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Relationship between Benjamin–Feir instability and recurrence in the nonlinear Schrödinger equation

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A simple relationship between the Benjamin–Feir instability associated with uniform solutions of the nonlinear Schrödinger equation and the long time evolution of the unstable solution is reported. The number of modes which actively participate in the energy sharing process associated with the instability is governed by the number of harmonics of the initial disturbance which lie within the unstable region as predicted by the Benjamin–Feir analysis. Generalization of this observation implies that equations which possess high wavenumber cutoffs in the instability characteristics should not thermalize in the conventional sense when undergoing such an instability, since active modes are confined to a finite range of wavenumbers.

We consider the nonlinear Schrödinger equation, which has found application in many branches of physics for describing the evolution of weakly nonlinear wave envelopes

$$i\frac{\partial A}{\partial T}-\frac{1}{8}\frac{\partial^2 A}{\partial X^2}-\frac{1}{2}|A|^2A=0\ . \tag{1}$$

For initial conditions which decay sufficiently rapidly as $|x|\to\infty$, the initial value problem has been solved by Zakharov and Shabat[1] using the inverse scattering method. However, for initial conditions which do not decay with large $|x|$, the inverse scattering method cannot be applied and the properties of the solutions have not been fully explored.

We shall be concerned with solutions of (1) with spatially periodic boundary conditions. There are two known properties associated with these periodic solutions:

(i) The Benjamin and Feir instability

The uniform solution $A=a_0\exp[-(i/2)a_0^2T]$ is unstable to infinitesimal perturbations of the form $b_\pm\exp[i(\Delta X\pm\Omega T)]$, provided that the perturbation wavenumber Δ lies in the range

$$0<\Delta<\Delta_{cr}=2\sqrt{2}a_0\ . \tag{2}$$

The maximum instability occurs when $\Delta=\Delta_{max}=2a_0$ as shown in Benjamin and Feir.[2] (The stability diagram is given in Fig. 4.)

(ii) Fermi–Pasta–Ulam recurrence

It has been found that the unstable modulations to the uniform solution would first grow at an exponential rate as predicted by Benjamin and Feir, but eventually the solution would demodulate and return to a near-uniform state. The energy in the system, which is initially confined to a few low modes, would spread to many higher modes due to the nonlinear instability, but would eventually regroup into the original low modes. This process repeats periodically in time. The recurrence or repetition may not be perfect. This phenomenon is known as the Fermi–Pasta–Ulam recurrence, since it was first reported by Fermi *et al.*[3] when they studied nonlinear lattice vibration numerically. Its occurrence in the nonlinear Schrödinger equation in connection with deep water waves has been verified experimentally by Lake *et al.*[4] and Yuen *et al.*[5]

One would naturally ask if there exists any relationship between the initial instability and the recurrence phenomenon. In the following, we shall report some findings which tend to answer the question in the affirmative.

TWO TYPES OF RECURRENCE: SIMPLE AND COMPLEX

We consider the long time evolution of the solution to (1) with initial conditions of the form

$$A=a_0(1-0.1\cos 2\pi\Delta X)\ . \tag{3}$$

Numerical solution indicates that for some values of Δ, the evolution is "simple." The prescribed modulation initially grows exponentially as predicted by the instability results, but the modulation eventually subsides and a near-uniform solution is reconstructed. This process repeats itself periodically in time. Both the reconstruction and the repetition are almost perfect. In Fig. 1(a) we show a plot of the amplitude $|A|$ against time for one modulational period (the prescribed modulational period). In Fig. 1(b) we plot the time evolution of the amplitude departure norm $\epsilon(T)$ defined as

$$\epsilon(T)=\frac{1}{N}\sum_{j=1}^{N}|a(j,T)-a(j,0)|\ , \tag{4}$$

where N is the number of grid points used in the computation, and $a(j,T)$ is the amplitude of the solution at the jth point at time T. It can be seen that the initial condition is well reconstructed periodically, and a well-defined recurrence time T_r can be deduced from the plot. In Fig. 1(c) we show the amplitude of the Fourier trans-

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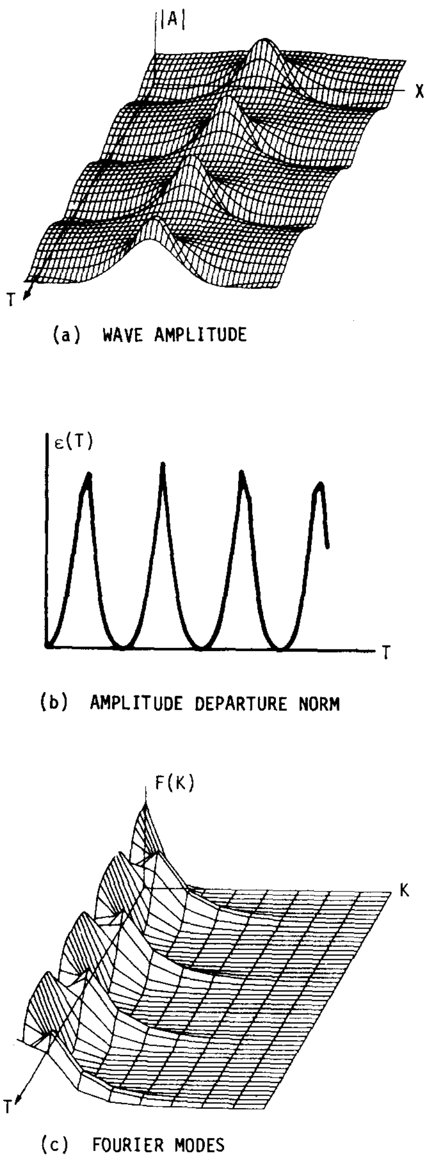

FIG. 1. Example of a simple evolution. (a) Envelope amplitude $|A|$ plotted against X and T. (b) Amplitude departure norm [defined in (4)] against time. (c) Fourier modes (first eight modes) against time.

form of the solution. One sees that the energy contained in the zeroth and the first modes spreads to higher modes and then regroups periodically in time.

On the other hand, for some other values of Δ, the evolution is relatively "complex" as illustrated in Fig. 2. From the amplitude plot, we see that although a near uniform state is achieved during many times in the evolution, many modulations other than the one prescribed have appeared [Fig. 2(a)]. The plot of $\epsilon(T)$ indicates that a single value for T_r cannot characterize the evolution, and in fact, it would be a matter of judgment to pinpoint at which time a "recurrence" should have occurred [Fig. 2(b)] Consistent with these features, the Fourier modes exhibit complex sharing and regrouping of energy [Fig. 2(c)].

Further investigations have led us to conclude that there are basically two types of evolution, namely, simple and complex, and their occurrences are closely connected to the value of Δ through the Benjamin and Feir instability results. Specifically, we have found that for values of Δ in the range

$$\tfrac{1}{2}\Delta_{cr} \leq \Delta < \Delta_{cr} , \tag{5}$$

the evolution is simple in the sense described here; whereas for values of Δ in the range

$$0 < \Delta < \tfrac{1}{2}\Delta_{cr} , \tag{6}$$

the evolution is complex. This division at $\frac{1}{2}\Delta_{cr}$ is not arbitrary for the following reason: For modulations with values of Δ in the range $\frac{1}{2}\Delta_{cr} \leq \Delta < \Delta_{cr}$ (corresponding to simple evolution), all the higher harmonics of the prescribed modulation are stable according to the Benjamin and Feir stability results. The generated harmonics always appear as forced oscillations, being phase-locked to the fundamental mode (which is the prescribed mode). This is illustrated in Fig. 3 which

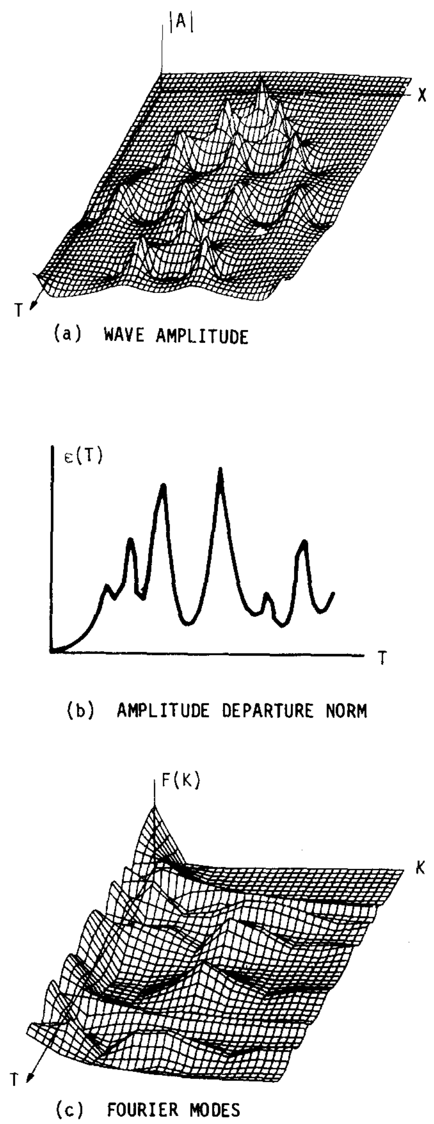

FIG. 2. Example of a complex evolution. (a) Envelope amplitude $|A|$ plotted against X and T. (b) Amplitude departure norm [defined in (4)] against time. (c) Fourier modes (first eight modes) against time.

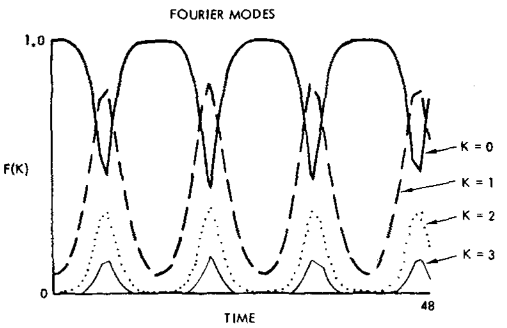

FIG. 3. Normalized energy content for the first four Fourier modes.

shows the energy content in the first four modes for a simple evolution. For modulations with values in the range of $0 < \Delta < \frac{1}{2}\Delta_{cr}$, however, at least one higher harmonic of the prescribed mode lies in the unstable region of the Benjamin and Feir instability diagram (Fig. 4). Apparently, these unstable harmonics tend to grow independently in an exponential rate when triggered by the forced oscillation. Infact, the complex evolution appears to be a composition of the evolution of all the unstable harmonics of the prescribed mode, with each and every harmonic taking its turn dominating the solution profile. To illustrate this, we have computed several cases in which different numbers of harmonics fall within the unstable range. We use the notation that case (n) corresponds to a case in which n modes, including the fundamental, fall within the unstable region and thus actively participate in the evolutionary process. The values of Δ for each case ($1 \leq n \leq 5$) are given in Fig. 4 and the results shown in Fig. 5. The number of free wave modes actively participating in the evolution can be checked against the number of spatial "humps" present in the amplitude plots. The modal energy content corresponding to case 4 is shown in Fig. 2(c) as an example.

Since these results have been obtained with a particularly simple set of initial conditions, one may ask whether the conclusions drawn are sensitive to a change

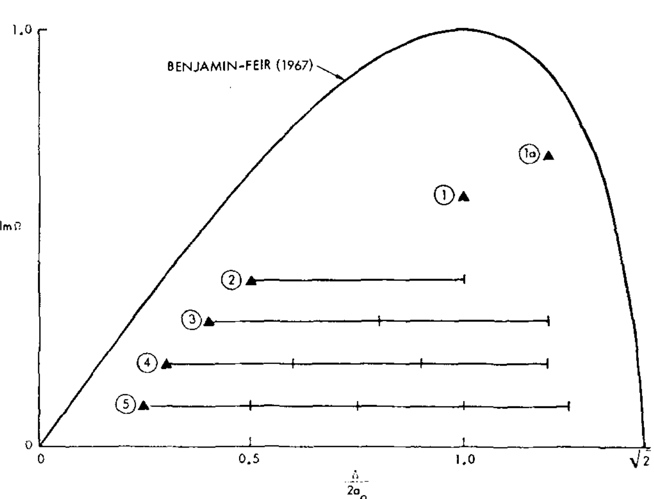

FIG. 4. Instability diagram of Benjamin–Feir.[2] Growth rate $\text{Im}\,\Omega$ plotted against normalized perturbation wavenumber $\Delta/2a_0$. ▲ indicates values of $\Delta/2a_0$ used for cases with case numbers in circles.

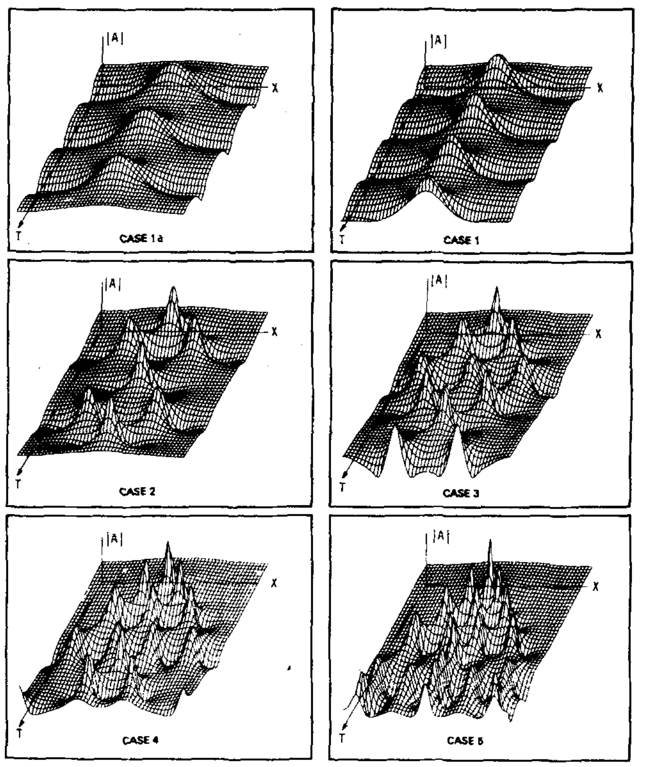

FIG. 5. Examples of simple and complex evolutions. Envelope amplitude $|A|$ plotted against X and T. Case (n) refers to case in which n modes, including the fundamental, are within the Benjamin–Feir unstable region (see Fig. 4 for values of $\Delta/2a_0$ corresponding to each case shown). Notice that the number of "humps" appearing in evolution is consistent with the number of unstable modes.

in initial conditions. To examine this we have computed with initial conditions of the form

$$A = a_0(1 - p\cos 2\pi\Delta X)\exp[ip\cos(2\pi\Delta X + q)] , \qquad (7)$$

where p is a small parameter controlling the magnitude of the perturbation and q is an arbitrary phase. Without presenting the details, we merely report that the changes in the initial condition do not affect the conclusion regarding the type of evolution and the number of active modes participating, although the details of the evolution (e.g., the sequence in which the various modes dominate the wave form) do alter. Further studies in this and other related questions are still under way.

DISCUSSION

We have reported what we think are some remarkable properties of the periodic solutions of the nonlinear Schrödinger equation, namely, a simple relationship between the Benjamin–Feir instability and the Fermi–Pasta–Ulam recurrence. Although it is clear that we have not explored the full subtleties of the phenomenon of recurrence, yet the simplicity of the findings has encouraged us to make a conjecture on the long time behavior of a limited class of initial conditions for weakly nonlinear, dispersive systems in general. We propose that with periodic boundary conditions, a near-uniform solution undergoing long-wave instability will not thermalize if there is a high-wavenumber cutoff in the instability characteristics. This is because a generalization of the findings here indicates that the active participants in the energy sharing process are always confined to the unstable low wavenumbers. Thus, there cannot be a permanent, irreversible leakage of energy to high modes, a process which is necessary for thermalization. This proposition is untested. However, we hope that it can at least serve as a center of attraction for criticism and opinions which will help stimulate further interest in the fascinating subject of thermalization, which, incidently, has been the very question that motivated the findings of Fermi *et al*.

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