MA30060 PS3: Detailed bifurcation analysis

Feedback hand-in: Friday 29 Oct 2021

1. Bifurcation analysis

Show that the following maps $x_{n+1} = F(x_n, \mu)$ have a fixed point at $x = 0$ and examine analytically the bifurcations of this fixed point as μ varies. Consider both positive and negative μ.

i $F(x,\mu) = \mu \sin x$,
ii $F(x,\mu) = \mu \sinh x$.

Sketch the maps F for different values of μ to identify additional fixed points and describe the bifurcations that arise. You do not need to consider these cases analytically. Use this information to sketch the associated bifurcation diagram.

2. A details bifurcation analysis I

Consider the map $x_{n+1} = 4\mu - (\mu+3)x_n + x_n^2$.

- Find the fixed points.
- Determine the ranges of μ for which they are stable.
- Sketch their locations in the (μ,x) plane (i.e. sketch a bifurcation diagram), indicating stability.
- Use your sketch to informally classify the bifurcations that occur at $\mu = 2$, $\mu = 4$ and $\mu = 6$.

Once you have done the above, formally classify each bifurcation in the following way. Perform a coordinate shift which sends the bifurcation point to the origin, e.g. using

$$y \equiv x - x_0 \qquad \lambda \equiv \mu - \mu^*$$

for fixed point, x_0, and bifurcation parameter μ^*.

Using this substitution, write down a simplified form of the map near the bifurcation point and use the criteria on the Taylor series coefficients in Chaps 4–5 to classify the points formally.

3. A detailed bifurcation analysis II

Consider the map

$$x_{n+1} = \mu - 2 + (\mu+1)x_n + 3x_n^2 + x_n^3.$$

i Show that there is a fixed point at $x = -1$ and find any other fixed points.
ii Determine the ranges of μ for which the fixed points are stable.
iii Hence informally classify the bifurcation that occurs at $\mu = 3$.

Similar to Q2, formally classify the bifurcation at $\mu = 3$ by shifting coordinates $(x,\mu) \to (y,\lambda)$ to place the bifurcation point at $y = \lambda = 0$ and put the map into canonical form. Hence sketch a bifurcation diagram showing the behaviour near $\mu = 3$.