MA30044/MA40044/MA50181, Mathematical Methods 1, 2021
Problem Sheet 8: Inhomogeneous quasilinear first-order PDEs and characteristics

Feedback hand-in: Wednesday 1 Dec 2021 4pm

Your should attempt Questions 1,2 and 5

1. Let $\Omega = \{(x, y) : x > 0, y > 0\}$. Suppose that $u = u(x, y)$ solves the first-order linear equation

$$yu_x(x, y) + xu_y(x, y) = xy^2, \qquad (x, y) \in \Omega. \tag{1}$$

If u is subject to the condition that

$$u(s, 0) = \frac{s^2}{2}, \quad s > 0,$$

derive the solution

$$u(x, y) = \frac{x^2 - y^2}{2} + \frac{y^3}{3}.$$

Verify that u satisfies (1).

2. Let $\Omega = \{(x; y) : x > 0, y > 0\}$. Suppose that $u = u(x, y)$ satisfies the first-order linear equation

$$x^2u_x(x, y) + y^2u_y(x, y) = (x - y)u(x, y), \qquad (x, y) \in \Omega.$$

(i) If u is subject to the condition

$$u(s, 1) = 1, \quad s > 0,$$

determine u. Verify your result by a direct calculation.

(ii) Now assume that $u(x, y) = 1$ on the straight half-line

$$y = x = s, \quad s > 0,$$

What can we say about u in this case?

3. Suppose that f is a positive continuous function. Consider its primitive F, i.e.

$$F(x) = \int_0^x f, \qquad x \in \mathbb{R}.$$

Show that if $u = u(x, y)$ solves the quasilinear PDE

$$f\big(u(x, y)\big)\big(xu_x(x, y) - yu_y(x, y)\big) = 2(x^2 + y^2), \qquad (x, y) \in \mathbb{R}^2,$$

and satisfies the condition

$$u(s,s) = 0, \quad s \in \mathbb{R},$$

then

$$u(x,y) = F^{-1}(x^2 - y^2).$$

***4.** Solve the quasilinear PDE

$$\big(u(x,y)\big)^4 \big(x u_x(x,y) - y u_y(x,y)\big) = 2(x^2 - y^2), \quad (x,y) \in \mathbb{R}^2,$$

under the conditionthat

$$u(s,s) = 0, \quad s \in \mathbb{R}.$$

5. (i) Find the characteristics $x = x(s,\cdot)$, $y = y(s,\cdot)$, $v = v(s,\cdot)$, $s \in \mathbb{R}$, of the equation

$$u_y(x,y) + u(x,y) u_x(x,y) + \frac{1}{2} u(x,y) = 0 \tag{2}$$

under the condition that $u(s,0) = \sin(s)$, $s \in \mathbb{R}$.

(ii) By considering the envelope of the characteristics, show that there is a single-valued solution to (2) in the region

$$\Omega = \big\{(x,y) \in \mathbb{R}^2 : -2\log(3/2) < y < 2\log(2)\big\}.$$