Fluid Dynamics Research 12 (1993) 281–291
North-Holland

FLUID DYNAMICS
RESEARCH

Competition between dispersion and nonlinearity in deep water waves

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Received 10 December 1992
Accepted 11 May 1993

Abstract. A simple wave system consisting of a fundamental wave and its second harmonic has been studied for infinitely deep water waves. Two competing effects of the waves, nonlinearity and dispersion, were investigated. These two effects make up the behavior of the second harmonic complex. From a stability consideration in phase space it was clarified that trajectories displaced with a small amount deviate from the original ones. The asymptotic aspect of the Stokes wavetrain, which is the steady wave solution, was also discussed.

1. Introduction

Surface waves traveling on infinitely deep water are strongly dispersive in comparison with shallow water ones. An infinitesimal wave of the angular frequency ω and the wavenumber k must satisfy the well-known dispersion relation

$$\omega = \sqrt{k}, \tag{1}$$

in a properly nondimensionalized unit. This wave travels with the phase velocity

$$c = \omega/k = 1/\sqrt{k}. \tag{2}$$

Accordingly, the second harmonic of the wavenumber $2k$ and frequency 2ω travels with $1/\sqrt{2}$ times slower velocity than the fundamental wave. It is clear that the wavenumber and frequency cannot satisfy the dispersion relation (1). This implies that the second harmonic does not have larger amplitude than the fundamental wave. On the other hand, nonlinearity generates the harmonic. It is thus expected that generation due to the nonlinear effect and the suppression effect based on the dispersion relation compete with each other and give rise to the complex behavior of the wave. One of the objectives of the present paper is to clarify the behavior.

It is well known that the Stokes wavetrain is the steady wavetrain for deep water waves (Stokes, 1847; see also Kinsman, 1965). This uniform wavetrain becomes unstable when a resonant condition is satisfied for four specially selected waves (Benjamin and Feir, 1967; see also Phillips, 1974). This is called the Benjamin–Feir instability, which arises from the third order nonlinearity. However, the second harmonic is generated by the lowest order nonlinearity, i.e., the second order one. *Although our equations derived in the present paper include not only the second order term but also the third order ones, our main concern is not the instability caused by the third order terms.* For this reason we omit the four-wave interaction leading to the instability and confine ourselves to

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0169-5983/93/$03.75

the discussion of a fundamental wave and its harmonic. In such a situation the Stokes wavetrain retains its initial wave form if the initial wave is the Stokes one. However, for given initial conditions far from the Stokes wave, our knowledge of how the wave deforms is poor. The second objective is to investigate the development of the generated second harmonic.

2. Derivation of interaction equations

The equation governing the irrotational flow of an incompressible inviscid fluid is written in terms of the Cartesian coordinates (x, y):

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{3}$$

where ϕ is the velocity potential. The coordinate x is measured horizontally to the right and y vertically upwards. At the free surface, given by $y = \xi(x, t)$, the velocity potential must satisfy the boundary conditions:

$$\frac{\partial \phi}{\partial y} = \frac{\partial \xi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \xi}{\partial x}, \tag{4}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] + \xi = 0, \tag{5}$$

where t is time; all the quantities have been normalized in terms of a characteristic length L, which will be specified later, and a characteristic velocity $\sqrt{gL}$, g being the acceleration due to gravity. For infinitely deep water, the remaining boundary condition is that the velocity decreases to zero as $y \to -\infty$.

In the present paper, we investigate the generation and suppression of the second harmonic. To derive coupled equations for a wave of the fundamental frequency and its second harmonic, we expand the dependent variables ϕ and ξ as

$$\phi(x, y, t) = \sum_{n=0}^{\infty} a_n(t)\, \mathrm{e}^{\mathrm{i}nkx}\, \mathrm{e}^{nky} + \mathrm{CC}, \tag{6}$$

$$\xi(x, t) = \sum_{n=1}^{\infty} X_n(t)\, \mathrm{e}^{\mathrm{i}nkx} + \mathrm{CC}, \tag{7}$$

where CC stands for the complex conjugate to the preceding expression and a_n and X_n are complex functions of time. Each term of the right-hand side of eq. (6) satisfies the Laplace equation (3). Inserting the expansions (6) and (7) into eqs. (4) and (5) and arranging in $\mathrm{e}^{\mathrm{i}nkx}$, we have, from eq. (4),

$$\mathrm{e}^{\mathrm{i}kx}\!: \quad ka_1 = \frac{\mathrm{d}X_1}{\mathrm{d}t} + \tfrac{1}{2}k^3 a_1^* X_1^2 + k^2 X_2 a_1^* - 2k^2 a_2 X_1^* - k^3 |X_1|^2 a_1, \tag{8}$$

$$\mathrm{e}^{2\mathrm{i}kx}\!: \quad 2ka_2 = \frac{\mathrm{d}X_2}{\mathrm{d}t} - 2k^2 a_1 X_1, \tag{9}$$

and from eq. (5),

$$\mathrm{e}^{0\mathrm{i}kx}\!: \quad \frac{\mathrm{d}a_0}{\mathrm{d}t} + k\left(\frac{\mathrm{d}a_1}{\mathrm{d}t}X_1^* + \frac{\mathrm{d}a_1^*}{\mathrm{d}t}X_1\right) + 2k^2|a_1|^2 = 0, \tag{10}$$

$$\mathrm{e}^{\mathrm{i}kx}\colon\quad (1 + k^2|X_1|^2)\frac{\mathrm{d}a_1}{\mathrm{d}t} + (X_2 + \tfrac{1}{2}kX_1^2)k\frac{\mathrm{d}a_1^*}{\mathrm{d}t} + 2kX_1^*\frac{\mathrm{d}a_2}{\mathrm{d}t}$$

$$+ 4k^3|a_1|^2X_1 + 4k^2a_2a_1^* + X_1 = 0, \tag{11}$$

$$\mathrm{e}^{2\mathrm{i}kx}\colon\quad k\frac{\mathrm{d}a_1}{\mathrm{d}t}X_1 + \frac{\mathrm{d}a_2}{\mathrm{d}t} + X_2 = 0, \tag{12}$$

where the asterisk denotes the complex conjugate.

In a situation where no resonance occurs, it may be plausible to assume that the higher harmonics have smaller amplitude. It is thus predicted that when the fundamental wave is of order ε in magnitude, the generated second harmonic is estimated to be of order ε^2. In the above five equations we have omitted the terms of the order higher than ε^4. These equations yield a closed set for the present problem whenever higher harmonics have smaller amplitude than lower ones. Elimination of a_1 and a_2 gives

$$\frac{\mathrm{d}^2X_1}{\mathrm{d}t^2} + kX_1 = k^3|X_1|^2X_1 + k^2X_1^*\left(\frac{\mathrm{d}X_1}{\mathrm{d}t}\right)^2 - 2k\frac{\mathrm{d}X_2}{\mathrm{d}t}\frac{\mathrm{d}X_1^*}{\mathrm{d}t}$$

$$+ 2k^2X_2X_1^* - 2k^2\frac{\mathrm{d}X_1}{\mathrm{d}t}\frac{\mathrm{d}X_1^*}{\mathrm{d}t}X_1, \tag{13}$$

$$\frac{\mathrm{d}^2X_2}{\mathrm{d}t^2} + 2kX_2 = -2k^2X_1^2. \tag{14}$$

Equation (13) can be rewritten in a simpler form by supposing that the fundamental wave X_1 travels only in the positive x-direction. Then the equation for X_1 is simplified as

$$\frac{\mathrm{d}^2X_1}{\mathrm{d}t^2} + kX_1 = -2k^3|X_1|^2X_1 + 2k^2X_2X_1^* - 2\mathrm{i}k^{3/2}\frac{\mathrm{d}X_2}{\mathrm{d}t}X_1^*. \tag{15}$$

The set of eqs. (13) and (14) or of eqs. (15) and (14) is our desired set of equations from which we can draw the two competing effects, namely nonlinearity and dispersion. Equation (10) is regarded as the relation which determines a_0 with a_1 and X_1 determined.

The coupled equations derived above are somewhat simpler than the original one but still difficult to handle analyticaly. Thus, we try to solve them numerically. However, before carrying out the computation which demonstrates the competing effects, we first derive a simplest wave solution corresponding to the steady wave solution to validate our formulation. The solution is easily obtained by letting

$$X_1 = A_1\mathrm{e}^{-\mathrm{i}\omega t}, \tag{16}$$

$$X_2 = A_2\mathrm{e}^{-2\mathrm{i}\omega t}, \tag{17}$$

where A_1 and A_2 are complex constants. From eqs. (13) and (14) or eqs. (15) and (14), it is easily seen that the above forms are valid only when the following relations are satisfied:

$$\omega^2 = k + 4k^3|A_1|^2, \tag{18}$$

$$A_2 = kA_1^2. \tag{19}$$

These relations coincide with those of the Stokes wavetrain (see, for example, Kinsman, 1965).

3. Two competing effects

The dominant behavior may be described by the following equations:

$$\frac{d^2X_1}{dt^2} + kX_1 = 0, \tag{20}$$

$$\frac{d^2X_2}{dt^2} + 2kX_2 = -2k^2X_1^2. \tag{21}$$

In demonstrating the competing effect, it is useful to choose the initial values of the second harmonic such that $X_2 = dX_2/dt = 0$ when the fundamental wave exists, because this initial state is far from the Stokes wavetrain. As time passes the second harmonic is generated immediately by the fundamental wave. The generation is, however, suppressed by the strongly dispersive nature of deep water waves, which corresponds to the difference between the natural frequency of the harmonic $\sqrt{2k}$ and the forced frequency $2\sqrt{k}$. Hence the energy of the harmonic decreases. After that, it is clear that the energy increases again. Such a process is repeated quasi-periodically. The energy $X_2X_2^*$ has three different radian frequencies, $2\sqrt{2k}$ and $2\sqrt{k} \pm \sqrt{2k}$. This process is a linear one, but somewhat complex because the energy varies quasi-periodically. However, in a more exact formulation, there exists a nonlinear feedback loop between X_1 and X_2 through the coupling terms appearing in eq. (13) or (15). These nonlinear terms give rise to a slow change of the amplitude of the fundamental wave. This is the change that makes the behavior remarkably complex. It is difficult to investigate such a complex behavior in an analytical manner.

In order to show the influence of the wave amplitude on this system explicitly, we change the two dependent variables X_n ($n = 1$ or 2) into $\tilde{X}_n$ such that

$$X_n = \varepsilon\tilde{X}_n, \tag{22}$$

where ε is a measure of smallness of the amplitude. In terms of the new variables $\tilde{X}_n$s eqs. (13)–(15) are rewritten. In what follows, we will neglect the tilde for simplicity.

Next, we solve initial value problems numerically to obtain an understanding of the two competing effects. We choose the initial values for a_1 and b_1 as

$$a_1 = 0.5, \quad b_1 = 0,$$

where a_n and b_n ($n = 1$ or 2) are the real and imaginary parts of X_n, respectively. The second harmonic was assumed to be absent initially, and the derivatives for a_n and b_n were determined using the linear relation for the wave traveling in the positive x-direction,

$$\frac{dX_n}{dt} = -i\sqrt{nk}\,X_n. \tag{23}$$

We solved eight ordinary differential equations of order unity for real dependent variables instead of the two equations of the second order for the complex variables. The set of equations is briefly written as

$$\frac{dY_i}{dt} = f_i(Y_1, Y_2 \dots, Y_8), \quad (i = 1, 2, 3, \dots, 8), \tag{24}$$

where f_i are real functions of Y_i. The computation was carried out with fifth-order Runge–Kutta method. In all the computations the time step Δt was chosen as $0.002 \times \pi$ and the wavenumber k was chosen as unity. The above choice of the wavenumber means that the characteristic length L is specified to be $\lambda/2\pi$, λ being the wavelength of the fundamental wave. Thus the nonlinear parameter ε is regarded as a ratio of the amplitude a to the characteristic length L, i.e., $\varepsilon = 2\pi a/\lambda$.

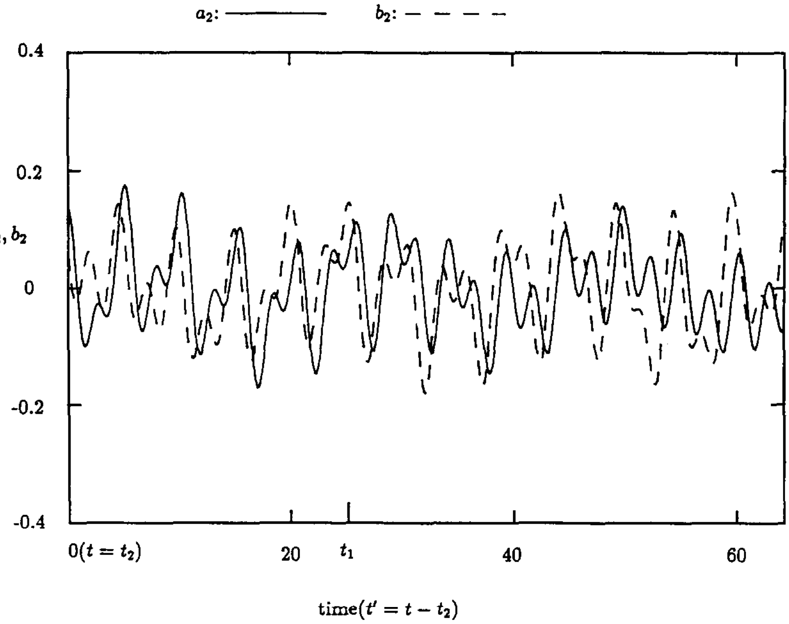


Fig. 1. Second harmonic components a_2 (solid line) and b_2 (broken line) as functions of time between 20000 Δt ($= t_2$) and 30000 Δt. The nonlinear parameter ϵ is equal to 0.7.

First, we consider the time variations of a_2 and b_2. In fig. 1, the variations are shown in the time interval between 20000 Δt and 30000 Δt, when $\varepsilon = 0.7$. This value of ε ($= 0.7$) implies that $a/\lambda \approx 0.1$. The value seems to be too large to break our assumption that nonlinearity is weak. However, such a large value was often chosen in the present computations, in order to save computation time. More time is necessary for smaller ε to draw out the recognizable characteristic. Figure 1 shows that the variation is complicated and the same form is never repeated. To make this complex behavior clearer, we investigated it in phase space of eight dimensions. For clarifying such a complicated behavior it is suitable to draw Poincaré sections. Four sections were plotted in fig. 2. The center one gives the plane $\mathrm{d}a_2/\mathrm{d}t = 0$ with which the trajectory intersects when $\mathrm{d}^2a_2/\mathrm{d}t^2 > 0$. The other three plots correspond to the planes $a_2 = 0$, $a_2 + 0.05 = 0$, and $a_2 + 0.1 = 0$. All the plots are shown for the points where the trajectory in phase space intersects with the plane only with the condition $\mathrm{d}a_2/\mathrm{d}t > 0$. The solid lines on the plane $a_2 = 0$ correspond to the case described in eqs. (20) and (21). It is clear that the complex behavior comes from the nonlinear terms in eq. (13).

In a certain kind of dissipative system it is known that all the trajectories passing through a certain domain of phase space are attracted to a geometric object called an attractor (Bergé et al., 1986). Figure 2 also shows that the trajectory passes through a confined domain of phase space in a complex manner. In contrast to those systems, however, in our system the relative rate of change of a volume V in phase space is zero, i.e.,

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \sum_{i=1}^{8} \frac{\mathrm{d}f_i}{\mathrm{d}Y_i} = 0, \tag{25}$$

as can easily be derived from eq. (24). Furthermore, the present dispersive case is strongly dependent on the initial values. For example, if the Stokes wavetrain is chosen as an initial one, the corresponding sections would become simpler than in the present initial condition because

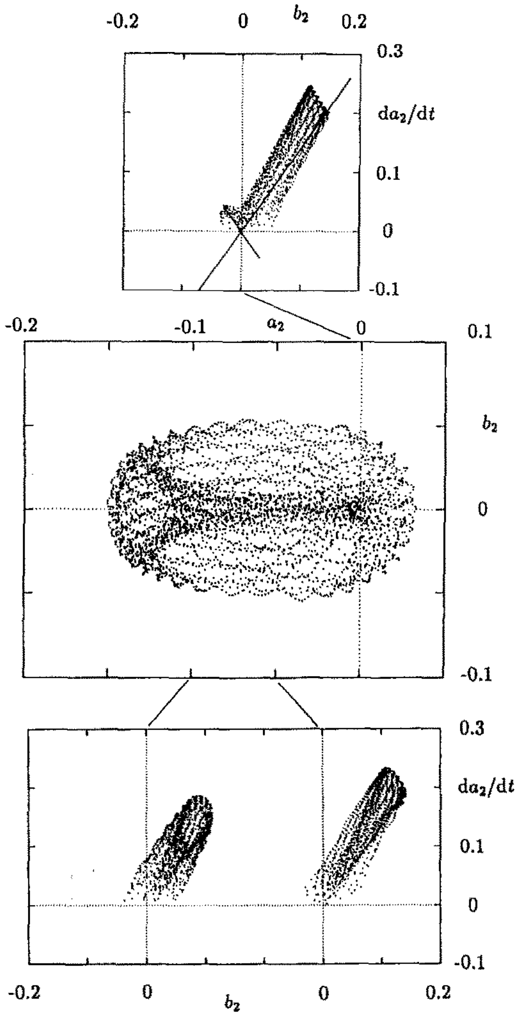


Fig. 2. Four Poincaré sections for $\varepsilon = 0.3$, $a_2 = 0$, $a_2 + 0.05 = 0$, $a_2 + 0.1 = 0$, and $da_2/dt = 0$. In each section, 2000 points are plotted. The solid line drawn in the top right corresponds to the linear case described in eqs. (20) and (21).

the train is stable. In this respect, the two systems, the dissipative and our present dispersive ones, are quite different.

We may observe a certain small structure in the sections. It was difficult to identify it, but it is clear that it results not from the coupling terms between X_1 and X_2 in eq. (13) but from the cubic nonlinear terms such as $|X_1|^2 X_1$ and $|dX_1/dt|^2 X_1$. Some calculations carried out by omitting the cubic terms in eq. (13) gave almost uniform distribution of points.

One of our main aim is to obtain an understanding of what happens in this wave system when a disturbance is added. Will the train approach the Stokes one? Is the Stokes wavetrain an asymptotic solution as time passes indefinitely? To answer these questions, stability analysis of this system was performed. Three computations were carried out by choosing $\varepsilon = 0.7$. Two cases were for eqs. (13) and (14), and the other was for the linear equation omitting the right-hand side of eq. (13) together with eq. (14), i.e., eqs. (20) and (21). A disturbance is added only to a_2 at $t_1 = 24000\,\Delta t$ for the linear and nonlinear cases, and also at $t_2 = 20000\,\Delta t$ for the other nonlinear

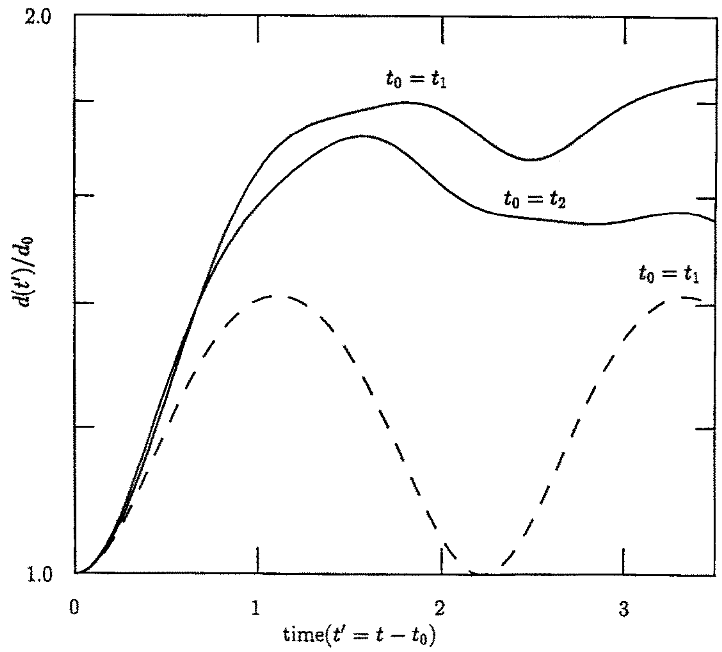
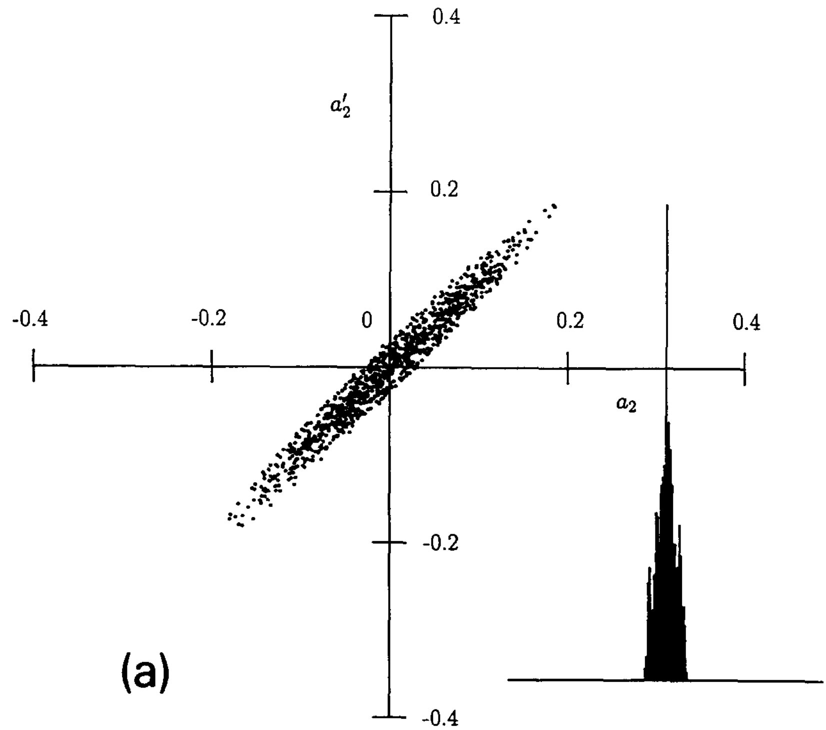


Fig. 3. Deviation of trajectories for $\varepsilon = 0.7$ on a log scale arising from a small displacement whose magnitude $d_0 = 10^{-4}$ as functions of t' ($= t - t_1$ or $t - t_2$). The displacement was made at $t_2 = 20000\,\Delta t$ and $t_1 = 24000\,\Delta t$. Two solid lines and a broken line show the nonlinear and linear cases, respectively.

one. These two times, t_1 and t_2, are also marked in fig. 1. A deviation $d(t)$ normalized in terms of the initial one d_0 is shown in fig. 3 when $d_0 = 10^{-4}$. The deviation is calculated as a Euclid norm in eight-dimensional space. The linear system gives only a periodic deviation, and hence it does not deviate unlimitedly. Both the linear and nonlinear cases seem to give similar initial growth rates for the distance between two neighboring trajectories. However, further calculation shows some difference between the two. For $\varepsilon = 0.5$ a similar procedure to the one described above starting at 100 000 points in phase space was carried out, and the exponential growth rate, which corresponds to the maximum Lyapunov exponent, was estimated at the average level. Other calculations for 100 000 points were also made. An exact estimation of the value was difficult, but it was certain that the nonlinear case always gave positive values. In a general sense, the trajectory is unstable. This exponential growth could not be detected in fig. 3, because of the smallness of the rate, being of order 10^{-2} or less. It results from our present situation where the nonlinearity is weak and the lowest order approximation to eq. (24) gives only an oscillatory motion. On the other hand, for the linear case the values were quite small and negative values were often calculated. Hence, the linear case is thought to be stable. As is shown in fig. 3, in the nonlinear case the deviation rapidly increases soon after the slow deviation. The growth is nearly exponential. It is thus clear that nonlinearity takes the trajectory away from the one with no disturbance.

In order to identify the behavior of the disturbed trajectory, a long-term computation was carried out when a displacement was made at $t = t_1$. Two trajectories with and without displacement were traced when $\varepsilon = 0.7$ and $d_0 = 10^{-3}$. In figs. 4a and 4b, the correlations between the original a_2 and the disturbed a_2' are shown together with their histograms for the deviations. These figures suggest that the correlation becomes weaker as time passes, which implies the loss of memory of initial conditions.



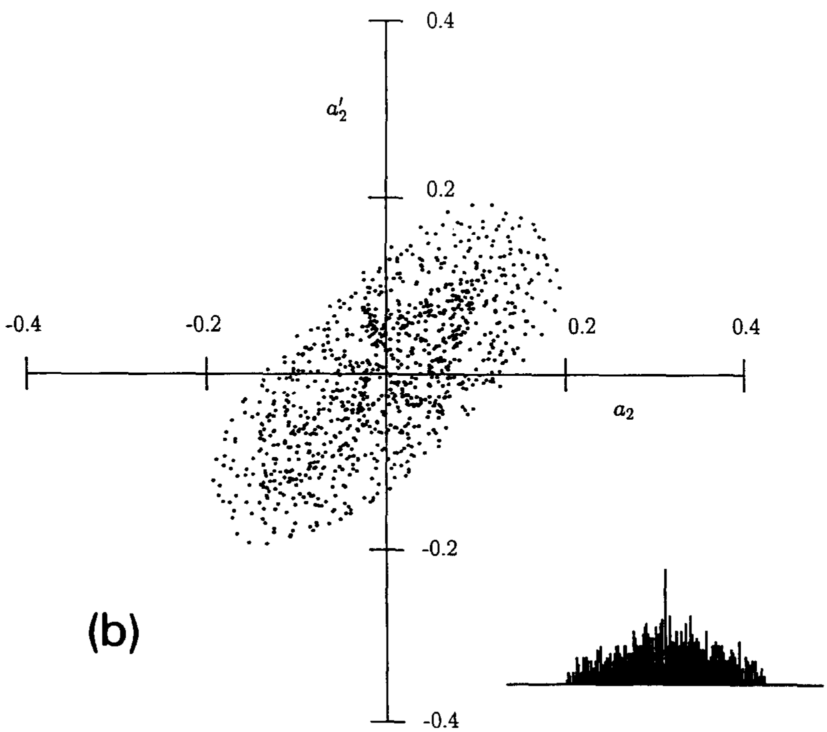


Fig. 4. Plots for a'_2 versus a_2. A disturbance of the magnitude 10^{-3} is added to a'_2 only at 24000 Δt when $\varepsilon = 0.7$. The initial plots in (a) and (b) are at 400000 Δt and 2000000 Δt, respectively. Histograms for the deviations are also plotted.

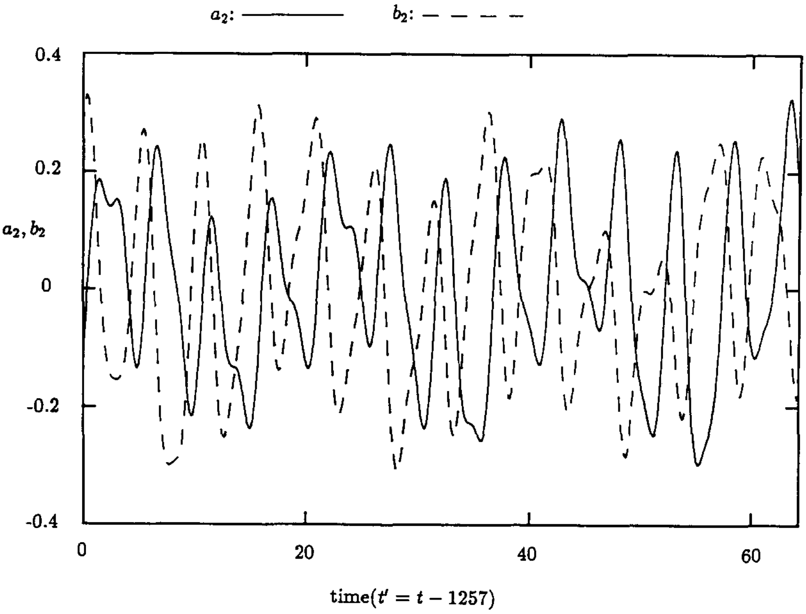


Fig. 5. Temporal variations of a_2 (solid line) and b_2 (broken line) starting at 200000 Δt for $\varepsilon = 0.7$. Gaussian noise with standard deviation 10^{-4} is added to a_2 at all the time steps.

As mentioned above, the trajectory is unstable. This raises a question: Will the trajectory be attracted to a fixed simple object when disturbances are added continually? To answer this question further computations were carried out by adding Gaussian noise with standard deviation $\sigma = 10^{-4}$ only to a_2 at each time step. In fig. 5, the time variations of $a_2(t)$ and $b_2(t)$ beginning at $200000\,\Delta t$ are shown. Small ripples become few, and the variation seems to be simpler than in the case without disturbance (see fig. 1). The variation also seems to be somewhat similar to the Stokes wavetrain in the phase relation between a_2 and b_2, but it shows larger amplitude than that of the Stokes one. To investigate it in more detail we calculate wave deformations in x–t plane. The second harmonic is to be expressed as

$$\xi_2(x,t) = X_2(t)\,\mathrm{e}^{2\mathrm{i}x} + X_2^*(t)\,\mathrm{e}^{-2\mathrm{i}x}. \tag{26}$$

In fig. 6, the time and space variations of $\xi_2(x,t)$ are drawn at two different time intervals. The corresponding initial waves shown in figs. 6a, and 6b are at $20000\,\Delta t$ and $200000\,\Delta t$, respectively. From fig. 6a, a general characteristic inherent in the second harmonic is easily seen. When the amplitude of the second harmonic is small, nonlinearity based on the fundamental wave increases it until the dispersion begins to take effect, while the dispersion effect suppresses the growth when the amplitude is large. Such generation and suppression occur repeatedly as a result of the two competing effects. This process at the initial stage is essentially the same as that described simply by eqs. (20) and (21). The appearance of the harmonic is almost periodic or quasi-periodic. This pattern might be repeated *ad infinitum* if no disturbances were added. In these plots, two traces are also drawn outside the main parts for two particular waves, i.e., the wave with the linear phase velocity c_2 of the wavenumber 2 predicted by the linear dispersion relation (2) and the Stokes wave with the velocity c_s. The propagation velocity is faster than the linear phase velocity c_2, while it is slower than that of the Stokes wavetrain. Figure 6b shows that the amplitude of the harmonic increases by absorbing the energy of the noise, although the possibilities that the energy will be gained and released are even. Furthermore, it is remarkable that a chaotic behavior of the

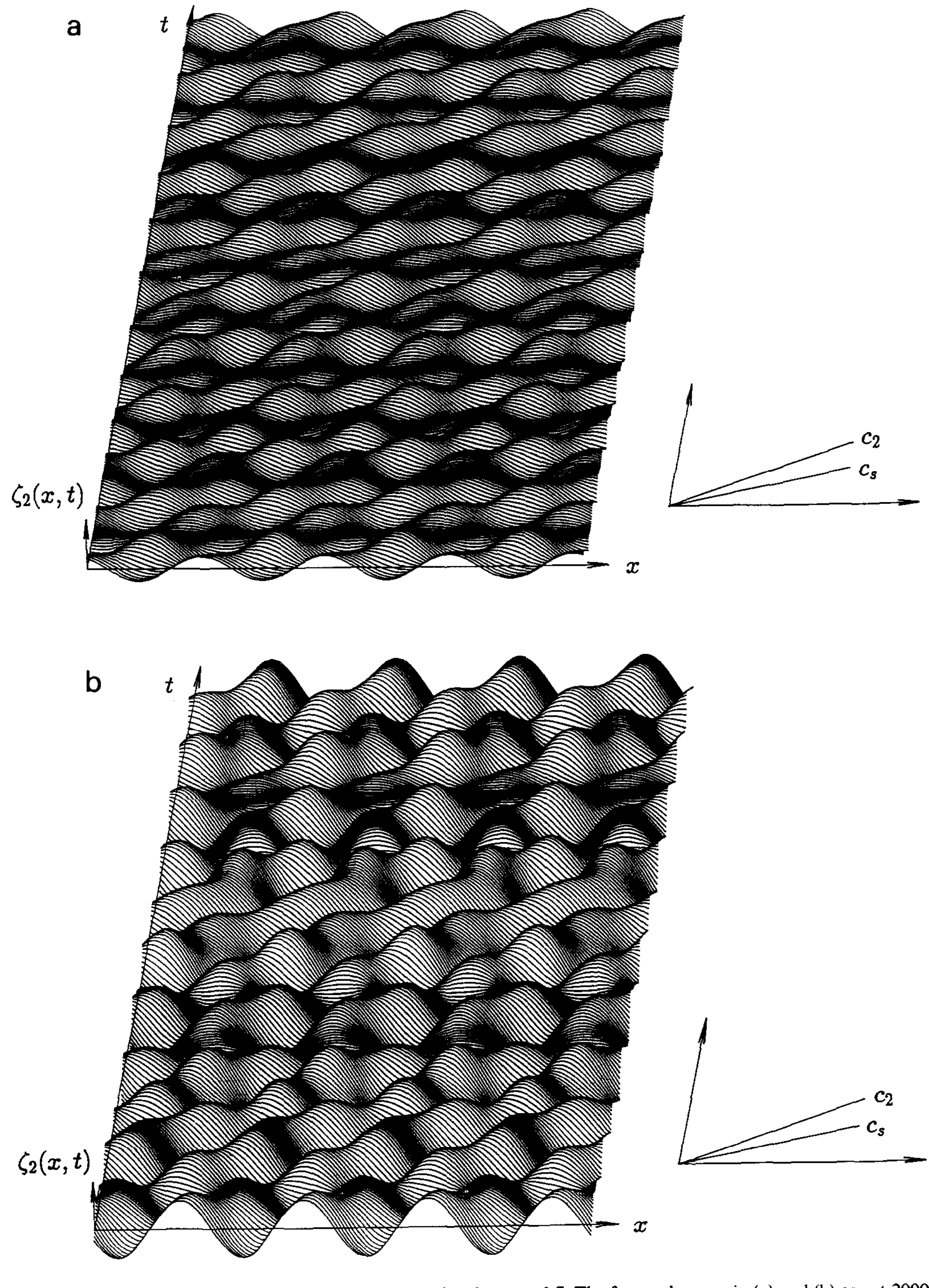


Fig. 6. Space and time variations of the second harmonic when $\varepsilon = 0.7$. The forward waves in (a) and (b) are at 20000Δ and $200000\Delta t$, respectively. All the wave forms are drawn by 20 time steps.

train appears. It should also be noted that wavetrains traveling in the negative x-direction are recognizable in short intervals of time, which reflects the fact that our formulation can describe waves traveling in both the positive and negative x-directions. In most parts of the variation the traveling velocity has a positive value less than that shown in fig. 6a. The wave velocity deviates from the velocity of the Stokes wavetrain. Accordingly, it is concluded that one never finds a uniform wavetrain.

4. Concluding remarks

In general wave systems of finite amplitude, any waves will deform due to non-linearity. This deformation is equivalent to the generation of higher harmonics. In the present paper, a simplified system consisting of two waves, namely a fundamental wave and the second harmonic, has been investigated for deep water waves. It is found that the behavior of the second harmonic is complicated. A solution starting at a certain point in phase space shows that the trajectory is unstable to a small displacement and gradually deviates from the original trajectory, indicating a gradual loss of memory. This deviation in the present dispersive system is not so drastic as that in dissipative systems, for example, in the Lorenz model (Lorenz 1963).

No experimental evidence of such a phenomenon has been presented yet. The loss of memory may appear more remarkably in the case where initial waves deviate further from the Stokes ones. To find the evidence, any generation of waves different from the fundamental wave must be suppressed initially.

Our last conclusion is concerned with asymptotic behavior of deep waves. It is very interesting to ask where the disturbed system will go. Will the system approach the Stokes wavetrain as a steady wave solution? The answer is 'no'. The trajectories deviate from the Stokes one within the present frame of considerations. It is plausible that the Stokes wavetrain can be observed only when the initial values are set exactly equal to those of the Stokes one.

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