MA30044/MA40044/MA50181, Mathematical Methods 1, 2021
Problem Sheet 7: Homogeneous quasilinear first-order PDEs

Feedback hand-in: Wednesday 24 Nov 2021 4pm

Your should attempt Questions 1,3 and 4.

1. (i) Find, and sketch, families of contour lines for the solution of the PDE

$$u_x + x^2 u_y = 0$$

for a function $u = u(x, y)$.

(ii) Hence, determine u given the initial data

$$u(s, 0) = \cos(s), \quad s \in \mathbb{R}, \tag{1}$$

verifying by a direct calculation that your answer is correct.

(iii) Similarly, suppose instead that u satisfies the data

$$u(s, s^3/3) = g(s), \quad s \in \mathbb{R}.$$

Show that this admits a solution if and only if g is a constant fucntion, and that in this case the solution is not defined everywhere on $\mathbb{R}^2$. 6

Hint: First determine the relevant data curves. For this, note for example that the statement in (1) *is equivalent to providing initial data on the line* $\Gamma = \{(x, y) : x = s, y = 0, s \in \mathbb{R}\}$, *i.e. the three-dimensional data curve is*

$$\tilde{\Gamma} = \Big\{(x, y, u) : x = s, y = 0, u = u_\Gamma(s) = \cos(s), \quad s \in \mathbb{R}\Big\}.$$

2. (i) Find, and sketch, families of contour lines for the PDE

$$y u_x + x u_y = 0$$

for a function $u = u(x, y)$.

(ii) Let $u(s, 0) = g(s)$ for $s > 0$. Show that

$$u(x, y) = g(\sqrt{x^2 - y^2}), \qquad x > |y|.$$

(iii) Determine u if it has the data $u(1, s) = g(s), s > 0$, discussing the range of validity of your solution. 6

3. (i) Suppose that $u = u(x, y)$ is a solution to the PDE

$$2xyu_x + (y^2 - x^2 + 1)u_y = 0.$$

Show that the related contour lines satisfy the equation

$$(x - a)^2 + y^2 = a^2 - 1,$$

where $a > 1$ is constant on each contour line.

Hint: Differentiate the function $v(t) = (x(t) - a)^2 + y(t)^2 - a^2 + 1$, where $(x(t), y(t))$ are on a contour line, and deduce that $v(t) = 0$ for all t.

Plot the set of contours.

(ii) Using your calculation of the contours in (i), and finding a or otherwise, find the solution $u(x, y)$ of the PDE in the region

$$\{(x, y) : x > 1, y \in \mathbb{R}\},$$

given that

$$u(s, 0) = 1 - s, \quad s > 1.$$

4. (i) Find the contour lines for the PDE

$$xu_x - yu_y + zxy^2u_z = 0 \tag{2}$$

for a function $u = u(x, y, z)$.

Note: Here the approach we took for equations satisfied by functions of two variables x, y needs to be extended to the three-dimensional setup. As a result, the contour lines are curves in the three-dimensional Euclidean space, and the graph of the function u is a (three-dimensional) "hypersurface" in the four-dimensional space of the variables x, y, z, u.

(ii) Write down the general formula for a solution, and verify that it satisfies the PDE (2).