MA30044/MA40044/MA50181, Mathematical Methods 1, 2021
Problem Sheet 2: Advanced Sturm-Liouville Equations

Feedback hand-in: Wednesday 20 Oct 2021 4pm

You should attempt the non-starred parts of Questions 1,2,3.

1. The **Legendre differential equation**

$$-\left((1-x^2)u'(x)\right)' = \lambda u(x), \qquad x \in (-1,1), \tag{1}$$

arises in the study of the solutions of Laplace's equation in spherical coordinates. As this is a singular problem, it does not have boundary conditions in the sense we have discussed in the lectures. These are replaced by the conditions that $u(x)$ is bounded as $|x| \to 1$. Its eigenfunctions, the so-called *Legendre polynomials*, have numerous applications in applied mathematics and numerical analysis.

(i) Show that the following polynomials are eigenfunctions of (1) and find the corresponding eigenvalues $\lambda_0, \lambda_1, \lambda_2$:

$$\phi_0(x) = 1, \quad \phi_1(x) = x, \quad \phi_2(x) = \frac{1}{2}(3x^2 - 1).$$

(ii) Check directly the orthogonality of the system $\{\phi_0, \phi_1, \phi_2\}$ with weight function $r(x) = 1$.

(iii) By finding appropriate values for a, b, d find λ_3, ϕ_3 and λ_4, ϕ_4 given that

$$\phi_3(x) = x^3 + ax \quad \text{and} \quad \phi_4(x) = x^4 + bx^2 + d.$$

*(iv) The eigenfunctions of any SL system have the zero interlacing property. This means that ϕ_n has precisely n zeros in $(-1,1)$ and each of its zeros lies between two adjacent zeros of ϕ_{n+1}. Calculate the zeros of $\phi_1, \phi_2, \phi_3, \phi_4$ and verify these properties.

NOTE: The zeros of the Legendre polynomials play a central role in a range of numerical techniques and lie at the heart of quadrature, collocation, and finite element methods.

2. The **Chebyshev differential equation** arises in approximation theory and is given by

$$-(1-x^2)u''(x) + xu'(x) = \lambda u(x), \quad x \in (-1,1),$$

with the condition that $u'(x)$ is bounded as $|x| \to 1$.

(i) Put this equation into Sturm-Liouville form and show that $r(x) = 1/\sqrt{1-x^2}$.

(ii) Show that the differential equation has linearly independent solutions given by $\cos(\sqrt{\lambda}\cos^{-1}(x))$ and $\sin(\sqrt{\lambda}\sin^{-1}(x))$.

(iii) Assume that the eigenfunctions take the form,

$$u(x) = a\cos(\sqrt{\lambda}\cos^{-1}(x)),$$

where a is constant and $\lambda \geq 0$.

(a) Determine conditions that would guarantee that u' is bounded at $x=\pm 1$.

(b) Show that, with the boundedness condition on u', this equation has eigenfunctions and eigenvalues given by

$$T_n(x)=\cos\big(n\cos^{-1}(x)\big), \quad \lambda_n=n^2, \qquad n=0,1,2,...$$

(iv) By using a suitable substitution, show directly that the functions T_n are orthogonal with weight $r(x)=1/\sqrt{1-x^2}$.

(v) The functions $T_n(x)$ are Chebyshev polynomials. Use trigonometric identities to show that $T_0(x)=1$, $T_1(x)=x$, $T_2(x)=2x^2-1$, and find polynomial expressions for T_3 and T_4.

3. The **Hermite differential equation** arises in quantum mechanics, statistics, and also in problems connected to heat conduction. It takes the form

$$-u''(x)+\frac{x}{2}u'(x)=\lambda u(x), \quad x\in\mathbb{R}. \tag{2}$$

(i) Put (2) into SL form, and hence show that the associated weight function is

$$r(x)=e^{-x^2/4}, \qquad x\in\mathbb{R}. \tag{3}$$

(ii) The eigenfunctions are the *Hermite polynomials* H_n. Given that $H_n(x)=x^n+p(x)$, where $p\in\mathbb{P}_{n-1}$ is a polynomial of degree $n-1$, show that

$$\lambda_n=\frac{n}{2}, \quad n=0,1,2,...$$

Hence calculate H_0,H_1,H_2,H_3.

(iv) Using integration by parts, show that H_0,H_1,H_2 are orthogonal with respect to r given by (3).

*(v) Verify that the zeros of H_0,H_1,H_2,H_3 interlace.

***4.** An orthogonal system $\{\phi_n\}_{n\in\mathbb{N}}$ with continuous weight function $r>0$ is said to be *closed* on $[a,b]$ if for any piecewise continuous function f defined over $[a,b]$ the equalities

$$\langle f,\phi_n\rangle_r=0 \quad \forall n\in\mathbb{N}$$

imply $f=0$. Show that if $\{\phi_n\}_{n\in\mathbb{N}}$ is complete, then it is closed.