

Q1.

$$(i) \quad -u''(x) - 6u'(x) = \lambda u(x), \quad x \in (-\infty, \infty),$$

Here, in terms of the notation used in the lecture,

$$\alpha(x) = 1, \quad \beta(x) = 6, \quad \gamma(x) = 0, \quad \delta(x) = 1.$$

Hence, using the formulae obtained therein,

$$p(x) = e^{6x}, \quad q(x) = 0, \quad r(x) = e^{6x},$$

and therefore the SL form of the original equation is

$$\boxed{-\left(e^{6x}u'(x)\right)' = \lambda e^{6x}u(x).} \quad (1)$$

$$(ii) \quad -x(1-x)u'' + 2xu'(x) = \lambda xu(x), \quad x \in [0, 1].$$

Here, in terms of the notation used in the lecture,

$$\alpha(x) = x(1-x), \quad \beta(x) = -2x, \quad \gamma(x) = 0, \quad \delta(x) = x.$$

Hence, using the formulae obtained therein,

$$p(x) = \exp \left\{ \int_0^x \left(-\frac{2y}{y(1-y)} \right) dy \right\} = \exp \left\{ -2 \int_0^x \frac{dy}{1-y} \right\}.$$

Making the change of variables $z = 1 - y$, so $dz = -dy$, we proceed to conclude

$$p(x) = \exp \left(-2 \int_{1-x}^1 \frac{dz}{z} \right) = \exp(2 \log(1-x)) = (1-x)^2.$$

Furthermore,

$$q(x) = 0, \quad r(x) = \frac{x(1-x)^2}{x(1-x)} = 1-x,$$

and therefore the SL form of the original equation is

$$\boxed{-\left((1-x)^2u'(x)\right)' = \lambda(1-x)u(x).} \quad (2)$$

$$(iii) \quad -x^2 u''(x) - xu'(x) = \lambda \frac{u(x)}{x}, \quad x \in (0, \infty).$$

Here, in terms of the notation used in the lecture,

$$\alpha(x) = x^2, \quad \beta(x) = x, \quad \gamma(x) = 0, \quad \delta(x) = \frac{1}{x}.$$

Hence, using the formulae obtained therein,

$$p(x) = \left(\int_1^x \frac{dy}{y} \right) = \exp(\log(x)) = x, \quad q(x) = 0, \quad r(x) = x \frac{1/x}{x^2} = \frac{1}{x^2},$$

and therefore the SL form of the original equation is

$$\boxed{-(xu'(x))' = \lambda \frac{u(x)}{x^2}.} \quad (3)$$

Q2.

$$(i) \quad -u''(x) = \lambda u(x), \quad x \in [0, \pi], \quad (4)$$

$$u(0) = u(\pi) = 0, \quad (5)$$

All solutions to (4) vanishing at zero (first condition in (5)) have the form $u(x) = \sin(\sqrt{\lambda}x)$. Using the second condition in (5), we obtain $u(\pi) = 0$, and hence $\sqrt{\lambda}\pi = n\pi$, $n = 1, 2, \dots$. Hence,

$$\boxed{u(x) = \sin(\sqrt{\lambda}x), \quad \lambda = n^2, \quad n \in \mathbb{N}.} \quad (6)$$

$$(ii) \quad -u''(x) = \lambda u(x), \quad x \in [a, b], \quad (7)$$

$$u'(a) = u'(b) = 0, \quad (8)$$

Note that we would typically write the general solution as

$$u(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x),$$

but because of the boundary conditions at $x = a, b$, it is instead better to either write

$$u(x) = A \cos(\sqrt{\lambda}(x + B)) \quad \text{or} \quad u(x) = A \sin(\sqrt{\lambda}(x + B)).$$

We do this because then B is easily selected. In this case, the derivative condition of $u'(a) = 0$ suggests that we choose a cosine expression and $B = -a$ thus

$$u(x) = \cos(\sqrt{\lambda}(x - a)),$$

(setting the constant pre-factor to 1). Note that you did not necessarily have to do this but it makes the algebra a bit easier (since now the eigenfunction is a single cosine).

Applying the boundary condition then requires $u'(x) = -\sqrt{\lambda} \sin(\sqrt{\lambda}(x-a))$. Now the second condition in (8) yields $\sqrt{\lambda} \sin(\sqrt{\lambda}(b-a)) = 0$, and hence $\sqrt{\lambda}(b-a) = n\pi$, $n = 1, 2, \dots$, or $\lambda = 0$. Summarising, the sought eigenfunction-eigenvalue pairs are given by

$$\boxed{u(x) = \cos\left(n\pi \frac{x-a}{b-a}\right), \quad \lambda = \frac{n^2\pi^2}{(b-a)^2}, \quad n = 0, 1, 2, \dots} \quad (9)$$

$$(iii) \quad -u''(x) = \lambda u(x), \quad x \in \left[0, \frac{\pi}{2}\right], \quad (10)$$

$$u(0) = u'\left(\frac{\pi}{2}\right) = 0, \quad (11)$$

Using (10) and the first boundary condition in (11), we obtain $u(x) = \sin(\sqrt{\lambda}x)$, and hence $u'(x) = \sqrt{\lambda} \cos(\sqrt{\lambda}x)$. Now from the second condition in (8) we have

$$u'\left(\frac{\pi}{2}\right) \sqrt{\lambda} \sin\left(\sqrt{\lambda} \frac{\pi}{2}\right) = 0,$$

and hence $\sqrt{\lambda}(b-a) = n\pi$, $n = 1, 2, \dots$, or $\lambda = 0$. Therefore,

$$\boxed{u(x) = \sin((2n+1)x), \quad \lambda = (2n+1)^2, \quad n = 0, 1, 2, \dots} \quad (12)$$

$$(iv) \quad -(e^{-x}u'(x))' = \lambda e^{-x}u(x), \quad x \in \left[0, \frac{\pi}{2}\right], \quad (13)$$

$$u(0) = u(1) = 0, \quad (14)$$

Differentiating and dividing by $\exp(-x)$, the equation (13) is written in the form

$$-u'' + u' = \lambda u. \quad (15)$$

This is an equation with constant coefficients, which generically have solutions of the form $\exp(\alpha x)$. Substituting this as an ansatz into (15) gives $\alpha^2 - \alpha + \lambda = 0$, from which

$$\alpha_{\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

[Note: To make sense of the last expression, strictly speaking we need to specify the branch of the square root. For example, we can stipulate that $\arg(\sqrt{z}) \in (-\pi/2, \pi/2]$. However, the final answer will not depend on this choice, of course.]

The general solution to (13) is given by

$$u(x) = Ae^{\alpha_+x} + Be^{\alpha_-x} = e^{x/2} \left(A \exp\left(\sqrt{\frac{1}{4} - \lambda}x\right) + B \exp\left(-\sqrt{\frac{1}{4} - \lambda}x\right) \right)$$

Using the first condition in (14) implies $A + B = 0$, so

$$u(x) = Ae^{x/2} \left(\exp \left(\sqrt{\frac{1}{4} - \lambda} x \right) - \exp \left(\sqrt{\frac{1}{4} - \lambda} x \right) \right) \quad (16)$$

The second condition in (14) yields

$$\exp \left(\sqrt{\frac{1}{4} - \lambda} \right) - \exp \left(\sqrt{\frac{1}{4} - \lambda} \right) = 0,$$

or equivalently

$$\sin \left(\sqrt{\lambda - \frac{1}{4}} \right) = 0,$$

which implies

$$\lambda - \frac{1}{4} = n^2 \pi^2, \quad n = 1, 2, \dots \quad (17)$$

For the values λ given by (17), the expression (16) is of the form

$$u(x) = 2i\tilde{A}e^{x/2} \sin(n\pi x).$$

[The relation between \tilde{A} and A (in fact $\tilde{A} = A$ or $\tilde{A} = -A$) depends on which one of the two possibilities

$$\sqrt{\frac{1}{4} - \lambda} = i\sqrt{\lambda - \frac{1}{4}}, \quad \sqrt{\frac{1}{4} - \lambda} = -i\sqrt{\lambda - \frac{1}{4}}$$

is realised under our choice of the branch of the square root.

Summarising, the eigenfunction-eigenvalue pairs are given by

$$\boxed{u(x) = e^{x/2} \sin(n\pi x), \quad \lambda = \frac{1}{4} + n^2 \pi^2, \quad n \in 1, 2, \dots} \quad (18)$$

Q3.

$$-(pu'(x))' + qu(x) = \lambda ru(x), \quad x \in [a, b].$$

(i) Multiplying both sides by $\overline{u(x)}$, we write

$$-(pu')'\overline{u} + q|u|^2 = \lambda r|u|^2. \quad (19)$$

Let (λ, ϕ) be an eigenvalue-eigenfunction pair. Set $u = \phi$ in (19) and integrate over $[a, b]$:

$$-\int_a^b (p\phi')'\overline{\phi} + \int_a^b q|\phi|^2 = \lambda \int_a^b r|\phi|^2.$$

Integrating by parts, we obtain

$$-[p\phi'\overline{\phi}]_a^b + \int_a^b p|\phi'|^2 + \int_a^b q|\phi|^2 = \lambda \int_a^b r|\phi|^2,$$

and therefore

$$\lambda \int_a^b r|\phi|^2 = \int_a^b p|\phi'|^2 + \int_a^b q|\phi|^2 - [p\phi'\overline{\phi}]_a^b. \quad (20)$$

(ii) In the case (a), we have $\phi'(a) = \phi'(b) = 0$, and therefore $[p\phi'\overline{\phi}]_a^b = 0$. In the case (b), we have $\phi(a) = \phi(b) = 0$, and therefore $[p\phi'\overline{\phi}]_a^b = 0$ again.

Combining this with (20) yields

$$\lambda \int_a^b r|\phi|^2 = \int_a^b p|\phi'|^2 + \int_a^b q|\phi|^2.$$

Using the fact that

$$\int_a^b r|\phi|^2 > 0, \quad \int_a^b (p|\phi'|^2 + q|\phi|^2) \geq 0,$$

we obtain $\lambda \geq 0$.

Q4.

$$-x^2 u''(x) - xu'(x) = \lambda u(x), \quad x > 0. \quad (21)$$

(i) Substituting the ansatz $u(x) = x^\alpha$ into (21), we obtain

$$-x^2 \alpha(\alpha - 1)x^{\alpha-2} - x\alpha x^{\alpha-1} = \lambda x^\alpha \quad \forall x > 0,$$

and hence

$$\alpha(\alpha - 1) + \alpha + \lambda = 0,$$

or equivalently

$$\alpha^2 + \lambda = 0.$$

(ii) Using the notation from lectures, we note that for the equation (21) we have

$$\alpha(x) = x^2, \quad \beta(x) = x, \quad \gamma(x) = 0, \quad \delta(x) = 1$$

Hence, by the theorem about writing equations in the SL form, we have

$$p(x) = \exp\left(\int_1^x \frac{dy}{y}\right) = \exp(\log(x)) = x, \quad q(x) = 0, \quad r(x) = \frac{x}{x^2} = \frac{1}{x}.$$

Therefore, the SL form of (21) is

$$-(xu'(x))' = \frac{\lambda}{x}u(x), \quad x > 0.$$

(iii) Using Q3(ii), case (b), it follows from $u(1) = u(e) = 0$ that $\lambda \geq 0$. Suppose first that $\lambda > 0$. By the result of part (i), the general solution is

$$u(x) = c_+ x^{i\sqrt{\lambda}} + c_- x^{-i\sqrt{\lambda}} \equiv c_+ e^{i\sqrt{\lambda} \log(x)} + c_- e^{-i\sqrt{\lambda} \log(x)}. \quad (22)$$

Setting first $x = 1$, then $x = e$, we obtain

$$\begin{cases} c_+ + c_- = 0, \\ c_+ e^{i\sqrt{\lambda}} + c_- e^{-i\sqrt{\lambda}} = 0. \end{cases} \quad (23)$$

This system has a non-trivial solution pair (c_+, c_-) if and only if its determinant vanishes, which gives the condition $\sin(\sqrt{\lambda}) = 0$, and hence $\lambda = n^2\pi^2$, $n = 1, 2, \dots$. If $\lambda = 0$, then the general solution is given by $u(x) = c_+ + c_- \log(x)$, which does not satisfy the boundary conditions for any c_+, c_- . So, the final answer is

$$\boxed{\lambda = n^2\pi^2, \quad n = 1, 2, \dots} \quad (24)$$

(iv) The eigenfunctions corresponding to the eigenvalues (24) are found by combining the formula (22) for the general solution with the fact that for solutions of (23) we have $c_- = -c_+$:

$$u(x) = c_+ e^{i\pi n \log(x)} - c_+ e^{-i\pi n \log(x)} = c_+ \sin(n\pi \log(x)).$$

So the (non-normalised) eigenfunctions are given by

$$\boxed{\phi_n(x) = \sin(n\pi \log(x)), \quad n = 1, 2, \dots}$$

$$(v) \quad \int_1^e \phi_n(x) \phi_m(x) \frac{dx}{x} = \int_1^e \sin(n\pi \log(x)) \sin(m\pi \log(x)) \frac{dx}{x}.$$

Changing the variable of integration: $z = \log(x)$, so $dz = dx/x$, we obtain

$$\int_0^1 \sin(n\pi z) \sin(m\pi z) dz = 0, \quad n \neq m,$$

as was shown in the lectures.

Q5.

$$-Eu^{(4)} = \lambda u, \quad -L < x < L, \quad (25)$$

$$u(-L) = u(L) = u''(-L) = u''(L) = 0. \quad (26)$$

(i) Setting $u(x) = \exp(i\omega x)$ in (25), we obtain $E\omega^4 + \lambda = 0$, so the four possible values for ω are

$$\omega_j = \left| \frac{\lambda}{E} \right|^{1/4} \alpha_j, \quad j = 1, 2, 3, 4,$$

where α_j , $j = 1, 2, 3, 4$, are the four roots of -1 of degree 4:

$$\alpha_{1,2} = \exp\left(\pm i \frac{\pi}{4}\right), \quad \alpha_{3,4} = \exp\left(\pm i \frac{3\pi}{4}\right),$$

In other words,

$$\omega_{1,2} = \frac{R}{\sqrt{2}}(1 \pm i), \quad \omega_{3,4} = \frac{R}{\sqrt{2}}(-1 \pm i), \quad R := \left| \frac{\lambda}{E} \right|^{1/4}.$$

(ii) The corresponding four exponential solutions of (25) are

$$\tilde{\phi}_{1,2} = \exp\left(\frac{R}{\sqrt{2}}(1 \pm i)x\right), \quad \tilde{\phi}_{3,4} = \exp\left(\frac{R}{\sqrt{2}}(-1 \pm i)x\right).$$

These can be combined to obtain four linearly independent real-valued solutions:

$$\begin{aligned} \phi_1(x) &= \frac{1}{2}(\tilde{\phi}_1(x) + \tilde{\phi}_2(x)) = \exp\left(\frac{R}{\sqrt{2}}x\right) \cos\left(\frac{R}{\sqrt{2}}x\right), \\ \phi_2(x) &= \frac{1}{2}(\tilde{\phi}_1(x) - \tilde{\phi}_2(x)) = \exp\left(\frac{R}{\sqrt{2}}x\right) \sin\left(\frac{R}{\sqrt{2}}x\right), \\ \phi_3(x) &= \frac{1}{2}(\tilde{\phi}_3(x) + \tilde{\phi}_4(x)) = \exp\left(-\frac{R}{\sqrt{2}}x\right) \cos\left(\frac{R}{\sqrt{2}}x\right), \\ \phi_4(x) &= \frac{1}{2}(\tilde{\phi}_3(x) - \tilde{\phi}_4(x)) = \exp\left(-\frac{R}{\sqrt{2}}x\right) \sin\left(\frac{R}{\sqrt{2}}x\right). \end{aligned}$$

(iii) Differentiating twice the functions ϕ_j , $j = 1, 2, 3, 4$, we obtain

$$u''(x) = -\frac{R^2}{2}(a\phi_1(x) + b\phi_2(x) + c\phi_3(x) + d\phi_4(x))$$

The boundary conditions (26) lead to:

$$u(L) = 0 :$$

$$\left(a \exp\left[\frac{RL}{\sqrt{2}}\right] + c \exp\left[-\frac{RL}{\sqrt{2}}\right]\right) \cos\left[\frac{RL}{\sqrt{2}}\right] + \left(b \exp\left[\frac{RL}{\sqrt{2}}\right] + d \exp\left[-\frac{RL}{\sqrt{2}}\right]\right) \sin\left[\frac{RL}{\sqrt{2}}\right] = 0, \quad (27)$$

$$u(-L) = 0 :$$

$$\left(a \exp\left[-\frac{RL}{\sqrt{2}}\right] + c \exp\left[\frac{RL}{\sqrt{2}}\right]\right) \cos\left[\frac{RL}{\sqrt{2}}\right] - \left(b \exp\left[-\frac{RL}{\sqrt{2}}\right] + d \exp\left[\frac{RL}{\sqrt{2}}\right]\right) \sin\left[\frac{RL}{\sqrt{2}}\right] = 0, \quad (28)$$

$$u''(L) = 0 :$$

$$\left(a \exp\left[\frac{RL}{\sqrt{2}}\right] - c \exp\left[-\frac{RL}{\sqrt{2}}\right]\right) \cos\left[\frac{RL}{\sqrt{2}}\right] + \left(b \exp\left[\frac{RL}{\sqrt{2}}\right] + d \exp\left[-\frac{RL}{\sqrt{2}}\right]\right) \sin\left[\frac{RL}{\sqrt{2}}\right] = 0, \quad (29)$$

$$u(-L) = 0 :$$

$$\left(a \exp\left[-\frac{RL}{\sqrt{2}}\right] - c \exp\left[\frac{RL}{\sqrt{2}}\right]\right) \cos\left[\frac{RL}{\sqrt{2}}\right] - \left(b \exp\left[-\frac{RL}{\sqrt{2}}\right] - d \exp\left[\frac{RL}{\sqrt{2}}\right]\right) \sin\left[\frac{RL}{\sqrt{2}}\right] = 0. \quad (30)$$

Together (27)–(30) form a system of four linear equations for a, b, c, d . The dependency on λ is implicit, since $r = |\lambda/E|^{1/4}$.