MA30060 PS5: Topological conjugacy

Feedback hand-in: Friday 12 Nov 2021, 4:00pm

1. Topological conjugacy and homeomorphisms

Let $X,Y \subseteq \mathbb{R}$ be closed intervals and let $F : X \to X$ and $G : Y \to Y$ be continuous maps that are topologically conjugate via a homeomorphism $h : X \to Y$.

Show that if F has an N-cycle $\{x_0, x_1, ..., x_{N-1}\}$ then so does G, and write down the N-cycle for G explicitly.

2. Topological conjugacy and homeomorphisms

Consider the two maps on $\mathbb{R}$:

i $x_{n+1} = F(x_n) = 4x_n(1-x_n)$ on $X = [0,1]$, and
ii $y_{n+1} = G(y_n) = 2 - y_n^2$ on an interval $Y \subset \mathbb{R}$.

a) Find the fixed points of F and G and sketch both functions. Show that F and G are topologically conjugate by constructing a homeomorphism $h : X \to Y$ for a suitable choice of Y. For the construction you may wish to use the result of the previous question (fixed points must map to fixed points) and try an affine form for $y = h(x)$, writing $h(x) = ax + b$ and making suitable choices for a and b.
b) Let us study a generalisation. Consider $F(x,\mu) = \mu x(1-x)$ and $G(y,\lambda) = \lambda - y^2$. Find a homeomorphism between F and G by generalising the h you constructed previously. Determine the relationship between λ and μ required for this homeomorphism to work and use this relationship to find the range of λ (and μ) over which F and G are topologically conjugate.

3. Topological conjugacy and homeomorphisms

Consider the two maps $F(x) = 2x$ and $G(y) = 3y$ defined on the half-line $[0,\infty)$.

Construct a suitable conjugacy map h to show that F and G are topologically conjugate. You may wish to consider what is required from h at $x = 0$ and then develop h from the affine form used in the previous question. Explain why h is a homeomorphism but not a diffeomorphism.

Remark: this example is typical in the sense that topologically conjugate maps are almost never conjugate by a diffeomorphism.

4. Topological conjugacy and SDIC

Consider the continuous maps $F : \mathbb{R}_+ \to \mathbb{R}_+$ given by $F(x) = 2x$, and the map $G : \mathbb{R} \to \mathbb{R}$ given by $G(y) = y + \log 2$. Show that they are conjugate via the map $h(x) = \log(x)$. Hence show that SDIC is not preserved under conjugacy.