MA30044/MA40044/MA50181, Mathematical Methods I, 2020

Problem Sheet 8: Inhomogeneous quasilinear first-order PDEs and method of characteristics

Solutions

Q1.

The characteristic equations are

$$\dot{x} = y, \quad \dot{y} = x, \quad \dot{u} = xy^2, \tag{1}$$

and

$$\Gamma = \{(s, 0),\ s \in \mathbb{R}\}.$$

As in Question 2 (i), (ii), Problem Sheet 7, we argue that

$$x(t) = s\cosh(t), \qquad y(t) = s\sinh(t),$$

and in particular $s = \sqrt{x^2 - y^2}$. Note that $y < x$, i.e. the solution only exists in the domain D shown in Fig. 1.

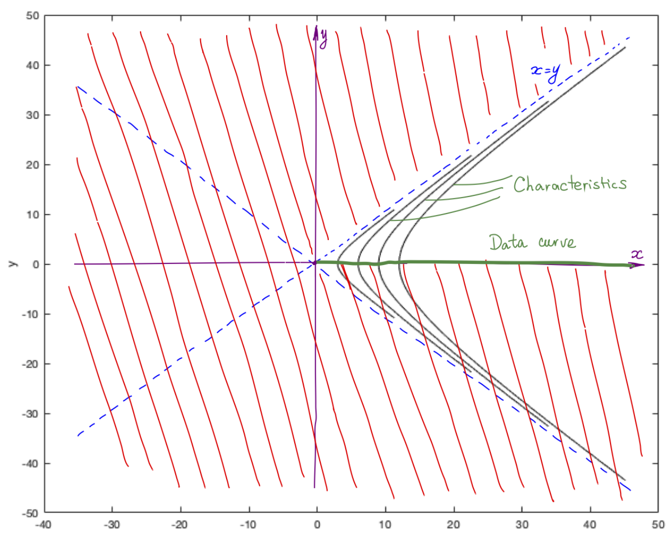

Figure 1: The characteristics and data curve in Question 1.

Furthermore, from (1), we obtain

$$\dot{v} = xy^2 = s^3\cosh(t)\sinh(t),$$

and hence

$$v(t) = \frac{s^3}{3}\sinh^3(t) + v(0).$$

Next, we use the initial condition for v in the data curve: for each s, we have

$$v(0) = u(s,0) = \frac{s^2}{2},$$

Summarising,

$$v(t) = \frac{s^3}{3}\sinh^3(t) + \frac{s^2}{2} = \frac{y^3}{3} + \frac{x^2 - y^2}{2}.$$

Verifying:

$$yu_x + xu_y = yx + xy^2 - xy = xy^2.$$

Q2.

(i) The characteristic equations are

$$\dot{x} = x^2, \quad \dot{y} = y^2, \quad \dot{v} = (x - y)v. \tag{2}$$

From the first equation we obtain

$$\frac{dx}{x^2} = dt,$$

and therefore

$$x(t) = \frac{1}{(x(0))^{-1} - t}. \tag{3}$$

Similarly, we obtain

$$y(t) = \frac{1}{(y(0))^{-1} - t}. \tag{4}$$

Suppose that the point $(x(0), y(0))$ lies on the data line, i.e. $x(0) = s$, $y(0) = 1$. Then the formulae (3), (4) yield

$$x(t) = \frac{s}{1 - st}, \qquad y(t) = \frac{1}{1 - t}, \tag{5}$$

see Fig. 2.

Combining this with the third equation in (2), we obtain

$$\dot{v} = (x - y)v = \left(\frac{s}{1 - st} - \frac{1}{1 - t}\right)v,$$

which we can integrate with respect to t :

$$\frac{dv}{v} = \frac{sdt}{1 - st} - \frac{dt}{1 - t},$$

$$\log\left(\frac{v(t)}{v(0)}\right) = -\log(1 - st) + \log(1 - t)$$

(Note that $1 - st > 0$ and $1 - t > 0$, as $(x, y) \in \Omega$.) Therefore,

$$v(t) = v(0)\frac{1 - t}{1 - st} = \frac{1 - t}{1 - st},$$

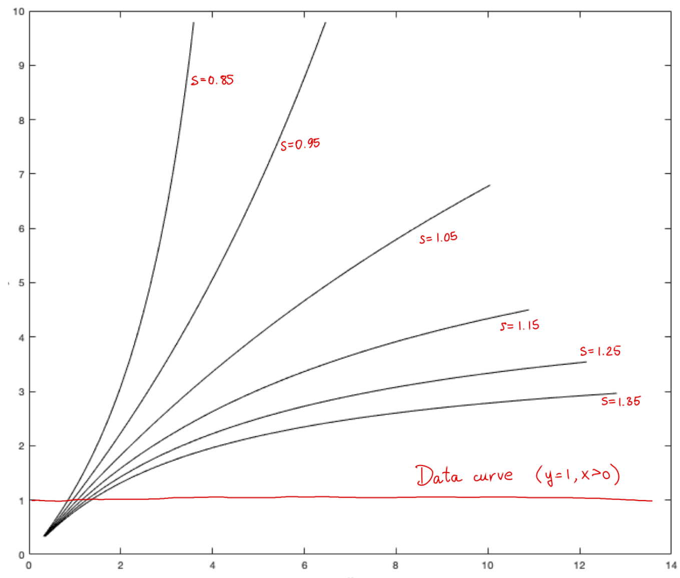

Figure 2: The characteristics and data curve in Question 2.

as $v(0) = 1$ from the initial condition.

Next, solving the second equation in (5) for t, we obtain

$$t = \frac{y-1}{y}$$

and so the first equation in (5) yields

$$s = \frac{x}{1+tx} = \frac{xy}{y+(y-1)x}.$$

Finally, $u(x(s,t), y(s,t)) = v(s,t)$, and therefore

$$u(x,y) = \frac{x}{sy} = \frac{x}{y}\frac{y+(y-1)x}{xy} = \frac{1}{y} + \frac{x}{y} - \frac{x}{y^2}.$$

Verifying: on the one hand,

$$u_x = \frac{1}{y} - \frac{1}{y^2} = \frac{y-1}{y^2} \qquad u(y) = -\frac{1}{y^2} - \frac{x^{\,2}}{y} + 2\frac{x}{y^3} = \frac{-y-xy+2x}{y^3},$$

and hence

$$x^2u_x + y^2u_y = \frac{x^2(y-1)}{y^2} + \frac{-y-xy+2x}{y} = \frac{x^2y - x^2 - y^2 - xy^2 + 2xy}{y^2}.$$

On the other hand,

$$(x-y)u = (x-y)\frac{y-xy-x}{y^2} = \frac{xy - y^2 + x^2y - xy^2 - x^2 + xy}{y^2} = \frac{x^2y - x^2 - y^2 - xy^2 + 2xy}{y^2},$$

as required.

(ii) The line $y = x = s$ is a characteristic, and the condition $u(s, s) = 1$ is consistent with the third equation in (2). Therefore, there are infinitely many solutions satisfying this condition.[1]

Q3

2021–22: Was not a huge fan of this solution method since it requires some atypical manipulations. A very similar question appears in Q4, and we have re-written the solution there. For the moment, we have added more detail into the below solution in blue.

The given PDE can be written in the form

$$\frac{xu_x}{2(x^2+y^2)} - \frac{yu_y}{2(x^2+y^2)} = \frac{1}{f(u)}, \tag{6}$$

and therefore the characteristics are solutions to

$$\dot{x} = \frac{x}{2(x^2+y^2)}, \qquad \dot{y} = \frac{-y}{2(x^2+y^2)} \qquad \dot{v} = \frac{1}{f(v)}. \tag{7}$$

The equation for v implies

$$f(v)dv = dt,$$

and hence

$$t = \int_0^v f = F(v),$$

which can also be written as $v = F^{-1}(t)$, since F is invertible.

Multiply the first equation in (7) by $2x$ and the second equation by $2y$ and then add them together. This gives

$$2x\dot{x} - 2y\dot{y} = \frac{x^2}{2(x^2+y^2)} + \frac{x^2}{2(x^2+y^2)} = 1. \tag{8}$$

However note the LHS can be simplified and the expression is then:

$$\frac{d}{dt}(x^2 - y^2) = 1. \tag{9}$$

At $t = 0$, we are on the data line $x(0) = y(0) = 1$, hence integrating (9), we obtain $x^2 - y^2 = t$. Finally,

$$u(x, y) = F^{-1}t = F^{-1}(x^2 - y^2).$$

Q4.

2021–22: Re-wrote this solution to be more clear/straightforward.

[1] 2021–22 note: this is not meant to be obvious without extra explanation. Here is a very short explanation. Take any curve C that passes through the line $x = y$ and is non-tangential. Suppose that it crosses at the point P. Now specify a new initial data problem with $\Gamma = \{(x, y, u) : (x, y) \in C, u(P) = 1\}$. In other words the initial data can be for any set values of u so long as it agrees with $u = 1$ at the point P. Had you solved this problem, you would derive a new surface $u(x, y)$ that nevertheless satisfies $u = 1$ along the curve $x = y$ (ask yourself why). Since Γ was chosen almost arbitrarily, there are infinitely many solutions to the problem.

The given PDE can be written in the form

$$\frac{xu_x}{2(y^2-x^2)} - \frac{yu_y}{2(y^2-x^2)} = -\frac{1}{u^4},$$

and therefore the characteristics are solutions to

$$\dot{x} = \frac{x}{2(y^2-x^2)}, \qquad \dot{y} = \frac{-y}{2(y^2-x^2)} \qquad \dot{u} = -\frac{1}{u^4}. \tag{10}$$

From (10), we divide the equation for y with the equation for x, giving

$$\frac{dy}{dx} = -\frac{y}{x} \Longrightarrow \int \frac{dy}{y} = -\int \frac{dx}{x}, \tag{11}$$

which solved gives

$$\log y = -\log x + D \Longrightarrow y = \frac{C}{x}. \tag{12}$$

Now we know that by the initial conditions, $x = s$ and $y = s$, so $C = s^2$ and we have the following form for the characteristics.

$$y = \frac{s^2}{x}. \tag{13}$$

It will be convenient for us to find the t-dependence relations of the characteristics. We then substitute the above into one of the characteristic equations. This gives

$$\frac{dx}{dt} = \frac{x}{2\left(\frac{s^4}{x^2} - x^2\right)}.$$

Separate both sides gives

$$\int \left(\frac{s^4}{x^3} - x\right) dx = \int \frac{dt}{2}.$$

Solve this, and simplify everything and back-substitute $s^2 = xy$ to get

$$-y^2 - x^2 = t + D = t - 2s. \tag{14}$$

Again, apply the condition that at $t = 0$, $x = y = s$ to get $D = -2s$. We have then developed the two equations for (x, y) or (s, t) via (13) and (14). It remains to deal with the equation for u. We have

$$\frac{du}{dt} = -\frac{1}{u^4} \Longrightarrow u = [-5(t + K)]^{1/5}.$$

Apply the condition that at $t = 0$, $u = 0$ to get $K = 0$. Conclude that

$$u(s,t) = (-5t)^{1/5}.$$

Finally, we solve for t exclusively from (13) and (14), giving $t = -(x-y)^2$. Thus finally

$$u(x,y) = \left(5(x-y)^2\right)^{1/5}.$$

Q5.

The characteristic equations are

$$\dot{x} = v, \qquad \dot{y} = 1, \qquad \dot{v} = -\frac{v}{2}. \tag{15}$$

Since $x(0) = s$, $y(0) = 0$, we have $y(t) = t$. Also, from the last equation in (15) we obtain

$$\frac{dv}{v} = -\frac{dt}{2},$$

and hence $v(s,t) = v(s,0)\mathrm{e}^{-t/2}$. From the initial condition $u(s,0) = \sin(s)$, we have $v(0) = \sin(s)$, and hence

$$v(s,t) = \sin(s)\mathrm{e}^{-t/2}.$$

Substituting this into the right-hand side of the first equation of (15), we obtain

$$\dot{x} = \sin(s)\mathrm{e}^{-t/2},$$

and hence, upon integrating,

$$x(s,t) = x(s,0) + 2\sin(s)\big(1 - \mathrm{e}^{-t/2}\big) = s + 2\sin(s)\big(1 - \mathrm{e}^{-t/2}\big).$$

In summary, we have

$$x(s,t) = x(s,0)+2\sin(s)\big(1-\mathrm{e}^{-t/2}\big) = s+2\sin(s)\big(1-\mathrm{e}^{-t/2}\big) \qquad y(s,t) = t, \qquad v(s,t) = \sin(s)\mathrm{e}^{-t/2}. \tag{16}$$

(ii) On the envelope, one has

$$\frac{x_s}{x_t} = \frac{y_s}{y_t},$$

which since $y_s = 0$, results in $x_s = 0$. Writing this condition explicitly on the basis of the first formula in (16) yields

$$\cos(s) = -\frac{1}{2\big(1 - \mathrm{e}^{-t/2}\big)}. \tag{17}$$

Viewed as an equation for s for a given t, the equation (17) has no real solutions if

$$-1 < -2\big(1 - \mathrm{e}^{-t/2}\big) < 1.$$

Therefore, since $y = t$ on a characteristic, the domain of influence is contained in

$$\Big\{(x,y) : y \in \big[-2\log(3/2), 2\log(2)\big]\Big\},$$

i.e. in the unshaded strip in Fig. 3.

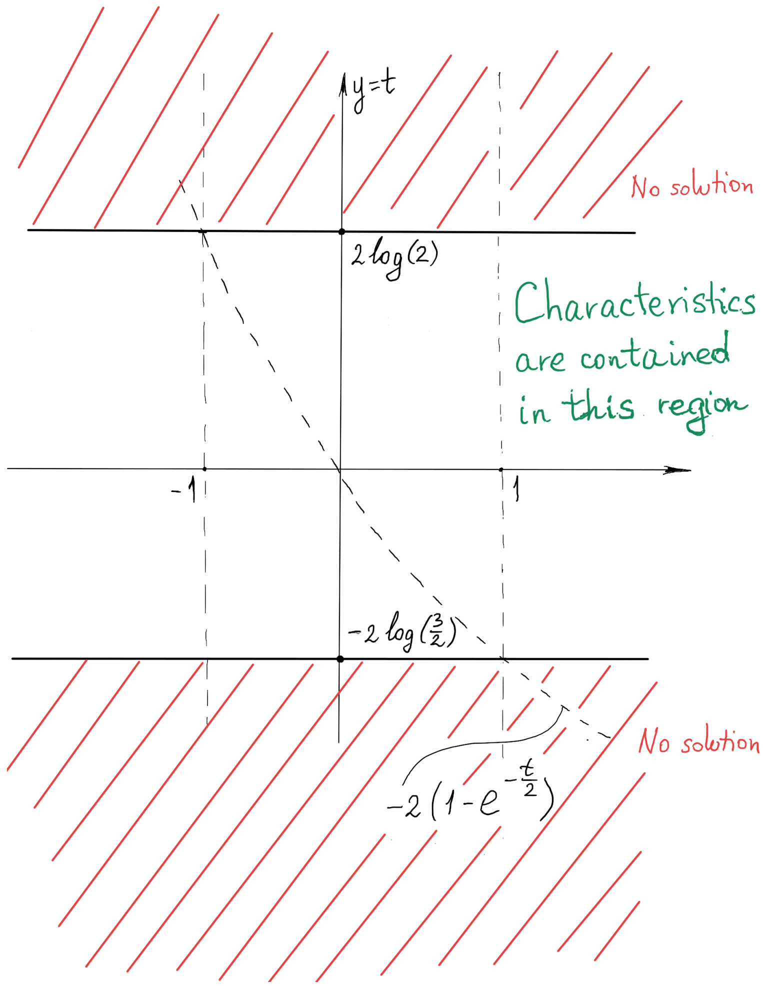

Figure 3: The domain of the solution in Question 5 is contained in the unshaded region, as the lines $y = 2\log(2)$ and $y = -2log(3/2)$ bound the envelope of characteristics.