MA30044/MA40044/MA50181, Mathematical Methods 1, 2021
Problem Sheet 5: Convolutions, δ-function, and linear ODEs

Feedback hand-in: Wednesday 10 Nov 2021 4pm

You should attempt questions 1, 2, 3, 4(i, ii), 6.

1. Let

$$f(x) = \begin{cases} \dfrac{2x}{3}, & 0 \le x \le 3, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} 1, & -1 \le x \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $f * g$.

2. (i) Suppose that

$$f(x) = \begin{cases} 1, & -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$(f * f)(x) = \begin{cases} 2 + x, & -2 < x < 0, \\ 2 - x, & 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Compute the Fourier transform $\mathcal{F}[f * f]$ directly.

(iii) Show directly that $\mathcal{F}[f * f] = (\mathcal{F}[f])^2$.

3. Suppose that

$$\mathcal{F}[u](\omega) = \frac{1}{(1 + \mathrm{i}\omega)^2}, \qquad \omega \in \mathbb{R}.$$

Using Convolution Theorem, show that

$$u(x) = \begin{cases} x\mathrm{e}^{-x}, & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

4. Define the step function H by

$$H(x) = \begin{cases} 1, & x > 0, \\ 1/2, & x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Let $u(x) = u'(x) = 0$ for $x \le 0$, and suppose that u satisfies the differential equation

$$u''(x) + 5u'(x) + 6u(x) = H(x)\mathrm{e}^{-x}, \quad x > 0.$$

Using the convolution theorem and decomposition into partial fractions, determine u.

(ii) Similarly calculate $u(x)$ if it satisfies the differential equation

$$u''(x) + 7u'(x) + 12u(x) = H(x)\cos(x)\mathrm{e}^{-2x}, \qquad x \in \mathbb{R}.$$

*(iii) Finally, if

$$u(x) = u'(x) = v(x) = v'(x) = 0 \qquad x < 0,$$

using convolution theorem and/or partial fractions (and also the result given in Q3, find u and v if

$$v'' + 3v' + 2v = u, \qquad\qquad u''(x) + 4u'(x) + 3u(x) = H(x)\mathrm{e}^{-4x}, \quad x \in \mathbb{R}.$$

Note: This question describes an electrical system in which u is the output of a system with input $H(x)\mathrm{e}^{-x}$, $x \in \mathbb{R}$, which is then fed as input into another system that has output v.

***5.** Let $f(x) = H(x)\mathrm{e}^{-x}$, $x \in \mathbb{R}$ and $g(x) = \sin(x)/x$, $x \in \mathbb{R}$. Show that

$$(f * g)(x) = \frac{1}{2}\int_{-1}^{1} \frac{\mathrm{e}^{\mathrm{i}\omega x}}{1 + \mathrm{i}\omega}\, d\omega.$$

6. (i) Let f be a periodic function of period L. By considering the complex Fourier Series of f, show that

$$\mathcal{F}[f](\omega) = \sum_{n=-\infty}^{\infty} 2\pi c_n \delta(\omega - 2\pi n/L), \qquad\qquad c_n = \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{e}^{-2\pi \mathrm{i} n x/L} f(x)dx, \quad n \in \mathbb{Z}.$$

(ii) Hence, use Convolution Theorem to show that for $f(x) = \sin(x)$, $x \in \mathbb{R}$, and $g(x) = H(x)\mathrm{e}^{-x}$, $x \in \mathbb{R}$. one has

$$(f * g)(x) = \frac{\sin(x)}{2} - \frac{\cos(x)}{2}, \qquad x \in \mathbb{R}.$$

Hint: Evaluate the Inverse Fourier Transform of $\mathcal{F}[f]\mathcal{F}[g]$ directly, using the properties of the δ-function.

(iii) Hence or otherwise, find the bounded function u that solves the equation

$$u'(x) + u(x) = \sin(x), \quad x \in \mathbb{R}.$$

****7.** Let

$$\delta_\epsilon(x) = \frac{\epsilon}{\pi(\epsilon^2 + x^2)}, \quad \epsilon > 0,$$

be an approximation to the δ-function. Show that for all bounded continuous functions f, one has

$$\int_{-\infty}^{\infty} \delta_\epsilon(x - a)f(x)dx \to f(a), \qquad \text{as } \epsilon \to 0.$$

Hint: Consider a small interval $I = a - \alpha < x < a + \alpha$, $\alpha \ll 1$, and consider the integral above over I and its complement separately as $\epsilon \to 0$.