MA30060 PS7: Coverings, SSFT, piecewise-linear maps

Feedback hand-in: Friday 10 Dec 2021, 4:00pm

Deadline likely to be amended dependent on lecture progression

1. Coverings and SSFT

Let F be a piecewise-linear map $F: I \to \mathbb{R}$ with a 4-cycle $\{x_0, x_1, x_2, x_3\}$ such that $x_{n+1} = F(x_n)$ if indices are taken mod 4. Consider the following orderings of the points $\{x_i\}$ in $\mathbb{R}$ and closed intervals I_0, I_1, I_2 given in (i)–(iii) below,

i. $x_0 < x_1 < x_2 < x_3$ with $I_0 = [x_0, x_1]$, $I_1 = [x_1, x_2]$, $I_2 = [x_2, x_3]$
ii. $x_1 < x_0 < x_2 < x_3$ with $I_0 = [x_1, x_0]$, $I_1 = [x_0, x_2]$, $I_2 = [x_2, x_3]$
iii. $x_3 < x_1 < x_2 < x_0$ with $I_0 = [x_3, x_1]$, $I_1 = [x_1, x_2]$, $I_2 = [x_2, x_0]$
iv. $x_1 < x_2 < x_0 < x_3$ with $I_0 = [x_1, x_2]$, $I_1 = [x_2, x_0]$, $I_2 = [x_0, x_3]$.

a. For each of the above orderings:
   - Draw the (minimal) graph Γ of F-covering relations for I_0, I_1, I_2.
   - Consider these graphs, and where necessary, the induced graphs of compositions of F^2, F^3, and so forth to determine which orderings (if any) imply that F is chaotic.

b. Consider the ordering of the points $\{x_i\}$ and closed intervals I_0, I_1, I_2 given in (iii).
   - Draw the piecewise-linear graph of F and F^2. Determine the explicit functional form of F and F^2.
   - Use the transition graph to write down the transition matrix A.
   - Given that $F|_A$ is semiconjugate to the SSFT $\sigma_A: \Sigma_{3,A} \to \Sigma_{3,A}$, compute the number of 1,2,3,4 and 5-cycles that are guaranteed to exist under $F|_A$. Write down the explicit symbol sequences in $\Sigma_{3,A}$ that correspond to these cycles. For simplicity, you may assume that any cycle found in $\Sigma_{3,A}$ corresponds to a cycle of F.

2. Coverings and SSFT

Let F be a continuous map $F: I \to \mathbb{R}$ and let $x_4 < x_2 < x_0 < x_6 < x_1 < x_3 < x_5$ be the members of a 7-cycle such that $x_{n+1} = F(x_n)$ with indices taken mod 7. Define the closed intervals

$$I_0 = [x_4, x_2], \quad I_1 = [x_2, x_0], \quad I_2 = [x_0, x_6], \tag{1}$$

$$I_3 = [x_6, x_1], \quad I_4 = [x_1, x_3], \quad I_5 = [x_3, x_5]. \tag{2}$$

i. Draw the (minimal) graph Γ of F-covering relations that must exist for the intervals $I_0, \ldots, I_5$.
ii. Show, by considering distinct paths in Γ, that F is guaranteed to have N-cycles for $N = 1,2,4,6$ and for all $N \geq 8$, but does not have either a 3-cycle or a 5-cycle.
iii. Construct a map F that has the properties specified in (a), and sketch it.

3. A study of piecewise-linear maps

Consider the piecewise-linear continuous map F mapping the unit interval to itself defined by:

$$F(x)=\begin{cases} 2x+\frac{1}{2} & 0\le x\le\frac{1}{4} \\ \frac{3}{2}-2x & \frac{1}{4}\le x\le\frac{3}{4} \\ 2x-\frac{3}{2} & \frac{3}{4}\le x\le 1 \end{cases} \tag{3}$$

It can be proved that, if a map has a horseshoe then it has periodic points if every period.

i. Determine the functional form of the piecewise linear function F^2. Sketch graphs of $F(x)$ and $F^2(x)$.
ii. Consider your sketches to show that F has a fixed point and N-cycles for all even N. Use your sketches to assess the characteristics of an orbit of F of period 3 and so conclude that F does not have N-cycles for any odd $N\ge 3$.

4. The skewed tent map

Consider the skewed tent map

$$F(x)=\begin{cases} F_L(x)=\mu x & 0\le x\le a \\ F_R(x)=\frac{\mu a}{1-a}(1-x) & a\le x\le 1 \end{cases} \tag{4}$$

where $0<a<1$ is a fixed parameter.

i. Sketch the graph of $F(x)$ for a typical value of a. Find the range of μ for which F maps [0,1] into itself. Find the non-trivial fixed point x_0 of F. Determine the range of values of μ for which x_0 exists and is stable.
ii. Assume $\mu>1$ and $a\mu\le 1$. Construct F^2 (it consists of four straight-line sections). Sketch the graphs of F and F^2 on the same axes.
iii. *(Non-examinable)* Indicate on your graphs the non-trivial fixed point of F, x_0 and its two nearest preimages under F^2, $x_{-1}<x_0$ and $x_{-2}>x_0$ (i.e. $F^2(x_{-1})=x_0=F^2(x_{-2})$). Find an expression for x_{-1} in terms of μ. Show that F is chaotic if

$$\mu^2>\frac{1-a}{a^2}.$$