MA30044/MA40044/MA50181, Mathematical Methods 1, 2021
Problem Sheet 6: Plancherel's Theorem

Feedback hand-in: Wednesday 17 Nov 2021 4pm

You should attempt Questions 1,2 and 5.

1. Using Plancherel's Theorem for the function

$$u(x) = \begin{cases} x, & x \in [-1,1], \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

show that

$$\int_{-\infty}^{\infty} \frac{\big(\sin(x) - x\cos(x)\big)^2}{x^4} dx = \frac{\pi}{3}. \tag{2}$$

Hint: The labelling of the variable of integration in (2) in unimportant – for example one can use ω instead of x.

2. We showed in the lectures that if a sufficiently regular $u = u(x,t)$ satisfies the heat equation

$$u_t(x,t) = u_{xx}(x,t), \qquad x \in \mathbb{R}, \quad t \geq 0,$$

with

$$u(x,0) = f(x), \qquad x \in \mathbb{R},$$

then it is given by

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \mathrm{e}^{-(x-y)^2/4t} f(y)\, dy, \qquad x \in \mathbb{R}, \quad t > 0.$$

(i) Suppose that $f(x) = \delta(x-a) + \delta(x+a)$, for $a > 0$, where δ is the δ-function. Calculate the function $u(\cdot,t)$ for $t > 0$ and sketch it for a series of values of increasing t.
(ii) Show that the integral

$$\int_{-\infty}^{\infty} u(x,t)\, dx$$

is constant in $t \geq 0$, irrespective of the choice of f.

3. The convection-diffusion equation

$$u_t(x,t) + cu_x(x,t) = \epsilon^2 u_{xx}(x,t), \quad x \in \mathbb{R}, \quad t \geq 0,$$

where $\epsilon \neq 0$, describes the flow of a slowly diffusing pollutant with density $u = u(x,t)$ in a cross-wind of speed $c > 0$. This equation can model spreading of smoke from chimneys.

(i) Assume that $u = u(x,t)$ has a Fourier Transform with respect to x and that $u(x,0) = f(x)$, $x \in \mathbb{R}$. Using Fourier Transform, show that

$$u(x,t) = \frac{1}{2\epsilon\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-(x-y-ct)^2/4\epsilon^2 t} f(y)\, dy, \qquad x \in \mathbb{R}, \quad t > 0.$$

(ii) Suppose that $f(x) = \delta(x-a)$, $a \in \mathbb{R}$, where δ is the delta-function. Using results from Question 2, calculate and sketch $u(\cdot,t)$ for increasing values of t.

4. On Problem Sheet 3, we looked at the equation for heating an object of length L in a microwave oven. Imagine now an infinitely long object with temperature $u = u(x,t)$ satisfying

$$u_t(x,t) = u_{xx}(x,t) + \gamma u(x,t), \qquad x \in \mathbb{R}, \quad t \geq 0,$$
$$u(x,0) = f(x), \qquad x \in \mathbb{R},$$

where $\gamma > 0$. Using Fourier Transform, write down an expression for $u(x,t)$ in terms of f, for $x \in \mathbb{R}$, $t > 0$.

5. (i) Suppose $u = u(x,t)$ satisfies the initial-value problem for a first-order wave equation

$$u_t(x,t) = cu_x(x,t), \qquad x \in \mathbb{R}, \quad t \geq 0,$$
$$u(x,0) = f(x), \qquad x \in \mathbb{R},$$

where $c \in \mathbb{R}$ is fixed. Using Fourier Transform and Shift Theorems, show that

$$u(x,t) = f(x+ct), \qquad x \in \mathbb{R}, \quad t \geq 0.$$

(ii) Small waves in a large ocean traveling at speed $c > 0$ satisfy the second-order wave equation

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \qquad x \in \mathbb{R}, \quad t \geq 0.$$

Suppose that at time $t = 0$ the displacement and velocity are given by

$$u(x,0) = f(x), \qquad u_t(x,0) = 0, \qquad x \in \mathbb{R}.$$

Using the Fourier Transforms with respect to x, show that

$$u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right), \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{3}$$

Note: The formula (3) represents the so-called d'Alembert's solution to the wave equation. We will return to it later in the unit.