Mathematics of fluid dynamics

University of Bath 25-26, MA32051/MA52129

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Dept. of Math. Sci., University of Bath

December 29, 2025

Website: Course website

Official description

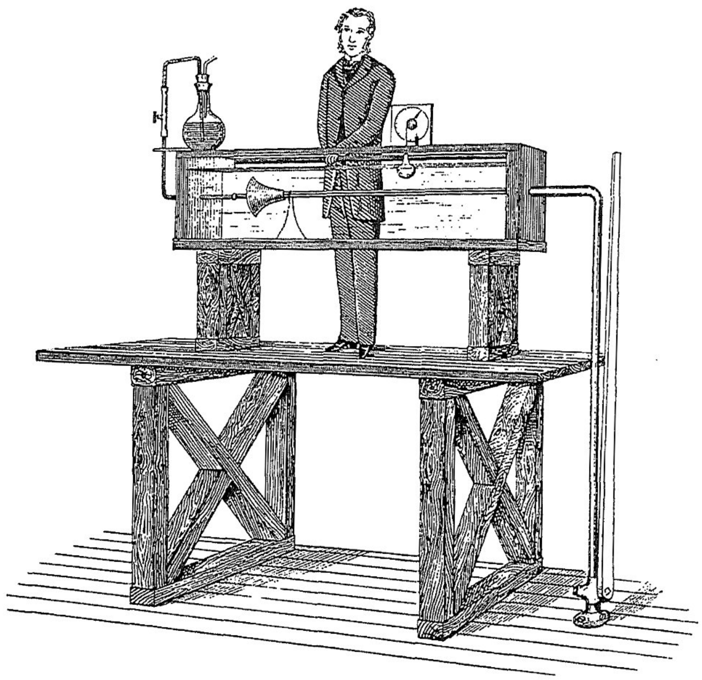


Figure 0.0.1 Rayleigh's 1883 experiment on turbulence, as duplicated in World of Flosws (Darrigol, 2005).

The description of this unit in the official catalogue is the following:

Aims In this unit you will explore the mathematical theory of fluid dynamics, with a view towards applications to physical phenomena such as flight, vortex motion and water waves. You will study the mathematics of conservation laws and the derivation of governing fluid dynamical equations. This unit will provide you with a foundation for further study of more advanced theory of fluid dynamics and continuum mechanics, and its application in scientific areas including engineering, physics and biology.

Outcomes

(i) Demonstrate an understanding of the principles of mathematical fluid dynamics; (ii) discuss and apply techniques from vector calculus and complex variable theory to analyse and solve fluid flow problems; (iii) give a qualitative and quantitative account of a range of phenomena in fluid dynamics.

Content Complex analysis primer: Cauchy-Riemann equations; harmonic functions; complex maps; residue integration. The mathematics of fluid phenomena and its applications: derivation and interpretation of governing equations; reduction of governing equations to equations of simpler formulation; potential flow; vortical flow. Two-dimensional incompressible and irrotational flow: velocity potential; stream function; complex potential. Conformal mapping. Vortex motion: vortex lines and tubes; Kelvin circulation theorem; Helmholtz' principal. Water waves: free surfaces; harmonic waves; finite depth; instability; group velocity. Computational fluid dynamics.

History of the unit

Previously at Bath in the Mathematical Sciences, there were two units meant to teach continuum and fluid mechanics (or dynamics) to students. Prior to 2025, there was the MA30253 Continuum Mechanics module. This was then continued into the MA40255 Viscous fluid dynamics module.

As part of the curriculum transformation (with the first change to Year 3 in 2025), we are attempting to unify these two treatments, providing a more streamlined teaching of elementary fluid dynamics, which is oriented towards a broad range of styles of emphasis, from applied mathematics, to physics and engineering. The hope is that this new course on Fluid Dynamics provides you with a strong foundation in different basic fluid flows and their mathematical formulation and study.

Related units at Bath

We will only mention units from Year 2 onwards in this. Apart from the key pre-requisites of MA22016 (Differential equations and vector calculus) and/ or MA20223 (the older Vector calculus and partial differential equations), we make an effort to keep the material in the module self-contained. You are recommended to have taken MA22021 (partial differential equations).

- *MA22016: Differential equations and vector calculus.*

  This unit forms a standard second-year module on differential equations and vector calculus, and is a key pre-requisite for this module. In addition to teaching and reviewing basic techniques for solving ordinary differential equations, you will learn about some of the core methods in vector calculus (directional derivatives; gradients; potentials; line integrals; divergence; curl; surface and volume integrals; curvilinear coordinates; integral theorems).

  Note prior to curriculum transformation, this would have been part of the MA20223 unit (with additional material from the below MA22021).

- *MA22021: Partial differential equations.*

  This module teaches basic techniques and theory for the core PDEs (Laplace, heat, wave equations). Generally, we will make with your broad familiarity of PDE different equation types and terminology (e.g. boundary conditions). This unit will be useful, as it will teach you some basic familiarity with partial differential equations. However, the current fluid dynamics module assumes you may not have taken it, and attempts to fill in any necessary gaps.

- *MA32045: Complex analysis.*

  This module covers some of the theory and applications behind complex-valued functions. You will have encountered complex functions, e.g. $f(z) = z^2, \mathrm{e}^z, \log z, \ldots$ in an *ad-hoc* way, perhaps in earlier courses on Analysis. Again, we will attempt to cover all the necessary pre-requisites, and also provide you with helpful references.

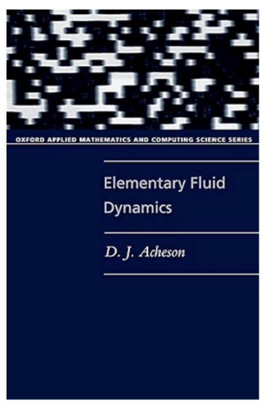
Moodle and other references

Besides this document, the main resource for this unit is the Moodle page. Links to the video recordings, course notes, and other resources are collected there.

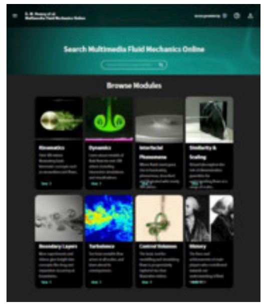
There are countless fluid mechanics or fluid dynamics courses and textbooks, and for the most part, the development of a *first course* on fluid dynamics tends to be quite similar between universities and treatments. If you would like additional references, here are a few useful ones.

However, note that our goal is to be as self-sufficient as possible via the lecture notes.

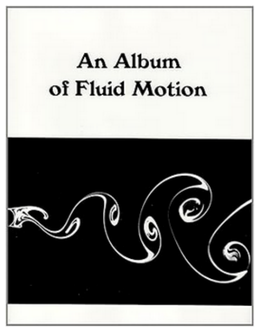
- David Acheson's (1990) book *Elementary fluid dynamics* [4]: a significant part of this course follows some of the now-classic treatments that would have been developed simultaneous to the design of this book by Acheson (often used by Oxford UG students). It is written in quite an informal style.



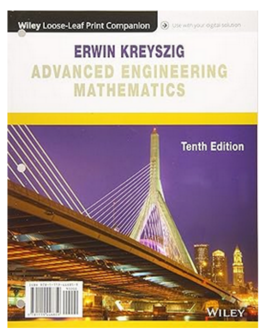
- Multimedia Fluid Mechanics Online, edited by G. M. Homsy [1]: a collection of videos and explanations of various fluid mechanics phenomena. This is an online resource available through the University of Bath library system.



- Milton Van Dyke's (1982) book "An album of fluid motion" [3]: a classic album showing beautiful black and white images of fluid motion. Published by an iconic private press and sold (by design by Van Dyke) at affordable prices!



- Kreyszig, E. (2007) book "Advanced engineering mathematics" [2] covers all the necessary essentials in terms of Vector Calculus and Complex Variables. This is one of my favourite reference texts for mathematical methods just on account of how straightfoward it is. Despite the "engineering" in the title, the style of presentation here fits in well with the style of UK applied mathematics.



Contents

Chapter 1

Introduction

> Now I think hydrodynamics is to be the root of all physical science, and is at present second to none in the beauty of its mathematics.
>
> —William Thompson (Lord Kelvin), Dec. 1857.

As evidenced by the quotation above, during the great Victorian era, Lord Kelvin had predicted that the field of hydronamics (predominantly, the study of liquids) would reign over all the physical sciences. Today that is not quite true, in the sense that hydrodynamics is only one of many sub-branches of the wider study of fluid motion. However, Kelvin's prediction is certainly true in spirit: there are very few branches in the physical sciences, from biology and chemistry, to engineering and physics, where the study of fluid mechanics does not possess strong historical and scientific connections. The principles, language, and techniques of fluid mechanics begin from the fundamental laws of conservation; in their form, one can argue that all of nature can be derived--this grandiose point is, to some extent, what Kelvin had meant when he referred to being the ”root of all physical science”.

Moreover, countless areas of mathematics, from the 19th century and onwards, have been developed as a direct consequence of the need to investigate the beautiful nature of fluid motion. These include everything from the theory of ordinary and partial differential equations, calculus, complex analysis, mathematical methods and approximation theory, geometry and topology, analysis, numerical methods and numerical analysis, industrial and applied mathematics, and so forth and so on.

Remark 1.0.1 Fluid mechanics vs. fluid dynamics. Why is this course called fluid dynamics and yet fluid mechanics? What is the difference?

In general, fluid mechanics includes both ”statics” (fluids at rest) and ”dynamics” (things in motion). This is a distinction that, in the authors' opinions, mathematicians rarely use (we probably refer to ”fluid mechanics” more commonly so as to avoid being specific); these distinctions are perhaps more important in engineering. For example, the study of the shape of a soap film or bubble between a wire hoop is a question of statics; but the behaviour of the soap or bubble if it moves or pops is a question of dynamics!

A tree diagram of different subfields and their applications might be made as follows.

```
Fluid Mechanics
 □□□ Fluid Statics
 □     □□□ Hydrostatics → Pressure in tanks and dams
```

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□     □□□ Buoyancy → Ship and submarine design
□     □□□ Manometry → Pressure measurement in pipelines
□
□□□ Fluid Dynamics
      □□□ Inviscid Flow
      □     □□□ Potential Flow → Aerodynamics of airfoils
      □     □□□ Compressible Flow (ideal gases) → Supersonic jet nozzles
      □
      □□□ Viscous Flow
      □     □□□ Internal Flows (pipe/duct flow) → Water distribution networks
      □     □□□ External Flows (boundary layers) → Drag on cars and airplanes
      □     □□□ Lubrication Theory → Bearings in machines
      □
      □□□ Turbulence
      □     □□□ Atmospheric Turbulence → Weather prediction
      □     □□□ Industrial Turbulence → Mixing in chemical reactors
      □
      □□□ Multiphase Flow
            □□□ Gas-Liquid Flow → Oil & gas pipelines
            □□□ Liquid-Solid Flow → Slurry transport in mining
            □□□ Gas-Solid Flow → Fluidized bed reactors
```

It is not uncommon for some scientists and mathematicians to devote their entire careers to an entire subbranch of fluid mechanics. Some university departments will focus solely on certain subbranches as well! Hence fluid mechanics is for many, a lifelong pursuit!

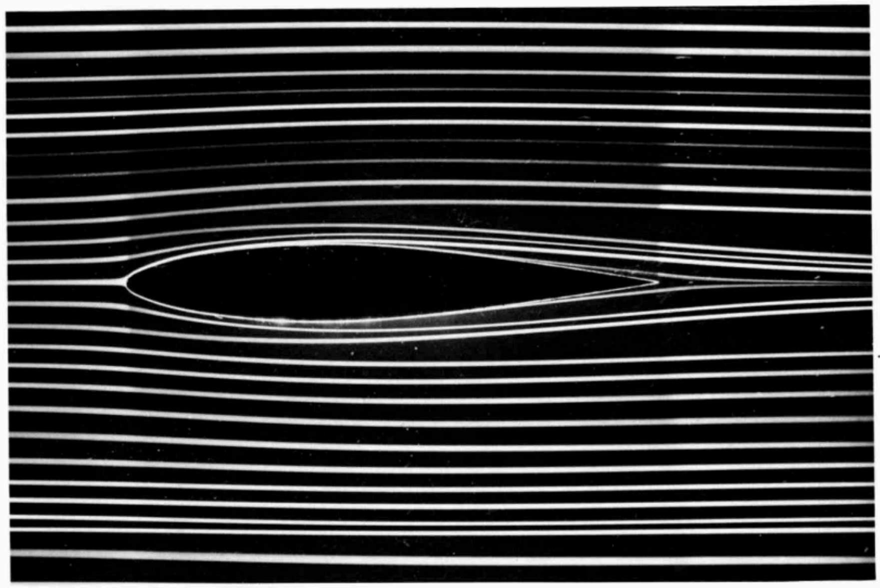
What is the course about?

Our first task, starting in Chapter 2 and Chapter 3 is to derive the so-called Euler equation, given by

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right) = -\nabla p + \rho\boldsymbol{g}.$$

Euler's equation relates the velocity of a fluid, $\boldsymbol{u}$, with its density ρ, pressure p, and associated forces, $\boldsymbol{g}$. It is an expression of the conservation of momentum of fluid, and is completed with an accompanying equation for conservation of mass. Together, the two equations are solved with the specification of additional boundary conditions to describe many liquids, from the water in your bathtub, to the water in the ocean as a ship travels through the surface.

Slowly, we will begin to appreciate the use of conservation laws, and the mathematics necessary, to derive the above equations (compressing some hundred years of scientific work into only a handful of lectures!). We will also appreciate that, despite their compact form, the above equations are certainly not easy to solve!



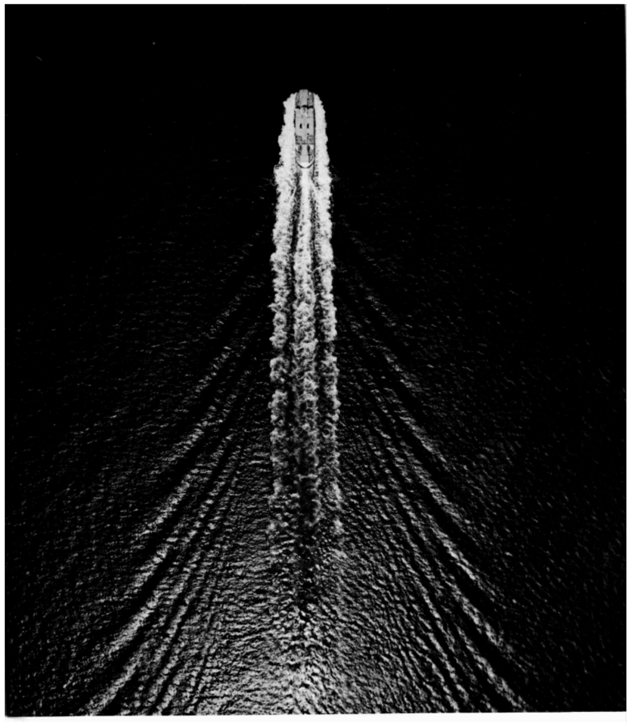
23. Symmetric plane flow past an airfoil. An NACA 64A015 profile is at zero incidence in a water tunnel. The Reynolds number is 7000 based on the chordlength. Streamlines are shown by colored fluid introduced upstream. The flow is evidently laminar and appears to be unseparated, though one might anticipate a small separated region near the trailing edge. *ONERA photograph, Werlé 1974*

Figure 1.0.2 Flow around an airfoil is an example of potential flow theory. From Van Dyke, An Album of Fluid Motion.

Our next task, starting in Chapter 4 is to study a particular simplification of the Euler equation that yields the flow of an *ideal* or *potential* fluid. Under such constraints, instead of solving the above difficult equations, we solve a much simpler equation:

$$\nabla^2 \phi = 0,$$

for a so-called *velocity potential*, ϕ, related to the velocity by $\boldsymbol{u} = \nabla\phi$. Potential flow theory would have occupied some of the greatest minds of our time, from Euler, Lagrange, Bernoulli, d'Alembert, and Laplace of the 18th century; to the great Victorian scientists of the 19th century: Kelvin, Green, Stokes, Helmholtz; then towards the modern 20th century workers such as Prandtl, Lighthill, Milne-Thompson, and so forth. It is the simplest framework for studying the motion of a liquid, such as water, but enjoys incredibly deep connections with the beautiful theory of complex variables. We will study some of these connections, leading to learning about things such as conformal mapping theory.



201. Wave pattern of a ship. An aerial photograph from directly overhead shows, away from the breaking wave from the bow and its turbulent wake, the asymptotic pattern deduced by Kelvin in 1887. The waves are confined to an angle of 19½° on either side of the path of the ship, in agreement with the theory, with an effective origin that is displaced approximately one ship length ahead of the bow. *Newman 1970*

Figure 1.0.3 In 1887, Kelvin famously predicted that the waves trailing a ship produce a wedge of approximately $2 \times 19.5°$. This theory belongs to the analysis of linear water waves (though it is unlikely we will have time to reproduce Kelvin's approximation!) From Van Dyke, An Album of Fluid Motion.

Water waves is probably the application that was at the forefront of Lord Kelvin's thoughts when he discussed the state of hydrodynamics in the quotation that begins in Chapter 5. The study of the surface motion of water introduces a seemingly minor complexity that is responsible for great heartache: the fluid is now bounded by an unknown free surface, which much now be solved as part of the problem! The theory of water waves was at the heart of many minds in the 19th century, given its importance in all applications naval and oceanic. The study of water waves begins with so-called linear wave theory, and moves towards numerical solutions of the full Euler equations.

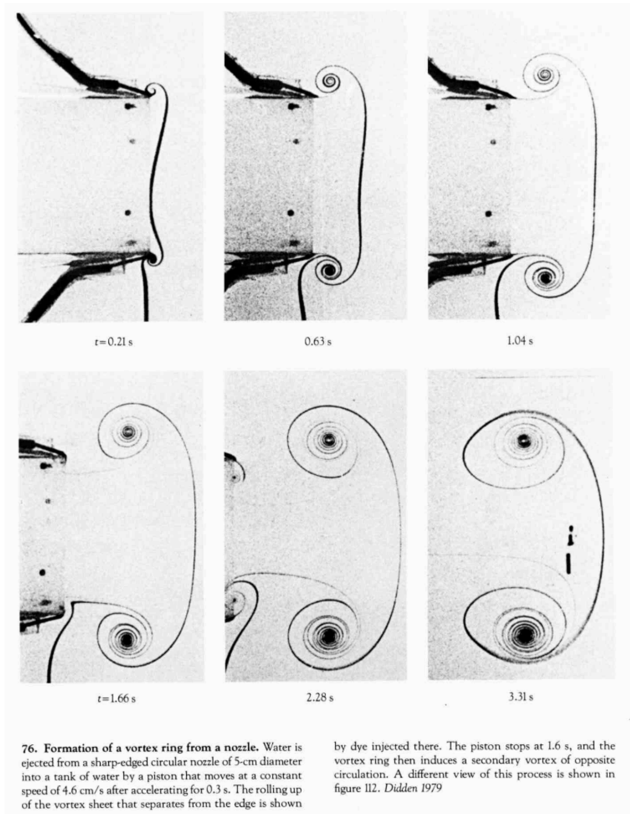
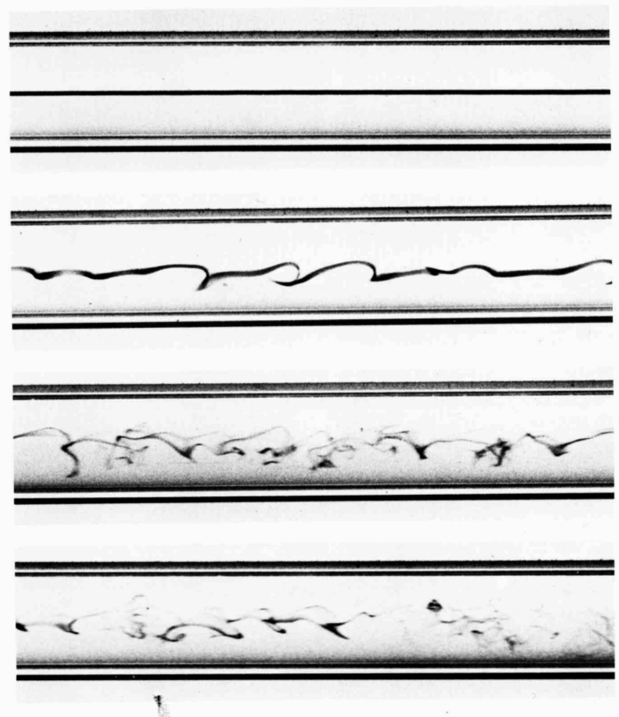


Figure 1.0.4 Water ejected from a nozzle forms a vortex sheet that rolls up into two vortex rings. From Van Dyke (1982) "An Album of Fluid Motion" [3].

In the above applications, our attention will be focused on so-called irrotational flows: flows that either do not contain or do not introduce new rotational characteristics. Of course, real life fluids are rarely so well-behaved and indeed, the study of vortices is an important domain in its own right, leading to models for tornados, flight, and many other fluid phenomena. The principle character of Chapter **??** is the vorticity, $\omega = \nabla \times \boldsymbol{u}$. What is vorticity, how does one measure it, and what is its role in governing fluid motion: these are the topics of this chapter.



103. Repetition of Reynolds' dye experiment. Osborne Reynolds' celebrated 1883 investigation of stability of flow in a tube was documented by sketches rather than photography. However the original apparatus has survived at the University of Manchester. Using it a century later, N. H. Johannesen and C. Lowe have taken this sequence of photographs. In laminar flow a filament of colored water introduced at a bell-shaped entry extends undisturbed the whole length of the glass tube. Transition is seen in the second of the photographs as the speed is increased; and the last two photographs show fully turbulent flow. Modern traffic in the streets of Manchester made the critical Reynolds number lower than the value 13,000 found by Reynolds.

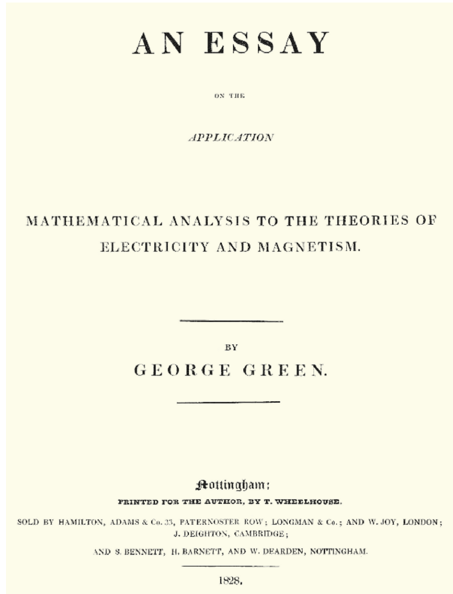
Figure 1.0.5 As illustrated in Figure 0.0.1, in 1883 Rayleigh performed a famous experiment of stability of flow in a tube, showing the emergence of an instability as the speed of the flow is increased. This leads to turbulence, which is predicted, in principle, from the Navier-Stokes equations of viscous flows. (From Van Dyke, An Album of Fluid Motion.

Finally, the last part of this course will open up to the important discpline of viscous flows in Chapter **??**. For this, the Euler equations are no longer sufficient, and we must include a dissipative component,

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right) = -\nabla p + \rho \boldsymbol{g} + \mu \nabla^2 \boldsymbol{u},$$

with μ being the *viscosity* of the fluid. The above equation, together with an equation for conservation of mass, forms the so-called *Navier-Stokes equations*. All fluids are viscous to some extent, though some moreso than others. You will learn about the connections between viscous flow theory and inviscid theory, and some of the simple viscous flows that are foundational in this subbranch.

1.1 A reminder of vector calculus


AN ESSAY

ON THE

APPLICATION

MATHEMATICAL ANALYSIS TO THE THEORIES OF
ELECTRICITY AND MAGNETISM.

BY

GEORGE GREEN.

Nottingham:
PRINTED FOR THE AUTHOR, BY T. WHEELHOUSE.
SOLD BY HAMILTON, ADAMS & Co. 33, PATERNOSTER ROW; LONGMAN & Co.; AND W. JOY, LONDON;
J. DEIGHTON, CAMBRIDGE;
AND S. BENNETT, H. BARNETT, AND W. DEARDEN, NOTTINGHAM.

1828.

Figure 1.1.1 George Green's monumental work on electricity and magnetism, making use of many new concepts in vector calculus, 1828.

During the first week, we will provide a very brief review of some of the necessities that you may require in terms of vector calculus. Many of you will have taken the MA20223 Vector Calculus and Partial Differential Equations module, and a version of the 2024-25 lecture notes has been updated for easy reference on Moodle.

We will assume that you are familiar enough with how to interpret many of the vector calculus identities found in Sec. 10 of the University of Bath book of tables, which can be access on Moodle or via this link.

In general, over the next few weeks, you will want to be familiar with recalling/looking up concepts like:

- The use of identities like div curl $= 0$ and curl grad $= 0$. Some of these are found on p.24 of the above tables.
- The notion of line integrals, surface integrals, and volume integrals.
- The divergence theorem and Stokes' theorem. (p.24)
- Conversion of vector operations and integrals into different coordinate systems (p.25)

1.2 A reminder of complex variables

In Chapter 4, we will leverage the power of complex variables to study certain problems in fluids (flow of a potential flow). One concept that you may be unfamiliar with at this stage is the concept of a *branch cut*.

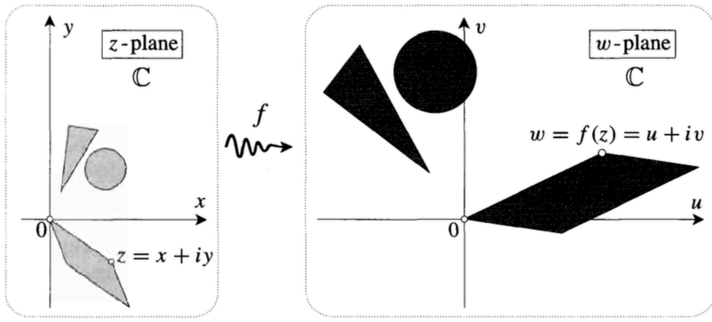


Figure 1.2.1 An image from Tristan Needham's "Visual Complex Analysis" [5] showing how complex mappings transform shapes from one region to another.

1.2.1 Basic complex representations

Generally, we write the Cartesian and polar form of a complex number as,

$$z = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta},$$

for magnitude $r > 0$ and angle θ. Below, we will consistently refer to $z \in \mathbb{C}$. The decomposition of the complex exponential is given by Euler's identity:

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta. \tag{1.2.1}$$

The usual trigonometric functions can be extended to the complex plane by considering their definition in terms of complex exponentials and Euler's identity. For example, we have

$$\cos z = \frac{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}{2} \quad \text{and} \quad \sin z = \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}}. \tag{1.2.2}$$

Another important function we shall consider is the complex logarithm, defined as

$$\log z \equiv \log r + \mathrm{i}\theta, \tag{1.2.3}$$

where $z = r\mathrm{e}^{\mathrm{i}\theta}$. That this definition is sensible is verified by checking that the logarithm is the inverse of the exponential. That is,

$$\mathrm{e}^{\log z} = \mathrm{e}^{\log r + \mathrm{i}\theta} = \mathrm{e}^{\log r}\mathrm{e}^{\mathrm{i}\theta} = z.$$

However, the definition (1.2.3) is troubling because it is not single-valued. For example, writing $z = 1\mathrm{e}^{\mathrm{i}\cdot 0}$ and $z = 1\mathrm{e}^{\mathrm{i}\cdot 2\pi}$ gives two different possible values of $\log z$ for the same value of $z = 1$. We dig deeper into this issue.

1.2.2 Complex functions

A complex function maps points on the complex plane to points on the complex plane. For instance, the square function,

$$f(z) = z^2 = r^2\mathrm{e}^{2\mathrm{i}\theta},$$

can be better understood by its effect on points on the unit circle, $|z| = 1$.

Consider a particle that orbits around the unit circle in the $z-$plane at unit speed. If the particle rotates by half a revolution, with $\theta = \pi$, then in the image plane, the image particle has rotated by a full revolution, with $f(z) = (\mathrm{e}^{\mathrm{i}\pi})^2$ in this same unit time. This is illustrated by the image in Figure 1.2.2.

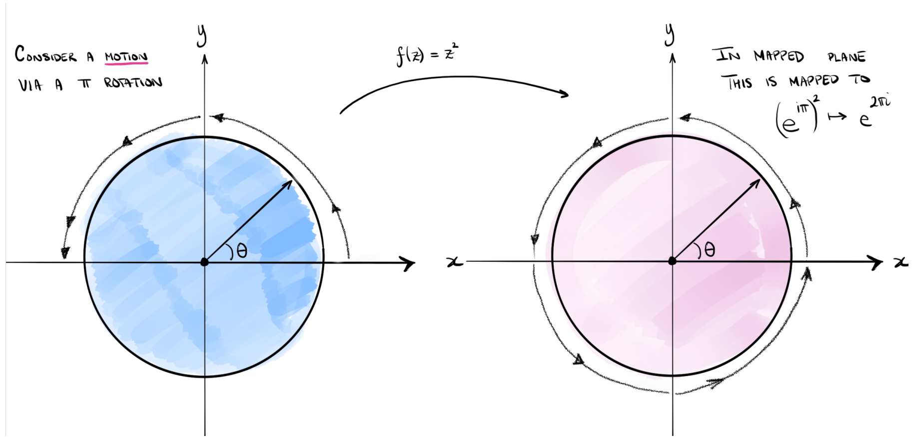


Figure 1.2.2 A revolution of π in the pre-image produces a full rotation in the image plane.

Now we continue rotating around the unit circle in the $z-$plane, performing an additional π rotation. Within the image plane, the particle has now completed another full rotation around the unit circle. This is shown in Figure 1.2.3.

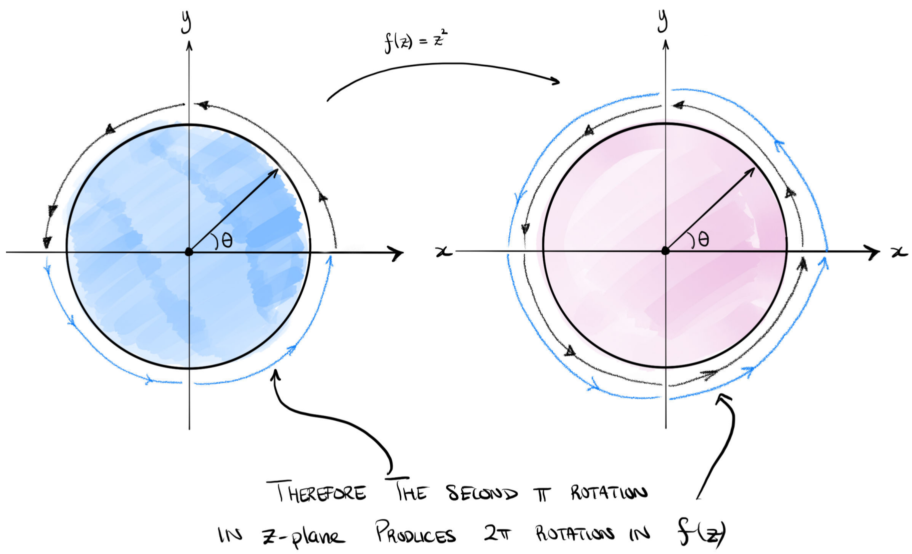


Figure 1.2.3 A revolution of π in the pre-image produces a full rotation in the image plane.

1.2.3 Multifunctions, branch cuts, and Riemann sheets

Now consider the inverse function of the above, given by the square root function,

$$f(z) = z^{1/2}.$$

Visually, we can simply consider the same figures as before, but now with the mapping proceeding from the right subfigure to the left subfigure. Observe that there is now an ambiguity, because for each point in the original $z-$plane, there are two possible images to assign for $z^{1/2}$, corresponding to either the top semicircle of the left figure, or the bottom semicircle.

That is, it is unclear of whether we should define:

$$f(z) = z^{1/2} = r^{1/2}\mathrm{e}^{\mathrm{i}\theta/2},$$

or

$$f(z) = z^{1/2} = -r^{1/2}\mathrm{e}^{\mathrm{i}\theta/2}.$$

Moreover there is a problem, for if we allow a ”motion” of the z values such that z rotates more than a complete revolution around the origin, then $f(z)$ is no longer well-defined and takes on multiple possible values.

This leads to the following restriction. We define a *branch cut* of the $z-$plane, and restrict the possible argument values. For example, we may choose

$$0 \leq \theta < 2\pi,$$

hence imposing that the branch cut is along the positive real axis. This is illustrated in Figure 1.2.4 where the branch cut is shown as a wavy line.

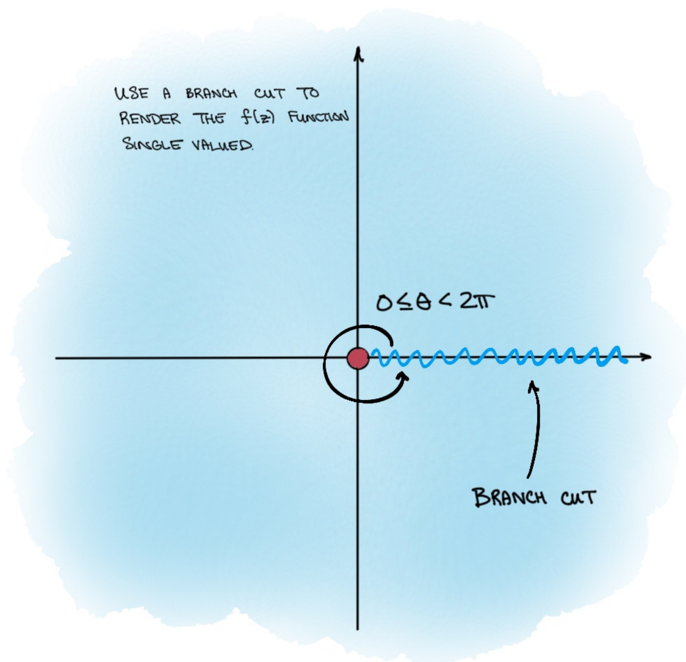


Figure 1.2.4 Branch cut

Alternatively, we could have equally chosen

$$-\pi \leq \theta < \pi,$$

hence taken the branch cut along the negative real axis.

Once the branch cut has been selected, the previous multi-function is restricted to one of the two possible definitions above. This leads to the definition as follows.

Definition 1.2.5 Branches of the square root function. The positive branch of the square root, with branch cut taken along the positive real axis, is defined by

$$f(z) = f_1(z) = z^{1/2} = r^{1/2}\mathrm{e}^{\mathrm{i}\theta/2},$$

for $r > 0$ and $0 \leq \theta < 2\pi$. Other branch cut choices can be taken in an analogous manner (curve that extends from $z = 0$ to $z = \infty$).

Then the above f with z restricted as given is a well-defined single-valued function.

There is an analogous negative branch defined as

$$f(z) = f_2(z) = z^{1/2} = -r^{1/2}\mathrm{e}^{\mathrm{i}\theta/2},$$

Therefore, f_1 and f_2 make up the two "layers" of the square root function.

We often refer to each individual "layer" as a *Riemann sheet*. The critical point $z = 0$ is referred to as a *branch point*.

The collection of Riemann sheets is referred to as a *Riemann surface*. $\diamondsuit$

In Exercise 1.3.1, you will plot the Riemann surface that corresponds to $f(z) = z^{1/2}$. The result might look like this version created using Python graphing tool:

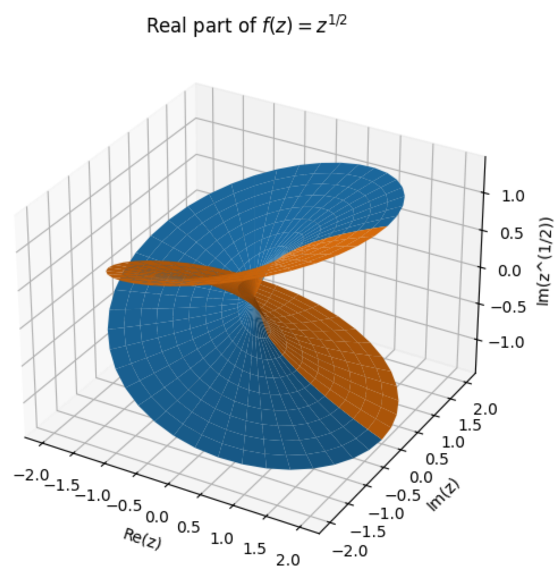


Figure 1.2.6 Riemann surface for the square root function. This shows the real part of the square root, which consists of both a positive branch and a negative branch. In the above case, we have placed the branch cut along the negative real axis.

A similar argument would indicate that the function

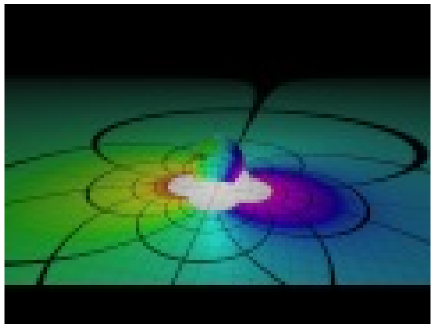
$$f(z) = (z+1)^{1/2}(z-1)^{1/2}, \tag{1.2.4}$$

requires two branch cuts in general, each cut originating from the two branch points at $z = \pm 1$. You will study this function in more detail in Exercise 1.3.2.

Also in Exercise 1.3.3, you will study the branch structure of the complex logarithm in (1.2.3).

1.2.4 Other functions and visualisations

The theory of complex analysis is rich in different kinds of visualisations. Another way to visualise a complex function is by considering its effect on a gridded pattern in the original z-plane, and then to imagine the function as warping this pattern. With this interpretation, it will be seen clearly that the operation of $f(z) = z^2$ essentially rotates and expands the (x, y) plane. This is seen in the following video.



Standalone

However, you will also encounter some of these more complex mappings, notably the inversion map, $f(z) = 1/z$ in Chapter 4.

The theory in the video (visualisation on the Riemann sphere) is not necessary for this course; it is presented here just out of interest (and because the video is beautiful!).

Remark 1.2.7 It is sensible to ask: *what does this have to do with fluid mechanics?*. In Chapter 4, we will see that the use of complex functions can map a region of fluid to another.

1.2.5 Differentiation of complex functions

In this chapter, we will only cover the basic necessities of visualising and studying complex functions. In Chapter 4, we will need additional theory on the differentiation of complex functions.

1.2.6 Summary

Certain complex functions are only well-defined with appropriate branch cuts chosen. However, once such restrictions are made (and an individual Riemann sheet chosen), the complex function is well defined. The branch cut will correspond to locations where the function is nonsmooth (in its real and/or imaginary components).

1.3 Exercises

The main function of this chapter was to briefly review complex functions and also review/introduce you to the notion of branch cuts. Complex functions will be used in the potential theory of Chapter 4 and wave theory of Chapter 5.

1. **Plotting a Riemann surface.** Select the branch cut of $f(z) = z^{1/2}$ that runs along the positive real axis.

   (a) Consider a contour that starts from $z = 1$, then encircles the origin (anticlockwise) and returns to $z = 1$. What is the jump in the value of $f(z)$ at the end of the contour as compared to the start?

   Hint. Let $z = \mathrm{e}^{\mathrm{i}\theta}$ and consider θ ranging from the initial value to a final value.

   Solution. Let $z = \mathrm{e}^{\mathrm{i}\theta}$. For $\theta = 0$, $f(z) = r^{1/2}$ if we choose the positive branch of the square root (by convention). At the other side of the branch cut, $\theta = 2\pi$ and $f(z) = r^{1/2}e^{\pi i} = -r^{1/2}$. Therefore there is a jump in value of $-2r^{1/2}$.

(b) By hand, plot the Riemann surface as visualised in $(x, y, \operatorname{Im} f(x + iy))$-space, where $\operatorname{Im} f(z) = r^{1/2}\sin(\theta/2)$. You may also confirm your sketch with a computational tool, if desired.

Hint. It is useful to first consider the plot of $\sin(\theta/2)$, and then separately, what happens for the magnitude variation that depends on $r^{1/2}$.

Solution. A sketch of the imaginary part of the square root function is shown below. The two key features to capture is the dependence on θ and the dependence on r.

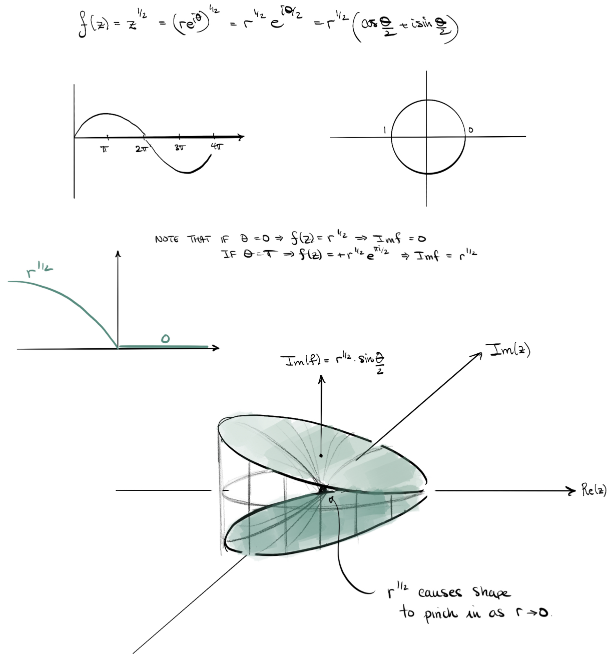


Figure 1.3.1 Sketch of the imaginary part of the square root function

2. **A function with two branch points.** Consider the function (1.2.4):

$$f(z) = (z + 1)^{1/2}(z - 1)^{1/2}.$$

(a) Choose the branch cut from $z = 1$ in the positive real direction. Choose the branch cut from $z = -1$ in the negative real direction. Write either $z = r_1 e^{i\theta_1}$ or $z = r_2 e^{i\theta_2}$ for θ_1 and θ_2 defined as relative angles from the two branch points.

Show that: (i) when $z = 1$ is encircled by a complete revolution, the function jumps in value by a factor of $e^{i\pi}$; (ii) that there is a similar jump in value when $z = -1$ is encircled. Finally what happens if (iii) $z = 0$ is encircled?

Draw a picture of the final z-plane, showing the branch cuts.

Solution. (i) We have that

$$f(z) = (r_1 r_2)^{1/2} e^{i\theta_1/2} e^{i\theta_2/2}.$$

Let $\theta_1 \in [0, 2\pi)$ be the angle about the point $z = 1$. Similarly let $\theta_2 \in [-\pi, \pi)$ be the angle about the point $z = -1$.

Considering firstly a revolution around $z = 1$ (that does not also enclose $z = -1$). Let the initial point be denoted "A", with $\theta_1 = 0, \theta_2 = 0$. And the final point be "B", with $\theta_1 = 2\pi$ and $\theta_2 = 0$.

Then

$$f(B) - f(A) = (r_1 r_2)^{1/2} e^{2\pi i/2} - (r_1 r_2)^{1/2} e^0 = -2(r_1 r_2)^{1/2}.$$

so indeed there is a jump in the function about the branch cut along $z > 1$.

(ii) We would similarly verify that for a contour around the branch point $z = -1$ there is a jump. Let the initial point be denoted "A" with $\theta_1 = \pi, \theta_2 = -\pi$. And the final point be "B" with $\theta_1 = \pi, \theta_2 = \pi$. Then

$$f(B) - f(A) = (r_1 r_2)^{1/2} e^{(\pi+\pi)i/2} - (r_1 r_2)^{1/2} e^{(\pi-\pi)i/2} = -2(r_1 r_2)^{1/2}.$$

so indeed there is also a jump about the branch cut that runs $z < -1$.

(iii) For the centre point, there are two cases to consider. The first case is if only $z = 0$ is encircled and none of the other branch points. This was the situation originally envisioned in the question. For example, consider the circle of radius $1/2$, i.e. $z = (1/2)e^{i\theta}$ with $\theta \in [0, 2\pi)$. You can verify that for this circle the start and end points agree.

If along with $z = 0$, one of the branch points is encircled, then there would be a discontinuity. If both branch points are encircled, there is no discontinuity.

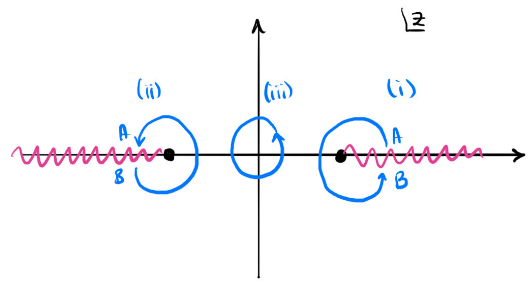


Figure 1.3.2 Branch cut configuration for the double square root. The original choice of branches is shown on top. Through the analysis, we see that a circle (i) around $z = 1$ does not produce a discontinuity. Hence only the second picture of the cut arrangement is needed.

(b) Consider now a branch cut from $z = -1$ that tends in the positive real direction and the branch cut from $z = 1$ tends in the positive real direction as well. Repeat the experiment above, considering (i)-(iii). Conclude that there is no jump in value along the region $z > 1$ and hence the branch cuts required only extends between $z = \pm 1$.

Draw a picture of the final z-plane, showing the branch cuts.

Solution. For this situation, we would define the ranges of $\theta_1 \in [0, 2\pi)$ and $\theta_2 \in [0, 2\pi)$. One main difference is the analysis around the point $z = 1$. Consider the similar loop to the above with,

$$f(B) - f(A) = (r_1 r_2)^{1/2} \mathrm{e}^{(2\pi+2\pi)\mathrm{i}/2} - (r_1 r_2)^{1/2} \mathrm{e}^{(0+0)\mathrm{i}/2} = 0,$$

so in this case, notice that there is no jump due to the fact the total variation of angle is $2\pi + 2\pi$.

In essence, because both branch cuts are running in the positive real direction, when we orbit across $z > 1$, we jump through both branches, hence returning to the original. There is no required branch cut for $z > 1$.

The analysis of parts (ii) and (iii) are identical, with the exception of the angle range. However, the final result, of whether there exists a jump is the same.

In the end, the final branch cut picture is shown in the figure below.

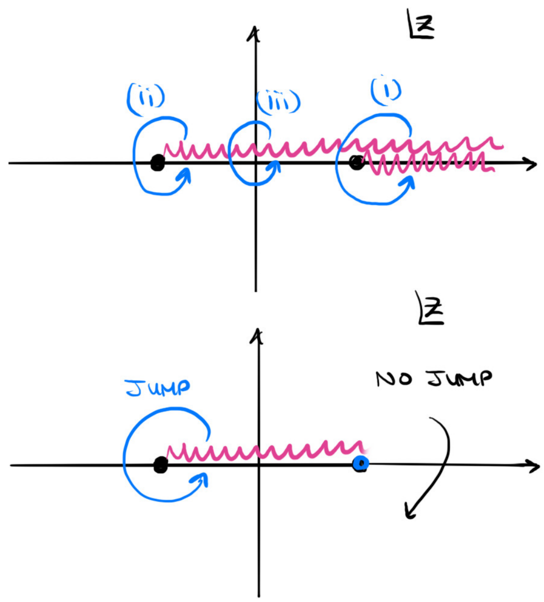


Figure 1.3.3 Branch cut configuration for the double square root

(c) (Challenging). If you consider a plot of $(x, y, \operatorname{Re} f(x + \mathrm{i}y))$ or $(x, y, \operatorname{Im} f(x + \mathrm{i}y))$, what will the Riemann surface look like? You can attempt to plot this using any tool.

Solution. This is certainly not an easy function to imagine! There are two features you may want to keep in mind. First, in examining the imaginary part of the function, if $z > 1$ on the real axis, then the imaginary part is zero. Second, if $z < -1$ on the real axis, then again the function is zero. Finally, for the case of the branch selection in part (b), the there is a cut along $[-1, 1]$. A generated plot is shown below for the imaginary part.

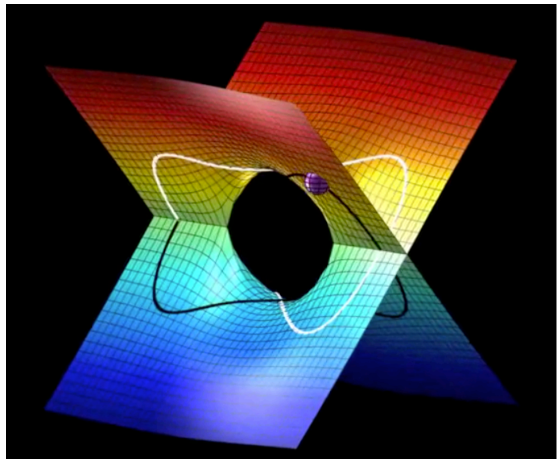


Figure 1.3.4 The imaginary part of $f(z) = (z-1)^{1/2}(z+1)^{1/2}$

3. Branch cuts of the complex logarithm. Consider the complex logarithm as defined in (1.2.3).

(a) Explain why there must be a branch cut imposed, originating from the branch point at $z = 0$.

Solution. The logarithm will have a jump in the imaginary part every rotation of 2π in the argument.

(b) Take the branch cut along the positive real axis. Do your best to draw the Riemann surface (consisting of the distinct Riemann sheets) of the logarithm, as visualised in the space $(x, y, \text{Im}\, f(x+\text{i}y))$.

Solution.

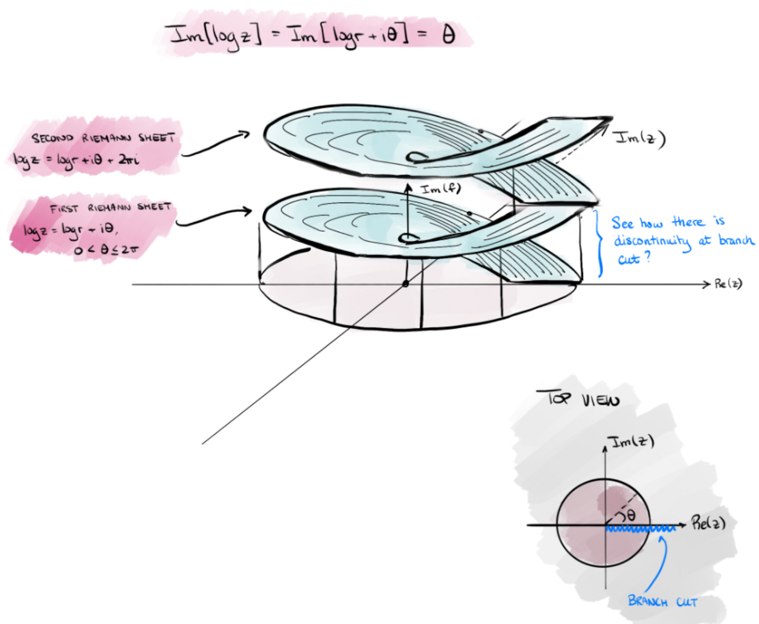


Figure 1.3.5 The imaginary part of $f(z) = \log z$.

(c) Again, you may find it useful to confirm your work above by plotting the function using a computational tool.

Solution. There is a picture of the complex logarithm on Wikipedia.

Chapter 2

Kinematics

Fluid *kinematics* refers to the study of fluid motions without considering the associated forces (or energies) that cause such a fluid to move. In a nutshell, it relates to studying e.g. the general velocity and acceleration fields of a fluid, visualising the fluid trajectories, and so forth.

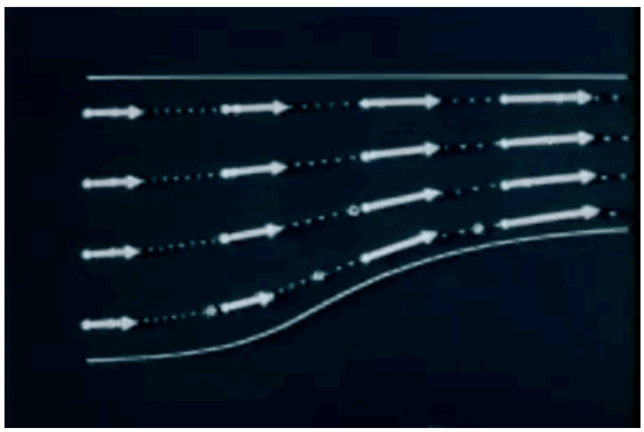


Figure 2.0.1 Eulerian description of flow from the NCFMF video.

There are some unintuitive challenges when it comes to defining even the most basic quantities in fluid dynamics. Consider the following two thought experiments.

In the first experiment, we consider a tube filled with water, where the bottom is released at time $t = 0$. One can imagine that the entire column of fluid will fall under the effect of gravity.

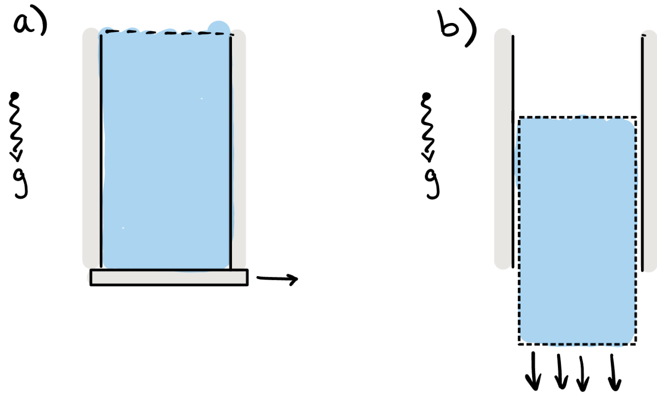


Figure 2.0.2 (a) Tube filled with fluid with a plate to be removed; (b) fluid falls under the effect of gravity with uniform acceleration everywhere.

Each particle within the fluid is accelerated downwards at a constant rate (gravity). Therefore the velocity of the fluid is given by $\boldsymbol{u} = (0, 0, -U(t))$. In this case, we imagine that acceleration can be defined simply as the partial derivative, in time, of the velocity field, or

$$\boldsymbol{a} = \frac{\partial \boldsymbol{u}}{\partial t} = (0, 0, -\dot{U}(t)) = (0, 0, -g),$$

where we have used the over-dot notation for a derivative in time.

However, consider now the second thought experiment of fluid being pumped through the vessel shown in the figure below. The fluid has reached a steady-state (so the streamlines of the flow, if you can imagine them, are constant and travelling from left-to-right).

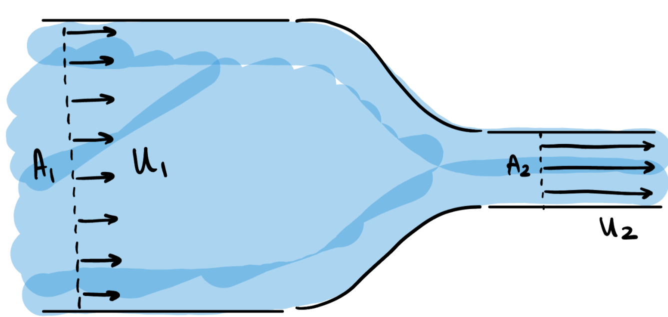


Figure 2.0.3 Fluid pumped through a tube with variable cross-sectional area, A_1, A_2 and corresponding velocities U_1, U_2.

Upstream of the bottleneck, the volume flowrate is equal to the velocity times area, or

$$\text{Upstream flow rate (area times velocity)} = A_1 U_1.$$

This is the amount of fluid that is flowing through a cross section of the pipe upstream. Similarly, we have

$$\text{Downstream flow rate (area times velocity)} = A_2 U_2.$$

However, since mass must be conserved, that fact that there is a smaller cross-sectional area downstream must mean that the velocity is higher downstream. Therefore, $U_2 > U_1$, and to achieve this, the fluid must therefore have been accelerated somewhere within the channel.

However, a thought to the situation might convince you that the velocity field in the channel is time-independent: it only changes as a function of its position and is therefore of the form $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$. Indeed, at every fixed point in the channel, the velocity *at that point* is not changing in time. Therefore, if we use our previous definition that $\boldsymbol{a} = \dot{\boldsymbol{u}}$ we see that the acceleration would be zero everywhere!

The crux of the issue, and the difference between the two thought experiments, is that velocity is both a function of space and time, i.e. $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$. The change in velocity (acceleration) can therefore derive from changes in $\boldsymbol{u}$ at fixed space and variable time; but also due to changes in $\boldsymbol{u}$ at fixed time and variable space. Disentangling this leads us to define the nature of Eulerian and Lagrangian coordinates, presented next.

2.1 Eulerian and Lagrangian coordinates

There are essentially two natural ways to think of motion in a fluid. We can imagine positioning ourselves at a fixed point in space, $\boldsymbol{x} = (x, y, z)$. At this point, we then attempt to measure a fluid quantity such as the density, $\rho(\boldsymbol{x}, t)$, or temperature, $T(\boldsymbol{x}, t)$. This is essentially the *Eulerian frame*. One can imagine, for example, fixing sensor station into the ocean bottom, and obtaining measurements of the water temperature.

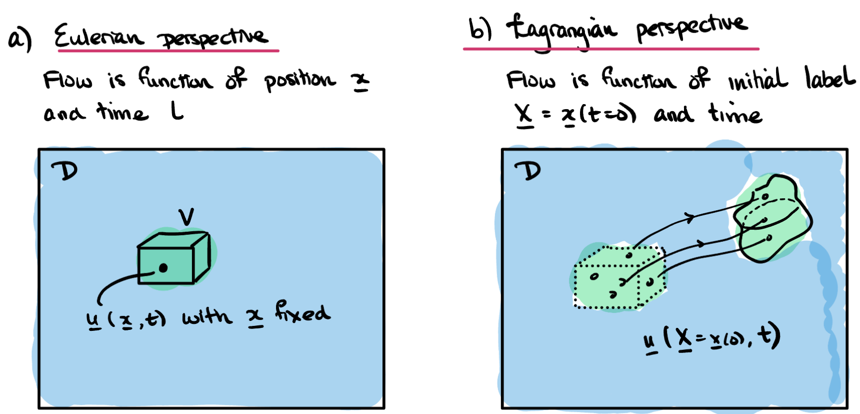


Figure 2.1.1 (a) The Eulerian interpretation; (b) the Lagrangian interpretation.

Alternatively, we can imagine tracking of a single fixed particle (or a fluid element) within the flow. The particle begins at some position. Let us define a label to describe the particle's initial position. For example, if the particle's position is given by

$$(x, y, z) = (x_1(t), y_1(t), z_1(t)),$$

we can define the corresponding *Lagrangian label* as

$$\boldsymbol{X} = (X, Y, Z) = (x_1(0), y_1(0), z_1(0)).$$

We then ask for the corresponding measurement of the fluid quantity that corresponds to the label $\boldsymbol{X}$. For example, this is equivalent to tagging a free-floating buoy in the ocean with the label $\boldsymbol{X}$, then measuring the temperature of the water as the buoy drifts in the ocean. This Lagrangian temperature could be written $\mathcal{T}(\boldsymbol{X}, t)$, where $\boldsymbol{X}$ is simply a fixed quantity for the particular buoy.

We are now in a position to define the Eulerian velocity field of a fluid.

Definition 2.1.2 The Eulerian velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity of the fluid at the point with spatial coordinates $\boldsymbol{x}$ at time t. Note that, in physical terms this velocity is the average velocity at the time t of the fluid particles (e.g. molecules, ions) in a small box centred on the point $\boldsymbol{x}$. See also Remark 3.0.3 for a discussion of the continuum assumption. $\Diamond$

It will be useful to introduce the concept of steady flow.

Definition 2.1.3 A velocity field $\boldsymbol{u}$ is defined as steady if it can be written $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$. $\Diamond$

Note that steady flow does not mean that the fluid particles are not moving. It simply means that at a fixed point in space, the Eulerian velocity does not change in time. The Lagrangian velocity of a fluid particle will generally change in time, even in a steady flow.

A simple example of the conversion between the Eulerian and Lagrangian reference frames is in Exercise 2.3.3.

2.1.1 The convective derivative

Let us now be more specific. We wish to consider how different quantities in our flow changes with time, but the matter is made complicated by the two above perspectives (fixed or following the flow).

Again, let us consider a scalar property of the fluid (for example, its density, temperature, velocity component, pressure, etc.), and let us suppose that this quantity is a function of both position, $\boldsymbol{x}$, and time, t, and denote it by $f(\boldsymbol{x}, t)$. This is the *Eulerian description* of the property since it is defined by specifying a fixed position in space. Fixing $\boldsymbol{x}$ and then measuring f is akin to standing in the fluid at a fixed location and measuring the property value in time.

We can alternatively write the property by its Lagrangian description. That is, given a label $\boldsymbol{X}$, we obtain the current position of the particle associated with this label, $\boldsymbol{x}(\boldsymbol{X}, t)$, then obtain its property value. This we can write as the following:

$$F(\boldsymbol{X}, t) \equiv f(\boldsymbol{x}(\boldsymbol{X}, t), t).$$

Now, fixing $\boldsymbol{X}$ and changing t corresponds to tracking the scalar property at a material point in the flow, or, equivalently, how f changes as we move with the particle along the deforming fluid.

There are thus two ways of considering time derivatives.

Definition 2.1.4 We use the normal partial derivative notation to refer to an *Eulerian time derivative*, considered at a fixed point in space:

$$\frac{\partial}{\partial t} \equiv \left.\frac{\partial}{\partial t}\right|_{\boldsymbol{x}} = \text{rate of change with } \boldsymbol{x} \text{ held constant.}$$

On the other hand, the *Lagrangian time derivative* is defined at a fixed material point in the fluid.

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \left.\frac{\partial}{\partial t}\right|_{\boldsymbol{X}} = \text{rate of change with } \boldsymbol{X} \text{ held constant.}$$

We often refer to the Lagrangian time derivative as the *convective derivative* or the *material derivative*. $\Diamond$

Remark 2.1.5 The reason why the above derivatives are introduced is because, for the purpose of much of fluid dynamics, it is easier to work with Eulerian coordinates and quantities. However, for the purpose of deriving many governing equations, it turns out to be much easier to work with Lagrangian variables.

This is because physical forces act on physical particles, or material elements, of the fluid.

The natural question is how the two derivatives relate to one another. This is given by the following theorem.

Theorem 2.1.6 The material/convective derivative. *The* material or convective derivative *can be defined in terms of Eulerian derivative in the following way:*

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (\boldsymbol{u} \cdot \nabla). \tag{2.1.1}$$

Proof. This is a result of the chain rule. For a scalar function $f = f(\boldsymbol{x}, t)$, we have the fact that

$$\begin{aligned} \frac{\mathrm{D}f}{\mathrm{D}t} &= \frac{\mathrm{D}t}{\mathrm{D}t}\frac{\partial f}{\partial t} + \frac{\mathrm{D}x}{\mathrm{D}t} \cdot \nabla f \\ &= \frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \nabla f. \end{aligned}$$

■

The proof to Theorem 2.1.6 seems to use magic vector operations! In Exercise 2.3.2, we ask you to check this more carefully by expanding the vector operations explicitly.

We can now apply the above result to the question of how to calculate the acceleration within the fluid (more specifically, we are enquiring about the acceleration of a volume or particle within the fluid). The acceleration is given by the convective or material derivative of the velocity:

$$\boldsymbol{a} \equiv \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}. \tag{2.1.2}$$

You will practice using this formula in Exercise 2.3.4 of the problem set.

Remark 2.1.7 Vector gradient. In the formula for the acceleration in (2.1.2), the quantity

$$\nabla \boldsymbol{u}$$

appears. This is a tensor (matrix). One mustn't be too intimidated as it is just a convenient notation.

It is worth considering what this must be by considering each individual element of the acceleration. The acceleration is calculated simply by taking the velocity,

$$\boldsymbol{u} = (u, v, w) = (u_1, u_2, u_3).$$

and working out the material derivative for each individual component. So we have

$$\frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial u_i}{\partial t} + \boldsymbol{u} \cdot \nabla u_i. \tag{2.1.3}$$

So the above gives each of the three components of $\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}$.

You may prefer to see the above written in terms of the notation for $\boldsymbol{u} = (u, v, w)$, so it is

$$\begin{aligned} \frac{\mathrm{D}u}{\mathrm{D}t} &= u_t + (uu_x + vu_y + wu_z) \\ \frac{\mathrm{D}v}{\mathrm{D}t} &= v_t + (uv_x + vv_y + wv_z) \\ \frac{\mathrm{D}w}{\mathrm{D}t} &= w_t + (uw_x + vw_y + ww_z) \end{aligned}$$

You can also re-arrange the above in something closer to "matrix" form. So

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial}{\partial t}(u, v, w) + (u, v, w) \cdot \begin{pmatrix} \nabla u \\ \nabla v \\ \nabla w \end{pmatrix},$$

and where the last quantity corresponds to

$$\nabla \boldsymbol{u} \equiv \begin{pmatrix} \nabla u \\ \nabla v \\ \nabla w \end{pmatrix} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}. \tag{2.1.4}$$

And hence the vector gradient can be defined as a matrix, where each row of the matrix is the gradient of the elements of the vector.

The author's opinionated note it is that it is easier simply to remember that the material derivative is applied via (2.1.3) then it is to try and untangle the multiplication of matrix via (2.1.4).

2.2 Flow visualisation, fluxes, and forces

There are different ways to visualise the dynamics of a fluid. Given the velocity, $\boldsymbol{u}(\boldsymbol{x}, t)$, we can plot a vector field at each point in space, and at a fixed moment in time. Little arrows are used to indicate the direction and the length of the arrow can be chosen to represent the magnitude. Joining these up at a fixed moment in time into smooth curves gives the *streamlines* of the flow. This is often the easiest type of visualisation to perform mathematically, but the hardest experimentally.

Another representation of the flow is using *particle paths* or *pathlines*. Given a point and time, the particle path is the trajectory that would result if a particle were dropped into the flow at that chosen point and time. It is thus found by solving an equation where at every point on the trajectory, the particle's velocity is the specified velocity of the fluid.

A third representation is a *streakline*. If dye were continuously released into a fluid from a fixed chosen point, the streakline at a given time is the line that would be made by the dye. It is thus found by finding the current position of those particles whose pathline has visited the chosen point at any past time. This is often the easiest type of visualisation to perform experimentally, but the hardest to perform mathematically.

Note that in a steady flow, the streamlines, pathlines and streaklines all coincide. However, in an unsteady flow, they are all different. In Exercise 2.3.1 you will study a video showing this concept.

2.2.1 Definitions of streamlines, pathlines, streaklines

In the definition below, we define these concepts more concretely.

Definition 2.2.1 Particle streamlines. Consider a fixed time, $t = t_1$.

Select an initial point, $\boldsymbol{x}_1$ at this time.

The *streamline*, $\boldsymbol{x} = \boldsymbol{x}(s)$ through the above point is given by solving the parametric equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \boldsymbol{u}(\boldsymbol{x}(s), t_1), \quad \text{with } \boldsymbol{x}(s = 0) = \boldsymbol{x}_1,$$

where s is a parameter along the streamline. Choosing a variety of different initial points, $\boldsymbol{x}_1$, and solving the above equation gives a family of streamlines at time t_1. $\lozenge$

Basically, given a velocity field, we freeze time. The streamlines are those curves that are traced out by the velocity field in the "snapshot".

Definition 2.2.2 Pathline or particle path. Consider now a particle that begins at the location $\boldsymbol{x}_2$ at time $t = 0$.

We consider the *partical path* or *pathline* of the particle, given by the curve $\boldsymbol{x} = \boldsymbol{x}(t)$ and found by solving the equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}(t), t), \quad \text{with } \boldsymbol{x}(0) = \boldsymbol{x}_2.$$

Choosing a variety of initial points, $\boldsymbol{x}_2$, yields a family of pathlines. $\lozenge$

The pathline or particle path from an initial point is what we would physically expect if we were to dye the point with a colour and follow the dye colour as time increases.

Definition 2.2.3 Streakline. Consider now fixing a location $\boldsymbol{x}_3$.

The *streakline* for a point $\boldsymbol{x}_3$ is given by solving the equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}(t), t), \qquad \text{with } \boldsymbol{x}(t_3) = \boldsymbol{x}_3,$$

for a variety of values of t_3. This gives the current position of all particles that have passed through the point $\boldsymbol{x}_3$ at any time t_3 in the past. $\lozenge$

If it is the case that the velocity is time independent, i.e. $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$, then the three above definitions coincide.

You will practice the theory of these concepts in Exercise 2.3.5 and do a worked example in Exercise 2.3.6 of the problem set.

2.2.2 Examples of streamlines, pathlines, and streaklines

Let us practice these concepts.

Example 2.2.4 Stagnation point flow. Consider a fluid described by the two-dimensional velocity field

$$\boldsymbol{u}(\boldsymbol{x}, t) = (x, -y, 0).$$

Derive and plot the streamlines of the flow. Discuss what occurs with particle paths and streaklines.

Solution. There are many online applications, such as this one that will allow you plot a two-dimensional direction field. It is also good to do it yourself by hand.

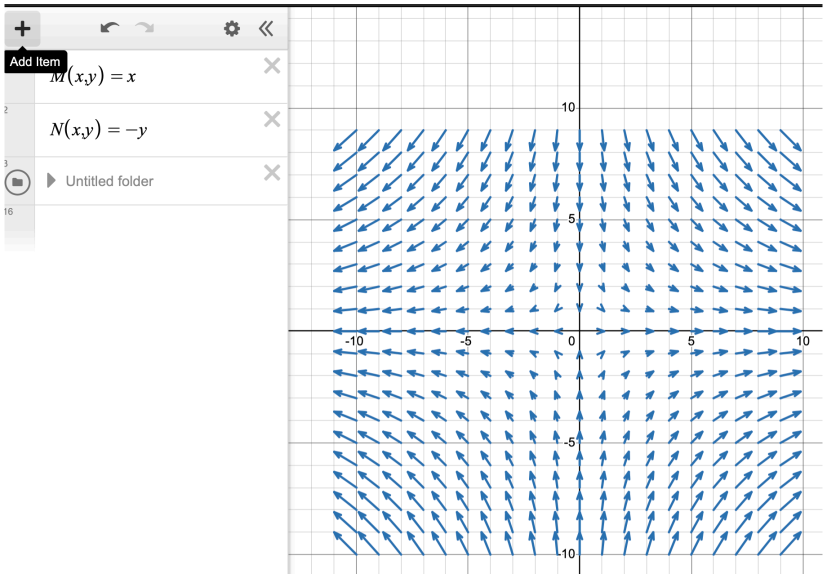


Figure 2.2.5 An example of a direction field

The streamlines follow from Definition 2.2.1. We seek to solve the equations

$$\frac{\mathrm{d}x}{\mathrm{d}s} = x,$$
$$\frac{\mathrm{d}y}{\mathrm{d}s} = -y,$$
$$\frac{\mathrm{d}z}{\mathrm{d}s} = 0.$$

Solving thus gives $x = A\mathrm{e}^s, y = B\mathrm{e}^{-s}, z = C$, for constants A, B, C.

You can put initial conditions to determine the constant and plot the trajectories for different values of s.

However, in this case, it is easier to remove the time-like variable, s, entirely. Notice that

$$xy = AB = \text{constant}.$$

Therefore, the trajectories lie along hyperbolae.

Notice that in this case, the velocity field is time-independent, and therefore the particle paths coincide with the streamlines and streaklines.

For instance, the particle path through a particle at $\boldsymbol{x}_2 = (1, 1, 0)$ is given by (replacing s with t):

$$\boldsymbol{x}(t) = (\mathrm{e}^t,\ \mathrm{e}^{-t},\ 0).$$

Similarly, the *streakline* through the point $(1, 1, 0)$ is precisely the set of points above. □

Example 2.2.6 Straight streamlines and circular pathlines. Consider the unsteady flow given by

$$\boldsymbol{u}(\boldsymbol{x}, t) = (\cos t, \sin t, 0).$$

Plot the streamlines of the flow on the plane $z = 0$, and also the particle trajectories. What occurs with the streaklines?

Answer. In this case, the velocity field is changing in time. Consider firstly the concept of the streamline in Definition 2.2.1.

We solve the governing equations for the streamlines $\boldsymbol{x} = \boldsymbol{x}(s)$,

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos t, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \sin t, \quad \frac{\mathrm{d}z}{\mathrm{d}s} = 0,$$

for fixed t.

This gives

$$x(s) = A + s\cos t, \quad y(s) = B + \sin t, \quad z(s) = C,$$

for constants A, B, C. Therefore the streamline are given by straight lines in the (x, y) plane, if $z = C$ is fixed.

Consider instead the definition of particle paths via Definition 2.2.2. We seek to solve

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \cos t, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \sin t, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = 0,$$

yielding

$$x(t) = A + \sin t, \quad y(t) = B - \cos t, \quad z(t) = C,$$

for constants A, B, C. Therefore, we see that the particle paths are closed circles (in the xy-plane) of unit radius encircling the point $(A, B, 0)$.

The fact that the streamlines are straight lines while the particle paths are circular can be visualised in Figure 2.2.7

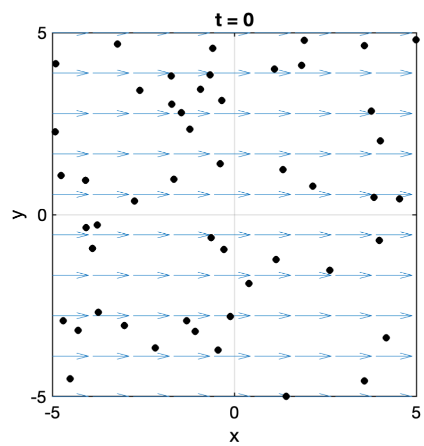


Figure 2.2.7 Streamlines and particle paths

In this case, considering the streakline via Definition 2.2.3, we conclude that the streakline coincides with the particle path. Can you reason why this must be the case in this situation? What is necessary in order for this not to be true? □

Example 2.2.8 An oscillating hose/plate. Water flows out of an oscillating sprinkler head, held along the edge $y = 0$, such that the velocity field produced is given by

$$\boldsymbol{u} = [u_0 \sin[\omega(t - y/v_0)], v_0],$$

where u_0 and v_0 are constants.

Determine the streamline that passes through the origin at $t = 0$ and $t = \pi/(2\omega)$.

Determine the pathline of the particle that was at the origin at $t = 0$; at $t = \pi/2$.

Qualitatively describe the shape of the streakline that passes through the origin?

Answer.

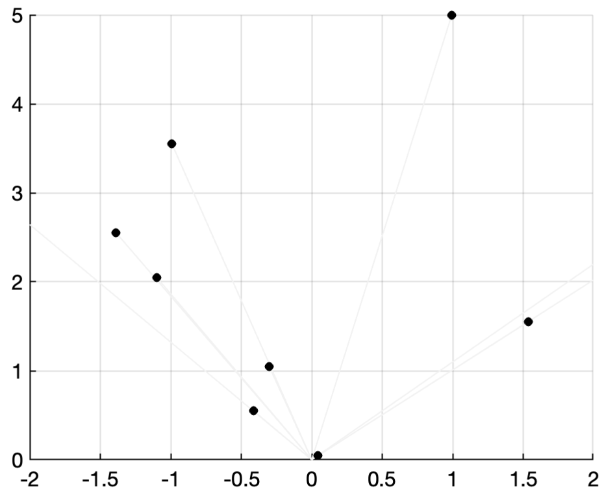


Figure 2.2.9 Isolated pathlines show that particles move along straight paths.

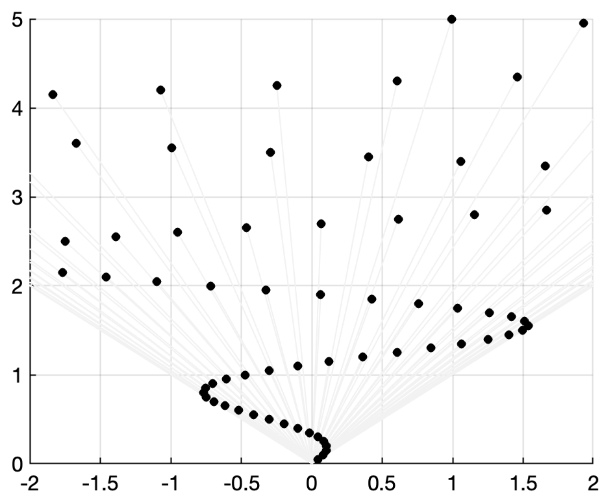


Figure 2.2.10 However, viewed in terms of the streakline, the visible pattern is oscillatory.

□

2.2.3 Fluxes and forces

Before we go on, we remind the reader of two important quantities that will be used in the following chapters. These are typicaly introduced in the prior modules on vector calculus.

The first is the notion of *flux through a surface*. Given a surface S and a velocity field $\boldsymbol{u}$, the flux through the surface is the amount of flow through the surface per unit time. It is given by the integral

$$\text{flux through } S = \iint_S \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{2.2.1}$$

and $\boldsymbol{n}$ is the outer unit normal.

In 2D, the flux due to a 2D velocity field can also be written as

$$\text{flux through } C = \int_C \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}s. \tag{2.2.2}$$

In the two above formulae, refer to your previous vector calculus notes for the procedures to calculate the area element $\mathrm{d}S$ or the line element $\mathrm{d}s$.

Finally, we may like to also calculate the total force on a surface or on a contour. If $\boldsymbol{F}$ is the pointwise force applied at every point, the total force is given by

$$\text{Total force} = \iint_S \boldsymbol{F} \, \mathrm{d}S. \tag{2.2.3}$$

The above remains a vector quantity.

2.3 Exercises

The kinematics chapter covered the basic essentials about the measurement and computation of fluid velocities and acceleration. We examined the difference between Eulerian and Lagrangian coordinates and derivatives. Examples of streamlines and velocity fields were examined.

1. **Flow visualisation.** Watch the first 13 and half minutes of the video "Flow visualisation" from the NCFMF archives.

   (a) Name a few ways in which a fluid flow can be visualised (i.e. how does one produce, in an experiment, such a visualisation?)

   (b) Define, in words, what *pathline*, *streakline*, *timeline*, and *streamline* means. At 6:30, the author comments that "there is no way to make a streamline visible". Discuss this point.

   (c) Give the example, shown in the video, where the pathline, streakline, and streamlines all coincide. Draw pictures to illustrate the concept.

   (d) Give an example, shown in the video, where the pathline, streakline, and streamlines do not coincide. Draw pictures to illustrate the concept.

   (e) In the video, an intuitive explanation is given for what it means for the velocity to be *incompressible*. What is this explanation? (near 8:25)

2. **Derivation of the material derivative.** Prove the Eulerian representation of the material derivative, as presented in Theorem 2.1.6 by manually expanding the components of the function. That is, consider a scalar function $f(\boldsymbol{x}, t)$. Let the spatial coordinate follow $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$ and hence is defined by a specified material label $\boldsymbol{X}$. Then derive the identity for

   $$\frac{\mathrm{D}f}{\mathrm{D}t} = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{X}},$$

   according to Definition 2.1.4.

   Solution. This is the chain rule. Denote the three components of $\boldsymbol{x} = (x_1, x_2, x_3)$. Then we differentiate through both outer arguments of f:

   $$\left.\frac{\partial f(\boldsymbol{x}(\boldsymbol{X}, t), t)}{\partial t}\right|_{\boldsymbol{X}} = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{x}} + \sum_{i=1}^{3} \left.\frac{\partial x_i(\boldsymbol{X}, t)}{\partial t}\right|_{\boldsymbol{X}} \frac{\partial f}{\partial x_i}$$

The factor,

$$\left.\frac{\partial x_i}{\partial t}\right|_{\boldsymbol{X}} = \frac{\mathrm{d}x_i}{\mathrm{d}t} \equiv u_i,$$

corresponds to the velocity component in each direction. Then

$$\left.\frac{\partial f(\boldsymbol{x}(\boldsymbol{X},t),t)}{\partial t}\right|_{\boldsymbol{X}} = \frac{\partial f}{\partial t} + \boldsymbol{u}\cdot\nabla f,$$

as desired.

3. **Eulerian and Lagrangian descriptions.** A velocity field is described in Eulerian terms in Cartesian coordinates by $\mathbf{u} = (-y, x)$.

(a) What is the Lagrangian position of the particle that starts at the point (x_0, y_0)? Describe its path. What is its velocity?

Solution. A particle moves with the velocity of the fluid. Then it must satisfy the equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y(t), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = x(t) \quad \Rightarrow \quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{\mathrm{d}y}{\mathrm{d}t} = -x(t).$$

Trying solutions of the form $x(t) = \mathrm{e}^{\lambda t}$ gives $\lambda = \pm i$, so the general solution is

$$x(t) = Ae^{it} + Be^{-it},$$

or, equivalently,

$$x(t) = C\cos t + D\sin t,$$

where C and D are constants, and

$$y(t) = -\frac{\mathrm{d}x}{\mathrm{d}t} = C\sin t - D\cos t.$$

Setting $x(0) = x_0$ and $y(0) = y_0$ gives $C = x_0$ and $D = -y_0$, so

$$\boldsymbol{x}(t) = (x_0\cos t - y_0\sin t, x_0\sin t + y_0\cos t).$$

This is a circular path of radius $\sqrt{x_0^2 + y_0^2}$ going anticlockwise around the origin. The velocity is

$$\boldsymbol{u}(t) = (-x_0\sin t - y_0\cos t, x_0\cos t - y_0\sin t).$$

(b) Express the position and velocity in polar coordinates. What do you notice?

Hint. Note that

$$\hat{\boldsymbol{r}} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}, \quad \hat{\boldsymbol{\theta}} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j},$$

see also (**??**), (**??**) and (**??**).

Solution. The calculation works out more simply in polar coordinates; the velocity field is

$$\boldsymbol{u} = r\boldsymbol{e}_\theta,$$

where $\boldsymbol{e}_\theta$ is the angular unit vector given in (**??**). Then the equation for particle paths is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}r\theta}{\mathrm{d}t} = r \quad \Rightarrow \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = 1.$$

Thus $r(t) = r_0 = \sqrt{x_0^2 + y_0^2}$ and $\theta(t) = \theta_0 + t$, where $\theta_0 = \tan^{-1}(y_0/x_0)$. The velocity is $\boldsymbol{u}(t) = r_0\boldsymbol{e}_\theta(t)$.

Note that the expression for the velocity in polar coordinates is simpler than that in Cartesian coordinates, being in one coordinate direction only. Performing this conversion makes it easier to find pathlines and streamlines.

This flow is an example of *rigid-body* rotation, that is the fluid is moving with a velocity field that would be attainable for a rigid body, and there is no *relative* motion of the fluid particles.

4. **Eulerian and Lagrangian descriptions.** A fluid flows through the nozzle shown from $x = 0$ to $x = L$ with one-dimensional velocity

$$u = U\left(1 + \frac{x}{L}\right)$$

in the x-direction, where U and L are constants. Note that, in reality the flow would be two- or three-dimensional, but we ignore this here for simplicity.

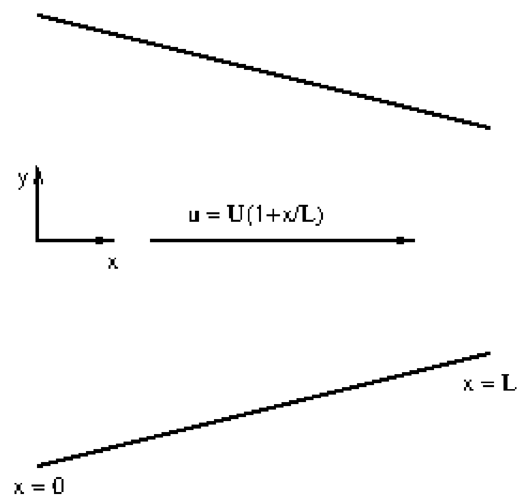


Figure 2.3.1 Sketch of nozzle.

(a) What is the particle acceleration?

Solution.

$$\begin{aligned}
\boldsymbol{a} =& \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \\
=& \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \\
=& 0 + \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\boldsymbol{e}_x + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\boldsymbol{e}_y + \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\boldsymbol{e}_z \\
=& u\frac{\partial u}{\partial x}\boldsymbol{e}_x \\
=& U\left(1 + \frac{x}{L}\right)\frac{U}{L}\boldsymbol{e}_x \\
=& \frac{U^2}{L}\left(1 + \frac{x}{L}\right)\boldsymbol{e}_x
\end{aligned}$$

(b) If a particle starts at $x = 0$ at time $t = 0$, what is its position at time t?

Solution. We have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u(x) = U\left(1 + \frac{x}{L}\right).$$

Solving by separation of variables gives,

$$\begin{aligned} \int_0^x \frac{dx}{1 + x/L} &= \int_0^t U\,dt \\ \Rightarrow \quad L\ln\left(1 + \frac{x}{L}\right) &= Ut \\ \Rightarrow \quad x(t) = L\left(\mathrm{e}^{Ut/L} - 1\right). \end{aligned}$$

(c) What is its Lagrangian velocity as a function of time?

Solution. The velocity is

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} =& \frac{\mathrm{d}}{\mathrm{d}t}\left(L\left(\mathrm{e}^{Ut/L} - 1\right)\right) \\ =& L\frac{U}{L}\mathrm{e}^{Ut/L} \\ =& U\mathrm{e}^{Ut/L}. \end{aligned}$$

(d) What is its Lagrangian acceleration as a function of time?

Solution. The acceleration is

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} =& \frac{\mathrm{d}}{\mathrm{d}t}\left(U\mathrm{e}^{Ut/L}\right) \\ =& \frac{U^2}{L}\mathrm{e}^{Ut/L}. \end{aligned}$$

(e) Write the Lagrangian velocity and acceleration as a function of x. Compare with the corresponding Eulerian velocity and particle acceleration. What do you notice?

Solution. The velocity is

$$U\mathrm{e}^{Ut/L} = U\left(1 + \frac{x}{L}\right),$$

as expected. This is the same as the Eulerian velocity at the particle position. The acceleration is

$$\frac{U^2}{L}\mathrm{e}^{Ut/L} = \frac{U^2}{L}\left(1 + \frac{x}{L}\right),$$

which is the same as the particle acceleration, as expected from the defintion.

(f) How long does it take for a particle to travel from $x = 0$ to $x = L$?

Solution. The particle reaches $x = L$ when

$$\begin{aligned} L\left(\mathrm{e}^{Ut/L} - 1\right) &= L \\ \Rightarrow \quad \mathrm{e}^{Ut/L} &= 2 \\ \Rightarrow \quad t &= \frac{L}{U}\ln(2). \end{aligned}$$

5. Particle paths and streamlines.

(a) Define the particle paths and streamlines for a velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$. When do these coincide?

Answer. Particle paths are the trajectories of individual fluid particles, which are found by solving the ODE

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}, t).$$

Streamlines are curves that are instantaneously tangent to the velocity field, and therefore satisfy

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \boldsymbol{u}(\boldsymbol{x}, t),$$

where s is a parameter along the streamline. The two coincide if the flow is steady, i.e. $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x})$.

(b) By drawing a sketch of a pathline and considering the points $\boldsymbol{x}$ where the particle is at time t and $\boldsymbol{x} + \mathrm{d}\boldsymbol{x}$ at time $t + \mathrm{d}t$, show that a quantity $f(\boldsymbol{x}, t)$ is preserved following the flow if

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \nabla f = 0.$$

Hint. We can write $\mathrm{d}\boldsymbol{x} = \boldsymbol{u}(\boldsymbol{x}, t)\mathrm{d}t + \ldots$, where "..." indicates smaller corrections. We need to find

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \lim_{\mathrm{d}t \to 0} \frac{f(\boldsymbol{x} + \mathrm{d}\boldsymbol{x}, t + \mathrm{d}t) - f(\boldsymbol{x}, t)}{\mathrm{d}t}.$$

Now write $f(\boldsymbol{x} + \mathrm{d}\boldsymbol{x}, t + \mathrm{d}t)$ as a Taylor series about $f(\boldsymbol{x}, t)$ and use this to simplify the above expression.

Answer.

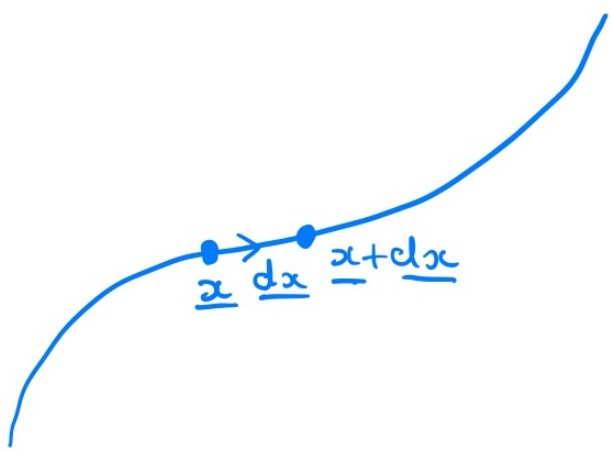


Figure 2.3.2 Pathline, showing the particle positions at times t and $t + \mathrm{d}t$.

Figure 2.3.2 shows the points $\boldsymbol{x}(t)$ and $\boldsymbol{x}(t + dt) = \boldsymbol{x}(t) + \mathrm{d}\boldsymbol{x}$, and the equation for pathlines gives $\mathrm{d}\boldsymbol{x} = \boldsymbol{u}(\boldsymbol{x}, t)\mathrm{d}t$. We follow the value of f following a particle:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{\mathrm{d}t \to 0} \frac{1}{\mathrm{d}t} \left(f(\boldsymbol{x} + \mathrm{d}\boldsymbol{x}, t + \mathrm{d}t) - f(\boldsymbol{x}, t) \right)$$

$$= \lim_{\mathrm{d}t\to 0} \frac{1}{\mathrm{d}t}\left(f(\boldsymbol{x},t) + \mathrm{d}\boldsymbol{x}\cdot\nabla f(\boldsymbol{x},t) + \mathrm{d}t\frac{\partial f}{\partial t}(\boldsymbol{x},t) + \dots - f(\boldsymbol{x},t)\right)$$
$$= \lim_{\mathrm{d}t\to 0} \frac{1}{\mathrm{d}t}\left(\boldsymbol{u}(\boldsymbol{x},t)\mathrm{d}t\cdot\nabla f(\boldsymbol{x},t) + \mathrm{d}t\frac{\partial f}{\partial t}(\boldsymbol{x},t) + \dots \right)$$
$$= \frac{\partial f}{\partial t} + \boldsymbol{u}\cdot\nabla f = \frac{\mathrm{D}f}{\mathrm{D}t},$$

(''...'' denotes higher order terms). Thus the value is preserved if $\frac{\mathrm{D}f}{\mathrm{D}t} = 0$.

(c) In a similar way, show that f is constant along streamlines if

$$\boldsymbol{u}\cdot\nabla f = 0.$$

Hint. In this case we can consider the points $\boldsymbol{x}$ and $\boldsymbol{x}+\mathrm{d}\boldsymbol{x}$ on the streamline, where $\mathrm{d}\boldsymbol{x} = \boldsymbol{u}(\boldsymbol{x},t)\mathrm{d}s + \dots$, where "..." indicates smaller corrections. We need to find

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \lim_{\mathrm{d}s\to 0} \frac{f(\boldsymbol{x}+\mathrm{d}\boldsymbol{x},t) - f(\boldsymbol{x},t)}{\mathrm{d}t}.$$

Now write $f(\boldsymbol{x}+\mathrm{d}\boldsymbol{x},t)$ as a Taylor series about $f(\boldsymbol{x},t)$.

Answer. Now suppose that in Figure 2.3.2, we have shown instead the situation of a streamline.

In this case, the points $\boldsymbol{x}(s)$ and $\boldsymbol{x}(s+ds) = \boldsymbol{x}(s) + \mathrm{d}\boldsymbol{x}$ are shown, and the equation for streamlines gives $\mathrm{d}\boldsymbol{x} = \boldsymbol{u}(\boldsymbol{x},t)\mathrm{d}s$. We follow the value of f along the streamline:

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \lim_{\mathrm{d}s\to 0}\frac{1}{\mathrm{d}s}\left(f(\boldsymbol{x}+\mathrm{d}\boldsymbol{x},t) - f(\boldsymbol{x},t)\right)$$
$$= \lim_{\mathrm{d}s\to 0}\frac{1}{\mathrm{d}s}\left(f(\boldsymbol{x},t) + \mathrm{d}\boldsymbol{x}\cdot\nabla f(\boldsymbol{x},t) - f(\boldsymbol{x},t)\right)$$
$$= \lim_{\mathrm{d}s\to 0}\frac{1}{\mathrm{d}s}\left(\boldsymbol{u}(\boldsymbol{x},t)\mathrm{d}s\cdot\nabla f(\boldsymbol{x},t)\right)$$
$$= \boldsymbol{u}\cdot\nabla f.$$

Thus the value of f is preserved if $\boldsymbol{u}\cdot\nabla f = 0$.

6. **Streamlines, pathlines and streaklines.** The velocity of a two-dimensional fluid flow in Cartesian coordinates is $u = 1$, $v = te^{-4x}$. Calculate and sketch the following:

(a) the streamlines at a fixed time $t > 0$,

Answer. The streamlines are given by

$$\frac{\partial y}{\partial x} = te^{-4x} \quad \Rightarrow \quad y = y_0 - \frac{t}{4}e^{-4x},$$

for any constant y_0. Different values of y_0 give different streamlines. See Figure 2.3.3 for a sketch.

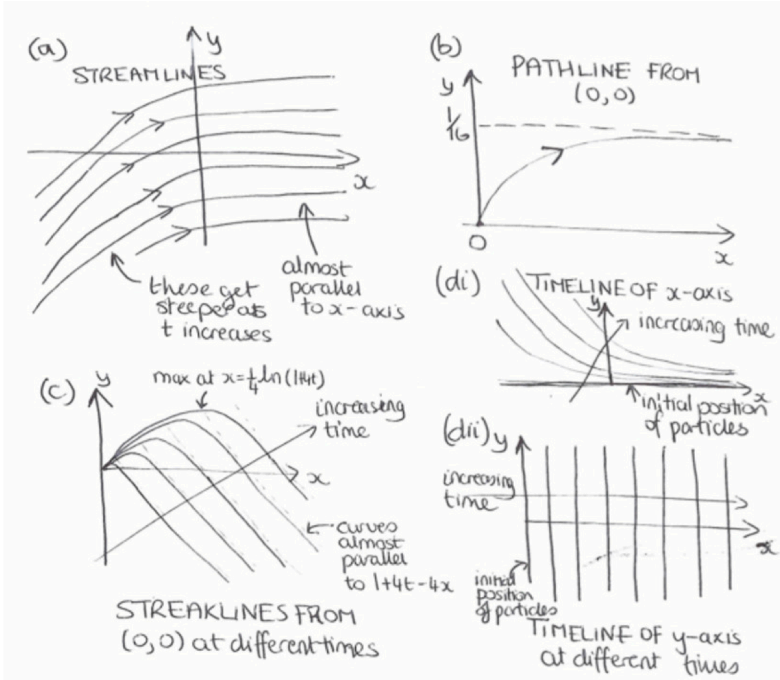


Figure 2.3.3 Sketch of streamlines, pathline and streaklines.

(b) the pathline of the particle starting from (x_0, y_0). Hence find and sketch the pathline of the particle that starts from $(0, 0)$,

Answer. The pathlines for a particle starting from (x_0, y_0) are given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = te^{-4x}$$
$$\Rightarrow \quad x = x_0 + t, \quad y = y_0 + \frac{1}{16}e^{-4x_0} - \frac{1+4t}{16}e^{-4(x_0+t)}.$$

Eliminating t:

$$y = y_0 + \frac{1}{16}e^{-4x_0} - \frac{1+4x-4x_0}{16}e^{-4x}.$$

This is a general pathline. For the particular particle that starts from $(0, 0)$:

$$y = \tfrac{1}{16}\left(1 - (1+4x)e^{-4x}\right).$$

See Figure 2.3.3 for a sketch.

(c) and the streaklines passing through $(0, 0)$.

Hint. The expression for a general pathline starting from (x_0, y_0) is given by

$$x = x_0 + t, \quad y = y_0 + \frac{1}{16}e^{-4x_0} - \frac{1+4t}{16}e^{-4(x_0+t)}.$$

Answer. Suppose the particle that started at (x_0, y_0) hits the special point $(0, 0)$ at time t_1. From the pathline equation:

$$0 = x_0 + t_1, \quad 0 = y_0 + \tfrac{1}{16}e^{-4x_0} - \tfrac{1+4t_1}{16}e^{-4(x_0+t_1)}.$$

Solving for x_0 and y_0:

$$x_0 = -t_1, \quad y_0 = -\tfrac{1}{16}\left(e^{4t_1} - 1 - 4t_1\right).$$

At time t the particle is at:

$$x = t - t_1, \quad y = \tfrac{1}{16}\left((1+4t_1) - (1+4t)e^{-4(t-t_1)}\right).$$

Eliminating t_1 gives the streakline at time t:

$$y = \tfrac{1}{16}\left((1+4t)(1-e^{-4x}) - 4x\right).$$

See Figure 2.3.3 for a sketch. Note that the streamlines, pathlines and streaklines are all very different.

(d) For a line/curve in the (x, y), its timeline at time t is the locus of the material elements at time t that started on the line/curve at $t = 0$. In experiments this can be visualised using a streak of dye placed initally along the line/curve. Tracking this over time gives the timelines.

Calculate and sketch the timelines of the line (i) $y = 0$, (ii) $x = 0$.

Answer. From the pathline equation, a particle starting from $(x_0, 0)$ reaches

$$x = x_0 + t, \quad y = \tfrac{1}{16}e^{-4x_0} - \tfrac{1+4t}{16}e^{-4(x_0+t)}$$

at time t. Eliminating x_0 gives the timeline:

$$y = \tfrac{1}{16}\left(e^{4t} - 1 - 4t\right)e^{-4x}.$$

See Figure 2.3.3 for a sketch.

From the pathline equation, a particle starting from $(0, y_0)$ reaches

$$x = t, \quad y = y_0 + \tfrac{1}{16} - \tfrac{1+4t}{16}e^{-4t}$$

at time t. Since y_0 can take any value, the timeline is just the line $x = t$. See Figure 2.3.3 for a sketch.

7. An experiment for the material derivative. Watch around 12:00 to around 19:00 of the video "Eulerian and Lagrangian Descriptions of Fluid Mechanics" from the NCFMF archives.

(a) In the video, the authors describe an experiment that one can setup to measure decay of a substance, say $C(x, t)$ along a 1D segment in a river.

In the first situation, it is assumed that C is distributed uniformly in the river, but is a naturally decaying substance with some uniform rate of change, $\frac{\mathrm{d}C}{\mathrm{d}t}$. Explain what is the (material) change in C that would be measured via two sensors, one upstream ane one downstream.

(b) Next, the authors imagine a situation where C is not uniformly distributed. They give a visual and analytical explanation of the material derivative,

$$\frac{\mathrm{D}C}{\mathrm{D}t} = \frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x},$$

where u is the horizontal velocity within the infinitessimal 1D section. Write your own explanation of the above.

Chapter 3

Equations of motion

In this chapter, we develop the basic equations of fluid dynamics. The equations are derived from applying principles of conservation of mass, momentum, and energy. In the simplest scenario, this leads to Euler's equation for a perfect (or ideal) fluid. Eventually, we relax these assumptions so as to incorporate the effects of viscosity.

Let D be a region in a 2D or 3D region filled with fluid. The fluid is in motion and we wish to describe this motion.

Let $\boldsymbol{x} \in D$ be a point in D and consider the particle of fluid that moves through the point $\boldsymbol{x}$ at time t. In the figure below, we sketch the trajectory that this particle might take within the fluid. Also at the point $\boldsymbol{x}$ at time t, we let $\boldsymbol{u}(\boldsymbol{x}, t)$ be the velocity of the particle. At each fixed time, we can imagine the velocity vectors drawn at each point in the fluid; thus we call $\boldsymbol{u}$ the *velocity field* of the fluid.

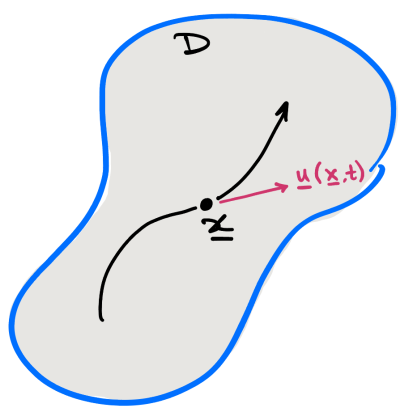


Figure 3.0.1 Image of the fluid domain D.

At each point in space and moment in time, we assume that the fluid has a well-defined mass density, $\rho(\boldsymbol{x}, t)$. Let $V \subseteq D$ be any subregion of D. Then the mass of the fluid in V at time t is given by

$$m(t) = \iiint_{V(t)} \rho(\boldsymbol{x}, t)\, \mathrm{d}V. \tag{3.0.1}$$

Remark 3.0.2 Smoothness and well-posedness. In most of what follows in this chapter, we shall always assume that the functions $\boldsymbol{u}$, ρ, and similarly for others describing fluid quantities are sufficiently smooth that the standard calculus operations can be applied to them. *Can you think of some situations* *in fluid dynamics where smoothness might not be guaranteed?*

Remark 3.0.3 Continuum assumption. The assumption that the fluid can be described by a smooth scalar density field, ρ, is a *continuum assumption*. This is indeed one of the core assumptions that underlies this course, but it is worth noting that this is only an assumption (albeit a widely accepted one, applicable to most real-life scenarios). On the other extreme of this view is the assumption that the fluid is composed of a discrete set of molecules, all bouncing around, and hence fluid dynamics might be posed as the study of the kinetic motion of molecules!

For example, consider the fluid density, ρ. In the continuum approximation, this needs to be a smooth mathematical function of position and time. We define it at a point $\boldsymbol{x}$ by taking a small volume δV about $\boldsymbol{x}$ and then define ρ as mass of the fluid particles in δV divided by δV. But how big should δV be? It needs to be big enough that the effects of individual particles will be smoothed out and small enough that the resulting function ρ captures macroscopic density variations in the fluid (which would be averaged out if δV is too large).

An illustration is shown in Figure 3.0.4, which shows a sketch of how the function ρ defined as particle mass in δV divided by δV might vary as δV varies (note the x-axis of the graph is on a logarithmic scale). The ideal choice of the volume δV is of the order of 10^{-9} mm^3 for liquids such as water, but note that the value of the density is pretty stable for a significant range around 10^{-9} mm^3, and that this is necessary in order that our definition of the density makes physical sense (because the exact size of δV shouldn't affect the density). For volumes much smaller than 10^{-9} mm^3, we can see the effects of individual molecules. For volumes much larger, we see macroscopic effects. For liquids, a volume of 10^{-9} mm^3 contains about 3×10^7 particles.

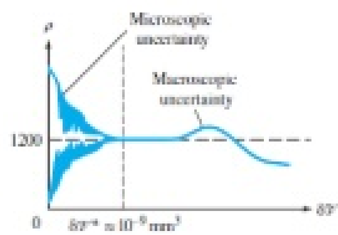


Figure 3.0.4 Sketch illustrating how the fluid density at a point might behave, where the fluid density is defined as mass of fluid in volume δV divided by δV, and this is plotted against the volume δV. Note that the x-axis is on a logarithmic scale. Picture from Fig 1.2 of [8], which you can view in the University Library's online collection

As well as the definition of the density, we need to do a similar procedure to define the velocity, and potentially other variables (pressure, stress, temperature etc). In this course, we will generally assume the procedure of choosing δV and averaging has already been carried out.

This approach implies there is a minimum size of the fluid body required for this approach to be valid (if we consider smaller sizes of fluid, the motion of individual particles becomes important). Specifically, a region containing about 3×10^7 molecules must be small in comparison all lengthscales in the problem:

- For gases at standard temperature and pressure (STP): the minimum volume is around 10^{-18} m$^3 = 1$ μm^3 (this contains around 3×10^7 molecules).

- For liquids: the minimum volume of fluid is smaller as the particles are closer together. For water, the minimum volume is around 10^{-21} m^3 (this contains around 3×10^7 molecules).
- For complex fluids: the minimum volume needed to invoke the continuum approximation is often larger, e.g. blood contains red blood cells, which significantly affect its mechanics (the mechanics of blood plasma approximates that of water) and the volume of a single red blood cell is around 10^{-16} m^3.

In Exercise 3.6.1, you will consider whether salt flow can be considered under a continuum assumption.

Figure 3.0.5 An AI-generated image of salt flowing out of a large orifice.

The derivation of the governing equations for a fluid is then based on three basic principles of physics:

1. *Conservation of mass:* mass is neither created nor destroyed.
2. *Conservation of momentum:* the rate of change of momentum of a portion of the fluid is equal to the applied force (Newton's second law).
3. *Conservation of energy:* energy is neither created nor destroyed.

Before starting, though, our first task is to derive some helpful results on the differentiation of quantities such as (3.0.1).

3.1 Reynolds' Transport Theorem

3.1.1 Jacobian of Lagrangian to Eulerian

Consider a fluid volume, $V \subset D$, that is initially dyed a certain colour. The packet of fluid is initially located at $V(0)$. As the fluid evolves in time, it then occupies the volume $V(t)$ with $t > 0$.

As an example, we may express the density of the fluid as

$$\rho = \rho(\boldsymbol{x}, t), \qquad \boldsymbol{x} \in V(t)$$

i.e. at every point in space and moment in time, the above retrieves the density of the fluid within a designated region (which may be changing). This is a natural quantity to study in a fixed frame of the fluid.

Alternatively, we can write

$$\rho = \rho(\boldsymbol{X}, t), \qquad \boldsymbol{X} \in V(0)$$

which is the density of fluid for those particles making up an originally chosen volume, $V(0)$. This is a natural quantity to study if we were to move with the fluid; for instance, if we were to colour the volume $V(0)$ with a dye and track its density in time.

The correspondence between the Euclidean and Lagrangian coordinates is written as

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t).$$

Theorem 3.1.1 Jacobian of Lagrangian to Eulerian. *Since we assume the medium is continuous, then we would generally require that the mapping from Lagrangian to Eulerian coordinates is continuous and one-to-one; then the map assigns every element (label) in the original reference configuration, denoted $\boldsymbol{X}$, a unique position, $\boldsymbol{x}$, in the deformed state.*

From Multivariable Calculus, a sufficient condition for this to be true is that the Jacobian of the transformation*,*

$$J = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \frac{\partial(x, y, z)}{\partial(X, Y, Z)} = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix}$$

is finite and non-zero:

$$0 < J < \infty.$$

The following requires a bit of algebra, and you are not required to prove the result.

Theorem 3.1.2 Euler's identity. *The material derivative of the Jacobian of the transformation $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$ is given by*

$$\frac{\mathrm{D}J}{\mathrm{D}t} = J\nabla \cdot \boldsymbol{u}, \tag{3.1.1}$$

where $\boldsymbol{u} = \frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t}$.

Proof. The proof follows from direct differentiation on the determinant and use of the identity of the material derivative. ■

3.1.2 Reynolds' Transport Theorem

We are now ready to derive a key result that eases our path towards developing the governing equations for a fluid. The result is as follows.

Theorem 3.1.3 Reynolds' Transport Theorem. *Consider a time-dependent volume, $V(t)$, that is transported by the fluid so that it always consists of the same fluid particles. Then, for any function, $f(\boldsymbol{x}, t)$, continuously differentiable with respect to its arguments,* Reynolds' Transport Theorem *is as follows:*

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} f \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{V(t)} \left[\frac{\partial f}{\partial t} + \nabla \cdot (f\boldsymbol{u})\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{3.1.2}$$

Proof. We transform the integral in Euclidean coordinates to Lagrangian coordinates, integrating in the label space:

$$I(t) \equiv \iiint_{V(t)} f \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{V(0)} fJ \,\mathrm{d}X\mathrm{d}Y\mathrm{d}Z,$$

and notice that we now only need to integrate over the fixed volume as defined in Lagrangian space, at the expense of adding the Jacobian factor. We now write

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \iiint_{V(0)} \frac{\mathrm{D}(fJ)}{\mathrm{D}t} \,\mathrm{d}X\mathrm{d}Y\mathrm{d}Z$$

and the material derivative passes through since the domain is fixed. The Theorem 2.1.6 now allows us to convert the material derivative to regular partial derivatives. By the chain rule:

$$\begin{aligned}\frac{\mathrm{D}(fJ)}{\mathrm{D}t} &= \frac{\mathrm{D}f}{\mathrm{D}t}J + f\frac{\mathrm{D}J}{\mathrm{D}t} \\ &= \left[\frac{\partial f}{\partial t} + \boldsymbol{u}\cdot\nabla f\right]J + f\left[J\nabla\cdot\boldsymbol{u}\right] \\ &= \left[\frac{\partial f}{\partial t} + \nabla\cdot(f\boldsymbol{u})\right]J\end{aligned}$$

and we have differentiated the Jacobian via Euler's identity Theorem 3.1.2 in the second line. The last line follows from the chain rule applied to the vector identity:

$$\nabla\cdot(f\boldsymbol{u}) = f\nabla\cdot\boldsymbol{u} + \nabla f\cdot\boldsymbol{u}.$$

We can now revert from Lagrangian to Eulerian integration, and this thus completes the proof of the Reynolds' Transport Theorem. ■

Note 3.1.4 It is helpful for you to convince yourself that the vector identity used in the proof is the only possible arrangement of operations that makes sense, i.e. in order for $\nabla\cdot(f\boldsymbol{u})$ to return a scalar.

In summary: the Reynolds' Transport Theorem thus gives an identity for how time differentiation can pass through the integral when the domain of integration is changing in time!

Remark 3.1.5 A 1D version of the Reynolds' Transport Theorem. In 1D, Reynolds' Transport Theorem reduces to an identity known as Leibniz's rule. This is presented as an exercise in Exercise 3.6.3.

Remark 3.1.6 RTT and conserved quantities. Reynolds' Transport Theorem is particularly useful when the integral quantity

$$B = \iiint_{V(t)} f \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

is a *conserved quantity*. A conserved quantity is something for which we can apply a *principle of conservation*. In practice for this course, this is usually mass or linear momentum:

- Mass: $f = \rho$ and the left-hand side of (3.1.2) is zero because mass is conserved.
- Linear momentum: $\boldsymbol{f} = \rho\boldsymbol{u}$ is now a vector quantity and, by *Newton's second law*, the left-hand side of (3.1.2) equals the sum of the all the forces acting on the fluid in the control volume V, which are:

- Surface forces: Forces acting on the surfaces of the fluid, which could be reaction forces from the walls of a container, or forces due to pressure or stress from surrounding fluid. (Note that we will cover stress in Chapter **??**.)
- Body forces: Forces acting over the interior of the fluid, and in this course you will usually only need to consider the force of gravity, although in principle other body forces could act such as an electromagnetic force. In addition, if working in a noninertial frame of reference, other apparent forces need to be added (forces due to linear acceleration, angular acceleration and the Coriolis and centrifugal forces).

Remark 3.1.7 General statement of RTT. *Non-examinable:*

A more general statement of the Reynolds' Transport Theorem is as follows: We define a *control volume* $V(t)$. This is a region of the fluid that we specify; it could be any fluid-containing volume, and in particular it doesn't need to follow the fluid particles (unlike (3.1.2)). Then the rate of change of B in the system of particles equals the integral of the rate of change of f over the control volume plus the net flux of the f through the surface of the control volume:

$$\frac{\mathrm{d}B_{\text{syst}}}{\mathrm{d}t} = \frac{\mathrm{d}B_{CV}}{\mathrm{d}t} + \dot{B}_{CS}. \tag{3.1.3}$$

Here, B_{syst} is the amount of B in the system of particles, that is the Lagrangian derivative of B, B_{CV} is the amount of B in the control volume and $\dot{B}_{CS}$ is the flux of B across the surface of the control volume. Note that, if B is conserved then we can write the left-hand side in terms of other things (zero if B is mass and the sum of the forces if B is linear momentum).

This is in a form that can be used to solve many problems in engineering. The key point is that these problems are about quantities on the scale of the *control volume* and not about the details of the flow at particular points.

Thus, for example we could consider a tank with inlets and outlets and, using the fluid in the tank as the control volume, we can apply the conservation of mass to find the volume flow rates in the inlets and outlets and the level of fluid in the tank.

We can use momentum conservation to find the force on a section of a pipe or the dynamics of a piston moving a fluid to which a known force is applied.

3.2 Conservation of mass

Our task from this section is to prove the following equation for the conservation of mass of a fluid:

Theorem 3.2.1 Continuity equation. *The differential form of the law of conservation of mass, otherwise known as the* continuity equation *is*

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0, \tag{3.2.1}$$

where $\rho = \rho(\boldsymbol{x}, t)$ *is the density of the fluid and* $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ *is the velocity of the fluid.*

The above form is equivalent, by the definition of the convective derivative, to

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho(\nabla\cdot\boldsymbol{u}) = 0. \tag{3.2.2}$$

In fact, as it turns out, the proof of this result is trivial if we use the

Reynolds' Transport Theorem and Lagrangian formulation following Theorem 3.1.3.

Proof. We consider the time differentiation of the mass integral,

$$\iiint_{V(t)} \rho \, \mathrm{d}V,$$

for an arbitrary material volume $V(t)$ in the fluid.

By Theorem 3.1.3, we can use the Reynolds' Transport Theorem to pass the time derivative through the integral. This gives:

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} \rho \mathrm{d}V = \iiint_{V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) \right] \mathrm{d}V,$$

which is satisfied for any volume $V \subseteq D$ where the fluid is defined. Since the result is true for any such possible volume, then the integrand of the right hand-side, itself, must be zero. This gives immediately (3.2.1).

The proof of (3.2.2) follows immediately from application of the definition of the convective derivative (2.1.1). ∎

Within the above proof, we use an idea used throughout this chapter, which is that if an integral of a quantity (the integrand) is zero for all possible domains of integration, then the integrand, itself, is zero. This is sometimes referred to as the "Bump lemma".

Lemma 3.2.2 Bump lemma. *Let $f(\boldsymbol{x})$ be a sufficiently smooth function defined on $\Omega \subseteq \mathbb{R}^n$. Suppose it is the case that*

$$\int_V f(\boldsymbol{x}) \, \mathrm{d}V = 0,$$

for all $V \subseteq \Omega$. Then

$$f(\boldsymbol{x}) \equiv 0$$

in Ω.

In Exercise 3.6.2, you will prove the above lemma.

3.2.1 Derivation of mass conservation using Eulerian methods

The derivation we have just shown for Theorem 3.2.1, using the Lagrangian viewpoint and the Reynolds' Transport Theorem is misleadingly simple, and it can be instructive to see how the result is proved purely from the perspective of Eulerian coordinates.

For this, let us consider V to be a *fixed and closed* subregion of the overall fluid, D that does not change with time. An illustration of this is shown in Figure 3.2.3.

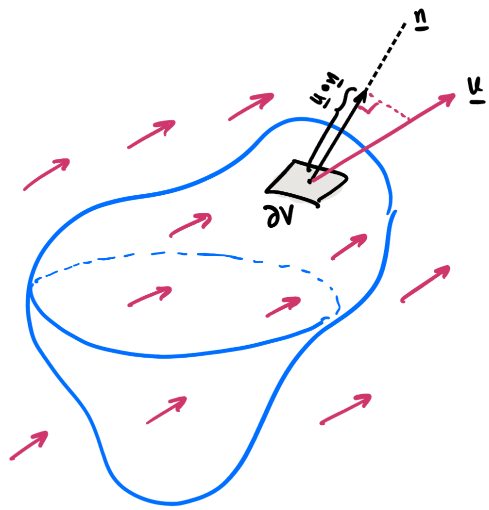


Figure 3.2.3 Picture of the fluid volume, V, shown in blue, with a small surface element, ∂V, and the outwards flux.

We want to prove the following result, which essentially equates the change in mass, due to density changes, to the flow of mass in or out of the volume.

Theorem 3.2.4 Integral form of the law of conservation of mass. *Given a fixed volume element V and boundary ∂V, the integral form of the law of mass conservation is*

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho \, \mathrm{d}V = -\iint_{\partial V} \rho \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S. \tag{3.2.3}$$

Proof. The rate of change of mass in V is

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho(\boldsymbol{x}, t) \, \mathrm{d}V = \iiint_V \frac{\partial \rho}{\partial t} \, \mathrm{d}V,$$

and note the derivative passes through the integral since the volume, V, does not change with time.

Let the boundary of V be given by ∂V, and let $\boldsymbol{n}$ denote the outward unit normal defined along the boundary ∂V. At each point on the boundary, the volume flow rate (known as the *flux*) across the boundary is given by $\boldsymbol{u} \cdot \boldsymbol{n}$ and therefore the mass flow rate is $\rho \boldsymbol{u} \cdot \boldsymbol{n}$.

We now sum the total mass flow across the entire boundary. This is given by the surface integral

$$\text{rate of mass change across } \partial V = \iint_{\partial V} \rho \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S.$$

The flux out of the boundary is also sketched in Figure 3.2.3.

Mass conservation is now applied. Therefore, the rate of change of pass in the volume V is equal to the rate at which mass enters the boundary in the *inwards* direction. ■

We now want to transform the integral form in (3.2.3) into the form of a partial differential equation. To do this, apply the *Divergence theorem* to the right hand-side of the above integral, converting the surface integral to a volume integral. Moving all quantities to the left hand side now yields

$$\iiint_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u})\right] \mathrm{d}V = 0.$$

Since the above integral equation holds for all possible V, it must be equivalent to the integrand equated to zero (Lemma 3.2.2). This yields our final result leading to Theorem 3.2.1.

3.2.2 Corollary of the Transport Theorem

Corollary 3.2.5 *There is a useful corollary to the transport theorem in Theorem 3.1.3 in the case where the desired function of integration, f, is proportional to the density, i.e. $f = \rho h$ for any continuously differentiable h. Indeed consider a moving volume $V(t)$, and a density ρ that satisfies the continuity equation (3.2.1) or (3.2.2).*

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho h \,\mathrm{d}V = \iiint_{V(t)} \rho \frac{\mathrm{D}h}{\mathrm{D}t}\,\mathrm{d}V. \tag{3.2.4}$$

Proof. This proof just follows from expansion. Pass the derivative through the integral and use the Reynolds' transport theorem formula. Then expand the quantity:

$$\nabla \cdot (f\boldsymbol{u}) = \nabla f \cdot \boldsymbol{u} + f \nabla \cdot \boldsymbol{u}.$$

where $f = \rho h$. Finally use the continuity equation on ρ. ∎

3.3 Momentum balance

With mass conservation now handled, we turn our eyes towards a law that governs the conservation of momentum. This is *Newton's second law* the rate of change of momentum of a body is equal to the sum of all forces acting on the body.

Our task from this section is to prove the following equation for the conservation of mass of a fluid:

Theorem 3.3.1 Momentum equation. *The differential form of the law of conservation of momentum, is*

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\nabla p + \rho \boldsymbol{g} \tag{3.3.1}$$

where $\rho = \rho(\boldsymbol{x}, t)$ is the density of the fluid, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity of the fluid, $p = p(\boldsymbol{x}, t)$ is the pressure, and $\boldsymbol{g}$ is the acceleration due to a body force (typically gravity).

Note that above, we have used the notation $\nabla \boldsymbol{u}$ for the gradient of a vector. This is explained in Remark 2.1.7.

3.3.1 Proof of the momentum equation for an inviscid fluid

Consider again a material volume $V(t)$ of the fluid. Then the rate of change of net momentum of this volume is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho \boldsymbol{u}(\boldsymbol{x}, t)\,\mathrm{d}V. \tag{3.3.2}$$

It is helpful to remember that the mass of a volume element is $\rho \mathrm{d}V$ and the acceleration is $\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}$, so the above is similar to mass times acceleration. However, we work with the more general form above to allow for changes in the density throughout the fluid, and also for variable fluid elements, $V(t)$.

We must equate the above to the sum of all surface and body forces applied to the fluid element.

An example of a body force is the force of gravity. For a small volume element of mass $\rho \mathrm{d}V$, the force of gravity is equal to $(\rho \mathrm{d}V)\boldsymbol{g}$. Therefore, the total external force due to gravity on the volume is equal to

$$\iiint_V \rho \boldsymbol{g}\, \mathrm{d}V. \tag{3.3.3}$$

There may be other external body forces. For example, if your fluid is electrically conductive (like a plasma), there may be electromagnetic forces that must be considered. In any case, $\boldsymbol{g}$ can be considered as the analogous body force.

The final type of forces we should consider are *surface forces*, which are applied to the boundary of the fluid element, denoted ∂V. Let us assume that at each point on the boundary, there is a per-surface area surface force, $\boldsymbol{F}$, that decomposes into component normal to the boundary, $F_n \boldsymbol{n}$, and a component tangential to the boundary, $F_t \boldsymbol{t}$. So the surface forces, summed over the boundary, will be

$$\iint_{\partial V} [F_n \boldsymbol{n} + F_t \boldsymbol{t}]\, \mathrm{d}S. \tag{3.3.4}$$

In the above, the interpretation is that the force on a small patch of surface area $\mathrm{d}S$ is equal to the per-area force, say F_n, multiplied by the area, then directed into the normal direction, $\boldsymbol{n}$.

At this point, we make a key assumption that is applied in the particular case of *inviscid fluids*.

Theorem 3.3.2 Forces in inviscid fluids. *In the case of* inviscid fluids*, we assume that the surface pressure exerted on (interior) volume elements is accounted solely by a pressure, p, which acts in the inward normal direction at each point, with*

$$F_n \boldsymbol{n} = -p\boldsymbol{n}.$$

Consequently, the surface force, given by (3.3.4), is

$$\iint_{\partial V} (-p)\boldsymbol{n}\, \mathrm{d}S = \iiint_V (-\nabla p)\, \mathrm{d}V. \tag{3.3.5}$$

In particular, for the case of inviscid fluids, we ignore tangential internal forces.

Proof. The result follows by applying the divergence theorem to each component of,

$$F_i = -p\boldsymbol{e}_i, \qquad i = 1, 2, 3.$$

Applying it once, we have

$$\iiint_V \nabla \cdot F_i\, \mathrm{d}V = \iint_{\partial V} (-p)\boldsymbol{e}_i \cdot \boldsymbol{n}\, \mathrm{d}S.$$

Note that the LHS is:

$$\iiint_V (-\nabla p)_i\, \mathrm{d}V$$

the ith element of $-\nabla p$.

Now add up the above for the three elements of $i = 1, 2, 3$, and use the shorthand notation for the divergence of a vector field:

$$\nabla \cdot \boldsymbol{F} = \sum_{i=1}^{3} (\nabla \cdot F_i)\boldsymbol{e}_i = (-\nabla p)$$

hence establishing the theorem. ■

Finally, it follows by summation of the forces above that Newton's law states that

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} \rho \boldsymbol{u}(\boldsymbol{x}, t)\, \mathrm{d}V = -\iiint_{V} (\nabla p)\, \mathrm{d}V + \iiint \rho \boldsymbol{g}\, \mathrm{d}V.$$

We can now use the corollary to the transport theorem Corollary 3.2.5, with the assignment of $f = \rho u_x, \rho u_y, \rho u_z$ and this allows us to the conclude that

$$\iiint_{V} \left(\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \nabla p - \rho \boldsymbol{g} \right) \mathrm{d}V = 0.$$

Again, the above holds for all material volumes V and therefore it must follow that the integrand is zero. We conclude thus with the following result as stated in Theorem 3.3.1:

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \rho \boldsymbol{g}.$$

In Exercise 3.6.4, you will practice the derivation of the mass and momentum equations.

3.4 The Euler equations

3.4.1 Incompressible fluids and the Euler equations

Recall from Theorem 3.1.2 that the Jacobian relates infinitessimal volumes in Eulerian and Lagrangian frames via

$$\mathrm{d}x\mathrm{d}y\mathrm{d}z = J\mathrm{d}X\mathrm{d}Y\mathrm{d}Z. \tag{3.4.1}$$

So the Jacobian is a measure of the local expansion or contraction of a fluid (relative to its original state). If we use Euler's identity in the theorem, we are led to an important interpretation for what it means for a fluid to satisfy $\nabla \cdot \boldsymbol{u} = 0$. This leads us to defining the notion of an *incompressible fluid.* (You already previously encountered an intuitive definition for an incompressible flow as part of Exercise 2.3.1).

Theorem 3.4.1 Incompressible fluids. *A fluid is said to be* incompressible *if it preserves infinitessimal volumes. Since such volumes are related via (3.4.1), then this is equivalent to*

$$\frac{\mathrm{D}J}{\mathrm{D}t} \equiv 0,$$

within the relevant fluid (and temporal) domain.

The following equivalence (iff) then follows:

$$\textit{Fluid is incompressible} \quad \Longleftrightarrow \quad \nabla \cdot \boldsymbol{u} = 0. \tag{3.4.2}$$

Proof. Note that if $\frac{\mathrm{D}J}{\mathrm{D}t} = 0$, then indeed the an infinitessimal volume element must be of static volume for all time according to (3.4.1). Then by Euler's identity in Theorem 3.1.2, this implies $\nabla \cdot \boldsymbol{u} = 0$. Conversely, if $\nabla \cdot \boldsymbol{u} = 0$, then again by the identity the fluid is incompressible. ■

So in conclusion, for the case of an incompressible fluid, it suffices to solve the equation

$$\nabla \cdot \boldsymbol{u} = 0, \tag{3.4.3}$$

instead of the more complicated equation in Theorem 3.2.1.

By the way, what happens to the mass conservation equation in Theorem 3.2.1 if the fluid is incompressible? This leads to the following corollary.

Corollary 3.4.2 Constant density along streamlines. *For an incompressible fluid, the density is constant along streamlines, i.e.*

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + (\boldsymbol{u} \cdot \nabla)\rho = 0.$$

Proof. This relies on setting $\nabla \cdot \boldsymbol{u} = 0$ in Theorem 3.2.1. ■

Let us return to consider the total sum of equations and unknowns. We have introduced the scalar mass conservation equation found in Theorem 3.2.1 (or alternatively the more simplified equation (3.4.3) for incompressible fluids). Also the vector momentum equation found in Theorem 3.3.1. This yields four scalar equations for five unknowns: the pressure p, density ρ, and the three velocity components $\boldsymbol{u}$.

One way of proceeding is to attempt to establish or to inuit a relationship between pressure and density. For instance, the assumption of an ideal gas law can be derived from kinetic theory, which leads to the empirical law that $p = RT\rho$, relating pressure in a linear fashion to density, and depending on the temperature, T, and a (gas) constant R. Other assumptions of the form $p = p(\rho)$ are possible, and this is involved in the study of gases and *compressible fluids*.

However, empirically, we observe that in most liquids, the density only varies within a few percent under typical variations of temperature and pressure. Therefore, it is common to assume:

Note 3.4.3 Constant density assumption. We often assume that in the situation of liquids, the density is taken to be constant, $\rho =$ constant.

Note, then that in the case of constant density fluids, if we consider the mass conservation equation (3.2.1), it follows that $\nabla \cdot \boldsymbol{u} = 0$. Therefore, from the definition Theorem 3.4.1, we conclude that the fluid is indeed incompressible.

Remark 3.4.4 Confusingly, in many references, authors define a fluid to be incompressible if ρ is constant. However, we see this is not necessarily the case. A fluid can be incompressible and therefore $\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$ without ρ being constant.

We are finally ready to state the *Euler equations*.

Definition 3.4.5 Euler equations. The *Euler equations* consist of the continuity equation (3.4.3) and momentum equations (3.3.1), considered in the situation of an incompressible fluid:

$$\nabla \cdot \boldsymbol{u} = 0, \tag{3.4.4}$$

$$\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\frac{1}{\rho}\nabla p + \boldsymbol{g}. \tag{3.4.5}$$

Note this is then four scalar equations for the three unknown velocities in $\boldsymbol{u}$ and pressure p. We shall assume in the course that incompressible fluids have constant density, ρ. The above Euler equations include the (gravitational) body force $\boldsymbol{g}$. ♢

Simple applications of the Euler equations are given in Exercise 6.5.1, Exercise 3.6.5 and Exercise 3.6.6

You will consider the derivation of the Euler equations as part of Exercise 3.6.4.

Example 3.4.6 Computer graphics and fluid simulations. There has always been an intimate link between the research field of fluid mechanics and the entertainment field, where cutting-edge applications in animation, movie, and video-game graphics use ideas from fluid mechanics to model fluids. Many fluid simulators will use things like the Euler equations (or "bastardised version") to simulate fluids. In some cases, such simulations can even happen in real time. Here is an example of a real-time Euler equation solver coded in Javascript. □

3.4.2 Boundary conditions

A great deal of the complexity of fluid motion comes from the conditions that we must consider between the fluid and its bounding surfaces. For water in the ocean, a bounding surfaces might include the ocean floor bottom and the body of a boat, down to the vegetation in the water or the sand on a beach. *This can get quite complex!* In this module, and when we first learn fluid dynamics, we only consider simple bounded fluid regions (fluid above a plate, fluid in a box, fluid in a channel, etc.)

In the case of a water wave, a bounding surface will also include the free surface of the wave, itself, which interacts with the surrounding atmospheric gas. This leads to the situation of free-boundary or free-surface conditions.

For the situation of fluid at a solid boundary, this leads us to formulate the following.

Note 3.4.7 No-flux condition through a boundary. For a fluid in contact with a fixed rigid boundary, ∂V, then we require that the normal velocity of the fluid there must be zero:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial V, \tag{3.4.6}$$

where $\boldsymbol{n}$ is the unit normal to ∂V. This condition states that the fluid cannot flow through ∂V or separate from ∂V (hence leaving a vacuum).

It turns out that the above no-flux condition is sufficient to provide well-posed boundary conditions for most inviscid fluids. Note that in particular, we have not constrained the tangential velocity of the fluid, i.e. $\boldsymbol{u} \cdot \boldsymbol{t}$, where $\boldsymbol{t}$ is the tangential vector at the boundary.

Later in Chapter **??**, you will study the situation of a *viscous fluid*, where it will be important to consider the tangential fluid velocity.

3.5 Bernoulli's equation

There is a reformulation of the momentum equations that proves to be useful. In essence, it emerges from attempting to integrate the momentum equation (3.4.5) and yields the famous *Bernoulli equation(s).*

We will first need a vector identity.

Identity 3.5.1 *Note the following identity:*

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nabla \left(\frac{1}{2} |\boldsymbol{u}|^2 \right) + (\nabla \times \boldsymbol{u}) \times \boldsymbol{u}. \tag{3.5.1}$$

Proof. The proof follows from direct expansion of both sides. ■

It will also be useful for us to introduce the notion of vorticity.

Definition 3.5.2 Vorticity of a vector field. The *vorticity*, ω, of a vector field is defined by

$$\omega \equiv \nabla \times \boldsymbol{u}. \tag{3.5.2}$$

The vorticity is a measure of the local rotation of the flow. ◊

Let us recall the definition of a *conservative force*.

Definition 3.5.3 Conservative forces. A force $\boldsymbol{F}$ is a *conservative force* if and only if there exists a potential χ, such that

$$\boldsymbol{F} = -\nabla\chi,$$

in a simply connected region where the quantities are defined. ◊

Note the distinction about a simply-connected neighbourhood. The above is not quite the definition of a conservative force (typically defined to be a force for which the work done on an object between two points is independent of path). For now, this is not an important distinction since we will focus on fluid regions that are free of holes. Until told otherwise, we will always assume that the fluid is simply connected and therefore the above serves as a definition of conservative forces.

Theorem 3.5.4 Bernoulli's equation for steady flow. *Bernoulli's equation (theorem) for steady flow states that*

$$B(\boldsymbol{x}) \equiv \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + \chi = \textit{constant along streamlines}, \tag{3.5.3}$$

where we have assumed that the body force is conservative*, i.e. it can be written as*

$$\boldsymbol{g} = -\nabla\chi, \tag{3.5.4}$$

for a potential function, χ *.*

Proof. Start from the momentum equation (3.3.1),

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{1}{\rho}\nabla p - \nabla\chi$$

and use the vector identity (3.5.1) gives

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla\times\boldsymbol{u})\times\boldsymbol{u} = -\nabla\left(\frac{1}{\rho}\nabla p + \frac{1}{2}|\boldsymbol{u}|^2 + \chi\right). \tag{3.5.5}$$

Next, the flow is steady, and therefore we can zero the first term. This leaves

$$\boldsymbol{\omega}\times\boldsymbol{u} = -\nabla B.$$

We now take the dot product of both sides of the equation with $\boldsymbol{u}$. We use the fact that,

$$\boldsymbol{u}\cdot(\boldsymbol{\omega}\times\boldsymbol{u}) = 0,$$

since it is a triple scalar product with two repeated entries. It therefore results in the fact that

$$\boldsymbol{u}\cdot\nabla B = 0.$$

Notice that this is the steady component that comes from the material derivative, $\frac{\mathrm{D}B}{\mathrm{D}t} = 0$. So we conclude that B is constant along streamlines of the flow.

It is useful to note that if we use the definition of vorticity in (3.5.2), and the definition of B in (3.5.3), we have from the above that

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} = -\nabla B, \tag{3.5.6}$$

a form that will be useful, shortly. ■

Remark 3.5.5 For the typical gravity force directed in the $-z$ direction, we can write $\boldsymbol{g} = -g[0,0,1] = -g\boldsymbol{e}_z$. So in this case, the (gravitational) potential is $\chi = gz$.

The proof of Theorem 3.5.4 introduced a useful form of the momentum equation using the vorticity function, resulting in (3.5.6). This leads to the so-called *vorticity equation* form of the momentum equation.

Theorem 3.5.6 The vorticity equation for incompressible flow. *The momentum equation for incompressible flow, reposed in terms of the vorticity is:*

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}, \tag{3.5.7}$$

or, equivalently,

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}, \tag{3.5.8}$$

and is known as the vorticity equation*.*

In the proof, it is useful to use the vector identity for the curl of a cross product:

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{v}) = (\nabla \cdot \boldsymbol{v})\boldsymbol{u} - (\nabla \cdot \boldsymbol{u})\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}. \tag{3.5.9}$$

Proof. Crucially, we recall the result that ”curl grad equals zero” for a vector field. So from (3.5.6), we take the curl of both sides to obtain

$$\nabla \times \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} \right) = 0.$$

Expanding the outer cross product, the first term can be simplified by swapping differentiation in space with differentiation in time. The second term requires the cross product identity (3.5.9). For the second term, we get

$$-\nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) = 0 - 0 + (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega},$$

with the first zero resulting in the fact that ”div curl equals zero”, while the second results from incompressibility.

Thus we have,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u},$$

which is the main result. ■

You will prove this result from first principles in Exercise 3.6.7.

3.5.1 Potential flow

We are interested in the simplest scenarios that will result in reducing the governing equations. Previously, we introduced the notion of incompressible flows, which results in the Euler equations in Definition 3.4.5 and in particular, the reduction of the mass conservation equation to $\nabla \cdot \boldsymbol{u} = 0$.

We attempt to reduce further by assuming that the fluid is *irrotational*. This lends to the following definition.

Definition 3.5.7 Irrotational flows. A flow is said to be *irrotational* if the vorticity is identically zero:

$$\text{flow is irrotational} \iff \boldsymbol{\omega} = \nabla \times \boldsymbol{u} \equiv 0. \tag{3.5.10}$$

$\Diamond$

Theorem 3.5.8 Equivalence of irrotational and potential flow. *If a flow with velocity vector /bu(/bx, t) is irrotational, there exists a function $\phi(\boldsymbol{x}, t)$ such that*

$$\boldsymbol{u} = \nabla\phi.$$

The function ϕ is called the potential.

Proof. Any function $\boldsymbol{f}(\boldsymbol{x}, t)$ whose curl is everywhere zero $\nabla \times \boldsymbol{f} = 0$ can be written as a gradient of a scalar function. In the case of irrotational flow, the velocity field is such a function. ■

For an irrotational and incompressible flow, there is a more powerful version of Bernoulli's equation.

Theorem 3.5.9 Bernoulli's equation for steady irrotational flow. *For an incompressible and irrotational flow, Bernoulli's theorem states that*

$$B(\boldsymbol{x}) \equiv \frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + \chi = \textit{constant everywhere}, \tag{3.5.11}$$

where $\boldsymbol{g} = -\nabla\chi$ is the body force.

Proof. The proof of this relies on returning to the proof of the unsteady Bernoulli's equation in Theorem 3.5.4. In the proof, we arrived at the result in (3.5.3) that

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} = -\nabla B.$$

For steady flow, the first term disappears. If we have the additional assumption that the flow is irrotational, then we have that $\boldsymbol{\omega} = 0$, and therefore we have

$$\nabla B = 0.$$

But the only way that all spatial derivatives is zero is if the function is constant. ■

You will practice an application of Bernoulli's equation in Exercise 3.6.8.

Example 3.5.10 Airflow into or out of a room. During the lecture, just for fun, we will play this video by Matthias Wandel about airflow out of a room. This is somewhat related to Bernoulli's equation (the connection between pressure and velocity), though the situation is considerably more difficult. □

Example 3.5.11 Pascal's bursting barrel. For fun during the lecture, we will play this video demonstration done at Princeton demonstrating the power of hydrostatic pressure. □

Example 3.5.12 Urination across species. During the lecture, we will play this video describing the work by Yang et al. on "Duration of urination does not change with body size". The title is not quite accurate, and the article describe that for most mammals tested greater than about 3kg, this is true. Bernoulli's equation is used. □

3.6 Exercises

This chapter focused on deriving the key Euler equations for an inviscid and incompressible flow, starting from first principles, and deriving the conservation laws using vector calculus. Eventually, we studied different variations of the Euler equations and its consequences.

1. **Continuum approximation.** Domestic salt flows quite well out of a container with a hole in the bottom. Following Remark 3.0.3, consider whether salt matches up to the properties quoted for a fluid

   What is the size of a typical salt particle? What size of hole would required so that a continuum model of the salt flow is a reasonable assumption? (Based on Q1, p. 42 by Paterson [10].)

   Solution. Salt is not like a fluid because you can make it into a pile, whereas the surface of a resting fluid is always horizontal. However, some aspects of salt flow are similar to the flow of a fluid. A typical salt particle is approximately a cube of side 0.5 mm, and therefore has a volume of approximately 10^{-10} m^3. We need a size of hole that admits a very large number of particles. A reasonable value of the volume V appearing in the continuum approximation might be the volume of 3×10^7 salt grains (this was the number of gas or liquid molecules quoted in the notes), i.e. 3×10^{-3} m^3, which is a cube of side length approximately 10 cm. We need a hole that is many times this distance, perhaps a hole with a diameter of one metre or so might be reasonable! See also Remark 3.0.3.

2. **Bump lemma.** Within our derivations of the governing equations, we often make use of the so-called "Bump lemma" given in Lemma 3.2.2.

   Give a proof of the lemma.

   Solution. This proof assumes that the function f is not pathological (e.g. it cannot oscillate infinitely fast, take different values for rational vs. irrational, etc.). That's what we mean by "sufficiently smooth".

   Assume that there exists a point $\boldsymbol{x}_0 \in \Omega$ where $f(\boldsymbol{x}_0) \neq 0$. Suppose without loss of generality $f > 0$ here. Since f is sufficiently smooth (and in particular, it is continuous), then there exists a non-trivial neighbourhood near $\boldsymbol{x}_0$ where $f > 0$ everywhere in this neighbourhood, say V_0. Then $\int_{V_0} f \,\mathrm{d}V = 0$, contradicting the assumption. Therefore f must be zero everywhere.

3. **Leibnitz's rule.** The Reynolds' Transport Theorem Theorem 3.1.3, which relates to the passage of a time derivative through an integral, is a general form of the 1D Leibnitz rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x \\ = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,\mathrm{d}x + f(b(t),t)\dot{b}(t) - f(a(t),t)\dot{a}(t).\end{aligned} \tag{3.6.1}$$

   (a) Examine the figure in Figure 3.6.1. Can you associate the quantities in Leibnitz's rule with the circled elements in the figure?

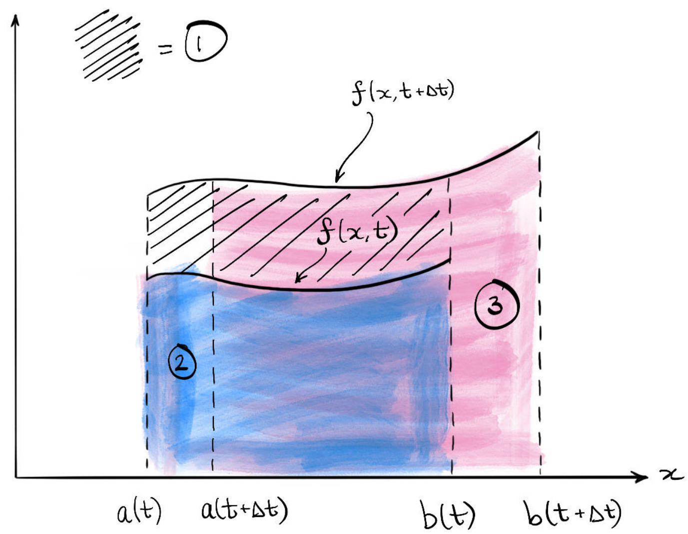


Figure 3.6.1 Visual "proof" of Leibnitz's rule

Solution. The first circled element is associated with the quantity:

$$\frac{1}{\Delta t}\int_{a(t)}^{b(t)} [f(x,t+\Delta t) - f(x,t)]\,\mathrm{d}x.$$

This is essentially the finite-difference version of the first term on the RHS.

The second circled element is associated with the quantity:

$$f(x,t)[a(t+\Delta t) - a(t)]\frac{1}{\Delta t},$$

and is associated with the area of the rectangle.

The third circled element is associated with

$$f(x,t+\Delta t)[b(t+\Delta t) - b(t)]\frac{1}{\Delta t},$$

and is associated with the area of the rectangle. Notice that for small Δt, the rectangle height of $f(x,t+\Delta t)$ is approximately the same as the rectangle height of $f(x,t)$.

(b) A nice proof of the Leibnitz rule is given in this YouTube video by Brian Storey. Watch the video and follow the derivation, making your own notes.

(c) Can you now connect the form of the rule in (3.6.1) to the Reynolds' Transport Theorem in (3.1.2)?

Solution. Show that the Reynolds Transport Theorem in one dimension is equivalent to the Leibniz rule. Write the second and third terms on the RHS as

$$\int_{a(t)}^{b(t)} \frac{\mathrm{d}}{\mathrm{d}x}(f(x,t)u(x,t))\,\mathrm{d}x.$$

4. Derivation of governing equations. *This is a re-derivation of what is already presented in the notes, giving you valuable practice to understand the concepts.*

Starting from conservation of mass and momentum for a material volume moving in an inviscid fluid, use Reynolds' Transport Theorem (3.1.2) to derive the equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{3.6.2}$$

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \boldsymbol{g}. \tag{3.6.3}$$

What additional assumptions are required to transform the set of equations to the Euler equations of Definition 3.4.5? Apply these assumptions and conclude with the Euler equations.

Solution. The proof of the continuity equation is given in Theorem 3.2.1 and the proof of the momentum equation is given in Theorem 3.3.1, so make sure you can follow these arguments and produce your own notes.

The derivation of the Euler equations, as given in Definition 3.4.5 requires the additional assumption of an incompressible fluid, following Theorem 3.4.1. Incompressibility implies that $\nabla \cdot \boldsymbol{u} = 0$, and this equation then replaces the continuity equation.

5. Pressure field in uniform flow. An ideal (incompressible and irrotational) fluid in a gravitational field with density ρ and acceleration vector $\boldsymbol{g} = -g\boldsymbol{k}$ (with $\boldsymbol{k}$ being the unit vector in the positive z-direction) has uniform velocity

$$\boldsymbol{u} = (a + bt,\ t^3,\ e^t).$$

The pressure at the origin is fixed at p_0.

(a) Using the Euler equations in Definition 3.4.5, find the pressure field in terms of $x, y, z, t, a, b, g, \rho$ and p_0.

Solution. The fluid has a uniform velocity, which means that all components of $\nabla \boldsymbol{u}$ vanish. The Euler equation simplifies to

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p - \rho g \boldsymbol{k},$$

where

$$\frac{\partial \boldsymbol{u}}{\partial t} = (b,\ 3t^2,\ e^t).$$

Hence, the pressure is given by solving the equation,

$$\nabla p = -\rho \frac{\partial \boldsymbol{u}}{\partial t} - \rho g \boldsymbol{k} = -\rho(\, b,\ 3t^2,\ e^t + g\,),$$

which is a vector equation. Splitting into components, we obtain

$$\frac{\partial p}{\partial x} = -\rho b, \quad \frac{\partial p}{\partial y} = -3\rho t^2, \quad \frac{\partial p}{\partial z} = -\rho(e^t + g).$$

Solving sequentially, the first equation gives

$$p = -\rho b x + f_1(y, z, t).$$

Substituting into the second equation,

$$\frac{\partial f_1}{\partial y} = -3\rho t^2 \quad \Rightarrow \quad f_1 = -3\rho t^2 y + f_2(z, t),$$

$$p = -\rho b x - 3\rho t^2 y + f_2(z,t).$$

The third equation then implies

$$\frac{\partial f_2}{\partial z} = -\rho(e^t + g) \quad \Rightarrow \quad f_2 = -\rho(e^t+g)z + f_3(t),$$

$$p = -\rho b x - 3\rho t^2 y - \rho(e^t+g)z + f_3(t).$$

Using $p = p_0$ at the origin gives $f_3(t) = p_0$ for all t. Hence

$$p = p_0 - \rho\big(bx + 3t^2 y + (e^t+g)z\big).$$

(b) What can you tell about the surfaces of constant pressure? Sketch this at a fixed time t, and hence describe the pressure field.

Solution. The pressure is constant on parallel planes normal to the vector $(b,\ 3t^2,\ e^t+g)$, as illustrated below:

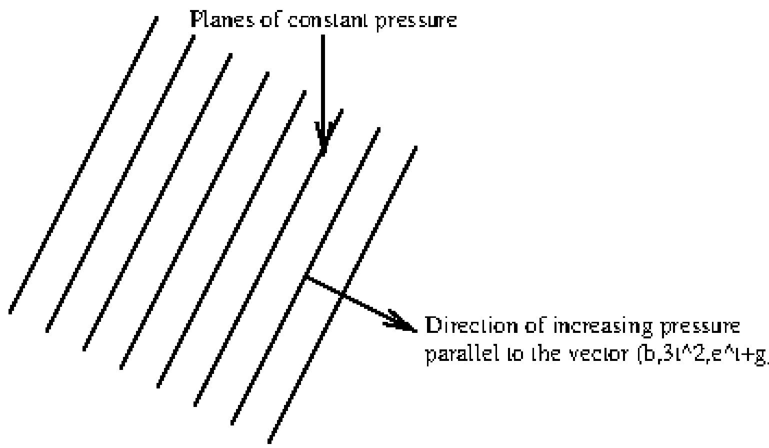


Figure 3.6.2 Contours of the pressure field at fixed t.

6. **Piston problem.** The piston shown in the diagram below is pushed with a force $F(t)$ into a pipe of length L and cross-sectional area A containing incompressible and irrotational fluid of density ρ.

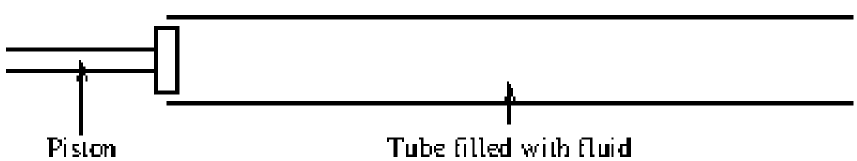


Figure 3.6.3 Schematic of piston and pipe system.

Assume that the pressure at the open end is held at atmospheric pressure, $p = p_{\mathrm{atm}}$.

Assume the fluid moves as a rigid body and neglect friction and the mass of the piston. Using the Euler equations in Definition 3.4.5, write down a differential equation governing the position, $X(t)$, of the piston at time t. You may neglect gravity. You do not need to solve the equation (it could be solved using Python or similar).

Solution. We assume that the fluid pressure at the open end of the pipe is atmospheric pressure. Since the fluid moves as a rigid body, the velocity field has no spatial gradients, and hence the Euler equation reduces to

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p,$$

where the velocity is uniformly $\boldsymbol{u} = (\dot{X}(t), 0)$.

Solving for p and using the fact that p is atmospheric pressure at $x = L$, we have

$$p(x,t) = \rho \frac{\mathrm{d}^2 X}{\mathrm{d}t^2}(L - x) + p_{\mathrm{atm}}.$$

At the piston $x = X$, note that atmospheric pressure applies in addition to F, meaning that the force on the fluid is

$$F + Ap_{\mathrm{atm}},$$

and hence the fluid pressure is

$$\frac{F}{A} + p_{\mathrm{atm}}.$$

Then

$$\frac{F}{A} + p_{\mathrm{atm}} = \rho \frac{\mathrm{d}^2 X}{\mathrm{d}t^2}(L - X) + p_{\mathrm{atm}} \quad \Rightarrow \quad \frac{\mathrm{d}^2 X}{\mathrm{d}t^2} = \frac{F}{A\rho(L - X)}.$$

Note that the atmospheric pressure cancels in this problem, and this is common to problems of this type, because atmospheric pressure acts uniformly on all surfaces exposed to the atmosphere and therefore does not lead to a resultant force or affect the dynamics. This motivates the introduction of a gauge pressure, p_{gauge}, whereby a uniform and constant baseline pressure is chosen, in this case p_{atm}, with the gauge pressure given by $p_{\mathrm{gauge}} = p - p_{\mathrm{atm}}$. Since the pressure only appears as a gradient in the governing equations, using the gauge pressure does not affect the result. Introducing a gauge pressure can make the algebra simpler.

7. **The vorticity equation.** Consider an incompressible fluid, with constant density ρ, subject to a conservative body force (i.e. $\boldsymbol{g} = -\nabla\chi$ for some potential function χ).

(a) Starting from the Euler equations, show that the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ satisfies

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}.$$

Note that, in this question, you are expected to do a long-form calculation. The method given in the notes uses an alternative expression for the momentum equation combined with a vector identity to get the same result with less effort.

Solution. The Euler equation reads:

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \right) = -\nabla p - \rho \nabla \chi.$$

Since the curl of a gradient is zero, ρ is constant and we can commute derivatives, taking the curl of both sides gives

$$\rho \left(\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \right) = 0.$$

We have

$$\begin{aligned}
&\nabla \times (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \\
=&\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \boldsymbol{u} \cdot \nabla u & \boldsymbol{u} \cdot \nabla v & \boldsymbol{u} \cdot \nabla w \end{vmatrix}, \\
=&\boldsymbol{i}\left(\boldsymbol{u} \cdot \nabla \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \frac{\partial \boldsymbol{u}}{\partial y} \cdot \nabla w - \frac{\partial \boldsymbol{u}}{\partial z} \cdot \nabla v\right) \\
&+ \boldsymbol{j}\left(\boldsymbol{u} \cdot \nabla \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \frac{\partial \boldsymbol{u}}{\partial z} \cdot \nabla u - \frac{\partial \boldsymbol{u}}{\partial x} \cdot \nabla w\right) \\
&+ \boldsymbol{k}\left(\boldsymbol{u} \cdot \nabla \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial \boldsymbol{u}}{\partial x} \cdot \nabla v - \frac{\partial \boldsymbol{u}}{\partial y} \cdot \nabla u\right),
\end{aligned}$$

At this point, note that we can see $\boldsymbol{u} \cdot \nabla \boldsymbol{\omega}$ appearing in the components, but it is not so easy to see how to make out the terms of $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$. Since we know what we are aiming for, we instead calculate

$$\begin{aligned}
&\nabla \times (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}) \\
=&\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} \\
&+ \boldsymbol{i}\left(\frac{\partial u}{\partial y}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\frac{\partial v}{\partial z}\right) \\
&+ \boldsymbol{j}\left(\frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial u}{\partial z} - \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) \\
&+ \boldsymbol{k}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right), \\
=&\boldsymbol{i}\left(\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\frac{\partial u}{\partial y} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial z}\right) \\
&+ \boldsymbol{j}\left(\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial z}\right) \\
&+ \boldsymbol{k}\left(\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\frac{\partial w}{\partial x} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\frac{\partial w}{\partial y} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial w}{\partial z}\right) \\
&+ \boldsymbol{i}\left(\frac{\partial u}{\partial y}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\frac{\partial v}{\partial z}\right) \\
&+ \boldsymbol{j}\left(\frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial u}{\partial z} - \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) \\
&+ \boldsymbol{k}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right), \\
=&\boldsymbol{\omega}\nabla \cdot \boldsymbol{u}, \\
=&0,
\end{aligned}$$

where the final equality follows from incompressibility. Hence,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = 0 \quad \Rightarrow \quad \frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u},$$

as required.

(b) Deduce that, in two-dimensional incompressible flow, ω is conserved following the flow.

Solution. If the flow is the (x, y)-plane, we have $w = 0$ and all z-derivatives are zero. Hence the only non-zero component of the vorticity is the z-component, $\omega_z = \partial v/\partial x - \partial u/\partial y$. Thus

$$\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = \omega_z \frac{\partial \boldsymbol{u}}{\partial z} = 0.$$

Hence

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = 0,$$

meaning that $\boldsymbol{\omega}$ is conserved following the flow.

8. **Using streamlines: The clepsydra.** One of the earliest means for measuring the passage of time, invented by the ancient Egyptians, was the **clepsydra** (or 'water thief'): a large jar with a hole in its base is filled with water. The shape of the jar was such that the interval of time taken for the water surface to pass two equally-spaced markers on the side of the jar is constant.

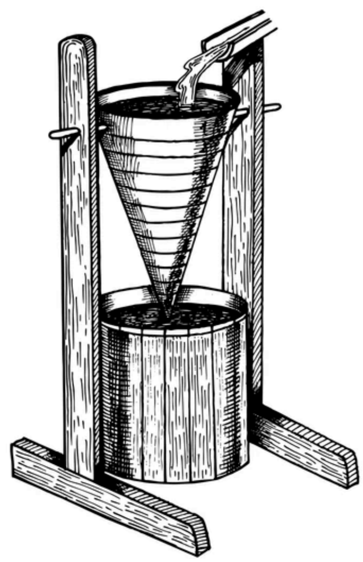


Figure 3.6.4 Clepsyndra

In this question you will determine the shape of jar required to achieve this. In particular, we denote the (axisymmetric) jar radius a height z above the hole by $r = a(z)$. The radius of the hole, $a_h \neq a(0)$, in general.

(a) Explain why the curve $r = a(z)$ is a streamline.

Solution. The fluid velocity at the surface of the jar satisfies $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, so that no fluid leaves or enters the jar through its sides. This means that the flow next to the boundary of the jar must be parallel to this boundary, meaning that streamlines are parallel to the boundary. Equivalently, the boundary $r = a(z)$ is a streamline.

(b) If the surface of the water lies at $z = h(t)$, use an appropriate form of Bernoulli's principle to calculate the speed of liquid, u, leaving the jar at $z = 0$.

[You may assume that the desired surface speed $u_S = |\dot{h}| \ll u$, so that the flow is approximately steady and that the fluid pressure is atmospheric at $z = 0$.]

Solution. We apply Bernoulli's equation to equate the energy at a point at the edge of the free surface at $z = h(t)$ and the fluid leaving the hole at $z = 0$. At the free surface, the pressure is atmospheric, $p = p_{atm}$, and the speed is $u_S\sqrt{1 + a'^2}$. At the hole, the pressure is also atmospheric, $p = p_{atm}$, and the velocity is u. Thus, Bernoulli's equation for the streamline joining the two points gives

$$\frac{p_{atm}}{\rho} + \frac{1}{2}u_S^2(1 + a'^2) + gh = \frac{p_{atm}}{\rho} + \frac{1}{2}u^2.$$

Assuming that $u_S \ll u$, this simplifies to

$$gh = \frac{1}{2}u^2.$$

(c) Use the principle of conservation of mass to link u_S, u, a_h and $a(h(t))$; thereby determine the correct shape for a clepsydra.

Solution. The volume flux out of the hole is $\pi a_h^2 u$, while the rate of decrease of the volume of fluid in the jar is $\pi a(h(t))^2 u_S$. Conservation of mass therefore gives

$$\pi a^2 u_S = \pi a_h^2 u,$$

or

$$u_S = \frac{a_h^2}{a^2}u.$$

Using the result from part (b) to eliminate u gives

$$u_S = \frac{a_h^2}{a^2}\sqrt{2gh}.$$

We require u_S to be independent of time so that equally spaced intervals are traversed in equal times. Hence,

$$a = \left(\frac{a_h^2\sqrt{2g}}{u_S}\right)^{1/2} h^{1/4}.$$

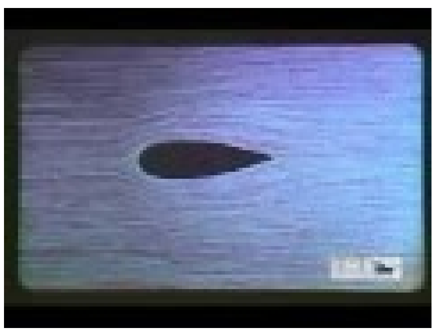
The terms in the brackets are constant, so we have that the radius of the jar must vary as the fourth root of the height.

Chapter 4

Potential flows

Typically, the most basic introduction on fluid dynamics usually involves the study of two-dimensional incompressible and irrotational flows, otherwise known as *potential flows*. There is a good reason for this: the theory is elegant, visualisable, and uses to its benefit the enormous power of complex-variable theory.

Previously, we defined the concept of vorticity via $\omega = \nabla \times \boldsymbol{u}$, which serves as a measure of the local angular velocity of the flow. It turns out that in 2D, irrotational flows, with $\omega = 0$, provide a powerful restricton to the complexity of flows, allowing the development of the above *potential flow theory.*.



Standalone

Figure 4.0.1 From "Fundamental Principles of Flows" and "Characteristics of the Laminar and Turbulent Flows" by Hunter Rouse. Courtesy of IIHR, University of Iowa.

4.1 The velocity potential

In this chapter, we focus on 2D flows where the velocity vector is given by

$$\boldsymbol{u}(x, y, t) = u(x, y, t)\boldsymbol{i} + v(x, y, z)\boldsymbol{j} = [u, v].$$

With the velocity given as above, the vorticity is then

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \boldsymbol{k}. \tag{4.1.1}$$

If we assume that the flow is irrotational according to Definition 3.5.7, then $\nabla \times \boldsymbol{u} = 0$ and

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{4.1.2}$$

Further, we know that if the flow is irrotational, then there exists a *velocity potential*, ϕ, such that $\boldsymbol{u} = \nabla\phi$. Thus the velocities are expressed as

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y}. \tag{4.1.3}$$

The above result about irrotational flows is a standard result in Vector Calculus, but we will re-state the result here for reference, and provide a review of its proof.

Theorem 4.1.1 Existence of a potential. *Consider a three-dimensional time-dependent velocity field,* $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ *defined on a simply connected domain* $\boldsymbol{x} \in D \subset \mathbb{R}^3 \times \mathbb{R}^+$.

Then $\boldsymbol{u}$ *is irrotational, i.e.* $\nabla \times \boldsymbol{u} = 0$ *if and only if there exists a scalar potential, defined on* $D \times \mathbb{R}^+$*, such that* $\boldsymbol{u} = \nabla\phi$.

Proof. Define

$$\phi(\boldsymbol{x}, t) \equiv \phi_0(t) + \int_C \boldsymbol{u} \cdot \mathrm{d}x,$$

where C is any contour connecting an arbitrary origin point to the point $\boldsymbol{x}$ (changing the origin point will change the "constant" of integration $\phi_0(t)$).

We can verify, using the definition of differentiation, and the fundamental theorme of calculus, applied along each of the three coordinate directions, that $\nabla\phi = \boldsymbol{u}$ as desired.

The key is to prove that the above definition is unique, regardless of the choice of contour C. To this end, consider two contours, C_1 and C_2, both with the same origin point, O, and end point P. Then the contour $C_1 - C_2$ is a close contour beginning and ending at O.

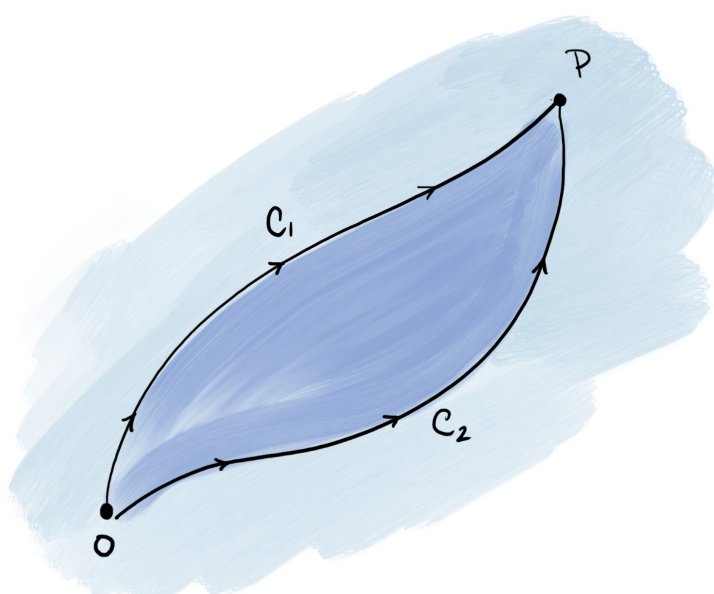


Figure 4.1.2 Proof of the uniqueness of the potential

By Stokes' theorem,

$$\int_{C_1 - C_2} \boldsymbol{u} \cdot \mathrm{d}x = \iint_S (\nabla \times \boldsymbol{u})\, \boldsymbol{n}\, \mathrm{d}S,$$

where S is any surface with bounding curve $C_1 - C_2$, and with unit normal $\boldsymbol{n}$ positively oriented with the bounding curve. However, the right hand-side is zero by irrotationality, and therefore

$$\int_{C_1} \boldsymbol{u} \cdot \mathrm{d}x = \int_{C_2} \boldsymbol{u} \cdot \mathrm{d}x,$$

and our choice of curve in the definition of the potential is irrelevant.

The converse direction of the theorem follows directly from the fact that "curl grad equals zero", i.e. $\nabla \times \nabla\phi = 0$. ■

Let us return to discussing the setting of potential flow.

In addition to being irrotational, we furthermore have assumed that the flow is incompressible. Therefore from Theorem 3.4.1,

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{4.1.4}$$

This is the crucial result, which is that in potential flows, we need only solve the Laplace equation:

$$\nabla^2 \phi = 0, \tag{4.1.5}$$

within the flow region. This is effectively a single linear equation for the single unknown ϕ. However, for different problems, the boundary conditions can render even this "simple" problem difficult.

Once the velocity potential ϕ has been solved, the velocities in the flow can be recovered from the relationship (4.1.3). The pressure in the flow also follows from Bernoulli's equation. For the situation of a steady potential flow, following Theorem 3.5.9, it is

$$\frac{p}{\rho} + \frac{1}{2}|\nabla\phi|^2 + \chi = \text{constant}. \tag{4.1.6}$$

4.1.1 Elementary flows

The next three examples will introduce you to the elementary flows consisting of uniform flow, stagnation point flow, and line source/sink flows. You will also investigate the notion of a source "strength".

Example 4.1.3 Uniform flow. Consider the potential given by the linear function

$$\phi(x, y) = Ux \cos\alpha + Uy \sin\alpha,$$

with constants U and α. Then by differentiation we have that the velicity is

$$\boldsymbol{u} = U[\cos\alpha, \sin\alpha].$$

The image shown below shows the streamlines of the flow.

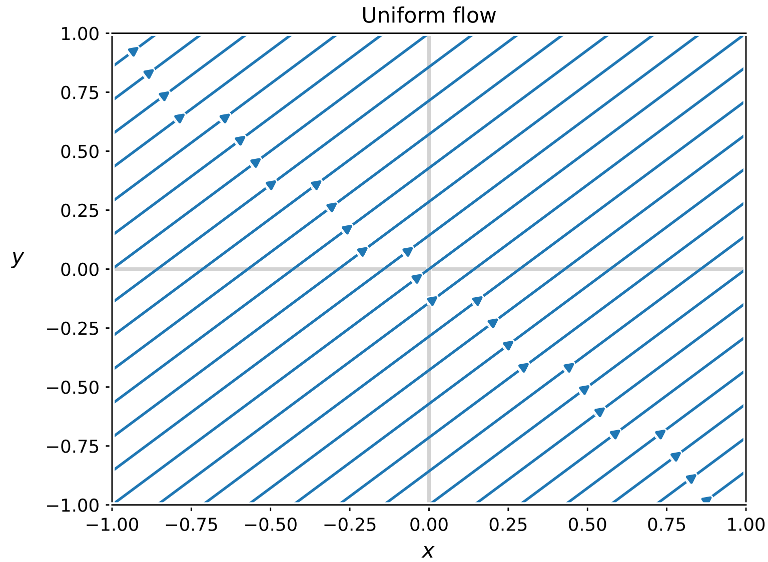


Figure 4.1.4 Streamlines (or velocity field) of uniform flow with $U = 1$ and $\alpha = \pi/4$.

□

Example 4.1.5 Stagnation point flow. We can verify that the velocity potential

$$\phi = \frac{1}{2}(x^2 - y^2),$$

satisfies Laplace's equation. The corresponding velocity field is given by

$$[u, v] = [x, -y].$$

and corresponds to *stagnation point flow*.

The streamlines (or velocity field) is shown below.

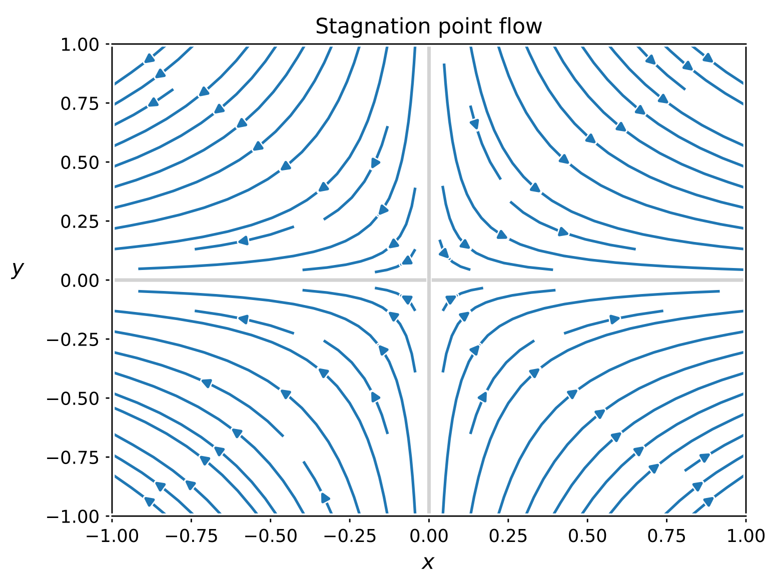


Figure 4.1.6 Streamlines (or velocity field) of stagnation point flow.

□

Example 4.1.7 Line source. We aim to derive the potential and velocity for a *line source*, imagined as the flow consisting of a point source or point sink that ejects/drains fluid from a point in space. Since it would be expected for the potential to be axisymmetric, we attempt to solve $\nabla^2\phi = 0$ in plane polar coordinates. This is given by

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.$$

We assume that the potential takes the form $\phi = \phi(r)$. Then direct integration gives

$$\phi = \frac{Q}{2\pi}\log r,$$

where we have set an additional constant of integration to zero without loss of generality. The leading constant has been set to $Q/(2\pi)$ so that Q can be later identified with a physical quantity.

The velocity then follows from consideration of the gradient in polar form,

$$\boldsymbol{u} = \nabla\phi = \frac{\partial\phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{e}_\theta = \frac{Q}{2\pi r}\boldsymbol{e}_r,$$

where the unit vectors written in the Cartesian basis are $\boldsymbol{e}_r = [\cos\theta, \sin\theta]$ and $\boldsymbol{e}_\theta = [-\sin\theta, \cos\theta]$. Thus we can write the velocity as

$$\boldsymbol{u} = \frac{Q}{2\pi r^2}r[\cos\theta, \sin\theta] = \frac{Q}{2\pi r^2}[x, y].$$

The above corresponds to a velocity field directed radially outwards from the origin. The flow is a called a *line source* because fluid is ejected from the origin (a source). It refers to a "line" because in (x, y, z), the source runs parallel to the z-axis.

The streamlines (or velocity field) are shown below.

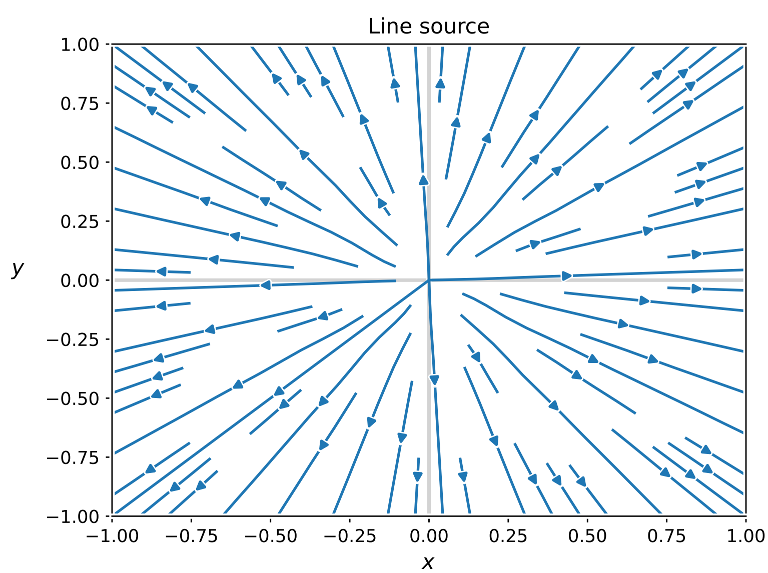


Figure 4.1.8 Streamlines (or velocity field) of line source flow.

Let us also identify the *strength* of this line source. Consider a closed contour C around the origin. Then the quantity

$$\int_C \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}s,$$

is the flux (the flow per unit time) of fluid crossing the contour, with $\boldsymbol{n}$ denoting the unit normal along C.

For simplicity, let us take the contour C to be a circle of constant radius $r = a$. Then since the unit normal is precisely $\boldsymbol{e}_r$, we have that

$$\int_C \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}s = \int_0^{2\pi} \frac{Q}{2\pi a}\boldsymbol{e}_r\cdot\boldsymbol{e}_r\,(a\,\mathrm{d}\theta) = Q.$$

In computing the above integral, remember that the conversion following the polar Jacobian is $\mathrm{d}s = r\mathrm{d}\theta$ where $r = a$.

Therefore, Q is the rate at which fluid is produced from the line source. If $Q < 0$, we refer to the flow as a *line sink*. □

Crucially, because the governing fluid mechanical equation is only Laplace's equation: this is a linear partial differential equation, and therefore the summation of elementary flows also produces an admissible flow.

Example 4.1.9 Line source in a uniform flow. For instance, we may combine a uniform flow in the x-direction with a line source:

$$\phi = Ux + \frac{Q}{2\pi}\log r = Ux + \frac{Q}{2\pi}\log\sqrt{x^2+y^2}.$$

We can then obtain the velocity field as

$$\boldsymbol{u} = [U, 0] + \frac{Q}{2\pi(x^2+y^2)}[x, y].$$

The streamlines (or velocity field) is shown below.

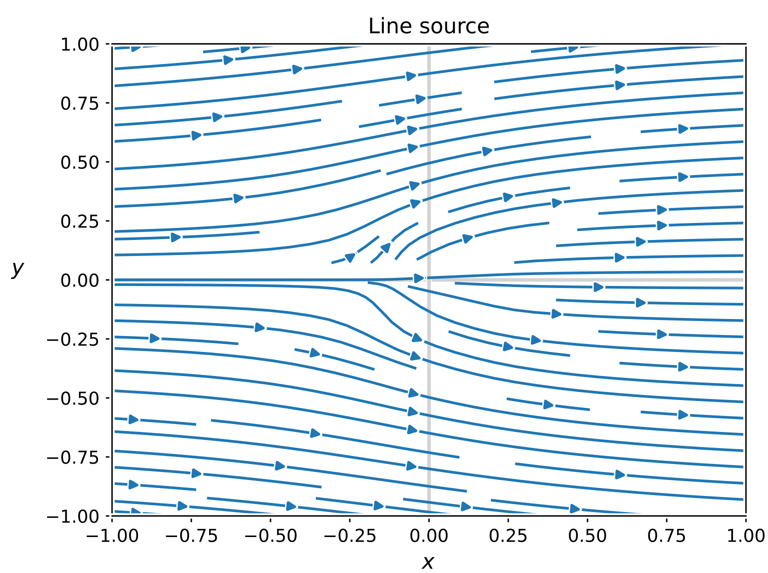


Figure 4.1.10 Streamlines (or velocity field) of a line source in a uniform flow with $U = 1$ and $Q = 1$.

Where do you think the stagnation point lies in this flow? □

4.2 The streamfunction

Our next task is to introduce the concept of the *streamfunction*. Remember that in 2D, the irrotational flow led to the equation (4.1.2) and this led to the existence of the potential function. If we begin with incompressibility, however, we have (4.1.4), which can be written as

$$\nabla \times [-v, u] = 0. \tag{4.2.1}$$

And we can deduce the existence of an analogous function, the *streamfunction*, $\psi(x, y, t)$ satisfying,

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \tag{4.2.2}$$

Alternatively and more conveniently, we can write

$$\boldsymbol{u} = \nabla \times (\psi \boldsymbol{k}). \tag{4.2.3}$$

To establish the existence of the streamfunction, we follow a similar proof as in Theorem 4.1.1 but now with the definition that

$$\psi(x, y, t) = \psi_0(t) + \int_0^{\boldsymbol{x}} (u \mathrm{d}y - v \mathrm{d}x), \tag{4.2.4}$$

where again $\psi_0(t)$ is an arbitrary function of t. The proof is otherwise identical, relying on establishing the independence of path of the integral using Stokes' theorem.

Why all this work? The streamfunction has an intuitive intepretation via the following result.

Theorem 4.2.1 The streamfunction is constant along streamlines. *The streamfunction, $\psi(x, y, t)$, is constant along streamlines of the flow (i.e. the trajectory formed by a particle in the flow).*

Proof. The proof follows simply by the fact that

$$\boldsymbol{u} \cdot \nabla \psi = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right) \cdot \left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} \right) = 0,$$

and for which the substitution in the first equality follows from (4.2.2).

The above equality indicates that the velocity vector, $\boldsymbol{u}$, is orthogonal to the vector pointing along $\nabla \psi$. However, it is known from Vector Calculus that $\nabla \psi$ runs along curves of steepest descent/ascent of ψ---these must hence be orthogonal to the level sets of ψ. Therefore, the level sets of ψ are tangential to $\boldsymbol{u}$ (the definition of a streamline). ■

A graphical depiction of the property in Theorem 4.2.1 is shown below.

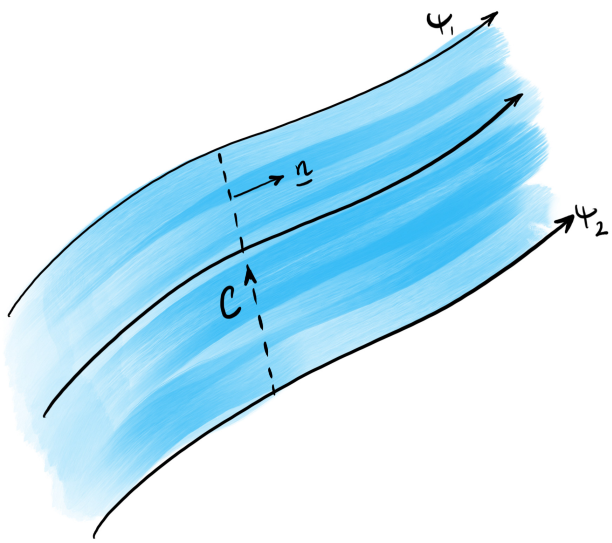


Figure 4.2.2 Flow between two streamlines, where the streamlines ψ_1 and ψ_2 are constant. Later, we will consider the flux through the contour C illustrated in the figure along with its unit normal $\boldsymbol{n}$.

The streamfunction is thus constant on streamlines. Consider two streamlines. The following theorem relates the flux between the streamlines to the streamline values.

Theorem 4.2.3 Flux between streamlines. *Consider two streamlines $\psi = \psi_1$ and $\psi = \psi_2$. We assume that the streamlines pass through the points A and B respectively. Consider a contour C connecting A and B. The* flux *(net flow of fluid) through C and hence between the streamlines is*

$$|\mathit{flux}| = |\psi_2 - \psi_1|$$

We have not specified the sign of the flux as it is subject to the considered direction through C.

Proof. By definition, the flux is given by the integral

$$\int_C \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}s,$$

where C is any smooth path joining the two streamlines with unit normal $\boldsymbol{n}$, as shown in Figure 4.2.2.

Note that given the curve, and in consideration of a small arclength element $\mathrm{d}s$, the tangent and normal are given by

$$\boldsymbol{t} = \frac{[\mathrm{d}x, \mathrm{d}y]}{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}} = \frac{[\mathrm{d}x, \mathrm{d}y]}{\mathrm{d}s}, \qquad \boldsymbol{n} = \frac{[-\mathrm{d}y, \mathrm{d}x]}{\mathrm{d}s}.$$

Therefore we can write $\boldsymbol{n}\mathrm{d}s = [\mathrm{d}y, -\mathrm{d}x]$. Then the flux is re-written as the following:

$$\begin{aligned}
\text{flux} &= \int_C \left[\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right] \cdot [\mathrm{d}y, \mathrm{d}x], \\
&= \int_C \left(\frac{\partial \psi}{\partial x} \mathrm{d}x + \frac{\partial \psi}{\partial y}\right), \\
&= [\psi]_C,
\end{aligned}$$

$$
\begin{aligned}
&= \psi(B) - \psi(A), \\
&= \psi_2 - \psi_1,
\end{aligned}
$$

In the third line above, $[\psi]_C$ is the change of ψ across the contour. Note that there is somewhat an arbitrary choice of direction for the contour C, as related to the selection of the normal direction $\boldsymbol{n}$, and the positivity or negativity of the flux. To be safe, we have taken the absolute value in the problem. ■

An example of the use of the above theorem is given in Exercise 4.7.2.

Example 4.2.4 Let the velocity of a two-dimensional flow be given by the formula

$$\boldsymbol{u} = [3ax^2 - 3ay^2,\ -6axy]$$

where a is a positive constant. By finding the streamfunction, ψ, calculate the volume flux across a curve connecting points $A = (0, 0)$ and $B = (1, 1)$ using Theorem 4.2.3.

Solution. (Done in problem class) We can check, using the integration of velocity components that

$$\psi = 3ax^2 y - ay^3 + C.$$

Hence the flux is

$$Q = \psi(B) - \psi(A) = 2a.$$

We can verify this flux should be positive if the normal is oriented to the right of the curve from A to B. □

Finally, note that the velocity potential was governed by Laplace's equation, $\nabla^2\phi = 0$. The streamfunction is also governed by the same Laplace's equation.

Theorem 4.2.5 Streamfunction satisfies Laplace's equation. *Like the velocity potential, the streamfunction satisfies Laplace's equation:*

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \nabla^2\psi = 0. \tag{4.2.5}$$

Proof. Substitute the relationship (4.2.2) into (4.1.2). ■

Like in the situation of certain flows, e.g. the line source in Example 4.1.7, it is easier to work in alternative coordinate systems to study the streamfunction. Since we know that $\boldsymbol{u} = \nabla \times (\psi\boldsymbol{k})$ by (4.2.3), we can use the conversion of the curl to polar coordinates in (**??**) to give

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad u_\theta = -\frac{\partial\psi}{\partial r} \tag{4.2.6}$$

which allows us to relate the radial and angular velocities to the streamfunction.

4.2.1 Elementary flows

Let us return to each of the examples in Section 4.1 and reconsider their corresponding streamfunctions.

Example 4.2.6 Uniform flow. From Example 4.1.3, we can directly integrate $u = \psi_y$ and $v = -\psi_x$ to get

$$\psi = Uy\cos\alpha - Ux\sin\alpha,$$

up to an arbitrary constant. Therefore, lines of constant ψ correspond to

$$y = x\tan\alpha + \text{constant},$$

which indeed yields the image seen in Figure 4.1.4. □

Example 4.2.7 Stagnation point flow. Now turning to Example 4.1.5, we integrate $u = y = \psi_x$ and $v = x = \psi_y$. This gives

$$\psi = xy,$$

up to an arbitrary constant. Indeed curves of constant ψ match the hyperbole shown in Figure 4.1.6. □

Example 4.2.8 Line source. For the situation of the line source in Example 4.1.7, we want to work with polar coordinates. From the previous work, we have for this situation the potential $\phi = Q/(2\pi logr$. It then follows from the polar version of the gradient in (**??**), that the velocity is entirely radial and

$$\boldsymbol{u} = \frac{\partial \phi}{\partial r}\boldsymbol{e}_r + 0 = \frac{Q}{2\pi r}\boldsymbol{e}_r. \tag{4.2.7}$$

We use the formulae in (4.2.6) and integrate $\frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{Q}{2\pi r}$ yielding

$$\psi = \frac{Q}{2\pi}\theta.$$

Then indeed note that the lines of constant ψ are given by the radial lines of constant θ, matching the illustration in Figure 4.1.8. □

Another remark concerns the fact that ψ is a multi-valued function in the example of the line source Example 4.2.8, gaining a jump of Q every time the origin is encircled. This is indeed a warning that the standard proof, analogous to Theorem 4.1.1, leading to the existence of a unique streamfunction, via (4.2.4) would not apply since the velocity field Example 4.2.8 is not defined at the origin. However, despite this, we see that the streamfunction provides well-defined predictions of streamlines on the *cut* plane with e.g. $\theta \in [0, 2\pi)$.

Example 4.2.9 Line source in a uniform flow. Like the case of the potential function in Example 4.1.9, the linearity of the equation governing the streamfunction implies that we can consider the combination of those streamfunctions for a line source with a uniform flow. This yields

$$\psi = Uy + \frac{Q}{2\pi}\theta,$$

for the case of uniform flow of speed U in the positive x-direction. The streamlines are then given by

$$Uy + \frac{Q}{2\pi}\theta = \frac{Q}{2\pi}C,$$

having designed the constant combination on the right hand-side for convenience. Then using $y = r\sin\theta$, we have

$$r = \left(\frac{Q}{2\pi U}\right)\frac{C-\theta}{\sin\theta}.$$

□

Our previous examples were reliant on considering the flows generated by the velocity potentials studied previously. However, we can also consider "fundamental solutions" of the equation $\nabla^2\psi = 0$ in their own right. Recall that in deriving the velocity potential of the line source in Example 4.1.7, we considered the solution of an axi-symmetric problem, where $\phi = \phi(r)$ is only dependent on the distance from the origin. A similar argument must imply that the analogous axi-symmetric streamfunction is a permissible solution. And this leads us to the following example.

Example 4.2.10 Line vortex. The fundamental solution for the streamfunction, in plane polar coordinates, is the axisymmetric solution,

$$\psi = -\frac{\Gamma}{2\pi}\log r, \tag{4.2.8}$$

defined up to a constant, and corresponds to a *line vortex*.

The streamlines of such a flow correspond to circular trajectories with r constant, and are visualised in

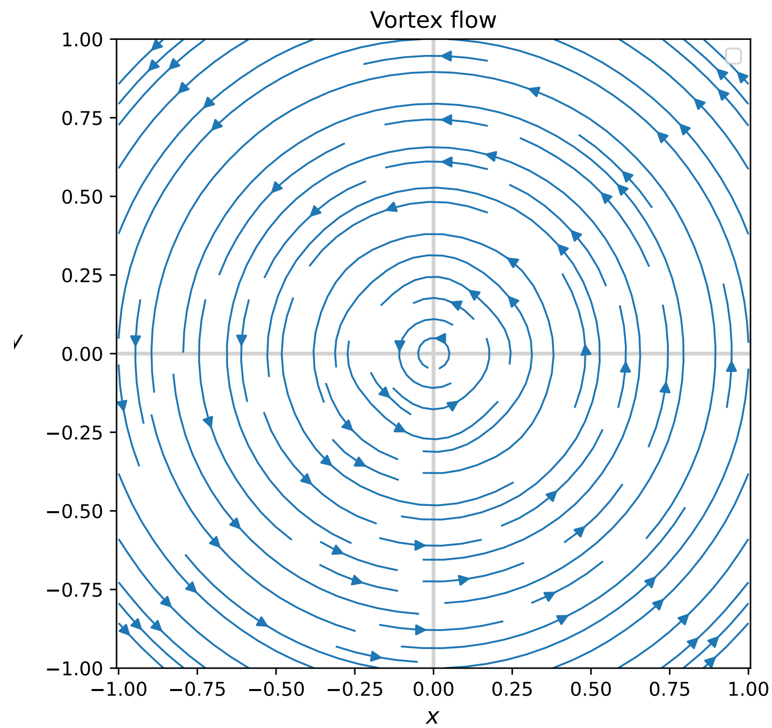


Figure 4.2.11 Streamlines (or velocity field) of a line vortex with $\Gamma = 1$.

Recall that the radial and angular velocities are given by (4.2.6). Therefore, we see that the velocity vector is given by

$$\boldsymbol{u} = -\frac{\partial\psi}{\partial r}\boldsymbol{e}_\theta = \frac{\Gamma}{2\pi r}\boldsymbol{e}_\theta, \tag{4.2.9}$$

and thus this flow corresponds to entirely circular trajectories orbiting the origin, and with angular velocity increasing as $r \to 0$.

The quantity Γ is called the *vortex strength*, analogous to the source strength Q in Example 4.1.7. Let us consider the amount of circulation around a contour C that contains the origin:

$$\oint_C \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{x} = \oint_C (u\,\mathrm{d}x + v\,\mathrm{d}y),$$

i.e. one envisions encircling the origin along C, adding up each of the velocity components tangential to the path. This is the circulation. By Stokes' theorem it is equal to the flux of vorticity of the corresponding bounding surface.

Choosing C to be the circle of radius $r = a$, we have $\mathrm{d}\boldsymbol{x} = a\boldsymbol{e}_\theta \mathrm{d}\theta$. Then the circulation is given by

$$\oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \int_0^{2\pi} \frac{\Gamma}{2\pi a} \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta a \mathrm{d}\theta = \Gamma. \tag{4.2.10}$$

So indeed this gives us an intuitive understanding of Γ. Notice that $\Gamma > 0$ corresponds to flow in the anticlockwise sense, and $\Gamma < 0$ corresponds to flow in the clockwise sense.

In Exercise 4.7.1, you will practice the computation of fluxes using line integrals. □

4.3 The complex potential

In the previous two sections, we studied the properties and utility of the velocity potential ϕ and streamfunction ψ in the context of two-dimensional potential flows (inviscid, incompressible, irrotational). We did this with techniques from real-valued Vector Calculus. As it turns out, there is a much more elegant and powerful framework for studying two-dimensional flow which leverages the significant power of complex analysis and complex variables.

In fact, you may have already noticed this on an intuitive level, given the intimate relationships between ϕ and ψ, seeming to exhibit a certain kind of symmetry in formulae and operations. Complex analysis is the language in which we can make this kind of ”symmetry” transparent.

We begin by reviewing (or in some cases, introducing) you to some key theorems about the properties of well-behaved (analytic) complex functions.

In his section, we will refer to a complex function generically as

$$f(z) = u(x, y) + \mathrm{i}v(x, y)$$

where u and v and the real-valued decompositions of the complex-valued function f. Also, $z = x + \mathrm{i}y$.

A complex function is differentiated in much the same way as real-valued functions, with the definition that

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}. \tag{4.3.1}$$

The crucial difference with real-valued differentiation is that the above limit is required to hold whilst approaching the point z in any direction of the complex plane.

Remark 4.3.1 As long as you stay away from exceptional points of a function, the ”calculus” of complex functions is largely the same as for real-valued functions, e.g.

$$\frac{\mathrm{d}(z^m)}{\mathrm{d}z} = mz^{m-1}, \qquad \frac{\mathrm{d}\mathrm{e}^z}{\mathrm{d}z} = \mathrm{e}^z,$$
$$\frac{\mathrm{d}(\log(z+1))}{\mathrm{d}z} = \frac{1}{z+1}, \quad \frac{\mathrm{d}\sin(z^2)}{\mathrm{d}z} = 2z\cos(z^2).$$

and the usual rules of algebraic manipulations hold. There are some caveats, however, which are dealt with on an individual manner.

Remark 4.3.2 Examinable content. The proofs of all theorems from this section are non-examinable, but you are expected to understand the theorems and their relevance to the theory of potential flow.

Below, we will often use susbcripts for partial differentiation, e.g. $u_x = \frac{\partial u}{\partial x}$.

4.3.1 Cauchy's theorem and harmonic functions

We will follow the reference text by [2] and [6] and introduce the basic notions of complex functions that we will need.

Definition 4.3.3 Analyticity. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be analytic in a domain if $f(z)$ is defined and differentiable at all points in the domain. The function is said to be *analytic at a point* $z = z_0$ if it is analytic in a neighbourhood of z_0.

When we refer to an *analytic function*, we mean a function that is analytic in some domain of $\mathbb{C}$ (often clear by context).

The function is *holomorphic* if it is analytic; the terms are synonyms.

An analytic function is *entire* if its region of analyticity includes all points in $\mathbb{C}$, including infinity. ♢

Note that above, we have stated that a function is analytic if it is well-defined and differentiable *once*(!) As it turns out, the requirement that a complex-valued function is differentiable is a strong condition. An analytic function turns out to be infinitely differentiable by consequence!

Remark 4.3.4 In this module, we will not be concerned with formalities when they are not relevant. For example, in our definition of analyticity, we do not specify if the relevant domains are open or closed. The functions we work with are generally non-pathological---they may have isolated singularities or exceptional points, but generally the application and context will make it clear the limits of our results.

Now we have one of the most important theorems of complex analysis.

Theorem 4.3.5 Cauchy-Riemann equations I. *If $f(z) = u(x, y) + \mathrm{i}v(x, u)$ is differentiable, then the Cauchy-Riemann equations, given by*

$$u_x = v_y \quad \textit{and} \quad u_y = -v_x, \tag{4.3.2}$$

hold.

In particular, if f is analytic on a domain, then its real and complex parts must satisfy the Cauchy-Riemann equations (4.3.2) (in that domain).

Proof. (Non-examinable)

This follows by considering the definition of the derivative (4.3.1) when the point z is approached from the x or y directions.

With $\Delta z = \Delta x$ real, we can verify from applying the definition that

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}.$$

On the other hand, approaching with $\Delta z = \mathrm{i}\Delta y$ gives

$$f'(z) = -\mathrm{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Equating the two results then yields the Cauchy-Riemann equations. ■

It turns out that the Cauchy-Riemann equations are not only necessary to an analytic function, but are actually sufficient as well.

Theorem 4.3.6 Cauchy-Riemann equations II. *If two real-valued continuous functions, $u(x, y)$ and $v(x, y)$ of two real variables x and y have continuous* *firsst partial derivatives that satisfy the Cauchy-Riemann equations (4.3.2) in some domain, then $f(z) = u(x, y) + \mathrm{i}v(x, y)$ is analytic in that domain.*

Proof. (Non-examinable)

The proof is not difficult, but we will refer students to [2] for its proof. It relies on constructing the derivative of f along any direction using the decomposition into the two Cartesian directions. ■

Theorem 4.3.7 *If f is analytic, it is differentiable to all orders.*

Our last step involves relating complex functions to the solution of Laplace's equation(s), i.e. (4.1.5) or (4.2.5), that govern potential flow.

Theorem 4.3.8 Analyticity and Laplace's equation. *If $f(z) = u(x, y) + \mathrm{i}v(x, y)$ is analytic in a domain D, then u and v satisfy Laplace's equation,*

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \quad \textit{and} \quad \nabla^2 v = v_{xx} + v_{yy} = 0,$$

in D, and have continuous second partial derivatives in D.

Proof. Since f is analytic, then it follows from Theorem 4.3.5 that $u_x = v_y$ and $u_y = -v_x$. It furthermore follows from Theorem 4.3.7 that f is differentiable to all orders; therefore, we can take derivatives to obtain

$$u_{xx} = v_{yx} \quad \text{and} \quad -u_{yy} = v_{xy}.$$

The second derivatives are continuous and therefore $v_{xy} = v_{yx}$. Therefore $u_{xx} + u_{yy} = 0$. Laplace's equation is analogously proved for v. ■

The above theorem is truly a remarkable result; it would not be a stretch to state that this single result was at the forefront of why complex analysis played such an important role in the development of applied mathematics and physics in the 18th, 19th, and 20th centuries.

Since Laplace's equation, $\nabla^2\phi = 0$, is such an important equation in physics, occuring in theories of gravitation, electrostatics, fluid mechanics, etc. the theorem establishes that there exists a parallel theory in the language of complex variables for the specific case of two-dimensional applications.

A potential fluid, for example, can be studied by manipulating complex functions of the form

$$f(z) = \phi(x, y) + \mathrm{i}\psi(x, y).$$

One can envision all kinds of different forms of f---polynomials, cosines and sines, exponentials, logarithms, etc. As long as the function is locally differentiable, it is thus analytic and therefore can be associated with (some kind of) fluid flow.

4.3.2 Examples of elementary flows

Let us return to the examples of flows studied in Section 4.1 and re-interpret them using the theory of analytic functions.

We will associate the velocity potential, ϕ, and streamfunction ψ with an analytic function in the following way.

Definition 4.3.9 Let $\phi(x, y)$ and $\psi(x, y)$ be the respective velocity potential and streamfunction for some potential flow. We define

$$f(z) = \phi(x, y) + \mathrm{i}\psi(x, y), \tag{4.3.3}$$

and call f the *complex potential*.

Note that by differentiating in the horizontal and vertical directions, we have

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial}{\partial x}(\phi + \mathrm{i}\psi) = \frac{\partial}{\partial y}(\phi + \mathrm{i}\psi).$$

Therefore it follows that the horizontal and vertical velocity components of the flow, related via $\boldsymbol{u} = [u, v]$, are given by

$$\frac{\mathrm{d}f}{\mathrm{d}z} = u - \mathrm{i}v. \tag{4.3.4}$$

♢

Example 4.3.10 Uniform flow. We can verify that the complex potential for uniform flow is given from

$$f(z) = U\mathrm{e}^{-\mathrm{i}\alpha}z.$$

This can be compared to Example 4.1.3. □

Example 4.3.11 Stagnation point flow. We can verify that the complex potential for a stagnation point flow is given by

$$f(z) = \frac{z^2}{2}.$$

This can be compared to Example 4.1.5. □

Example 4.3.12 Line source. We can verify that the complex potential for a line source flow is given by

$$f(z) = \frac{Q}{2\pi}\log z \tag{4.3.5}$$

This can be compared to Example 4.1.7.

The complex logarithm is an example of a function that is not analytic at the isolated point $z = 0$ where it possesess a *branch point*. However, it still provides a permissible analytic function away from the origin.

The evaluation of the complex logarithm can be performed via the definition

$$\log z = \log r + \mathrm{i}\theta, \tag{4.3.6}$$

where $z = r\mathrm{e}^{\mathrm{i}\theta}$ is the polar form representation of z. In particular *logz* is a multi-function with a *branch point* at the origin.

With the above decomposition of the logarithm in mind, notice that we can then conclude that the velocity potential and streamfunction are given by

$$\phi = \mathrm{Re}[f] = \frac{Q}{2\pi}\log r \quad \text{and} \quad \psi = \mathrm{Im}[f] = \frac{Q}{2\pi}\theta.$$

Indeed the streamlines are along the rays θ constant.

From (4.3.5), we can also compute the velocities using the relationship (4.3.4). We thus have

$$u - \mathrm{i}v = \frac{Q}{2\pi}\frac{1}{z} = \frac{Q}{2\pi}\frac{x - \mathrm{i}y}{(x^2 + y^2)},$$

once we have multiplied the top and bottom by the conjugate of z. □

Example 4.3.13 Line vortex flow. We can verify that the complex potential for a line vortex is given by

$$f(z) = -\frac{\mathrm{i}\Gamma}{2\pi} \log z. \tag{4.3.7}$$

This can be compared to Example 4.2.10.

Again, using the definition of the complex logarithm, via (4.3.6), we can write f in terms of its real and complex components as

$$f(z) = \frac{\Gamma}{2\pi}(\theta - \mathrm{i} \log r),$$

where $z = r\mathrm{e}^{\mathrm{i}\theta}$.

Therefore, the streamfunction is given by $\psi = -(\Gamma/2\pi) \log r$ and is constant along circular trajectories with constant distance from the origin, r. □

4.4 The method of images

The preceeding sections would give the misleading impression that solving potential-flow problems for two-dimensional flows is easy. This is not the case, and the primary reason is due to the presence of *boundary conditions*. The elementary flows we have previously considered were unconfined and/or we did not consider additonal constraints on their behaviours at infinity. In reality, a real physical fluid, whether in the ocean, the air, or in a container, is confined in some direction, and we must often consider subtle questions about the mechanism that produces the fluid motion.

In this section, we consider the situation of solving for the potential flow in a fluid region with boundaries. Recall that this is equivalent to finding a velocity potential satisfying $\nabla^2\phi = 0$ or an analytic complex potential, $f(z)$.

Recall from Note 3.4.7 that on solid boundaries, we must impose the no-flux condition that

$$\boldsymbol{u} \cdot \boldsymbol{n} = \nabla\phi \cdot \boldsymbol{n} = 0 \quad \text{on } \partial V.$$

4.4.1 Planar boundaries: a half-plane

Consider the situation illustrated in Figure 4.4.1.

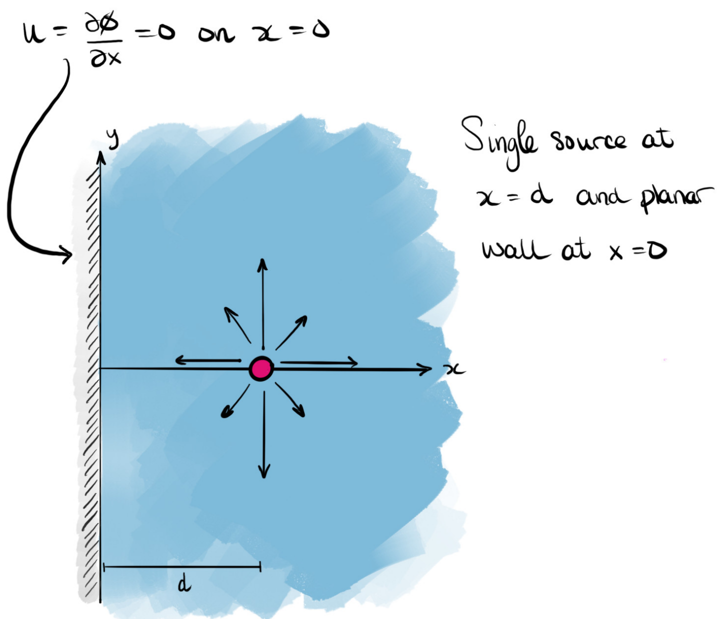


Figure 4.4.1 A flow region in the right half-plane, with a single source placed at $x = d$.

We envision a semi-infinite region of fluid bounded on the left by a wall at $x = 0$. A single (line) source of strength Q is placed at $x = d$. Therefore from (4.3.5), we would expect that at least near $x = d$, the complex potential behaves as

$$f(z) \sim \frac{Q}{2\pi} \log(z - d) \quad \text{as } z \to d.$$

However the above solution does not satisfy the required boundary conditions at $x = 0$ since it corresponds to a velocity field for which the horizontal velocity penetrates through $x = 0$. This can be verified via inspection. For example, we can inspect the velocity or the streamlines; this is part of Exercise 4.7.4.

Rephrased in terms of the streamlines, the boundary condition at $x = 0$ is equivalent to the constraint that

$$\operatorname{Im} f(z) = \psi = \text{constant at } x = 0.$$

Our inspired solution to the above problem is referred to as *the method of images*.

Note 4.4.2 Method of images. Given potential flow problem, we consider the superposition of elementary sinks/sources, i.e.

$$f(z) = \frac{1}{2\pi} \sum_{j=0}^{N-1} Q_j \log(z - z_j),$$

and/or vortices,

$$f(z) = -\frac{\mathrm{i}}{2\pi} \sum_{j=0}^{N-1} \Gamma_j \log(z - z_j).$$

The strengths and locations of the individual contributions are chosen so that boundary conditions on the required boundaries (including at infinity) can be met.

Notice that the *linearity* of the potential flow problem is crucial: any analytic function is associated with a velocity potential that satisfies Laplace's equation, $\nabla^2\phi = 0$, and therefore the superposition of such functions also yields a permissible complex potential, f.

We consider the addition of a "fictitious" image source, with the same strength at the reflected point $x = -d$, which lies outside of the posited fluid region. This gives the complex potential of

$$f(z) = \frac{Q}{2\pi}\log(z-d) + \frac{Q}{2\pi}\log(z+d).$$

This yields the illustration of the flow in Figure 4.4.3

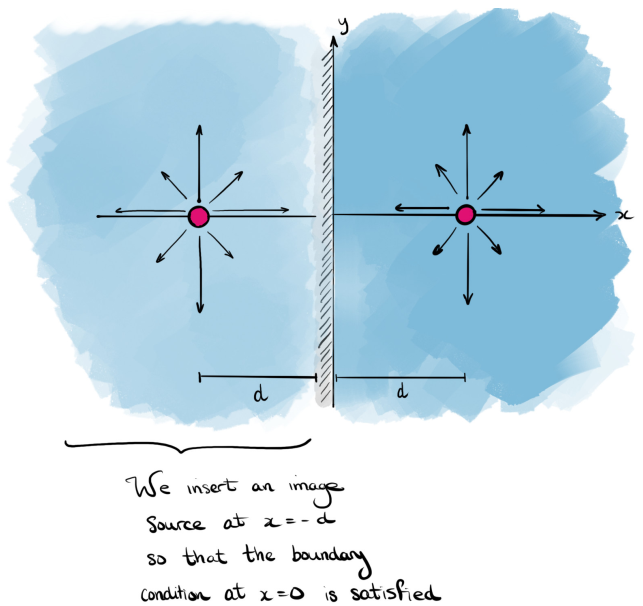


Figure 4.4.3 Placement of the image source at $x = -d$ makes it so the boundary condition on $x = 0$ can be satisfied.

The corresponding complex velocity is given by

$$u - \mathrm{i}v = f'(z) = \frac{Q}{2\pi}\frac{z}{z^2 - d^2}.$$

So indeed, on the central boundary, we have $z = \mathrm{i}y$, and

$$u - \mathrm{i}v = -\mathrm{i}\frac{Q}{\pi}\frac{y}{y^2 + d^2},$$

and the velocity is entirely vertical. So indeed, the condition that $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on the planar boundary is satisfied.

In order to study the complex velocity, $f(z)$, and develop an equation for the streamlines of the flow, we must first navigate the fact that the complex logarithm function is only well-defined in a slit complex plane. First, let

$$z - d = r_1 e^{\mathrm{i}\theta_1} \quad \text{and} \quad z + d = r_2 e^{\mathrm{i}\theta_2}.$$

Using the definition of the complex logarithm (4.3.6), we have

$$f(z) = \frac{Q}{2\pi}\left[\log r_1 + \log r_2\right] + \mathrm{i}\frac{Q}{2\pi}\left[\theta_1 + \theta_2\right].$$

The definitions of r_1, r_2 and θ_1, θ_2, are shown in the below figure. In order for each logarithm to be well defined, the angles θ_1 and θ_2 must be restricted to be less than a complete revolution. We thus restrict $\theta_1 \in [0, 2\pi)$ and $\theta_1 \in [-\pi, \pi)$.

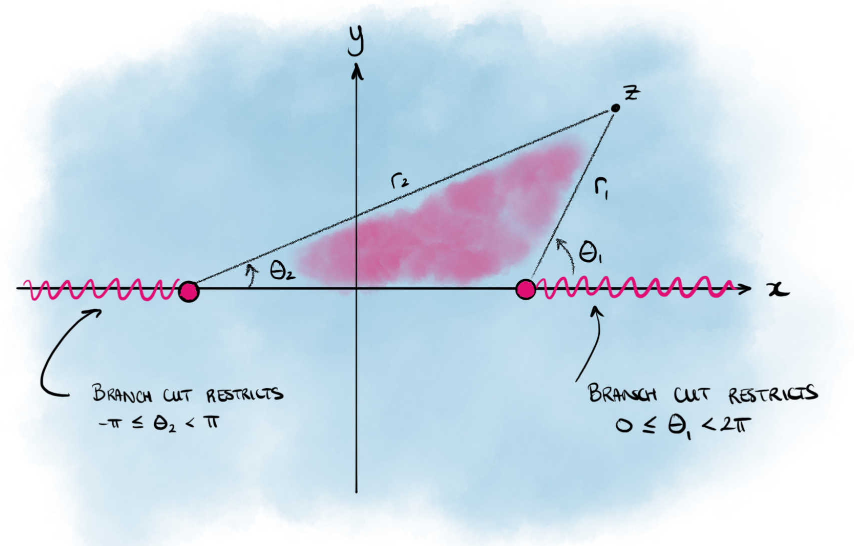


Figure 4.4.4 When considering the evaluation of the flow, we must take care of the fact that the logarithm is multi-valued. A branch cut from each of the two branch points is imposed.

In Exercise 4.7.5, you will be asked to develop an equation for the streamlines of this flow.

The above ideas can be extended to the situation of a line vortex in a half plane. Again, we are interested in describing the flow due to a line vortex at $z = d$, and therefore we expect that near this point,

$$f(z) \sim -\frac{\mathrm{i}\Gamma}{2\pi} \log(z - d) \quad \text{as } z \to d.$$

However, the above potential does not satisfy the necessary zero-flux condition at $x = 0$.

In this case, the approach is to add an image vortex at $z = -d$, but opposite in direction:

$$f(z) = -\frac{\mathrm{i}\Gamma}{2\pi} \log(z - d) + \frac{\mathrm{i}\Gamma}{2\pi} \log(z + d).$$

Therefore, this flow is composed by a line vortex circulating anticlockwise on the right, and a line vortex circulating clockwise on the left. It can be verified that the complex velocity is given by

$$u - \mathrm{i}v = -\frac{\mathrm{i}\Gamma d}{\pi(z^2 - d^2)}$$

and indeed the velocity at $x = 0$ is entirely vertical and there is no flux through the boundary.

There is an exercise in Exercise 4.7.6.

Remark 4.4.5 Uniqueness of solutions. You may be wondering: if a permissible potential function is found that satisfies the necessary boundary conditions, can we be certain it is the unique solution in the problem (up to a constant)? You may understand the construction of potentials, via the method of images, but perhaps irked that it involves the insertion of these so-called 'fictitious' points. The answer, at least for most non-pathological problems in potential flow theory (i.e. all the problems you study) is *yes*, the solution you have found is assured to be the only solution (up to a constant).

This is, to some extent, related to the uniqueness of analytic continuation. In a nutshall, the relevant theorem states that given two admissible complex potentials, say $f_1(z)$ and $f_2(z)$, that agree on the line $x = 0$ (in the case of the above situation), it is the case that $f_1 = f_2$ everywhere (where they are analytic).

Therefore, you can be certain that solutions you find via the trick of method of images are the only solutions.

4.5 Conformal mapping

The essential idea of conformal mapping is as follows. Suppose that we are given a two-dimensional potential fluid flow problem in a region, $R \subseteq \mathbb{C}$, with impermeable boundary ∂R. There may be singularities in R corresponding to sinks, sources, vortices, etc. We then seek a *conformal mapping* from the z-plane to the ζ-plane via

$$\zeta = g(z),$$

so that the region R is mapped to a new region $\hat{R} \subseteq \mathbb{C}$, as shown in Figure 4.5.1.

The hope is that within the ζ-plane, the fluid region is sufficiently simple that a complex potential, say $F(\zeta)$, can be found. This task is aided by virtue of the fact that sinks/sources and vortices are preserved by the conformal map. Typically, we wish for $\hat{R}$ to be e.g. the upper half-plane or the unit disc, with $\partial\hat{R}$ to be the real axis or circumferance of the unit disc, respectively. Once found, the complex potential in the z-plane is then obtained simply by inverting the conformal map, i.e.

$$f(z) = F(g(\zeta)) = \phi(x, y) + \mathrm{i}\psi(x, y).$$

This simple idea turns out to yield many insights to potential flows in two dimensions.

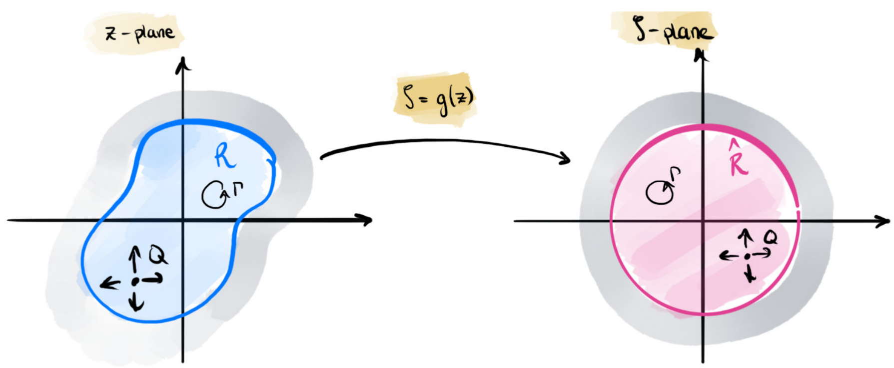


Figure 4.5.1 A general conformal mapping from the z-plane to the ζ-plane. The object is to map the region R to the region $\hat{R}$, which is geometrically simpler.

4.5.1 Source in a wedge

Consider fluid contained in a wedge with walls at $\theta = 0$ and $\theta = \alpha > 0$, and with the fluid in $0 < \theta < \alpha$. A source of strength Q is placed somewhere within the flow, say at the point $z = c$.

Consider then the map

$$\zeta = g(z) = z^{\pi/\alpha}. \tag{4.5.1}$$

It can be verified that this map transforms the fluid region to the upper half-ζ-plane. Indeed the ray $\theta = 0$ is mapped to the positive real axis and the ray $\theta = \alpha$ is mapped to the negative real axis. This is shown in Figure 4.5.2.

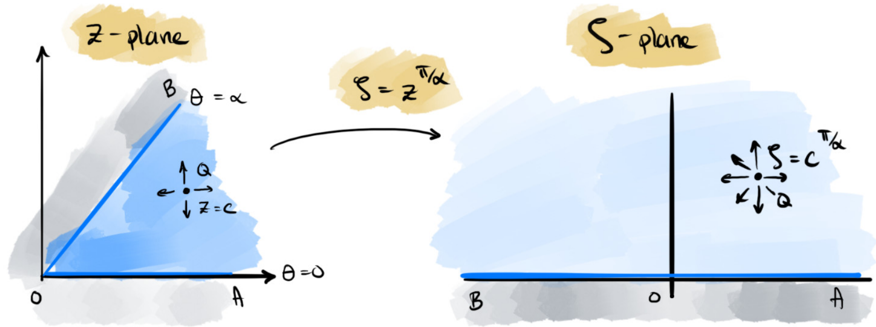


Figure 4.5.2 The map from the wedge-shaped region in the z-plane (left) and the upper half-ζ-plane (right).

In the ζ-plane, the fluid problem thus consists of solving for flow in the upper half-plane with a source of strength Q at the location $\zeta = c^{\pi/\alpha} = \zeta_c$, with an impermeable boundary on the real ζ-axis. Indeed, from the previous section, we know this can be solved using the method of images, with a source placed at both the point ζ_c and its complex conjugate point, $\overline{\zeta_c}$. It then follows that the complex potential is

$$F(\zeta) = \frac{Q}{2\pi} \log(\zeta - \zeta_c) + \frac{Q}{2\pi} \log(\zeta - \overline{\zeta_c}). \tag{4.5.2}$$

We can then invert the above formula, expressing the complex potential in the z-plane as

$$f(z) = F(g(z)) = \frac{Q}{2\pi} \log(z^{\pi/\alpha} - c^{\pi/\alpha}) + \frac{Q}{2\pi} \log(z^{\pi/\alpha} - \overline{c^{\pi/\alpha}}). \tag{4.5.3}$$

We can verify with a computational plot that this complex potential indeed seems to duplicate the necessary fluid flow within the wedge.

4.5.2 The conformal mapping method

How does it work?

We can say that the conformal mapping method is dependent on a number of key properties of conformal maps.

Definition 4.5.3 Conformal map. Let us specifically define a *conformal map* as a mapping, $\zeta = g(z)$, where g is analytic in a region R and also that $\frac{\mathrm{d}g}{\mathrm{d}z} \neq 0$ in R. ◊

The following properties hold for conformal maps.

Proposition 4.5.4 Properties of conformal maps.

1. *Conformal maps preserve angles.*
2. *If the boundary $\partial\hat{R}$ is a streamline in the ζ-plane, then the corresponding boundary ∂R is a streamline in the z-plane (and vice versa).*
3. *A source (or vortex) of strength Q at $\zeta = g(c) \in \hat{R}$ in the ζ-plane*

corresponds to a source (or vortex) of the same strength Q at $z = c \in R$ in the z-plane (and vice versa).

4.5.3 Standard conformal maps

The *exponential map* is used to map a channel to a half-space. Consider a channel of width h in the region $0 < \operatorname{Im} z < h$ in the z-plane. Then

$$\zeta = g(z) = \mathrm{e}^{\pi z/h}, \tag{4.5.4}$$

maps this channel to the upper half-ζ-plane. The correspondence of critical points and points at infinity in the pre-image and the image is shown in (4.5.4). It is good to see the map as essentially 'unfolding' the infinite channel, sending points AD to the origin, while sending B to negative infinity and C to positive infinity. The conformal map will preserve the orientation of the boundary, so as we traverse along ABCD, the fluid region is always on the left.

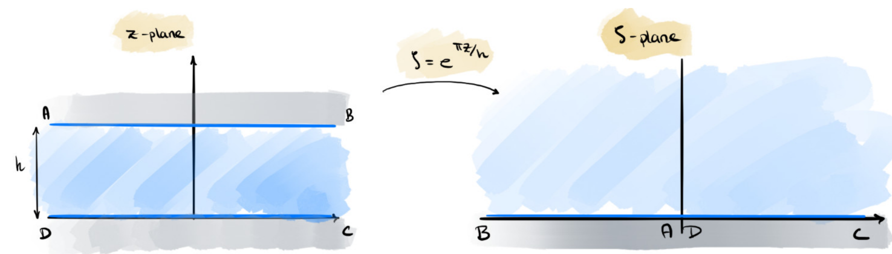


Figure 4.5.5 The exponential map maps the infinite strip of height h to the upper half-plane.

Trigonometric maps are used to map semi-infinite channels into a half space. Consider for example, the region R given in the following diagram in (4.5.5). We can then see that the semi-infinite channel of width $2a$ has been mapped to the upper half-plane. The two corners at $z = \pm a$ have been mapped to $\zeta = \pm 1$, respectively.

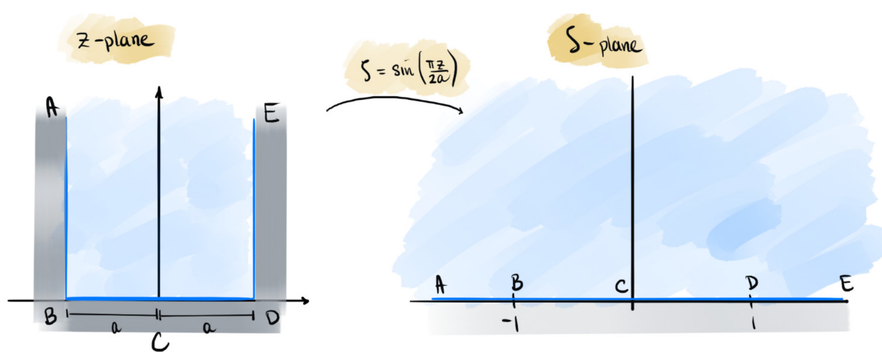


Figure 4.5.6 The sinusoidal transformation maps the semi-infinite strip of width $2a$ to the upper half-plane.

The above map is given by

$$\zeta = g(z) = \sin\left(\frac{\pi z}{2a}\right). \tag{4.5.5}$$

Example 4.5.7 Vortex flow in a semi-infinite channel. Consider the channel shown in the left of Figure 4.5.8. Insert a vortex of strength Γ at the point $z = d \in \mathbb{R}^+$. Verify that an appropriate conformal map is given by

$$\zeta = g(z) = \sinh\left(\frac{\pi z}{2a}\right), \tag{4.5.6}$$

and find where the map sends the relevant critical points of the pre-image.

Using the conformal map, find the complex velocity of the flow.

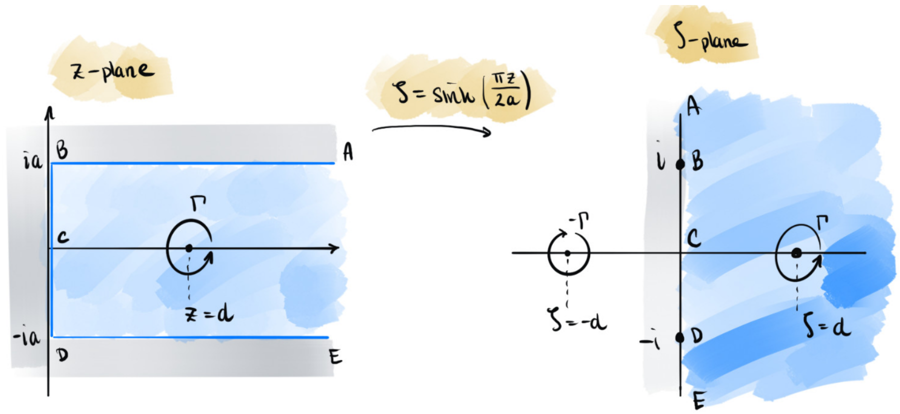


Figure 4.5.8 The sinh transformation maps the semi-infinite channel with height $2a$ to the right half-plane.

□

4.6 Flow past an aerofoil

4.6.1 Circle maps

Previously we used the method of images to construct flows past polygonal and straight boundaries.

Theorem 4.6.1 Milne-Thomson's Circle Theorem. *Let $f(z)$ be a given velocity potential such that any singularities in f occur in $|z| > a$. Then we can construct the following potential:*

$$w(z) = f(z) + \overline{f(a^2/\overline{z})}, \tag{4.6.1}$$

and this potential has the following properties: (i) it has the asme singularities as f in the region $|z| > a$; and (ii) the circle $|z| = a$ is a streamline.

Proof. (i) if $|z| > a$, then $|a^2/\overline{z}| < a$. Therefore the argument of the second term of w is within the circle of radius $|a|$ and by assumption f is not singular there.

(ii) On the circle itself, $z = ae^{i\theta}$. Then $a^2/\overline{z} = ae^{i\theta} = z$. Therefore, we have

$$w(z) = f(z) + \overline{f(z)} = 2\,\mathrm{Re}[f(z)].$$

Therefore the complex potential is entirely real. It follows that $\mathrm{Im}\,w = 0$ on $|z| = a$, establishing that the circle is streamline. ■

The above theorem then allows us to easily construct certain flows past circular cylinders. Here is an example.

Example 4.6.2 Uniform flow past a circular cylinder. We begin with the complex potential given by $f(z) = Uz$, which corresponds to horizontal uniform flow. We then consider inserting a circular cylinder with boundary $|z| = a$. Clearly, f satisfies the requirements of the Circle Theorem since it is non-singular for all z. Thus, we can construct the following result

Theorem 4.6.3 *Potential flow of consisting of uniform horizontal flow of speed U at infinity, past a circular cylinder of radius a (placed at the origin) is given by the complex potential*

$$w(z) = \phi + \mathrm{i}\psi = Uz + \frac{\overline{Ua^2}}{\overline{z}} = Uz + \frac{Ua^2}{z}. \tag{4.6.2}$$

The streamlines of the flow, found by taking the imaginary part of the above, are shown in Figure 4.6.4.

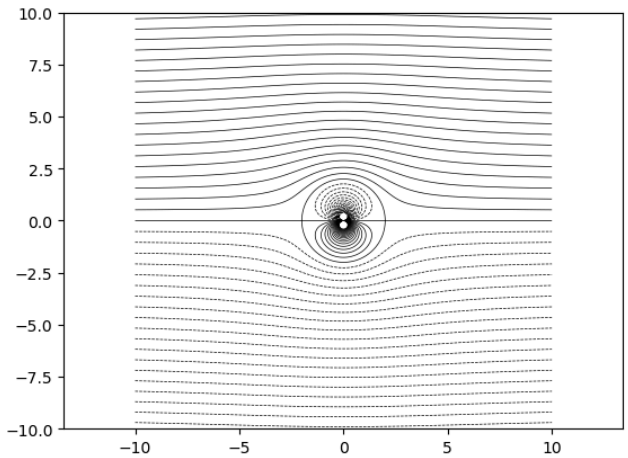


Figure 4.6.4 Uniform flow past a circle

□

The next example will be studied during the problem class.

Example 4.6.5 Problem class example. Previously, it was demonstrated, via (4.6.2) that uniform horizontal flow past a circle (circular cylinder) of radius a can be found via the complex potential function

$$w(z) = Uz + \frac{Ua^2}{z}, \tag{4.6.3}$$

corresponding to uniform horizontal flow at infinity of speed U.

(a) Confirm, by use of the streamfunction ψ, that the circle boundary is a streamline of the flow.

It will be useful to remind yourself that

$$\cos\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}. \tag{4.6.4}$$

Solution. We set $z = a\mathrm{e}^{\mathrm{i}\theta}$ into the streamfunction. This gives

$$\phi + \mathrm{i}\psi = U\left(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}\right) = 2U\cos\theta.$$

Therefore $\psi = \mathrm{Im}\, w = 0$ on the surface of the cylinder.

(b) Derive the complex velocity, $u - \mathrm{i}v$ and use it to show that on the surface of the cylinder, the velocity is given by

$$u - \mathrm{i}v = 2U\mathrm{e}^{\mathrm{i}(\pi/2-\theta)}\sin\theta,$$

where $z = a\mathrm{e}^{\mathrm{i}\theta}$.

It is helpful for you to remember that

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{4.6.5}$$

Solution. We need to take the derivative to obtain the complex velocity. So we have

$$w'(z) = u - iv = U\left(1 - \frac{a^2}{z^2}\right).$$

So again, substitution of $z = ae^{i\theta}$ and simplifying gives

$$u - iv = Ue^{-i\theta}\left(e^{i\theta} - e^{-i\theta}\right) = 2Ue^{i(\pi/2-\theta)}\sin\theta.$$

(c) Conclude that on the surface of the cylinder, the maximum velocity is $|\boldsymbol{u}| = 2U$. Where does this maximum velocity occur? Where do stagnation points in the flow (where the velocity is zero) occur?

Solution. From the above, we clearly see that the magnitude is $|u - iv| = 2U\sin\theta$ so the maximum speed on the cylinder is $2U$. This value is obtained at the top and bottom with $\theta = \pm\pi/2$. The stagnation point is found where the speed is zero, and this is at the fore and aft points, $\theta = \pi, 0$.

(d) Show that on the surface of the cylinder, the pressure force is given by

$$p = p_\infty + \frac{1}{2}\rho U^2(1 - 4\sin^2\theta),$$

where p_∞ is a constant value.

Solution. We know from Bernoulli's equation, without gravity, that we have (Theorem 3.5.9):

$$\left(p + \frac{\rho}{2}|\boldsymbol{u}|^2\right)_{\text{cylinder}} = \left(p_\infty + \frac{\rho}{2}U^2\right)_{\text{infinity}}$$

where we have set the constant on the RHS of the formula to be p_∞/ρ, setting a reference pressure. We now only need to substitute the speed on the cylinder, $|\boldsymbol{u}| = U(4\sin^2\theta)$, giving

$$p = p_\infty + \frac{\rho}{2}U^2(1 - 4\sin^2\theta)$$

as desired.

(e) It is possible to add a line vortex to the interior of the circular cylinder, centred at $z = 0$ by writing in addition to (4.6.3), an additional term:

$$w(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\log z.$$

Verify again that the above modification does not change the streamline properties of $|z| = a$.

Does the above flow satisfy the condition that the flow is of uniform speed U far upstream?

Solution. The addition of the line vortex does not change the streamline patterns on the cylinder since it is a linear addition. Indeed, on the surface, we have from the addition:

$$-\frac{i\Gamma}{2\pi}(\log a + i\pi\theta) = -\frac{\Gamma}{2\pi}(-\pi\theta + i\log a).$$

Therefore the imaginary part is still a constant, and hence the boundary of the cylinder is still a streamline of the flow.

To verify that the upstream flow is not altered, we can take a derivative to obtain the complex velocity:

$$w'(z) = u - \mathrm{i}v = U\left(1 - \frac{a^2}{z^2}\right) - \frac{\mathrm{i}}{2\pi z}.$$

So in the limit $|z| \to \infty$, we see indeed that the velocity is horizontal with speed U.

(f) By studying the velocity of the flow with the rotation element, verify that the complex velocity on the surface is given by

$$u - \mathrm{i}v = \mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}\left(2U\sin\theta - \frac{\Gamma}{2\pi a}\right).$$

Conclude that a point on the cylinder that satisfies the equation

$$\sin\theta = \frac{\Gamma}{4\pi a U},$$

is a stagnation point of the flow.

Solution. We had previously simplified the left terms in brackets:

$$u - \mathrm{i}v = 2U\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}\sin\theta - \frac{\Gamma\mathrm{i}}{2\pi a}\mathrm{e}^{-\mathrm{i}\theta} = \mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}\left(2U\sin\theta - \frac{\Gamma}{2\pi a}\right).$$

A stagnation point requires that $|u - \mathrm{i}v| = 0$, so setting the terms in brackets to zero implies

$$\sin\theta = \frac{\Gamma}{4\pi a U},$$

as desired.

(g) (Not examinable) Use a computational tool to investigate the streamlines of the flow for $\Gamma = 0, 2\pi aU, 4\pi aU, and 6\pi aU$.

The fact that the stagnation point shifts, moving on the topside or bottomside of the circular cylinder, dependent on the value of Γ will result in an asymmetry in the pressure, causing net force on one side of the cylinder. This is related to the well-known *Magnus effect*, which is used by athletes to cause a ball's trajectory to move from a straight path.

Solution. The point of the previous exercise is to note that when $\Gamma \neq 0$, this shifts the location of the stagnation point. For example, if $\Gamma = 4\pi aU$, then the stagnation points are found at

$$\sin\theta = 1 \Longrightarrow \theta = \pi/2,$$

i.e. the top of the cylinder.

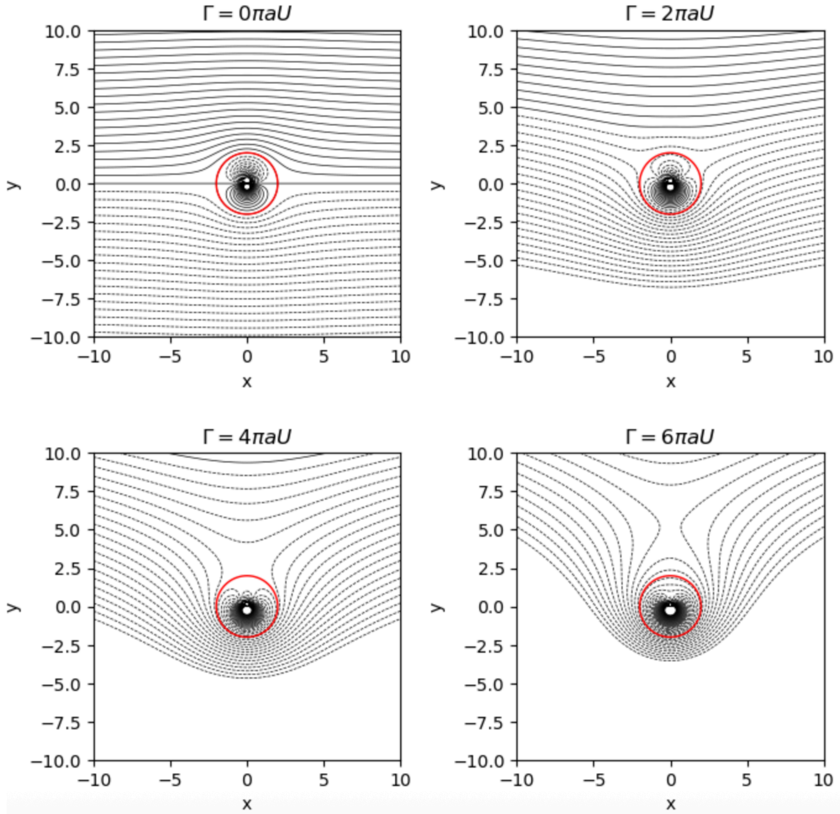


Figure 4.6.6 Flow past a circular cylinder with interior line vortex of different strengths.

□

4.6.2 The Joukowski map

We shall study the following mapping.

Definition 4.6.7 The *Joukowski transformation* is the map

$$z = G(\zeta) = \zeta + \frac{a^2}{\zeta} \tag{4.6.6}$$

where $a > 0$ is a parameter. The mapping is conformal at all points except $\zeta = 0$ (pole) and $\zeta = \pm a$ where $G'(\zeta) = 0$. ♢

Consider a circle in the ζ-plane. Let $\zeta = r\mathrm{e}^{\mathrm{i}\theta}$. Then in the z-plane, this is mapped to

$$z = r\mathrm{e}^{\mathrm{i}\theta} + \frac{a^2}{r}\mathrm{e}^{-\mathrm{i}\theta} = \left(r + \frac{a^2}{r}\right)\cos\theta + \mathrm{i}\left(r - \frac{a^2}{r}\right)\sin\theta.$$

Provided that $r > a$, this the equation of an ellipse with principle radii $r + a^2/r$ and $r - a^2/r$. The orientation of circles is preserved, with an anticlockwise rotation in ζ corresponding to anticlockwise in z. The exterior of the circle $|\zeta| = r$ is mapped to the exterior of the ellipse in z.

In addition note that if $r \to a$, the ellipses tends to the line segment

$$S = \{z = 2a\cos\theta\,|0 \le \theta \le \pi\},$$

which then ranges from $z = 2a$ for $\theta = 0$ to $z = -2a$ for $\theta = \pi$. As θ further ranges in $[\pi, 2\pi)$, this traverses the horizontal plate again. One can interpret it as traversing the top or the bottom side of the plate (a view strengthened from considering the limiting images). This is shown in Figure 4.6.8.

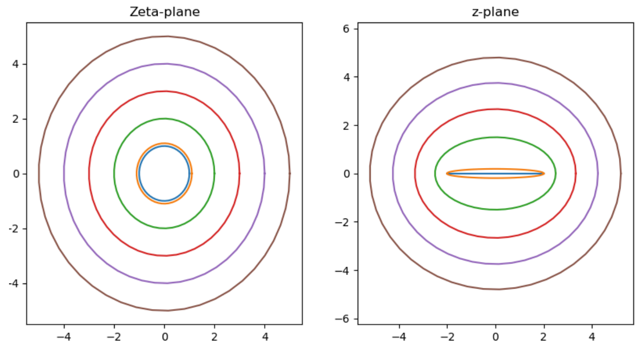


Figure 4.6.8 The Joukowski transformation in (4.6.6) sends circles with radius $r \geq 1$ to ellipses in the $z-$plane. In the limit $r \to a^+$, this approaches a horizontal plate of length $4a$.

If on the other hand, $r < a$, then we can verify that the mapping still produces ellipses, but now with principal radii $r + a^2/r$ and $a^2/r - r$. The orientation is now reversed, with anticlockwise orientation in ζ mapped to clockwise orientation. Therefore, it is the case that the interior of the discs with $|\zeta| = r$ are mapped to the exterior of ellipses in z. Again, in the limit that $r \to a^-$ the images approach a flat plate of length $4a$ on the axis.

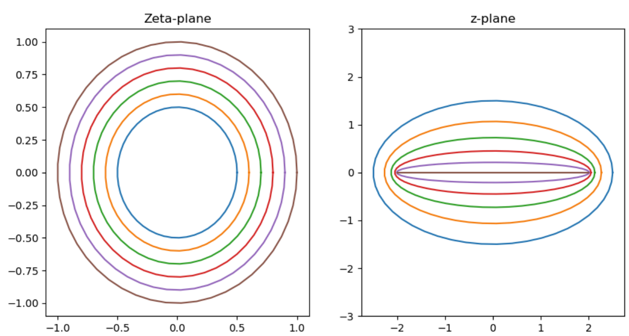


Figure 4.6.9 The Joukowski transformation in (4.6.6) sends circles with radius $r \leq 1$ to ellipses in the $z-$plane. As $r \to \infty$, these tend to infinitely large circles. The values here are $r = 0.5, 0.6, 0.7, 0.8, 0.9, 1$.

We can invert (4.6.6), and this gives

$$\zeta = \frac{1}{2}\left(z \pm \sqrt{z^2 - 4a^2}\right) = \frac{1}{2}\left(z \pm \sqrt{z - 2a}\sqrt{z + 2a}\right). \tag{4.6.7}$$

As noted in Subsection 1.2.3, we must take care to define the proper branch cut structure for the two branch points at $z = \pm 2a$.

As noted in the exercises of Exercises 1.3, if we let

$$z + 2a = r_1 \mathrm{e}^{\mathrm{i}\theta_1} \quad \text{and} \quad z - 2a = r_2 \mathrm{e}^{\mathrm{i}\theta_2},$$

and take $\theta_1, \theta_2 \in [0, 2\pi)$, this corresponds to a single branch cut between $z = -2a$ and $z = 2a$. This branch cut choice is the most convenient since then it directly coincides with the central axis of the ellipse.

Once the proper branch structure has been chosen, we can define the two possible inverses of the mapping, corresponding to the positive and negative branches. We have

$$\zeta = g_+(z) = \frac{1}{2}\left(z + \sqrt{z^2 - 4a}\right), \tag{4.6.8}$$

$$\zeta = g_-(z) = \frac{1}{2}\left(z - \sqrt{z^2 - 4a}\right). \tag{4.6.9}$$

Let us return to the perspective of images from the ζ-plane to the z-plane. We can now consider the images of circles in the ζ-plane that have either been horizontally shifted and/or vertically shifted. The two critical points (points where the map is not conformal) mentioned above correspond to either $\zeta = \pm a$ or $z = \pm 2a$.

Firstly in Figure 4.6.10, we see that horizontal shifts of the circle will corresponding to shifting and deforming the ellipse such that, as a critical point is approached, this forms a point of non-conformality in the image (a cusp).

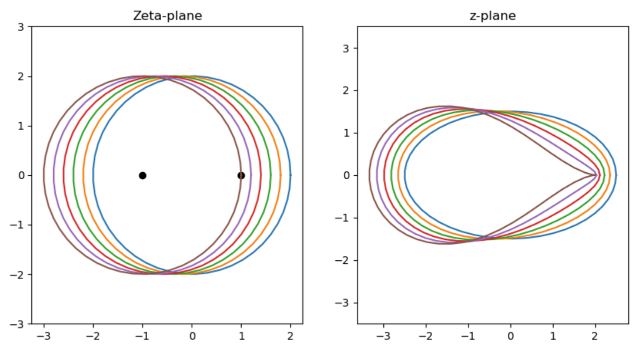


Figure 4.6.10 We now consider shifting the circle with $|\zeta| = 2$ to the left, so that the circumferance passes through the critical point at $\zeta = a = 1$. This produces an image that resembles an aerofoil with a sharp trailing edge.

Top-bottom asymmetry can also be introduced by considering vertical shifts of the preimages.

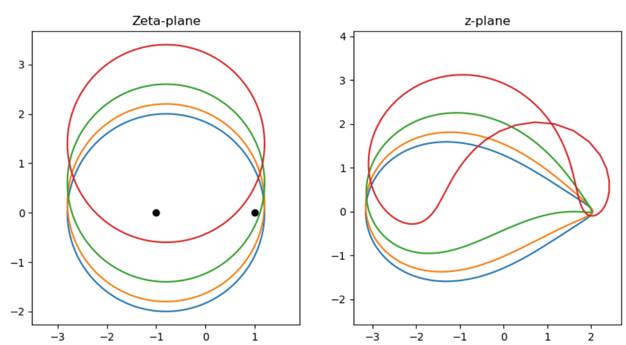


Figure 4.6.11 Vertical shifts correspond to changing the top-bottom symmetry of the aerofoil. In this case, notice that if the circle in the left plane passes through a critical point, this produces a self-intersecting shape.

4.7 Exercises

There is an excellent website at www.potentialflows.com that allows you to plug in different flow elements (sources/sinks, vortices, etc.) into a potential flow and observe the streamlines and potential-flow lines. In your exercises, use this to help you visualise the flow.

Remark 4.7.1 2025-26 note. After having administered this problem set, we realised how long it was! It was not our intention to make the exercises too long, but we had wanted to make sure you had plenty of examples of potential flows and conformal maps.

To aid your studying, we would highlight the following exercises that might be prioritised:

(a) Q1. Basic calculations is useful to do a few to learn the ins and outs, though note it does become tedious. I would suggest on first attempt to ignore the request to calculate fluid forces.

(b) Q2. This is nice to do. It teaches you that much of the "pain" of Q1 can be handled by the streamfunction theorem.

(c) Q4 and Q5 can be covered on the first pass. These are quick and get you practice on method of images.

(d) Q9 and Q10 can be done on a first pass. Q10 was done almost entirely in lectures but students can use this to go through the ideas.

Potential flows, part 1. These exercises cover approximately around sections 4.1 (the velocity potential) to 4.3 (the complex potential).

1. Basic calculations. The following question relates to two-dimensional potential flow. Remember that the fluid flux through a surface given by contour C is given by (2.2.2), or

$$\int_C \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}s.$$

You will get some practice on calculating this quantity below.

We can also calculate the total force on the surface specified by C by using the integral in (2.2.3). Since the force in potential flow is given by Theorem 3.3.2, then the total force is

$$\text{total force} = \int_C (-p\boldsymbol{n}) \, \mathrm{d}s$$

where p is the pressure force given by Bernoulli's equation:

$$p = p_0 - \frac{\rho}{2}|\boldsymbol{u}|^2,$$

and p_0 is a reference value and we ignore gravity.

For each of the following elementary flows, state or calculate:

- the complex potential, $f(z)$;
- the velocity vector, written in vector form $\boldsymbol{u} = [u, v]$;
- the flux and fluid force on a surface consisting of a circle of unit radius;

  Hint: the unit normal for the circle is $\boldsymbol{n} = [\cos\theta, \sin\theta]$; when converting to polar coordinates remember that $\mathrm{d}s = r\mathrm{d}\theta$.
- the flux and fluid force on a surface consisting of a plate given by the line $y = -x + 1$ with $0 \leq x \leq 1$.

(a) Uniform flow of velocity U oriented at an angle of $\pi/4$ to the horizontal.

Solution. (i) complex potential given in the notes is

$$f(z) = U\mathrm{e}^{-\mathrm{i}\alpha}z = U\mathrm{e}^{-\mathrm{i}\pi/4}z;$$

(ii) complex velocity is given by $f'(z) = u - \mathrm{i}v = U\mathrm{e}^{-\mathrm{i}\pi/4}$. Writing out the velocity components in vector form, we have

$$\boldsymbol{u} = U\Big[\cos(\pi/4),\, \sin(\pi/4)\Big] = U\frac{\sqrt{2}}{2}\Big[1, 1\Big].$$

(iii) We expect the flux past a circle will be zero (since the flow will enter one side and exit the other). The flux is given by

$$\int_{r=1} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}s = U\frac{\sqrt{2}}{2}\int_0^{2\pi} [1,1]\cdot[\cos\theta, \sin\theta]\mathrm{d}\theta = 0.$$

The force is given by

$$\boldsymbol{F}_{\mathrm{total}} = \left(\frac{\rho}{2}U^2 - p_0\right)\int_0^{2\pi} [\cos\theta, \sin\theta]\,\mathrm{d}\theta = [0, 0].$$

The total force again is zero since the force on the two hemispheres will cancel themselves out.

(iv) The normal of the plate is given by

$$\boldsymbol{n} = \frac{[1,1]}{\sqrt{2}}$$

(here we assume this normal is pointing 'out'). We can parameterise the plate using a vector equation for the position vector:

$$\boldsymbol{r}(t) = [t, -t+1], \quad 0 \le t \le 1,$$

since the equation of the line is $y = -x + 1$. Then

$$|\boldsymbol{r}'(t)| = |[1, -1]| = \sqrt{2}.$$

So the surface conversion is

$$\mathrm{d}s = |\boldsymbol{r}'(t)|\,\mathrm{d}t = \sqrt{2}\mathrm{d}t.$$

The flux is then

$$U\frac{\sqrt{2}}{2}\int_{t=0}^{1} \frac{[1,1]}{\sqrt{2}}\cdot[1,1]\sqrt{2}\mathrm{d}t = \sqrt{2}U.$$

The above makes perfect sense. The fluid is entirely normal to the plate, and the plate has length $\sqrt{2}$.

For the force, remember that the pressure is constant, since the speed is constant. Thus

$$\boldsymbol{F}_{\mathrm{total}} = \left(\frac{\rho}{2}U^2 - p_0\right)\int_0^1 \frac{[1,1]}{\sqrt{2}}\cdot\sqrt{2}\,\mathrm{d}t = \left(\frac{\rho}{2}U^2 - p_0\right)[1, 1].$$

(b) A line source of strength Q located at the origin.

Solution.

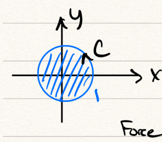
(b) Line source about $z=0$

(i) $\boxed{f(z) = \frac{Q}{2\pi}\log z}$

(ii) $f'(z) = \frac{Q}{2\pi}\cdot\frac{1}{z} = \frac{Q}{2\pi}\cdot\frac{1}{r}e^{-i\theta} = \frac{Q}{2\pi r}(\cos\theta - i\sin\theta)$

$$\boxed{\therefore\ [u,v] = \frac{Q}{2\pi r}[\cos\theta, \sin\theta]}$$

(iii) $\text{Flux} = \int_C \underline{u}\cdot\underline{n}\, ds = \int_{\theta=0}^{2\pi} \frac{Q}{2\pi r}[\cos\theta, \sin\theta]\cdot\Big\{\underbrace{[\cos\theta, \sin\theta]}_{\hat{n}}\Big\}\, r\cdot d\theta$



$$= \frac{Q}{2\pi}\int_0^{2\pi} d\theta = \boxed{Q}$$

$$\text{Force} = \int_C -p\,\underline{n}\cdot ds = \int_{\theta=0}^{2\pi}\left\{\rho|\underline{u}|^2 - p_0\right\}[\cos\theta, \sin\theta]\cdot r\cdot d\theta$$

$$|\underline{u}|^2 = \left(\frac{Q}{2\pi r}\right)^2(\cos^2\theta + \sin^2\theta) = \frac{Q^2}{4\pi r^2}$$

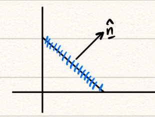
$$\text{Force} = \int_0^{2\pi}\left\{\rho\frac{Q^2}{4\pi r^2} - p_0\right\}[\cos\theta, \sin\theta]\, r\cdot d\theta$$

(set $r=1$)

$$= \left[\rho\cdot Q^2/4\pi - p_0\right]\cdot\int_0^{2\pi}[\cos\theta, \sin\theta]\cdot d\theta$$

$$\boxed{= (0,0)}$$

The part about integrating the flux and forces for the little diagonal plate segment has some annoy integrals. You are not expected to obtain the final values of the integrals, but you will see in the next question that they can be predicted rather easily using a theorem about the streamfunctions.

(iv) 

Compute $\text{Flux} = \int_{t=0}^{1} \underline{u} \cdot \frac{(1,1)}{\sqrt{2}} \sqrt{2}\, dt$

$\underline{r} = (t, 1-t), \quad 0 \le t \le 1$, tangent $= \underline{t} = \frac{(1,-1)}{\sqrt{2}}$, $\hat{\underline{n}} = \frac{(1,1)}{\sqrt{2}}$

$ds = \left|\frac{d\underline{r}}{dt}\right| \cdot dt = |(1,-1)|\, dt = \sqrt{2} \cdot dt$

$$\text{so Flux} = \int_{t=0}^{1} \frac{Q}{2\pi r} \cdot [\cos\theta, \sin\theta] \cdot [1,1] \cdot dt$$

Better to write this as following:

$$\text{Flux} = \int_0^1 \frac{Q}{2\pi r^2} [x, y] \cdot [1,1] \cdot dt = \frac{Q}{2\pi} \int_0^1 \frac{x+y}{x^2+y^2} \cdot dt$$

$$= \frac{Q}{2\pi} \cdot \underbrace{\int_0^1 \frac{dt}{t^2 + (1-t)^2}}_{\pi/2 \text{ [not expected to do]}} = \frac{Q}{2\pi}\left(\frac{\pi}{2}\right) = \boxed{\frac{Q}{4}}$$

$$\text{Force} = \int_{t=0}^{t=1} \left\{\frac{\rho Q^2}{4\pi r^2} - p_0\right\} \frac{(1,1)}{\sqrt{2}} \sqrt{2} \cdot dt = \int_0^1 \left\{\frac{\rho Q^2}{4\pi [t^2 + (1-t)^2]} - p_0\right\} \times (1,1) \cdot dt$$

This one can be done but we shall leave integral as is!

(c) Stagnation point flow with a stagnation point at the origin.

Solution.

c) STAGNATION POINT:

(i) $\boxed{f(z) = \frac{z^2}{2}}$

(ii) $f'(z) = z = x + yi \Rightarrow \boxed{[u, v] = [x, -y]}$

(iii) Circle of radius 1

$$\text{Flux} = \int_{(\text{Circle } r=1)} [x, -y] \cdot [\cos\theta, \sin\theta] \cdot r \cdot d\theta$$

$$= \int_{\theta=0}^{2\pi} r[r\cos\theta, -r\sin\theta] \cdot [\cos\theta, \sin\theta] \cdot d\theta$$

$$= \int_0^{2\pi} (\cos^2\theta - \sin^2\theta)\, d\theta \quad \boxed{= 0}$$

$$\text{Force} = \int_0^{2\pi} \left\{\rho |\underline{u}|^2 - p_0\right\} [\cos\theta, \sin\theta] \cdot r \cdot d\theta$$

$$= \int_0^{2\pi} \left\{\rho(x^2 + y^2) - p_0\right\} [\cos\theta, \sin\theta]\, r\, d\theta$$

$$(r = 1) \quad = \left\{\rho - p_0\right\} \cdot \int_0^{2\pi} [\cos\theta, \sin\theta]\, d\theta$$

$$\boxed{\text{Force} = [0, 0]}$$

The integrals for the flux and force on the plate are less annoying than in the previous geometries, but again we have left the final force calculuation un-evaluated (out of laziness).

(iv) Plate: $ds = |\underline{r}'(t)| \cdot dt = \sqrt{2} \cdot dt$, $\hat{\underline{n}} = \frac{[1,1]}{\sqrt{2}}$

$$\text{Flux} = \int_{t=0}^{1} [x,-y] \cdot \frac{[1,1]}{\sqrt{2}} \cdot \sqrt{2} \cdot dt$$

$$= \int_{t=0}^{1} (x-y) \cdot dt \qquad [\text{use } x=t,\ y=1-t]$$

$$= \int_0^1 [t-(1-t)] \cdot dt = \int_0^1 (2t-1) \cdot dt = t^2 - t\Big|_0^1 = 0$$

$$\text{Force} = \int_0^1 \{\rho \cdot |\underline{u}|^2 - p_0\} [1,1] \cdot dt$$

$$= \int_0^1 \{\rho(x^2+y^2) - p_0\} [1,1] \cdot dt \qquad [x=t,\ y=1-t]$$

$$\boxed{\text{Force} = \int_0^1 \left\{\rho\left[t^2+(1-t)^2\right] - p_0\right\} [1,1] \cdot dt}$$

Can be evaluated but won't bother

(d) A line vortex flow of strength Γ placed at the origin.

Solution.

d) Line vortex

(i) $f(z) = \frac{\Gamma}{2\pi i} \log z = \boxed{-\frac{i\Gamma}{2\pi} \log z}$

(ii) $f'(z) = \frac{\Gamma}{2\pi i} \cdot \frac{1}{z} = -\frac{i\Gamma}{2\pi} \cdot \frac{1}{r} [\cos\theta - i\sin\theta]$

$$= \frac{\Gamma}{2\pi r} \cdot [-\sin\theta - i\cos\theta].$$

so $\boxed{[u, v] = \frac{\Gamma}{2\pi r} [-\sin\theta, \cos\theta].}$

(iii) Circle flux:

$$\text{Flux} = \int_{r=1} \underline{u} \cdot \hat{\underline{n}} \, ds = \int_{\theta=0}^{2\pi} \frac{\Gamma}{2\pi r} [-\sin\theta, \cos\theta] \cdot [\cos\theta, \sin\theta] \cdot r \, d\theta$$

$$\boxed{= 0.}$$

$$\text{Force} = \int_0^{2\pi} \left\{ \rho \left(\frac{\Gamma}{2\pi r} \right)^2 (\sin^2\theta + \cos^2\theta) - p_0 \right\} [\cos\theta, \sin\theta] \cdot r \, d\theta$$

$$= \int_0^{2\pi} \left\{ \rho \left(\frac{\Gamma}{2\pi} \right)^2 \cdot \frac{1}{r} - p_0 r \right\} [\cos\theta, \sin\theta] \cdot d\theta \quad (r = 1)$$

$\boxed{= 0}$

(iv) Plate : $\underline{r}(t) = [t, 1-t]$, $0 \le t \le 1$

$$ds = \sqrt{2} \cdot dt$$

$$\underline{n} = \frac{[1,1]}{\sqrt{2}}$$

$$\text{Flux} = \int_0^1 \frac{\Gamma}{2\pi r}[-\sin\theta, \cos\theta] \cdot [1,1] \cdot dt \quad \text{where } r^2 = t^2 + (1-t)^2$$

$$= \int_0^1 \frac{\Gamma}{2\pi} \cdot \frac{1}{r^2}[-y, x] \cdot [1,1] \cdot dt = \int_0^1 \frac{\Gamma}{2\pi} \frac{[t-(1-t)]\,dt}{(t^2 + (1-t^2))} =$$

$$\boxed{= \int_0^1 \frac{\Gamma}{2\pi} \cdot \frac{(2t-1)}{t^2 + (1-t^2)}\,dt = 0}$$

$$\text{Force} = \int_{t=0}^1 \left\{ \rho \underbrace{\left(\frac{\Gamma}{2\pi r}\right)^2 \sqrt{\sin^2\theta + \cos^2\theta}}_{|\underline{u}|^2} - p_0 \right\} [1,1] \cdot dt$$

$$= \int_{t=0}^1 \left\{ \rho \left(\frac{\Gamma}{2\pi}\right)^2 \frac{1}{\sqrt{t^2 + (1-t)^2}} - p_0 \right\} [1,1] \cdot dt$$

You can do with trig subst.
but won't bother!

2. Evaluation of the flux. Return to the previous question and, instead of directly calculating the flux via a line integral, use Theorem 4.2.3 to calculate the flux for the two geometries (unit circle and straight plate) for each of the flows given.

Solution. This is a nice question to do because there is a simple recipe. For each of the complex potentials in the previous question, find the imaginary part to obtain the streamfunction. Then evaluate the difference in streamfunction values at the start and end values of the desired curve.

For uniform flow, $\psi(r,\theta) = Ur\sin(\theta - \alpha)|$. For the case of a circle, the flux is:

$$|\psi(1, 2\pi) - \psi(1, 0)| = 0.$$

For the case of a plate,

$$|\psi(1, 0) - \psi(1, \pi/2)| = U|\sin(-\pi/4) - \sin(\pi/4)| = \sqrt{2}U.$$

For source flow, $\psi = (Q/2\pi)\theta$ once you convert the logarithm into its real and complex parts. Then for the circle:

$$|\psi(1, 2\pi) - \psi(1, 0)| = Q.$$

For the case of the plate,

$$|\psi(1, 0) - \psi(1, \pi/2)| = \frac{Q}{4}.$$

(Actually calculating the above might convince you the route of calculating flux by integrals is possible!).

For the stagnation point, $\psi = (r^2/2)\sin 2\theta$. For the circle,

$$|\psi(1, 2\pi) - \psi(1, 0)| = 0.$$

For the case of the plate,

$$|\psi(1, 0) - \psi(1, \pi/2)| = 0.$$

For the case of the line vortex, $\psi = -\Gamma/(2\pi)\log r$, and for the circle,

$$|\psi(1, 2\pi) - \psi(1, 0)| = 0,$$

while for the plate,

$$|psi(1, 0) - \psi(1, \pi/2)| = 0.$$

3. **Doublet.** A line source of strength Q is at $z = a$ and a line sink of the same strength is at $z = -a$ where $a > 0$.

(a) Write down the complex potential, $f(z)$. Find $f'(z)$. Locate any stagnation points and derive an equation for the streamlines of the flow. Finally, use a plotter, such as the one at the potential flow simulator to sketch the streamlines.

Answer. The complex potential is the sum of a line source and line sink:

$$f(z) = \frac{Q}{2\pi}[\log(z - a) - \log(z + a)].$$

Therefore the complex velocity is given by

$$f'(z) = \frac{Q}{2\pi}\left[\frac{1}{z - a} - \frac{1}{z + a}\right].$$

Stagnation points are where $f'(z) = 0$. There are no stagnation points. The streamfunction is given by

$$\operatorname{Im} f = \frac{Q}{2\pi}(\theta_1 - \theta_2),$$

where θ_1 and θ_2 are angles measured relative to the positions about $\pm a$. Thus,

$$\theta_{1,2} = \tan^{-1}\left(\frac{y}{x \mp a}\right).$$

The streamlines are given where ψ is constant. The streamlines are shown below.

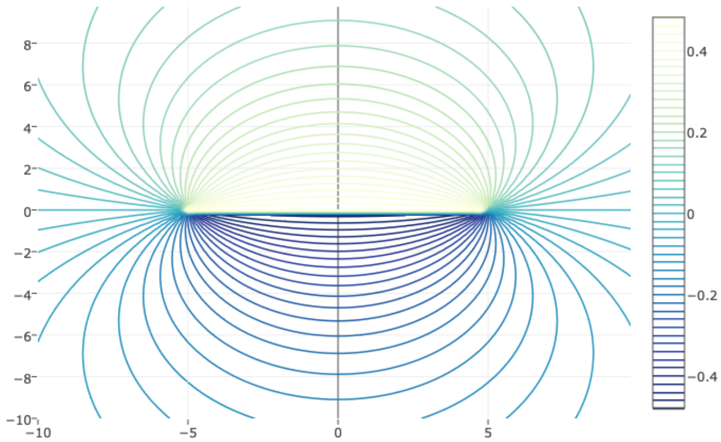


Figure 4.7.2 Streamlines for a sink and source

(b) Let $a \to 0$ and $Q \to \infty$ while keeping the product aQ fixed. This gives the flow due to a *doublet*. Show that its complex potential is μ/z where μ is expressed in terms of a and Q. It will be useful for you to use the fact that

$$\log(1+\alpha) = \alpha + O(\alpha^2)$$

considered as an appropriate approximation when α is small. Show that the streamlines are circles through the origin with centres on the y-axis.

Solution. Expanding the logarithms for small a gives

$$\log z - \log z + \log(1 - a/z) - \log(1 + a/z) \sim -\frac{2a}{z}.$$

Thus in the limit $a \to 0$, we have

$$f(z) \sim -\frac{Qa}{\pi}\frac{1}{z}.$$

Therefore, we let both $a \to 0$ and $Q \to \infty$ in a fashion such that $m = aQ$ is fixed. We then have the potential

$$f(z) = \frac{\mu}{z}$$

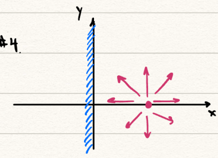
where $\mu = -Qa/\pi$.

Potential flows, part 2. These exercises will cover the second part of Chapter 4, from sec. 4.4 (the method of images).

4. Single source in a semi-infinite flow. Verify that a single source of strength Q placed at the point $z = d > 0$ is insufficient to describe flow bounded in the semi-infinite region, $x > 0$, with a planar boundary at $x = 0$. What is the horizontal and vertical velocities on the boundary? Find an equation for the streamlines and sketch the flow.

Solution.

#4.



Source at $z = d$: $f(z) = \frac{Q}{2\pi} \log(z-d)$

Velocity $\Rightarrow f'(z) = \frac{Q}{2\pi} \frac{1}{z-d} = \frac{Q}{2\pi} \frac{(x-d-iy)}{[(x-d)+iy][(x-d)-iy]}$

$$= \frac{Q}{2\pi} \frac{(x-d) + iy}{(x-d)^2 + y^2}$$

So on $x = 0$:

$$f'(z) = \frac{Q}{2\pi} \frac{-d+iy}{d^2+y^2} \Rightarrow u = \frac{Q}{2\pi}\left(\frac{1}{d^2+y^2}\right)(-d)$$

$$v = -\frac{Q}{2\pi} \cdot \frac{1}{d^2+y^2} y$$

Equation of streamline

$$\Rightarrow \psi = \operatorname{Im} f(z) = \operatorname{Im}\left[\frac{Q}{2\pi} \log(z-d)\right]$$

Let $z - d = re^{i\theta}$ $\Rightarrow \psi = \frac{Q}{2\pi} \cdot \theta$ where θ is measured about $z = d$

5. **A source in a semi-infinite flow.** Consider the situation of two point sources of identical strength, Q, placed at $z = \pm d$, with $d > 0$. Develop equations for the complex potential, $f(z) = \phi + \mathrm{i}\psi$, and complex velocity, $u - \mathrm{i}v$.

Demonstrate that the streamlines are given by hyperbolae and develop the equation for their form.

Solution.

#5. $f(z) = \dfrac{Q}{2\pi}\left\{\log(z-d) + \log(z+d)\right\} = \dfrac{Q}{2\pi}\log\left[z^2 - d^2\right]$

Best to measure via $z - d = r_1 e^{i\theta_1}$ and $z + d = r_2 e^{i\theta_2}$

but if you insist can do it in $x + iy$ coordinates

$$f'(z) = \frac{Q}{2\pi}\cdot\left\{\frac{1}{z-d} + \frac{1}{z+d}\right\} = \frac{Q}{2\pi}\left\{\frac{2z}{z^2 - d^2}\right\}$$

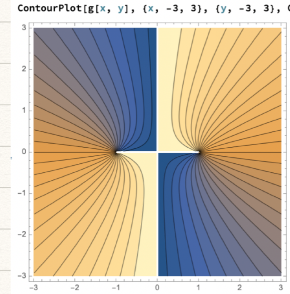
If you want this in terms of (x, y)

$$f(z) = \phi + i\psi = \frac{Q}{2\pi}\cdot\left\{\log\left|(x+iy)^2 - d^2\right| + i\,\mathrm{Arg}\left[(x+iy)^2 - d^2\right]\right.$$

The point is this is awkward to evaluate whatever the form.

Here is a graph of $\psi = \dfrac{Q}{2\pi}\mathrm{Arg}\left[(x+iy)^2 - d^2\right]$ to convince you.

```
g[x_, y_] = Arg[-1^2 + (x + ⅈ y)^2];
ContourPlot[g[x, y], {x, -3, 3}, {y, -3, 3}, Contours → 30]
```



6. Two vortices and a dividing boundary. Consider the situation of two point vorticies of identical strength, but opposite direction, placed at $z = \pm d$, with $d > 0$. Develop equations for the complex potential, $f(z) = \phi + \mathrm{i}\psi$, and complex velocity, $u - \mathrm{i}v$.

Demonstrate that the streamlines are given by

$$\psi = \frac{\Gamma}{2\pi}\log\left(\frac{r_2}{r_1}\right).$$

Solution.

#6. $f(z) = -\frac{i\Gamma}{2\pi}\left\{\log(z-d) - \log(z+d)\right\} = \phi + i\psi$

$f'(z) = -\frac{i\Gamma}{2\pi}\left\{\frac{1}{z-d} - \frac{1}{z+d}\right\} = u - iv$

Stream Function? Let $z-d = r_1 e^{i\theta_1}$

$z+d = r_2 e^{i\theta_2}$

Then need the real part of $\log(z-d) - \log(z+d) = \log r_1 - \log r_2$

$\therefore \text{Im} f(z) = -\frac{\Gamma}{2\pi}(\log r_1 - \log r_2) = -\frac{\Gamma}{2\pi}\log\left(\frac{r_1}{r_2}\right) = \frac{\Gamma}{2\pi}\log\left(\frac{r_2}{r_1}\right)$

7. A line source in a flow; stagnation points. Incompressible inviscid fluid occupies the region $y > 0$, and there is a rigid plane wall at $y = 0$. There is a uniform flow, speed U, in the positive x-direction, and a line source of strength Q at $(0, a)$, where $a > 0$. Find the complex potential $f(z)$ and calculate $f'(z)$. Let $\beta = Q/(2\pi aU)$. Show that if $\beta > 1$ there are two stagnation points, both on the wall, while if $\beta < 1$ there is only one, in the fluid, a distance a from the origin. Try to sketch the streamlines in either case

Solution. The potential is given by

$$f(z) = Uz + \frac{Q}{2\pi}[\log(z - a\mathrm{i}) + \log(z + a\mathrm{i})] = Uz + \frac{Q}{2\pi}\log(z^2 + a^2).$$

The complex velocity is then

$$f'(z) = U + \frac{Qz}{\pi(z^2 + a^2)}.$$

A stagnation point requires $f'(z) = 0$ and so we solve the resultant quadratic to obtain

$$z = \frac{-Q \pm \sqrt{Q^2 - (2a\pi U)^2}}{2\pi U}.$$

Setting $\beta = Q/(2\pi aU)$, there are two real roots if $\beta > 1$. Hence this is the case of two stagnation points along the axis. If $\beta < 1$, then there two complex conjugate points, and hence only one stagnation point in the flow field.

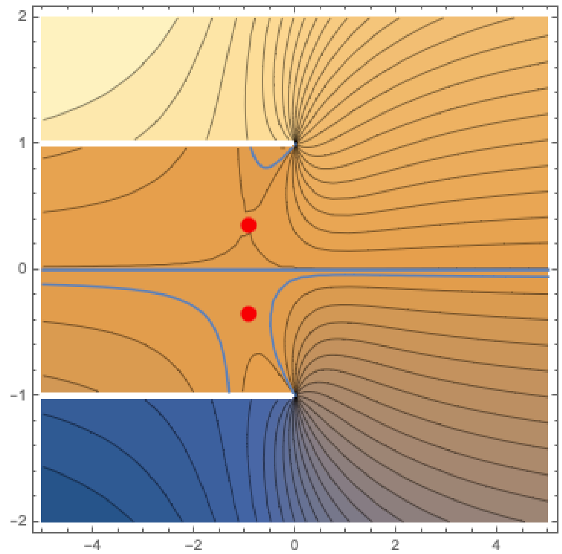


Figure 4.7.3 $\beta = 0.94$

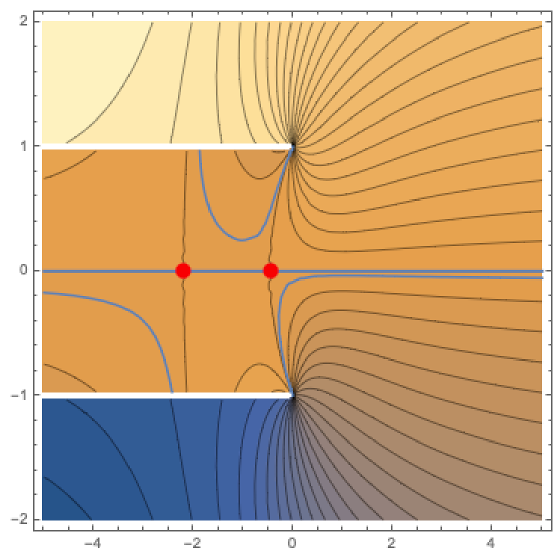


Figure 4.7.4 $\beta = 1.3$

8. **The exponential and sinusoidal conformal maps.** Check where each of the labeled points in the associated diagrams for the exponential map (4.5.4) and (4.5.5) are sent, from the z-plane to the ζ-plane. In each of these cases, verify that an interior (within the fluid domain) is sent to the upper half-plane.

   Construct the complex potential that corresponds to a single line source placed within the fluid in each of these cases (an infinite strip of height h and a semi-infinite strip).

Solution. For the exponential map, $\zeta = \mathrm{e}^{\pi z/h}$, we want to check the outputs of points A, B, C, D listed on the figure. These are as follows.

Table 4.7.5 Exponential map

z-plane	ζ-plane
$-\infty + \mathrm{i}h$	0^-
$\infty + \mathrm{i}h$	$-\infty$
∞	$+\infty$
$-\infty$	0^+

When thinking of the map, the most important step is to just verify that the line, $z = t + \mathrm{i}h$, is mapped to $\zeta = -\mathrm{e}^{\pi t/h}$, so the effect of shifting vertically by $\mathrm{i}h$ is to negate the values as compared to evaluating on $z = t$. To verify the region the fluid is in, simply "walk" from ABCD in the z-plane, and remark on the orientation of the fluid relative to your journey (the fluid is on the right). The direction is preserved also in the ζ-plane.

The potential that corresponds to a single source is found by using method of images in the ζ plane and mapping backwards. It is

$$F(\zeta) = \frac{Q}{2\pi}\left[\log(\zeta - d) + \log(\zeta + \overline{d})\right],$$
$$f(z) = \frac{Q}{2\pi}\left[\log(\mathrm{e}^{\pi z/h} - d) + \log(\mathrm{e}^{\pi z/h} + \overline{d})\right],$$

For the trigonometric map, $\zeta = \sin(\pi z/2a)$, the work is similar.

Table 4.7.6 Trigonometric map

z-plane	ζ-plane
$-a + \mathrm{i}\infty$	$-\infty$
$-a$	-1
0	0
a	1
$a + \mathrm{i}\infty$	∞

The potential that corresponds to a single source is found by using method of images in the ζ plane and mapping backwards. It is

$$F(\zeta) = \frac{Q}{2\pi}\left[\log(\zeta - d) + \log(\zeta + \overline{d})\right],$$
$$f(z) = \frac{Q}{2\pi}\left[\log(\sin(\pi z/2a) - d) + \log(\sin(\pi z/2a) + \overline{d})\right],$$

9. Elementary conformal maps. Define the term *conformal map*.

Write down the conformal maps from the wedge $0 < \arg(z) < \alpha$ into the upper half-plane. Find all the points at which the map is not conformal.

Solution. A conformal map is a mapping $\zeta = g(z)$ where g is analytic and its derivative is nonzero in some region of the complex plane.

The map from the wedge to the upper half-plane is given in the notes near (4.5.1). It is $\zeta = z^{\pi/\alpha}$. The points where the mapping is not conformal are the critical points listed in the figure Figure 4.5.2. Point A at infinity is mapped directly to +infinity. The origin is mapped to the origin. And the point, B, found via $z = r\mathrm{e}^{\mathrm{i}\alpha}$, is sent to $\zeta = -r^{\pi/\alpha}$ and then afterwards $r \to \infty$.

10. Flow past a flat plate. Show that the circle $|\zeta| = a$ is mapped to a line segment

$$S = \{z : \operatorname{Im} z = 0, -2a \le \operatorname{Re} z \le 2a\}$$

along the real z-axis by the Joukowski transformation

$$z = \zeta + \frac{a^2}{\zeta}.$$

Deduce that the exterior of the line segment is mapped to the exterior of the circle $|\zeta| = a$ by the transformation

$$\zeta = \frac{1}{2}\left(z + \sqrt{z^2 - 4a^2}\right).$$

Carefully define the function $\sqrt{z^2 - 4a^2}$ and determine where the mapping above is conformal.

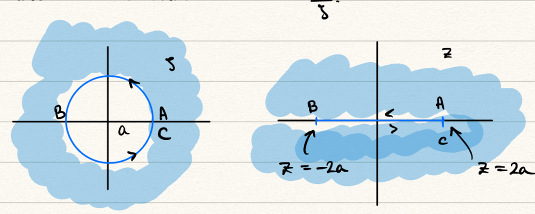
Constuct a flow, by providing the complex potential $\phi + i\psi$ that consists of the following elements: (i) uniform flow oriented an angle α to the horizontal at infinity; (ii) a line source of strength Q placed in the flow; and (iii) a horizontal flat plate of length $2a$ on the axis, $-2a < z < 2a$.

Sketch what you believe will be the streamlines of such a flow.

Solution. This problem was covered in the Week 5, Lecture 2 set, so do have a look at the video recordings to remind yourself. But is essentially broken up as follows. First, verify that there is a map from the outside of the circle (in ζ) to a flat plate (in z).

#5. Flow past a plate

Consider $|\zeta| = a$ under $z = \zeta + \frac{a^2}{\zeta}$.



Let $\zeta = ae^{i\theta}$, $\theta \in [0, 2\pi)$ $\Rightarrow$ $z = ae^{i\theta} + \frac{a^2}{(ae^{i\theta})} = ae^{i\theta} + ae^{-i\theta}$

use $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ $\Rightarrow$ $z = a\,2\cos\theta$

note also $\theta = 0 \Rightarrow z = 2a$ and $\theta = \pi \Rightarrow z = -2a$

To get inverse mapping:

$$z = \zeta + \frac{a^2}{\zeta} = \frac{\zeta^2 + a^2}{\zeta} \Rightarrow \zeta^2 - \zeta z + a^2 = 0$$

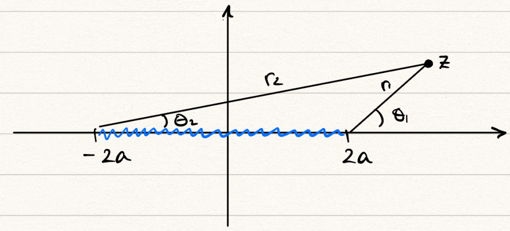
$$\Rightarrow \zeta = z \pm \sqrt{z^2 - 4a^2}$$

and we take the positive sign (WLOG)

Next, make sure the function $z = f(\zeta)$ is well defined via specification of the branch structure.

To define the function, just write $\zeta = z \pm (z-2a)^{1/2}(z+2a)^{1/2}$

and define $z - 2a = r_1 e^{i\theta_1}$, $z + 2a = r_2 e^{i\theta_2}$



and define $\zeta = z + (r_1 e^{i\theta_1})^{1/2} (r_2 e^{i\theta_2})^{1/2}$

Conformal maps done wk 5-2

The flow past a circle follows from using circle theorem

$$W(\zeta) = F(\zeta) + \overline{F(a^2/\bar{\zeta})}$$

where F is your desired map

Finally, construct a generic map of uniform flow plus a source, called $F(\zeta)$. Use the circle map theorem to obtain the corresponding flow past a circular cylinder with radius a. Finally use your connection to z to get the final solution.

(i) Uniform flow at angle α. $F(\zeta) = Ue^{-i\alpha}\zeta$

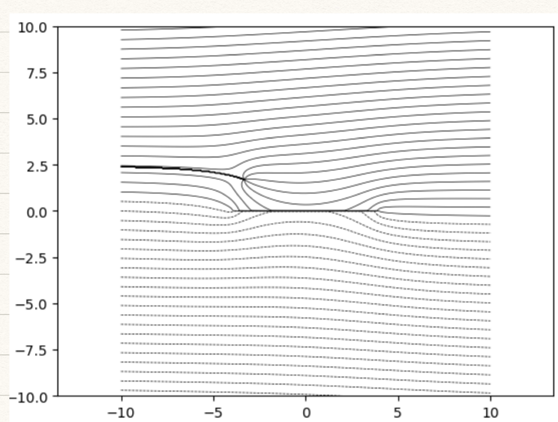
(ii) Add a line source: $F(\zeta) = Ue^{-i\alpha}\zeta + \frac{Q}{2\pi}\log(\zeta - d)$

(iii) Form the circle map:

$$W(\zeta) = F(\zeta) + \overline{F(a/\bar{\zeta})}$$

(iv) Finally map to z-plane by setting $\zeta = g(z)$

$$= z + \sqrt{z-2a}\,\sqrt{z+2a}$$



11. The Magnus effect and flow past a circular cylinder. During the problem class, we will study Example 4.6.5 corresponding to verifying different properties of uniform flow past a circular cylinder, ending with studying the *Magnus effect* due to adding an initial line vortex of strength Γ into the flow. Please review the derivations and make your own notes on the example so that you are confident with the problem and its conclusions.

Solution. There is no solution needed since you are asked just to review what is already written in the notes in Example 4.6.5.

Chapter 5

Water waves

In the mid-nineteenth century, George Gabriel Stokes made significant contributions to the study of water waves, establishing much of the theoretical foundation that underpins modern fluid mechanics. His analysis of periodic, finite-amplitude waves on the surface of an incompressible fluid led to what are now known as *Stokes waves*, which describe how real waves deviate from purely sinusoidal motion as their amplitude increases. Stokes' work marked a shift from qualitative observation to quantitative description, providing a systematic framework for understanding the balance of gravity, inertia, and nonlinearity in wave motion.

Our main task in this chapter is to introduce the fundamental principles governing water waves and the mathematical models that describe their behavior across different regimes.

Note 5.0.1 As part of our lecture, we will study the related video on the NCFMF database on Waves in Fluids. The particular section of relevance is at the start, concerning the particle trajectories in waves of deep water.

5.1 Governing equations of water waves

We shall consider the case of waves on the free surface of a two-dimensional potential fluid (hence inviscid, incompressible, and irrotational). The free surface bounds the water from above, and is assumed to be given by

$$y = \eta(x,t) \qquad -\infty < x < \infty.$$

Since the fluid satisfies the assumptions of potential flow, there exists a velocity potential $\boldsymbol{u} = \nabla\phi$ where

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad \text{in the fluid.}$$

What makes water-wave problems unique (and challenging) is that not only do we need to solve for the potential (and hence velocities) within the fluid, as given by Laplace's equation above, but that we must do so in an *a priori* unknown domain, where the free-surface location $y = \eta(x,t)$ must also be solved in parallel. Key considerations, then, must relate to the precise boundary conditions to impose.

5.1.1 Boundary conditions

5.1.1.1 Bottom boundary conditions

If we consider waves on a fluid of infinite (or deep) depth, then far below the free surface, we expect that the velocity tends to zero. Hence a suitable boundary condition is

$$\boldsymbol{u} = \nabla\phi \to 0 \quad \text{as } y \to -\infty. \tag{5.1.1}$$

On the other hand, for the case of fluid bounded below by a flat bed, say at $y = -H$, we have

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \Longrightarrow \frac{\partial\phi}{\partial y} = 0 \qquad \text{on } y = -H. \tag{5.1.2}$$

5.1.1.2 Dynamic boundary conditions

There are now two boundary conditions to consider at the free surface. The *dynamic boundary condition* corresponds to the requirement of satisfying a force balance at the free surface. At the free surface, the pressure within the liquid must be equal to the pressure within the air. This leads to the dynamic boundary condition that

$$p = p_{\text{atm}} \qquad \text{at } y = \eta(x,t), \tag{5.1.3}$$

tv where p_{atm} is the atmospheric pressure above the fluid (assumed to be constant).

Typically, we need the above pressure condition expressed in terms of ϕ, and this suggests using Bernoulli's equation. We require yet another version of Bernoilli's equation from a previous chapter, this one for unsteady flow and posed in terms of a potential function. This involves returning to an earlier manipulation of the momentum equation giving (3.5.5):

Theorem 5.1.1 Bernoulli's equation for unsteady potential flow. *Bernoulli's equation for unsteady potential flow states that*

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + \chi = F(t), \tag{5.1.4}$$

for an arbitrary function $F(t)$, and where $\boldsymbol{g} = -\nabla\chi$ is the conservative body force.

Proof. Returning to (3.5.5), we re-write in terms of the potential function and set the vorticity $\omega = \nabla \times \boldsymbol{u} = 0$, giving

$$\frac{\partial}{\partial t}(\nabla\phi) = -\nabla\left(\frac{1}{\rho}\nabla p + \frac{1}{2}|\nabla\phi|^2 + \chi\right).$$

We can interchange the order of differentiation on the left

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{\rho}\nabla p + \frac{1}{2}|\nabla\phi|^2 + \chi\right) = 0.$$

We now integrate each of the spatial components and this yields the desired result. ■

Below, we will set the body force indeed to be gravity. Thus if we write

$$\chi = gy \Longrightarrow \boldsymbol{g} = -\nabla\chi = \Big[0, -g\Big],$$

and gravity is then directed in the negative y-direction.

We now evaluate Bernoulli's equation (5.1.4) on the free surface, $y = \eta$. This gives

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p_{\mathrm{atm}}}{\rho} + g\eta = F(t), \qquad \text{on } y = \eta.$$

It is convenient to set the arbirary function $F(t) = \frac{p_{\mathrm{atm}}}{\rho}$, yielding the final result:

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0, \qquad \text{on } y = \eta,$$

which is a re-phrasing of the necessary dynamic boundary condition on the free surface.

5.1.1.3 Kinematic boundary conditions

In the case of fluids near solid surfaces, we require a *no-penetration* condition, which enforces that the fluid velocity is zero in a direction normal to the surface, $\boldsymbol{u}\cdot\boldsymbol{n} = 0$. For the case of a free boundary, such as the one at $y = \eta(x,t)$, there is a similar constraint, and it may be shown that this is equivalent to imposing that the material fluid fluid elements on the free surface must remain on the free surface.

Thus, if $y = \eta$ for some fluid particle at time t, then $y = \eta$ for the same particle for all time. Thus we need

$$\frac{\mathrm{D}(y-\eta)}{\mathrm{D}t} = 0, \qquad \text{on } y = \eta.$$

Expansion and use of the material derivative then implies the *kinematic boundary condition*

$$v = \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x}, \qquad \text{at } y = \eta. \tag{5.1.5}$$

5.1.2 Summary of the governing equations

Let us summarise the governing water-wave equations in one section.

Theorem 5.1.2 Water wave equations in potential flow. *Consider two-dimensional free-surface flow of a potential (inviscid, incompressible, irrotational) fluid. The fluid satisfies Laplace's equation,*

$$\nabla^2\phi = 0, \qquad \textit{in the fluid domain.}$$

On the free surface, we have the kinematic boundary condition and dynamic boundary conditions:

$$\frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} = v, \qquad y = \eta(x,t) \tag{5.1.6}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0, \qquad y = \eta(x,t). \tag{5.1.7}$$

Finally there is typically a condition in the far field, either at infinity in the case of infinite depth, or a condition on a channel bottom in the case of finite depth. For example, in the case of infinite depth and an otherwise motionless *fluid at infinity, we would require*

$$\nabla\phi \to 0, \qquad \textit{as } y \to -\infty.$$

While for the case of a finite channel bottom at $y = -h$, we would require

$$\frac{\partial \phi}{\partial y} = 0 \qquad \textit{on } y = -h.$$

5.2 Wave parameters

A general travelling wave is written as

$$\eta = a \sin\left[kx - \omega t\right]. \tag{5.2.1}$$

An additional phase, e.g. in $\sin(kx - \omega t - \beta)$ can be included but this can be removed through a shift in space or time. What do the quantities k and ω represent?

We can factor out the k and obtain

$$\eta = a \sin\left[k\left(x - \frac{\omega}{k}t\right)\right].$$

The quantity k is called the *wavenumber.* If we define it instead to be

$$k \equiv \frac{2\pi}{\lambda}, \tag{5.2.2}$$

then we see that shifting x by $\pm\lambda$ does not change the wave. The quantity λ is called the *wavelength.*

Let us now turn to the quantity ω/k. If we introduce

$$c \equiv \frac{\omega}{k}, \tag{5.2.3}$$

then c is called the *phase speed.* It corresponds to the speed at which the "phase" of the wave (the crests and troughs) propagate.

At a fixed location in space, what is the time that it takes for the condition at this point to repeat itself? It is the time required for the wave to travel one wavelength, λ. Hence this is defined as *the period* of the wave:

$$T \equiv \frac{\lambda}{c}. \tag{5.2.4}$$

These concepts can be generalised to wave in a three-dimensional fluid, where we would write instead

$$\eta = a \sin(kx + ly + mz - \omega t) = a \sin(\boldsymbol{K} \cdot \boldsymbol{x} - \omega t). \tag{5.2.5}$$

In this case, $\boldsymbol{K} = (k, l, m)$ is the *wavenumber vector* and has magnitude $K = \sqrt{k^2 + l^2 + m^2}$. The wavelength is then

$$\lambda = \frac{2\pi}{K}. \tag{5.2.6}$$

The phase velocity is given by

$$\boldsymbol{c} = \frac{\omega}{K}\frac{\boldsymbol{K}}{K}. \tag{5.2.7}$$

5.3 Stokes waves

5.3.1 Linearisation of the water-wave equations

Our task in this section is to linearise the water-wave equations given in Theorem 5.1.2. The culprit to the difficulty in seeking solutions concerns two main issues. The first is the nonlinearity in Bernoulli's equation (i.e. the term $|\nabla\phi|^2$). The second is to do with the specification of the unknown free surface, $\eta(x,t)$, and the need to solve the extra equations on this unknown location.

The idea of linearisation relies upon the consideration that if disturbances in the fluid are small, then the quadratic terms in the free-surface conditions can be neglected. One way to see this is to introduce a scaling on all the associated quantities. For example, we consider the free surface as written as

$$\eta(x,t) = \delta\overline{\eta}(x,t), \tag{5.3.1}$$

where $\delta \ll 1$ (δ is a small number). The parameter δ is simply a notational device to help remind us of the smallness of the free surface.

Now since we assume the motion in the fluid is also small, then we introduce a similar scaling on the potential function. We assume

$$\phi(x,y,t) = \delta\overline{\phi}(x,y,t), \tag{5.3.2}$$

within the fluid. As a consequence, all velocities are small and of order δ.

Now consider the evaluation of the vertical velocity on the surface. By Taylor's formula, we can expand:

$$v = \frac{\partial\phi(x,\eta,t)}{\partial y} = \frac{\partial\phi(x,0,t)}{\partial y} + \frac{\partial^2\phi(x,0,t)}{\partial y^2}\eta + \text{higher-order terms.}$$

In fact, if we use our assumptions (5.3.1) and (5.3.2), we can see the size of the vertical velocity more explicitly:

$$v = \delta\frac{\partial\overline{\phi}(x,0,t)}{\partial y} + \delta^2\frac{\partial^2\overline{\phi}(x,0,t)}{\partial y^2}\overline{\eta} + O(\delta^3).$$

We will essentially ignore all terms of δ^2 or smaller.

Let us consider now the kinematic condition (5.1.6). Since the term with the horizontal velocity $u\frac{\partial\eta}{\partial x} = O(\delta^2)$, then we see that the kinematic approximation now yields

$$\delta\frac{\partial\overline{\eta}}{\partial t} = \delta\frac{\partial\overline{\phi}}{\partial y} + O(\delta^2), \qquad \text{on } y = 0.$$

As we indicated, the use of δ is only to aid with ordering. We see that if we return back to the original variable scalings, then the above indicates that the approximate kinematic condition to be considered is:

$$\frac{\partial\eta}{\partial t} \sim \frac{\partial\phi}{\partial y} \qquad \text{on } y = 0. \tag{5.3.3}$$

It is typical, when dealing with linear water waves, to dispense with the notation indicating "approximate", i.e. $\sim$ here, on the understanding that the solutions are satisfied only approximately.

Remark 5.3.1 The development of the linearised equation(s) seem quite lengthy, as presented above, but with understanding, you will see that they are quite simple. In essence, you have replaced the exact free-surface condition of

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v, \qquad \text{on } y = \eta(x,t),$$

with

$$\frac{\partial \eta}{\partial t} = v, \qquad \text{on } y = 0,$$

on the assumption the flow is linearised about $\phi = 0$ and $\eta = 0$.

You will be able to practice this linearisation in the exercises.

The linearisation of Bernoulli's equation in (5.1.7) is done similarly, and the end result is that the term,

$$|\nabla \phi|^2 = \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2,$$

is assumed to be small compared to the other terms of the equation.

Using the linearisation techniques, similar to the kinematic equation and its linear variation (5.3.3), Bernoulli's equation in (5.1.7) is then approximated as

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + g\eta = 0 \qquad \text{on } y = 0.$$

This yields the set of governing linear equations as follows.

Theorem 5.3.2 Linear water wave equations. *Consider two-dimensional potential flow (inviscid, incompressible, irrotational) of fluid. A model for linear water waves is as follows:*

$$\nabla^2 \phi = 0 \qquad -\infty < y < 0 \tag{5.3.4}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + g\eta = 0 \qquad \textit{on } y = 0 \tag{5.3.5}$$

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v \qquad \textit{on } y = 0 \tag{5.3.6}$$

where the free surface is located at $y = \eta(x,t)$.

In water of infinite depth, we require that

$$\nabla \phi \to (0,0) \qquad \textit{as } y \to -\infty, \tag{5.3.7}$$

while for the case of finite depth, we have

$$\frac{\partial \phi}{\partial y} = 0 \qquad \textit{on } y = -h. \tag{5.3.8}$$

5.3.2 Solving for linear waves in infinite depth

We now seek to solve the linear water wave equations in Theorem 5.3.2.

We assume that the wave takes the form of a simple harmonic wave:

$$\eta(x,t) = A\cos(kx - \omega t). \tag{5.3.9}$$

Examining the free-surface conditions in (5.3.5) and (5.3.6) suggests that the potential should take the form of a sinusoidal:

$$\phi(x,y,t) = f(y)\sin(kx - \omega t), \tag{5.3.10}$$

which we shall verify is appropriate.

Firstly, substitution of (5.3.10) into Laplace's equation (5.3.4) yields

$$f''(y) - k^2 f(y) = 0. \tag{5.3.11}$$

Turning now to the two free-surface conditions (5.3.5) and (5.3.6), as well as the condition of infinite depth, we have the following necessary constraints:

$$f'(0) = \omega A, \tag{5.3.12}$$

$$\omega f(0) = gA, \tag{5.3.13}$$

$$f(-\infty) \to 0. \tag{5.3.14}$$

The solution of (5.3.11) yields

$$f(y) = B\mathrm{e}^{ky} + D\mathrm{e}^{-ky},$$

and so the infinite depth condition (5.3.7) yields $D \equiv 0$. We substitute now $f(y) = B\mathrm{e}^{ky}$ into the two remaining conditions and simplify, yielding the matrix equation:

$$\begin{pmatrix} \omega & -k \\ g & -\omega \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, in order for there to be nontrivial solutions, the determinant of the matrix must ber zero, and this yields

$$\omega^2 = gk,$$

or alternatively if we take the positive root,

$$\omega = \sqrt{gk}. \tag{5.3.15}$$

The above expression is known as the *dispersion relation* for water waves in deep water.

5.4 Generalisations of water waves

During the lectures and exercises, you will cover some of the following variants.

- Linear water waves in finite depth: part of Exercise 5.5.2 and done in Problem Class Week 6.

  In lectures and the exercise, you will confirm that the dispersion relation in finite depth is modified to:

  $$\omega = \omega_{\text{finite}} = \sqrt{gk\tanh(kh)}.$$

  In the limit $h \to \infty$, since $\tanh(kh) \to 1$, this approaches the deep-water dispersion relation of

  $$\omega_{\text{deep}} = \sqrt{gk},$$

  as shown in (5.3.15).

- Linear water waves in a current: part of Exercise 5.5.3 and done in Week 6 Thursday video.

  In these problems, you will modify the setup to include a horizontal current at infinity, with $\phi \sim Ux$ as $x \to -\infty$. The main difference in this case is that there are additional terms to be included in the linear kinematic and dynamic conditions. As shown in the exercise, the dispersion relation is modified to

  $$\omega = \omega_{\text{current}} = Uk \mp \sqrt{gk\tanh(kh)}.$$

  There are now two distinct wavespeeds. In the version where there is zero current, $U = 0$, the wavespeeds reduce to a single one (i.e. a single $|c|$, with both left- and right-travelling waves; remember that we usually drop the $\pm$).

- Linear water waves in two fluids of different densities: part of Exercise 5.5.4 and done in Week 7 Problem Class.

  Consider that the interface $y-\eta$ separates two fluids of different densities, say ρ_1, lying in the lower portion, $y < \eta$, and ρ_2 in the upper portion, with $y > \eta$. The main modification in the governing equations happens in Bernoulli's equation (5.1.4).

  In this case, the linear Bernoulli equation on the surface is

$$\rho_1 \left(\frac{\partial \phi_1}{\partial t} + g\eta \right) = \rho_2 \left(\frac{\partial \phi_2}{\partial t} + g\eta \right), \qquad \text{on } y = 0. \tag{5.4.1}$$

  The procedure is the same for this version, but the algebra is noticeably more involved. In this case, the dispersion relation is

$$\omega = \pm \sqrt{\left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \right) gk}.$$

- Linear water waves with surface tension: part of Exercise 5.5.5 and done in Week 7 Problem Class.

  For the case of deep water waves, this modifies the dispersion relation to

$$\omega = \sqrt{gk + \frac{Tk^3}{\rho}}.$$

5.5 Exercises

1. **Derivation of the kinematic condition.** In this question, you will derive the kinematic boundary condition on a free surface, which ensures that fluid particles on the free surface must remain on the free surface. Consider a vector that points to a location on a 2D free surface given by

$$\boldsymbol{r}(s,t) = (x, \eta(x,t)).$$

Introduce the curve $F(x,y,t) = y - \eta(x,t)$. For any particle on the surface, it must be the case that $F(x,y,t) \equiv 0$. By applying the material derivative show that the condition that

$$\frac{\mathrm{D}F}{\mathrm{D}t} = 0,$$

for particles on the free surface is equivalent to the kinematic condition

$$v = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} \qquad \text{on } y = \eta(x,t).$$

Solution. By the definition of the material derivative, we have that

$$\frac{\mathrm{D}(y-\eta)}{\mathrm{D}t} = \frac{\partial (y-\eta)}{\partial t} + (u,v) \cdot \nabla (y-\eta) = -\frac{\partial \eta}{\partial t} + (u,v) \cdot (-\eta_x, 1).$$

Equating the above to zero for the case of $y = \eta(x,t)$ yields the desired result.

2. **Stokes waves in finite depth.** Consider small 2D water waves on the free surface of a potential (incompressible and irrotational) fluid with velocity potential $\phi(x, y, t)$, with ϕ satisfying $\nabla^2\phi = 0$. Let the free surface be at $y = \eta(x, t)$ and assume the water is confined above a flat channel of depth $y = -h$. Show that on the free surface, $y = \eta(x, t)$, the kinematic equation is given by

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x},$$

while the dynamic (Bernouilli) equation is

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0.$$

Show that when the problem is linearised by ignoring quadratic terms, then instead the boundary conditions can be simplified to be

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t},$$
$$\frac{\partial \phi}{\partial t} + g\eta = 0,$$

and now applied on $y = 0$.

Show that travelling harmonic waves, with $\eta = A\cos(kx - \omega t)$ and $\phi = f(y)\sin(kx - \omega t)$, are possible and derive the associated dispersion relation,

$$\omega^2 = gk\tanh(kh).$$

In deriving the above, it will be convenient for you to use the hyperbolic cosine and sine functions:

$$\cosh z = \frac{\mathrm{e}^z + \mathrm{e}^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{2},$$

and note that the cosh function is even at $z = 0$ (and therefore has a zero derivative). The use of hyperbolic functions is used in the solution of $f(y)$ and is a common trick when solving boundary-value problems with exponentials.

Solution. Note that the finite-depth linear wave theory was derived as part of the problem class, so you can examine the video there for more details. Firstly, we look to linearise the two surface boundary conditions,

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x},$$
$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0,$$

which are imposed on $y = \eta(x)$. The quadratic terms can be ignored by assuming that the fluid is linearised about a small disturbance. Hence η is assumed to be small. Similarly, the velocity terms are expanded about $y = 0$. For example,

$$\left.\frac{\partial \phi}{\partial x}\right|_{y=\eta} = \left.\frac{\partial \phi}{\partial x}\right|_{y=0} + \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{y=0}\eta + \dots.$$

Thus we can approximate this quantity to leading order by

$$\left.\frac{\partial \phi}{\partial x}\right|_{y=\eta} \sim \left.\frac{\partial \phi}{\partial x}\right|_{y=0}.$$

Therefore, both kinematic and dynamic conditions are simplified in this manner, yielding

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t},$$
$$\frac{\partial \phi}{\partial t} + g\eta = 0,$$

now imposed on $y = 0$.

We substitute now the ansatzes

$$\eta = A\cos(kx - \omega t), \qquad \phi = f(y)\sin(kx - \omega t),$$

into the governing equations.

Written in terms of f, we must now solve the following system:

$$f'' + k^2 f = 0,$$
$$f'(-h) = 0,$$
$$f'(0) = (\omega)A,$$
$$-(\omega)f(0) + gA = 0.$$

Now instead of writing the solution for f in terms of the two exponentials $\mathrm{e}^{\pm ky}$, it is better to write the general solution of the second-order ODE as

$$f(y) = B\cosh[k(y - y_0)],$$

where y_0 is constant. We can then see that if $y_0 = -h$, then the boundary condition $f'(-h) = 0$ is satisfied.

The last two boundary conditions, evaluated on $y = 0$ yield

$$\begin{pmatrix} \omega & -k\sinh(kH) \\ g & -(\omega)\cosh(kh) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0. \tag{5.5.1}$$

For there to be non-trivial solutions, we thus need

$$(\omega)^2 = gk\tanh(kh),$$

and therefore there are two possible wave speeds, namely

$$c = \mp\left(\frac{g}{k}\tanh(kh)\right)^{1/2},$$

and both are simply negatives of one another; so represent left and rightwards propagating waves of the same speed.

3. **Linear waves in a current.** We now consider linear waves in a channel of depth h, as in the previous question. This time, however, assume that there is a uniform flow of speed U at infinity.

(a) Begin by writing down the boundary conditions that you expect at $y = -h$ and also far upstream, with $x \to -\infty$.

Therefore, this time, we shall consider the velocity as expressed as

$$u = U + \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y},$$

and again linearise the necessary equations.

Solution. We begin with the governing equations as shown in Theorem 5.1.2.

On the channel bottom, we expect that the vertical velocity is zero, hence

$$\frac{\partial \phi}{\partial y} = 0, \qquad y = -h,$$

while at upstream infinity, we expect the the velocity to be uniform and of speed U. Hence in terms of the potential,

$$\phi \sim Ux, \qquad x \to -\infty.$$

Indeed, this suggests that we should linearise about the uniform flow, with

$$\phi = Ux + \hat{\phi},$$

where $\hat{\phi}$ is the approximation, which we expect to be small (as is standard in linear problens, we often ignore the hat notation).

(b) Linearise the associate free-surface equations, and show that this time, they produce

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x}, \qquad \text{on } y = 0,$$

and

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + g\eta = 0, \qquad \text{on } y = 0.$$

When considering the second of the two equations above (Bernoilli's) return to the original Bernoulli equation:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + \frac{p}{\rho} + gy = F(t),$$

and choose your constant F more strategically.

Solution. The linearisation process for the kinematic condition remains largely the same, except we would produce:

$$\frac{\partial \eta}{\partial t} + \left[U + \frac{\partial \hat{\phi}}{\partial x}\right]\frac{\partial \eta}{\partial x} = \frac{\partial \hat{\phi}}{\partial y}.$$

Again quadratic terms are ignored, and we see that the requisite condition becomes (dropping the hat):

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial y}, \qquad y = \eta(x,t).$$

The linearisation of Bernoulli's equation is done similarly.

When considering the Bernoulli equation,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + \frac{p}{\rho} + gy = F(t),$$

it is prudent to take the choice of $F(t)$ so that the inertial term is removed. Specifically, if we take

$$F(t) = \frac{U^2}{2} + p_{\text{atm}}\rho,$$