MA30044/MA40044/MA50181, Mathematical Methods 1, 2021
Problem Sheet 3: Applications of Sturm-Liouville Theory
Feedback hand-in: Wednesday 27 Oct 2021 4pm

You should attempt questions 1,2,3.

1. Consider the set of functions $\{\psi_n\}$ given by $\psi_n(x) = \cos((2n+1)x)$, for $n = 0, 1, 2, \ldots$ and $x \in [0, \pi]$. The weight function in this problem is $r(x) = 1$, $x \in [0, \pi]$.

(i) Show that the set $\{\psi_n\}$ is orthogonal and use it to obtain an orthonormal set $\{\phi_n\}$.

(ii) Expand the function

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \dfrac{\pi}{2}, \\ -1 & \text{for } \dfrac{\pi}{2} < x \le \pi, \end{cases}$$

in terms of ϕ_n, i.e. find the values α_n such that

$$f(x) \approx f_N(x) = \sum_{n=0}^{N} \alpha_n \phi_n(x), \qquad x \in [0, \pi],$$

where the symbol $\approx$ is understood in the sense of minimising the "error" $\|f - f_N\|_r$.

(iii) Assuming that the set $\{\phi_n\}$ is complete, use Parseval's identity to show that

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

2. Suppose that u satisfies the inhomogeneous SL problem

$$u''(x) + 2u'(x) + \lambda u(x) = f(x), \quad x \in [0, \pi], \quad u(0) = u(\pi) = 0.$$

(i) Put this equation into the Sturm-Liouville form

$$-Lu + \lambda r u = h,$$

where you should find the function h and the weight function r.

(ii) Find all *orthonormal* eigenfunction/eigenvalue pairs (λ_n, ϕ_n) of the homogeneous Sturm-Liouville system

$$Lu = \lambda r u.$$

You can assume that the boundary conditions can only be satisfied if $\lambda > 1$.

(iii) Use the Fredholm Alternative to show that the inhomogeneous system has no solution if

$$f(x) = e^{-x} \sin(2x), \quad x \in [0, \pi], \qquad \text{and} \qquad \lambda = 5.$$

(iv) Use the Fredholm Alternative to determine the solution if

$$f(x) = \sqrt{\frac{2}{\pi}} e^{-x} \sin(x), \quad x \in [0, \pi], \qquad \text{and} \qquad \lambda = 4.$$

Hint: Express h in terms of ϕ_n and the weight function r.

3. If some initially chilled food stuff (such as potato mash) is placed in a microwave oven, then the microwaves penetrate the food and heat it up. As they penetrate, the microwave energy is in turn attenuated by the food, exponentially with penetration depth into the food. We will consider the food to be in a dish, with $x = 0$ being the surface of the food and $x = L$ the base. If the food is in a container of depth L, then heat is lost by convection at the surface and we will assume that there is no heat loss at the base. The resulting equation describing the temperature $T(x,t)$ of the food is then

$$\begin{aligned} cT_t &= \kappa T_{xx} + \Lambda e^{-\beta x}, \\ T_x(0,t) &= BT(0,t), \quad T_x(L,t) = 0, \quad T(x,0) = 0. \end{aligned} \tag{1}$$

(i) Find the steady state temperature $T_S(x)$ of the food which satisfies the equation $(T_S)_t = 0$.

(ii) Let $\theta(x,t) = T(x,t) - T_S(x)$. Show that

$$c\theta_t = \kappa\theta_{xx}, \quad \theta_x(0) = B\theta(0,t), \quad \theta_x(L,t) = 0, \quad \theta(x,0) = -T_S(x). \tag{2}$$

Show further that (2) admits a separable solution of the form

$$\theta(x,t) = e^{-\alpha t}\phi(x) \quad \text{where} \quad \phi_{xx} = -\lambda^2\phi, \quad \phi_x(0) = B\phi(0), \quad \phi_x(L) = 0,$$

and give an expression for λ in terms of α, c, κ.

(iii) Find the eigensolutions (λ, ϕ) of the SL equation above, and hence show that λ must satisfy the transcendental equation

$$\lambda \sin(\lambda L) = B\cos(\lambda L).$$

Taking $L = B = 1$ show, by drawing a suitable graph, that this equation has an infinite number of solutions λ_n, $n = 0, 1, 2, \dots$ with corresponding eigenfunctions ϕ_n which are orthogonal with weight $r(x) = 1$.

(iv) Hence show that

$$T(x,t) = T_S(x) - \sum_{n=0}^{\infty} A_n e^{-\kappa\lambda_n^2 t/c}\phi_n(x)$$

where

$$A_n = \left(\int_0^L \phi_n^2\right)^{-1}\left(\int_0^L T_S\phi_n\right), \quad n = 0, 1, 2, \dots$$

4. The **Generating function for the Chebyshev polynomials**. Show that the generating function for the Chebyshev polynomials T_n from Problem Sheet 2 is given by

$$g(x,t) = \sum_{n=0}^{\infty} T_n(x)t^n = \frac{1 - tx}{1 - 2tx + t^2}. \tag{3}$$

For this proceed as follows:

(i) Show that the function $g(x,t)$ satisfes the equation

$$(1 - x^2)\frac{\partial^2 g}{\partial x^2} - x\frac{\partial g}{\partial x} + t\frac{\partial}{\partial t}\left(t\frac{\partial g}{\partial t}\right) = 0.$$

(ii) Use this and the series expansion (3) to show that the functions T_n satisfy the Chebyshev equation

$$-(1 - x^2)T_n''(x) + xT_n'(x) = \lambda_n T_n(x), \quad \lambda_n = n^2.$$