Mathematics of fluid dynamics

University of Bath 25-26, MA32051/MA52129

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Dept. of Math. Sci., University of Bath

October 11, 2025

Website: Course website (`trinh.github.io/BathMAFluids/`)

Official description

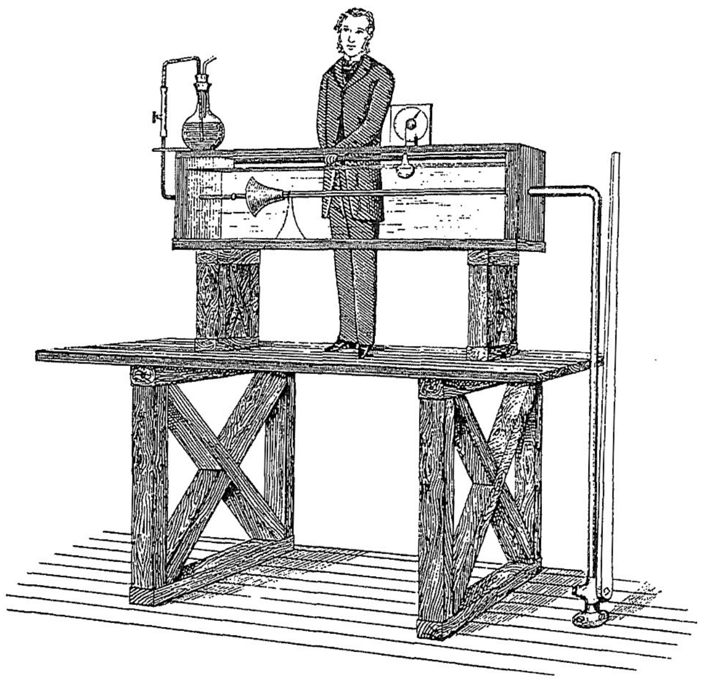


Figure 0.0.1 Rayleigh's 1883 experiment on turbulence, as duplicated in World of Flosws (Darrigol, 2005).

The description of this unit in the official catalogue is the following:

Aims In this unit you will explore the mathematical theory of fluid dynamics, with a view towards applications to physical phenomena such as flight, vortex motion and water waves. You will study the mathematics of conservation laws and the derivation of governing fluid dynamical equations. This unit will provide you with a foundation for further study of more advanced theory of fluid dynamics and continuum mechanics, and its application in scientific areas including engineering, physics and biology.

Outcomes (i) Demonstrate an understanding of the principles of mathematical fluid dynamics; (ii) discuss and apply techniques from vector calculus and complex variable theory to analyse and solve fluid flow problems; (iii) give a qualitative and quantitative account of a range of phenomena in fluid dynamics.

Content Complex analysis primer: Cauchy-Riemann equations; harmonic functions; complex maps; residue integration. The mathematics of fluid phenomena and its applications: derivation and interpretation of governing equations; reduction of governing equations to equations of simpler formulation; potential flow; vortical flow. Two-dimensional incompressible and irrotational flow: velocity potential; stream function; complex potential. Conformal mapping. Vortex motion: vortex lines and tubes; Kelvin circulation theorem; Helmholtz' principal. Water waves: free surfaces; harmonic waves; finite depth; instability; group velocity. Computational fluid dynamics.

History of the unit

Previously at Bath in the Mathematical Sciences, there were two units meant to teach continuum and fluid mechanics (or dynamics) to students. Prior to 2025, there was the MA30253 Continuum Mechanics (`https://www.bath.ac.uk/catalogues/2024-2025/ma/MA30253.html`) module. This was then continued into the MA40255 Viscous fluid dynamics (`https://www.bath.ac.uk/catalogues/2024-2025/ma/MA40255.html`) module.

As part of the curriculum transformation (with the first change to Year 3 in 2025), we are attempting to unify these two treatments, providing a more streamlined teaching of elementary fluid dynamics, which is oriented towards a broad range of styles of emphasis, from applied mathematics, to physics and engineering. The hope is that this new course on Fluid Dynamics provides you with a strong foundation in different basic fluid flows and their mathematical formulation and study.

Related units at Bath

We will only mention units from Year 2 onwards in this. Apart from the key pre-requisites of MA22016 (Differential equations and vector calculus) and/or MA20223 (the older Vector calculus and partial differential equations), we make an effort to keep the material in the module self-contained. You are recommended to have taken MA22021 (partial differential equations).

- *MA22016: Differential equations and vector calculus.*

  This unit (`www.bath.ac.uk/catalogues/2024-2025/ma/MA22016.html`) forms a standard second-year module on differential equations and vector calculus, and is a key pre-requisite for this module. In addition to teaching and reviewing basic techniques for solving ordinary differential equations, you will learn about some of the core methods in vector calculus (directional derivatives; gradients; potentials; line integrals; divergence; curl; surface and volume integrals; curvilinear coordinates; integral theorems).

  Note prior to curriculum transformation, this would have been part of the MA20223 unit (with additional material from the below MA22021).

- *MA22021: Partial differential equations.*

  This module (`www.bath.ac.uk/catalogues/2024-2025/ma/MA22021.html`) teaches basic techniques and theory for the core PDEs (Laplace, heat, wave equations). Generally, we will make with your broad familiarity of PDE different equation types and terminology (e.g. boundary conditions). This unit will be useful, as it will teach you some basic familiarity with partial differential equations. However, the current fluid dynamics module assumes you may not have taken it, and attempts to fill in any necessary gaps.

- *MA32045: Complex analysis.*

  This module (`www.bath.ac.uk/catalogues/2025-2026/ma/MA32045.html`) covers some of the theory and applications behind complex-valued functions. You will have encountered complex functions, e.g. $f(z) = z^2, \mathrm{e}^z, \log z, \ldots$ in an *ad-hoc* way, perhaps in earlier courses on Analysis. Again, we will attempt to cover all the necessary pre-requisites, and also provide you with helpful references.

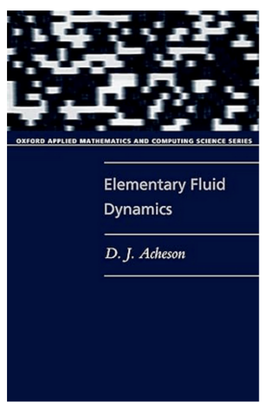
Moodle and other references

Besides this document, the main resource for this unit is the Moodle page. Links to the video recordings, course notes, and other resources are collected there.

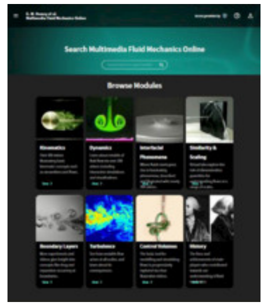
There are countless fluid mechanics or fluid dynamics courses and textbooks, and for the most part, the development of a *first course* on fluid dynamics tends to be quite similar between universities and treatments. If you would like additional references, here are a few useful ones.

However, note that our goal is to be as self-sufficient as possible via the lecture notes.

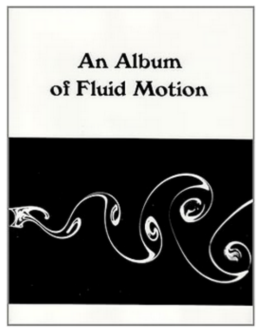
- David Acheson's (1990) book *Elementary fluid dynamics* [4]: a significant part of this course follows some of the now-classic treatments that would have been developed simultaneous to the design of this book by Acheson (often used by Oxford UG students). It is written in quite an informal style.



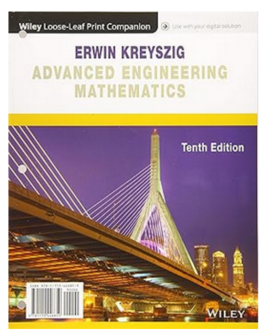
- Multimedia Fluid Mechanics Online, edited by G. M. Homsy [1]: a collection of videos and explanations of various fluid mechanics phenomena. This is an online resource available through the University of Bath library system.



- Milton Van Dyke's (1982) book "An album of fluid motion" [3]: a classic album showing beautiful black and white images of fluid motion. Published by an iconic private press and sold (by design by Van Dyke) at affordable prices!



- Kreyszig, E. (2007) book "Advanced engineering mathematics" [2] covers all the necessary essentials in terms of Vector Calculus and Complex Variables. This is one of my favourite reference texts for mathematical methods just on account of how straightfoward it is. Despite the "engineering" in the title, the style of presentation here fits in well with the style of UK applied mathematics.



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1 Introduction

1.3 Exercises

1.3.1. Plotting a Riemann surface. Select the branch cut of $f(z) = z^{1/2}$ that runs along the positive real axis.

(a) Consider a contour that starts from $z = 1$, then encircles the origin (anticlockwise) and returns to $z = 1$. What is the jump in the value of $f(z)$ at the end of the contour as compared to the start?

Hint. Let $z = \mathrm{e}^{\mathrm{i}\theta}$ and consider θ ranging from the initial value to a final value.

Solution. Let $z = \mathrm{e}^{\mathrm{i}\theta}$. For $\theta = 0$, $f(z) = r^{1/2}$ if we choose the positive branch of the square root (by convention). At the other side of the branch cut, $\theta = 2\pi$ and $f(z) = r^{1/2}\mathrm{e}^{\pi\mathrm{i}} = -r^{1/2}$. Therefore there is a jump in value of $-2r^{1/2}$.

(b) By hand, plot the Riemann surface as visualised in $(x, y, \operatorname{Im} f(x + \mathrm{i}y))$-space, where $\operatorname{Im} f(z) = r^{1/2}\sin(\theta/2)$. You may also confirm your sketch with a computational tool, if desired.

Hint. It is useful to first consider the plot of $\sin(\theta/2)$, and then separately, what happens for the magnitude variation that depends on $r^{1/2}$.

Solution. A sketch of the imaginary part of the square root function is shown below. The two key features to capture is the dependence on θ and the dependence on r.

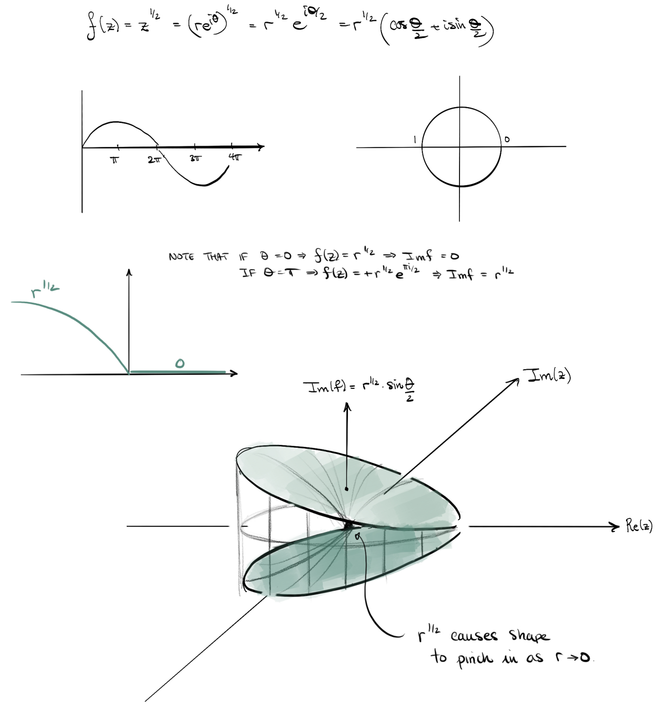

Figure 1.3.1 Sketch of the imaginary part of the square root function

1.3.2. A function with two branch points. Consider the function (1.2.4):

$$f(z) = (z+1)^{1/2}(z-1)^{1/2}.$$

(a) Choose the branch cut from $z = 1$ in the positive real direction. Choose the branch cut from $z = -1$ in the negative real direction. Write either $z = r_1 \mathrm{e}^{\mathrm{i}\theta_1}$ or $z = r_2 \mathrm{e}^{\mathrm{i}\theta_2}$ for θ_1 and θ_2 defined as relative angles from the two branch points.

Show that: (i) when $z = 1$ is encircled by a complete revolution, the function jumps in value by a factor of $\mathrm{e}^{\mathrm{i}\pi}$; (ii) that there is a similar jump in value when $z = -1$ is encircled. Finally what happens if (iii) $z = 0$ is encircled?

Draw a picture of the final z-plane, showing the branch cuts.

Solution. (i) We have that

$$f(z) = (r_1 r_2)^{1/2} \mathrm{e}^{\mathrm{i}\theta_1/2} \mathrm{e}^{\mathrm{i}\theta_2/2}.$$

Let $\theta_1 \in [0, 2\pi)$ be the angle about the point $z = 1$. Similarly let $\theta_2 \in [-\pi, \pi)$ be the angle about the point $z = -1$.

Considering firstly a revolution around $z = 1$ (that does not also enclose $z = -1$). Let the initial point be denoted ”A”, with $\theta_1 = 0, \theta_2 = 0$. And the final point be ”B”, with $\theta_1 = 2\pi$ and $\theta_2 = 0$.

Then

$$f(B) - f(A) = (r_1 r_2)^{1/2} \mathrm{e}^{2\pi \mathrm{i}/2} - (r_1 r_2)^{1/2} \mathrm{e}^0 = -2(r_1 r_2)^{1/2}.$$

so indeed there is a jump in the function about the branch cut along $z > 1$.

(ii) We would similarly verify that for a contour around the branch point $z = -1$ there is a jump. Let the initial point be denoted "A" with $\theta_1 = \pi, \theta_2 = -\pi$. And the final point be "B" with $\theta_1 = \pi, \theta_2 = \pi$. Then

$$f(B) - f(A) = (r_1 r_2)^{1/2} e^{(\pi+\pi)i/2} - (r_1 r_2)^{1/2} e^{(\pi-\pi)i/2} = -2(r_1 r_2)^{1/2}.$$

so indeed there is also a jump about the branch cut that runs $z < -1$.

(iii) For the centre point, there are two cases to consider. The first case is if only $z = 0$ is encircled and none of the other branch points. This was the situation originally envisioned in the question. For example, consider the circle of radius $1/2$, i.e. $z = (1/2)e^{i\theta}$ with $\theta \in [0, 2\pi)$. You can verify that for this circle the start and end points agree.

If along with $z = 0$, one of the branch points is encircled, then there would be a discontinuity. If both branch points are encircled, there is no discontinuity.

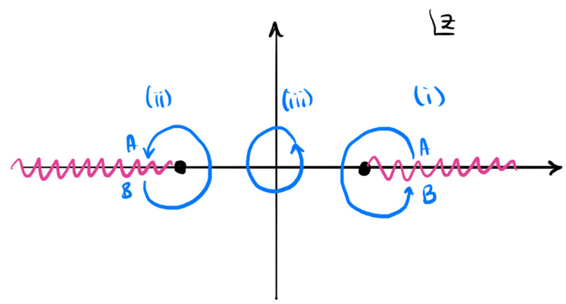

Figure 1.3.2 Branch cut configuration for the double square root. The original choice of branches is shown on top. Through the analysis, we see that a circle (i) around $z = 1$ does not produce a discontinuity. Hence only the second picture of the cut arrangement is needed.

(b) Consider now a branch cut from $z = -1$ that tends in the positive real direction and the branch cut from $z = 1$ tends in the positive real direction as well. Repeat the experiment above, considering (i)-(iii). Conclude that there is no jump in value along the region $z > 1$ and hence the branch cuts required only extends between $z = \pm 1$.

Draw a picture of the final z-plane, showing the branch cuts.

Solution. For this situation, we would define the ranges of $\theta_1 \in [0, 2\pi)$ and $\theta_2 \in [0, 2\pi)$. One main difference is the analysis around the point $z = 1$. Consider the similar loop to the above with,

$$f(B) - f(A) = (r_1 r_2)^{1/2} e^{(2\pi+2\pi)i/2} - (r_1 r_2)^{1/2} e^{(0+0)i/2} = 0,$$

so in this case, notice that there is no jump due to the fact the total variation of angle is $2\pi + 2\pi$.

In essence, because both branch cuts are running in the positive real direction, when we orbit across $z > 1$, we jump through both branches, hence returning to the original. There is no required branch cut for $z > 1$.

The analysis of parts (ii) and (iii) are identical, with the exception of the angle range. However, the final result, of whether there exists a jump is the same.

In the end, the final branch cut picture is shown in the figure below.

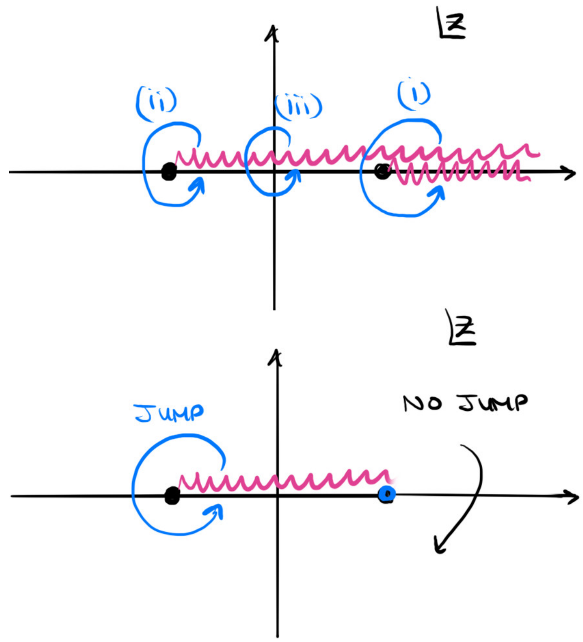

Figure 1.3.3 Branch cut configuration for the double square root

(c) (Challenging). If you consider a plot of $(x, y, \operatorname{Re} f(x+iy))$ or $(x, y, \operatorname{Im} f(x+iy))$, what will the Riemann surface look like? You can attempt to plot this using any tool.

Solution. This is certainly not an easy function to imagine! There are two features you may want to keep in mind. First, in examining the imaginary part of the function, if $z > 1$ on the real axis, then the imaginary part is zero. Second, if $z < -1$ on the real axis, then again the function is zero. Finally, for the case of the branch selection in part (b), the there is a cut along $[-1, 1]$. A generated plot is shown below for the imaginary part.

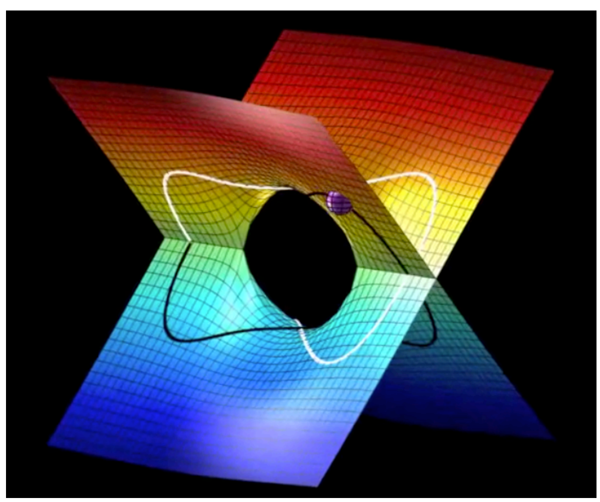

Figure 1.3.4 The imaginary part of $f(z) = (z-1)^{1/2}(z+1)^{1/2}$

1.3.3. Branch cuts of the complex logarithm. Consider the complex logarithm as defined in (1.2.3).

(a) Explain why there must be a branch cut imposed, originating from the branch point at $z = 0$.

Solution. The logarithm will have a jump in the imaginary part every rotation of 2π in the argument.

(b) Take the branch cut along the positive real axis. Do your best to draw the Riemann surface (consisting of the distinct Riemann sheets) of the logarithm, as visualised in the space $(x, y, \operatorname{Im} f(x + \mathrm{i}y))$.

Solution.

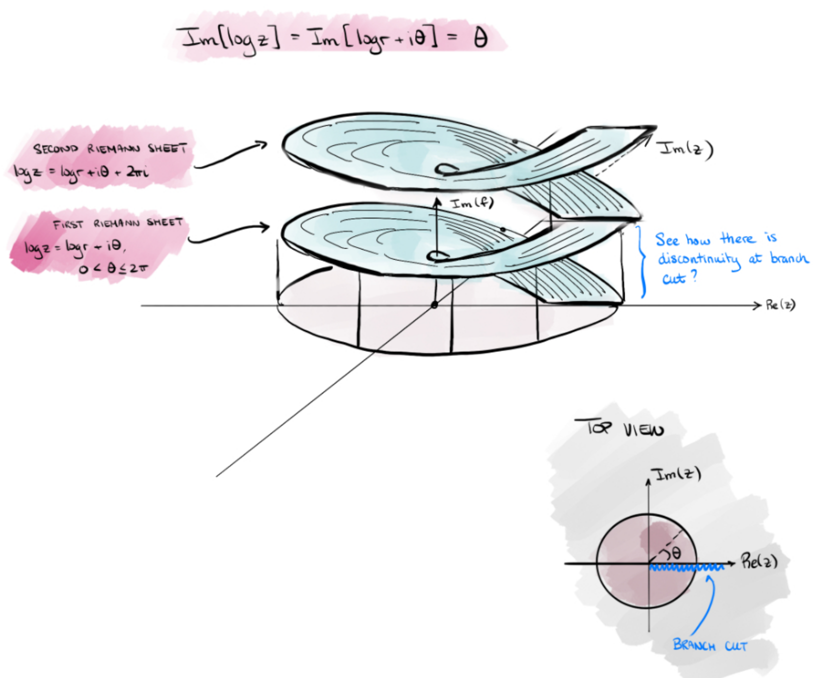

Figure 1.3.5 The imaginary part of $f(z) = \log z$.

(c) Again, you may find it useful to confirm your work above by plotting the function using a computational tool.

Solution. There is a picture of the complex logarithm on Wikipedia (`en.wikipedia.org/wiki/Complex_logarithm`).

Appendix A

Vector calculus

We write scalar functions as, e.g. $\phi($

Appendix B

List of Symbols

Symbol	Description	Page

References and further reading

[1] Homsy, G. M. (Ed.) (2019). Multimedia Fluid Mechanics Online. Cambridge University Press.

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[8] White, F. M. and Xue H. (2021). Fluid mechanics (ninth edition). McGraw--Hill, New York.

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[10] Paterson, A. R. (1983). A first course in fluid dynamics. Cambridge University Press.