MA30044/MA40044/MA50181, Mathematical Methods 1, 2021
Problem Sheet 9: Linear second-order PDEs

Feedback hand-in: Wednesday 8 Dec 2021 4pm

You should attempt Questions 1,2 and 3.

1. Solve the second-order PDE

$$u_{xx} + 3u_{xy} + 2u_{yy} = 0,$$

subject to the initial conditions on the line $y = 0$:

$$u(x,0) = \cos(x), \quad u_y(x,0) = x, \qquad x \in \mathbb{R}.$$

2. Solve the second-order PDE

$$u_{xx} + 3u_{xy} - 4u_{yy} = 0,$$

subject to the initial conditions on the line $x = 0$:

$$u(0,y) = 0, \quad u_x(0,y) = y, \qquad y \in \mathbb{R}.$$

3. Let u satisfy the boundary value problem,

$$\begin{aligned} u_{xy} &= 0, \\ u(x,0) &= \phi(x), \quad u_y(x,0) = \psi(x), \qquad x \in \mathbb{R}. \end{aligned}$$

(i) By finding the general solution of $u_{xy} = 0$ show that this system has no solution unless ψ is a constant function.

(ii) Show further that if ψ is constant then the system has an infinite number of solutions.

4. Consider the PDE with constant coefficients,

$$au_{xx} + bu_{xy} + cu_{yy} = 0, \qquad a \neq 0.$$

Recall from the notes that this is of *parabolic type* if

$$b^2 - 4ac = 0.$$

Assuming this to hold, let λ_* be the unique root of the quadratic equation

$$a\lambda_*^2 + b\lambda_* + c = 0.$$

(i) Show that if we set

$$s = x, \quad t = y + \lambda_* x, \quad v(s,t) = u(x,y),$$

then v satisfies the equation

$$v_{ss} = 0.$$

(ii) Integrating this equation twice, show that the solution of the original parabolic PDE is

$$u(x,y) = f(y + \lambda_* x) + x g(y + \lambda_* x),$$

where f and g are arbitrary functions.