MA30044/MA40044/MA50181, Mathematical Methods I, 2020

Problem Sheet 1: Introduction to Sturm-Liouville Equations

SOLUTIONS

Q1.

(i) $$-u''(x) - 6u'(x) = \lambda u(x), \quad x \in (-\infty, \infty),$$

Here, in terms of the notation used in the lecture,

$$\alpha(x) = 1, \quad \beta(x) = 6, \quad \gamma(x) = 0, \quad \delta(x) = 1.$$

Hence, using the formulae obtained therein,

$$p(x) = \mathrm{e}^{6x}, \quad q(x) = 0, \quad r(x) = \mathrm{e}^{6x},$$

and therefore the SL form of the original equation is

$$\boxed{-\big(\mathrm{e}^{6x}u'(x)\big)' = \lambda \mathrm{e}^{6x}u(x).} \tag{1}$$

(ii) $$-x(1-x)u'' + 2xu'(x) = \lambda x u(x), \quad x \in [0,1].$$

Here, in terms of the notation used in the lecture,

$$\alpha(x) = x(1-x), \quad \beta(x) = -2x, \quad \gamma(x) = 0, \quad \delta(x) = x.$$

Hence, using the formulae obtained therein,

$$p(x) = \exp\left\{\int_0^x \left(-\frac{2y}{y(1-y)}\right) dy\right\} = \exp\left\{-2\int_0^x \frac{dy}{1-y}\right\}.$$

Making the change of variables $z = 1-y$, so $dz = -dy$, we proceed to conclude

$$p(x) = \exp\left(-2\int_{1-x}^1 \frac{dz}{z}\right) = \exp\big(2\log(1-x)\big) = (1-x)^2.$$

Furthermore,

$$q(x) = 0, \quad r(x) = \frac{x(1-x)^2}{x(1-x)} = 1-x,$$

and therefore the SL form of the original equation is

$$\boxed{-\big((1-x)^2u'(x)\big)' = \lambda(1-x)u(x).} \tag{2}$$

$$\text{(iii)} \qquad -x^2u''(x) - xu'(x) = \lambda\frac{u(x)}{x}, \quad x \in (0,\infty).$$

Here, in terms of the notation used in the lecture,

$$\alpha(x) = x^2, \quad \beta(x) = x, \quad \gamma(x) = 0, \quad \delta(x) = \frac{1}{x}.$$

Hence, using the formulae obtained therein,

$$p(x) = \left(\int_1^x \frac{dy}{y}\right) = \exp(\log(x)) = x, \quad q(x) = 0, \quad r(x) = x\frac{1/x}{x^2} = \frac{1}{x^2},$$

and therefore the SL form of the original equation is

$$\boxed{-\big(xu'(x)\big)' = \lambda\frac{u(x)}{x^2}.} \tag{3}$$

Q2.

$$\text{(i)} \qquad -u''(x) = \lambda u(x), \quad x \in [0,\pi], \tag{4}$$

$$u(0) = u(\pi) = 0, \tag{5}$$

All solutions to (4) vanishing at zero (first condition in (5)) have the form $u(x) = \sin(\sqrt{\lambda}x)$. Using the second condition in (5), we obtain $u(\pi) = 0$, and hence $\sqrt{\lambda}\pi = n\pi$, $n = 1, 2, \dots$ Hence,

$$\boxed{u(x) = \sin(\sqrt{\lambda}x), \qquad \lambda = n^2, \quad n \in \mathbb{N}.} \tag{6}$$

$$\text{(ii)} \qquad -u''(x) = \lambda u(x), \quad x \in [a,b], \tag{7}$$

$$u'(a) = u'(b) = 0, \tag{8}$$

Note that we would typically write the general solution as

$$u(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x),$$

but because of the boundary conditions at $x = a, b$, it is instead better to either write

$$u(x) = A\cos(\sqrt{\lambda}(x+B)) \qquad \text{or} \quad u(x) = A\sin(\sqrt{\lambda}(x+B)).$$

We do this because then B is easily selected. In this case, the derivative condition of $u'(a) = 0$ suggests that we choose a cosine expression and $B = -a$ thus

$$u(x) = \cos(\sqrt{\lambda}(x-a)),$$

(setting the constant pre-factor to 1). Note that you did not necessarily have to do this but it makes the algebra a bit easier (since now the eigenfunction is a single cosine).

Applying the boundary condition then requires $u'(x) = -\sqrt{\lambda}\sin\big(\sqrt{\lambda}(x-a)\big)$. Now the second condition in (8) yields $\sqrt{\lambda}\sin\big(\sqrt{\lambda}(b-a)\big) = 0$, and hence $\sqrt{\lambda}(b-a) = n\pi$, $n = 1, 2, \ldots$, or $\lambda = 0$. Summarising, the sought eigenfunction-eigenvalue pairs are given by

$$\boxed{u(x) = \cos\left(n\pi\frac{x-a}{b-a}\right), \qquad \lambda = \frac{n^2\pi^2}{(b-a)^2}, \quad n \in 0, 1, 2\ldots} \tag{9}$$

(iii)

$$-u''(x) = \lambda u(x), \quad x \in \left[0, \frac{\pi}{2}\right], \tag{10}$$

$$u(0) = u'\left(\frac{\pi}{2}\right) = 0, \tag{11}$$

Using (10) and the first boundary condition in (11), we obtain $u(x) = \sin\big(\sqrt{\lambda}x\big)$,and hence $u'(x) = -\sqrt{\lambda}\cos\big(\sqrt{\lambda}x\big)$. Now from the second condition in (8) we have

$$u'\left(\frac{\pi}{2}\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) = 0,$$

and hence $\sqrt{\lambda}(b-a) = n\pi$, $n = 1, 2, \ldots$, or $\lambda = 0$. Therefore,

$$\boxed{u(x) = \sin\big((2n+1)x\big), \qquad \lambda = (2n+1)^2, \quad n \in 0, 1, 2\ldots} \tag{12}$$

(iv)

$$-\left(e^{-x}u'(x)\right)' = \lambda e^{-x}u(x), \quad x \in \left[0, \frac{\pi}{2}\right], \tag{13}$$

$$u(0) = u(1) = 0, \tag{14}$$

Differentiating and dividing by $\exp(-x)$, the equation (13) is written in the form

$$-u'' + u' = \lambda u. \tag{15}$$

This is an equation with constant coefficients, which generically have solutions of the form $\exp(\alpha x)$. Substituting this as an ansatz into (15) gives $\alpha^2 - \alpha + \lambda = 0$, from which

$$\alpha_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}.$$

We should address the three possibilities of $1/4 - \lambda > 0$, $1/4 - \lambda = 0$, and $1/4 - \lambda < 0$. Because of the boundary conditions $u(0) = u(1) = 0$, this suggests that only the last case will be applicable and solutions are oscillatory.

Case 1: $1/4 - \lambda > 0$. Write $\alpha_{\pm} = a \pm b$. Then the general solution can be written $u = e^{ax}[c_1 e^{bx} + c_2 e^{-bx}]$ or alternatively as

$$u = e^{ax}[C_1 \cosh(bx) + C_2 \sinh(bx)].$$

Then $u(0) = 0$ implies $C_1 = 0$. And $u(1) = 0$ implies $C_2 = 0$. Only trivial solution.

Case 2: $1/4 - \lambda = 0$. You can verify for yourself that in this case, solutions are given by $u = Ax + B$ and the only solution that results is trivial.

Case 3: $1/4 - \lambda < 0$. Now write $\alpha_{\pm} = a \pm ib$. Here $a = 1/2$ and $b = \sqrt{\lambda - 1/4}$. Again, it is convenient to write solutions as a single sinusoidal with a phase shift. The general solution is

$$u = e^{ax} A \sin[b(x - B)]$$

where A and B are arbitrary real constants. The first boundary condition, $u(0) = 0$ can be most easily satisfied by $B = 0$, which centres the sinusoidal at the origin. This leaves $u(1) = 0$ or

$$\sin(b) = 0 \implies b = \sqrt{\lambda - \frac{1}{4}} = n\pi,$$

for any integer n. Solving thus yields

$$\boxed{u(x) = \mathrm{e}^{x/2} \sin(n\pi x), \qquad \lambda = \frac{1}{4} + n^2\pi^2, \quad n \in 1, 2 \ldots} \tag{16}$$

Q3.

$$-\big(pu'(x)\big)' + qu(x) = \lambda r u(x), \quad x \in [a, b].$$

(i) Multiplying both sides by $\overline{u(x)}$, we write

$$-(pu')'\overline{u} + q|u|^2 = \lambda r|u|^2. \tag{17}$$

Let (λ, ϕ) be an eigenvalue-eigenfunction pair. Set $u = \phi$ in (17) and integrate over $[a, b]$:

$$-\int_a^b (p\phi')'\overline{\phi} + \int_a^b q|\phi|^2 = \lambda \int_a^b r|\phi|^2.$$

Integrating by parts, we obtain

$$-\big[p\phi'\overline{\phi}\big]_a^b + \int_a^b p|\phi'|^2 + \int_a^b q|\phi|^2 = \lambda \int_a^b r|\phi|^2,$$

and therefore

$$\lambda \int_a^b r|\phi|^2 = \int_a^b p|\phi'|^2 + \int_a^b q|\phi|^2 - \big[p\phi'\overline{\phi}\big]_a^b. \tag{18}$$

(ii) In the case (a), we have $\phi'(a) = \phi'(b) = 0$, and therefore $\left[p\phi'\overline{\phi}\right]_a^b = 0$. In the case (b), we have $\phi(a) = \phi(b) = 0$, and therefore $\left[p\phi'\overline{\phi}\right]_a^b = 0$ again.

Combining this with (18) yields

$$\lambda \int_a^b r|\phi|^2 = \int_a^b p|\phi'|^2 + \int_a^b q|\phi|^2.$$

Using the fact that

$$\int_a^b r|\phi|^2 > 0, \qquad \int_a^b \left(p|\phi'|^2 + q|\phi|^2\right) \geq 0,$$

we obtain $\lambda \geq 0$.

Q4.

$$-x^2 u''(x) - xu'(x) = \lambda u(x), \qquad x > 0. \tag{19}$$

(i) Substituting the ansatz $u(x) = x^\alpha$ into (19), we obtain

$$-x^2\alpha(\alpha-1)x^{\alpha-2} - x\alpha x^{\alpha-1} = \lambda x^\alpha \qquad \forall x > 0,$$

and hence

$$\alpha(\alpha-1) + \alpha + \lambda = 0,$$

or equivalently

$$\alpha^2 + \lambda = 0.$$

(ii) Using the notation from lectures, we note that for the equation (19) we have

$$\alpha(x) = x^2, \qquad \beta(x) = x, \qquad \gamma(x) = 0, \qquad \delta(x) = 1$$

Hence, by the theorem about writing equations in the SL form, we have

$$p(x) = \exp\left(\int_1^x \frac{dy}{y}\right) = \exp\bigl(\log(x)\bigr) = x, \qquad q(x) = 0, \qquad r(x) = \frac{x}{x^2} = \frac{1}{x}.$$

Therefore, the SL form of (19) is

$$-\bigl(xu'(x)\bigr)' = \frac{\lambda}{x}u(x), \qquad x > 0.$$

(iii) Using Q3(ii), case (b), it follows from $u(1) = u(\mathrm{e}) = 0$ that $\lambda \geq 0$. Suppose first that $\lambda > 0$. By the result of part (i), the general solution is

$$u(x) = c_+ x^{\mathrm{i}\sqrt{\lambda}} + c_- x^{-\mathrm{i}\sqrt{\lambda}} \equiv c_+ \mathrm{e}^{\mathrm{i}\sqrt{\lambda}\log(x)} + c_- \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}\log(x)}. \tag{20}$$

Setting first $x = 1$, then $x = e$, we obtain

$$\begin{cases} c_+ + c_- = 0, \\ c_+ \mathrm{e}^{\mathrm{i}\sqrt{\lambda}} + c_- \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}} = 0. \end{cases} \tag{21}$$

This system has a non-trivial solution pair (c_+, c_-) if and only if its determinant vanishes, which gives the condition $\sin(\sqrt{\lambda}) = 0$, and hence $\lambda = n^2\pi^2$, $n = 1, 2, \ldots$ If $\lambda = 0$, then the general solution is given by $u(x) = c_+ + c_- \log(x)$, which does not satisfy the boundary conditions for any c_+, c_-. So, the final answer is

$$\boxed{\lambda = n^2\pi^2, \qquad n = 1, 2, \ldots} \tag{22}$$

(iv) The eigenfunctions corresponding to the eigenvalues (22) are found by combining the formula (20) for the general solution with the fact that for solutions of (21) we have $c_- = -c_+$:

$$u(x) = c_+ \mathrm{e}^{\mathrm{i}\pi n \log(x)} - c_+ \mathrm{e}^{-\mathrm{i}\pi n \log(x)} = c_+ \sin\big(n\pi \log(x)\big).$$

So the (non-normalised) eigenfunctions are given by

$$\boxed{\phi_n(x) = \sin\big(n\pi \log(x)\big), \qquad n = 1, 2, \ldots}$$

(v) $$\int_1^e \phi_n(x)\phi_m(x)\frac{dx}{x} = \int_1^e \sin\big(n\pi\log(x)\big)\sin\big(m\pi\log(x)\big)\frac{dx}{x}.$$

Changing the variable of integration: $z = \log(x)$, so $dz = dx/x$, we obtain

$$\int_0^1 \sin(n\pi z)\sin(m\pi z)dz = 0, \qquad n \neq m,$$

as was shown in the lectures.

Q5.

$$-Eu^{(4)} = \lambda u, \quad -L < x < L, \tag{23}$$

$$u(-L) = u(L) = u''(-L) = u''(L) = 0. \tag{24}$$

(i) Setting $u(x) = \exp(\mathrm{i}\omega x)$ in (23), we obtain $E\omega^4 + \lambda = 0$, so the four possible values for ω are

$$\omega_j = \left|\frac{\lambda}{E}\right|^{1/4}\alpha_j, \qquad j = 1, 2, 3, 4,$$

where α_j, $j = 1, 2, 3, 4$, are the four roots of -1 of degree 4:

$$\alpha_{1,2} = \exp\left(\pm\mathrm{i}\frac{\pi}{4}\right), \qquad \alpha_{3,4} = \exp\left(\pm\mathrm{i}\frac{3\pi}{4}\right),$$

In other words,

$$\omega_{1,2} = \frac{R}{\sqrt{2}}(1 \pm \mathrm{i}), \quad \omega_{3,4} = \frac{R}{\sqrt{2}}(-1 \pm \mathrm{i}), \qquad R := \left|\frac{\lambda}{E}\right|^{1/4}.$$

(ii) The corresponding four exponential solutions of (23) are

$$\widetilde{\phi}_{1,2} = \exp\left(\frac{R}{\sqrt{2}}(1 \pm \mathrm{i})x\right), \qquad \widetilde{\phi}_{3,4} = \exp\left(\frac{R}{\sqrt{2}}(-1 \pm \mathrm{i})x\right).$$

These can be combined to obtain four linearly independent real-valued solutions:

$$\phi_1(x) = \frac{1}{2}\big(\widetilde{\phi}_1(x) + \widetilde{\phi}_2(x)\big) = \exp\left(\frac{R}{\sqrt{2}}x\right)\cos\left(\frac{R}{\sqrt{2}}x\right),$$

$$\phi_2(x) = \frac{1}{2}\big(\widetilde{\phi}_1(x) - \widetilde{\phi}_2(x)\big) = \exp\left(\frac{R}{\sqrt{2}}x\right)\sin\left(\frac{R}{\sqrt{2}}x\right),$$

$$\phi_3(x) = \frac{1}{2}\big(\widetilde{\phi}_3(x) + \widetilde{\phi}_4(x)\big) = \exp\left(-\frac{R}{\sqrt{2}}x\right)\cos\left(\frac{R}{\sqrt{2}}x\right),$$

$$\phi_4(x) = \frac{1}{2}\big(\widetilde{\phi}_3(x) - \widetilde{\phi}_4(x)\big) = \exp\left(-\frac{R}{\sqrt{2}}x\right)\sin\left(\frac{R}{\sqrt{2}}x\right).$$

(iii) Differentiating twice the functions ϕ_j, $j = 1, 2, 3, 4$, we obtain

$$u''(x) = -\frac{R^2}{2}\big(a\phi_1(x) + b\phi_2(x) + c\phi_3(x) + d\phi_4(x)\big)$$

The boundary conditions (24) lead to:

$u(L) = 0:$

$$\left(a\exp\left[\frac{RL}{\sqrt{2}}\right] + c\exp\left[-\frac{RL}{\sqrt{2}}\right]\right)\cos\left[\frac{RL}{\sqrt{2}}\right] + \left(b\exp\left[\frac{RL}{\sqrt{2}}\right] + d\exp\left[-\frac{RL}{\sqrt{2}}\right]\right)\sin\left[\frac{RL}{\sqrt{2}}\right] = 0, \tag{25}$$

$u(-L) = 0:$

$$\left(a\exp\left[-\frac{RL}{\sqrt{2}}\right] + c\exp\left[\frac{RL}{\sqrt{2}}\right]\right)\cos\left[\frac{RL}{\sqrt{2}}\right] - \left(b\exp\left[-\frac{RL}{\sqrt{2}}\right] + d\exp\left[\frac{RL}{\sqrt{2}}\right]\right)\sin\left[\frac{RL}{\sqrt{2}}\right] = 0, \tag{26}$$

$u''(L) = 0:$

$$\left(a\exp\left[\frac{RL}{\sqrt{2}}\right] - c\exp\left[-\frac{RL}{\sqrt{2}}\right]\right)\cos\left[\frac{RL}{\sqrt{2}}\right] + \left(b\exp\left[\frac{RL}{\sqrt{2}}\right] + d\exp\left[-\frac{RL}{\sqrt{2}}\right]\right)\sin\left[\frac{RL}{\sqrt{2}}\right] = 0, \tag{27}$$

$u(-L) = 0:$

$$\left(a\exp\left[-\frac{RL}{\sqrt{2}}\right] - c\exp\left[\frac{RL}{\sqrt{2}}\right]\right)\cos\left[\frac{RL}{\sqrt{2}}\right] - \left(b\exp\left[-\frac{RL}{\sqrt{2}}\right] - d\exp\left[\frac{RL}{\sqrt{2}}\right]\right)\sin\left[\frac{RL}{\sqrt{2}}\right] = 0. \tag{28}$$

Together (25)–(28) form a system of four linear equations for a, b, c, d. The dependency on λ is implicit, since $r = \left|\lambda/E\right|^{1/4}$.