MA30044/MA40044/MA50181, Mathematical Methods I, 2020

Problem Sheet 4: Introduction to Fourier Transform

SOLUTIONS

Q1.

$$\text{(i)} \qquad \mathcal{F}[u](\omega) = \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}\omega x}\mathrm{e}^{2x}dx + \int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x}\mathrm{e}^{-x}dx = \frac{1}{2-\mathrm{i}\omega} + \frac{1}{1+\mathrm{i}\omega};$$

$$\text{(ii)} \qquad \mathcal{F}[u](\omega) = \int_{3}^{4} \mathrm{e}^{-\mathrm{i}\omega x}dx + 2\int_{4}^{5} \mathrm{e}^{-\mathrm{i}\omega x}dx = \frac{\mathrm{e}^{-4\mathrm{i}\omega} - \mathrm{e}^{-3\mathrm{i}\omega}}{-\mathrm{i}\omega} + 2\frac{\mathrm{e}^{-5\mathrm{i}\omega} - \mathrm{e}^{-4\mathrm{i}\omega}}{-\mathrm{i}\omega} = \frac{\mathrm{i}}{\omega}\Big(2\mathrm{e}^{-5\mathrm{i}\omega} - \mathrm{e}^{-4\mathrm{i}\omega} - \mathrm{e}^{-3\mathrm{i}\omega}\Big);$$

$$\begin{aligned}
\text{(iii)} \qquad \mathcal{F}[u](\omega) &= \int_{-2}^{0}\Big(1+\frac{x}{2}\Big)\mathrm{e}^{-\mathrm{i}\omega x}dx + \int_{0}^{2}\Big(1-\frac{x}{2}\Big)\mathrm{e}^{-\mathrm{i}\omega x}dx \\
&= \frac{1}{-\mathrm{i}\omega}\Big(1+\frac{x}{2}\Big)\mathrm{e}^{-\mathrm{i}\omega x}\Big|_{-2}^{0} + \frac{1}{-\mathrm{i}\omega}\Big(1-\frac{x}{2}\Big)\mathrm{e}^{-\mathrm{i}\omega x}\Big|_{0}^{2} + \frac{1}{2\mathrm{i}\omega}\int_{-2}^{0}\mathrm{e}^{-\mathrm{i}\omega x}dx - \frac{1}{2\mathrm{i}\omega}\int_{0}^{2}\mathrm{e}^{-\mathrm{i}\omega x}dx \\
&= \frac{1}{2\omega^2}\mathrm{e}^{-\mathrm{i}\omega x}\Big|_{-2}^{0} - \frac{1}{2\omega^2}\mathrm{e}^{-\mathrm{i}\omega x}\Big|_{0}^{2} = \frac{1}{2\omega^2}\Big(1 - \mathrm{e}^{2\mathrm{i}\omega} - \mathrm{e}^{-2\mathrm{i}\omega} + 1\Big) = 2\frac{\sin^2(\omega)}{\omega^2};
\end{aligned}$$

$$\begin{aligned}
\text{(iv)} \qquad \mathcal{F}[u](\omega) &= \int_{0}^{\infty}\mathrm{e}^{-x}\cos(x)\mathrm{e}^{-\omega x} = \frac{1}{2}\int_{0}^{\infty}\Big(\mathrm{e}^{-x+\mathrm{i}x-\mathrm{i}\omega x} + \mathrm{e}^{-x-\mathrm{i}x-\mathrm{i}\omega x}\Big)dx \\
&= \frac{1}{2}\int_{0}^{\infty}\Big(\mathrm{e}^{(-1+\mathrm{i}-\mathrm{i}\omega)x} + \mathrm{e}^{(-1-\mathrm{i}-\mathrm{i}\omega)x}\Big)dx \\
&= \frac{1}{2}\Big(\frac{1}{1+\mathrm{i}(\omega-1)} + \frac{1}{1+\mathrm{i}(\omega+1)}\Big) = \frac{1+\mathrm{i}\omega}{(1+\mathrm{i}\omega)^2+1}.
\end{aligned}$$

Q2.

$$\text{(i)} \qquad \mathcal{F}[u](\omega) = \int_{-\infty}^{0}\mathrm{e}^{\alpha x}\mathrm{e}^{-\mathrm{i}\omega x}dx + \int_{0}^{\infty}\mathrm{e}^{-\alpha x}\mathrm{e}^{-\mathrm{i}\omega x}dx = \frac{1}{\alpha-\mathrm{i}\omega} - \frac{1}{-\alpha-\mathrm{i}\omega} = \frac{2\alpha}{\alpha^2+\omega^2};$$

$$\text{(ii)} \qquad \mathcal{F}[u](0) = \frac{2\alpha}{\alpha^2} = \frac{2}{\alpha} \to \infty \quad \text{as} \quad \alpha \to 0; \tag{1}$$

$$\mathcal{F}[u](\omega) = \frac{2\alpha}{\alpha^2+\omega^2} < \frac{2\alpha}{\omega^2} \to 0 \quad \text{as} \quad \alpha \to 0 \quad \text{if} \quad \omega \neq 0; \tag{2}$$

$$\int_{-\infty}^{\infty}\mathcal{F}[u] = \int_{-\infty}^{\infty}\frac{2\alpha d\omega}{\alpha^2+\omega^2} = 2\tan^{-1}\Big(\frac{\omega}{\alpha}\Big)\Big|_{-\infty}^{\infty} = 2\pi;$$

(iii) Notice that $u(x) \to 1$ as $\alpha \to 0$ for all $x \in \mathbb{R}$. The properties (1) and (2) imply that the Fourier transform of the function $\mathbb{1}(x) = 1$, $x \in \mathbb{R}$, is equal to $2\pi\delta$, where δ is the "delta-function" discussed in the lectures.

Q3.

$$\text{(i)} \qquad \mathcal{F}[u](\omega) = 3\int_{-2}^{2} \mathrm{e}^{-\mathrm{i}\omega x}\,dx = \frac{3}{-\mathrm{i}\omega}\left(\mathrm{e}^{-2\mathrm{i}\omega} - \mathrm{e}^{2\mathrm{i}\omega}\right) = \frac{6\sin(2\omega)}{\omega}.$$

(ii) Using First Shift Theorem:

$$\mathcal{F}[v](\omega) = \mathcal{F}\left[\mathrm{e}^{-2\mathrm{i}x}u(x)\right] = \mathcal{F}[u](\omega+2) = \frac{6\sin(2\omega+4)}{\omega+2}.$$

Direct calculation:

$$\mathcal{F}[v](\omega) = 3\int_{-2}^{2} \mathrm{e}^{-2\mathrm{i}x}\mathrm{e}^{-\mathrm{i}\omega x}\,dx = \frac{1}{-\mathrm{i}(\omega+2)}\left(\mathrm{e}^{-4\mathrm{i}-2\omega} - \mathrm{e}^{4\mathrm{i}+2\omega}\right) = \frac{6\sin(2\omega+4)}{\omega+2}.$$

$$\text{(iii)} \quad u(\omega) = \frac{6}{10+2\omega+\omega^2} = \frac{6}{(\omega+1)^2+9} = \frac{2\alpha}{(\omega+1)^2+\alpha^2},$$

for $\alpha = 3$. Hence, $u(\omega) = \widetilde{u}(\omega+1)$, where $\widetilde{u}$ is the Fourier transform of $\exp(-3|x|)$, $x \in \mathbb{R}$. Hence, u is the Fourier Transform of $\exp(-\mathrm{i}x)\exp(-3|x|)$, $x \in \mathbb{R}$.

Q4.

(i) Notice that

$$w(x) = \mathrm{e}^{-(x+4)}u(x+4) = v(x+4), \qquad x \in \mathbb{R},$$

and therefore

$$\mathcal{F}[w](\omega) = \mathrm{e}^{4\mathrm{i}\omega}\mathcal{F}[v](\omega) = \frac{\mathrm{e}^{4\mathrm{i}\omega}}{1+\mathrm{i}\omega}, \qquad \omega \in \mathbb{R}.$$

(ii) We know that the function $\sin(2\omega)/\omega$ is the Fourier Transform of

$$g(x) = \begin{cases} 3, & |x| < 2, \\ \frac{3}{2}, & |x| = 2, \\ 0, & |x| > 2, \end{cases}$$

hence u is the Fourier Transform of

$$g(x-4) = \begin{cases} 3, & 2 < x < 6, \\ \frac{3}{2}, & \text{for } x = 2 \text{ or } x = 6, \\ 0, & x < 2 \text{ or } x > 6. \end{cases}$$

Q5.

$$\text{(i)} \quad \frac{dF}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2}\mathrm{e}^{-\mathrm{i}\omega x}\,dx = \int_{-\infty}^{\infty} \frac{d}{d\omega}\left(\mathrm{e}^{-ax^2}\mathrm{e}^{-\mathrm{i}\omega x}\right)dx = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x}x\mathrm{e}^{-ax^2}\,dx$$

$$\text{(ii)} \quad \frac{dF}{d\omega} = \frac{\mathrm{i}}{2a}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x}(-2ax)\mathrm{e}^{-ax^2}\,dx = \frac{\mathrm{i}}{2a}\left\{\left[\mathrm{e}^{-\mathrm{i}\omega x}\mathrm{e}^{-ax^2}\right]_{-\infty}^{\infty} + \mathrm{i}\omega\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x}\mathrm{e}^{-ax^2}\,dx\right\}$$

Using the fact that

$$\left[\mathrm{e}^{-\mathrm{i}\omega x}\mathrm{e}^{-ax^2}\right]_{-\infty}^{\infty} = 0$$

and

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega x}\mathrm{e}^{-ax^2}\,dx = \mathcal{F}[u](\omega) = F(\omega),$$

we obtain

$$\frac{dF}{d\omega} = -\frac{\omega}{2a}F(\omega), \tag{3}$$

as required.

(iii) Rewrite (3) in the form

$$\frac{dF}{F} = -\frac{\omega d\omega}{2a}.$$

Integrating both sides of the last equation, we obtain

$$\log\frac{F(\omega)}{F(0)} = -\frac{\omega^2}{4a},$$

and hence

$$F(\omega) = F(0)\exp\left(-\frac{\omega^2}{4a}\right) = \sqrt{\frac{\pi}{a}}\exp\left(-\frac{\omega^2}{4a}\right).$$

Q6.

$$\text{(i)} \qquad I = \int_0^\infty \frac{\sin(ax)}{x}dx = \int_0^\infty \frac{\sin(ax)}{ax}d(ax) = \int_0^\infty \frac{\sin(y)}{y}d(y)$$

$$\begin{aligned}
\text{(ii)} \qquad J(y) = \int_0^\infty e^{-xy}\sin(x)dx &= \frac{1}{2\mathrm{i}}\int_0^\infty \left(\mathrm{e}^{-xy+\mathrm{i}x} - \mathrm{e}^{-xy-\mathrm{i}x}\right)dx \\
&= \frac{1}{2\mathrm{i}}\left\{\frac{1}{\mathrm{i}-y}\mathrm{e}^{-xy+\mathrm{i}x}\Big|_{x=0}^{x=\infty} - \frac{1}{-\mathrm{i}-y}\mathrm{e}^{-xy-\mathrm{i}x}\Big|_{x=0}^{x=\infty}\right\} \\
&= \frac{1}{2\mathrm{i}}\left\{\frac{1}{\mathrm{i}-y} - \frac{1}{-\mathrm{i}-y}\Big|_{x=0}^{x=\infty}\right\} = \frac{1}{1+y^2}.
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} \qquad \int_0^\infty J &= \int_0^\infty\int_0^\infty e^{-xy}\sin(x)dxdy = \int_0^\infty\left(\int_0^\infty e^{-xy}\sin(x)dy\right)dx \\
&= \int_0^\infty\left[-\frac{\sin(x)}{x}\mathrm{e}^{-xy}\right]_{x=0}^{x=\infty}dx = \int_0^\infty \frac{\sin(x)}{x}dx = I.
\end{aligned}$$

At the same time,

$$\int_0^\infty dy 1 + y^2 = \tan^{-1}(x)\big|_0^\infty = \frac{\pi}{2}.$$

Therefore, $I = \pi/2$.