MA30044/MA40044/MA50181, Mathematical Methods I, 2021

Problem Sheet 10: Wave Equation

Solutions

Q1.

(i) Substituting $\phi(x) = \sin(x)$, $\psi(x) = 1$ into the d'Alembert's formula (see lecture notes), we obtain

$$u(x,t) = \frac{1}{2}\big(\sin(x+ct)+\sin(x-ct)\big)+\frac{1}{2c}\int_{x-ct}^{x+ct} dz = \sin(x)\cos(ct)+\frac{1}{2c}\big((x+ct)-(x-ct)\big) = \sin(x)\cos(ct)+t.$$

(ii) Substituting

$$\phi(x) = \begin{cases} 1, & -1 < x < 1, \\ 0 & \text{otherwise}, \end{cases} \qquad\qquad \psi(x) = 0,$$

into d'Alembert's formula, we obtain

$$u(x,t) = \frac{1}{2}\big(\phi(x+ct)+\phi(x-ct)\big) = \frac{1}{2}\left(\left\{\begin{array}{ll} 1, & -1-ct < x < 1-ct, \\ 0 & \text{otherwise} \end{array}\right\} + \left\{\begin{array}{ll} 1, & -1+ct < x < 1+ct, \\ 0 & \text{otherwise} \end{array}\right\}\right).$$

Hence, the solution is described by two "rectangular" functions of height 1/2 travelling in opposite directions, see Fig. 1.

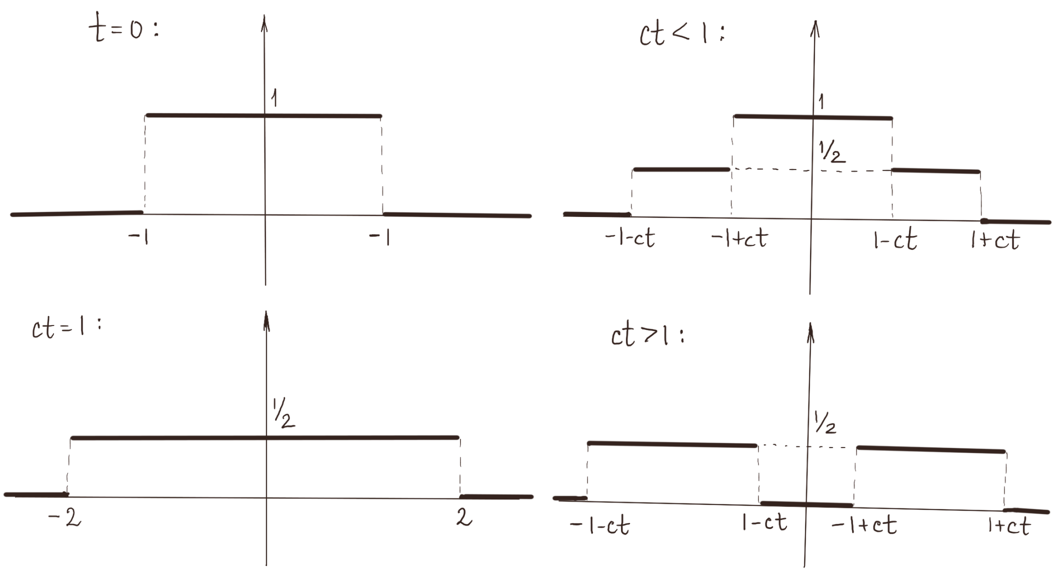

Figure 1: The dynamics of the solution to Q1(ii).

(iii) Substituting $\phi(x) = 0$, $\psi(x) = \delta(x)$ into d'Alembert's formula, we obtain

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \delta = \begin{cases} \frac{1}{2c}, & x - ct < 0 < x + ct, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2c}, & -ct < x < ct, \\ 0 & \text{otherwise.} \end{cases}$$

Q2. Using the values $\phi(x) = 0$ in the d'Alembert's formula, we obtain

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi. \tag{1}$$

Due to the fact that $\psi(x) = 0$ for $|x| \geq x_0$ the integral in (1) vanishes if the interval $(-x_0, x_0)$ does not overlap with the interval $(x - ct, x + ct)$. This is the case for $x - ct \geq x_0$ or $x + ct \leq -x_0$, which is equivalent to $x \geq x_0 + ct$ or $-x \geq x_0 + ct$, in other words $|x| \geq x_0 + ct$, see Fig. 2.

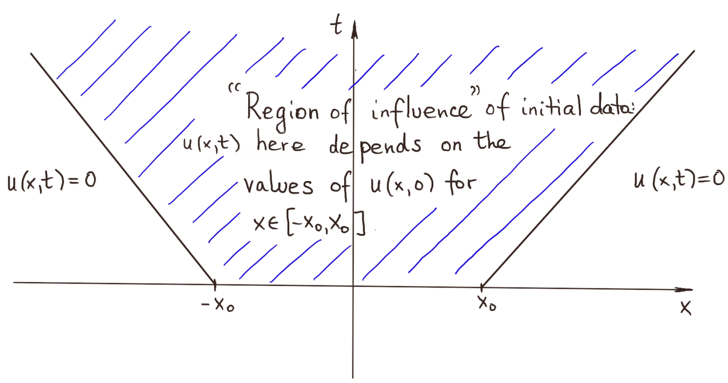

Figure 2: "Region of influence" of non-zero initial data: outside this "cone" the solution vanishes.

(ii) Consider a fixed $x \in \mathbb{R}$. Then, for $t \to \infty$, we have

$$u(x,t) \to \lim_{t\to\infty} \frac{1}{2c}\int_{x-ct}^{x+ct} \psi = \frac{1}{2c}\int_{-\infty}^{\infty} \psi = \frac{1}{2c}\int_{-x_0}^{x_0} \psi,$$

as required.

Q3. This question was completely treated in Lectures 31 and 32 in 2021–21!