MA30044/MA40044/MA50181, Mathematical Methods I, 2020

Problem Sheet 5: Convolutions and Distributions

SOLUTIONS

Q1.

Note first that

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{x-3}^{x+1} f. \tag{1}$$

Furthermore, the function f takes non-zero values only for $y \in [0,3]$, hence the integral (1) is zero unless the interval $[0,3]$ overlaps with the interval $[x-3, x+1]$, i.e. $(f*g)(x) = 0$ for $x \leq -1$ and for $x \geq 6$. For $x \in (-1,6)$, there are 3 possibilities:

$$\begin{aligned}
-1 < x < 2 \quad &\Longrightarrow \quad [0,3] \cap [x-3, x+1] = [0, x+1] \\
&\Longrightarrow \quad (f*g)(x) = \int_0^{x+1} \frac{2y}{3} dy = \left.\frac{x^2}{3}\right|_0^{x+1} = \frac{1}{3}(x+1)^2; \\
2 \leq x < 3 \quad &\Longrightarrow \quad [0,3] \cap [x-3, x+1] = [0,3] \\
&\Longrightarrow \quad (f*g)(x) = \int_0^{3} \frac{2y}{3} dy = \left.\frac{x^2}{3}\right|_2^{3} = 3; \\
3 \leq x < 6 \quad &\Longrightarrow \quad [0,3] \cap [x-3, x+1] = [x-3, 3] \\
&\Longrightarrow \quad (f*g)(x) = \int_{x-3}^{3} \frac{2y}{3} dy = \left.\frac{x^2}{3}\right|_{x-3}^{3} = \frac{1}{3}x(6-x).
\end{aligned}$$

In conclusion,

$$(f*g)(x) = \begin{cases} \frac{1}{3}(x+1)^2, \quad -1 \leq x < 2, \\ 3, \quad 2 \leq x < 3, \\ \frac{1}{3}x(6-x), \quad 3 \leq x < 6, \\ 0, \quad x \in (-\infty, -1) \cup [6, \infty). \end{cases}$$

Q2.

(i) Similarly to Q1, we write

$$(f * f)(x) = \int_{-\infty}^{\infty} f(y)f(x-y)dy = \int_{-1}^{1} f(x-y)dy \tag{2}$$

and notice that $f(x-y) \neq 0$ of $-1 \leq < 0$, i.e. $x-1 \leq y < x+1$. It follows that the integral (2) vanishes if the intervals $[-1,1]$ and $[x-1, x+1]$ do not overlap, i.e. $(f*f)(x) = 0$ for $x \leq -2$

and for $x \geq 2$. For those values of x for which the mentioned intervals do overlap, we have:

$$\begin{aligned}
-2 < x < 0 \quad &\Longrightarrow \quad [-1,1] \cap [x-1, x+1] = [-1, x+1] \\
&\Longrightarrow \quad (f * f)(x) = \int_{-1}^{x+1} dy = x + 2; \\
0 \leq x < 2 \quad &\Longrightarrow \quad [-1,1] \cap [x-1, x+1] = [x-1, 1] \\
&\Longrightarrow \quad (f * f)(x) = \int_{x-1}^{1} dy = 2 - x.
\end{aligned}$$

In conclusion,

$$(f * f)(x) = \begin{cases} 2 - |x|, & |x| < 2, \\ 0, & |x| \geq 2. \end{cases}$$

(ii) $$\begin{aligned}
\mathcal{F}[f * f](\omega) &= \int_{-2}^{0} (2+x)\mathrm{e}^{-\mathrm{i}\omega x} dx + \int_{0}^{2} (2-x)\mathrm{e}^{-\mathrm{i}\omega x} dx \\
&= 2\int_{-2}^{2} \mathrm{e}^{-\mathrm{i}\omega x} dx + 2\int_{-2}^{0} x\mathrm{e}^{-\mathrm{i}\omega x} dx - \int_{0}^{2} x\mathrm{e}^{-\mathrm{i}\omega x} dx \\
&= 2\frac{1}{-\mathrm{i}\omega}\left(\mathrm{e}^{-2\mathrm{i}\omega} - \mathrm{e}^{2\mathrm{i}\omega}\right) + \frac{1}{-\mathrm{i}\omega} x\mathrm{e}^{-\mathrm{i}\omega x}\Big|_{-2}^{0} - \frac{1}{-\mathrm{i}\omega} x\mathrm{e}^{-\mathrm{i}\omega x}\Big|_{0}^{2} + \frac{1}{\mathrm{i}\omega}\int_{-2}^{0} \mathrm{e}^{-\mathrm{i}\omega x} dx - \frac{1}{\mathrm{i}\omega}\int_{0}^{2} \mathrm{e}^{-\mathrm{i}\omega x} dx \\
&= \frac{4\sin(2\omega)}{\omega} - \frac{2}{\mathrm{i}\omega}\mathrm{e}^{2\mathrm{i}\omega} + \frac{2}{\mathrm{i}\omega}\mathrm{e}^{-2\mathrm{i}\omega} + \frac{1}{\mathrm{i}\omega}\left\{\frac{1}{-\mathrm{i}\omega}\mathrm{e}^{-\mathrm{i}\omega x}\Big|_{-2}^{0} - \frac{1}{-\mathrm{i}\omega}\mathrm{e}^{-\mathrm{i}\omega x}\Big|_{0}^{2}\right\} \\
&= \frac{4\sin(2\omega)}{\omega} - \frac{4\sin(2\omega)}{\omega} - + \left[1 - \mathrm{e}^{2\mathrm{i}\omega} - \mathrm{e}^{-2\mathrm{i}\omega} + 1\right] = \frac{4\sin^2(\omega)}{\omega^2}.
\end{aligned} \tag{3}$$

(iii) $$\mathcal{F}[f](\omega) = \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}\omega x} dx + \frac{1}{-\mathrm{i}\omega}\left(\mathrm{e}^{-\mathrm{i}\omega} - \mathrm{e}^{\mathrm{i}\omega}\right) = \frac{2\sin(\omega)}{\omega}.$$

Comparing this with (3), we obtain $\mathcal{F}(f * f)(\omega) = (\mathcal{F}[f])^2(\omega)$, as required.

Q3.

We know that the Fourier Transform of the function

$$w(x) = \begin{cases} \mathrm{e}^{-x}, & x \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

is given by

$$\mathcal{F}[w](\omega) = \frac{1}{1 + \mathrm{i}\omega}.$$

It follows hat

$$\mathcal{F}[u](\omega) = \big(\mathcal{F}[w](\omega)\big)^2,$$

and therefore

$$u(x) = (w * w)(x) = \int_{-\infty}^{\infty} w(y)w(x-y)dy = \int_{0}^{\infty} \mathrm{e}^{-y} w(x-y)dy. \tag{4}$$

Suppose first that $x > 0$. As $w(x-y) = 0$ for $x - y < 0$, i.e. $y > x$, the formula (4) implies

$$u(x) = \int_0^x \mathrm{e}^{-y}\mathrm{e}^{-(x-y)}dy = \mathrm{e}^{-x}\int_0^x dy = x\mathrm{e}^{-x}.$$

If $x < 0$ then $w(x-y) = 0$, since $x - y < 0$ for $y > 0$, and $w(z) = 0$ for $z < 0$.

Finally, the function u is continuous at zero, so $u(0) = 0$. This concludes the derivation of the required formula.

Q4.

(i) Notice that on the one hand

$$\begin{aligned}\mathcal{F}[u'' + 5u' + 6u](\omega) &= -\omega^2\mathcal{F}[u](\omega) + 5\mathrm{i}\omega\mathcal{F}[u](\omega) + 6\mathcal{F}[u](\omega)\\ &= (-\omega^2 + 5\mathrm{i}\omega + 6)\mathcal{F}[u](\omega) = -(\omega - 2\mathrm{i})(\omega - 3\mathrm{i})\mathcal{F}[u](\omega),\end{aligned}$$

and on the other hand,

$$\mathcal{F}\Big[H(x)\mathrm{e}^{-x}\Big] = \frac{1}{1+\mathrm{i}\omega}.$$

Hence,

$$\mathcal{F}[u](\omega) = \frac{1}{(1+\mathrm{i}\omega)(2+\mathrm{i}\omega)(3+\mathrm{i}\omega)} = \frac{1/2}{1+\mathrm{i}\omega} - \frac{1}{2+\mathrm{i}\omega} + \frac{1/2}{3+\mathrm{i}\omega},$$

and therefore, taking the Inverse Fourier Transform,

$$u(x) = H(x)\left[\frac{\mathrm{e}^{-x}}{2} - \mathrm{e}^{-2x} + \frac{\mathrm{e}^{-3x}}{2}\right].$$

(ii) Taking the Fourier Transform of both sides of the equation, we obtain

$$(-\omega^2 + 7\mathrm{i}\omega + 12)\mathcal{F}[u](\omega) = \frac{1}{2}\left(\frac{1}{2+\mathrm{i}(\omega-1)} + \frac{1}{2+\mathrm{i}(\omega+1)}\right).$$

Hence,

$$\begin{aligned}\mathcal{F}[u](\omega) &= \frac{1}{2}\left(\frac{1}{(\mathrm{i}\omega+3)(\mathrm{i}\omega+4)(2+\mathrm{i}(\omega-1))} + \frac{1}{(\mathrm{i}\omega+3)(\mathrm{i}\omega+4)(2+\mathrm{i}(\omega+1))}\right)\\ &= \frac{-1/2}{\mathrm{i}\omega+3} + \frac{2/5}{\mathrm{i}\omega+4} + \frac{1}{2(1+\mathrm{i})(2+\mathrm{i})}\cdot\frac{1}{2+\mathrm{i}(\omega-1)} + \frac{1}{2(1-\mathrm{i})(2-\mathrm{i})}\cdot\frac{1}{2+\mathrm{i}(\omega+1)}.\end{aligned} \tag{5}$$

Taking the Inverse Fourier Transform of both sides of (5), yields

$$\begin{aligned}u(x) &= -\frac{1}{2}\mathrm{e}^{-3x} + \frac{2}{5}\mathrm{e}^{-4x} + \frac{\mathrm{e}^{\mathrm{i}x}\mathrm{e}^{-2x}}{2(1+\mathrm{i})(2+\mathrm{i})} + \frac{\mathrm{e}^{-\mathrm{i}x}\mathrm{e}^{-2x}}{2(1-\mathrm{i})(2-\mathrm{i})}\\ &= -\frac{1}{2}\mathrm{e}^{-3x} + \frac{2}{5}\mathrm{e}^{-4x} + \frac{\mathrm{e}^{-2x}}{2}\left[\frac{\mathrm{e}^{\mathrm{i}x}}{1+3\mathrm{i}} + \frac{\mathrm{e}^{-\mathrm{i}x}}{1-3\mathrm{i}}\right]\\ &= -\frac{1}{2}\mathrm{e}^{-3x} + \frac{2}{5}\mathrm{e}^{-4x} + \frac{\mathrm{e}^{-2x}}{10}\big[\cos(x) + 3\sin(x)\big].\end{aligned}$$

(iii) Consider first the equation

$$u''(x) + 4u'(x) + 3u(x) = H(x)\mathrm{e}^{-4x}$$

As before, we write

$$(\omega^2 + 4\mathrm{i}\omega + 3)\mathcal{F}[u](\omega) = \frac{1}{4 + \mathrm{i}\omega},$$

and as $-\omega^2 + 4\mathrm{i}\omega + 3 = (\mathrm{i}\omega + 1)(\mathrm{i}\omega + 3)$, we obtain

$$\mathcal{F}[u](\omega) = \frac{1}{(4 + \mathrm{i}\omega)(\mathrm{i}\omega + 1)(\mathrm{i}\omega + 3)} = \frac{1/3}{4 + \mathrm{i}\omega} + \frac{1/6}{1 + \mathrm{i}\omega} + \frac{-1/2}{3 + \mathrm{i}\omega}. \tag{6}$$

It follows that

$$u(x) = H(x)\left[\frac{\mathrm{e}^{-4x}}{3} + \frac{\mathrm{e}^{-x}}{6} - \frac{\mathrm{e}^{-3x}}{2}\right].$$

Now consider the equation

$$v''(x) + 3v'(x) + 2v(x) = u(x).$$

Similarly to the above, and using the formula (6) for the Fourier Transform of the function u, we write

$$(-\omega^2 + 3\mathrm{i}\omega + 2)\mathcal{F}[u](\omega) = \frac{1/3}{4 + \mathrm{i}\omega} + \frac{1/6}{1 + \mathrm{i}\omega} + \frac{-1/2}{3 + \mathrm{i}\omega},$$

and therefore

$$\mathcal{F}[v](\omega) = \frac{1/3}{(4 + \mathrm{i}\omega)(1 + \mathrm{i}\omega)(3 + \mathrm{i}\omega)} + \frac{1/6}{(1 + \mathrm{i}\omega)^2(2 + \mathrm{i}\omega)} - \frac{1/2}{(3 + \mathrm{i}\omega)(1 + \mathrm{i}\omega)(2 + \mathrm{i}\omega)}.$$

The first and third terms are straightforward, since all the factors in the factors in each denominator are different. For the second terms, we have

$$\frac{1}{(1 + \mathrm{i}\omega)^2(2 + \mathrm{i}\omega)} = \frac{1}{2 + \mathrm{i}\omega} - \frac{\mathrm{i}\omega}{(1 + \mathrm{i}\omega)^2} = \frac{1}{2 + \mathrm{i}\omega} - \frac{\mathrm{i}\omega}{1 + \mathrm{i}\omega} + \frac{\mathrm{i}\omega}{(1 + \mathrm{i}\omega)^2}.$$

From Q3 we know that $1/(1 + \mathrm{i}\omega)$ is the Fourier Transform of

$$u(x) = \begin{cases} x\mathrm{e}^{-x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

and hence

$$\mathcal{F}^{-1}[u](x) = \begin{cases} x\mathrm{e}^{-2x} - \mathrm{e}^{-x} + x\mathrm{e}^{-x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Q5.

Notice first that

$$\mathcal{F}[f](\omega) = \frac{1}{1 + \mathrm{i}\omega} =: F(\omega).$$

Furthermore, for

$$G(\omega) = \begin{cases} \pi, & -1 \leq x < 0, \\ 0, & \text{otherwise}, \end{cases}$$

we have

$$\mathcal{F}^{-1}[G](x) = \frac{\pi}{2\pi}\int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega x} = \frac{\pi}{2\pi}\frac{1}{\mathrm{i}x}\Big(\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}\Big) = \frac{\sin(x)}{x},$$

hence G is the Fourier Transform of g.

Finally,

$$(f * g)(x) = \mathcal{F}^{-1}[FG] = \frac{1}{2\pi}\int_{-1}^{1} \frac{\pi \mathrm{e}^{\mathrm{i}\omega x}}{1+\mathrm{i}\omega} d\omega = \frac{1}{2}\int_{-1}^{1} \frac{\mathrm{e}^{\mathrm{i}\omega x}}{1+\mathrm{i}\omega} d\omega,$$

as required.

Q6.

(i) Using First Shift Theorem, we have

$$\int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} n x/L} \mathrm{e}^{-\mathrm{i}\omega x} dx = 2\pi\delta\left(\omega - \frac{2\pi n}{L}\right)$$

Therefore,

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}\omega x}dx = \sum_n c_n \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} n x/L}\mathrm{e}^{-\mathrm{i}\omega x}dx = \sum_n c_n 2\pi\delta\left(\omega - \frac{2\pi n}{L}\right).$$

(ii) As $f(x) = \sin(x) = \big(\exp(\mathrm{i}x) - \exp(-\mathrm{i}x)\big)/(2\mathrm{i})$, we have

$$\mathcal{F}[f](\omega) = \frac{\pi}{\mathrm{i}}\big(\delta(\omega-1) - \delta(\omega+1)\big),$$

and since $\mathcal{F}[g](\omega) = 1/(1+\mathrm{i}\omega)$, we conclude that

$$(f*g)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega x}\frac{\pi}{\mathrm{i}}\big(\delta(\omega-1) - \delta(\omega+1)\big)\frac{d\omega}{1+\mathrm{i}\omega} = \frac{1}{2\mathrm{i}}\left(\frac{\mathrm{e}^{\mathrm{i}x}}{1+\mathrm{i}} - \frac{\mathrm{e}^{\mathrm{i}x}}{1-\mathrm{i}}\right) = \frac{1}{2}\big(\sin(x) - \cos(x)\big).$$

(iii) Taking the Fourier Transform of both sides of the equation, we obtain

$$(\mathrm{i}\omega + 1)\mathcal{F}[u](\omega) = \frac{\pi}{\mathrm{i}}\big(\delta(\omega-1) - \delta(\omega+1)\big),$$

and hence

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega x}\left(\frac{\delta(\omega-1)}{1+\mathrm{i}\omega} - \frac{\delta(\omega+1)}{1+\mathrm{i}\omega}\right)d\omega = \frac{1}{2}\big(\sin(x) - \cos(x)\big),$$

where for the last equality we used the result of (ii).

Q7.

Notice that

$$\int_{-\infty}^{\infty} \delta_\varepsilon(x-a)f(x)dx = \int_{-\infty}^{a-\alpha} \delta_\varepsilon(x-a)f(x)dx + \int_{a-\alpha}^{\infty} \delta_\varepsilon(x-a)f(x)dx + \int_{a-\alpha}^{a+\alpha} \delta_\varepsilon(x-a)f(x)dx. \quad (7)$$

We know consider the three terms in the last expression in turn:

$$\left|\int_{-\infty}^{a-\alpha}\delta_\varepsilon(x-a)f(x)dx\right| \le \int_{-\infty}^{a-\alpha}\delta_\varepsilon(x-a)\big|f(x)\big|dx \le M\int_{-\infty}^{a-\alpha}\delta_\varepsilon(x-a)dx = M\int_{-\infty}^{a-\alpha}\delta_\varepsilon(x)dx$$

$$= \frac{M}{\pi}\tan^{-1}\left(\frac{x}{\varepsilon}\right)\Big|_{-\infty}^{-\alpha} = \frac{M}{2}\left(\pi - 2\tan^{-1}\left(\frac{\alpha}{\varepsilon}\right)\right) \to 0 \quad \text{as } \varepsilon\to 0,$$

since

$$\lim_{\varepsilon\to 0}\tan^{-1}\left(\frac{\alpha}{\varepsilon}\right) = \frac{\pi}{2}.$$

Similarly, we obtain

$$\int_{a+\alpha}^{\infty}\delta_\varepsilon(x-a)f(x)dx \to 0 \quad \text{as } \varepsilon\to 0.$$

Furthermore, we write the middle term on the right-hand side of (7) as a sum of two:

$$\int_{a-\alpha}^{\infty}\delta_\varepsilon(x-a)f(x)dx = \int_{a-\alpha}^{a+\alpha}\delta_\varepsilon(x-a)\big(f(x)-f(a)\big)dx + f(a)\int_{a-\alpha}^{a+\alpha}\delta_\varepsilon(x-a)f(x)dx,$$

and estimate each of them separately:

$$\begin{aligned}\left|\int_{a-\alpha}^{a+\alpha}\delta_\varepsilon(x-a)\big(f(x)-f(a)\big)dx\right| &\le \int_{a-\alpha}^{a+\alpha}\delta_\varepsilon(x-a)\big|f(x)-f(a)\big|dx\\ &\le L\int_{a-\alpha}^{a+\alpha}\delta_\varepsilon(x-a)|x-a|dx = 2L\int_0^{\alpha}\delta_\varepsilon(x)dx\\ &= \frac{L\varepsilon}{\pi}\int_0^{\alpha}\frac{2x}{\varepsilon^2+x^2}dx = \frac{L\varepsilon}{\pi}\log(\varepsilon^2+x^2)\Big|_0^{\alpha} = \frac{L\varepsilon}{\pi}\log\frac{\varepsilon^2+\varepsilon^2}{\varepsilon^2}\\ &= \frac{L\varepsilon}{\pi}\log\left(1+\frac{\varepsilon^2}{\varepsilon^2}\right) \to 0 \quad \text{as } \varepsilon\to 0,\end{aligned}$$

and

$$\begin{aligned}\int_{a-\alpha}^{a+\alpha}\delta_\varepsilon(x-a)f(x)dx &= \frac{1}{\pi}\int_{-\alpha}^{\alpha}\frac{\varepsilon}{x^2+\varepsilon^2}dx = \frac{1}{\pi}\tan^{-1}\left(\frac{x}{\varepsilon}\right)\\ &= \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha}{\varepsilon}\right) \to \frac{2}{\pi}\cdot\frac{\pi}{2} = 1 \quad \text{as } \varepsilon\to 0.\end{aligned}$$

It follows that

$$\int_{a-\alpha}^{\infty}\delta_\varepsilon(x-a)f(x)dx \to f(a) \quad \text{as } \varepsilon\to 0.$$

Putting everything together yields

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\delta_\varepsilon(x-a)f(x)dx = f(a).$$