MA30044/MA40044/MA50181, Mathematical Methods I, 2020

Problem Sheet 1: Introduction to Sturm-Liouville Equations

<small>SOLUTIONS</small>

Q1.

(i)
$$-u''(x) - 6u'(x) = \lambda u(x), \quad x \in (-\infty, \infty),$$

Here, in terms of the notation used in the lecture,

$$\alpha(x) = 1, \quad \beta(x) = 6, \quad \gamma(x) = 0, \quad \delta(x) = 1.$$

Hence, using the formulae obtained therein,

$$p(x) = e^{6x}, \quad q(x) = 0, \quad r(x) = e^{6x},$$

and therefore the SL form of the original equation is

$$\boxed{-\left(e^{6x}u'(x)\right)' = \lambda e^{6x}u(x).} \tag{1}$$

(ii)
$$-x(1-x)u'' + 2xu'(x) = \lambda x u(x), \quad x \in [0, 1].$$

Here, in terms of the notation used in the lecture,

$$\alpha(x) = x(1-x), \quad \beta(x) = -2x, \quad \gamma(x) = 0, \quad \delta(x) = x.$$

Hence, using the formulae obtained therein,

$$p(x) = \exp\left\{\int_0^x \left(-\frac{2y}{y(1-y)}\right) dy\right\} = \exp\left\{-2\int_0^x \frac{dy}{1-y}\right\}.$$

Making the change of variables $z = 1 - y$, so $dz = -dy$, we proceed to conclude

$$p(x) = \exp\left(-2\int_{1-x}^1 \frac{dz}{z}\right) = \exp\left(2\log(1-x)\right) = (1-x)^2.$$

Furthermore,

$$q(x) = 0, \quad r(x) = \frac{x(1-x)^2}{x(1-x)} = 1 - x,$$

and therefore the SL form of the original equation is

$$\boxed{-\left((1-x)^2 u'(x)\right)' = \lambda(1-x)u(x).} \tag{2}$$

(iii)
$$-x^2 u''(x) - xu'(x) = \lambda \frac{u(x)}{x}, \quad x \in (0, \infty).$$

Here, in terms of the notation used in the lecture,

$$\alpha(x) = x^2, \quad \beta(x) = x, \quad \gamma(x) = 0, \quad \delta(x) = \frac{1}{x}.$$

Hence, using the formulae obtained therein,

$$p(x) = \left(\int_1^x \frac{dy}{y} \right) = \exp(\log(x)) = x, \quad q(x) = 0, \quad r(x) = x \frac{1/x}{x^2} = \frac{1}{x^2},$$

and therefore the SL form of the original equation is

$$\boxed{-\left(xu'(x) \right)' = \lambda \frac{u(x)}{x^2}.} \tag{3}$$

Q2.

(i)
$$-u''(x) = \lambda u(x), \quad x \in [0, \pi], \tag{4}$$
$$u(0) = u(\pi) = 0, \tag{5}$$

All solutions to (4) vanishing at zero (first condition in (5)) have the form $u(x) = \sin(\sqrt{\lambda}x)$. Using the second condition in (5), we obtain $u(\pi) = 0$, and hence $\sqrt{\lambda}\pi = n\pi$, $n = 1, 2, \ldots$ Hence,

$$\boxed{u(x) = \sin(\sqrt{\lambda}x), \qquad \lambda = n^2, \quad n \in \mathbb{N}.} \tag{6}$$

(ii)
$$-u''(x) = \lambda u(x), \quad x \in [a, b], \tag{7}$$
$$u'(a) = u'(b) = 0, \tag{8}$$

Using (7) and the first boundary condition in (8), we obtain $u(x) = \cos\left(\sqrt{\lambda}(x - a)\right)$, and hence $u'(x) = -\sqrt{\lambda}\sin\left(\sqrt{\lambda}(x - a)\right)$. Now the second condition in (8) yields $\sqrt{\lambda}\sin\left(\sqrt{\lambda}(b - a)\right) = 0$, and hence $\sqrt{\lambda}(b - a) = n\pi$, $n = 1, 2, \ldots$, or $\lambda = 0$. Summarising, the sought eigenfunction-eigenvalue pairs are given by

$$\boxed{u(x) = \cos\left(n\pi \frac{x - a}{b - a} \right), \qquad \lambda = \frac{n^2 \pi^2}{(b - a)^2}, \quad n \in 0, 1, 2 \ldots} \tag{9}$$

(iii)
$$-u''(x) = \lambda u(x), \quad x \in \left[0, \frac{\pi}{2}\right], \tag{10}$$
$$u(0) = u'\left(\frac{\pi}{2}\right) = 0, \tag{11}$$

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Using (10) and the first boundary condition in (11), we obtain $u(x) = \sin(\sqrt{\lambda}x)$, and hence $u'(x) = -\sqrt{\lambda}\cos(\sqrt{\lambda}x)$. Now from the second condition in (8) we have

$$u'\left(\frac{\pi}{2}\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) = 0,$$

and hence $\sqrt{\lambda}(b-a) = n\pi$, $n = 1, 2, \ldots$, or $\lambda = 0$. Therefore,

$$\boxed{u(x) = \sin\big((2n+1)x\big), \qquad \lambda = (2n+1)^2, \quad n \in 0, 1, 2 \ldots} \tag{12}$$

(iv) $$-\big(e^{-x}u'(x)\big)' = \lambda e^{-x}u(x), \quad x \in \left[0, \frac{\pi}{2}\right], \tag{13}$$

$$u(0) = u(1) = 0, \tag{14}$$

Differentiating and dividing by $\exp(-x)$, the equation (13) is written in the form

$$-u'' + u' = \lambda u. \tag{15}$$

This is an equation with constant coefficients, which generically have solutions of the form $\exp(\alpha x)$. Substituting this as an ansatz into (15) gives $\alpha^2 - \alpha + \lambda = 0$, from which

$$\alpha_{\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

[Note: To make sense of the last expression, strictly speaking we need to specify the branch of the square root. For example, we can stipulate that $\arg(\sqrt{z}) \in (-\pi/2, \pi/2]$. However, the final answer will not depend on this choice, of course.]

The general solution to (13) is given by

$$u(x) = Ae^{\alpha_+ x} + Be^{\alpha_- x} = e^{x/2}\left(A\exp\left(\sqrt{\frac{1}{4} - \lambda}x\right) + B\exp\left(\sqrt{\frac{1}{4} - \lambda}x\right)\right)$$

Using the first condition in (14) implies $A + B = 0$, so

$$u(x) = Ae^{x/2}\left(\exp\left(\sqrt{\frac{1}{4} - \lambda}x\right) - \exp\left(\sqrt{\frac{1}{4} - \lambda}x\right)\right) \tag{16}$$

The second condition in (14) yields

$$\exp\left(\sqrt{\frac{1}{4} - \lambda}\right) - \exp\left(\sqrt{\frac{1}{4} - \lambda}\right) = 0,$$

or equivalently

$$\sin\left(\sqrt{\lambda - \frac{1}{4}}\right) = 0,$$

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which implies

$$\lambda - \frac{1}{4} = n^2\pi^2, \qquad n = 1, 2 \dots. \tag{17}$$

For the values λ given by (17), the expression (16) is of the form

$$u(x) = 2i\widetilde{A}e^{x/2}\sin(n\pi x).$$

[The relation between $\widetilde{A}$ and A (in fact $\widetilde{A} = A$ or $\widetilde{A} = -A$) depends on which one of the two possibilities

$$\sqrt{\frac{1}{4} - \lambda} = i\sqrt{\lambda - \frac{1}{4}}, \qquad \sqrt{\frac{1}{4} - \lambda} = -i\sqrt{\lambda - \frac{1}{4}}$$

is realised under our choice of the branch of the square root.
   Summarising, the eigenfunction-eigenvalue pairs are given by

$$\boxed{u(x) = e^{x/2}\sin(n\pi x), \qquad \lambda = \frac{1}{4} + n^2\pi^2, \quad n \in 1, 2 \dots} \tag{18}$$

Q3.

$$-\big(pu'(x)\big)' + qu(x) = \lambda r u(x), \quad x \in [a, b].$$

(i)   Multiplying both sides by $\overline{u(x)}$, we write

$$-(pu')'\overline{u} + q|u|^2 = \lambda r |u|^2. \tag{19}$$

Let (λ, ϕ) be an eigenvalue-eigenfunction pair. Set $u = \phi$ in (19) and integrate over $[a, b]$:

$$-\int_a^b (p\phi')'\overline{\phi} + \int_a^b q|\phi|^2 = \lambda \int_a^b r|\phi|^2.$$

Integrating by parts, we obtain

$$-\big[p\phi'\overline{\phi}\big]_a^b + \int_a^b p|\phi'|^2 + \int_a^b q|\phi|^2 = \lambda \int_a^b r|\phi|^2,$$

and therefore

$$\lambda \int_a^b r|\phi|^2 = \int_a^b p|\phi'|^2 + \int_a^b q|\phi|^2 - \big[p\phi'\overline{\phi}\big]_a^b. \tag{20}$$

(ii)   In the case (a), we have $\phi'(a) = \phi'(b) = 0$, and therefore $\big[p\phi'\overline{\phi}\big]_a^b = 0$. In the case (b), we have $\phi(a) = \phi(b) = 0$, and therefore $\big[p\phi'\overline{\phi}\big]_a^b = 0$ again.
   Combining this with (20) yields

$$\lambda \int_a^b r|\phi|^2 = \int_a^b p|\phi'|^2 + \int_a^b q|\phi|^2.$$

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Using the fact that

$$\int_a^b r|\phi|^2 > 0, \qquad \int_a^b \left(p|\phi'|^2 + q|\phi|^2\right) \geq 0,$$

we obtain $\lambda \geq 0$.

Q4.

$$-x^2 u''(x) - x u'(x) = \lambda u(x), \qquad x > 0. \tag{21}$$

(i)   Substituting the ansatz $u(x) = x^\alpha$ into (21), we obtain

$$-x^2 \alpha(\alpha - 1)x^{\alpha-2} - x\alpha x^{\alpha-1} = \lambda x^\alpha \qquad \forall x > 0,$$

and hence

$$\alpha(\alpha - 1) + \alpha + \lambda = 0,$$

or equivalently

$$\alpha^2 + \lambda = 0.$$

(ii)   Using the notation from lectures, we note that for the equation (21) we have

$$\alpha(x) = x^2, \qquad \beta(x) = x, \qquad \gamma(x) = 0, \qquad \delta(x) = 1$$

Hence, by the theorem about writing equations in the SL form, we have

$$p(x) = \exp\left(\int_1^x \frac{dy}{y}\right) = \exp\left(\log(x)\right) = x, \qquad q(x) = 0, \qquad r(x) = \frac{x}{x^2} = \frac{1}{x}.$$

Therefore, the SL form of (21) is

$$-\left(xu'(x)\right)' = \frac{\lambda}{x}u(x), \qquad x > 0.$$

(iii)   Using Q3(ii), case (b), it follows from $u(1) = u(e) = 0$ that $\lambda \geq 0$. Suppose first that $\lambda > 0$. By the result of part (i), the general solution is

$$u(x) = c_+ x^{i\sqrt{\lambda}} + c_- x^{-i\sqrt{\lambda}} \equiv c_+ e^{i\sqrt{\lambda}\log(x)} + c_- e^{-i\sqrt{\lambda}\log(x)}. \tag{22}$$

Setting first $x = 1$, then $x = e$, we obtain

$$\begin{cases} c_+ + c_- = 0, \\ c_+ e^{i\sqrt{\lambda}} + c_- e^{-i\sqrt{\lambda}} = 0. \end{cases} \tag{23}$$

This system has a non-trivial solution pair (c_+, c_-) if and only if its determinant vanishes, which gives the condition $\sin(\sqrt{\lambda}) = 0$, and hence $\lambda = n^2\pi^2$, $n = 1, 2, \ldots$ If $\lambda = 0$, then the general

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solution is given by $u(x) = c_+ + c_- \log(x)$, which does not satisfy the boundary conditions for any c_+, c_-. So, the final answer is

$$\boxed{\lambda = n^2\pi^2, \qquad n = 1, 2, \ldots} \tag{24}$$

(iv)   The eigenfunctions corresponding to the eigenvalues (24) are found by combining the formula (22) for the general solution with the fact that for solutions of (23) we have $c_- = -c_+$:

$$u(x) = c_+ e^{i\pi n \log(x)} - c_+ e^{-i\pi n \log(x)} = c_+ \sin\big(n\pi \log(x)\big).$$

So the (non-normalised) eigenfunctions are given by

$$\boxed{\phi_n(x) = \sin\big(n\pi \log(x)\big), \qquad n = 1, 2, \ldots}$$

(v) $$\int_1^e \phi_n(x)\phi_m(x)\frac{dx}{x} = \int_1^e \sin\big(n\pi \log(x)\big)\sin\big(m\pi \log(x)\big)\frac{dx}{x}.$$

Changing the variable of integration: $z = \log(x)$, so $dz = dx/x$, we obtain

$$\int_0^1 \sin(n\pi z)\sin(m\pi z)dz = 0, \qquad n \neq m,$$

as was shown in the lectures.

Q5.

$$-Eu^{(4)} = \lambda u, \quad -L < x < L, \tag{25}$$

$$u(-L) = u(L) = u''(-L) = u''(L) = 0. \tag{26}$$

(i)   Setting $u(x) = \exp(i\omega x)$ in (25), we obtain $E\omega^4 + \lambda = 0$, so the four possible values for ω are

$$\omega_j = \left|\frac{\lambda}{E}\right|^{1/4}\alpha_j, \qquad j = 1, 2, 3, 4,$$

where α_j, $j = 1, 2, 3, 4$, are the four roots of -1 of degree 4:

$$\alpha_{1,2} = \exp\left(\pm i\frac{\pi}{4}\right), \qquad \alpha_{3,4} = \exp\left(\pm i\frac{3\pi}{4}\right),$$

In other words,

$$\omega_{1,2} = \frac{R}{\sqrt{2}}(1 \pm i), \quad \omega_{3,4} = \frac{R}{\sqrt{2}}(-1 \pm i), \qquad R := \left|\frac{\lambda}{E}\right|^{1/4}.$$

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(ii)   The corresponding four exponential solutions of (25) are

$$\widetilde{\phi}_{1,2} = \exp\left(\frac{R}{\sqrt{2}}(1 \pm \mathrm{i})x\right), \qquad \widetilde{\phi}_{3,4} = \exp\left(\frac{R}{\sqrt{2}}(-1 \pm \mathrm{i})x\right).$$

These can be combined to obtain four linearly independent real-valued solutions:

$$\phi_1(x) = \frac{1}{2}\left(\widetilde{\phi}_1(x) + \widetilde{\phi}_2(x)\right) = \exp\left(\frac{R}{\sqrt{2}}x\right)\cos\left(\frac{R}{\sqrt{2}}x\right),$$

$$\phi_2(x) = \frac{1}{2}\left(\widetilde{\phi}_1(x) - \widetilde{\phi}_2(x)\right) = \exp\left(\frac{R}{\sqrt{2}}x\right)\sin\left(\frac{R}{\sqrt{2}}x\right),$$

$$\phi_3(x) = \frac{1}{2}\left(\widetilde{\phi}_3(x) + \widetilde{\phi}_4(x)\right) = \exp\left(-\frac{R}{\sqrt{2}}x\right)\cos\left(\frac{R}{\sqrt{2}}x\right),$$

$$\phi_4(x) = \frac{1}{2}\left(\widetilde{\phi}_3(x) - \widetilde{\phi}_4(x)\right) = \exp\left(-\frac{R}{\sqrt{2}}x\right)\sin\left(\frac{R}{\sqrt{2}}x\right).$$

(iii) Differentiating twice the functions ϕ_j, $j = 1, 2, 3, 4$, we obtain

$$u''(x) = -\frac{R^2}{2}\left(a\phi_1(x) + b\phi_2(x) + c\phi_3(x) + d\phi_4(x)\right)$$

The boundary conditions (26) lead to:

$u(L) = 0:$

$$\left(a\exp\left[\frac{RL}{\sqrt{2}}\right] + c\exp\left[-\frac{RL}{\sqrt{2}}\right]\right)\cos\left[\frac{RL}{\sqrt{2}}\right] + \left(b\exp\left[\frac{RL}{\sqrt{2}}\right] + d\exp\left[-\frac{RL}{\sqrt{2}}\right]\right)\sin\left[\frac{RL}{\sqrt{2}}\right] = 0, \quad (27)$$

$u(-L) = 0:$

$$\left(a\exp\left[-\frac{RL}{\sqrt{2}}\right] + c\exp\left[\frac{RL}{\sqrt{2}}\right]\right)\cos\left[\frac{RL}{\sqrt{2}}\right] - \left(b\exp\left[-\frac{RL}{\sqrt{2}}\right] + d\exp\left[\frac{RL}{\sqrt{2}}\right]\right)\sin\left[\frac{RL}{\sqrt{2}}\right] = 0, \quad (28)$$

$u''(L) = 0:$

$$\left(a\exp\left[\frac{RL}{\sqrt{2}}\right] - c\exp\left[-\frac{RL}{\sqrt{2}}\right]\right)\cos\left[\frac{RL}{\sqrt{2}}\right] + \left(b\exp\left[\frac{RL}{\sqrt{2}}\right] + d\exp\left[-\frac{RL}{\sqrt{2}}\right]\right)\sin\left[\frac{RL}{\sqrt{2}}\right] = 0, \quad (29)$$

$u(-L) = 0:$

$$\left(a\exp\left[-\frac{RL}{\sqrt{2}}\right] - c\exp\left[\frac{RL}{\sqrt{2}}\right]\right)\cos\left[\frac{RL}{\sqrt{2}}\right] - \left(b\exp\left[-\frac{RL}{\sqrt{2}}\right] - d\exp\left[\frac{RL}{\sqrt{2}}\right]\right)\sin\left[\frac{RL}{\sqrt{2}}\right] = 0. \quad (30)$$

Together (27)–(30) form a system of four linear equations for a, b, c, d. The dependency on λ is implicit, since $r = |\lambda/E|^{1/4}$.