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Parametrically excited, progressive cross-waves

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The variational formulation of the nonlinear wavemaker problem, previously applied (Miles 1988) to cross-waves in a short tank, is extended to allow for slow spatial, as well as slow temporal, variation of cross-waves in a long tank. The resulting evolution equations for the envelope of the cross-waves are equivalent to those derived by Jones (1984) and may be combined to obtain a cubic Schrödinger equation in a semi-infinite domain. The corresponding criterion for the stability of plane waves (i.e. for the temporal decay of cross-waves) agrees with Jones but differs from Mahony (1972). Weak damping is incorporated, and those stationary envelopes that are evanescent at large distances from the wavemaker are determined through analytical approximations and numerical integration and compared with the experimental observations of Barnard & Pritchard (1972) and the numerical calculations of Lichter & Chen (1987). These comparisons suggest that stationary envelopes with either no or one maximum are stable for sufficiently small amplitudes (solutions with multiple maxima may be stable but more difficult to attain) and evolve into limit cycles for somewhat larger amplitudes, but the analytical question of stability remains open.

1. Introduction

Following Mahony (1972) and Jones (1984), we consider the excitation of gravity waves of free-surface displacement ζ in a semi-infinite rectangular channel of width b and depth d in response to the symmetric wavemaker motion

$$x = \chi(z,t) = af(z)\sin 2\omega t \quad (0 < y < b, \quad -d < z < \zeta) \tag{1.1}$$

on the assumptions that

$$\epsilon \equiv ka \ll 1, \quad kd \gg 1 \quad \left(k \equiv \frac{\pi}{b}\right) \tag{1.2}$$

and that the frequency of excitation approximates twice one of the resonant frequencies of the transverse modes. Our formulation follows that for a short tank (Miles 1987, hereinafter referenced by I followed by the appropriate equation number therein), but we now assume that the basic wave is progressive, rather than standing, and allow for slow spatial, in addition to slow temporal, modulation of the cross-wave.

It is evident from symmetry that the boundary-value problem admits a plane-wave (y-independent) solution; however, nonlinearity may break that symmetry and couple energy into cross-waves if ω approximates one of the natural frequencies

$$\omega_n = (ngk)^{\frac{1}{2}} \quad (n = 1, 2, \ldots, kd \gg 1). \tag{1.3}$$

We assume that ω approximates ω_1 according to

$$\frac{\omega^2-\omega_1^2}{\omega^2} = 1-\frac{k}{\kappa} = O(\epsilon^2) \quad \left(\kappa \equiv \frac{\omega^2}{g}\right), \tag{1.4}$$

which determines the bandwidth of the hypothetical resonance (cf. Mahony 1972). If $\omega \approx \omega_n$ ($n = 2, 3, \ldots$) it is necessary only to replace k by nk in (1.2) and subsequently. The dominant effect of a small surface tension T is to raise the natural frequency, with the result that ω_1^2 and k in (1.4) are multiplied by $1+\hat{T}$, where

$$\hat{T} \equiv \frac{k^2 T}{\rho g} \ll 1. \tag{1.5}$$

This correction is significant in (1.4) if $\hat{T} = O(\epsilon^2)$.

We pose the free-surface displacement of the hypothetical cross-wave, which is superimposed on the plane wave, in the form

$$\zeta = a\mathscr{R}\{(p+\mathrm{i}q)\,\mathrm{e}^{-\mathrm{i}\omega t}\}\sqrt{2}\cos ky + O(\epsilon a), \tag{1.6}$$

where $p+\mathrm{i}q$ is a dimensionless, slowly varying complex amplitude with the time and length scales $1/\epsilon^2\omega$ and $1/\epsilon k$ (cf. Mahony 1972 and Jones 1984). We derive the evolution equations for p and q from an extension (I, §2) of Luke's (1967) variational principle. Our trial functions are based on Havelock's (1929) solution of the wavemaker problem and Rayleigh's (1915) solution of the nonlinear standing-wave problem as in I, §3, but it now is necessary to include the self-interaction of the plane wave and the interaction between the plane wave and the cross-wave. The resulting partial differential equations for p and q (§5) are equivalent to those obtained by Jones (1984) and may be combined to obtain a cubic Schrödinger equation for $p+\mathrm{i}q$. Our stability criterion for plane-wave motion without damping agrees with Jones but differs from that of Mahony (1972); we also incorporate weak damping. In §6, we examine the stationary (p and q independent of τ) solutions of the Schrödinger equation, including weak damping, and obtain analytical approximations and numerical results for the amplitude and phase of the cross-waves for various values of the tuning and damping parameters. These results are qualitatively similar to the experimental results of Barnard & Pritchard (1972) and the numerical results of Lichter & Chen (1987); however, Barnard & Pritchard do not obtain stationary waves (although they comment that their cross-waves 'came close to maintaining a steady amplitude when the wavemaker motion was only slightly larger than that at the margin of [plane-wave] stability'), and Lichter & Chen obtain them only in a few cases. J. L. Hammack (personal communication) reports that stationary cross-waves are observed for sufficiently small amplitudes, but that they become unstable for larger amplitudes. Lichter & Chen's results suggest that this instability corresponds to a Hopf bifurcation to a limit cycle. This, in turn suggests, although we know of no direct evidence, that there could be further bifurcations to chaotic motion in some parametric regime.

2. Variational formulation

The assumption of motion started from rest in an incompressible, inviscid fluid in the wave tank described in §1 leads to the boundary-value problem

$$\nabla^2\phi = 0 \quad (\chi < x < \infty, \quad 0 < y < b, \quad -d < z < \zeta), \tag{2.1}$$

$$\phi_z = \zeta_t + \nabla\phi\cdot\nabla\zeta, \quad \phi_t + \tfrac{1}{2}(\nabla\phi)^2 + g\zeta = 0 \quad (z = \zeta), \tag{2.2a,b}$$

$$\phi_y = 0 \quad (y = 0, b), \quad \phi_z = 0 \quad (z = -d), \tag{2.3 a, b}$$

$$\phi_x = \chi_t + \nabla\phi \cdot \nabla\chi \quad (x = \chi), \tag{2.4}$$

$$\phi_x \to 0 \quad (x \uparrow \infty) \tag{2.5}$$

for the velocity potential $\phi(x, y, z, t)$ and the free-surface displacement $\zeta(x, y, t)$, where the subscripts x, y, z, t signify partial differentiation. The boundary condition (2.3*b*) is imposed at $z = -\infty$ (deep-water approximation) in §§3–6. The null condition (2.5) may be replaced by a radiation condition.

The boundary-value problem (2.1)–(2.5) may be deduced through Hamilton's principle from the Lagrangian (Luke 1967; I, §2),

$$\hat{L} = -\iiint [\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + gz]\, \mathrm{d}V, \tag{2.6}$$

where the volume integral is over the semi-infinite domain bounded by the wavemaker ($x = \chi$), the free surface ($z = \zeta$) and the fixed boundaries ($y = 0, b$ and $z = -d$). An equivalent form, which incorporates (2.3), is (I, §2)

$$L = \frac{1}{2}\int_0^b dy \left\{ \iint \phi\nabla^2\phi \, \mathrm{d}x\, \mathrm{d}z + \int_{x_0}^{\infty} [\phi(2\zeta_t - \phi_z + \nabla\phi\cdot\nabla\zeta) - g\zeta^2]_{z=\zeta}\, \mathrm{d}x \right. \\ \left. + \int_{-d}^{z_0} [\phi(\phi_x - \nabla\chi\cdot\nabla\phi - 2\chi_t) - gz^2\chi_z]_{x=\chi}\, \mathrm{d}z \right\}, \tag{2.7}$$

where $x_0(y, t)$ and $z_0(y, t)$ are the coordinates of the intersection of the wavemaker ($x = \chi$) and the free surface ($z = \zeta$).

3. Trial functions

Proceeding as in I, §3, but with different scaling† and allowing for slow spatial variation and interaction between the plane wave and the basic cross-wave, we posit

$$\frac{k\omega}{g}\phi = \epsilon\phi_0 + \epsilon\phi_1 + \epsilon^2(\phi_{00} + \phi_{01} + \phi_{11}) + O(\epsilon^3) \tag{3.1 a}$$

and

$$k\zeta = \epsilon\zeta_0 + \epsilon\zeta_1 + \epsilon^2(\zeta_{00} + \zeta_{01} + \zeta_{11}) + O(\epsilon^3), \tag{3.1 b}$$

where the dimensionless variables (ϕ_0, ζ_0) represent the first-order (linear) plane-wave solution of (2.1)–(2.4) with (2.5) replaced by a radiation condition, (ϕ_1, ζ_1) represent the first-order cross-wave solution, (ϕ_{00}, ζ_{00}) represent the second-order interaction of (ϕ_0, ζ_0) with itself, (ϕ_{11}, ζ_{11}) represent the second-order interaction of (ϕ_1, ζ_1) with itself, and (ϕ_{01}, ζ_{01}) represent the second-order interaction between (ϕ_0, ζ_0) and (ϕ_1, ζ_1).

It suffices for the present calculation to know that the first-order plane-wave solution (cf. Havelock 1929) is independent of y, satisfies

$$\phi_{0xx} + \phi_{0zz} = 0, \tag{3.2}$$

$$\phi_{0x} = 2\kappa f(z) \cos 2\omega t \quad (x = 0), \quad \int_0^{\infty} \phi_{0z}|_{z=0}\, \mathrm{d}x = 2\kappa \int_{-\infty}^{0} f(z)\, \mathrm{d}z \cos 2\omega t, \tag{3.3 a, b}$$

† The scaling, which differs from that for the short tank (I, §3), follows from the usual procedures of multiple-scale asymptotics; cf. Mahony (1972) and Jones (1984).

and has the asymptotic representation

$$\phi_0 \sim FS_{42}\,\mathrm{e}^{4\kappa z}, \quad \zeta_0 \sim 2FC_{42} \quad (\kappa x \gg 1), \tag{3.4a, b}$$

where

$$F = 4\kappa \int_{-\infty}^{0} f(z)\,\mathrm{e}^{4\kappa z}\,\mathrm{d}z = f(0) - \int_{-\infty}^{0} f'(z)\,\mathrm{e}^{4\kappa z}\,\mathrm{d}z, \tag{3.5}$$

and, here and subsequently,

$$C_{mn} \equiv \cos(m\kappa x - n\omega t), \quad S_{mn} \equiv \sin(m\kappa x - n\omega t). \tag{3.6a, b}$$

Substituting the plane-wave components of (3.1) into the expansion of (2.2) about $z = 0$ and invoking (3.6), we obtain the second-order terms

$$\phi_{00} \sim 0, \quad \zeta_{00} \sim 8F^2 C_{84} \quad (\kappa x \gg 1). \tag{3.7a, b}$$

It also is implicit that the second-order approximation $(\phi_0 + \epsilon\phi_{00}, \zeta_0 + \epsilon\zeta_{00})$ satisfies (2.4) to second order.

The second-order cross-wave solution of (2.1)–(2.3) for $k = \kappa$ may be inferred from Rayleigh's (1915) second-order solution for two-dimensional (y, z in the present context) standing waves. Matching Rayleigh's result to (1.6), we obtain (cf. I, (3.4, 5))

$$\phi_1 = \surd 2 A_{1\theta}(\theta; X, \tau)\cos ky\,\mathrm{e}^{kz}, \quad \zeta_1 = \surd 2 A_1(\theta; X, \tau)\cos ky, \tag{3.8a, b}$$

$$\phi_{11} = -A_1 A_{1\theta}, \quad \zeta_{11} = A_1^2 \cos 2ky, \tag{3.9a, b}$$

where

$$A_1(\theta; \tau) = p(X, \tau)\cos\theta + q(X, \tau)\sin\theta = \mathscr{R}\{(p + \mathrm{i}q)\,\mathrm{e}^{-\mathrm{i}\theta}\} \tag{3.10}$$

and

$$\theta = \omega t, \quad \tau = \epsilon^2 \omega t, \quad X = 2\epsilon\kappa x. \tag{3.11a, b, c}$$

Substituting (3.1) into (2.1)–(2.3), expanding (2.2) about $z = 0$, and invoking (3.4), (3.7), (3.8), (3.9) and (3.11), we obtain (cf. Jones 1984)

$$\phi_{01} \sim \surd 2F[\Phi_1(qC_{41} - pS_{41}) + \Phi_3(qC_{43} + pS_{43})]\cos ky\,\mathrm{e}^{\surd 17\kappa z}, \tag{3.12a}$$

$$\zeta_{01} \sim \surd 2F[Z_1(pC_{41} + qS_{41}) + Z_3(qS_{43} - pC_{43})]\cos ky, \tag{3.12b}$$

$$\Phi_1 = \tfrac{1}{4}(1 + \surd 17), \quad \Phi_3 = -\tfrac{3}{16}(9 + \surd 17), \quad Z_1 = \tfrac{1}{4}(11 - \surd 17), \quad Z_3 = \tfrac{1}{16}(9\surd 17 - 31). \tag{3.13}$$

It also is implicit that ϕ_{01} satisfies

$$\phi_{01x} = 0 \quad (x = 0) \tag{3.14}$$

and therefore makes a null contribution to the boundary condition (2.4) in the present approximation.

4. Average Lagrangian

The calculation of the average Lagrangian, following the substitution of the trial function (3.1) into (2.7), proceeds as in I, §4, but, in addition to the incorporation of (ϕ_{00}, ζ_{00}) and (ϕ_{01}, ζ_{01}), there are the following differences: (i) (2.1) is no longer satisfied exactly, and

$$\nabla^2\phi = \omega a[4\epsilon^2(\kappa\phi_{1XX} + \phi_{01xX}) + O(\epsilon^3)] \tag{4.1}$$

contributes to L; (ii) the various components of the integrand in L comprise terms that are either oscillatory with a lengthscale $1/k$ or non-oscillatory in x, but the contribution of the former to the integral is negligible compared with that of the latter by virtue of Riemann's lemma; (iii) L comprises both an integral over X and boundary ($X = 0$) terms, which are derived from the integral over the wavemaker and from the end point of the free-surface integral; (iv) ϕ_x must be replaced by $\phi_x + 2\epsilon\kappa\phi_X$ in the wavemaker integrand. The end result, after adopting X in place of x as a variable of integration and partial integration of the term $pp_{XX} + qq_{XX}$, which is derived from $\langle\phi\nabla^2\phi\rangle$, is (the calculation is straightforward in principle, but care must be taken to retain all terms of relevant order)

$$\mathscr{L} \equiv \frac{\langle L - L_0 \rangle}{ga^3b} = \frac{1}{4}\int_0^\infty [p_\tau q - pq_\tau + \beta(p^2+q^2) + \tfrac{1}{8}(p^2+q^2)^2 - (p_X^2 + q_X^2)]\,\mathrm{d}X + \tfrac{1}{2}B(pq)_{X=0} + O(\epsilon^2), \quad (4.2)$$

where L_0 is the plane-wave ($p = q = 0$) Lagrangian,

$$\beta = \frac{1}{2\epsilon^2}\left(1 - \frac{k}{\kappa}\right) - CF^2 = \frac{\omega^2 - \omega_1^2}{2\epsilon^2\omega^2} - CF^2, \quad (4.3)$$

F is given by (3.5), $C = \frac{1}{8}(4\sqrt{17} - 19) = 0.202$ is a measure of the plane-wave–cross-wave interaction terms, and

$$B = \int_{-\infty}^{0} [2\kappa f(z) + \tfrac{1}{2}f'(z)\,\mathrm{e}^{2\kappa z}]\,\mathrm{d}z - f(0). \quad (4.4)$$

5. Evolution equations

Invoking the variational principle $\delta\int \mathscr{L}\,\mathrm{d}\tau = 0$ for (4.2), we obtain

$$p_\tau + q_{XX} + \beta q + \tfrac{1}{4}(p^2+q^2)\,q = 0, \quad (5.1a)$$

$$-q_\tau + p_{XX} + \beta p + \tfrac{1}{4}(p^2+q^2)\,p = 0, \quad (5.1b)$$

and the boundary conditions

$$p_X = -Bq, \quad q_X = -Bp \quad (X = 0), \quad (5.2a, b)$$

together with a null condition at $X = \infty$ (see below). These are the counterparts of Jones's (1984) equations (38a, b) and (34c), respectively, after letting (Jones $\rightarrow$ present) $X \rightarrow \frac{1}{2}X$, $(C, D) \rightarrow \sqrt{2}(q, -p)$, $J \rightarrow -2\beta$ and $L \rightarrow B$ therein (the sign of λ should be changed in Jones's (36a), (37) and (39)). Weak damping may be incorporated by replacing ∂_τ by $\partial_\tau + \alpha$, where

$$\alpha \equiv \frac{\delta}{\epsilon^2}. \quad (5.3)$$

and δ is the ratio of actual to critical damping for the pure cross-wave, which (by hypothesis) would decay like $\exp(-\delta\omega_0 t)$ in the absence of external excitation.

Forming the complex equations (5.1b) + i(5.1a) and (5.2a) + i(5.2b), incorporating damping, and rescaling according to

$$p + \mathrm{i}q = 2\gamma^{\frac{1}{2}}\mathscr{A}(\xi, \eta), \quad \xi = \gamma^{\frac{1}{2}}X, \quad \eta = \gamma\tau, \quad (5.4a, b, c)$$

$$\beta + \mathrm{i}\alpha = \gamma\,\mathrm{e}^{\mathrm{i}\phi} \quad (0 < \phi < \pi), \quad B = \gamma^{\frac{1}{2}}\sigma, \quad (5.5a, b)$$

where $\gamma \equiv (\alpha^2 + \beta^2)^{\frac{1}{2}}$, we obtain the cubic Schrödinger equation

$$\mathscr{A}_{\xi\xi} + \mathrm{i}\mathscr{A}_\eta + (\mathrm{e}^{\mathrm{i}\phi} + |\mathscr{A}|^2)\,\mathscr{A} = 0 \quad (5.6)$$

and the boundary condition

$$\mathscr{A}_\xi = -\mathrm{i}\sigma\mathscr{A}^* \quad (\xi = 0), \quad (5.7)$$

where $\mathscr{A}^*$ is the complex conjugate of $\mathscr{A}$. Finally, we replace the null condition at $X = \infty$ by

$$\frac{\mathscr{A}_\xi}{\mathscr{A}} \sim \mathrm{i}\, \mathrm{e}^{\frac{1}{2}\mathrm{i}\phi} \quad (\xi \uparrow \infty). \tag{5.8}$$

(Note that (5.6)–(5.8) are invariant under the reflection $\mathscr{A} \to -\mathscr{A}$.) The parameter

$$\sigma \equiv B\gamma^{-\frac{1}{2}} = B\epsilon\left[\left(\frac{\omega^2-\omega_1^2}{2\omega^2} - \epsilon^2 CF^2\right)^2 + \delta^2\right]^{-\frac{1}{4}}, \tag{5.9}$$

in which the term $\epsilon^2 CF^2$ typically is negligible, is a measure of the nonlinear (parametric) forcing of the wavemaker divided by (the magnitude of) the impedance of the cross-wave.

It is obvious that (5.6)–(5.8) are satisfied by $\mathscr{A} = 0$, which is the counterpart of the fixed point at $p = q = 0$ in I, §5. To determine the stability of this solution, we pose the small perturbation

$$\mathscr{A} = \mathscr{A}_0 \exp\left(\lambda\eta - \mathrm{i}\sigma\, \mathrm{e}^{-2\mathrm{i}\psi_0}\xi\right) \quad (0 < \psi_0 < \tfrac{1}{2}\pi) \tag{5.10}$$

in (5.6) and (5.7), where, here and subsequently, the subscript zero implies $\xi = 0$ and $\psi_0 \equiv \arg \mathscr{A}_0$, and invoke (5.8) to obtain

$$\psi_0 = \tfrac{1}{4}\cos^{-1}\left(\frac{\cos\phi}{\sigma^2}\right), \quad \lambda = (\sigma^4 - \cos^2\phi)^{\frac{1}{2}} - \sin\phi \quad (\tfrac{1}{4}\pi < \psi_0 < \tfrac{1}{2}\pi). \tag{5.11}$$

It follows that the solution $\mathscr{A} = 0$ is unstable ($\lambda > 0$) if $\sigma > 1$. The corresponding perturbation grows like $\exp(\lambda\eta)$ in time and decays like $\exp[-2^{-\frac{1}{2}}(\sigma^2 - \cos\phi)^{\frac{1}{2}}\xi]$ in space.

The linearized solution of (5.6)–(5.8) for $\sigma^2 < |\cos\phi|$, for which the solution given by (5.10) and (5.11) fails, may be established through integral-transform techniques (cf. Mahony 1972) and proves to be stable for $\sigma < 1$. It follows that the stability boundary for plane-wave motion is given by $\sigma = 1$ or, equivalently,

$$\left(1 - \frac{\omega_1}{\omega} - \epsilon^2 CF^2\right)^2 + \delta^2 = (B\epsilon)^4. \tag{5.12}$$

Neglecting $\epsilon^2 CF^2$ and replacing B by

$$B' = \left|2\kappa \int_{-\infty}^{0} f(z)\,\mathrm{d}z - f(0)\right|, \tag{5.13}$$

we obtain Barnard & Pritchard's (1972) equation (4). B' is Mahony's counterpart of B (4.4) and differs only slightly therefrom for most $f(z)$.

6. Stationary solutions

The counterparts of the finite-amplitude fixed points in I, §5 are the stationary solutions of (5.6)–(5.8), for which $\mathscr{A}_\eta \equiv 0$. The corresponding differential equation

$$\mathscr{A}_{\xi\xi} + (\mathrm{e}^{\mathrm{i}\phi} + |\mathscr{A}|^2)\mathscr{A} = 0, \tag{6.1}$$

is autonomous, whence it is expedient to regard the logarithmic derivative of $\mathscr{A}$ as a function of the amplitude $|\mathscr{A}|$ or, as proves to be advantageous,

$$Z = \left(\frac{A}{A_*}\right)^2, \quad A_*^2 \equiv \tfrac{1}{4}(5 - 3\cos\phi), \tag{6.2a, b}$$

through the transformation

$$\mathscr{A} = A(\xi)\, \mathrm{e}^{\mathrm{i}\psi(\xi)}, \quad \frac{\mathrm{d}\log \mathscr{A}}{\mathrm{d}\xi} = \frac{\mathrm{d}\log A}{\mathrm{d}\xi} + \mathrm{i}\frac{\mathrm{d}\psi}{\mathrm{d}\xi} \equiv L + \mathrm{i}K. \tag{6.3a, b}$$

Transforming (5.6) and (5.7) and introducing

$$s \equiv \sin\tfrac{1}{2}\phi, \quad c \equiv \cos\tfrac{1}{2}\phi, \tag{6.4a, b}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}Z}\{Z[L^2 - s^2(1-Z)]\} = K^2 - c^2(1+\tfrac{1}{2}Z), \quad L\frac{\mathrm{d}(ZK)}{\mathrm{d}Z} = -sc, \tag{6.5a, b}$$

and

$$M \equiv (L^2+K^2)^{\frac{1}{2}} = \sigma, \quad \sin 2\psi_0 = -\frac{L}{\sigma} \quad (Z = Z_0). \tag{6.6a, b}$$

The system (6.5) has a singular point at $Z = 0$ ($\xi = \infty$), at which, from (5.8),

$$L = -s, \quad K = c \quad (Z = 0). \tag{6.7a, b}$$

The integration of (6.5) may be started from this singular point, thereby satisfying (5.8), and continued to that point at which (6.6*a*) is satisfied. This determines Z_0, after which ψ_0 is determined by (6.6*b*) and (6.3*b*) may be integrated to obtain

$$\xi = \frac{1}{2}\int_{Z_0}^{Z} \frac{\mathrm{d}Z}{ZL}, \quad \psi = \psi_0 + \frac{1}{2}\int_{Z_0}^{Z} \frac{K\,\mathrm{d}Z}{ZL}. \tag{6.8a, b}$$

6.1. *Undamped solutions*

Exact solutions of (6.5)–(6.8) are possible if either $\phi = 0$ ($\alpha = 0, \beta > 0$) and $\sigma > 1$ or $\phi = \pi$ ($\alpha = 0, \beta < 0$) and $\sigma < 1$. In the former case

$$L = 0, \quad K = (1+\tfrac{1}{2}Z)^{\frac{1}{2}}, \quad Z_0 = 2(\sigma^2-1), \quad \psi_0 = \tfrac{1}{2}\pi, \tag{6.9a, b, c, d}$$

$$A = A_0 = (\sigma^2-1)^{\frac{1}{2}}, \quad \psi = \psi_0 + \sigma\xi \quad (\phi = 0, \quad \sigma > 1), \tag{6.10a, b}$$

which describes a progressive wave of constant amplitude. But damping, however small, cannot be neglected, and (6.10) *qua* approximation for $\alpha \ll 1$, cannot be uniformly valid, as $\xi \uparrow \infty$; see (6.20).

In the latter case ($\alpha = 0, \beta < 0, \sigma < 1$),

$$L = -(1-Z)^{\frac{1}{2}}, \quad K = 0, \quad Z_0 = 1-\sigma^2, \quad \psi_0 = \pm\tfrac{1}{4}\pi, \tag{6.11a, b, c, d}$$

$$A = \surd 2\, \mathrm{sech}\,(\xi \pm \tanh^{-1}\sigma), \quad \psi = \psi_0 \quad (\phi = \pi, \quad \sigma < 1), \tag{6.12a, b}$$

which describes a pair of trapped solitary waves (cf. Miles 1985). The alternative signs are vertically ordered: the upper choice yields a monotonically decaying (in ξ) envelope for which the branch point at $Z = 1$ lies outside of the physical domain; the lower choice corresponds to a solution for which the integration is continued through the branch point, at which L changes sign and $A(\xi)$ has a maximum. The shape (6.12*a*) is at least qualitatively valid, but the approximation of constant phase (6.12*b*) is unrealistic, for small but finite damping; cf. (6.18).

6.2. *Analytical approximations*

An analytical approximation to the solution of (6.5)–(6.7) for $0 < \phi < \pi$ may be obtained through the expansion of L^2 and K^2 (rather than L and K) about $Z = 0$, which yields

$$L^2 = s^2(1-Z) + C_2 Z^2 + C_3 Z^3 + \dots, \quad K^2 = c^2(1+\tfrac{1}{2}Z) + 3C_2 Z^2 + 4C_3 Z^3 + \dots, \tag{6.13a, b}$$

where $$C_2 = \frac{15}{64}\left(\frac{\sin^2\phi}{5-4\cos\phi}\right), \quad C_3 = \frac{9}{128}\frac{(25-31\cos\phi)\sin^2\phi}{(5-4\cos\phi)(17-15\cos\phi)}. \tag{6.14a,b}$$

Invoking (6.6*a*), we obtain

$$5C_3 Z_0^3 + 4C_2 Z_0^2 + \tfrac{1}{4}(3\cos\phi - 1)Z_0 + 1 = \sigma^2 \tag{6.15}$$

for the determination of Z_0. If $\sigma > 1$ (in which domain the plane wave is unstable) (6.15) has one and only one positive root. If $\sigma < 1$ (in which domain the plane wave is stable) and $\phi > \cos^{-1}(\frac{1}{3}) = 70.5°$ (6.15) may have either zero or two positive roots. Comparison with the results of the numerical integration indicates that the approximations (6.13) and (6.15) are satisfactory for $\phi \lesssim \frac{1}{2}\pi$ (figure 1*a, b*), but they fail (except near $Z = 0$) for $\phi > \phi_* \doteqdot 127°$ (see figure 1*c*), in which domain L may have branch points within the domain of integration.

It is evident from (6.13*a*) that the smallest branch point of L for $\phi > \phi_*$ must be close to $Z = 1$, which suggests the approximation

$$L = -s(1-Z)^{\frac{1}{2}}. \tag{6.16a}$$

The corresponding truncation of (6.13*b*) is less satisfactory in that it fails to reproduce the branch point of K that necessarily accompanies that of L. We obtain a more suitable complement to (6.16*a*) by substituting that approximation into (6.5*b*), integrating from $Z = 0$, and invoking (6.7*b*), which yields

$$K = 2c[1+(1-Z)^{\frac{1}{2}}]^{-1}. \tag{6.16b}$$

We note that (6.16*b*) coincides with (6.13*b*) near $Z = 0$, where both may be approximated by $K = c(1+\frac{1}{4}Z)$. But if the solution is continued through the branch point at $Z = 1$ the sign of the radical must be changed, and (6.16*b*) then is singular at $Z = 0$. This singular point always lies outside the physical domain (see below) but is nevertheless significant; in particular, it renders the approximation (6.16) non-uniformly valid in the limit $\phi \uparrow \pi$.

Substituting (6.16) into (6.6) and (6.8) and remarking that the sign of $(1-Z)^{\frac{1}{2}}$ must be changed if the integration is continued through the branch point at $Z = 1$, we obtain

$$s^2(1-Z_0) + 4c^2[1 \pm (1-Z_0)^{\frac{1}{2}}]^{-2} = \sigma^2 \tag{6.17a}$$

and $$\sin 2\psi_0 = \pm\sigma^{-1}s(1-Z_0)^{\frac{1}{2}} \quad (\tfrac{1}{4}\pi \leqslant \pm\psi_0 \leqslant \tfrac{1}{2}\pi) \tag{6.17b}$$

for the determination of Z_0 and ψ_0, and

$$A = A_* \operatorname{sech}[s\xi \pm \operatorname{sech}^{-1} Z_0^{\frac{1}{2}}] = \frac{A_0}{\cosh s\xi \pm (1-Z_0)^{\frac{1}{2}}\sinh s\xi} \tag{6.18a}$$

and $$\psi = \psi_0 + c\left\{\xi + \left[\frac{1-(1-Z_0)^{\frac{1}{2}}}{1+(1-Z_0)^{\frac{1}{2}}}\right]^{\pm 1}\left(\frac{1-e^{-2s\xi}}{2s}\right)\right\}, \tag{6.18b}$$

wherein the alternative signs are vertically ordered. Letting $\phi \uparrow \pi$ in (6.17) and (6.18), we recover (6.11) and (6.12). Like (6.15), (6.17*a*) has only one positive root if $\sigma > 1$ and either zero or two positive roots if $\sigma < 1$ and $\phi > \cos^{-1}(\frac{1}{3})$. Comparison with the results of the numerical integration reveals that (6.17) and (6.18) are satisfactory approximations for all ϕ in $0 < Z_0 \lesssim \frac{1}{2}$ and $\sigma \lesssim 1.2$ (figure 1*a*) and for increasingly large ranges of Z_0 and σ as ϕ increases to π (see figure 1*b, c*).

It is evident from a comparison with (6.13) that the approximation (6.16) is

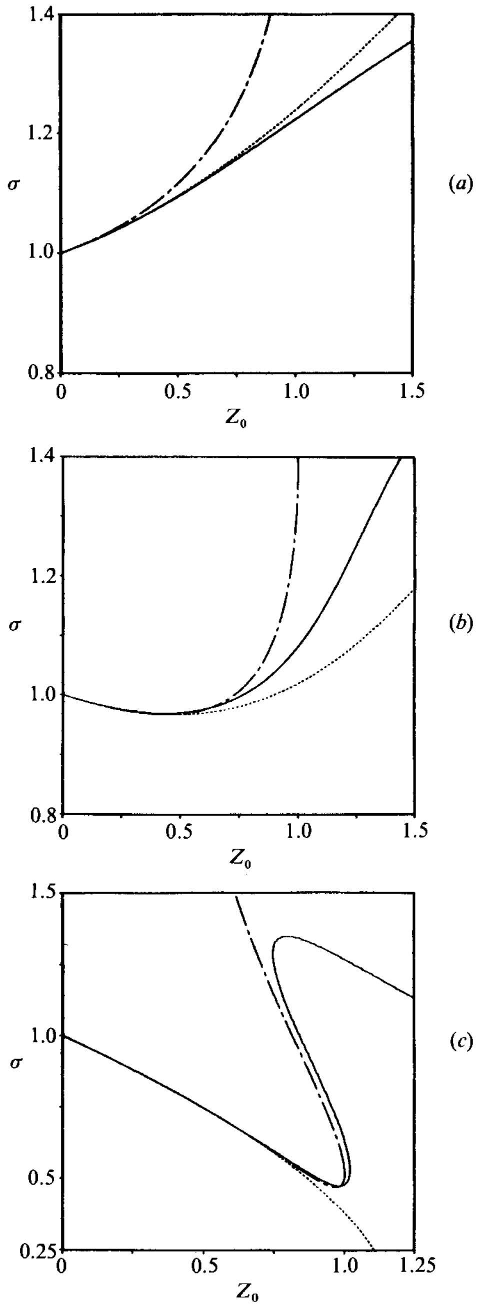

FIGURE 1. σ *vs.* Z_0 as given by (6.15) (—·—·—), (6.17*a*) (·····) and the numerical integration of (6.5) or (6.23) (——) for (*a*) $\phi = \frac{1}{4}\pi$, (*b*) $\frac{1}{2}\pi$ and (*c*) $\frac{5}{8}\pi$.

accurate for small ϕ if and only if $Z_0 \ll 1$, in which case (6.18) with the upper choice of signs may be approximated by

$$A = A_0\,\mathrm{e}^{-s\xi}[1+\tfrac{1}{4}Z_0(1-\mathrm{e}^{-2s\xi})], \quad \psi = \psi_0 + c[\xi + \tfrac{1}{8}s^{-1}Z_0(1-\mathrm{e}^{-2s\xi})] \quad (Z_0 \ll 1). \tag{6.19\,a,b}$$

Letting $\phi \downarrow 0$ and $\sigma \downarrow 1$, in which limit $Z_0 \to 2(\sigma^2-1)$, we obtain

$$A = (\sigma^2-1)^{\frac{1}{2}} \mathrm{e}^{-s\xi}, \quad \psi = \psi_0+\xi+\tfrac{1}{4}(\sigma^2-1)\, s^{-1}(1-\mathrm{e}^{-2s\xi}) \quad (\phi \downarrow 0, \sigma \downarrow 1), \tag{6.20a, b}$$

which are uniformly valid (with respect to ξ) counterparts of (6.10) in the neighbourhood of the plane-wave stability boundary ($\sigma = 1$).

The approximation (6.16) may be improved by substituting (6.16*b*) into (6.5*a*), integrating the result from $Z = 0$ to obtain

$$L = \mp\left\{s^2(1-Z)-c^2\left[\left(\frac{2}{1+R}\right)^2+\frac{8}{Z}\log\left(\tfrac{1}{2}(1+R)\right)+1+\tfrac{1}{4}Z\right]\right\}^{\frac{1}{2}}, \tag{6.21a}$$

$$R \equiv \pm(1-Z)^{\frac{1}{2}}, \tag{6.21b}$$

and proceeding by iteration. The resulting expansion evidently diverges as $Z \downarrow 0$ with $R = -(1-Z)^{\frac{1}{2}}$, but it does provide a valid approximation to L between its first (near $Z = 1$) and second (in $0 < Z < 1$) branch points.

6.3. *Numerical integration*

The numerical integration of (6.5) is straightforward if L remains negative, as proves to be the case if $\phi < \phi_* \doteqdot 127°$, and $A(\xi)$, as determined by (6.8*a*), then decreases monotonically as ξ increases from 0 to ∞ (Z decreases from Z_0 to 0). But if $\phi > \phi_*$, L has a sequence of branch points (cf. (6.16*a*)) at which (6.5*b*) is singular, and it then is expedient to introduce the alternative independent variable

$$Y = \tfrac{1}{2}c^{-1}ZK \tag{6.22}$$

and transform (6.5) and (6.8) to

$$s\frac{\mathrm{d}Z}{\mathrm{d}Y} = -2L, \quad sZ\frac{\mathrm{d}L}{\mathrm{d}Y} = s^2Z+L^2-s^2(1-Z)-c^2\left[\left(\frac{2Y}{Z}\right)^2-1-\tfrac{1}{2}Z\right] \tag{6.23a, b}$$

and

$$s\xi = \int_Y^{Y_0}\frac{\mathrm{d}Y}{Z}, \quad \psi = \psi_0+\frac{2c}{s}\int_Y^{Y_0}\frac{Y\,\mathrm{d}Y}{Z^2}. \tag{6.24a, b}$$

The integration of (6.23) may be started from $Y = 0$, where

$$Z = 0, \quad L = -s \quad (Y = 0). \tag{6.25a, b}$$

Expanding the solution of (6.23) and (6.25) in powers of Y, we obtain

$$Z = 2Y-Y^2+Z_3\,Y^3+Z_4\,Y^4+\dots \tag{6.26a}$$

and

$$L = s[-1+Y-\tfrac{3}{2}Z_3\,Y^2-2Z_4\,Y^3+\dots], \tag{6.26b}$$

where

$$Z_3 = \frac{5}{4}\left(\frac{c^2}{1+8s^2}\right), \quad Z_4 = \frac{1}{4}\frac{c^2(-8+51s^2)}{(1+8s^2)(1+15s^2)}. \tag{6.27a, b}$$

Numerical integrations of (6.5) and (6.23) were performed using a Gears routine for various values of ϕ. The end results for M and L *vs.* Z, which may be interpreted as plots of σ and $-\sigma\sin 2\psi_0$ *vs.* Z_0 (so that Z_0 and ψ_0 are determined for prescribed ϕ and σ), are plotted in figure 2. Both L and M *vs.* Z are single-valued if $\phi < \phi_*$, but if $\phi > \phi_*$ there is a sequence of branch points, $Z_2 < Z_1 < Z_4 < Z_3 \dots < Z_N$, where N increases with $\phi-\phi_*$, such that L exhibits a sequence of loops (see figure 2); in contrast, both L and Z are single-valued functions of Y (see e.g. figure 3). The

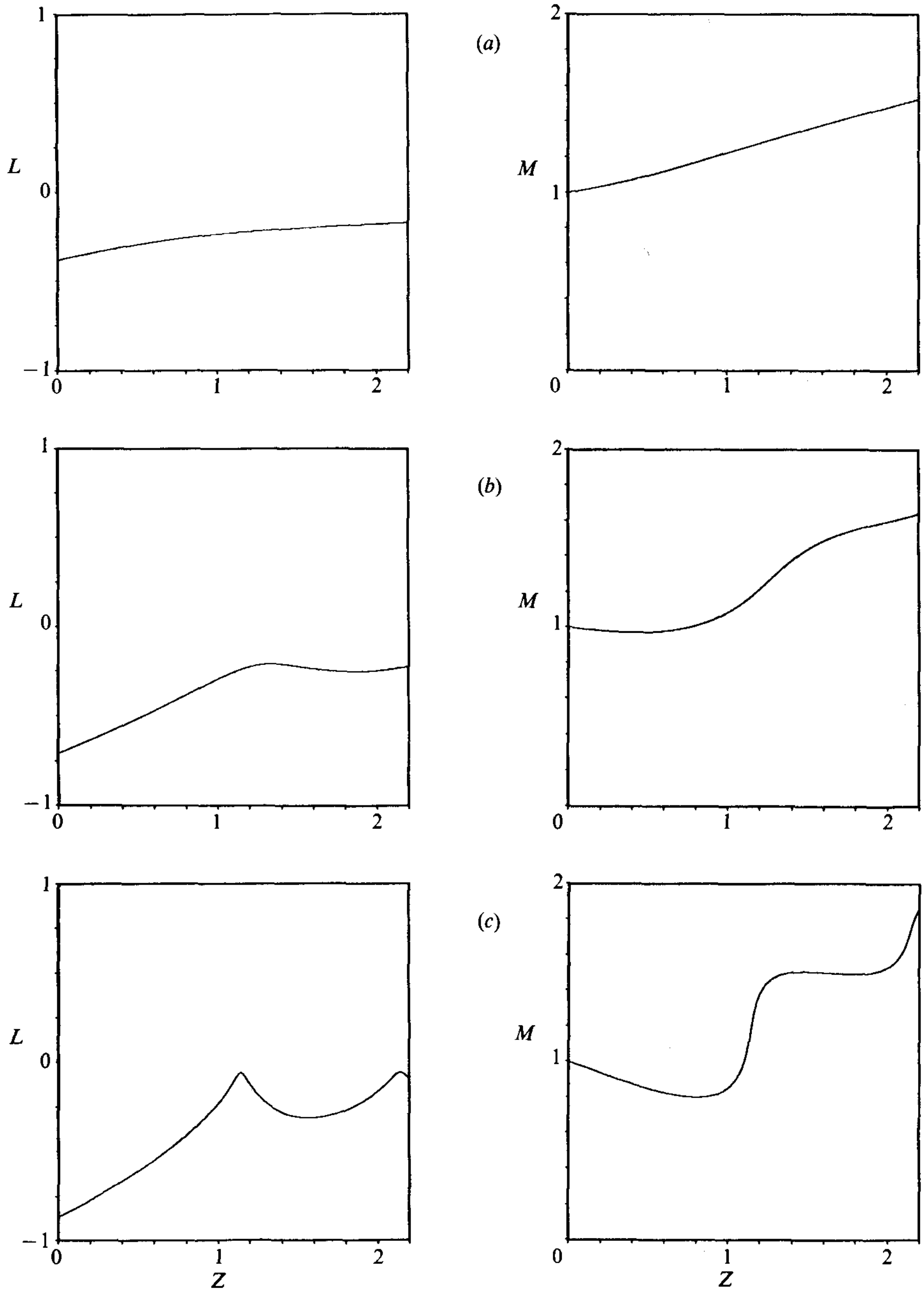

FIGURE 2 (*a–c*). For caption see next page.

direction of travel along $L(Z)$ is determined by the requirement that ξ decrease monotonically from ∞ at $Z = 0$ to 0 at $Z = Z_0$, and $A(\xi)$ has maxima at $Z_1, Z_3, \ldots$ and minima at $Z_2, Z_4, \ldots$. It follows from (6.5*b*) that K also has branch points, although it does not vanish, at $Z = Z_n$. M *vs.* Z increases monotonically if $\phi < 70.5°$ (figure 2*a*), exhibits a staircase-like behaviour but has bounded slope if $70.5° < \phi < 127°$ (figure 2*b, c*), and exhibits a staircase-like behaviour with rearward-facing risers and downward-sloping steps if $\phi > 127°$ (see figure 2*d, e, f*). We emphasize that L and K

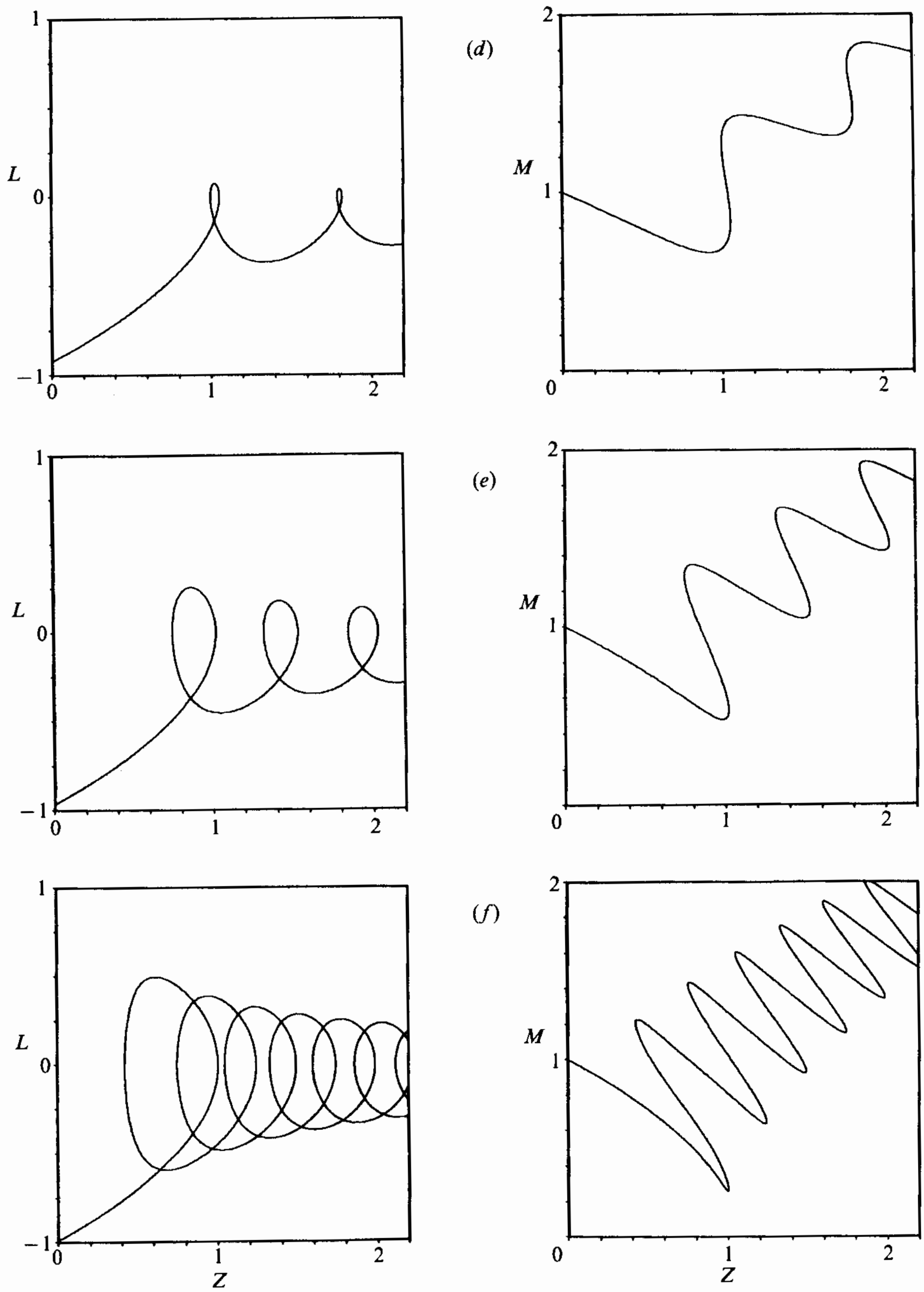

FIGURE 2. L *vs.* Z ($-\sigma \sin 2\psi_0$ *vs.* Z_0) and M *vs.* Z (σ *vs.* Z_0) for (a) $\phi = \frac{1}{4}\pi$, (b) $\frac{1}{2}\pi$, (c) $\frac{2}{3}\pi$, (d) $\frac{3}{4}\pi$, (e) $\frac{5}{6}\pi$ and (f) $\frac{11}{12}\pi$.

are independent of σ, which enters only through the determination of Z_0 and ψ_0. Results of the numerical integration for A/A_* and $\psi - \psi_0$ for $\sigma = \sqrt{2}$ and $1/\sqrt{2}$ are plotted in figures 4 and 5. It appears likely that those stationary solutions with more than one maximum may be either unstable or more difficult to attain (from any particular initial conditions) than those solutions with one or no maximum.

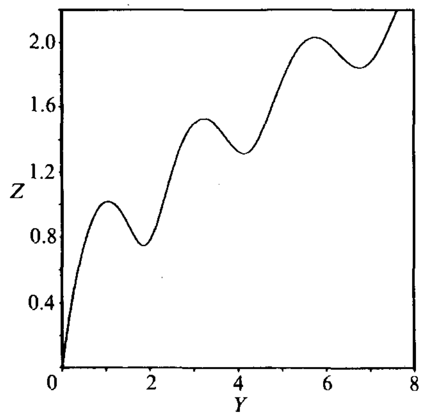

FIGURE 3. Z *vs.* Y for $\phi = \frac{5}{6}\pi$.

Lichter & Chen (1987) carry out numerical integrations of the equivalents of (5.6) and (5.7). Renormalizing their equations (10*a*, *b*) to (5.6) and (5.7), we obtain

$$B = \frac{R}{2\alpha}, \quad \gamma = \left(\frac{R}{2\alpha}\right)^2 (\lambda^2 + L^2)^{\frac{1}{2}}, \quad \sigma = (\lambda^2 + L^2)^{-\frac{1}{4}} \tag{6.28a, b, c}$$

in their notation, in which $\alpha\epsilon$ is equal to the present ϵ. Lichter & Chen set $\lambda = 2.666 \times 10^{-6}/\theta^2$ and $L = 5.695 \times 10^{-5}/\theta^2$, which imply $\phi = \frac{1}{2}\pi - 0.02$ and $\sigma = 132.4\theta$, where θ is the angular amplitude of the flapping wavemaker. They obtain stationary envelopes for $\sigma = 1.18$, 1.22 and $\sigma = 1.31$ (for which the decay in τ is very slow) and limit cycles for $\sigma = 1.48$ and 1.59. The numerical integration of our (6.5) and (6.8) for $(\phi, \sigma) = (\frac{1}{2}\pi, 1.18)$ reproduce the curves in Lichter & Chen's figures 6 and 7*a* for $(\phi, \sigma) = (0.49\pi, 1.18)$, to within the resolution of their plots (they do not give plots for the other stationary envelopes, $\sigma = 1.22$ and 1.31).

6.4. *Comparison with experiment*

The only published experimental data for parametrically excited cross-waves in a long tank appear to be those of Barnard & Pritchard (1972). These experiments employed a flap-type wavemaker with

$$f(z) = 1 + \frac{z}{d} \quad (-d \leqslant z \leqslant 0). \tag{6.29}$$

A comparison of (5.12) with their measured stability boundaries and Mahony's (1972) theory is presented in table 1 (note that in (5.12), $\epsilon^2 CF^2 = 8.3 \times 10^{-5}/6.1 \times 10^{-5}$ for $n = 2/3$ is negligible).

Barnard & Pritchard measured only slowly modulated cross-waves; however, the measured envelopes in their figure 6 for $n = 3$ were nearly steady ('variations in the amplitude of the cross-waves at a given position were less than about 5%') and may be compared with the present theory. Using their measured wavemaker amplitudes,† frequencies and damping coefficients in (4.4) and (5.5), we obtain $(\epsilon, \phi, \sigma) = (0.023, 0.65\pi, 1.20)$ and $(0.027, 0.65\pi, 1.43)$. Bearing in mind that the experimental errors may be of the order of 10% and that our results are sensitive to small changes in

† The wavemaker amplitudes given in the caption of Barnard & Pritchard's figure 6 should be divided by 10 (Pritchard agrees).

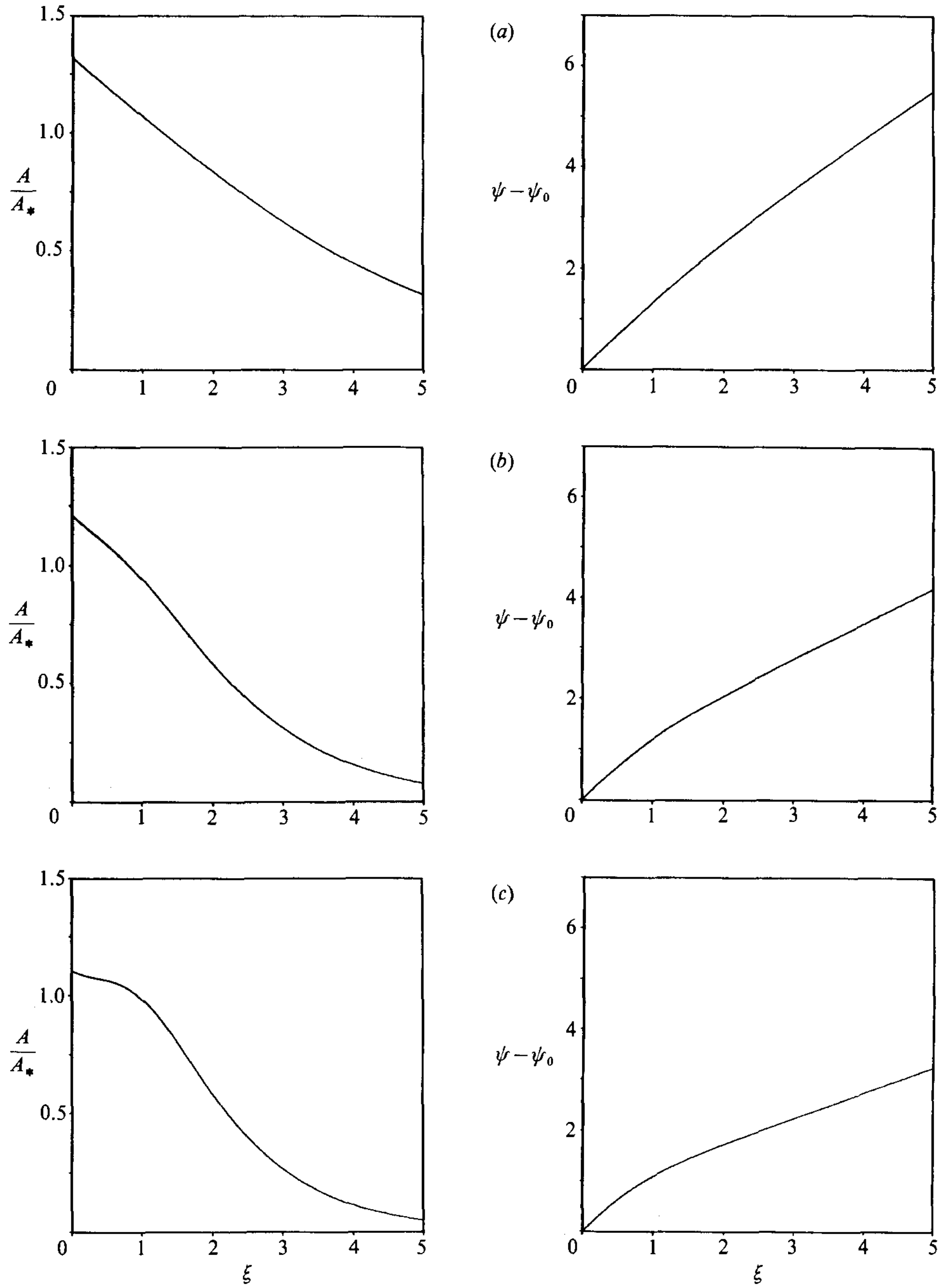

FIGURE 4 (a–c). For caption see facing page.

σ and ϕ owing to the proximity of ϕ and ϕ_* (see above), we compare our predicted envelopes for $(\epsilon, \phi, \sigma) = (0.023, \frac{2}{3}\pi, 1.2)$ and $(0.023, \frac{3}{4}\pi, 1.2)$ with Barnard & Pritchard's data for $(0.023, 0.65\pi, 1.20)$ in figure 6(a, b). The corresponding comparison for $(0.027, 0.65\pi, 1.43)$ is much less satisfactory, in part because of the non-uniqueness of our results for $\phi > \phi_*, \sigma = 1.4$, but presumably also because of the uncertainties in the values of ϕ and σ.

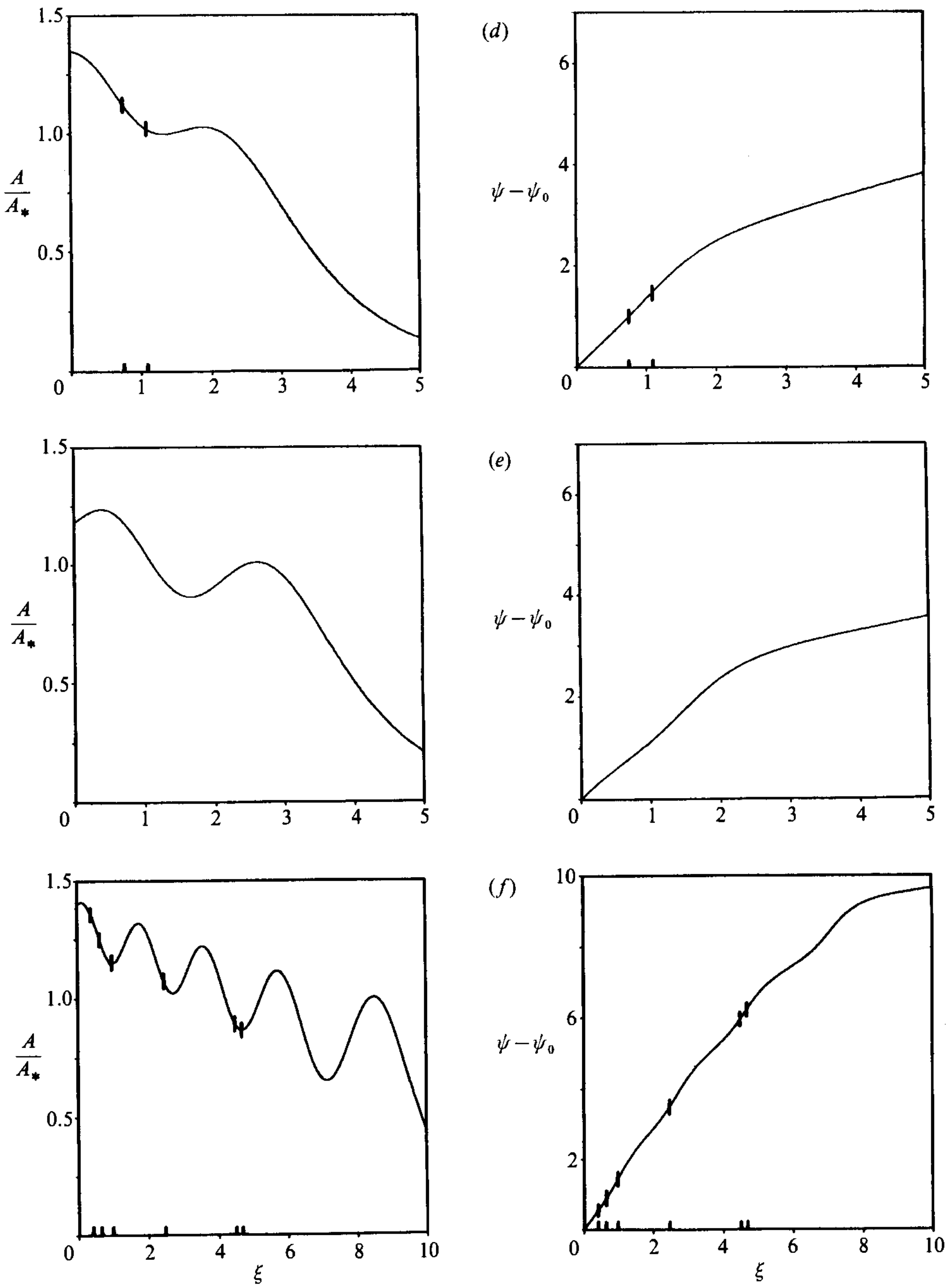

FIGURE 4. A/A_* and $\psi-\psi_0$ *vs.* ξ for $\sigma=\sqrt{2}$ and (*a*) $\phi=\frac{1}{4}\pi$, (*b*) $\frac{1}{2}\pi$, (*c*) $\frac{2}{3}\pi$, (*d*) $\frac{3}{4}\pi$, (*e*) $\frac{5}{6}\pi$ and (*f*) $\frac{11}{12}\pi$ (the scale of *f* differs from that of *a*–*e*). Multiple solutions exist for $\phi=\frac{3}{4}\pi$ and $\frac{11}{12}\pi$, and the complete profiles in (*d*) and (*f*) correspond to the largest values of Z_0. The positions of the wavemaker for the remaining solutions are indicated by vertical bars, with corresponding translations of the ξ-scale being implicit.

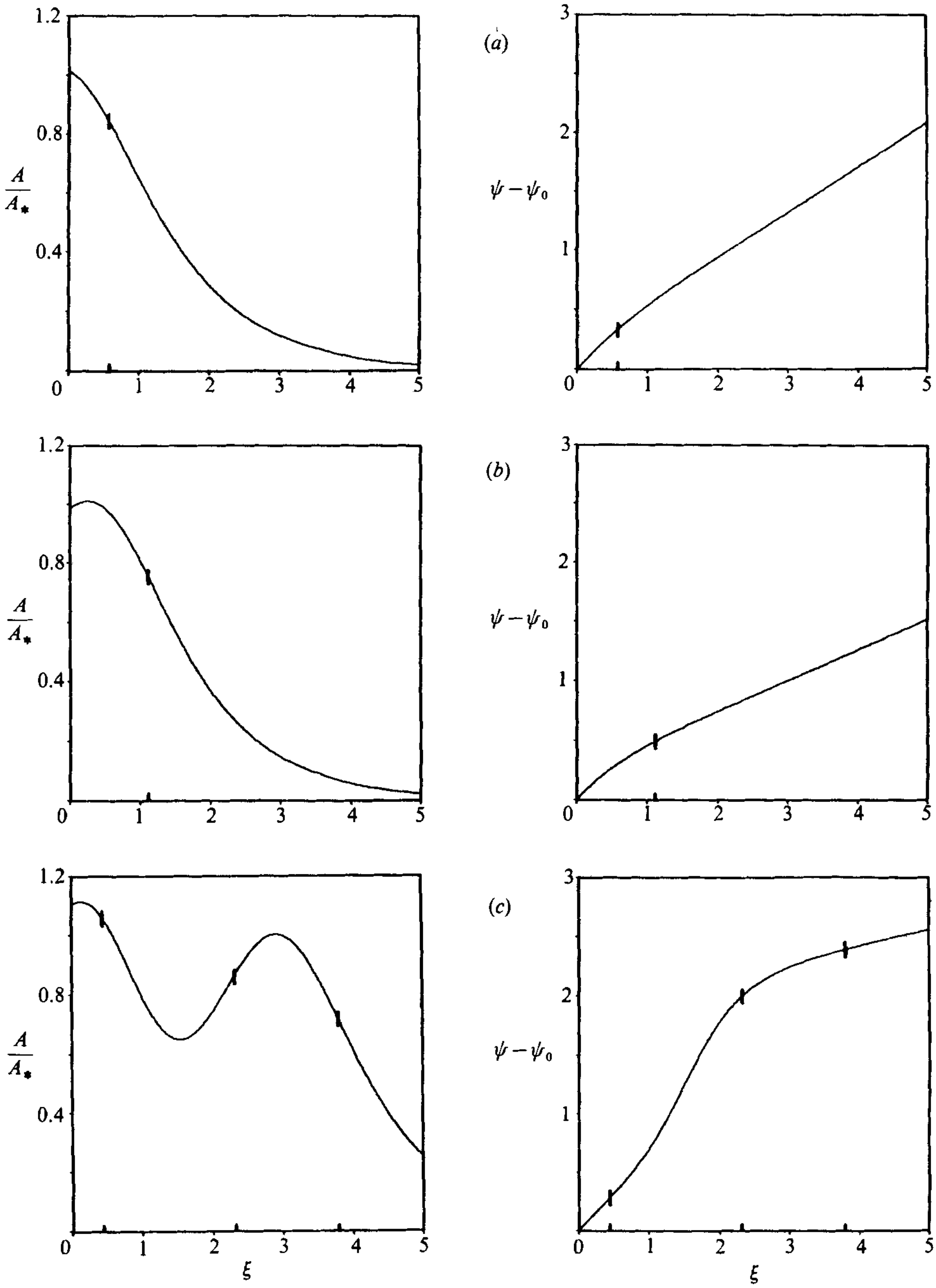

FIGURE 5. A/A_* and $\psi-\psi_0$ *vs.* ξ for $\sigma = 1/\sqrt{2}$ and (a) $\phi = \frac{3}{4}\pi$, (b) $\frac{5}{6}\pi$ and (c) $\frac{11}{12}\pi$. The vertical bars indicate the positions of the wavenumber for multiple solutions (see caption of figure 4).

Mode n	ϵ (measured)	α (measured)	$\alpha^{-\frac{1}{2}}B$	$\alpha^{-\frac{1}{2}}B'$
2	0.022	4.6	1.13	1.10
3	0.018	15.3	1.03	1.02

TABLE 1. The critical value of the stability parameter $\sigma = B/\alpha^{\frac{1}{2}}$, using Barnard & Pritchard's (1972) measured value of α and either the present or Mahony's (1972) calculated value, (4.4) and (5.12) respectively, for B. Both the present and Mahony's theories yield a critical value of $\sigma = 1$. Note that the reported value of $C_3 = 147$ in table 1 of Barnard & Pritchard (1972) as calculated from Mahony's (1972) theory is in error. It should read $C_3 = 195$.

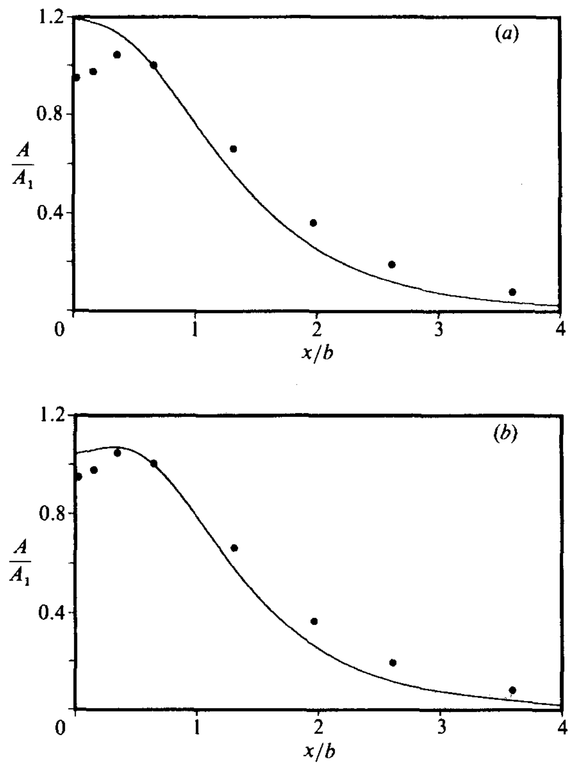

FIGURE 6. A/A_1, with $A_1 = A(x/b = 0.65)$, *vs.* x/b for $\sigma = 1.20$ and (*a*) $\phi = \frac{2}{3}\pi$ and (*b*) $\frac{3}{4}\pi$. The solid circles are the experimentally measured cross-wave envelope of Barnard & Pritchard's (1972) figure 6 for $(\epsilon, \sigma, \phi) = (0.023, 0.65\pi, 1.20)$.

6.5. *Stability*

The present analysis is less complete than that for standing cross-waves (I, §5) in that, although we have determined the stability of the progressive plane waves (in §5 above), we have left unresolved the question of stability of progressive cross-waves (i.e. of the stationary solutions for $\mathscr{A}$ in this section).† It seems likely that this question can be resolved only through further refinement of either the experimental work of Barnard & Pritchard (1972) or the numerical work of Lichter & Chen (1987).

† We are reminded of Watson's (1945) statement, in the introduction to his chapter XIII on infinite integrals, that 'In spite of the incompleteness of this chapter, its length must be contrasted unfavourably with the length of the chapter on finite integrals'.

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