# MA30044/MA40044/MA50181, Mathematical Methods 1, 2021
## Problem Sheet 10: Wave Equation

Feedback hand-in: Wednesday 15 Dec 2021 4pm

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*You should attempt Questions 1,2 and 4.*

**1. 1.** Let $u = u(x,t)$ satisfy the wave equation

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0.$$

Using the d'Alembert formula, find $u$ in the following three cases.

(i) $u(x,0) = \sin(x), \quad u_t(x,0) = 1.$

(ii)

$$u(x,0) = \begin{cases} 1, & x \in (-1,1), \\ 0 & \text{otherwise}, \end{cases} \qquad u_t(x,0) = 0.$$

(iii) $u(x,0) = 0, \quad u_t(x,0) = \delta(x)$, where $\delta$ is the delta-function.

*Note: Case (iii) corresponds to the situation of a musician instantaneously hitting a long string with a bow.*

**2.** Let the displacement $u(x,t)$ of an *infinite string* satisfy the wave equation

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0,$$

with the initial conditions $u(x,0) = 0$ and $u_t(x,0) = \psi(x)$, $x \in \mathbb{R}$. Suppose further that $\psi(x) = 0$ for $|x| \geq x_0$, where $x_0 > 0$ is fixed.

(i) Show that $u(x,t) = 0$ if $|x| \geq x_0 + ct$.

*Note: The region $|x| < x_0 + ct$ is the range of influence of the initial data.*

(ii) Show further that as $t \to \infty$ that the string comes to rest with a displacement

$$\frac{1}{2c} \int_{-x_0}^{x_0} \psi \tag{1}$$

so that $u(x,t)$ converges to the value (1) as $t \to \infty$ for all $x \in \mathbb{R}$.

**3.** Suppose the displacement $u = u(x,t)$ of a *semi-infinite string*, clamped at one end, satisfies the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad x \in [0,\infty), \quad t \geq 0,$$

subject to the *initial conditions*

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \qquad x \in \mathbb{R},$$

as well as the *boundary condition*

$$u(0,t) = 0, \quad t \geq 0.$$

For consistency, we set $\phi(0) = \psi(0) = 0$.

(i) First, set $\psi(x) = 0$, $x \in [0,\infty)$. By extending the problem to the whole line $\mathbb{R}$ in the variable $x$, with initial data

$$\Phi(x) = \begin{cases} \phi(x), & x \geq 0, \\ -\phi(-x), & x \leq 0, \end{cases}$$

show that the displacement of the semi-inifinite string is given by

$$u(x,t) = \begin{cases} \dfrac{1}{2}\big(\phi(x+ct) - \phi(ct-x)\big), & 0 \leq x \leq ct, \\ \dfrac{1}{2}\big(\phi(x+ct) + \phi(x-ct)\big), & x \geq ct. \end{cases}$$

(ii) Hence, for given $x_-, x_+$ such that $0 < x_- < x_+$, determine $u = u(x,t)$ if

$$\phi(x) = \begin{cases} 1, & x_- < x < x_+, \\ 0, & \text{otherwise.} \end{cases}$$

(iii) Show that if now $u_t(x,0) = \psi(x)$, $x \in \mathbb{R}$, is an arbitrary function, then for $0 < x \leq ct$ one has

$$u(x,t) = \frac{1}{2}\big(\phi(x+ct) - \phi(ct-x)\big) + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(z)\, dz.$$

What happens for $x,t$ such that $x > ct$?

**4.** Consider a *finite string*, fixed at $x = 0$ and $x = 1$, with displacement $u = u(x,t)$ satisfying

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad x \in [0,1], \quad t \geq 0, \\ u(0,t) &= u(1,t) = 0, \qquad t \geq 0. \end{aligned}$$

for initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = 0, \qquad x \in [0,1],$$

where the function $\phi$, such that $\phi(0) = \phi(1) = 0$, is assumed given.

(i) By separating variables and and applying the Sturm-Liouville theory, show that if

$$\phi(x) = \sum_{n=1}^{\infty} \beta_n \sin(n\pi x), \quad x \in [0,1], \qquad \beta_n \in \mathbb{R}, \quad n = 1, 2, \ldots,$$

then

$$u(x,t) = \sum_{n=1}^{\infty} \beta_n \cos(n\pi c t) \sin(n\pi x) \quad x \in [0,1], \ t \geq 0.$$

(ii) Hence show that

$$u(x,t) = \frac{1}{2}\big(\Phi(x - ct) + \Phi(x + ct)\big), \qquad x \in [0,1], \quad t \geq 0,$$

where $\Phi$ is the *odd periodic extension* of $\phi$ to the whole line $\mathbb{R}$.

(iii) Hence calculate $u(x,t)$, $x \in [0,1]$, $t \geq 0$, if

$$\phi(x) = \begin{cases} 1, & x \in (1/4, 3/4), \\ 0 & \text{otherwise.} \end{cases}$$