

You should attempt the non-starred parts of Questions 1,2,3.

1. The Legendre differential equation

$$-((1-x^2)u'(x))' = \lambda u(x), \quad x \in (-1,1), \quad (1)$$

arises in the study of the solutions of Laplace's equation in spherical coordinates. As this is a singular problem, it does not have boundary conditions in the sense we have discussed in the lectures. These are replaced by the conditions that $u(x)$ is bounded as $|x| \rightarrow 1$. Its eigenfunctions, the so-called *Legendre polynomials*, have numerous applications in applied mathematics and numerical analysis.

(i) Show that the following polynomials are eigenfunctions of (1) and find the corresponding eigenvalues $\lambda_0, \lambda_1, \lambda_2$:

$$\phi_0(x) = 1, \quad \phi_1(x) = x, \quad \phi_2(x) = \frac{1}{2}(3x^2 - 1).$$

(ii) Check directly the orthogonality of the system $\{\phi_0, \phi_1, \phi_2\}$ with weight function $r(x) = 1$.

(iii) By finding appropriate values for a, b, d find λ_3, ϕ_3 and λ_4, ϕ_4 given that

$$\phi_3(x) = x^3 + ax \quad \text{and} \quad \phi_4(x) = x^4 + bx^2 + d.$$

*(iv) The eigenfunctions of any SL system have the zero interlacing property. This means that ϕ_n has precisely n zeros in $(-1,1)$ and each of its zeros lies between two adjacent zeros of ϕ_{n+1} . Calculate the zeros of $\phi_1, \phi_2, \phi_3, \phi_4$ and verify these properties.

NOTE: The zeros of the Legendre polynomials play a central role in a range of numerical techniques and lie at the heart of quadrature, collocation, and finite element methods.

2. The Chebyshev differential equation arises in approximation theory and is given by

$$-(1-x^2)u''(x) + xu'(x) = \lambda u(x), \quad x \in (-1,1),$$

with the condition that $u'(x)$ is bounded as $|x| \rightarrow 1$.

(i) Put this equation into Sturm-Liouville form and show that $r(x) = 1/\sqrt{1-x^2}$.

(ii) Show that the differential equation has linearly independent solutions given by $\cos(\sqrt{\lambda}\cos^{-1}(x))$ and $\sin(\sqrt{\lambda}\sin^{-1}(x))$.

(iii) Show that, with the boundedness condition on u' , this equation has eigenfunctions and eigenvalues given by

$$T_n(x) = \cos(n\cos^{-1}(x)), \quad \lambda_n = n^2, \quad n = 0, 1, 2, \dots$$

(iv) By using a suitable substitution, show directly that the functions T_n are orthogonal with weight $r(x) = 1/\sqrt{1-x^2}$.

(v) The functions $T_n(x)$ are Chebyshev polynomials. Use trigonometric identities to show that $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, and find polynomial expressions for T_3 and T_4 .

3. The **Hermite differential equation** arises in quantum mechanics, statistics, and also in problems connected to heat conduction. It takes the form

$$-u''(x) + \frac{x}{2}u'(x) = \lambda u(x), \quad x \in \mathbb{R}. \quad (2)$$

(i) Put (2) into SL form, and hence show that the associated weight function is

$$r(x) = e^{-x^2/4}, \quad x \in \mathbb{R}. \quad (3)$$

(ii) The eigenfunctions are the *Hermite polynomials* H_n . Given that $H_n(x) = x^n + p(x)$, where $p \in \mathbb{P}_{n-1}$ is a polynomial of degree $n-1$, show that

$$\lambda_n = \frac{n}{2}, \quad n = 0, 1, 2, \dots$$

Hence calculate H_0, H_1, H_2, H_3 .

(iv) Using integration by parts, show that H_0, H_1, H_2 are orthogonal with respect to r given by (3).

*(v) Verify that the zeros of H_0, H_1, H_2, H_3 interlace.

***4.** An orthogonal system $\{\phi_n\}_{n \in \mathbb{N}}$ with continuous weight function $r > 0$ is said to be *closed* on $[a, b]$ if for any piecewise continuous function f defined over $[a, b]$ the equalities

$$\langle f, \phi_n \rangle_r = 0 \quad \forall n \in \mathbb{N}$$

imply $f = 0$. Show that if $\{\phi_n\}_{n \in \mathbb{N}}$ is complete, then it is closed.