MA30044/MA40044/MA50181, Mathematical Methods 1, 2021
Problem Sheet 10: Wave Equation

Feedback hand-in: Friday 18 Dec 2021 (ELECTRONICALLY)

You should attempt Questions 1, 2 and 3.

1. 1. Let $u = u(x,t)$ satisfy the wave equation

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0.$$

Using the d'Alembert formula, find u in the following three cases.

(i) $u(x,0) = \sin(x), \quad u_t(x,0) = 1.$

(ii)

$$u(x,0) = \begin{cases} 1, & x \in (-1,1), \\ 0 & \text{otherwise}, \end{cases} \qquad u_t(x,0) = 0.$$

(iii) $u(x,0) = 0, \quad u_t(x,0) = \delta(x)$, where δ is the delta-function.

Note: Case (iii) corresponds to the situation of a musician instantaneously hitting a long string with a bow.

2. Let the displacement $u(x,t)$ of an *infinite string* satisfy the wave equation

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0,$$

with the initial conditions $u(x,0) = 0$ and $u_t(x,0) = \psi(x)$, $x \in \mathbb{R}$. Suppose further that $\psi(x) = 0$ for $|x| \geq x_0$, where $x_0 > 0$ is fixed.

(i) Show that $u(x,t) = 0$ if $|x| \geq x_0 + ct$.

Note: The region $|x| < x_0 + ct$ is the range of influence of the initial data.

(ii) Show further that as $t \to \infty$ that the string comes to rest with a displacement

$$\frac{1}{2c} \int_{-x_0}^{x_0} \psi \tag{1}$$

so that $u(x,t)$ converges to the value (1) as $t \to \infty$ for all $x \in \mathbb{R}$.

3. Consider the wave equation $y_{tt} = c^2 y_{xx}$ for which

$$
\begin{aligned}
y(x,0) &= 0, \qquad -\infty < x < \infty \\
y_t(x,0) &= \begin{cases} 0 & x < -L, \\ \frac{vx}{L} & -L \le x \le L, \\ 0 & x > L. \end{cases}
\end{aligned}
$$

Construct the characteristic diagram in the (x,t)-plane for $t > 0$ and demonstrate the subdivision of the plane into six distinct regions. Determine the explicit form of the solution in each region.