MA30044/MA40044/MA50181, Mathematical Methods I, 2020

Problem Sheet 2: Advanced Sturm-Liouville Equations

SOLUTIONS

Q1.

(i)
$$\phi_0 = 1: \quad \phi_0' = 0, \lambda_0\phi_0 = 0 \implies \lambda_0 = 0;$$
$$\phi_1 = x: \quad \big((1-x^2)\phi_1'(x)\big)' + \lambda_1\phi_1(x) = (1-x^2)' + \lambda_1 x = -2x + \lambda_1 x \implies \lambda_1 = 2;$$
$$\phi_2 = \frac{1}{2}(3x^2-1): \quad \big((1-x^2)\phi_2'(x)\big)' + \lambda_2\phi_2(x) = 3\big((1-x^2)x\big)' + \lambda_2\frac{3x^2-1}{2}$$
$$= 3(1-3x^2) + \lambda_2\frac{3x^2-1}{2} - 6\frac{3x^2-1}{2} + \lambda_2\frac{3x^2-1}{2} \implies \lambda_2 = 6.$$

(ii)
$$\int_{-1}^{1}\phi_0(x)\phi_1(x)dx = \int_{-1}^{1} xdx = \frac{x^2}{2}\Big|_{-1}^{1} = 0;$$
$$\int_{-1}^{1}\phi_0(x)\phi_2(x)dx = \int_{-1}^{1} \frac{3x^2-1}{2}dx = \frac{x^3-x}{2}\Big|_{-1}^{1} = 0;$$
$$\int_{-1}^{1}\phi_1(x)\phi_2(x)dx = \int_{-1}^{1} \frac{3x^2-1}{2}xdx = \int_{-1}^{1} \frac{3x^3-x}{2}xdx = \left[\frac{3}{8}x^4 - \frac{1}{4}x^2\right]_{-1}^{1} = 0.$$

Hence, the functions ϕ_j, $j = 1, 2, 3$ are orthogonal to each other, and so the system $\{\phi_0, \phi_1, \phi_2\}$ is orthogonal.

(iii)
$$\phi_3(x) = x^3 + ax \implies \phi_3'(x) = 3x^2 + a,$$

and so

$$\big((1-x^2)\phi_3'(x)\big)' = \big((1-x^2)(3x^2+a)\big)' = \big(-3x^4 + (3-a)x^2 + a\big)' = -12x^3 + 2(3-a)x,$$

and

$$\big((1-x^2)\phi_3'(x)\big)' + \lambda_3\phi_3(x) = -12x^3 + 2(3-a)x + \lambda_3(x^3+ax) = 0.$$

Comparing the coefficients on x^3, we obtain $\lambda_3 = 12$, and comparing the coefficients on x, we obtain $2(3-a) + 12a = 0$, from which $a = -3/5$. Summarising,

$$\boxed{\phi_3(x) = \frac{1}{5}(5x^3 - 3x), \qquad \lambda_3 = 12.}$$

$$\phi_4(x) = x^4 + bx^2 + d \implies \phi_4'(x) = 4x^3 + 2bx,$$

and so

$$\big((1-x^2)\phi_4'(x)\big)' = \big((1-x^2)(4x^3+2bx)\big)' = \big(-4x^5 + (4-2b)x^3 + 2bx\big)' = -20x^4 + 6(2-b)x^2 + 2b,$$

and

$$((1-x^2)\phi_4'(x))' + \lambda_4\phi_4(x) = -20x^4 + 6(2-b)x^2 + 2b + \lambda_4(x^4 + bx^2 + d) = 0. \tag{1}$$

Comparing the coefficients on x^4, we obtain $\lambda_4 = 20$, and (1) takes the form

$$6(2-b)x^2 + 2b + 20bx^2 + 20d = 0. \tag{2}$$

Comparing the coefficients on x^2 in (2), we obtain $6(2-b)+20b = 0$, from which $b = -6/7$, and finally comparing the coefficients on x^0 in (2), we obtain $2b + 20d = 0$, from which $d = 3/35$. Summarising, we have

$$\boxed{\phi_4(x) = \frac{1}{35}(35x^4 - 30x^2 + 3), \qquad \lambda_4 = 20.}$$

(iv) The only zero of $\phi_1(x)$ is $x = 0$.

The zeros of $\phi_2 = 3x^2 - 1$ are $x = \pm 1/\sqrt{3} = \pm 0.577350$, and the zeros of $\phi_3(x) = 5x^3 - 3x$ are at $x = 0$ and those x on which $5x^2 - 3$ vanishes, so $x = \pm\sqrt{3/5} = \pm 0.774597$.

Finally, to determine the zeros of ϕ_4 set $z = x^2$ in $35x^4 - 30x^2 + 3 = 0$. This results in the equation

$$z^2 - \frac{6}{7}z + \frac{3}{35} = 0,$$

so

$$z = \frac{3}{7} \pm \sqrt{\frac{9}{49} - \frac{3}{35}} = \frac{1}{7}\left(3 \pm \sqrt{\frac{24}{5}}\right).$$

This gives four zeros for ϕ_4 :

$$x = \pm\sqrt{\frac{1}{7}\left(3 \pm \sqrt{\frac{24}{5}}\right)} = \begin{cases} \pm 0.861136, \\ \pm 0.339981. \end{cases}$$

Q2.

$$-(1-x^2)u''(x) + xu'(x) = \lambda u(x), \quad x \in (-1,1). \tag{3}$$

(i) Here, in terms of the notation used in the lectures, we have

$$\alpha(x) = 1 - x^2, \quad \beta(x) = -x, \quad \gamma(x) = 0, \quad \delta(x) = 1.$$

Hence,

$$p(x) = \exp\left(-\int_0^x \frac{ydy}{1-y^2}\right) = \exp\left(\frac{1}{2}\int_0^x \left(-\frac{ydy}{1-y^2}\right)\right) = \exp\left(\frac{1}{2}\log(1-x^2)\right) = \sqrt{1-x^2},$$

$$q(x) = 0, \qquad r(x) = \frac{\sqrt{1-x^2}}{1-x^2} = \frac{1}{\sqrt{1-x^2}}.$$

Therefore, the SL form of the equation (3) is

$$-\left(\sqrt{1-x^2}u'(x)\right)' = \frac{\lambda}{\sqrt{1-x^2}}u(x).$$

(ii) Consider the function $u(x) = \cos\left(\sqrt{\lambda}\cos^{-1}(x)\right)$. Applying the chain rule and using the fact that $\left(\cos^{-1}(x)\right)' = -1/\sqrt{1-x^2}$, we obtain

$$u'(x) = \sqrt{\lambda}\frac{1}{\sqrt{1-x^2}}\sin\left(\sqrt{\lambda}\cos^{-1}(x)\right), \tag{4}$$

and hence

$$\sqrt{1-x^2}u'(x) = \sqrt{\lambda}\sin\bigl(\sqrt{\lambda}\cos^{-1}(x)\bigr).$$

Taking derivative once again, we have

$$-\bigl(\sqrt{1-x^2}u'(x)\bigr)' = \frac{\lambda}{\sqrt{1-x^2}}u(x), \qquad x \in (-1,1).$$

The calculation for $u(x) = \sin\bigl(\sqrt{\lambda}\sin^{-1}(x)\bigr)$ is similar.

The linear independence of

$$\cos\bigl(\sqrt{\lambda}\cos^{-1}(x)\bigr), \qquad \sin\bigl(\sqrt{\lambda}\sin^{-1}(x)\bigr)$$

can be shown, for example, by considering the system

$$\begin{cases} \alpha\cos\bigl(\sqrt{\lambda}\cos^{-1}(x_1)\bigr) + \beta\sin\bigl(\sqrt{\lambda}\sin^{-1}(x_1)\bigr) = 0, \\ \alpha\cos\bigl(\sqrt{\lambda}\cos^{-1}(x_2)\bigr) + \beta\sin\bigl(\sqrt{\lambda}\sin^{-1}(x_2)\bigr) = 0, \end{cases}$$

with $x_1, x_2 \in (-1,1)$. As there are pairs of points x_1, x_2 for which the determinant of this system does not vanish, one necessarily has $\alpha = \beta = 0$, from which the claim follows.

(iii) Part of the difficulty of this question is to sort out boundedness. Consider solutions to (3) of the form

$$u(x) = a\cos\bigl(\sqrt{\lambda}\cos^{-1}(x)\bigr) \tag{5}$$

with some constant $a \neq 0$. The derivative of (5), given by (4), is bounded near $x = 1$. Indeed, setting $y = \cos^{-1}(x)$, we have

$$\sqrt{\lambda}\frac{\sin(\sqrt{\lambda}y)}{\sqrt{1-\cos^2(y)}} = \sqrt{\lambda}\frac{\sin(\sqrt{\lambda}y)}{\sin(y)}, \tag{6}$$

which is a bounded function near $y = \cos^{-1}(1) = 0$ [can be verified by L'Hopital's rule or similar].

It remains to analyse the behaviour of (4) near $x = -1$. Since $x \to -1$ means $y \to \pi$, then this suggests we should change the variables according to $y = \pi - \cos^{-1}(x)$. Now the critical point is $y = 0$.

This results in

$$\sqrt{\lambda}\frac{\sin\bigl(\sqrt{\lambda}(\pi-y)\bigr)}{\sqrt{1-\cos^2(\pi-y)}} = \sqrt{\lambda}\,\frac{\sin\bigl(\sqrt{\lambda}(\pi-y)\bigr)}{\sin(y)}. \tag{7}$$

rather than the expression (6). The expression (7) is bounded near $y = \pi - \cos^{-1}(-1) = 0$ if and only if the numerator vanishes at $y = 0$, which yields the condition $\sqrt{\lambda}\pi = n\pi$, $n \in \mathbb{Z}$, or equivalently $\lambda = n^2$, $n = 0, 1, 2\ldots$

(iv) Recall that $T_n(x) = \cos\bigl(n\cos^{-1}(x)\bigr)$, $x \in [-1,1]$. Taking the inner product of T_n and T_m with weight $1/\sqrt{1-x^2}$, we have

$$I_{mn} = \int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \int_{-1}^{1}\cos\bigl(n\cos^{-1}(x)\bigr)\cos\bigl(m\cos^{-1}(x)\bigr)\frac{dx}{\sqrt{1-x^2}}.$$

Making the change of variables $y = \cos^{-1}(x)$, so $dy = -dx/\sqrt{1-x^2}$, yields

$$I_{mn} = \int_0^{\pi}\cos(ny)\cos(my)dy = 0, \qquad n \neq m,$$

as required.

(v)

$$T_0(x) = \cos\bigl(0\cdot\cos^{-1}(x)\bigr) = \cos(0) = 1 \implies T_0(x) = 1;$$

$$T_1(x) = \cos\bigl(\cos^{-1}(x)\bigr) = x \implies T_1(x) = x.$$

Next, using the notation $\theta = \cos^{-1}(x)$, we have

$$T_2(x) = \cos\big(2\cos^{-1}(x)\big) = \cos(2\theta) = 2\cos^2(\theta) - 1 = 2x^2 - 1 \implies T_2(x) = 2x^2 - 1.$$

Similarly, using the formulae $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$ and $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$, we obtain

$$\begin{aligned} T_3(x) &= \cos\big(3\cos^{-1}(x)\big) = \cos(3\theta) = 4x^3 - 3x, \\ T_4(x) &= \cos\big(4\cos^{-1}(x)\big) = \cos(4\theta) = 8x^4 - 8x^2 + 1. \end{aligned}$$

Q3.

(i) Here, in terms of the notation used in the lectures, we have

$$\alpha(x) = 1, \quad \beta(x) = -\frac{x}{2}, \quad \gamma(x) = 0, \quad \delta(x) = 1.$$

Hence,

$$p(x) = \exp\left(-\frac{1}{2}\int_0^x y\,dy\right) = \exp\left(-\frac{x^2}{4}\right), \qquad q(x) = 0, \qquad r(x) = p(x) = \exp\left(-\frac{x^2}{4}\right).$$

Therefore, the SL form of the equation (3) is

$$-\left(\exp\left(-\frac{x^2}{4}\right)u'(x)\right)' = \lambda \exp\left(-\frac{x^2}{4}\right)u(x).$$

(ii) $H_n(x) = x^n + p(x)$, $p \in \mathbb{P}_{n-1}$ (the space of polynomials of degree not exceeding $n-1$). Notice first that

$$\begin{aligned} xH_n'(x) &= nx^n + p_1(x), \qquad p_1 \in \mathbb{P}_{n-1}, \\ H_n''(x) &\in \mathbb{P}_{n-2}. \end{aligned}$$

Therefore,

$$H_n''(x) - \frac{1}{2}xH_n'(x) + \lambda_n H_n(x) = -\frac{1}{2}nx^n + \lambda_n x^n + \left(H_n''(x) - \frac{1}{2}p_1(x) + \lambda_n p(x)\right), \tag{8}$$

where

$$H_n''(x) - \frac{1}{2}p_1(x) + \lambda_n p(x) \in \mathbb{P}_{n-1}.$$

In order for the expression (8) to vanish, one must have

$$-\frac{1}{2}n + \lambda_n = 0,$$

hence $\lambda_n = n/2$.

Setting the values $n = 0, 1, 2, 3$, we obtain

$$\lambda_0 = 0, \qquad H_0(x) = 1,$$

$$\lambda_1 = \frac{1}{2}, \qquad H_1(x) = x + a \implies H_1''(x) - \frac{1}{2}xH_1'(x) + \frac{1}{2}H_1(x) = -\frac{1}{2}x + \frac{1}{2}x + \frac{1}{2}a$$

$$\implies a = 0 \implies H_1(x) = x,$$

$$\lambda_2 = 1, \qquad H_2(x) = x^2 + ax + b \implies H_2''(x) - \frac{1}{2}xH_2'(x) + H_2(x)$$

$$= 2 - \frac{1}{2}(2x^2 + ax) + x^2 + ax + b = 2 - \frac{1}{2}ax + ax + b = 0$$

$$\implies a = 0, b = -2 \implies H_1(x) = x^2 - 2,$$

$$\lambda_3 = \frac{3}{2}, \qquad H_3(x) = x^3 + ax^2 + bx + c \implies H_3''(x) - \frac{1}{2}xH_3'(x) + \frac{3}{2}H_3(x)$$

$$= 6x + 2a - \frac{1}{2}(3x^3 + 2ax^2 + bx) + \frac{3}{2}(x^3 + ax^2 + bx + c)$$

$$= \left(-a + \frac{3}{2}a\right)x^2 + \left(6 - \frac{b}{2} + \frac{3}{2}b\right)x + 2a + \frac{3}{2}c = 0$$

$$\implies a = c = 0, b = -6 \implies H_1(x) = x^3 - 6x,$$

(iii)

$$\langle H_0, H_1 \rangle_r = \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{4}\right) dx = 0, \tag{9}$$

as the integrand is an odd function, and the integration interval is symmetric with respect to zero.

$$\langle H_1, H_2 \rangle_r = \int_{-\infty}^{\infty} x(x^2 - 2) \exp\left(-\frac{x^2}{4}\right) dx = 0,$$

in the same way as (9).

Finally, using integration by parts, we obtain

$$\langle H_0, H_2 \rangle_r = \int_{-\infty}^{\infty} (x^2 - 2) \exp\left(-\frac{x^2}{4}\right) dx = I_1 + I_2.$$

Let us study both integrals separately. We have,

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{4}\right) dx \\ &= -2 \int_{-\infty}^{\infty} x \frac{d}{dx}\left\{\exp\left(-\frac{x^2}{4}\right)\right\} dx \\ &= -2 \left[xe^{-x^2/4} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-x^2/4}\, dx \right] = 2 \int_{-\infty}^{\infty} e^{-x^2/4}\, dx = -I_2, \end{aligned}$$

since the first term is equal to zero. Therefore $I_1 + I_2 = -I_2 + I_2$.

(iv) $H_1(x)$ has one zero at $x = 0$, $H_2(x)$ has zeros at $x = \pm\sqrt{2}$, $H_3(x)$ has zeros at $x = \pm\sqrt{6}$.

Q4.

Suppose the system $\{\phi_n\}$ is complete. If there was $f \neq 0$ such that

$$\langle f, \phi_n \rangle_r = 0 \quad \text{for all} \quad n = 0, 1, 2, \ldots,$$

then one would have

$$\min E_N = \|f\|_r^2 - \sum_{n=0}^{N} \langle f, \phi_n \rangle_r^2 = \|f\|_r^2 \qquad \forall N,$$

and hence $\min E_N \to \|f\|_r^2 > 0$ as $N \to \infty$, which contradicts the completeness of $\{\phi_n\}$.