtained from the Zakharov integral equation in the following way: The integral Zakharov equation is

$$\begin{aligned} i \frac{\partial F(\mathbf{k}, t)}{\partial t} = \int \int \int_{-\infty}^{\infty} T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) F^*(\mathbf{k}_1, t) \\ \times F(\mathbf{k}_2, t) F(\mathbf{k}_3, t) \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \\ \times e^{i(\omega + \omega_1 - \omega_2 - \omega_3)t} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3. \end{aligned} \quad (23)$$

Here the kernel T is obtained by symmetrization over the last two arguments as described by Krasitskii.⁴

The first-order free-surface elevation is related to F through

$$\zeta(\mathbf{x}, t) = \int_{-\infty}^{\infty} \left(\frac{\omega(\mathbf{k})}{2g} \right)^{1/2} [F(\mathbf{k}, t) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} + \text{c.c.}] d\mathbf{k}. \quad (24)$$

Performing the analysis of the spectra around $\mathbf{k}_0 = (k_0, 0)$, we represent all wave numbers as $\mathbf{k} = \mathbf{k}_0 + \boldsymbol{\lambda}$, $\boldsymbol{\lambda} = (\boldsymbol{\lambda}, \mu)$. With the new variable $f(\boldsymbol{\lambda}, t) = F(\mathbf{k}, t) \times e^{-i[\omega(\mathbf{k}) - \omega_0]t}$, Eq. (23) gives

$$\begin{aligned} i \frac{\partial f(\boldsymbol{\lambda}, t)}{\partial t} - [\omega(\mathbf{k}) - \omega_0] f(\boldsymbol{\lambda}, t) \\ = \int \int \int_{-\infty}^{\infty} T(\mathbf{k}_0 + \boldsymbol{\lambda}, \mathbf{k}_0 + \boldsymbol{\lambda}_1, \mathbf{k}_0 + \boldsymbol{\lambda}_2, \mathbf{k}_0 + \boldsymbol{\lambda}_3) \\ \times f^*(\boldsymbol{\lambda}_1, t) f(\boldsymbol{\lambda}_2, t) f(\boldsymbol{\lambda}_3, t) \delta(\boldsymbol{\lambda} + \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_3) \\ \times d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2 d\boldsymbol{\lambda}_3. \end{aligned} \quad (25)$$

Equation (24) is expanded to the leading order in the spectral width:

$$\begin{aligned} \zeta(\mathbf{x}, t) = \left(\frac{\omega_0}{2g} \right)^{1/2} \left[e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t)} \int_{-\infty}^{\infty} f(\boldsymbol{\lambda}, t) e^{i\boldsymbol{\lambda} \cdot \mathbf{x}} d\boldsymbol{\lambda} + \text{c.c.} \right] \\ = \text{Re}[B(\mathbf{x}, t) e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t)}]. \end{aligned} \quad (26)$$

Here the wave envelope $B(\mathbf{x}, t)$ is the inverse Fourier transform of $f(\boldsymbol{\lambda}, t)$, multiplied by the factor $(2\omega_0/g)^{1/2}$. Multiplying (25) by $(2\omega_0/g)^{1/2}$ and taking its inverse Fourier transform results in

$$\begin{aligned} \frac{\partial B}{\partial t} + \frac{1}{4\pi^2} \int \int_{-\infty}^{\infty} i[\omega(\mathbf{k}_0 + \boldsymbol{\lambda}) - \omega_0] e^{i\boldsymbol{\lambda} \cdot (\mathbf{x} - \mathbf{y})} B(\mathbf{y}, t) d\mathbf{y} d\boldsymbol{\lambda} \\ = -i \left(\frac{2\omega_0}{g} \right)^{1/2} \int \int \int_{-\infty}^{\infty} T(\mathbf{k}_0 + \boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_3 - \boldsymbol{\lambda}_1, \\ \mathbf{k}_0 + \boldsymbol{\lambda}_1, \mathbf{k}_0 + \boldsymbol{\lambda}_2, \mathbf{k}_0 + \boldsymbol{\lambda}_3) f^*(\boldsymbol{\lambda}, t) f(\boldsymbol{\lambda}_2, t) \\ \times f(\boldsymbol{\lambda}_3, t) e^{i(\boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_3 - \boldsymbol{\lambda}_1) \cdot \mathbf{x}} d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2 d\boldsymbol{\lambda}_3. \end{aligned} \quad (27)$$

The next step is to expand the nonlinear terms on the right-hand side of (27), while keeping the linear terms on the left-hand side unchanged. The leading term of Taylor expansion of T is

$$T(\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0) = k_0^3. \quad (28)$$

Substitution of (28) into the right-hand side of (27) results in

$$\begin{aligned} \frac{\partial B}{\partial t} + \frac{1}{4\pi^2} \int \int_{-\infty}^{\infty} i[\omega(\mathbf{k}_0 + \boldsymbol{\lambda}) - \omega_0] e^{i\boldsymbol{\lambda} \cdot (\mathbf{x} - \mathbf{y})} B(\mathbf{y}, t) d\mathbf{y} d\boldsymbol{\lambda} \\ + \frac{i\omega_0 k_0^2}{2} |B|^2 B = 0. \end{aligned} \quad (29)$$

Stiassnie²⁰ further showed how the nonlinear terms of the fourth-order Dysthe⁸ can also be obtained from the third-order Zakharov integral equation.

IV. SIDEBAND INSTABILITY OF STOKES WAVES

To have an idea of the improved resolution in bandwidth afforded by the new equation, we consider the sideband instability for Stokes waves as a test case. Let us consider the problem normalized by the central wave number k_0 and frequency ω_0 ,

$$\begin{aligned} \frac{\partial B}{\partial t} + L \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) B + i\alpha_1 |B|^2 B + \alpha_2 |B|^2 \frac{\partial B}{\partial x} \\ + \alpha_3 B^2 \frac{\partial B^*}{\partial x} + i\alpha_4 \frac{\partial \bar{\phi}}{\partial x} \Big|_{z=0} B = 0, \end{aligned} \quad (30)$$

$$\frac{\partial \bar{\phi}}{\partial z} = \beta \frac{\partial}{\partial x} |B|^2 \quad \text{at } z=0, \quad (31)$$

$$\nabla^2 \bar{\phi} = 0 \quad \text{for } -\infty < z < 0, \quad (32)$$

$$\frac{\partial \bar{\phi}}{\partial z} = 0 \quad \text{at } z \rightarrow -\infty, \quad (33)$$

with

$$L(\partial_x, \partial_y) = i\{[(1 - i\partial_x)^2 - \partial_y^2]^{1/4} - 1\}. \quad (34)$$

A uniform wave solution is given by

$$B = B_0 e^{-i\alpha_1 B_0^2 t}. \quad (35)$$

The stability to sideband perturbations is investigated by assuming small perturbations in amplitude and phase of the form

$$B = B_0(1 + a + i\theta) e^{-i\alpha_1 B_0^2 t}, \quad (36)$$

having the plane wave solution

$$\begin{pmatrix} a \\ \theta \\ \bar{\phi} \end{pmatrix} = \begin{pmatrix} \hat{a} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} e^{i(\lambda x + \mu y - \Omega t)} + \text{c.c.} \quad (37)$$

The dispersion relation for the perturbation is given by

$$\begin{aligned} \Omega = P + \alpha_2 B_0^2 \lambda \\ \pm \sqrt{Q \left(Q - 2\alpha_1 B_0^2 + 2\alpha_4 \beta B_0^2 \frac{\lambda^2}{\sqrt{\lambda^2 + \mu^2}} \right) + \alpha_3^2 B_0^4 \lambda^2}. \end{aligned} \quad (38)$$

Here

$$P = -\frac{i}{2} (L(i\lambda, i\mu) + L^*(i\lambda, i\mu)), \quad (39)$$

$$Q = \frac{i}{2} (L(i\lambda, i\mu) - L^*(i\lambda, i\mu)). \quad (40)$$

The function $L^*(\cdot, \cdot)$ is the complex conjugate of the function $L(\cdot, \cdot)$. In the case of Eq. (34) we have

$$Q = 1 - \frac{1}{2}[(1 + \lambda)^2 + \mu^2]^{1/4} - \frac{1}{2}[(1 - \lambda)^2 + \mu^2]^{1/4}. \quad (41)$$

Exact four-wave resonance corresponds to neutral stability in the limit that $B_0 \rightarrow 0$, which implies that $Q = 0$. This is the “figure 8” resonant curve of Phillips.¹² We remark that the condition for four-wave resonance has then been recovered exactly for all bandwidths, even when only the leading nonlinear term is retained in (30).

In Fig. 1 we plot the instability region for a Stokes wave with amplitude $B_0 = 0.1$. A comparison is made between the NLS equation, the modified NLS equations of Dysthe⁸ and Trulsen and Dysthe,⁹ the present result with leading cubic nonlinearity, the present result with the additional nonlinear terms of Dysthe,⁸ and the exact result of McLean.¹³ It should be noted that McLean¹³ employed steepness 0.1 based on half the crest-to-trough height of the wave, while the results for the nonlinear Schrödinger equations are for steepness 0.1 based on the first-harmonic amplitude in the expansion (17). The difference between these two steepnesses is small in the present case and can be neglected (see Ref. 9).

We notice that the exact linear part together with the leading cubic nonlinearity is sufficient to contain the region of instability within the “figure 8” of Phillips.¹² However, to contain the region of instability within bandwidth less than unity, it is also necessary to include the additional nonlinear terms of Dysthe.⁸ For this reason we propose the equation of

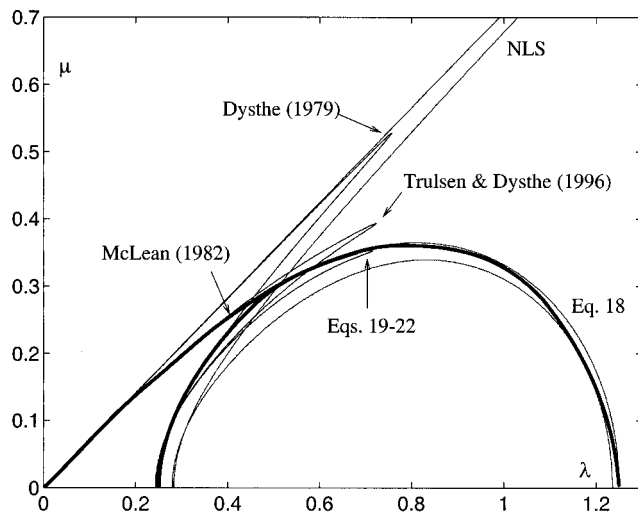


FIG. 1. Instability regions for $B_0 = 0.1$. Comparison between NLS, Dysthe (Ref. 8), Trulsen and Dysthe (Ref. 9), the present theory with leading cubic nonlinearity [Eq. (18)], the present theory with the nonlinear terms of Dysthe [Eqs. (19)–(22)]. The exact result of McLean (Ref. 13) is indicated by a thick curve.

Dysthe with exact linear dispersive part as particularly attractive for numerical simulation.

Figure 2 illustrates artificial instabilities suffered by the various equations in a wave number domain that extends to bandwidths greater than one. The new equation is seen to

eliminate the unphysical instabilities that appear for large modulation wave numbers in the cubic and modified NLS equations with truncated linear parts.

The most significant improvement offered by the new equation concerns bandwidths greater than those that are most important for type I sideband instability for Stokes waves. This is a clear advantage for simulation of realistic ocean waves, however, the full practical consequences of this cannot be revealed by a stability analysis of Stokes waves.

V. CONCLUSION

We have proposed a new model for the evolution of weakly nonlinear waves on deep water. The important new feature is that the linear dispersive behavior is captured exactly. The resolution in bandwidth is hence improved and the domain of validity is extended relative to the existing cubic and modified NLS equations. For broad-banded perturbations of Stokes waves, the unphysical instabilities of the conventional cubic and modified NLS equations are eliminated. The new equation can be solved numerically at no additional cost in comparison with the conventional cubic and modified NLS equations.

Extension of the above arguments and Eqs. (18) and (19)–(22) for finite depth or spatial evolution can also be done. We are presently using these extensions for numerical investigation of ocean wave evolution, and in particular, our simulations do not suffer from energy leakage.

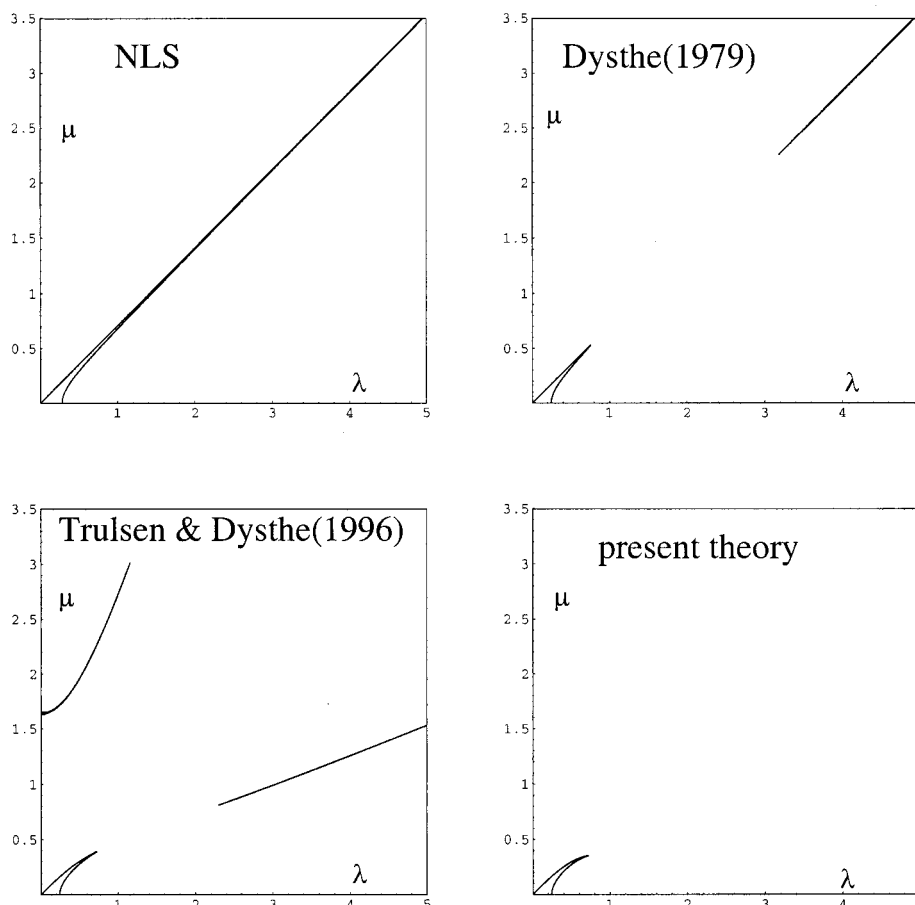


FIG. 2. Instability regions for $B_0 = 0.1$ in a large domain that extends to bandwidths greater than one. Comparison between NLS, Dysthe (Ref. 8), Trulsen and Dysthe (Ref. 9), and the present theory with the nonlinear terms of Dysthe [Eqs. (19)–(22)].

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