MA30060 PS6: The logistic diagram and the Cantor set

Feedback hand-in: Friday 19 Nov 2021, 4:00pm

1. Invariant sets and symbolic dynamics

Consider the map $x_{n+1}=H(x_n)$ where $H(x)$ is defined to be

$$H(x)=\begin{cases}3x & 0\leq x\leq \frac{1}{2}\\ 3x-2 & \frac{1}{2}<x\leq 1\end{cases} \tag{1}$$

i. Sketch the graph of $H(x)$. Find the set Λ_1 of points that remain in [0,1] for (at least) one iteration. Find the set Λ_2 of points that remain in [0,1] for (at least) two iterations. Hence, outline an inductive construction for the set Λ_n of points that remain in [0,1] for (at least) $n\geq 1$ iterations. Deduce that set Λ of points that remain in [0,1] for infinitely many iterations is non-empty.

ii. Describe Λ by considering the conditions base-3 expansions of points $x=0\cdot a_0a_1a_2\cdots$, where $a_i\in\{0,1,2\}$ must satisfy to remain in [0,1] for infinitely many iterations.

iii. Compute the point x that corresponds to the symbol sequence $0\cdot 002002002\cdots$ and verify, by iterating H, that this point lies on a 3-cycle.

2. The tent map and symbolic dynamics

The (full height) tent map $x_{n+1}=F(x_n)$ where $F:[0,1]\to[0,1]$ is defined by

$$F(x)=\begin{cases}2x & 0\leq x\leq \frac{1}{2}\\ 2(1-x) & \frac{1}{2}\leq x\leq 1\end{cases} \tag{2}$$

a. Sketch the graph of $F(x)$.

b. Explain why the action of F on points $x\in[0,1]$ is equivalent to the action of the modified shift map $\hat{\sigma}$ on the symbol space Σ_2, where $\hat{\sigma}$ is defined by

$$\hat{\sigma}(0\cdot a_0a_1a_2\cdots)=\begin{cases}0\cdot a_1a_2a_3\cdots \text{ if } a_0=0\\ 0\cdot \bar{a}_1\bar{a}_2\bar{a}_3\cdots \text{ if } a_0=1\end{cases} \tag{3}$$

where $\bar{a}_i:=1-a_i$.

c. By considering symbol sequences that are periodic under the action of $\hat{\sigma}$, which can switch the value of the binary digits as well as shifting left, find the two distinct 3-cycles of the tent map F.

3. Proof of Lemma in Chapter 11

Provide the proof to Lemma 11.2. That is, let $I=[0,1]$ be the closed unit interval and $F(x)=\mu x(1-x)$ be the usual logistic map. Show that if $\mu>2+\sqrt{5}$ then there exists $\lambda>1$ such that $|F'(x)|>\lambda$ for all $x\in I\cap F^{-1}(I)$.

4. Proof of Theorem in Chapter 11

The Theorem 11.1 from the notes includes a fairly long and complicated proof. During lectures, we shall discuss the key ideas and the precise steps for Steps 1 and 2 of the proof (establishing the conjugacy map h and injectivity of h).

Follow the proof of Steps 3 (surjectivity) and 4 (continuity) and produce your own set of notes for these two steps.