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Stokes' Expansion of Internal Deep Water Waves to the Fifth Order*

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Abstract: Stokes' expansion is applied to the internal waves of finite amplitude, which propagate on the interface between two layers of infinite thickness. Stream function, wave profile, phase velocity and mass transport velocity are given in the fifth order approximation. It is shown that (a) phase velocity increases with increase of wave steepness, (b) mass transport appears in the direction of the wave propagation in both layers as in the case of the surface waves, and (c) when the density difference is very small, the wave profile is flattened not only at the troughs but also at the crests.

1. Introduction

STOKES (1847) studied the properties of irrotational surface water waves of finite amplitude by using the technique of series expansion called Stokes' expansion. Thereafter many efforts have been devoted on this subject by various investigators (RAYLEIGH (1917), BURNSIDE (1916), LEVI CIVITA (1925), DE (1955), Красовский (1960), and so on). The following properties are confirmed for the surface gravitational waves of finite amplitude;

a) Wave velocity increases slightly with increase of wave steepness.
b) Wave profile is sharpened at crests and flattened at troughs.
c) Trajectories of water particles are not closed and water mass is transported in the direction of wave propagation.
d) Waves break at the crest angle of 120° and the highest wave steepness is about 1/7 (MICHELL (1893)).

For the internal waves which propagate on the interface between two homogeneous layers, HUNT (1961) applied Levi Civita's method and obtaind formulae of the wave profile and phase velocity to the third order. He also showed that the phase velocity increases with increase of wave steepness. In this paper, we apply Stokes' expansion to the internal waves for the case that the thickness of two layers is infinite, and calculate stream function, wave profile and phase velocity to the fifth order. It is shown that, when density difference of the two layers is sufficiently small, wave profile is flattened both at crests and troughs. In both sides of interface, water particles shift slightly forward and mass transport exists.

2. Formulation and calculation procedure

We consider finite amplitude internal waves of permanent type, which propagate with phase velocity c on the interface of two layers of inviscid incompressible fluid. We assume that thickness of each layer is infinite and that the density of the lower layer is ρ and that of the upper layer is ρ'. (We denote the quantities of the upper layer with prime hereafter). We consider only gravitational force as the restoring force and assume the motion to be irrotational. We observe the phenomenon from the coordinate system moving horizontally with velocity c. Then the motion is steady. We take the y-axis vertically upwards from the undisturbed interface and the x-axis horizontally to the right from one of the wave crests (see Figure 1).

Denoting stream functions by ψ, and ψ', we have

$$\nabla^2\psi=0 \tag{1-a}$$

$$\nabla^2\psi'=0 \tag{1-b}$$

for the lower and upper fluids, respectively. The kinematic boundary conditions at the

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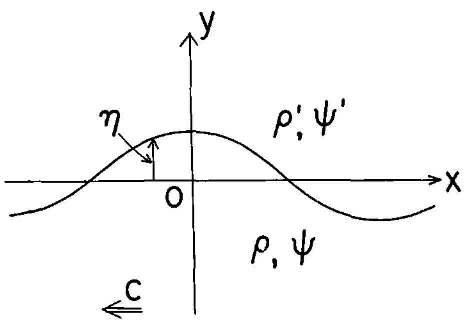

Fig. 1. Coordinate system.

interface are

$$\psi=\psi'\equiv 0 \quad \text{at} \quad y=\eta \qquad (2)$$

where η is the displacement of the interface measured from the undisturbed interface and

$$\int_0^L \eta\, dx=0 \qquad (3)^*$$

where L is the wave length under consideration. The dynamical boundary condition at the interface is

$$\rho\left[\frac{1}{2}\left\{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right\}+gy\right]$$
$$=\rho'\left[\frac{1}{2}\left\{\left(\frac{\partial\psi'}{\partial x}\right)^2+\left(\frac{\partial\psi'}{\partial y}\right)^2\right\}+gy\right]+Q$$
$$\text{at} \quad y=\eta \qquad (4)$$

where Q is the apparent pressure jump at the interface on the moving coordinate and has a constant value of $1/2\, c^2(\rho-\rho')$. Since the wave motions are concentrated near the interface, we have

$$\begin{cases}\psi\to -cy+cK \quad \text{for} \quad y\to -\infty & (5\text{-a})\\ \psi'\to -cy+cK' \quad \text{for} \quad y\to +\infty & (5\text{-b})\end{cases}$$

where K and K' are constants to be determined.

We expand stream functions ψ, ψ' as follows;

$$\frac{\psi}{c}=K-y+\sum_{j=1}^{\infty} a_j e^{jky}\cos jkx \qquad (6\text{-a})$$

$$\frac{\psi'}{c}=K'-y+\sum_{j=1}^{\infty} a'_j e^{-jky}\cos jkx \qquad (6\text{-b})$$

where k is the wave number defined by $k=2\pi/L$. These forms satisfy (1-a, b) and (5-a, b) respectively. We assume that $\{a_n k\}$ and $\{a_n' k\}$ are $O(\varepsilon^n)$ at most, where ε is a small quantity of the order of wave steepness. We determine coefficients K, K', $\{a_n\}$, $\{a_n'\}$, and the phase velocity c so that the boundary conditions (2), (3) and (4) may be satisfied.

By substituting (6-a) in (2), we obtain

$$k\eta=kK+\sum_{j=1}^{\infty} ka_j e^{jk\eta}\cos jkx$$
$$=kK+\sum_{j=1}^{\infty} ka_j\left\{1+jk\eta+\frac{1}{2!}(jk\eta)^2\right.$$
$$\left.+\frac{1}{3!}(jk\eta)^3+\frac{1}{4!}(jk\eta)^4\right\}\cos jkx+O(\varepsilon^6) \qquad (7)$$

We expand $k\eta$ as

$$k\eta=kA_1\cos kx+kA_2\cos 2kx+kA_3\cos 3kx$$
$$+kA_4\cos 4kx+kA_5\cos 5kx+O(\varepsilon^6) \qquad (8)$$

which is substituted into (7). Since ka_n, kA_n are of the order of ε^n for $n=1, 2, 3, \ldots,$ we can determine A_n as function of a_n to the fifth order. In the similar way, we can also represent A_n as the function of $\{a_n'\}$ from (6-b), (2) and (8).

Then, we have

$$\begin{cases} kA_1=ka_1+\frac{3}{2}k^2a_1a_2+\frac{5}{8}k^3a_1^3+\frac{229}{192}k^5a_1^5+\frac{37}{6}k^4a_1^3a_2+\frac{25}{8}k^3a_1^2a_3+\frac{21}{4}k^3a_1a_2^2+\frac{5}{2}k^2a_2a_3\\ \quad =ka_1'-\frac{3}{2}k^2a_1'a_2'+\frac{5}{8}k^3a_1'^3+\frac{229}{192}k^5a_1'^5-\frac{37}{6}k^4a_1'^3a_2'+\frac{25}{8}k^3a_1'^2a_3'+\frac{21}{4}k^3a_1'a_2'^2-\frac{5}{2}k^2a_2'a_3'\\ kA_2=ka_2+\frac{1}{2}k^2a_1^2+\frac{5}{6}k^4a_1^4+3k^3a_1^2a_2+2k^2a_1a_3\\ \quad =ka_2'-\frac{1}{2}k^2a_1'^2-\frac{5}{6}k^4a_1'^4+3k^3a_1'^2a_2'-2k^2a_1'a_3' \end{cases}$$

* This condition is not needed for the present problem. We set this condition for the symmetricity in the form of stream function. See Appendix.

$$
\left\{
\begin{aligned}
kA_3 &= ka_3+\frac{3}{8}k^3a_1{}^3+\frac{3}{2}k^2a_1a_2+\frac{409}{384}k^5a_1{}^5+\frac{89}{16}k^4a_1{}^3a_2+\frac{19}{4}k^3a_1{}^2a_3+\frac{25}{8}k^3a_1a_2{}^2+\frac{5}{2}k^2a_1a_4\\
&= ka_3'+\frac{3}{8}k^3a_1'^3-\frac{3}{2}k^2a_1'a_2'+\frac{409}{384}k^5a_1'^5-\frac{89}{16}k^4a_1'^3a_2'+\frac{19}{4}k^3a_1'^2a_3'+\frac{25}{8}k^3a_1'a_2'^2-\frac{5}{2}k^2a_1'a_4'\\
kA_4 &= ka_4+\frac{1}{3}k^4a_1{}^4+2k^3a_1{}^2a_2+2k^2a_1a_3+k^2a_2{}^2\\
&= ka_4'-\frac{1}{3}k^4a_1'^4+2k^3a_1'^2a_2'-2k^2a_1'a_3'-k^2a_2'^2\\
kA_5 &= ka_5+\frac{125}{384}k^5a_1{}^5+\frac{125}{48}k^4a_1{}^3a_2+\frac{25}{8}k^5a_1{}^2a_3+\frac{25}{8}k^3a_1a_2{}^2+\frac{5}{2}k^2a_1a_4+\frac{5}{2}k^2a_2a_3\\
&= ka_5'+\frac{125}{384}k^5a_1'^5-\frac{125}{48}k^4a_1'^3a_2'+\frac{25}{8}k^3a_1'^2a_3'+\frac{25}{8}k^3a_1'a_2'^2-\frac{5}{2}k^2a_1'a_4'-\frac{5}{2}k^2a_2'a_3'
\end{aligned}
\right. \tag{9}
$$

From the condition (3), K and K' must be expressed as follows:

$$
\left\{
\begin{aligned}
kK &= -\frac{1}{2}k^2a_1{}^2-k^2a_2{}^2-2k^3a_1{}^2a_2-\frac{1}{2}k^4a_1{}^4\\
kK' &= \frac{1}{2}k^2a_1'^2+k^2a_2'^2-2k^3a_1'^2a_2'+\frac{1}{2}k^4a_1'^4
\end{aligned}
\right. \tag{10}
$$

After expanding $\{e^{\pm jk\eta}\}$ and by considering that $a_nk,\ a_n'k=O(\varepsilon^n)$, we substitute (6-a, b) in (4) to obtain the dynamical boundary condition at the interface to the fifth order;

$$
\begin{aligned}
&\{BA_1-\rho F_1(a_1, a_2, a_3, a_4, a_5)+\rho' F_1(-a_1', -a_2', -a_3', -a_4', -a_5')\}\cos kx\\
&+\{BA_2-\rho F_2(a_1, a_2, a_3, a_4, a_5)+\rho' F_2(-a_1', -a_2', -a_3', -a_4', -a_5')\}\cos 2kx\\
&+\{BA_3-\rho F_3(a_1, a_2, a_3, a_4, a_5)+\rho' F_3(-a_1', -a_2', -a_3', -a_4', -a_5')\}\cos 3kx\\
&+\{BA_4-\rho F_4(a_1, a_2, a_3, a_4, a_5)+\rho' F_4(-a_1', -a_2', -a_3', -a_4', -a_5')\}\cos 4kx\\
&+\{BA_5-\rho F_5(a_1, a_2, a_3, a_4, a_5)+\rho' F_5(-a_1', -a_2', -a_3', -a_4', -a_5')\}\cos 5kx=0
\end{aligned} \tag{11}
$$

where

$$
B=\frac{2g}{c^2}(\rho-\rho') \tag{12}
$$

and

$$
\left\{
\begin{aligned}
&F_1(a_1, a_2, a_3, a_4, a_5)\\
&\quad =2ka_1+k^2a_1a_2-\frac{3}{4}k^3a_1{}^3+k^2a_2a_3-\frac{3}{2}k^3a_1a_2{}^2+\frac{7}{4}k^3a_1{}^2a_3-\frac{23}{6}k^4a_1{}^3a_2-\frac{83}{96}k^5a_1{}^5\\
&F_2(a_1, a_2, a_3, a_4, a_5)\\
&\quad =k^2a_1{}^2+4ka_2+4k^2a_1a_3-\frac{1}{3}k^4a_1{}^4\\
&F_3(a_1, a_2, a_3, a_4, a_5)\\
&\quad =6ka_3+5k^2a_1a_2+\frac{3}{4}k^3a_1{}^3+9k^2a_1a_2+\frac{21}{4}k^3a_1a_2{}^2+\frac{9}{2}k^3a_1{}^2a_3+\frac{15}{8}k^4a_1{}^3a_2+\frac{3}{64}k^5a_1{}^5\\
&F_4(a_1, a_2, a_3, a_4, a_5)\\
&\quad =8ka_4+4k^2a_2{}^2+10k^2a_1a_3+6k^3a_1{}^2a_2+\frac{2}{3}k^4a_1{}^4
\end{aligned}
\right.
$$

$$\begin{cases} F_5(a_1, a_2, a_3, a_4, a_5) \\ \quad =10ka_5+13k^2a_2a_3+17k^2a_1a_4+\dfrac{45}{4}k^3a_1a_2^2+\dfrac{55}{4}k^3a_1^2a_3+\dfrac{175}{24}k^4a_1^3a_2+\dfrac{125}{192}k^5a_1^5 \end{cases} \tag{13}$$

Since (11) should be satisfied for any value of x, each of the five brackets in (11) must vanish, so that we have five equations.

3. Wave profile, stream function and phase velocity to the fifth order

Now, we have fifteen equations (ten from (9) and five from (11)) for sixteen unknowns; $A_1, A_2, \dots, A_5, a_1, a_2, \dots, a_5, a_1', a_2', \dots, a_5'$, and c. So we can arbitrarily select one of these unknowns, and then the rest can be represented as its function*. By selecting $A=A_1$ as the arbitrary coefficient, we obtain the wave profile η as;

$$k\eta=\sum_{n=1}^{5} kA_n \cos nkx+O(\varepsilon^6) \tag{14}$$

where

$$\begin{cases} kA_1=kA \\ kA_2=\dfrac{1}{2}\dfrac{\rho-\rho'}{\rho+\rho'}\left(1+\dfrac{17\rho^2-38\rho\rho'+17\rho'^2}{12(\rho+\rho')^2}k^2A^2\right)k^2A^2 \\ kA_3=\dfrac{3\rho^2-10\rho\rho'+3\rho'^2}{8(\rho+\rho')^2}k^3A^3+\dfrac{459\rho^4-2468\rho^3\rho'+4130\rho^2\rho'^2-2468\rho\rho'^3+459\rho'^4}{384(\rho+\rho')^4}k^5A^5 \\ kA_4=\dfrac{(\rho-\rho')(\rho^2-6\rho\rho'+\rho'^2)}{3(\rho+\rho')^3}k^4A^4 \\ kA_5=\dfrac{125\rho^4-1516\rho^3\rho'+3118\rho^2\rho'^2-1516\rho\rho'^3+125\rho'^4}{384(\rho+\rho')^4}k^5A^5 \end{cases} \tag{15}$$

The stream functions ψ, ψ' are given by

$$\begin{cases} \dfrac{\psi}{c}=K-y+\sum\limits_{n=1}^{5} a_n e^{nky}\cos nkx+O(\varepsilon^6) \\ \dfrac{\psi'}{c}=K'-y+\sum\limits_{n=1}^{5} a_n' e^{-nky}\cos nkx+O(\varepsilon^6) \end{cases} \tag{16}$$

where

$$\begin{aligned} kK&=-\frac{1}{2}k^2A^2+\frac{\rho^2+6\rho\rho'-3\rho'^2}{8(\rho+\rho')^2}k^4A^4 \\ kK'&=\frac{1}{2}k^2A^2+\frac{3\rho^2-6\rho\rho'-\rho'^2}{8(\rho+\rho')^2}k^4A^4 \end{aligned} \tag{17}$$

and

$$\begin{cases} ka_1=kA+\dfrac{-5\rho+7\rho'}{8(\rho+\rho')}k^3A^3+\dfrac{-37\rho^3+218\rho^2\rho'-317\rho\rho'^2+100\rho'^3}{48(\rho+\rho')^3}k^5A^5 \end{cases}$$

* Calculations are not straightforward. We expand all coefficient to the fifth order and determine all terms successively from lower to higher orders.

$$
\left\{
\begin{aligned}
ka_1' &= kA + \frac{7\rho - 5\rho'}{8(\rho+\rho')}k^3A^3 + \frac{100\rho^3 - 317\rho^2\rho' + 218\rho\rho'^2 - 37\rho'^3}{48(\rho+\rho')^3}k^5A^5 \\
ka_2 &= -\frac{\rho'}{\rho+\rho'}k^2A^2 + \frac{6\rho^3 - 5\rho^2\rho' + 32\rho\rho'^2 - 29\rho'^3}{12(\rho+\rho')^3}k^4A^4 \\
ka_2' &= \frac{\rho}{\rho+\rho'}k^2A^2 + \frac{29\rho^3 - 32\rho^2\rho' + 5\rho\rho'^2 - 6\rho'^3}{12(\rho+\rho')^3}k^4A^4 \\
ka_3 &= -\frac{\rho\rho' - 3\rho'^2}{2(\rho+\rho')^2}k^3A^3 + \frac{4\rho^4 - 89\rho^3\rho' + 277\rho^2\rho'^2 - 587\rho\rho'^3 + 291\rho'^4}{48(\rho+\rho')^4}k^5A^5 \\
ka_3' &= \frac{3\rho^2 - \rho\rho'}{2(\rho+\rho')^2}k^3A^3 + \frac{291\rho^4 - 587\rho^3\rho' + 277\rho^2\rho'^2 - 89\rho\rho'^3 + 4\rho'^4}{48(\rho+\rho')^4}k^5A^5 \\
ka_4 &= -\frac{\rho'(\rho^2 - 7\rho\rho' + 8\rho'^2)}{3(\rho+\rho')^3}k^4A^4 \\
ka_4' &= \frac{\rho(8\rho^2 - 7\rho\rho' + \rho'^2)}{3(\rho+\rho')^3}k^4A^4 \\
ka_5 &= -\frac{\rho'(6\rho^2 - 73\rho^2\rho' + 196\rho\rho'^2 - 125\rho'^3)}{24(\rho+\rho')^4}k^5A^5 \\
ka_5' &= \frac{\rho(125\rho^3 - 196\rho^2\rho' + 73\rho\rho'^2 - 6\rho'^3)}{24(\rho+\rho')^4}k^5A^5
\end{aligned}
\right.
\tag{18}
$$

The phase velocity c is given as

$$
c^2 = \frac{g}{k}\,\frac{\rho-\rho'}{\rho+\rho'}\left\{1 + \frac{\rho^2+\rho'^2}{(\rho+\rho')^2}k^2A^2 + \frac{(\rho-\rho')^2(5\rho^2 - 14\rho\rho' + 5\rho'^2)}{4(\rho+\rho')^4}k^4A^4\right\} \tag{19}
$$

If we use the wave steepness $\delta = \frac{k}{2\pi}(2A_1 + 2A_3 + 2A_5)$, we have

$$
c^2 = \frac{g}{k}\,\frac{\rho-\rho'}{\rho+\rho'}\left\{1 + \frac{\rho^2+\rho'^2}{(\rho+\rho')^2}\pi^2\delta^2 + \frac{\rho^4 - 7\rho^3\rho' + 16\rho^2\rho'^2 - 7\rho\rho'^3 + \rho'^4}{2(\rho+\rho')^4}\pi^4\delta^4\right\} \tag{20}
$$

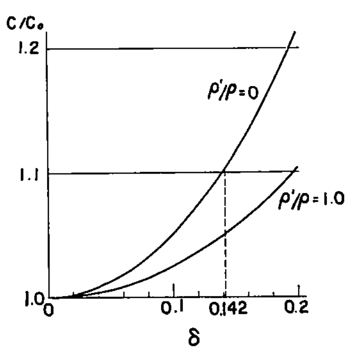

Fig. 2. The dependence of the phase velocity c on wave steepness. Two solid lines indicate two extreme cases that $\rho'/\rho=0$ and $\rho'/\rho=1$. In general, the phase velocity of the internal waves lies between these two. The phase velocity is normalized by $c_0=\sqrt{\frac{g}{k}\frac{\rho-\rho'}{\rho+\rho'}}$ which is the phase velocity for the infinitesimally small waves. $\delta=0.142$ corresponds the limiting steepness for the surface waves.

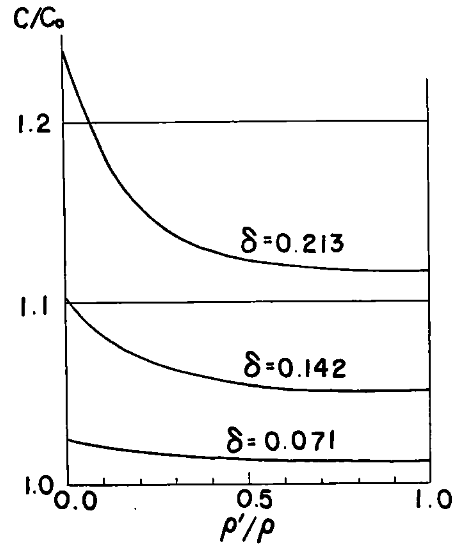

Fig. 3. The dependence of the phase velocity c on density ratio ρ'/ρ for the cases that $\delta=0.142$, and $\delta=0.071$.

The third term in the right hand side of (20) is positive for $0 \leqq \frac{\rho-\rho'}{\rho} \leqq 1$. Therefore the phase velocity of the internal wave always increases with increase of wave steepness. The dependence of the phase velocity c on wave steepness δ and density ratio ρ'/ρ is illustrated in Figures 2 and 3, respectively.

By application of Levi Civita's method to the internal finite amplitude wave, HUNT (1961) obtained formulae of wave profile and phase velocity to the third order. His kinematic boundary condition at the interface is not exact: he considered that the value of slope of interface represented as a function of velocity potential in the upper layer was equal to that in the lower layer for the same value of velocity potential. However, for the finite amplitude wave, velocity potential is not continuous at the interface except just at crests and at troughs. In spite of this, his results agree exactly with our results to the third order. The influence of his approximation will appear in higher orders.

If we put $\rho'=0$ in (15), (17), (18) and (19), we get the formulae of wave profile, stream function and phase velocity for the surface Stokes waves. As to the stream function, the resulted form differs from that given by previous authors (e.g. RAYLEIGH (1917), KINSMAN (1965)) due to the difference in the defined form of the stream function (see Appendix). It should be noted that, for the surface waves, each of the first terms of ka_2, ka_3, ka_4 and ka_5 turn out to be zero, and that ka_2 is of order of ε^4 and ka_3 is order of ε^5. Therefore for the surface wave correction of the stream function is needed only when calculations proceed to the fourth order. For the internal wave ka_n and ka_n' are always of order of ε^n and we need to correct the stream functions from the second order calculations.

In Figure 4, wave profiles for $\delta=0.14$ are shown for two cases of $\rho'/\rho=0$ and (i.e. surface wave) and $\rho'/\rho=0.9999$. The wave profile for $\rho'/\rho=0.9999$ with $\delta=0.14$ is compared with the sinusoidal wave $A\cos kx$ in Figure 5. We can recognize that the wave profile has the tendency to be flattened both at crests and troughs. When the density difference is very small ($\rho'\to\rho$), kA_2 and kA_4 in (14) tend to zero and may be neglected, while kA_3 becomes negative tending to $-\frac{1}{8}k^3A^3\left(1-\frac{7}{48}k^2A^2\right)$. Therefore, when the density difference is small, we cannot discuss the maximum steepness of the internal waves on the analogy of the surface Stokes waves, for which the wave crests are always sharpened with increase of wave steepness and the waves break at the crest angle of 120°. For the internal wave, the maximum value of wave steepness may be limited by shear instability at the interface, rather than the breaking condition at the crests. Moreover, neither the existence of the solution nor the convergency of the series are proved for the internal waves, while the existence of solution for surface waves has been shown if the maximum angle of inclination of wave profile is less than 30° (Красовский 1960)).

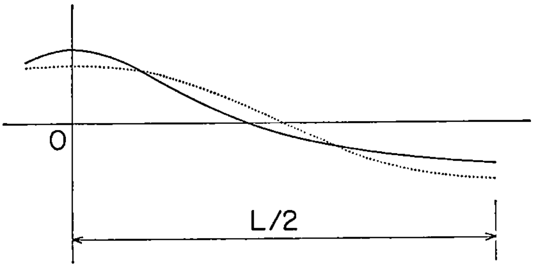

Fig. 4. Wave profiles of finite amplitude internal waves: solid line for $\rho'/\rho=0$ and $\delta=0.14$ and dotted line for $\rho'/\rho=0.9999$ and $\delta=0.14$.

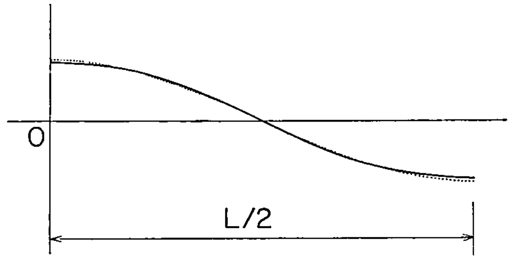

Fig. 5. Comparison of wave profile for $\rho'/\rho=0.9999$ and $\delta=0.14$ (full line) with a sinusoidal curve (dotted line).

4. Mass transport

The constant terms K and K' in (6) indicate that, due to the wave motion, the stationary mass transports are induced above and below the interface. The total transport M in the lower layer is;

$$M=-Kc$$
$$=\left\{\frac{1}{2}k^2A^2+\frac{-\rho^2-6\rho\rho'+3\rho'^2}{8(\rho+\rho')^2}k^4A^4\right\}c \quad \text{(20-a)}$$

and that M' in the upper layer is;

$$M'=K'c$$
$$=\left\{\frac{1}{2}k^2A^2+\frac{3\rho^2-6\rho\rho'-\rho'^2}{8(\rho+\rho')^2}k^4A^4\right\}c \quad \text{(20-b)}$$

Next, we will calculate the vertical distribution of mass transport velocity u^*. Consider a stream line whose value is $\bar{\psi}$. By substituting $\psi=\bar{\psi}$ in (6-a) and putting $\frac{\bar{\psi}}{c}+y-K=\zeta$, we have

$$\zeta=\sum_{j=1}^{\infty} a_j\, e^{jk(\zeta+K-\bar{\psi}/c)}\cos jkx$$
$$=\sum_{j=1}^{\infty} b_j\, e^{jk\zeta}\cos jkx \quad (21)$$

where

$$b_j=a_j\, e^{jk(K-\bar{\psi}/c)} \qquad (j=1,2,\dots)$$

If we solve ζ as the explicit function of x, and take the average for x, we have

$$k\bar{\zeta}=\frac{1}{2}k^2b_1{}^2+k^4b_1{}^4+2k^3b_1{}^2b_2+k^2b_2{}^2+O(\varepsilon^6)$$

which can be written in the original notations as

$$\bar{y}+\frac{\bar{\psi}}{c}-K=\frac{1}{2}ka_1{}^2\, e^{2k(K-\bar{\psi}/c)}$$
$$+(k^3a_1{}^4+2k^2a_1{}^2a_2+ka_2{}^2)\, e^{4k(K-\bar{\psi}/c)}+O(\varepsilon^6) \quad (22)$$

By the method of iteration, we obtain $\bar{\psi}$ as a form of explicit function of $\bar{y}$.

$$\frac{\bar{\psi}}{c}=K-y+\frac{1}{2}ka_1{}^2\exp\left\{2k\left(\bar{y}-\frac{1}{2}ka_1{}^2e^{2k\bar{y}}\right)\right\}$$
$$+(k^3a_1{}^4+2k^2a_1{}^2a_2+ka_2{}^2)e^{4k\bar{y}}+O(\varepsilon^6)$$
$$=K-\bar{y}+\frac{1}{2}ka_1{}^2e^{2k\bar{y}}$$
$$+\left(\frac{1}{2}k^3a_1{}^4+2k^2a_1{}^2a_2+ka_2{}^2\right)e^{4k\bar{y}}+O(\varepsilon^6) \quad (23)$$

By substituting (18) in (23) and differentiating (23) with respect to $\bar{y}$, we obtain the mass transport velocity u^* for the lower layer. By the same way, we can calculate u^* for the upper layer. The mass transport velocities in the fixed coordinate are;

$$u^*(\bar{y})=c\left[\left\{1+\frac{-5\rho+7\rho'}{4(\rho+\rho')}k^2A^2\right\}k^2A^2e^{2k\bar{y}}\right.$$
$$\left.+2\frac{\rho^2-2\rho\rho'-\rho'^2}{(\rho+\rho')^2}k^4A^4e^{4k\bar{y}}\right] \text{ for } \bar{y}\leqq 0. \quad \text{(24-a)}$$

$$u^*(\bar{y})=c\left[\left\{1+\frac{7\rho-5\rho'}{4(\rho+\rho')}k^2A^2\right\}k^2A^2\, e^{-2k\bar{y}}\right.$$
$$\left.+2\frac{-\rho^2-2\rho\rho'+\rho'^2}{(\rho+\rho')^2}k^4A^4e^{-4k\bar{y}}\right] \text{ for } \bar{y}\geqq 0 \quad \text{(24-b)}$$

By integrating (24-a, b) from 0 to $\pm\infty$ in each layer, we obtain

$$\int_{-\infty}^{0}u^*d\bar{y}=-Kc \quad\text{and}\quad \int_{0}^{\infty}u^*d\bar{y}=K'c$$

The mass transport velocity can be derived also by integrating orbital velocity along the stream lines. In Figure 6, we show an example of the orbits of water particles just above and just below the interface for the case $\rho'/\rho=1/1.1$ and $\delta=0.1$.

If we put $\rho'=0$ in (24-a), we can obtain the formula of mass transport for surface waves

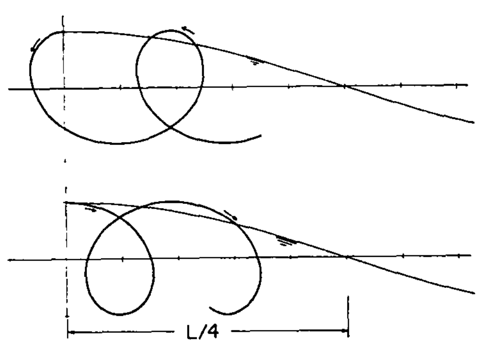

Fig. 6. Example of the orbits of water particles just above (upper figure) and just below (lower figure) the interface for $\rho'/\rho=1/1.1$ and $\delta=0.1$. The wave profile is shown by thin line.

to the forth order. In Figure 7, relationship between the wave steepness δ and the mass transport velocity of the water particle just at the surface is shown. The dotted and the solid lines in Figure 7 show the second and forth order approximations given by (24-a). Small circles in Figure 7 show the mass transport velocity given by K. SASAKI and T. MURAKAMI (1973) who conducted numerical integration of Levi Civita's equation for finite amplitude surface waves. The obtained mass transport velocity at the surface level in the fourth approximation shows good agreement with that obtained by K. SASAKI and T. MURAKAMI.

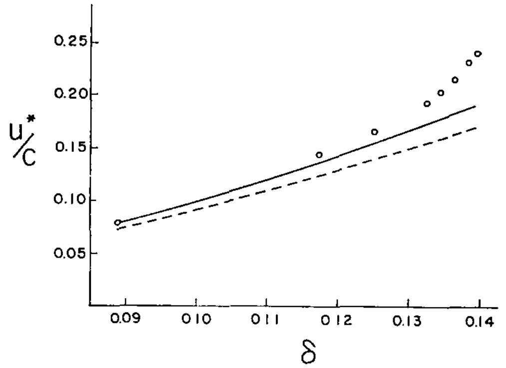

Fig. 7. Relationship between wave steepness and mass transport velocity of the surface waves at the surface level. Dashed line shows the second order approximation and full line shows the forth order approximation. Small circles show the values given by K. SASAKI and T. MURAKAMI (1973).

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Appendix

The expanded form of the stream function and mass transport velocity in the coordinate system in which x-axis is shifted up by K.

In the usual calculation of the surface Stokes waves (e.g. see KINSMAN (1965)), the stream function is assumed to have the form

$$\frac{\psi}{c}=-y+\sum_{j=1}^{\infty} a_j e^{jky} \cos jkx \qquad \text{(A-1)}$$

and the boundary condition for $y\to-\infty$ is set as

$$\psi\to-cy \qquad \text{(A-2)}$$

In this case,

$$\int_0^L \eta dx \neq 0 \qquad \text{(A-3)}$$

and the elevation of the mean surface level is interpreted afterward as the representation of mass transport. The only difference from our formulation is a vertical shift of the coordinate system, so that the expanded form of the profile and phase velocity are not affected. However, the expanded form of the stream function and mass transport velocity are considerably changed in higher order terms because of the nonlinear form of (6-a) and (A-1).

In order to see the correspondence to the surface Stokes waves, we can formulate

$$\begin{cases} \dfrac{\psi}{c}=-y+\sum\limits_{j=1}^{\infty} a_j e^{jky} \cos jkx \\ \dfrac{\psi'}{c}=K'-y+\sum\limits_{j=1}^{\infty} a_j e^{-jky} \cos jkx \end{cases} \qquad \text{(A-4)}$$

and

$$\begin{aligned} &\psi\to-cy && \text{for } y\to-\infty \\ &\psi'\to-cy+cK' && \text{for } y\to+\infty \end{aligned} \qquad \text{(A-5)}$$

Then we have the coefficients of expanded stream function as follows:

$$kK'=k^2A^2+\frac{\rho^2-6\rho\rho'+\rho'^2}{4(\rho+\rho')^2}k^4A^4$$

$$ka_1=kA+\frac{-9\rho+3\rho'}{8(\rho+\rho')}k^3A^3 +\frac{-10\rho^3+287\rho^2\rho'-308\rho\rho'^2+67\rho'^3}{48(\rho+\rho')^3}k^5A^5$$

$$ka_1'=kA+\frac{3\rho-9\rho'}{8(\rho+\rho')}k^3A^3 +\frac{67\rho^3-308\rho^2\rho'+287\rho\rho'^2-10\rho'^3}{48(\rho+\rho')^3}k^5A^5$$

$$ka_2 = -\frac{\rho'}{\rho+\rho'}k^2A^2 + \frac{6\rho^3+7\rho^2\rho'+56\rho\rho'^2-17\rho'^3}{12(\rho+\rho')^3}k^4A^4$$

$$ka_2' = \frac{\rho}{\rho+\rho'}k^2A^2 + \frac{17\rho^3-56\rho^2\rho'-7\rho\rho'^2-6\rho'^3}{12(\rho+\rho')^3}k^4A^4$$

$$ka_3 = \frac{-\rho'\rho+3\rho'^2}{2(\rho+\rho')^2}k^3A^3 + \frac{4\rho^4-53\rho^3\rho'+241\rho^2\rho'^2-767\rho\rho'^3+183\rho'^4}{48(\rho+\rho')^4}k^5A^5$$

$$ka_3' = \frac{3\rho^2-\rho\rho'}{2(\rho+\rho')^2}k^3A^3 + \frac{183\rho^4-767\rho^3\rho'+241\rho^2\rho'^2-53\rho\rho'^3+4\rho'^4}{48(\rho+\rho')^4}k^5A^5$$

$$ka_4 = -\frac{\rho'(\rho^2-7\rho\rho'+8\rho'^2)}{3(\rho+\rho')^3}k^4A^4$$

$$ka_4' = \frac{\rho(8\rho^2-7\rho\rho'+\rho'^2)}{3(\rho+\rho')^3}k^4A^4$$

$$ka_5 = -\frac{\rho'(6\rho^3-73\rho^2\rho'+196\rho\rho'^2-125\rho'^3)}{24(\rho+\rho')^4}k^5A^5$$

$$ka_5' = \frac{\rho(125\rho^3-196\rho^2\rho'+73\rho\rho'^2-6\rho'^3)}{24(\rho+\rho')^4}k^5A^5$$

And mass transport velocity in the fixed coordinate is given by

$$\bar{u^*(y)} = c\left[\left\{1+\frac{-9\rho+3\rho'}{4(\rho+\rho')}k^2A^2\right\}k^2A^2\,e^{2k\bar y} + \frac{-2\rho^2-4\rho\rho'+2\rho'^2}{(\rho+\rho')^2}k^4A^4\,e^{4k\bar y}\right]$$
$$\text{for } \bar y \leqq \frac{1}{2}kA^2 + \frac{\rho^2+6\rho\rho'-3\rho'^2}{8(\rho+\rho')^2}k^3A^4$$

$$u^*(\bar y) = c\left[\left\{1+\frac{11\rho-\rho'}{4(\rho+\rho')}k^2A^2\right\}k^2A^2e^{-2k\bar y} + \frac{2\rho^2-4\rho\rho'-2\rho'^2}{(\rho+\rho')^2}k^4A^4\,e^{-4k\bar y}\right]$$
$$\text{for } \bar y \geqq \frac{1}{2}kA^2 + \frac{\rho^2+6\rho\rho'-3\rho'^2}{8(\rho+\rho')^2}k^3A^4$$

These forms can be derived also by replacing each coefficient a_jk in (18) and (25) by a_jke^{-jkK} and by expanding the terms to the fifth order.

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内部深水波のストークス型の展開

都 司 嘉 宣, 永 田 豊

要旨: 理想流体の非回転運動における有限振幅表面波の理論として，ストークス波の理論がある．ここではそれぞれ無限の厚さをもつ密度の異なる二層の境界面を伝わる有限振幅の内部波についてストークス波と同様の展開を行ない，波形，波速，質量輸送等について第５次近似までを求めた．波の位相速度は表面波の場合と同様に波形勾配と共に増大すること，上下各層に，波の進む方向と同じ向きの質量輸送を伴うことが示される．表面波の場合と異なり上下層の密度差の小さい時には，波形は峰の所でも谷の所と同様に平らになる傾向をもつ．