MA50181

Asymptotic methods

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Contents

Chapter 1

Introduction

Aims: To introduce the basic methods of asymptotic analysis and apply these to interesting and relevant problems.

Asymptotic methods are used to find useful approximations to problems which are otherwise intractable, for example nonlinear partial differential equations. The usual application is to problems with a *"small (positive) parameter ϵ"* (or large parameter N) which are easier for $\epsilon \ll 1$ (or $N \gg 1$). Often we use those methods in parallel with numerical methods. There are basically two types of problems:

1. **Regular**: The solution changes in a small way as $\epsilon \to 0$.
2. **Singular**: The solution changes in a major way as $\epsilon \to 0$, i.e. it tends to ∞ and/or ceases to exist.

Example 1.1 (Stirling's formula)**.** If $N \gg 1$ (or $0 < \epsilon \ll 1$ where $\epsilon = 1/N$) then

$$N! \sim \sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N} \left(1 + \frac{1}{12N} + \dots\right) \tag{1.1}$$

Here "…" stands for additional terms which are of $\mathcal{O}(N^{-2})$. We will explain what exactly the symbol "$\sim$" means later, for the moment you can read it as "is approximately equal".

Example 1.2 (Quadratic equations I)**.**

$$x^2 - 5x + 6 + \epsilon = 0 \tag{1.2}$$

For $\epsilon = 0$ this has the two solutions $x = 2$, $x = 3$. For $0 < \epsilon \ll 1$ the two solutions are $x = 2 + \epsilon + \mathcal{O}(\epsilon^2)$, $x = 3 - \epsilon + \mathcal{O}(\epsilon^2)$. This is a **regular** problem.

Example 1.3 (Quadratic equations II)**.**

$$\epsilon x^2 + x - 1 = 0 \tag{1.3}$$

For $\epsilon = 0$ there is one solution $x = 1$. For $0 < \epsilon \ll 1$ there are two solutions $x = 1 - \epsilon + \mathcal{O}(\epsilon^2)$ and $x = -1/\epsilon + \mathcal{O}(1)$. This is a **singular** problem.

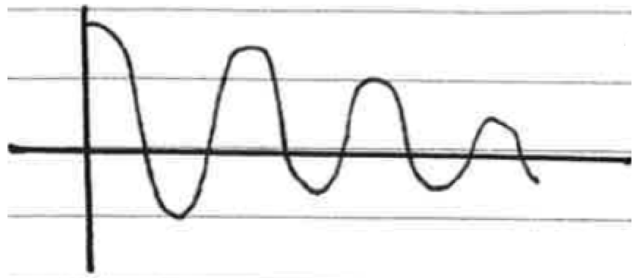

Figure 1.1: Solution of initial value problem in Example 1.4

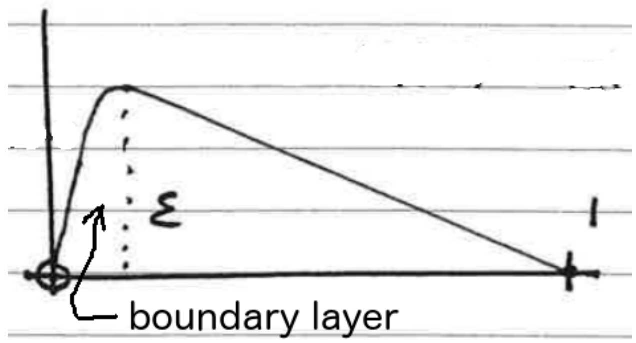

Figure 1.2: Solution of boundary value problem in Example 1.5

Example 1.4 (Initial value problem)**.**

$$\ddot{u} + \epsilon\dot{u} + 4u = 0, \qquad \text{with } u(0) = 1,\ \dot{u}(0) = 0. \tag{1.4}$$

Here ϵ is a small damping parameter. This is a **regular** problem.

Example 1.5 (Boundary value problem)**.**

$$\epsilon u'' + u' + 1 = 0 \qquad \text{with } u(0) = u(1) = 0 \tag{1.5}$$

A boundary layer of depth ϵ arises near the lower boundary $x = 0$ (see Fig. 1.2); this is a **singular** problem.

Example 1.6 (Incompressible Navier Stokes)**.**

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla P + \epsilon\nabla^2\boldsymbol{u}, \qquad\qquad \nabla\cdot\boldsymbol{u} = 0 \tag{1.6}$$

In the **singular** limit $\epsilon \to 0$ viscous boundary layers arise.

Example 1.7 (Porous media)**.**

$$uu'' = -xu' \qquad \text{with } u(0) = 1,\ u'(0) = -1/\epsilon \tag{1.7}$$

For $x \gg 0$ the solution approaches an exponentially small value $u(x) \to \exp\left[-\frac{1}{\epsilon^2}\right]$.

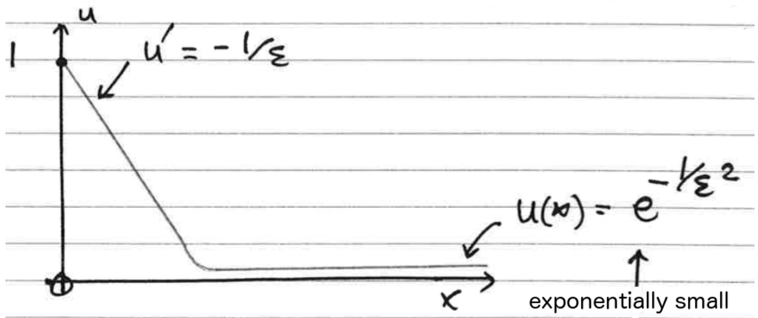

Figure 1.3: Solution of porous media problem in Example 1.7

Example 1.8 (Feynman diagrams)**.** Interactions of elementary particles can be described by a so-called Quantum Field Theory. This is a highly non-linear theory which requires complicated calculations which can not be carried out exactly. However, many processes such as electron-electron scattering can be written down as a series expansions in a small coupling parameter α, in physics this is known as perturbation theory. The calculation at each order of α requires the solution of certain loop integrals. Feynman diagrams can be used to depict this expansion, Fig. 1.4 shows the two leading order contributions to the electron scattering process. Each diagram corresponds to a particular integral which can be written down following certain rules. The beauty of this expansion is that it can also be interpreted as the propagation of particles. For example, the lowest order contribution in Fig. (a) can be interpreted as the exchange of a photon (the wiggly line) whereas the $\mathcal{O}(\alpha^2)$ corrections in (b) and (c), correspond to the exchange of two photons.

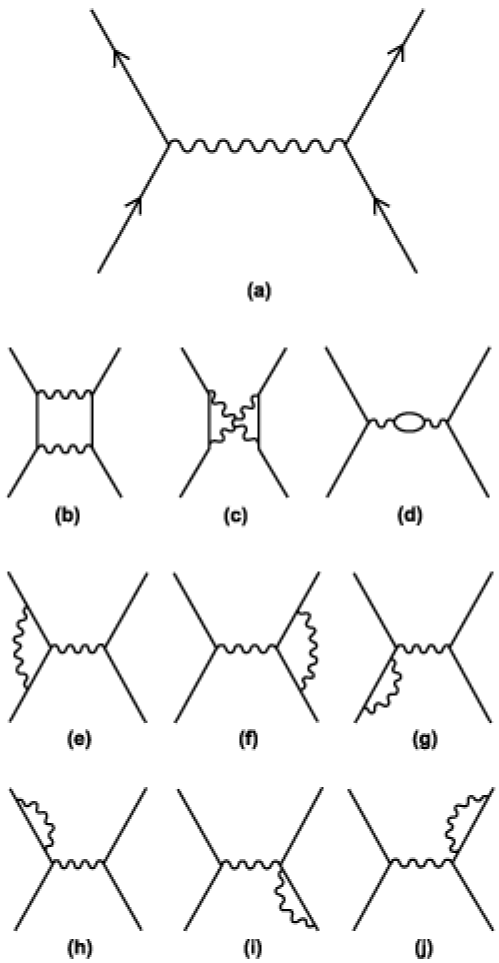

Figure 1.4: Leading order Feynman diagrams for electron scattering. The leading order $\mathcal{O}(\alpha)$ diagram is (a), the $\mathcal{O}(\alpha^2)$ corrections are shown in (b) to (j).

Chapter 2

Polynomial equations

The general polynomial equation of order n is

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0 \tag{2.1}$$

For general coefficients c_0, c_1, ..., c_n this can only be solved exactly for x if $n \leq 4$. Now assume that c_0, c_1, ..., c_n depend on ϵ. In the limit $\epsilon \to 0$ we can then use a perturbation method to find a solution. In general this solution is

- **Regular** if $\lim_{\epsilon\to 0} c_n(\epsilon) \neq 0$
- **Singular** if $\lim_{\epsilon\to 0} c_n(\epsilon) = 0$

2.1 Quadratic equations

Consider the equation from Example 1.3

$$\epsilon x^2 + x - 1 = 0 \tag{2.2}$$

which has the exact solution

$$x_\pm = \frac{1}{2\epsilon}\left(-1 \pm \sqrt{1+4\epsilon}\right). \tag{2.3}$$

We can use the Taylor expansion to write

$$\sqrt{1+4\epsilon} = (1+4\epsilon)^{1/2} = 1 + 2\epsilon - 2\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{2.4}$$

This allows the calculation of approximations to the exact solutions in Eq. (2.3) as

$$\begin{aligned} x_+ &= \frac{1}{2\epsilon}\left(-1 + 1 + 2\epsilon - 2\epsilon^2 + \mathcal{O}(\epsilon^3)\right) = 1 - \epsilon + \mathcal{O}(\epsilon^2) \\ x_- &= \frac{1}{2\epsilon}\left(-1 - 1 - 2\epsilon + 2\epsilon^2 + \mathcal{O}(\epsilon^3)\right) = -\frac{1}{\epsilon} - 1 + \epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \tag{2.5}$$

The natural question to ask is as to whether we can find the approximate solutions in Eq. (2.5) without using the Taylor expansion in (2.3) and how the method can be generalised to higher order polynomials for which there is no exact solution.

To achieve this we proceed in two steps:

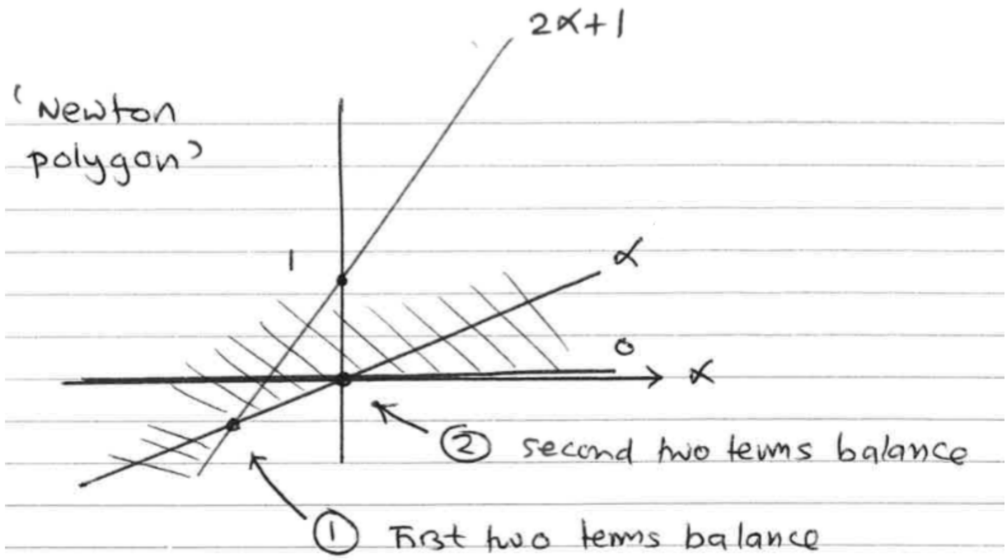

Figure 2.1: Newton polygon for Eq. (2.7)

Step I. We first need to find the 'size' of the solution as $\epsilon \to 0$. This is achieved by balancing terms in the equation as follows:

- Assume that to leading order the solution is of the form

$$x = A\epsilon^{\alpha} \tag{2.6}$$

  where A and α are real constants.

- Insert this ansatz into Eq. (2.2) to obtain

$$A^2\epsilon^{2\alpha+1} + A\epsilon^{\alpha} - 1 = 0 \qquad \text{(note that } 1 = \epsilon^0\text{)}. \tag{2.7}$$

- Find the values of α for which the terms in Eq. (2.7) balance, i.e. two or more terms are of the same size and the remaining terms are much smaller for $\epsilon \ll 1$.

To balance terms, draw a "Newton polygon" (see Fig. 2.1).

Look at the exterior polygon and take the intersection points which are

$$\alpha = 2\alpha + 1 \Rightarrow \alpha = -1, \qquad\qquad \alpha = 0. \tag{2.8}$$

This tells us that the leading order size of the solution is either ϵ^{-1} or $\epsilon^0 = 1$.

Step II. We next develop a series solution. This takes the form

$$x = \epsilon^{\alpha}\left(a_0 + a_1\epsilon + a_2\epsilon^2 + \dots\right). \tag{2.9}$$

By inserting this into Eq. (2.2) and comparing terms of different order in ϵ we can work out the coefficients a_0, a_1, ...

Let's see how this works in our example for $\alpha = -1$. In this case

$$\begin{aligned} x &= \frac{1}{\epsilon}\left(a_0 + a_1\epsilon + \dots\right) \\ x^2 &= \frac{1}{\epsilon^2}\left(a_0^2 + 2a_0a_1\epsilon + (a_1^2 + 2a_0a_2)\epsilon^2 + \dots\right). \end{aligned} \tag{2.10}$$

Inserting this into Eq. (2.2) we obtain

$$\epsilon \cdot \frac{1}{\epsilon^2}\left(a_0^2 + 2a_0a_1\epsilon + (a_1^2 + 2a_0a_2)\epsilon^2 + \dots\right) + \frac{1}{\epsilon}\left(a_0 + a_1\epsilon + a_2\epsilon^2 + \dots\right) - 1 = 0 \tag{2.11}$$

Now systematically compare terms of decreasing size (called **Gauge Expressions**).

- $\mathcal{O}(\epsilon^{-1})$ (leading order)
$$a_0^2 + a_0 = 0 \tag{2.12}$$
This gives $a_0 = -1$; we can ignore the second solution $a_0 = 0$ since this corresponds to the solution with $\alpha = 0$.
- $\mathcal{O}(\epsilon^0)$
$$2a_0a_1 + a_1 - 1 = 0 \tag{2.13}$$
Using the solution $a_0 = -1$ of Eq. (2.12) this implies that $-2a_1 + a_1 - 1 = 0 \Rightarrow a_1 = -1$.
- $\mathcal{O}(\epsilon)$
$$a_1^2 + 2a_0a_2 + a_2 = 0 \tag{2.14}$$
With $a_0 = -1$ and $a_1 = -1$ this results in $a_2 = 1$.

In summary we find that

$$x = \frac{1}{\epsilon}\left(-1 - \epsilon + \epsilon^2 + \mathcal{O}(\epsilon^3)\right) = -\frac{1}{\epsilon} - 1 + \epsilon + \mathcal{O}(\epsilon^2) \tag{2.15}$$

which is the same as the solution x_- in Eq. (2.5).

Student exercise: Find the other solution $x = a_0 + a_1\epsilon + a_2\epsilon^2 + \dots$ using the same ideas and compare it to x_+ from Eq. (2.5).

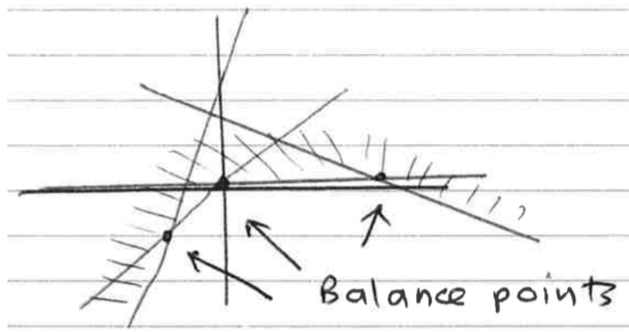

Figure 2.2: Sketch of a different Newton polygon

2.2 General equations

The same approach can be applied to the general polynomial equation

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0. \tag{2.16}$$

where each coefficient $c_k = c_k(\epsilon)$ with $k = 0, \ldots, n$ can depend on ϵ. For this proceed as follows:

- **Step I.** Set $x = A\epsilon^\alpha$ and insert it into Eq. (2.16). Draw the Newton polygon (see Fig. 2.2) to find values of α for which some of the terms balance and the other terms can be neglected.
- **Step II.** For each α set $x = \epsilon^\alpha \left(a_0 + a_1\epsilon + a_2\epsilon^2 + \ldots\right)$ and substitute into Eq. (2.16), expanding out all expressions in powers of ϵ.
- **Step III.** Write down separate equations for each power of ϵ. By solving those equations one by one work out each of the expansion coefficients a_0, a_1, ...

2.3 More complex expressions

The procedure fails when the equation has repeated roots to leading order. For example consider

$$(1-\epsilon)x^2 - 2x + 1 = 0 \tag{2.17}$$

which looks benign. To work out the leading order expression set $x = A\epsilon^\alpha$ to obtain

$$(1-\epsilon)A^2\epsilon^{2\alpha} - 2A\epsilon^\alpha + 1 = 0. \tag{2.18}$$

It is easy to see that the terms only balance if $\alpha = 0$. Proceed as before, i.e. set $x = a_0 + a_1\epsilon + a_2\epsilon^2 + \ldots$ and insert this into Eq. (2.17) to obtain

$$(1-\epsilon)\left(a_0^2 + 2a_0a_1\epsilon + (a_1^2 + 2a_0a_2)\epsilon^2 + \ldots\right) - 2\left(a_0 + a_1\epsilon + a_2\epsilon^2 + \ldots\right) + 1 = 0. \tag{2.19}$$

Writing down the equations at the lowest powers in ϵ gives

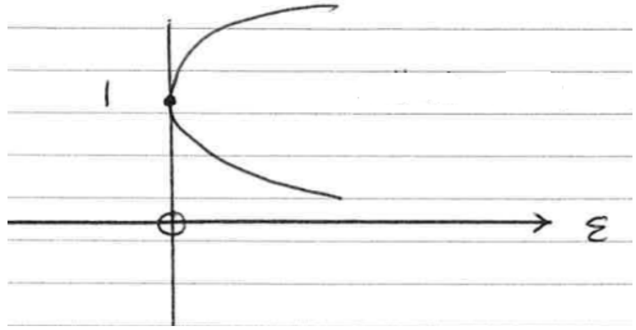

Figure 2.3: Fold bifurcation in the approximate solution Eq. (2.22)

- $\mathcal{O}(\epsilon^0)$: $a_0^2 - 2a_0 + 1 = (a_0 - 1)^2 = 0 \Rightarrow a_0 = 1$ (duplicate root)
- $\mathcal{O}(\epsilon)$: $-a_0^2 + 2a_0a_1 - 2a_1 = 0 \Rightarrow a_0^2 = 0$

Depending on which order we consider we get different values for a_0. The reason for this is the double root at leading order. For $\epsilon = 0$ Eq. (2.17) is $x^2 - 2x + 1 = (x-1)^2 = 0$ which has the repeated root $x = 1$.

In general this issue can usually be resolved by expanding in a fractional power. For the equation considered here set

$$x = a_0 + a_1\epsilon^{1/2} + a_2\epsilon + \dots. \tag{2.20}$$

Substituting the ansatz in (2.20) into Eq. (2.17) leads to

$$(1-\epsilon)\left(a_0^2 + 2a_0a_1\epsilon^{1/2} + (a_1^2 + 2a_0a_2)\epsilon + \dots\right) - 2(a_0 + a_1\epsilon^{1/2} + a_2\epsilon + \dots) + 1 = 0 \tag{2.21}$$

Considering the equations at different orders of ϵ separately gives

- $\mathcal{O}(\epsilon^0)$: $a_0^2 - 2a_0 + 1 = (a_0 - 1)^2 = 0 \Rightarrow a_0 = 1$
- $\mathcal{O}(\epsilon^{1/2})$: $2a_0a_1 - 2a_1 = 0$ Given $a_0 = 1$, a_1 can not be obtained from this equation, so look at the next order:
- $\mathcal{O}(\epsilon)$: $-a_0^2 + a_1^2 + 2a_0a_2 - 2a_2 = -1 + a_1^2 = 0$ and hence $a_1 = -1, +1$. Note that a_2 can not be determined at this order.

In summary we find that the solution to Eq. (2.17) can be expanded as

$$x = 1 \pm \epsilon^{1/2} + \mathcal{O}(\epsilon). \tag{2.22}$$

One often finds this in solutions of nonlinear systems at bifurcation points, see Fig. 2.3.

Chapter 3

Asymptotic series and integral expansions

3.1 Motivation: the exponential integral

Example 3.1 (Exponential integral)**.** Consider the following integral with lower boundary $x \in \mathbb{R}$

$$\mathrm{E}_1(x) = \int_x^\infty \frac{e^{-z}}{z}\, dz. \tag{3.1}$$

This integral is related to the **exponential integral** $\mathrm{Ei}(x)$ by $\mathrm{E}_1(x) = -\mathrm{Ei}(-x)$ and can not be calculated in closed form. However, for $x \gg 1$ one can systematically construct approximations to the integral in Eq. (3.1) as follows.

Apply integration by parts to obtain

$$\mathrm{E}_1(x) = \int_x^\infty \frac{e^{-z}}{z}\, dz = -\left[\frac{e^{-z}}{z}\right]_x^\infty - \int_x^\infty \frac{e^{-z}}{z^2}\, dz = \underbrace{\frac{e^{-x}}{x}}_{\phi_0} \underbrace{- \int_x^\infty \frac{e^{-z}}{z^2}\, dz}_{R_0} = \phi_0 + R_0. \tag{3.2}$$

This does not look like it has helped much, but we can estimate the last term in Eq. (3.2) by using $1/z^2 < 1/x^2$ for $z > x$

$$|R_0| = \int_x^\infty \frac{e^{-z}}{z}\, dz < \frac{1}{x^2} \int_x^\infty e^{-z}\, dz = \frac{e^{-x}}{x^2}. \tag{3.3}$$

This implies that

$$\frac{|R_0|}{|\phi_0|} < \frac{1}{x} \tag{3.4}$$

and hence $|R_0| \ll |\phi_0|$ for $x \gg 1$; the integral $\mathrm{E}_1(x)$ can be approximated by ϕ_0 in this limit.

This process can be repeated to calculate increasingly better approximations. To calculate the next term, apply integration by parts to the remainder R_0

$$R_0 = -\int_x^\infty \frac{e^{-z}}{z^2}\, dz = \left[\frac{e^{-z}}{z^2}\right]_x^\infty + 2\int_x^\infty \frac{e^{-z}}{z^3}\, dz = \underbrace{-\frac{e^{-x}}{x^2}}_{\phi_1} \underbrace{+2\int_x^\infty \frac{e^{-z}}{z^3}\, dz}_{R_1} = \phi_1 + R_1 \tag{3.5}$$

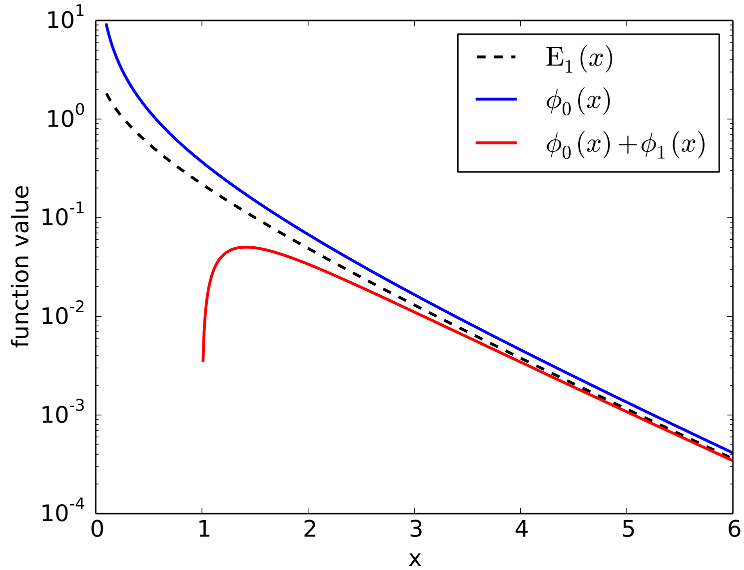

Figure 3.1: Integral $\mathrm{E}_1(x)$ defined Eq. (3.1) and two lowest order approximations for $x \gg 1$.

As above it is easy to see that both $|R_1| < \frac{2}{x^3}e^{-x} < \frac{2}{x}|\phi_1| \ll |\phi_1|$ and $|\phi_1| = \frac{1}{x}|\phi_0| \ll |\phi_0|$ for $x \gg 1$. In summary we find the approximation

$$\mathrm{E}_1(x) = \phi_0 + \phi_1 + R_1 = \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + \mathcal{O}(x^{-3}) \tag{3.6}$$

The two first approximations are plotted in Fig. 3.1. For $x = 6$ the relative difference between $\phi_0(x)$ and $\mathrm{E}_1(x)$ is 14.7%, if the next term is included in the expansion this relative error decreases to 4.4%. Note that the second order approximation $\phi_0(x) + \phi_1(x)$ is only better for large x. For $x \lesssim 2$ the lowest order approximation $\phi_0(x)$ is closer to the true value of the function.

By repeating this process one obtains[1]

$$\mathrm{E}_1(x) = \underbrace{\int_x^\infty \frac{e^{-z}}{z}\,dz = \frac{e^{-x}}{x}\left(1 - \frac{1}{x} + \frac{2}{x^2} - \frac{6}{x^3} + \cdots + (-1)^n\frac{n!}{x^n}\right)}_{S_n} + R_n \tag{3.7}$$

where

$$|R_n| < \frac{e^{-x}}{x^{n+2}}(n+1)! \tag{3.8}$$

Suppose now that we fix x and let $n \to \infty$. This implies that $\mathrm{E}_1(x)$ can be replaced by the infinite sum

$$\mathrm{E}_1(x) \mapsto \frac{e^{-x}}{x}\sum_{n=0}^\infty (-1)^n\frac{n!}{x^n} = \frac{e^{-x}}{x}\sum_{n=0}^\infty a_n x^{-n} \quad \text{with } a_n = (-1)^n n! \qquad \text{for } x > 1. \tag{3.9}$$

[1] This can be shown by induction over n.

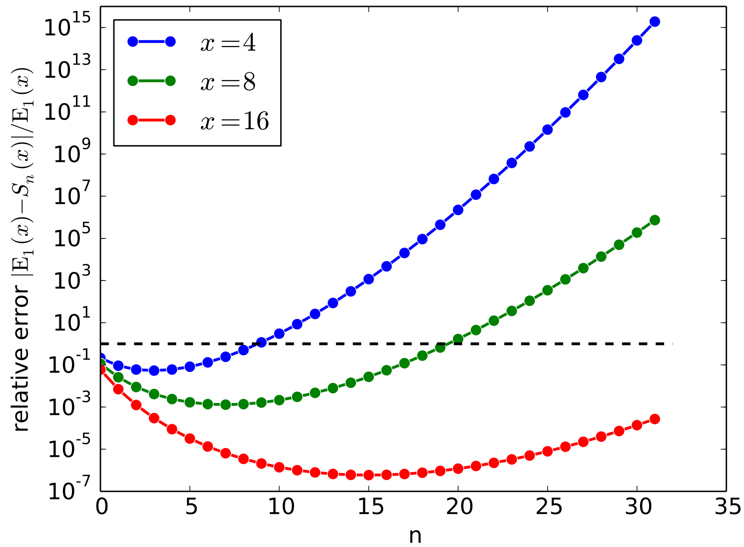

Figure 3.2: Relative error of the approximation $S_n(x)$ in Eq. (3.7) for different values of x.

There is, however, a serious problem: **The sum in Eq. (3.9) does not converge!** This initially sounds like a contradiction, since the above arguments and Fig. 3.1 seem to show that the approximation gets better if more terms are included in the sum. To resolve this paradox, let's look at the error term R_n. Following the same arguments as above it can be shown that

$$\begin{aligned} |R_n| &= \frac{(n+1)!}{x^{n+2}} e^{-x} + \mathcal{O}(x^{-(n+3)}) \qquad \text{and} \\ \frac{|R_n|}{|R_{n-1}|} &= \frac{(n+1)!}{n!} \frac{x^{n+1}}{x^{n+2}} \left(1 + \mathcal{O}(x^{-1})\right) = \frac{n+1}{x} + \mathcal{O}(x^{-2}). \end{aligned} \tag{3.10}$$

In other words, as long as $n < x - 1$, the error decreases as n increases, but if $n > x - 1$ the amplification factor is larger than 1 and the error grows.

This can be proven more rigorously by applying the ratio test to the series in Eq. (3.9). Since for fixed x

$$\rho_n = \frac{|a_{n+1} x^{-(n+1)}|}{|a_n x^{-n}|} = \frac{(n+1)!}{n!} \frac{1}{x} = \frac{n+1}{x} \tag{3.11}$$

and $\lim_{n\to\infty} \rho_n = \infty$ the sum does not converge. However, if we only include a small number of terms in the series, we still get a very good approximation of the function. More precisely, for given x the ideal number of terms is $n \approx x$, since this is the point where the amplification factor starts to grow. This is shown in Fig. 3.2.

The sum in Eq. (3.9) is an example of an **asymptotic series**.

Definition 3.1 (Asymptotic Series)**.** Let $0 < \epsilon \ll 1$ be a small parameter. The series

$$S_n(\epsilon) = \sum_{k=0}^{n} \phi_k, \qquad \text{with } \phi_k = \phi_k(\epsilon) \tag{3.12}$$

is an **asymptotic series** for a function $f(\epsilon)$ if

$$\lim_{\epsilon\to 0} \frac{|f(\epsilon) - S_n(\epsilon)|}{|\phi_n(\epsilon)|} = \lim_{\epsilon\to 0} \frac{|f(\epsilon) - \sum_{k=0}^{n} \phi_k(\epsilon)|}{|\phi_n(\epsilon)|} = 0. \tag{3.13}$$

To indicate that a function f can be approximated by an asymptotic series we write

$$f(\epsilon) \sim \sum_k \phi_k(\epsilon) \qquad \text{or simply} \quad f \sim \sum_k \phi_k. \tag{3.14}$$

Asymptotic series can be added and multiplied, for example

$$f \sim \sum_k \phi_k, \;\; g \sim \sum_k \psi_k \quad \Rightarrow \quad f + g \sim \sum_k \left(\phi_k + \psi_k\right). \tag{3.15}$$

In our example we have $\epsilon = 1/x$,

$$f(\epsilon) = \mathrm{E}_1(1/\epsilon), \qquad\qquad \phi_n(\epsilon) = (-1)^n n! \exp\left[-1/\epsilon\right] \epsilon^{n+1} \tag{3.16}$$

Using Eq. (3.8) we find that

$$\begin{aligned} & |f(\epsilon) - S_n(\epsilon)| = |R_n(\epsilon)| < (n+1)! \exp\left[-1/\epsilon\right] \epsilon^{n+2} \\ \Rightarrow \quad & \lim_{\epsilon\to 0} \frac{|f(\epsilon) - S_n(\epsilon)|}{|\phi_n(\epsilon)|} = \lim_{\epsilon\to 0} (n+1)\epsilon = 0 \end{aligned} \tag{3.17}$$

3.1.1 Convergent series

Note that it is also possible to write down a convergent series for $\mathrm{E}_1(x)$ by considering the following Taylor-expansion

$$\frac{d\,\mathrm{E}_1(x)}{dx} = -\frac{e^{-x}}{x} = -\frac{1}{x} + \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{(k+1)!} \tag{3.18}$$

Integrating this gives

$$\mathrm{E}_1(x) = C - \log(x) - \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k\, k!} \tag{3.19}$$

and it is easy to check that the infinite sum converges for fixed x by using the ratio test. It can also be shown that $C = -\gamma$, where $\gamma \approx 0.5772156649$ is the Euler-Mascheroni-constant.

However, for large x the infinite sum in (3.19) converges very slowly and needs many terms to reach a given accurary. In contrast, the asymptotic series in Eq. (3.9) is highly accurate even if it truncated after a small number of terms.

3.1.2 Optimal truncation

The obvious question to ask is where to truncate an asymptotic series to minimise the truncation error. We've already seen above that for $x \gg 1$ the optimal number n^* of terms satisfies $n^* + 1 = x = 1/\epsilon$ since the error starts growing after this point (see Eq. (3.10). Using this value for n and Eq. (3.8), we have for the error

$$|R_{n^*}| < \frac{e^{-x}}{x^{n^*+2}}(n^*+1)! = \frac{e^{-x}x!}{x^{x+1}} \tag{3.20}$$

We also use Stirling's approximation to $x!$ from Eq. (1.1)

$$x! \approx \sqrt{2\pi} x^{x+1/2} e^{-x} \tag{3.21}$$

to obtain

$$|R_{n^*}| < \sqrt{2\pi}\frac{e^{-2x}}{\sqrt{x}} = \sqrt{2\pi\epsilon}\exp\left[-2/\epsilon\right] \tag{3.22}$$

The error becomes very small as $\epsilon \to 0$. More precisely, we have that

$$\lim_{\epsilon\to 0}\frac{|R_{n^*}|}{\epsilon^m} = \sqrt{2\pi}\lim_{\epsilon\to 0}\left(\exp\left[-2/\epsilon\right]\epsilon^{1/2-m}\right) = 0 \tag{3.23}$$

for <u>any</u> m. We say that **"the error is beyond all orders in ϵ"**.

Let's look at some another example of an asymptotic series.

Example 3.2 (Euler-MacLaurin series)**.** Consider the integral of a smooth function f over the interval $[a, b]$ and assume that this interval is subdivided into n subintervals of length $h = \frac{b-a}{n}$. At lowest order the integral can then be calculated by the trapezoidal rule

$$\int_a^b f(x)\,dx = Tf + \mathcal{O}(h^2) = h\left(\frac{f_0}{2} + \sum_{i=1}^{n-1} f(x_i) + \frac{f_n}{2}\right) + \mathcal{O}(h^2) \tag{3.24}$$

where $f_0 = f(a)$, $f_n = f(b)$ and $f(x_i) = f(a+ih)$. By adding further terms, the error can be reduced to $\mathcal{O}(h^6)$

$$\int_a^b f(x)\,dx = Tf + \frac{h^2}{12}\left(f'(a) - f'(b)\right) - \frac{h^4}{720}\left(f'''(a) - f'''(b)\right) + \mathcal{O}(h^6). \tag{3.25}$$

Generalising this, the integral can be written as an asymptotic series

$$\int_a^b f(x)\,dx \sim Tf + \sum_{k=1}^{\infty} h^{2k}\frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(a) - f^{(2k-1)}(b)\right). \tag{3.26}$$

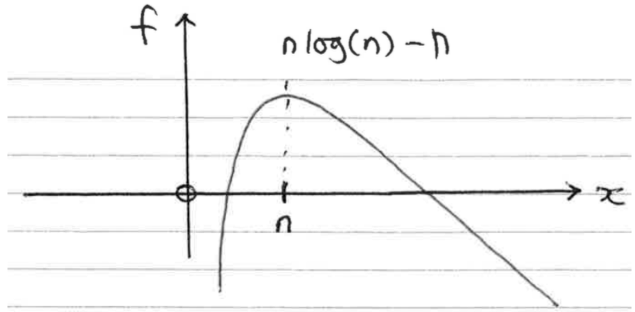

Figure 3.3: Function $f_n(x)$ defined in Eq. (3.32)

In this expression the **Bernoulli numbers** B_m can be defined by the generating function (see lecture notes for MA30044)

$$\frac{t}{1-e^{-t}} = \sum_{m=0}^{\infty} \frac{B_m}{m!} t^m \qquad \text{with } B_1 = \frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ \ldots . \tag{3.27}$$

3.2 Functions defined by integrals

It is common when using transform methods (e.g. Fourier, Laplace) to *define* a function by an integral e.g.

$$g(z) = \int_0^\infty e^{-zx} f(x)\, dx, \tag{3.28}$$

$$\tilde{g}(z) = \int_0^\infty \cos(zx) f(x)\, dx, \tag{3.29}$$

$$n! = \Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx. \tag{3.30}$$

Using asymptotic methods we can find expression for these integrals if z or n are large. The basic idea of these methods (called “stationary phase”, “saddle point” or “Watson Lemma”) is to identify the range of x which gives the leading order contribution to the integral.

We take the Γ-function in Eq. (3.30) as an example. For this rewrite

$$x^n e^{-x} = \exp\left(n \log(x) - x\right) \equiv \exp\left(f_n(x)\right). \tag{3.31}$$

Now consider the function

$$f_n(x) = n \log(x) - x \tag{3.32}$$

which is shown in Fig. 3.3.

For $n \to \infty$ this function is highly peaked. Since

$$f_n'(x) = \frac{n}{x} - 1 \tag{3.33}$$

the maximum is at $x = n$ with $f_n(n) = n\log(n) - n$. We also have

$$f_n''(x) = -\frac{n}{x^2}, \qquad f_n''(n) = -\frac{1}{n}. \tag{3.34}$$

Close to the maximum we can write

$$f_n(x) = n\log(n) - n - \frac{(x-n)^2}{2n} + \mathcal{O}\left((x-n)^3\right) \tag{3.35}$$

Inserting this approximation into the integral Eq. (3.30) gives

$$n! = \Gamma(n+1) = \int_0^\infty \exp\left(f_n(x)\right) \approx \int_{-\infty}^\infty \exp\left(n\log(n) - n\right)\exp\left(-\frac{(x-n)^2}{2n}\right)\, dx. \tag{3.36}$$

The first exponential does not depend on x and can be pulled out of the integral. Using the substitution $\tau = \frac{x-n}{\sqrt{n}}$, $dx = \sqrt{n}d\tau$ and evaluating the Gaussian integral gives the following approximation for $n \gg 1$

$$\begin{aligned} n! = \Gamma(n+1) &\approx e^{-n}n^n \int_{-\infty}^\infty \exp\left(-\frac{\tau^2}{2}\right)\sqrt{n}\, d\tau \\ &= \sqrt{2\pi}e^{-n}n^{n+1/2}. \end{aligned} \tag{3.37}$$

This is the famous Stirling's formula from Eq. (1.1). The reason this all works is that the exponential "magnifies" the peak; a more rigorous analysis can be found in Appendix A.

The same method can be used to evaluate more general integrals of the form

$$\int_{-\infty}^\infty \exp\left(\phi(x)/\epsilon\right) f(x)\, dx \sim \exp\left(\phi(x_s)/\epsilon\right) f(x_s)\sqrt{\frac{2\pi\epsilon}{|\phi''(x_s)|}} \tag{3.38}$$

Here $\phi(x_s)$ is the maximum value of the function $\phi(x)$. Again the idea is to expand the function $\phi(x)$ around it's maximum value as

$$\phi(x) = \phi(x_s) + \frac{1}{2}\phi''(x_s)(x-x_s)^2 + \ldots \tag{3.39}$$

and calculate the Gaussian integral which results from this approximation.

3.3 Method of stationary phase

The method of stationary phase is similar and relies on the following Lemma:

Lemma 3.1 (Riemann-Lebesgue). *Let $f(x)$ be a continuous function defined on an interval $[a,b]$. Then we have that*

$$\lim_{n\to\infty}\int_a^b f(x)\exp\left(inx\right)\, dx = 0. \tag{3.40}$$

In applications this allows us to ignore contributions to integrals which are highly oscillatory, i.e. the leading contribution comes from the least oscillatory part. More generally we want to calculate integrals such as

$$I(k) = \int_{-\infty}^{\infty} f(x) \exp\left(ik\phi(x)\right)\, dx \tag{3.41}$$

for $k \gg 1$. Note that this is similar to Eq. (3.38) but the phase is now imaginary. If $\phi(x)$ has stationary point at $x = x_s$ (i.e. $\phi'(x_s) = 0$) then the integral in Eq. (3.41) is

$$I_k \sim 2|f(x_s)|\sqrt{\frac{2\pi}{k\phi''(x_s)}} \exp\left(i\phi(x_s)k + i\frac{\pi}{4}\operatorname{sgn}(\phi''(x_s))\right). \tag{3.42}$$

3.4 The WKB method

Wentzel, Kramers and Brillouin used the following approximation to solve problems in quantum mechanics that are otherwise untractable. The method allows the calculation of approximate solutions $u(x)$ of the ODE

$$u'' + a(x)u = 0 \tag{3.43}$$

for a given function $a(x)$ which we assume to be positive at the moment. The method is particularly useful when $u(x)$ is highly oscillatory, in which case the numerical solution of Eq. (3.43) can be problematic. The key idea is to represent the solution as

$$u(x) = q(x)e^{ip(x)} \tag{3.44}$$

where $q(x) \in \mathbb{R}$ is the **amplitude** and $p(x) \in \mathbb{R}$ is the **phase**. We assume that the amplitude is slowly varying in x. Using the chain rule it follows that

$$\begin{aligned} u'(x) &= \left(q' + iqp'\right) e^{ip(x)} \\ u''(x) &= \left(q'' + (2q'p' + qp'')i - q(p')^2\right) e^{ip(x)} \end{aligned} \tag{3.45}$$

Inserting this into (3.43) and considering the real and imaginary parts separately gives

$$q'' - q(p')^2 + aq = 0 \ \ \text{(real part)} \tag{3.46}$$
$$2q'p' + qp'' = 0 \ \ \text{(imaginary part)} \tag{3.47}$$

Since we assume that the amplitude is slowly varying (more precisely, $|q''| \ll |a(x)q|$), q'' can be dropped in Eq. (3.46) to obtain $(p')^2 = a$ and hence there are two solutions

$$p(x) = \pm \int_{x_0}^{x} \sqrt{a(y)}\, dy. \tag{3.48}$$

for some lower bound x_0. From Eq. (3.47) we then have

$$\frac{q'}{q} = -\frac{p''}{2p'}, \qquad \Rightarrow \qquad \frac{d}{dx}\log(q) = -\frac{1}{2}\frac{d}{dx}\log(p') \tag{3.49}$$

and therefore by integration

$$q(x) = C/\sqrt{p'(x)} = Ca^{-1/4}. \tag{3.50}$$

We find the general solution

$$u(x) \sim C_+ a(x)^{-1/4} \exp\left(i\int_{x_0}^{x}\sqrt{a(y)}\,dy\right) + C_- a(x)^{-1/4} \exp\left(-i\int_{x_0}^{x}\sqrt{a(y)}\,dy\right) \tag{3.51}$$

where C_+ and C_- are constants which depend on boundary conditions. Note that the phase $p(x)$ is only real if $a(x) > 0$. However, the approximation can be extended to negative $a(x)$.

Example 3.3 (Quantum mechanics)**.** Consider the one-dimensional time-independent Schrödinger equation for the wave function $\psi(x)$

$$-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi. \tag{3.52}$$

This describes the motion of a particle with mass m and energy E in some potential $V(x)$; $\hbar$ is the Planck-constant. Rewriting Eq. (3.52) in the form (3.43) we find that

$$\psi'' + a(x)\psi = 0 \qquad \text{with } a(x) = \frac{2m}{\hbar^2}(E - V(x)) \tag{3.53}$$

Note that $a(x) > 0$ if $E > V(x)$, and in the following we will assume this to be true for all x. In quantum mechanics this corresponds to the scattering of a free particle. We consider the potential (see Fig. 3.4)

$$V(x) = \frac{V_0}{4}\left(1 + \tanh\left(\frac{x + x_0}{\sigma}\right)\right)\left(1 + \tanh\left(\frac{x_0 - x}{\sigma}\right)\right) \tag{3.54}$$

Note that $V(x) \to 0$ for $x \to \pm\infty$. In this limit we have that

$$\psi''_\infty + \frac{2mE}{\hbar^2}\psi_\infty = 0 \tag{3.55}$$

which has the solution

$$\psi_\infty(x) = A_+ e^{i(\delta_+ + p_\infty x)} + A_- e^{i(\delta_- - p_\infty x)} \qquad \text{with } p_\infty = \sqrt{\frac{2mE}{\hbar^2}} \tag{3.56}$$

where A_+, A_-, δ_+ and δ_- are real constants. We therefore choose the boundary conditions such that the solution to Eq. (3.53) is

$$\begin{aligned} \psi(x) &\to A_+ e^{ip_\infty x} && \text{as } x \to -\infty. \\ \psi(x) &\to A_+ e^{i(\delta_+ + p_\infty x)} && \text{as } x \to +\infty. \end{aligned} \tag{3.57}$$

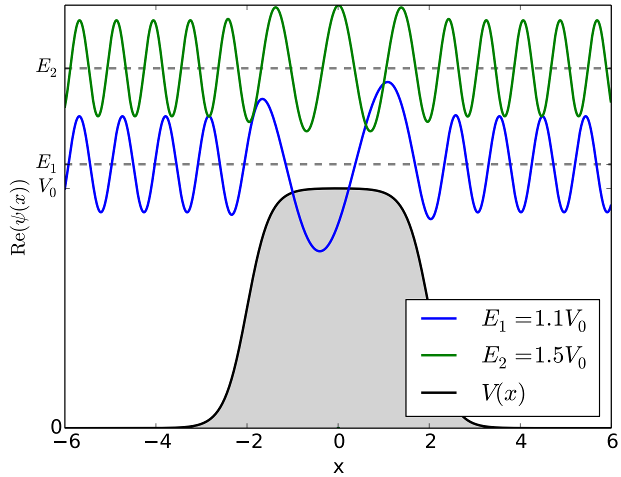

Figure 3.4: Solution $\psi(x)$ of the Schrödinger equation in Eq. (3.52). We chose $D = 2$, $\sigma = \frac{1}{2}$ for the potential in Eq. (3.54).

with $p_\infty = \sqrt{\frac{2mE}{\hbar^2}}$. This corresponds to the scattering of an incoming wave from the left and results in

$$C_+ = A_+ \left(\frac{2mE}{\hbar^2}\right)^{1/4} e^{ip_\infty x_0}, \qquad\qquad C_- = 0 \tag{3.58}$$

in Eq. (3.51)

$$\psi(x) \sim A_+ \left(\frac{E}{E - V(x)}\right)^{1/4} \exp\left(i\sqrt{\frac{2mE}{\hbar^2}}x_0 + i\int_{x_0}^{x} \sqrt{\frac{2m}{\hbar^2}(E - V(x))}\right) \qquad \text{with } x_0 \ll 0. \tag{3.59}$$

The real part of this solution is plotted for two different energies E in Fig. 3.4 (units are chosen such that $m/\hbar^2 = 10$). In Fig. 3.5 we also show the phase $p(x)$ and amplitude $q(x)$ given by

$$q(x) = A_+ \left(\frac{E}{E - V(x)}\right)^{1/4}, \qquad p(x) = \sqrt{\frac{2mE}{\hbar^2}}x_0 + \int_{x_0}^{x} \sqrt{\frac{2m}{\hbar^2}(E - V(x))}. \tag{3.60}$$

Note that the phase and amplitude are smooth functions whereas the solution $\psi(x)$ is highly oscillatory. This approximation breaks down, however, if $\sigma \to 0$, i.e. the slope of the potential gets very steep and hence the change in amplitude is very rapid (recall that one of the assumptions of the WKB approximation was that $q(x)$ varies slowly). In this case the Schrödinger equation can be solved exactly and it is found that part of the incoming wave is reflected back. This effect, which should also be present for non-zero σ is not captured in the WKB approximation. The WKB approximation is also bad if $E \to V_0$ since in this case the amplitude, which is $\propto \left(\frac{E}{E-V(x)}\right)^{1/4}$ will become very large.

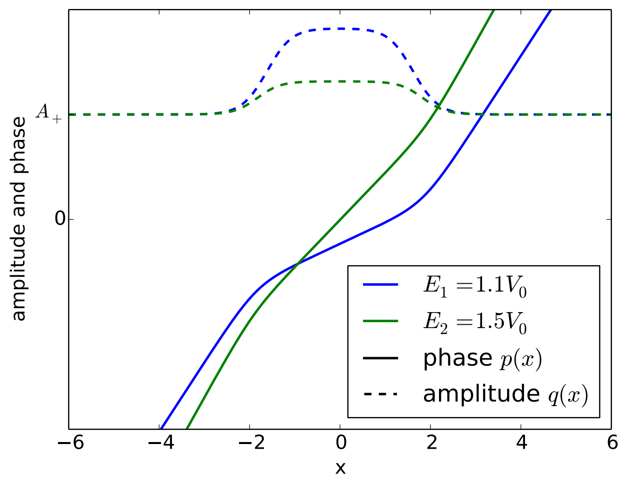

Figure 3.5: Amplitude and phase of the Schrödinger equation in Eq. (3.52)

3.4.1 Connection formulas

The case $a(x) < 0$ (which corresponds to $E < V(x)$ is Example 3.3) can be treated similarly and one obtains

$$u(x) \sim C'_+ |a(x)|^{-1/4} \exp\left(\int_{x_0}^{x} \sqrt{|a(y)|}\, dy\right) + C'_- |a(x)|^{-1/4} \exp\left(-\int_{x_0}^{x} \sqrt{|a(y)|}\, dy\right). \quad (3.61)$$

In this case the solution is the sum of an exponentially growing and an exponentially decreasing solution. However, what happens if the function $a(x)$ changes sign? In this case we somehow need to smoothly connect the solution in the region where $a(x) > 0$ and $u(x)$ is given by Eq. (3.51) to the solution in the region where $a(x)$ and $u(x)$ is given by Eq. (3.61). In other words, we need to find a relationship between C_+, C_- and C'_+, C'_- across the point where $a(x)$ changes sign. This is achieved by constructing an intermediate solution in the region where $a(x) \approx 0$ as follows:

Assume that the function $a(x)$ has exactly one zero at $x = 0$ and $a(x) > 0$ for $x < 0$ and $a(x) < 0$ for $x > 0$. For small x we can write

$$a(x) = -\lambda^2 x + \mathcal{O}(x^2) \qquad \text{with } \lambda^2 = -\left.\frac{da}{dx}\right|_{x=0} > 0 \quad (3.62)$$

and the differential equation in Eq. (3.43) reduces to

$$u'' - \lambda^2 x u = 0. \quad (3.63)$$

Using the substitution $z = \lambda^{2/3} x$, $v(z) = u(x)$, this equation becomes the Airy equation

$$\frac{d^2 v}{dz^2} - zv = 0 \quad (3.64)$$

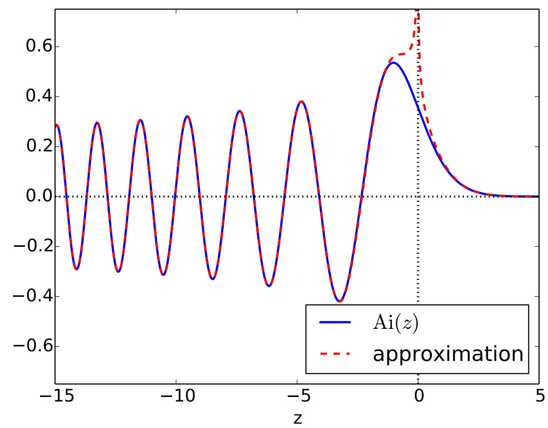

Figure 3.6: Airy function $\mathrm{Ai}(z)$ and aymptotic expansion in Eq. (3.66)

which is solved by the Airy functions $\mathrm{Ai}(z)$ (see Fig. 3.6) and $\mathrm{Bi}(z)$. In summary we obtain

$$u(x) = C_A \,\mathrm{Ai}(\lambda^{2/3}x) + C_B \,\mathrm{Bi}(\lambda^{2/3}x). \qquad \text{for } |x| \ll 1. \tag{3.65}$$

The crucial observation is now that the Airy functions have asymptotic expansions for $|z| \gg 1$

$$\begin{aligned} \mathrm{Ai}(z) &\sim \begin{cases} \frac{\exp\left(-\frac{2}{3}z^{3/2}\right)}{2\sqrt{\pi}z^{1/4}} & \text{for } z \gg 1 \\ \frac{\sin\left(\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right)}{\sqrt{\pi}(-z)^{1/4}} & \text{for } z \ll 1 \end{cases} \\ \mathrm{Bi}(z) &\sim \begin{cases} \frac{\exp\left(\frac{2}{3}z^{3/2}\right)}{2\sqrt{\pi}z^{1/4}} & \text{for } z \gg 1 \\ \frac{\cos\left(\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right)}{\sqrt{\pi}(-z)^{1/4}} & \text{for } z \ll 1 \end{cases} \end{aligned} \tag{3.66}$$

As Fig. 3.6 shows, those expansions are already very good approximations for $|z| \gtrsim 2$. On the other hand we can also write down Eqs. (3.51) and (3.61) for $a(x) = -\lambda^2 x$ to obtain

$$u(x) \sim \begin{cases} C'_+\lambda^{-1/2}x^{-1/4}\exp\left(\frac{2}{3}\lambda x^{3/2}\right) + C'_-\lambda^{-1/2}x^{-1/4}\exp\left(-\frac{2}{3}\lambda x^{3/2}\right) & \text{for } x > 0 \\ C_+\lambda^{-1/2}x^{-1/4}\exp\left(\frac{2}{3}i\lambda(-x)^{3/2}\right) + C_-\lambda^{-1/2}x^{-1/4}\exp\left(-\frac{2}{3}i\lambda(-x)^{3/2}\right) & \text{for } x < 0. \end{cases} \tag{3.67}$$

(here we have assumed that $x_0 = 0$ when calculating the integrals over y in the exponents). Using Eqs. (3.65), (3.66) and (3.67) gives after some straightforward algebra

$$C_+ = e^{i\frac{\pi}{4}}\left(C'_+ - iC'_-\right), \qquad\qquad C_- = e^{i\frac{\pi}{4}}\left(C'_+ + iC'_-\right). \tag{3.68}$$

Those relations are known as **connection formulas** since they relate the solution for positive and negative $a(x)$.

Chapter 4

Initial value problems (ODEs)

4.1 Overview

In this chapter we will consider initial value ODEs. Typically these are second order and take the form

$$\ddot{u} + f(t, u, \dot{u}; \epsilon) = 0 \qquad \text{with } u(0) = \alpha,\ \dot{u}(0) = \beta. \tag{4.1}$$

We are interested in the time-evolution of the solutions $u(t)$ when the parameer ϵ is small. We might also look at the limit of large times t.

Example 4.1 (Non-linear oscillator)**.** Consider

$$\ddot{u} + u + \epsilon f(u) = 0, \qquad \text{with } u(0) = 1,\ \dot{u}(0) = 0. \tag{4.2}$$

For $\epsilon = 0$ the solution is periodic with period $T = 2\pi$. To calculate the period $T(\epsilon) = 2\pi + C_1\epsilon + \mathcal{O}(\epsilon^2)$ for $0 < \epsilon \ll 1$ we can use the **Method of Strained Coordinates** in Section 4.2.

Example 4.2 (Damped oscillators)**.** Consider the weakly-damped harmonic oscillator

$$\ddot{u} + \epsilon\dot{u} + u = 0 \qquad \text{with } u(0) = 1,\ \dot{u}(0) = 0 \tag{4.3}$$

or the non-linear van-der-Pol oscillator

$$\ddot{u} + \epsilon(u^2 - 1)\dot{u} + u = 0 \qquad \text{with } u(0) = 1,\ \dot{u}(0) = 0. \tag{4.4}$$

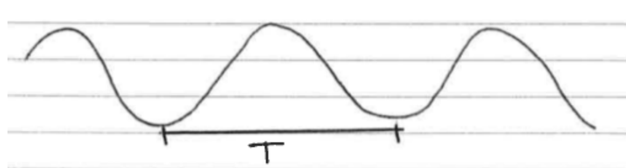

Figure 4.1: Amplitude of the non-linear oscillator in Example 4.1

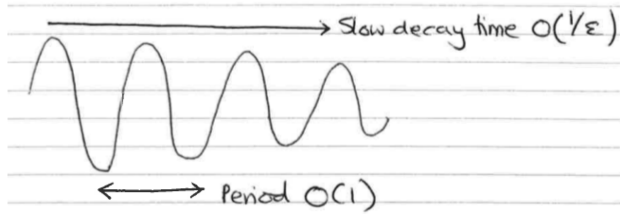

Figure 4.2: Amplitude of the damped oscillator in Example 4.2

In this case we typically see behaviour on two time scales e.g. for the damped harmonic oscillator in Eq. (4.3) there is a slow decay over a time $\mathcal{O}(1/\epsilon)$ and a faster oscillations with a period of $\mathcal{O}(1)$. To systematically treat this multiscale behaviour we can use the **Method of Multiple Scales** in Section 4.3.

4.2 Method of strained coordinates

This method is also called a "Uniformly valid asymptotic series".

Example 4.3 (Linearly perturbed harmonic oscillator)**.** Consider

$$\ddot{u} + u + \epsilon u = 0 \qquad \text{with } u(0) = 1,\ \dot{u}(0) = 0. \tag{4.5}$$

It is easy to see that the solution is

$$u(t;\epsilon) = \cos\left(\sqrt{1+\epsilon}\, t\right) \tag{4.6}$$

Observe that u is periodic with period

$$T = \frac{2\pi}{\sqrt{1+\epsilon}} = 2\pi - \pi\epsilon + \mathcal{O}(\epsilon^2). \tag{4.7}$$

We now attempt to deduce this solution from an asymptotic procedure.

Method 1 Try a solution of the form

$$u(t;\epsilon) = u_0(t) + \epsilon u_1(t) + \dots \tag{4.8}$$

and substitute this into the ODE in Eq. (4.5). Expand in ϵ and consider the equation at different orders of ϵ separately

$$\mathcal{O}(\epsilon^0): \qquad \ddot{u}_0 + u_0 = 0, \qquad \text{with } u_0(0) = 1,\ \dot{u}_0(0) = 0, \tag{4.9}$$
$$\mathcal{O}(\epsilon): \qquad \ddot{u}_1 + u_1 + u_0 = 0, \qquad \text{with } u_1(0) = \dot{u}_1(0) = 0. \tag{4.10}$$

Solving Eq. (4.9) gives

$$u_0(t) = \cos(t). \tag{4.11}$$

Using this, Eq. (4.9) can be rewritten as

$$\ddot{u}_1 + u_1 = -\cos(t), \qquad \text{with } u_1(0) = \dot{u}_1(0) = 0. \tag{4.12}$$

This has the solution

$$u_1(t) = -\frac{t}{2}\sin(t). \tag{4.13}$$

Note that this is a resonant solution with grows indefinitely as $t \to \infty$. Taking everything together using Eq. (4.8) we have

$$u(t;\epsilon) = \cos(t) - \frac{\epsilon t}{2}\sin(t) + \mathcal{O}(\epsilon^2). \tag{4.14}$$

Compare this to the expansion of the original solution in Eq. (4.6)

$$\begin{aligned} u(t;\epsilon) &= \cos\left(\sqrt{1+\epsilon}\; t\right) \\ &= \cos\left(t + \frac{\epsilon t}{2} + \mathcal{O}(\epsilon^2 t)\right) \qquad \text{for } \epsilon t \ll 1 \\ &= \cos(t) - \frac{\epsilon t}{2}\sin(t) + \mathcal{O}(\epsilon^2 t). \end{aligned} \tag{4.15}$$

The good news is that Eqs. (4.14) and (4.15) agree for small $\epsilon t \ll 1$. However, the asymptotic expansion in Eq. (4.14) is highly misleading since

- it breaks down for $t > 1/\epsilon$,
- it predicts a growing solution for $t \to \infty$,
- it does *not* predict a periodic solution.

Indeed the presence of an ϵt term alerts us to something wrong.

Solution. Consider u to be a function of a **strained coordinate** s with

$$u = u_0(s) + \epsilon u_1(s) + \mathcal{O}(\epsilon^2) \tag{4.16}$$
$$s = \left(1 + s_1\epsilon + \mathcal{O}(\epsilon^2)\right) t. \tag{4.17}$$

Using the chain rule we obtain

$$\begin{aligned} \dot{u} &= \frac{du}{ds}\cdot\frac{ds}{dt} = \left(1 + \epsilon s_1 + \mathcal{O}(\epsilon^2)\right) u_s \\ \ddot{u} &= \left(1 + \epsilon s_1 + \mathcal{O}(\epsilon^2)\right)^2 u_{ss} = \left(1 + 2\epsilon s_1 + \mathcal{O}(\epsilon^2)\right) u_{ss}. \end{aligned} \tag{4.18}$$

Substituting this into Eq. (4.6) we find

$$u_{ss} + 2\epsilon s_1 u_{ss} + u + \epsilon u = 0. \tag{4.19}$$

Expanding u according to Eq. (4.16) and writing down Eq. (4.19) for all orders of ϵ we obtain

$$\mathcal{O}(\epsilon^0): \qquad u_{0,ss} + u_0 = 0, \qquad \text{with } u_0(0) = 1,\ u_{0,s}(0) = 0, \tag{4.20}$$

$$\mathcal{O}(\epsilon): \qquad u_{1,ss} + u_0 + u_1 + 2s_1 u_{0,ss} = 0, \qquad \text{with } u_1(0) = u_{1,s}(0) = 0. \tag{4.21}$$

The equation at $\mathcal{O}(\epsilon^0)$ gives

$$u_0(s) = \cos(s) \tag{4.22}$$

Inserting this into the $\mathcal{O}(\epsilon)$ equation gives

$$u_{1,ss} + u_1 = (2s_1 - 1)\cos(s). \tag{4.23}$$

There will be a resonance (resulting in an unbounded solution for $t \to \infty$) unless $2s_1 - 1 = 0$. This can be avoided by setting $s_1 = \frac{1}{2}$ and $u_1 = 0$, resulting in the solution

$$u = \cos(s) + \mathcal{O}(\epsilon^2), \qquad s = \left(1 + \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2)\right) t. \tag{4.24}$$

The solution is now periodic and agrees with the exact solution in Eq. (4.6) up to corrections of $\mathcal{O}(\epsilon^2)$. The solution in Eq. (4.24) is periodic in s with period 2π and hence periodic in t with period T which satisfies

$$\begin{aligned} 2\pi &= \left(1 + \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2)\right)T \\ \Rightarrow \qquad T &= \frac{2\pi}{1 + \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2)} = 2\pi - \pi\epsilon + \mathcal{O}(\epsilon^2) \end{aligned} \tag{4.25}$$

Example 4.4 (Duffing oscillator)**.** Now consider the non-linear forcing term in

$$\ddot{u} + u - \epsilon u^3 = 0 \qquad \text{with } u(0) = 1,\ \dot{u}(0) = 0. \tag{4.26}$$

It can be shown that at lowest order the solution is periodic with period

$$T = 2\pi\left(1 + \frac{3}{8}\epsilon + \mathcal{O}(\epsilon^2)\right). \tag{4.27}$$

This is left as a student exercise. Hint: use

$$\cos^3(\theta) = \frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta). \tag{4.28}$$

4.3 Method of multiple scales

Example 4.5 (Damped oscillator)**.** For problems of the form

$$\ddot{u} + 2\epsilon\dot{u} + u = 0, \qquad \text{with } u(0) = 1,\ \dot{u}(0) = 0 \tag{4.29}$$

we have to adopt a different approach as the solution now evolves on two different time scales of $\mathcal{O}(1)$ and $\mathcal{O}(1/\epsilon)$.

Naive approach. Expand the solution in powers of ϵ

$$u(t;\epsilon) = u_0(t) + \epsilon u_1(t) + \mathcal{O}(\epsilon^2). \tag{4.30}$$

Inserting this into Eq. (4.29) and considering different orders of ϵ we find

$$\mathcal{O}(\epsilon^0): \qquad \ddot{u}_0 + u_0 = 0 \qquad \text{with } u_0(0) = 1,\ \dot{u}_0(0) = 0 \tag{4.31}$$

$\Rightarrow u_0(t) = \cos(t)$

$$\mathcal{O}(\epsilon): \qquad \ddot{u}_1 + u_1 = -2\dot{u}_0 = 2\sin(t) \qquad \text{with } u_1(0) = 0,\ \dot{u}_1(0) = 0 \tag{4.32}$$

$\Rightarrow u_1(t) = \sin(t) - t\cos(t)$. In summary we find

$$u(t) = \cos(t) + \epsilon\left(\sin(t) - t\cos(t)\right) + \mathcal{O}(\epsilon^2). \tag{4.33}$$

However, Eq. (4.29) has the exact solution

$$u(t) = e^{-\epsilon t}\left(\cos\left(\sqrt{1-\epsilon^2}\, t\right) + \frac{\epsilon}{\sqrt{1-\epsilon^2}}\sin\left(\sqrt{1-\epsilon^2}\, t\right)\right). \tag{4.34}$$

For small ϵt this agrees with Eq. (4.33), but the solution in Eq. (4.33) grows as $t \to \infty$ whereas the expression in Eq. (4.34) decays. However note that the exact solution in Eq. (4.34) has the form

$$u(t) = A(T)\left(u_0(t) + \epsilon u_1(t) + \mathcal{O}(\epsilon^2)\right) \qquad \text{where } T = \epsilon t. \tag{4.35}$$

This is an oscillating solution modulated by a slowly varying amplitude $A(T)$.

4.3.1 Fast and slow variables

This motivates the **Method of Multiple Scales**. The key idea is to write

$$u = u(\tau, T) \tag{4.36}$$

where $t = \tau$ is a **fast variable** and $T = \epsilon t \ll \tau$ is a **slow variable**. The crucial point is that we assume τ and T to be independent and write

$$\begin{aligned}
\dot{u} &= \frac{du}{dt} = \frac{d\tau}{dt}\frac{\partial u}{\partial \tau} + \frac{dT}{dt}\frac{\partial u}{\partial T} = \frac{\partial u}{\partial \tau} + \epsilon\frac{\partial u}{\partial T} \\
\ddot{u} &= \frac{d^2u}{dt^2} = \frac{\partial^2 u}{\partial \tau^2} + 2\epsilon\frac{\partial^2 u}{\partial T \partial \tau} + \epsilon^2\frac{\partial^2 u}{\partial T^2}.
\end{aligned} \tag{4.37}$$

Furthermore make the ansatz

$$u = u_0(\tau, T) + \epsilon u_1(\tau, T) + \mathcal{O}(\epsilon^2) \tag{4.38}$$

and insert this into Eq. (4.29) to obtain

$$u_{0,\tau\tau} + 2\epsilon u_{0,\tau T} + \epsilon u_{1,\tau\tau} + 2\epsilon u_{0,\tau} + u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2) = 0, \tag{4.39}$$

with $u_0(0) = 1$, $\dot{u}_0(0) = u_{0,\tau}(0) + \epsilon u_{0,T}(0) = 0$. At lowest order in ϵ we have that

$$u_{0,\tau\tau} + u_0 = 0. \qquad \text{with } u_0(0) = 1,\ u_{0,\tau}(0) = 0. \tag{4.40}$$

Since this equation only involves derivatives with respect to τ it can be solved by

$$u_0(\tau, T) = A(T)\cos(\tau) + B(T)\sin(\tau) \qquad \text{with } A(0) = 1,\ B(0) = 0. \tag{4.41}$$

Observe that both A and B are functions of the slow variable T. At the next order in ϵ we find

$$u_{1,\tau\tau} + u_1 + 2u_{0,\tau T} + 2u_{0,\tau} = 0. \tag{4.42}$$

From Eq. (4.41) we also have

$$u_{0,\tau} = -A\sin(\tau) + B\cos(\tau), \qquad u_{0,\tau T} = -A_T\sin(\tau) + B_T\cos(\tau) \tag{4.43}$$

and inserting this into Eq. (4.42) we get an equation for u_1

$$u_{1,\tau\tau} + u_1 = 2(A + A_T)\sin(\tau) - 2(B + B_T)\cos(\tau) \tag{4.44}$$

To ensure that the solution remains bounded as $t \to \infty$ we have to avoid resonant solutions. This can be achieved if

$$A_T + A = 0, \qquad B_T + B = 0 \tag{4.45}$$

With the initial conditions $A(0) = 1$, $B(0) = 0$ this is solved by

$$A(T) = e^{-T} = e^{-\epsilon t}, \qquad B(T) = 0 \tag{4.46}$$

and the lowest order solution to Eq. (4.29) is

$$u(t) = u_0 = e^{-\epsilon t}\cos(t) + \mathcal{O}(\epsilon). \tag{4.47}$$

4.3.2 Higher orders

To find the higher order corrections in Eq. (4.47) we proceed as above, but now make the ansatz

$$u = u_0(\tau, T) + \epsilon u_1(\tau, T) + \epsilon^2 u_2(\tau, T) + \mathcal{O}(\epsilon^3). \tag{4.48}$$

Inserting this into Eq. (4.29) and expanding up to (and including) $\mathcal{O}(\epsilon^2)$, we find for the $\mathcal{O}(\epsilon^2)$ terms

$$u_{0,TT} + 2u_{0,T} + 2u_{1,\tau T} + 2u_{1,\tau} + u_{2,\tau\tau} + u_2 = 0. \tag{4.49}$$

We also deduce from Eq. (4.44) that the $\mathcal{O}(\epsilon)$ solution u_1 is given by

$$u_1 = C(T)\sin(\tau) + D(T)\cos(\tau) \qquad \text{with } C(0) = 1,\ D(0) = 0. \tag{4.50}$$

The initial conditions $C(0) = 1$, $D(0) = 0$ follow from

$$\begin{aligned} 1 = u(0) &= \underbrace{u_0(0)}_{=1} + \epsilon \underbrace{u_1(0)}_{=0} + \mathcal{O}(\epsilon^2), \\ 0 = \dot{u}(0) &= \underbrace{\dot{u}_0(0)}_{=0} + \epsilon \underbrace{\left(u_{0,T}(0) + u_{1,\tau}(0)\right)}_{=-1+C(0)=0} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.51}$$

Inserting u_0 from Eq. (4.47) and u_1 from Eq. (4.50) into Eq. (4.49) gives

$$-2(D_T + D)\sin(\tau) + \left(2(C_T + C) - e^{-T}\right)\cos(\tau) + u_{2,\tau\tau} + u_2 = 0. \tag{4.52}$$

There are no resonances (in the fast variable τ) if

$$D_T + D = 0, \qquad\qquad C_T + C = \frac{1}{2}e^{-T}. \tag{4.53}$$

The solutions to those equations which are compatible with the initial conditions $C(0) = 1$ and $D(0) = 0$ are

$$C(T) = \left(1 + \frac{T}{2}\right)e^{-T}, \qquad\qquad D(T) = 0. \tag{4.54}$$

In summary we find for the solution to Eq. (4.29)

$$\begin{aligned} u(t) &= A(T)\cos(\tau) + \epsilon C(T)\sin(\tau) + \mathcal{O}(\epsilon^2) \\ &= e^{-\epsilon t}\left(\cos(t) + \epsilon\left(1 + \frac{\epsilon t}{2}\right)\sin(t)\right) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.55}$$

Up to terms of $\mathcal{O}(\epsilon^2)$ this agrees with the exact solution in Eq. (4.34). Note, however, that there is an issue here: The term proportional to $\epsilon^2 t$ in (4.55) formally arises from the expansion to $\mathcal{O}(\epsilon)$ and it grows linearly with t, in contrast to the exact solution which decays. However, the $\epsilon^2 t$ term only becomes significant for times $\gtrsim 1/\epsilon^2$, and hence the expansion is good for times $t \ll 1/\epsilon^2$. This inconsistency can be avoided by introducing a third, even slower, independent variable $\tilde{T} = \epsilon^2 t$ and using the ansatz

$$u = u_0(\tau, T, \tilde{T}) + \epsilon u_1(\tau, T, \tilde{T}) + \epsilon^2 u_2(\tau, T, \tilde{T}) + \mathcal{O}(\epsilon^3) \qquad \text{with } \tau = t,\ T = \epsilon\tau,\ \tilde{T} = \epsilon^2 t \tag{4.56}$$

instead of Eq. (4.48). We will not pursue this further here.

4.3.3 Other applications

While the exact solution is known for the damped linear oscillator in Eq. (4.29), we now consider a problem which does not have an analytic solution. In this case asymptotic methods can be used to find an approximate solution instead.

Example 4.6 (Van-der-Pol oscillator)**.** Consider the nonlinearly damped Van-der-Pol oscillator

$$\ddot{u} + \epsilon\dot{u}(u^2 - 1) + u = 0 \qquad \text{with } u(0) = 1,\ \dot{u}(0) = 0. \tag{4.57}$$

As above, make the ansatz $u(t) = u_0(\tau, T) + \epsilon u_1(\tau, T) + \mathcal{O}(\epsilon^2)$. Inserting this into Eq. (4.57) we find, using Eq. (4.37)

$$u_{0,\tau\tau} + 2\epsilon u_{0,\tau T} + \epsilon u_{1,\tau\tau} + \epsilon u_{0,\tau}(u_0^2 - 1) + u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2) = 0 \tag{4.58}$$

At lowest order ($\mathcal{O}(\epsilon^0)$) we get an equation for u_0

$$u_{0,\tau\tau} + u_0 = 0 \tag{4.59}$$

which, as for the linearly damped oscillator, is solved by

$$u_0(\tau, T) = A(T)\cos(\tau) + B(T)\sin(\tau) \qquad \text{with } A(0) = 1,\ B(0) = 0. \tag{4.60}$$

To find the slowly varying functions $A(T)$ and $B(T)$ we have to look at the terms in Eq. (4.58) which are proportional to ϵ. We find

$$2u_{0,\tau T} + u_{0,\tau}(u_0^2 - 1) + u_{1,\tau\tau} + u_1 = 0. \tag{4.61}$$

There is only a non-divergent solution for u_1 if there is no resonance. In other words the expression

$$U = 2u_{0,\tau T} + u_{0,\tau}(u_0^2 - 1) \tag{4.62}$$

must not contain any terms which are proportional to $\sin(\tau)$ or $\cos(\tau)$. This requirement will then lead to conditions on $A(T)$ and $B(T)$. Inserting the lowest order solution Eq. (4.60) into Eq. (4.62) we find after some tedious calculation, using the trigonometric identities in Tab. 4.1

$$\begin{aligned}
U &= 2(-A_T\sin(\tau) + B_T\cos(\tau)) + (-A\sin(\tau) + B\cos(\tau))\left[(A\cos(\tau) + B\sin(\tau))^2 - 1\right] \\
&= -2A_T\sin(\tau) + 2B_T\cos(\tau) - A^3\sin(\tau)\cos^2(\tau) + A^2B\cos^3(\tau) - 2A^2B\sin^2(\tau)\cos(\tau) \\
&\quad + \ 2AB^2\sin(\tau)\cos^2(\tau) - AB^2\sin^3(\tau) + B^3\sin^2(\tau)\cos(\tau) + A\sin(\tau) - B\cos(\tau) \\
&= \left(-2A_T - \frac{1}{4}A^3 - \frac{1}{4}AB^2 + A\right)\sin(\tau) + \left(2B_T + \frac{1}{4}B^3 + \frac{1}{4}A^2B - B\right)\cos(\tau) \\
&\quad + \ \text{other terms}
\end{aligned} \tag{4.63}$$

where “other terms” stands for all terms which are not proportional to $\sin(\tau)$ or $\cos(\tau)$. Hence, to avoid a resonance we need to have

$\sin^3(\theta)$	$=$	$\frac{3}{4}\sin(\theta) - \frac{1}{4}\sin(3\theta)$
$\sin(\theta)\cos^2(\theta)$	$=$	$\frac{1}{4}\sin(\theta) + \frac{1}{4}\sin(3\theta)$
$\sin^2(\theta)\cos(\theta)$	$=$	$\frac{1}{4}\cos(\theta) - \frac{1}{4}\cos(3\theta)$
$\cos^3(\theta)$	$=$	$\frac{3}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta)$

Table 4.1: Trigonometric identities used in Example 4.6.

$$\begin{aligned} 2A_T - A + \frac{1}{4}A(A^2 + B^2) &= 0 \\ 2B_T - B + \frac{1}{4}B(A^2 + B^2) &= 0 \end{aligned} \tag{4.64}$$

Since $B(0) = 0$ the second equation can be satisfied by $B(T) = 0$, leaving

$$\frac{dA}{dT} = \frac{1}{2}A\left(1 - \frac{1}{4}A^2\right). \tag{4.65}$$

For $T \to \infty$ this approaches a stationary solution A_∞ which satisfies

$$\frac{dA_\infty}{dT} = 0 = \frac{1}{2}A_\infty\left(1 - \frac{1}{4}A_\infty^2\right) \tag{4.66}$$

and hence[1] $A_\infty = 2$. In fact, it can be shown that the solution of Eq. (4.67) with initial condition $A(0) = 1$ is

$$A(T) = \frac{2}{\sqrt{1 + 3e^{-T}}}. \tag{4.67}$$

In summary we find the lowest order solution of the Van-der-Pol oscillator as

$$u(t) = A(T)\cos(\tau) + \mathcal{O}(\epsilon) = \frac{2}{\sqrt{1 + 3e^{-\epsilon t}}}\cos(t) + \mathcal{O}(\epsilon). \tag{4.68}$$

In Fig. 4.3 both the exact solution (obtained via numerical integration) and the envelope $A(T)$ of the approximate solution are shown. The agreement is very good. As expected, the asymptotic solution is reached over a time scale of $\sim 1/\epsilon \sim 10$.

[1] the other stationary solution $A_\infty = 0$ is not stable: for small amplitudes $A \ll 1$ we have $\frac{dA}{dt} \approx \frac{1}{2}A$ which is solved by $A(t) = A(0)\exp\left[\frac{1}{2}t\right]$, i.e. the solution grows exponentially.

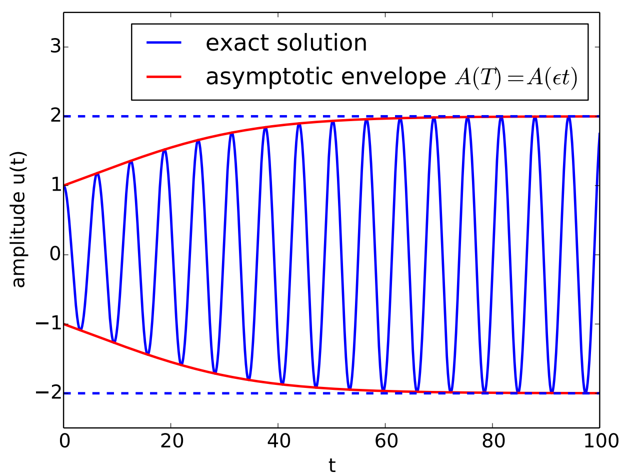

Figure 4.3: Exact solution of the Van-der-Pol oscilattor (blue) and envelope $A(T)$ of the lowest order approximation in Eq. (4.67) (red) for $\epsilon = 0.075 \ll 1$.

Chapter 5

Partial differential equations and boundary layers

5.1 Overview

A boundary value problem (BVP) is a differential equation with prescribed **boundary conditions**. It usually is the steady state of a evolutionary PDE problem. The typically encountered (second order) BVP which we study in this course is defined on the interval $[a, b]$ and takes the form

$$\epsilon u'' + f(x, u, u') = 0 \qquad \text{with } u(a) = \alpha,\ u(b) = \beta. \tag{5.1}$$

Observe that the highest order differential term is associated with the small parameter $0 < \epsilon \ll 1$. If we set $\epsilon = 0$ we get a first order PDE

$$f(x, u, u') = 0 \qquad \text{with } u(a) = \alpha,\ u(b) = \beta. \tag{5.2}$$

Since this is a first order PDE we can usually satisfy at most one of the two boundary conditions. We therefore have to use asymptotic methods to solve Eq. (5.1) for $\epsilon \to 0$. This allows us to introduce thin **boundary layers** of width $\mathcal{O}(\epsilon^\alpha)$ for some α close to the boundaries. The solution changes rapidly in those boundary layers, which allows satifying the boundary conditions for small ϵ. Those layers are physically relevant, e.g.

- Boundary behaviour close to aircraft wings
- Thin reaction zones in flames
- Skin effects in electromagnetism

5.2 Canonical examples

The solution procedure for PDEs of the form in Eq. (5.1) is to match asymptotic expansions as follows

- Construct on "outer" solution of Eq. (5.2) ($\epsilon = 0$) away from the boundaries.
- Rescale the independent variable $x = y\epsilon^{\alpha}$ close to the boundaries and construct an asymptotic "inner" solution in y close to the boundary.
- Match the two solutions in an "intermediate range".

To illustrate this we now look at two canonical examples.

Example 5.1. Consider

$$\epsilon u'' + u' + u = 1 \qquad \text{with } x \in [0,1] \text{ and } u(0) = u(1) = 0. \tag{5.3}$$

First we construct the outer expansion by setting $\epsilon = 0$. This requires solving

$$u' + u = 1 \qquad \text{with } u(0) = u(1) = 0. \tag{5.4}$$

The solution to Eq. (5.4) is

$$u_{\text{outer}}(x) = 1 + Ae^{-x}. \tag{5.5}$$

To find the constant A we use the boundary conditions

$$u_{\text{outer}}(0) = 1 + A, \qquad\qquad u_{\text{outer}}(1) = 1 + A/e \tag{5.6}$$

Unfortunately the lower boundary conditition $u(0) = 0$ gives $A = -1$ whereas the upper boundary condition $u(1) = 0$ gives $A = -e$, and hence we can not satisfy both. One idea to address this would be to expand the solution to the next order in ϵ, i.e. consider

$$u(x) = u_0(x) + \epsilon u_1(x) + \mathcal{O}(\epsilon^2). \tag{5.7}$$

Substituting into Eq. (5.3) and expanding gives

$$\begin{aligned} \mathcal{O}(\epsilon^0): &\qquad u_0' + u_0 = 1, \qquad \text{with } u_0(0) = u_0(1) = 0, \\ \mathcal{O}(\epsilon^1): &\qquad u_1' + u_1 = -u_0'', \qquad \text{with } u_1(0) = u_1(1) = 0. \end{aligned} \tag{5.8}$$

Unfortunately this does not solve the problem since the lowest order equation for u_0 is the same as Eq. (5.4).

To resolve this issue we consider the "inner" solution in the vincinity of the lower boundary $x = 0$ and assume that it changes rapidly. For this, introduce the new variable $y = x\epsilon^{-\alpha}$ where α is to be found. To re-write Eq. (5.3) in terms of this new variable we use

$$\frac{d}{dy} = \epsilon^{\alpha}\frac{d}{dx}, \qquad\qquad \frac{d^2}{dy^2} = \epsilon^{2\alpha}\frac{d^2}{dx^2} \tag{5.9}$$

and Eq. (5.3) becomes

$$\epsilon^{1-2\alpha}\frac{d^2u}{dy^2} + \epsilon^{-\alpha}\frac{du}{dy} + u = 1. \tag{5.10}$$

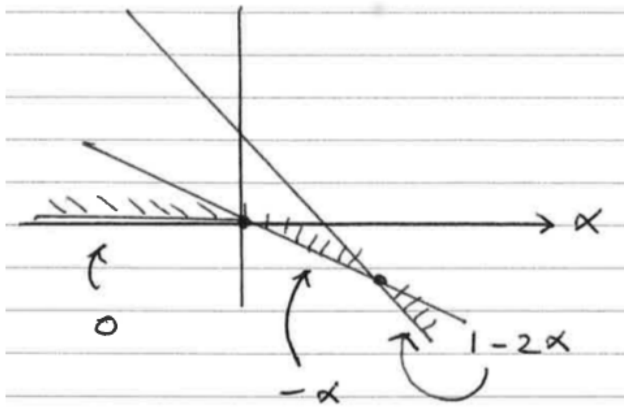

Figure 5.1: Newton polygon of Eq. (5.10).

The possible values of α are determined by balancing terms in Eq. (5.10) using the corresponding Newton polygon, which is shown in Fig. 5.1. The only non-trivial point satisfies $-\alpha = 1 - 2\alpha$, i.e. $\alpha = 1$. This corresponds to balancing the first and second derivative in Eq. (5.10). After multiplying the whole equation by ϵ this results in

$$\frac{d^2u}{dy^2} + \frac{du}{dy} + \epsilon u = \epsilon. \tag{5.11}$$

This equation can now be solved order-by-order. At leading order

$$\frac{d^2u}{dy^2} + \frac{du}{dy} = 0, \qquad \text{u(0)=0.} \tag{5.12}$$

This is solved by

$$u_{\text{inner}}(x) = C + De^{-y} = C + De^{-x/\epsilon}. \tag{5.13}$$

Note that we only specified a lower boundary condition in Eq. (5.12), at the upper boundary we require the solution to match the outer solution $u_{\text{outer}}(x)$ from Eq. (5.14). Using $u(0) = 0$ gives $C + D = 0$ and hence

$$u_{\text{inner}}(x) = C(1 - e^{-x/\epsilon}). \tag{5.14}$$

Note that as x increases, $y \to \infty$ and hence $u_{\text{inner}}(x) \to C$ as $x \to 1$. Coming back to the outer solution, the boundary condition at $x = 1$ can be satisfied if $A = -e$, i.e.

$$u_{\text{outer}}(x) = 1 - e^{1-x}. \tag{5.15}$$

Note that $u_{\text{outer}}(x) \to 1 - e$ for $x \to 0$. Comparing Eqs. (5.13) and (5.15) we see that the solutions match if we set $C = 1 - e$, i.e.

$$u_{\text{inner}}(x) = (1 - e)(1 - e^{-x/\epsilon}). \tag{5.16}$$

In summary we have

$$u(x) = \begin{cases} (1-e)(1-e^{-x/\epsilon}) & \text{for } x = \mathcal{O}(\epsilon), \\ 1-e^{1-x} & \text{for } x = \mathcal{O}(1). \end{cases} \tag{5.17}$$

This can be checked by finding the exact solution to Eq. (5.3) and expanding it in powers of ϵ. Making the ansatz $u(x) = 1 + ce^{\lambda x}$ we obtain an equation for λ

$$\epsilon\lambda^2 + \lambda + 1 = 0 \tag{5.18}$$

which has the two solutions

$$\lambda_1 = \frac{-1+\sqrt{1-4\epsilon}}{2\epsilon} = -1 + \mathcal{O}(\epsilon), \qquad \lambda_2 = \frac{-1-\sqrt{1-4\epsilon}}{2\epsilon} = -\frac{1}{\epsilon} + \mathcal{O}(1) \tag{5.19}$$

leading to

$$u(x) = 1 + c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \approx 1 + c_1 e^{-x} + c_2 e^{-x/\epsilon}. \tag{5.20}$$

The constants c_1 and c_2 are fixed by the boundary conditions

$$u(0) = 0 \Rightarrow 1 + c_1 + c_2 = 0, \qquad u(1) = 0 \Rightarrow 1 + c_1 e^{-1} + c_2 \underbrace{e^{-1/\epsilon}}_{\approx 0} = 0 \tag{5.21}$$

which gives $c_1 = -e$, $c_2 = e - 1$. The resulting solution

$$u(x) = 1 - e^{1-x} + (e-1)e^{-x/\epsilon}. \tag{5.22}$$

agrees with Eq. (5.17) for $x \to 0$ and $x \to 1$. The exact solution and the approximations from Eq. (5.17) are shown in Fig. 5.3. One obvious question to ask is as to why the boundary layer arises at $x = 0$ and not at the upper boundary ($x = 1$). To see this, construct a new variable z by stretching the region at $x \approx 1$ as $z = (1-x)\epsilon^{-1}$ with

$$\frac{du}{dz} = -\epsilon\frac{du}{dx}, \qquad \frac{d^2u}{dz^2} = \epsilon^2\frac{d^2u}{dx^2}. \tag{5.23}$$

Using this, Eq. (5.3) becomes

$$\frac{d^2u}{dz^2} - \frac{du}{dz} + \epsilon u = \epsilon. \tag{5.24}$$

Note that Eq. (5.24) differs from Eq. (5.11) by a minus sign in front of the first derivative term. Proceeding as above, we find that the leading order solution to Eq. (5.24) which satisfies the boundary condition at $z = 0$ is

$$\tilde{u}_{\text{inner}}(x) = \tilde{C}(1-e^{z}) = \tilde{C}(1-e^{(1-x)/\epsilon}). \tag{5.25}$$

However, now as $x \to 0$ we have $\tilde{u}_{\text{inner}}(x) \to -\tilde{C}e^{1/\epsilon}$ which blows up for small $\epsilon > 0$. We therefore can not match to the (bounded) outer solution in this case.

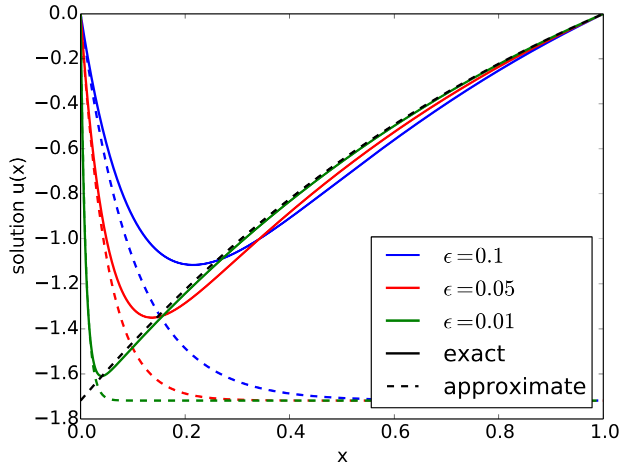

Figure 5.2: Exact and approximate solutions of Eq. (5.3)

Example 5.2. Consider

$$-\epsilon u'' + u = 1 \qquad \text{with u(0)=u(1)=0.} \tag{5.26}$$

For $\epsilon = 0$ the outer solution of Eq. (5.26) is readily found as

$$u_{\text{outer}}(x) = 1. \tag{5.27}$$

This does not satify any of the boundary conditions and we therefore need two boundary layers.

Lower boundary $(x = 0)$. Set $y = x\epsilon^{-\alpha}$, which results in

$$-\epsilon^{1-2\alpha}\frac{d^2u}{dy^2} + u = 1. \tag{5.28}$$

The two terms on the left hand side are balanced if $\alpha = \frac{1}{2}$, i.e. $x = \epsilon^{1/2}y$. Inserting this into Eq. (5.28) gives

$$-\frac{d^2u}{dy^2} + u = 1 \qquad \text{with } u(0) = 0. \tag{5.29}$$

The solution to this equation is

$$u_{\text{inner}}^{\text{lower}}(x) = 1 + Ae^{-y} + Be^{y} = 1 + A\exp\left[-\epsilon^{-1/2}x\right] + B\exp\left[\epsilon^{-1/2}x\right] \tag{5.30}$$

The upper boundary condition is treated by matching to the outer solution, i.e. by requiring that $u_{\text{inner}}^{\text{lower}}(x) \to u_{\text{outer}}(x) = 1$ for $x \to 1$, which corresponds to $y \to \infty$. This implies that $B = 0$. The lower boundary condition at $x = 0$ implies that $A = -1$ and hence the inner solution is

$$u_{\text{inner}}^{\text{lower}}(x) = 1 - e^{-y} = 1 - \exp\left[-\epsilon^{-1/2}x\right]. \tag{5.31}$$

Using a symmetry argument, the inner solution at the upper boundary is given by

$$u_{\text{inner}}^{\text{upper}}(x) = 1 - \exp\left[-\epsilon^{-1/2}(1-x)\right] \tag{5.32}$$

and in summary the approximate solution of Eq. (5.3) is

$$u(x) = \begin{cases} = 1 - \exp\left[-\epsilon^{-1/2}x\right] & \text{for } x = \mathcal{O}(\epsilon) \\ = 1 - \exp\left[-\epsilon^{-1/2}(1-x)\right] & \text{for } 1-x = \mathcal{O}(\epsilon) \\ = 1 & \text{otherwise.} \end{cases} \tag{5.33}$$

Again this can be checked by soluting Eq. (5.26) exactly, making the ansatz $u = 1 + ce^{\lambda x}$ this gives an algebraic equation for λ

$$-\epsilon\lambda^2 + 1 = 0 \tag{5.34}$$

which has the two solutions $\lambda_{1,2} = \pm\epsilon^{-1/2}$ and hence

$$u(x) = 1 + c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = 1 + c_1 \exp\left[-\epsilon^{-1/2}x\right] + c_2 \exp\left[\epsilon^{-1/2}x\right] \tag{5.35}$$

The upper boundary condition $u(1) = 0$ gives

$$0 = 1 + c_1 \exp\left[-\epsilon^{-1/2}\right] + c_2 \exp\left[\epsilon^{-1/2}\right] \approx 1 + c_2 \exp\left[\epsilon^{-1/2}\right] \tag{5.36}$$

where we have used the fact that $\exp\left[-\epsilon^{-1/2}\right] \to 0$ as $\epsilon \to 0$. This results in $c_2 \approx -\exp\left[-\epsilon^{-1/2}\right]$ The lower boundary condition $u(0) = 0$ gives

$$0 = 1 + c_1 + c_2 \approx 1 + c_1 - \exp\left[-\epsilon^{-1/2}\right] \approx 1 + c_1 \tag{5.37}$$

and hence $c_1 = -1$. Using those values for c_1 and c_2 in Eq. (5.35) gives

$$u(x) = 1 - \exp\left[-\epsilon^{-1/2}(1-x)\right] - \exp\left[-\epsilon^{-1/2}x\right] \tag{5.38}$$

which agrees with the approximate solution in Eq. (5.33) for $\epsilon \to 0$.

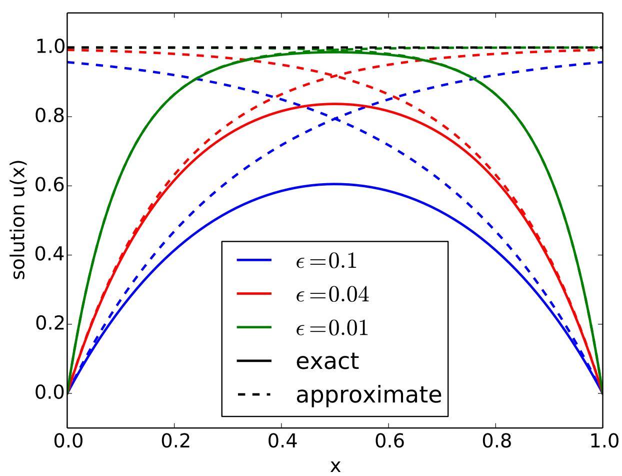

Figure 5.3: Exact and approximate solutions of Eq. (5.26)

Appendix A

Approximations in Stirling's formula

In this appendix we provide a more quantitative justification for the approximations in Section 3.2. Consider the function $f_n(x)$ defined in Eq. (3.32). Let $x = n + \sqrt{n}\tau$ and write

$$f_n(x) = f_n(n) + g_n(\tau) \tag{A.1}$$

where

$$g_n(\tau) = n \log\left(1 + \frac{\tau}{\sqrt{n}}\right) - \sqrt{n}\tau. \tag{A.2}$$

It can be shown that (see Fig. A.1)

- For negative τ the function $g_n(\tau)$ is bounded by

$$g_n(\tau) < -\frac{1}{2}\tau^2 \qquad \text{for } \tau < 0. \tag{A.3}$$

- For positive τ the function $g_n(\tau)$ is bounded by

$$g_n(\tau) \leq g_1(\tau) \qquad \text{for } n \geq 1 \text{ and } \tau > 0. \tag{A.4}$$

- $g_n(\tau)$ has a maximum at $\tau = 0$ and a Taylor-expansion around this point of

$$g_n(x) = -\frac{1}{2}\tau^2 + \mathcal{O}(\tau^3/\sqrt{n}). \tag{A.5}$$

The integral which we seek to calculate is

$$\int_0^\infty \exp\left(f_n(x)\right)\, dx = \exp\left(f_n(n)\right)\sqrt{n} \underbrace{\int_{-\sqrt{n}}^\infty \exp\left(g_n(\tau)\right)\, d\tau}_{I_n} \equiv I_n\sqrt{n}\exp\left(f_n(n)\right). \tag{A.6}$$

However, as $n \gg 1$ the contribution to I_n which comes from the intervals $[-\sqrt{n}, -n^{1/8}]$ and $[n^{1/8}, \infty]$ becomes exponentially small, as the following argument shows.

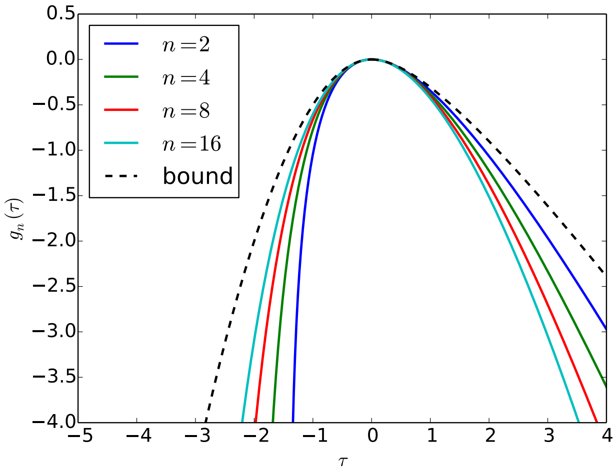

Figure A.1: Plot of the function $g_n(\tau)$ for different values of n. The upper bound is given by $-\frac{1}{2}\tau^2$ for $\tau < 0$ and $g_1(\tau)$ for $\tau > 0$.

For the interval $[-\sqrt{n}, -n^{1/8}]$ use Eq. (A.3) to bound $\exp(g_n(\tau))$ as follows

$$\begin{aligned}\int_{-\sqrt{n}}^{-n^{1/8}} \exp(g_n(\tau))\ d\tau &< \int_{-\infty}^{-n^{1/8}} \exp(g_n(\tau))\ d\tau < \int_{-\infty}^{-n^{1/8}} \exp\left(-\frac{\tau^2}{2}\right)\ d\tau \\ &= \int_{n^{1/8}}^{\infty} \exp\left(-\frac{\tau^2}{2}\right)\ d\tau < \int_{n^{1/8}}^{\infty} \tau \exp\left(-\frac{\tau^2}{2}\right)\ d\tau \\ &= \left[-\exp\left(-\frac{\tau^2}{2}\right)\right]_{n^{1/8}}^{\infty} = \exp\left(-\frac{n^{1/4}}{2}\right).\end{aligned} \tag{A.7}$$

For the interval $[n^{1/8}, \infty]$ use Eq. (A.4) to obtain

$$\begin{aligned}\int_{n^{1/8}}^{\infty} \exp(g_n(\tau))\ d\tau &< \int_{n^{1/8}}^{\infty} \exp(g_1(\tau))\ d\tau = \int_{n^{1/8}}^{\infty} (1+\tau)e^{-\tau}\ d\tau < 2\int_{n^{1/8}}^{\infty} \tau e^{-\tau}\ d\tau \\ &= 2\left[-(1+\tau)e^{-\tau}\right]_{n^{1/8}}^{\infty} - 2(1+n^{1/8})\exp\left(-n^{1/8}\right)\end{aligned} \tag{A.8}$$

However, in the remaining interval $[-n^{1/8}, n^{1/8}]$ the $\mathcal{O}(\tau^3/\sqrt{n})$ correction terms in Eq. (A.5) are $\mathcal{O}(n^{3/8}n^{-1/2}) = \mathcal{O}(n^{-1/8})$ and hence small for $n \gg 1$. This implies that we can replace $g_n(\tau) \to -\frac{\tau^2}{2}$ in the integral. It is also easy to see that after this replacement, extending the integral to $[-\infty, \infty]$ will only result in exponentially small corrections. In summary we find for the integral I_n

$$I_n \approx \int_{-n^{1/8}}^{n^{1/8}} (g_n(\tau))\ d\tau \approx \int_{-n^{1/8}}^{n^{1/8}} \left(-\frac{\tau^2}{2}\right)\ d\tau \approx \int_{-\infty}^{\infty} \left(-\frac{\tau^2}{2}\right)\ d\tau = \sqrt{2\pi}. \tag{A.9}$$