MA30060 PS1: Basic cobwebbing, stability, and bifurcation theory

Feedback hand-in: Friday 15 Oct 2021 4:00pm

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1. **(Logistic equation)**

Complete the calculations, sketched in lectures, for the dynamics of the logistic map

$$x_{n+1} = F(x_n) \equiv \mu x_n(1 - x_n),$$

as follows.

(a) By solving $x = F(x)$, find the two fixed points that exist in $\mu > 0$.

(b) Show that at $\mu = 3$ the derivative of F passes through -1 at one of these fixed points, call it x^*, i.e. that $F'(x^*) = -1$ when $\mu = 3$.

(c) Computing the points that lie on the 2-cycle by solving

$$x = F(F(x)).$$

This leads to a quartic polynomial to solve. Simplifying the highest power, it will be useful for you to express the above equation in the form of

$$(x - x_+)(x - x_-)(x^2 + bx + c)$$

where x_- and x_+ are the fixed points of F. Then expand the polynomial and match powers of x in order to solve for b and c. Show that the 2-cycle that emerges from x^* exists for $\mu > 3$ but not in $\mu < 3$.

(d) Consider the 'second iterate map' $G(x) := F^2(x) \equiv F(F(x))$. Write the derivative of G in terms of derivatives of F and hence show that the derivative of G is constant on the 2-cycle. Hence show that the 2-cycle is stable over the range $3 < \mu < 1 + \sqrt{6}$.

(e) Explain what happens at $\mu = 1 + \sqrt{6}$ and sketch a bifurcation diagram in the (μ, x) plane for $0 < \mu < 1 + \sqrt{6}$ to summarise the existence and stability of the invariant sets discussed in this question.

2. **(N-cycles)**

Generalise the first part of question 1(d) above to the case of an N-cycle; i.e. for a general map $F(x)$ define $G := F^N(x)$ and suppose that F has an N-cycle. Deduce an expression for $G'(x)$ in terms of F' and hence show that G' takes the same value at each point on the N-cycle.

3. **(Stability)**[1] There are two immediate kinds of stability we can define:

1. "Starts near, stays near": for a point sufficiently close to a fixed point, the orbits remain close.
2. "Eventually tends to": for a point sufficiently close to the fixed point, the orbits eventually tend to the fixed point.

By considering the graph of $F(x)$ for each of the maps below, decide whether the fixed point at the origin exhibits (1) or (2) above, or both. Explain and justify your answers. Draw 'cobweb' diagrams to help describe the dynamics near the origin in each case.

(a) $F(x) = -x$,

(b) $F(x) = x + x^2$,

(c) $F(x) = x - x^3$,

(d) $F(x) = x + x^3$.

[1] Re-written to be simpler in 21–22.