MA30044/MA40044/MA50181, Mathematical Methods 1, 2021
Problem Sheet 1: Introduction to Sturm-Liouville Equations
Feedback hand-in: Wednesday 13 Oct 2021 4pm

You should all attempt questions 1,2,3,4 and MA50181 students should also attempt question 5.

1. Rewrite each of the following differential equations in the form $Lu+\lambda ru=0$ where L is a Sturm-Liouville operator of the form $Lu=(pu')'+qu$:[1]

$$\text{(i)}\quad -u''(x)-6u'(x)=\lambda u(x),\quad x\in(-\infty,\infty),$$

$$\text{(ii)}\quad -x(1-x)u''(x)+2xu'(x)=\lambda xu(x),\quad x\in[0,1],$$

$$\text{(iii)}\quad -x^2u''(x)-xu'(x)=\frac{\lambda}{x}u(x),\quad x\in(0,\infty).$$

2. Find all eigenvalues λ and corresponding eigenfunctions ϕ for the following Sturm-Liouville systems: [8 points]

$$\text{(i)}\quad -u''(x)=\lambda u(x),\quad x\in[0,\pi],\qquad u(0)=u(\pi)=0,$$

$$\text{(ii)}\quad -u''(x)=\lambda u(x),\quad x\in[a,b],\qquad u'(a)=u'(b)=0,\qquad a,b\in\mathbb{R},$$

$$\text{(iii)}\quad -u''(x)=\lambda u(x),\quad x\in[0,\pi/2],\qquad u(0)=u'(\pi/2)=0,$$

$$\text{(iv)}\quad -\left(e^{-x}u'(x)\right)'=\lambda e^{-x}u(x),\quad x\in[0,1],\qquad u(0)=u(1)=0.$$

3. Let (λ,ϕ) be an eigenvalue-eigenfunction pair for the Sturm-Liouville equation[2]

$$\left(p(x)u'(x)\right)'+q(x)u(x)+\lambda ru(x)=0,\quad x\in[a,\,b],\tag{1}$$

with a weight function $r>0$.

(i) Set $u=\phi$. By multiplying (1) by $\phi(x)$ and integrating over $[a,b]$ by parts, show that

$$\lambda\int_a^b r\phi^2\mathrm{d}x=\int_a^b\left(p(\phi')^2-q|\phi|^2\right)-\left[p\phi'\phi\right]_a^b,$$

and hence establish the Rayleigh Quotient identity in the Big Theorem on SL equations in the lecture notes.

[1] 2021-22: to ensure consistency, we define a Sturm-Liouville problem of the form $(pu')'+qu+\lambda\sigma u=0$ (or here $\sigma=r$). Therefore keep all your coefficients positive. This is then consistent with the textbook by Haberman.

[2] 2021-22: this question was changed to be simpler and not require complex conjugation. It then matches the proof of the Rayleigh Quotient in §5.6 of Haberman. The signs were also corrected to be consistent.

(ii) If $p(x)>0$ and $q(x)\leq 0$ for all $x\in[a,\, b]$, and u satisfies the boundary conditions

$$\alpha u(a)+\beta u'(a)=\mu u(b)+\nu u'(b)=0,$$

deduce that $\lambda\geq 0$ in each of the two cases (a) $\alpha=\mu=0$ and (b) $\beta=\nu=0$.

NOTE: The positivity of the eigenvalues of a Sturm-Liouville problem is an important result. It is related to stability of associated partial differential equations and has implications for understanding the behaviour of the physics setup modelled by the Sturm-Liouville problem.

4. (i) The Cauchy-Euler equation

$$-x^2u''(x)-xu'(x)=\lambda u(x), \quad x>0 \tag{2}$$

arises in the study of partial differential equation posed in cylindrical domains (such as a drum or an organ pipe). Show that (2) has solutions of the form $u=x^\alpha$ provided that $\alpha^2+\lambda=0$.

(ii) Show that (2) can be put into the Sturm-Liouville form

$$-\big(xu'(x)\big)'=\frac{\lambda}{x}u(x), \qquad x>0. \tag{3}$$

(iii) We pose this equation in the domain $I=[1,e]$ where $e=2.71828..$ is the usual basis for natural logarithms, and set

$$u(1)=u(e)=0.$$

Show that (3) has no zero or negative eigenvalues, and that the positive eigenvalues are given by $\lambda_n=n^2\pi^2, n=1,2,3....$

(iv) Find the corresponding eigenfunctions ϕ_n, expressing them as real-valued functions.

(v) Verify directly the orthogonality result

$$\int_1^e \frac{\phi_n(x)\phi_m(x)}{x}dx=0, \quad n\neq m.$$

****5.** The equation for a deflecting beam of length $2L$, with clamped ends, is given by

$$-Eu^{(4)}(x)=\lambda u(x), \qquad -L<x<L, \tag{4}$$

with boundary conditions

$$u(-L)=u(L)=u''(-L)=u''(L)=0. \tag{5}$$

where $\lambda>0$ is the applied load and $E>0$ is a measure of the stiffness of the beam.

(i) Find the possible values of ω for which $u(x)=\exp(\omega x)$ satisfies the differential equation (4).

(ii) Hence, find four possible real solutions ϕ_j, $j=1,2,3,4$, of (4).

*(iii) By considering a linear combination

$$u=a\phi_1+b\phi_2+c\phi_3+d\phi_4$$

of these real solutions, construct equations which must be satisfied by a,b,c,d,λ for u to satisfy the boundary conditions (5). You do not have to attempt to solve these equations.