MA30044/MA40044/MA50181, Mathematical Methods 1, 2021
Problem Sheet 4: Introduction to the Fourier Transform

Feedback hand-in: Wednesday 3 Nov 2021 4pm

You should attempt questions 1,2,3,4,5.

1 The Fourier Transform $\mathcal{F}[u]$ of a piecewise continuous function u is defined by

$$\mathcal{F}[u](\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x} u(x)dx, \quad \omega \in \mathbb{R},$$

provided the integral on the right-hand side exists.

Determine $\mathcal{F}[u]$ for the following functions:

(i)

$$u(x) = \begin{cases} e^{-x}, & x \geq 0, \\ e^{2x}, & x < 0. \end{cases}$$

(ii)

$$u(x) = \begin{cases} 1, & 3 < x < 4, \\ 2, & 4 < x < 5, \\ 1/2, & x = 3, \\ 3/2, & x = 4, \\ 1, & x = 5, \\ 0 & \text{otherwise.} \end{cases}$$

(iii)

$$u(x) = \begin{cases} 1 - x/2, & 0 \leq x < 2, \\ 1 + x/2, & -2 < x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

(iv)

$$u(x) = \begin{cases} \mathrm{e}^{-x}\cos(x), & x > 0, \\ 1/2, & x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

2. (i) Show that if $u(x) = e^{-\alpha|x|}$, $x \in \mathbb{R}$, where $\alpha > 0$, then

$$\mathcal{F}[u](\omega) = \frac{2\alpha}{\alpha^2 + \omega^2}, \qquad \omega \in \mathbb{R}. \tag{1}$$

(ii) Show that $\mathcal{F}[u]$ in (1) has the following properties:

(a) $\mathcal{F}[u](0) \to \infty$ as $\alpha \to 0$;

(b) $\mathcal{F}[u](\omega) \to 0$ as $\alpha \to 0$, if $\omega \neq 0$;

(c) $\displaystyle\int_{-\infty}^{\infty} \mathcal{F}[u] = 2\pi.$

(iii) On the basis of the above properties, argue for what might be a "reasonable" formula for the Fourier Transform of the function $u(x) = 1$, $x \in \mathbb{R}$.

3. (i) Show that for

$$u(x) = \begin{cases} 3, & -2 < x < 2, \\ 3/2, & |x| = 2, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\mathcal{F}[u](\omega) = \frac{6\sin(2\omega)}{\omega}, \quad \omega \in \mathbb{R}.$$

(ii) Use the First Shift Theorem to determine the Fourier Transform of the function $v(x) = \exp(-2\mathrm{i}x)u(x)$, $x \in \mathbb{R}$. Verify this result by directly calculating the Fourier Transform of v.

(iii) Using the First Shift Theorem, calculate the inverse Fourier Transform of the function

$$u(\omega) = \frac{6}{10 + 2\omega + \omega^2}, \quad \omega \in \mathbb{R}.$$

4. (i) If $v(x) = \exp(-x)u(x)$, $x \in \mathbb{R}$, and $\mathcal{F}[v](\omega) = 1/(1 + \mathrm{i}\omega)$, $\omega \in \mathbb{R}$, use the Second Shift Theorem to find the Fourier Transform of the function $w(x) = u(x+4)\exp(-(x+4))$, $x \in \mathbb{R}$.

(ii) Using the Second Shift Theorem, find the inverse Fourier Transform of the function

$$u(\omega) = 6\mathrm{e}^{-4\mathrm{i}\omega}\frac{\sin(2\omega)}{\omega}, \quad \omega \in \mathbb{R}.$$

5. (i) Let $a > 0$ and define $u(x) = \exp(-ax^2)$, $x \in \mathbb{R}$. Denote by $F = F(\omega)$ the Fourier Transform $\mathcal{F}[u]$ of u. Show (without worrying about problems of interchanging integration with differentiation) that

$$\frac{dF}{d\omega} = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x} x \mathrm{e}^{-ax^2}\, dx, \qquad \omega \in \mathbb{R}.$$

(ii) Using integration by parts , show that

$$\frac{dF}{d\omega} = -\frac{\omega}{2a}F(\omega), \qquad \omega \in \mathbb{R}.$$

Hint: Consider the product of the functions $\exp(-\mathrm{i}\omega x)$ and $x\exp(-ax^2)$.

(iii) Using the result that $F(0) = \sqrt{\pi/a}$ show that

$$F(\omega) = \sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}, \quad \omega \in \mathbb{R}.$$

Remark: We will use this results when we use the Fourier Transform to solve the heat equation.

***6.** The function $\mathrm{sinc}(x) = \sin(x)/x$, $x \in \mathbb{R}$, plays a central role in the theory of the Fourier Transform.

(i) Show that

$$I := a\int_0^\infty \mathrm{sinc}$$

is independent of a.

(ii) Show that for $y > 0$

$$J(y) := \int_0^\infty \mathrm{e}^{-xy}\sin(x)\, dx = \frac{1}{1+y^2}.$$

(iii) By integrating J over the interval $[0, \infty)$ and interchanging the order of integration, show that

$$I = \frac{\pi}{2}.$$