3.7. Turunan Fungsi Logaritmis

Jika f(x) = ${}^{a}\log x$, maka f '(x) = $\frac{1}{x \ln a}$

Bukti : f(x) = ${}^{a}\log x \Rightarrow f(x+h) = {}^{a}\log (x+h)$

$f(x+h) - f(x) = {}^{a}\log (x+h) - {}^{a}\log x$

$= \frac{{}^{a}\log (x+h)}{{}^{a}\log x}$; $[\because {}^{a}\log (x+h) - {}^{a}\log x = \frac{{}^{a}\log (x+h)}{{}^{a}\log x}]$

$= {}^{a}\log (1 + h/x)$

$= {}^{a}\log (1 + h/x)^{h/x.x/h}$

$= h/x \,.\, {}^{a}\log (1 + h/x)^{x/h}$; $[\because \log x^{n} = n \log x]$

$f'(x) = \lim_{h \to 0} 1/h\,(h/x \,.\, {}^{a}\log (1 + h/x)^{x/h})$

$= \lim_{h \to 0} 1/x \,.\, {}^{a}\log (1 + h/x)^{x/h}$

$= \lim_{h \to 0} 1/x \,.\, \frac{\ln (1 + \frac{h}{x})^{x/h}}{\ln a}$; $[\because {}^{a}\log b = \frac{{}^{x}\log b}{{}^{x}\log a}]$

$= \lim_{h \to 0} (1/x \,.\, 1/\ln a \,.\, \ln (1 + h/x)^{x/h})$

$= \lim_{h \to 0} 1/x \lim_{h \to 0} 1/\ln a \,.\, \lim_{h \to 0} \ln (1 + h/x)^{x/h}$

$= 1/x \,.\, 1/\ln a \,.\, \ln (e)$

$= \frac{1}{x \ln a} \,.\, \ln e$

$= \frac{1}{x \ln a} \,.\, 1$

$= \frac{1}{x \ln a}$ ■

$\therefore f(x) = \log x \Rightarrow D (\log x) = \frac{1}{x \ln a}$

Contoh : Tentukan turunan pertama dari:

a. $f(x) = {}^{7}\log x$

b. $f(x) = {}^{4}\log x + {}^{6}\log x$

c. $f(x) = ({}^{5}\log x)(3 x^{2})$

d. $f(x) = \frac{{}^{3}\log x}{3x}$

Penyelesaian: a. $f(x) = {}^{7}\log x$

$f'(x) = \frac{1}{x \ln 7}$

$\therefore f(x) = {}^{7}\log x \Rightarrow f'(x) = \frac{1}{x \ln 7}$

b. $f(x) = {}^{4}\log x + {}^{6}\log x$

Misal $g(x) = {}^{4}\log x \Rightarrow g'(x) = \frac{1}{x \ln 4}$

Misal $h(x) = {}^{6}\log x \Rightarrow h'(x) = \frac{1}{x \ln 6}$

$\therefore f'(x) = \frac{1}{x \ln 4} + \frac{1}{x \ln 6}$; $[\because h(x) = f(x) + g(x) \Rightarrow h'(x) = f'(x) + g'(x)]$

c. $f(x = ({}^{3}\log x)(3x^2)$

Misal $g(x) = {}^{3}\log x \Rightarrow g'(x) = \frac{1}{x \ln 3}$

Misal $h(x) = 3x^2 \Rightarrow h'(x) = 6x$

$f'(x) = {}^{3}\log x \,.\, 6x + 3x^2 \,.\, \frac{1}{x \ln 3}$

$\therefore f'(x) = {}^{3}\log x \,.\, 6x + 3x^2 \,.\, \frac{1}{x \ln 3}$

d. $f(x) = \frac{{}^{3}\log x}{3x}$

Misal $g(x) = {}^{3}\log x \Rightarrow g'(x) = \frac{1}{x \ln 3}$

Misal $h(x) = 3x \Rightarrow h'(x) = 3$

$f'(x) = \frac{3x.\frac{1}{x \ln 3} - {}^{3}\log x .3}{(3x)^2}$

$\therefore f'(x) = \frac{3[x.\frac{1}{x \ln 3} - {}^{3}\log x]}{9x^2}$

3.7.1. Turunan Fungsi Logaritmis Naturalis

Jika $f(x) = \ln x$, maka $f'(x) = \frac{1}{x}$

Bukti : Karena $\ln x = {}^{e}\log x$, jika huruf "a" pada logaritma diganti dengan e, didapat

$f'(x) = \frac{1}{x \ln e} = \frac{1}{x}$ ■

$\therefore$ f(x) = ln x $\Rightarrow$ D(ln x) = $\frac{1}{x}$

Contoh : Tentukan turunan pertama dari:

a. f(t) = ln t

b. f(x) = (1/2 x^2) ln x

c. f(x) = $\frac{x}{\ln x}$

Penyelesaian: a. f(t) = ln t

f '(t) = $\frac{1}{t}$

b. f(x) = (1/2 x^2) ln x

Misal g(x) = 1/2 x^2 $\Rightarrow$ g'(x) = x

Misal h(x) = ln x $\Rightarrow$ h'(x) = $\frac{1}{x}$

f '(x) = (1/2 x^2). $\frac{1}{x}$ + ln x . x

$\therefore$ f '(x) = 1/2 x + x ln x

c. f(x) = $\frac{x}{\ln x}$

Misal g(x) = x $\Rightarrow$ g'(x) = 1

Misal h(x) = ln x $\Rightarrow$ h'(x) = $\frac{1}{x}$

f '(x) = $\frac{\ln x \,.1 - x \,.\frac{1}{x}}{(\ln x)^2}$

$\therefore$ f '(x) = $\frac{\ln x - 1}{(\ln x)^2}$

3.8. Turunan Fungsi Eksponensial

Jika f(x) = a^x , maka f '(x) = a^x ln a

Bukti : f(x) = a^x atau y = a^x

ln y = ln a^x [kedua ruas dioperasikan dengan ln]

ln y = x ln a

$\frac{d}{dx}$ ln y $= \frac{d}{dx}$ x ln a [kedua ruas diturunkan terhadap x]

$\frac{1}{y}\frac{dy}{dx}$ = x. 0 + ln a . 1

$\frac{1}{y}\frac{dy}{dx}$ = ln a ; atau

$\frac{dy}{dx}$ = y ln a [kedua ruas dikali dengan y dari kiri]

f '(x) = a^x ln a ■ [substitusi y dengan a^x]

∴ f(x) = a^x ⇒ f '(x) = D (a^x) = a^x ln a

Contoh : Tentukan turunan pertama dari:

a. f(x) = 9^x

b. f(x) = (5^x)(x^2 - 2x + 1)

c. f(x) = $\frac{2^x}{5x}$

Penyelesaian: a. f(x) = 9^x

∴ f '(x) = 9^x ln 9

b. f(x) = (5^x)(x^2 - 2x + 1)

Misal g(x) = 5^x ⇒ g'(x) = 5^x ln 5

Misal h(x) = (x^2 - 2x + 1) ⇒ h'(x) = 2x - 2

∴ f '(x) = (5^x)(2x - 2) + (x^2 - 2x + 1)(5^x ln 5)

c. f(x) = $\frac{2^x}{5x}$

Misal g(x) = 2^x ⇒ g'(x) = 2^x ln 2

Misal h(x) = 5x ⇒ h'(x) = 5

∴ f '(x) = $\frac{5x \,.2^x \ln 2 - 2^x.5}{(5x)^2}$

3.8.1. Turunan Fungsi Eksponensial Dengan Bilangan Pokok e

Jika f(x) = e^x , maka f '(x) = e^x

Bukti: Karena e = 2,72 ... situasi khusus dari a, maka

f '(x) = e^x ln e

$= e^x$ ■

$\therefore f(x) = e^x \Rightarrow f'(x) = e^x$

Contoh : Tentukan turunan pertama dari:

a. $f(x) = e^x + (2x - 7)$

b. $f(x) = (3x^2 - 7)(2e^x)$

c. $f(x) = \dfrac{3e^x}{3x}$

Penyelesaian : a. $f(x) = e^x + (2x - 7)$

Misal $g(x) = e^x \Rightarrow g'(x) = e^x$

Misal $h(x) = (2x - 7) \Rightarrow h'(x) = 2$

$\therefore f'(x) = e^x + 2$

b. $f(x) = (3x - 7)(2e^x)$

Misal $g(x) = (3x^2 - 7) \Rightarrow g'(x) = 6x$

Misal $h(x) = (2e^x) \Rightarrow h'(x) = (2e^x)$

$f'(x) = (3x^2 - 7)(2e^x) + (2e^x)(6x)$

$\therefore f'(x) = (2e^x)(3x^2 + 6x - 7)$

c. $f(x) = \dfrac{3e^x}{3x}$

Misal $g(x) = 3e^x \Rightarrow g'(x) = 3e^x$

Misal $h(x) = 3x \Rightarrow h'(x) = 3$

$f'(x) = \dfrac{3x\ .\ 3e^x - 3e^x .3}{(3x)^2}$

$\therefore f'(x) = \dfrac{e^x\ (x-1)}{x^2}$

3.9. Turunan Fungsi Hiperbolik

3.9.1. Turunan Fungsi Sinus Hiperbolik

Jika $f(x) = \sinh x$, maka $f'(x) = \cosh x$

Bukti: $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}$

dengan menerapkan turunan aturan hasil bagi diperoleh

$$f'(x) = \frac{2\,(e^{x} + e^{-x}) - (e^{x} - e^{-x}).0}{2^{2}}$$

$$= \frac{(e^{x} + e^{-x})}{2}$$

$$= \cosh(x) \quad \blacksquare$$

$\therefore f(x) = \sinh x \Rightarrow f'(x) = \cosh x$

Contoh : Tentukan turunan pertama dari:

a. f(x) = 10 sinh x

b. f(x) = (5 sinh x)(2x - 10)

c. $f(x) = \frac{3 \sinh x}{6 \sin x}$

Penyelesaian: a. f(x) = 10 sinh x

$\therefore$ f'(x) = 10 cosh x

b. f(x) = (5 sinh x)(2x - 10)

Misal g(x) = 5 sinh x ⇒ g'(x) = 5 cosh x

Misal h(x) = 2x - 10 ⇒ h'(x) = 2

f'(x) = (5 sinh x)2 + (5 cosh x)(2x - 10)

$\therefore$ f'(x) = 5(2 sinh x + (2x - 10)(cosh x))

c. $f(x) = \frac{3 \sinh x}{6 \sin x}$

Misal g(x) = 3 sinh x ⇒ g'(x) = 3 cosh x

Misal h(x) = 6 sin x ⇒ h'(x) = 6 cos x

$$f'(x) = \frac{(6 \sin x\,.\ 3 \cosh x) - 3 \sinh x.6 \cos x}{36 \sin^{2} x}$$

$$\therefore f'(x) = \frac{(18\,(\sin x\,.\ \cosh x) - (\sinh x.6 \cos x)}{36 \sin^{2} x}$$

3.9.2. Turunan Fungsi Cosinus Hiperbolik

Jika f(x) = cosh x, maka f '(x) = sinh x

Bukti: f(x) = cosh x = $\frac{(e^{x}+e^{-x})}{2}$

dengan menerapkan turunan aturan hasil bagi diperoleh

f '(x) = $\frac{2(e^{x}-e^{-x})-(e^{x}+e^{-x}).0}{2^{2}}$

= $\frac{2(e^{x}-e^{-x})}{2^{2}}$

= $\frac{(e^{x}-e^{-x})}{2}$

= sinh (x) ■

∴ f(x) = cosh x ⇒ f '(x) = sinh x

Contoh : Tentukan turunan pertama dari:

a. f(x) = 15 cosh x

b. f(x) = (5 cosh x)(5x - 10)

c. f(x) = $\frac{6\cosh x}{3\cos x}$

Penyelesaian: a. f(x) = 15 cosh x

∴ f'(x) = 15 sinh x

b. f(x) = (5 cosh x)(5x - 10)

Misal g(x) = 5 cosh x ⇒ g'(x) = 5 sinh x

Misal h(x) = 5x - 10 ⇒ h'(x) = 5

f '(x) = (5 cosh x)5 + (5 sinh x)(5x - 10)

∴ f '(x) = 5(5 cosh x + (5x - 10)(sinh x))

c. f(x) = $\frac{6\cosh x}{3\cos x}$

Misal g(x) = 6 cosh x ⇒ g'(x) = 6 sinh x

Misal h(x) = 3 cos x ⇒ h'(x) = -3 sin x

f '(x) = $\frac{(3\cos x .6\sinh x)-(-3\sin x .6\cosh x)}{(3\cos x)^{2}}$

f '(x) = $\frac{18(\cos x .\sinh x)+(\sin x .\cosh x)}{9\cos^{2} x}$

$\therefore$ f '(x) = $\frac{2\ (\cos\ x\ .\sinh\ x) + (\sin\ x\ .\cosh\ x)}{\cos^2\ x}$

3.9.3. Turunan Fungsi Tangen Hiperbolik

Jika f(x) = tanh x, maka f '(x) = $\frac{1}{\cosh^2 x}$ = sech^2 x

Bukti: f(x) = tanh x = $\frac{\sinh\ x}{\cosh\ x}$

dengan menerapkan turunan aturan hasil bagi diperoleh

f '(x) = $\frac{\cosh\ x.\cosh x - \sinh\ x.\sinh\ x}{(\cosh\ x)^2}$

f '(x) = $\frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$

diketahui $\cosh^2$ x – $\sinh^2$ x = 1, sehingga

f '(x) = $\frac{1}{\cosh^2 x}$ ■

$\therefore$ f(x) = tanh x $\Rightarrow$ f '(x) = $\frac{1}{\cosh^2 x}$ = sech^2 x

Contoh : Tentukan turunan pertama dari:

a. f(x) = 3 tanh x

b. f(x = (4 x + 6)(5 tanh x)

c. f(x) = $\frac{4x\text{-}4}{8 \tanh x}$

Penyelesaian : a. f(x) = 3 tanh x

$\therefore$ f'(x) = 3 sech x

b. f(x = (4 x + 6)(5 tanh x)

Misal g(x) = 4x + 6 $\Rightarrow$ g'(x) = 4

Misal h(x) = 5 tanh x $\Rightarrow$ h'(x) = 5 sech^2 x

f'(x) = (4x + 6)(5 sech^2 x) + (5 tanh x) 4

$\therefore$ f'(x) = 5((4x + 6)sech^2 x + 4 tanh x)

c. f(x) = $\frac{4x\text{-}4}{8 \tanh x}$

Misal $g(x) = 4x - 4 \Rightarrow g'(x) = 4$

Misal $h(x) = 8 \tanh x \Rightarrow h'(x) = 8 \text{ sech}^2 x$

$$f'(x) = \frac{(8 \tanh x \,.\, 4) - (\,(4x-4).\ 8 \text{ sech}^2 x)}{(\,8 \tanh x\,)^2}$$

$$f'(x) = \frac{8\,(4 \tanh x) - (\,(4x-4).\ \text{sech}^2 x)}{8(\,8 \tanh^2)}$$

$$\therefore f'(x) = \frac{(4 \tanh x) - (\,(4x-4).\ \text{sech}^2 x)}{(\,8 \tanh^2)}$$

3.9.4. Turunan Fungsi Cotangen Hiperbolik

Jika $f(x) = \coth x$, maka $f'(x) = -\frac{1}{(\sinh^2 x)} = -\text{ csch}^2\ x$

Bukti: $f(x) = \coth x = \frac{\cosh x}{\sinh x}$

dengan menerapkan turunan aturan hasil bagi diperoleh

$$f'(x) = \frac{\sinh x . \sinh x - (\cosh x . \cosh x)}{(\sinh^2 x)}$$

$$f'(x) = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x}$$

diketahui $\cosh^2 x – \sinh^2 x = 1$, sehingga

$$f'(x) = -\frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x}$$

$$f'(x) = -\frac{1}{\sinh^2 x}$$

$$= -\text{ csch}^2 x \quad \blacksquare$$

$\therefore f(x) = \coth x \Rightarrow f'(x) = -\text{ csch}^2 x$

Contoh : Tentukan turunan pertama dari:

a. $f(x) = 10 \coth x$

b. $f(x = (8x - 4)(4 \coth x)$

c. $f(x) = \frac{8x-4}{4 \coth x}$

Penyelesaian : a. $f(x) = 10 \coth x$

$\therefore f'(x) = -10 \text{ csch}^2 x$

b. f(x = (8x - 4)(4 coth x)

Misal g(x) = 8x - 4 ⇒ g'(x) = 8

Misal h(x) = 4 coth x ⇒ h'(x) = -4 csch^2 x

f'(x) = (8x - 4)(-4 csch^2 x) + (4 coth x) 8

∴ f'(x) = -4((8x - 4)csch^2 x - 8 coth x)

c. f(x) = $\frac{8x-4}{4 \coth x}$

Misal g(x) = 8x - 4 ⇒ g'(x) = 8

Misal h(x) = 4 coth x ⇒ h'(x) = -4 csch^2 x

f'(x) = $\frac{4 \coth x \, . \, 8-((8x-4).-4 \, \text{csch}^{2} x)}{(4 \coth x)^2}$

f'(x) = $\frac{4 \coth x \, . \, 8-((8x-4).-4 \, \text{csch}^{2} x)}{(4 \coth x)^2}$

f'(x) = $\frac{4(8 \coth x + ((8x-4). \, \text{csch}^{2} x)}{(4 \coth x)^2}$

∴ f '(x) = $\frac{(8 \coth x + ((8x-4). \, \text{csch}^{2} x)}{4 \coth^2 x}$

3.9.5. Turunan Fungsi Secant Hiperbolik

Jika f(x) = sech x, maka f '(x) = - sech x . tanh x

Bukti: f(x) = sech x atau f(x) = $\frac{1}{\cosh x}$

f '(x) = $\frac{\cosh x \, . \, 0-(1.\sinh x)}{\cosh^2 x}$

f '(x) = $\frac{-\sinh x}{\cosh^2 x}$

f '(x) = $-\frac{1}{\cosh x} \cdot \frac{\sinh x}{\cosh x}$

= - sech x . tanh x ■

∴ f(x) = sech x ⇒ f'(x) = - sech x . tanh x

Contoh : Tentukan turunan pertama dari:

a. f(x) = - 3 sech x

b. f(x = (x^2 - 2x + 1)(2x sech x)

c. f(x) = $\frac{4 \text{ sech } x}{2\,x^2}$

Penyelesaian: a. f(x) = - 3 sech x

∴ f'(x) = 3 sech x tanh x

b. f(x = (x^2 - 2x + 1)(2x sech x)

Misal g(x) = (x^2 - 2x + 1) ⇒ g'(x) = 2x - 2

Misal h(x) = 2x sech x ⇒ h'(x) = (-2x sech x tg x + 2 sech x)

∴ f '(x) = (x^2 - 2x + 1)(-2x sech x tg x + 2 sech x) + (2x sech x)(2x - 2)

c. f(x) = $\frac{4 \text{ sech } x}{2\,x^2}$

Misal g(x) = 4 sech x ⇒ g'(x) = -4 sech x tanh x

Misal h(x) = $2x^2$ ⇒ h'(x) = 4x

$$f'(x) = \frac{2\,x^2.-4 \text{ sech } x \tanh x - (4 \text{ sech } x .4x)}{(2x^2)^2}$$

$$f'(x) = \frac{4x(-2\,x.\ \text{sech } x \tanh x - (4 \text{ sech } x\ )}{4x^4}$$

$$\therefore f'(x) = \frac{(-2\,x^2.\ \text{sech } x \tanh x - (4 \text{ sech } x .x)}{x^3}$$

3.9.6. Turunan Fungsi Cosecant Hiperbolik

Jika f(x) = csch x, maka f '(x) = - csch x . coth x

Bukti: f(x) = csch x atau f(x) = $\frac{1}{\sinh x}$

$$f'(x) = \frac{\sinh x.\ 0-(1.\cosh x)}{\sinh^2 x}$$

$$= \frac{-\cosh x}{\sinh^2 x}$$

$$= -\frac{1}{\sinh x} . \frac{\cosh x}{\sinh x}$$

= - csch x . coth x ■

∴ f(x) = csch x ⇒ f'(x) = - csch x . coth x

Contoh : Tentukan turunan pertama dari:

a. f(x) = - 5 csch x

b. f(x = (2x + 1)(2x csch x)

c. f(x) = $\frac{4 \text{ csch } x}{3x^2}$

Penyelesaian: a. f(x) = - 5 csch x

∴ f'(x) = 5 csch x coth x

b. f(x = (2x + 1)(2x csch x)

Misal g(x) = (2x + 1) ⇒ g'(x) = 2

Misal h(x) = 2x csch x ⇒ h'(x) = -2x(csch x coth x)

∴ f '(x) = (2x + 1)(-2x(csch x coth x) + 2(2x csch x))

c. f(x) = $\frac{4 \text{ csch } x}{3x^2}$

Misal g(x) = 4 csch x ⇒ g'(x) = -4 csch x ctg x

Misal h(x) = 3 x^2 ⇒ h'(x) = 6x

f '(x) = $\frac{3x^2.{-4} \text{ csch } x \text{ ctg } x - (4 \text{ csch } x\,.6x)}{(3x^2)^2}$

f '(x) = $\frac{-12\,x^2.\text{csch } x \text{ ctg } x - (24 \text{ csch } x\,.\,x)}{9\,x^4}$

∴ f '(x) = $\frac{-\,4\,x.\text{csch } x \text{ ctg } x - (8 \text{ csch } x\,)}{3\,x^3}$

L A T I H A N

Tentukan turunan pertama dari

1. $f(x) = x^{-17}$

2. $f(x) = 5 + 2x + 3x^2$

3. $f(x) = g(x) - h(x)$ dengan $g(x) = x^2 - 2x$ dan $h(x) = 25$

4. $f(x) = ax^2 + \frac{b}{x} + c$

5. $f(x) = g(x).h(x)$ dengan $g(x) = (4x^2 - 3x)$ dan $h(x) = (2x - 1)$

6. $f(x) = (2x-1)(x^2 - 2x + 1)$

7. $f(x) = 1/x^3$

8. $f(x) = \frac{3}{1-2x}$

9. $f(x) = \frac{x+1}{2x+3}$

10. $f(x) = \sqrt[3]{3x^2}$

11. $f(x) = (1 - 3x)^{-2/3}$

12. $f(x) = \sqrt{\frac{1-x}{1+3x}}$

13. $f(x) = \log 3\,(x^3 - 4x^2 + x - 10)$

14. $f(x) = 1/4\,(x^3 - 3x^2 + 5)$

15. $f(x) = \sin x + 5x^2$

16. $f(x) = \tan x - \cos x$

17. $f(x) = (\sin x + 5)(\sec x - 4)$

18. $f(x) = \frac{7x}{3x\ \tan x}$

19. $f(x) = (1 - \cos 1/2\, x)^2$

20. $f(x) = 0.333 (\sin x - \csc x)$

21. $f(x) = \log x - 3$

22. $f(x) = \log (x + 2)(x-2)$

23. $f(x) = \log \frac{2x - 5}{3x + 3}$

24. $f(x) = \log (x + 5)^5$

25. $f(x) = 3x \ln x$

25. $f(x) = \frac{2x}{\ln x}$

26. $f(x) = \ln x^2$

27. $f(x) = a^{0.5 x}$

28. $f(x) = 5x \, . \, a^x$

29. $f(x) = \frac{a^x}{3 x}$

30. $f(x) = 2x \, . \, e^x$

31. $f(x) = \frac{9 x}{e^x}$

32. $f(x) = 5x \, . \, \sinh x$

33. $f(x) = 3 \sin x + 4 \cosh x$

34. $f(x) = 7 \tanh x \, . \, \cos x$

35. $f(x) = \cosh x \, . \, \tan x$