PS1: Basic cobwebbing, stability, and bifurcation theory

Feedback hand-in: Friday 15 Oct 2021 4:00pm

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1. **(Logistic equation)**

Complete the calculations, sketched in lectures, for the dynamics of the logistic map

$$x_{n+1} = F(x_n) \equiv \mu x_n(1 - x_n),$$

as follows.

(a) By solving $x = F(x)$, find the two fixed points that exist in $\mu > 0$.

(b) Show that at $\mu = 3$ the derivative of F passes through -1 at one of these fixed points, call it x^*, i.e. that $F'(x^*) = -1$ when $\mu = 3$.

(c) Computing the points that lie on the 2-cycle by solving

$$x = F(F(x)).$$

This leads to a quartic polynomial to solve. Simplifying the highest power, it will be useful for you to express the above equation in the form of

$$(x - x_+)(x - x_-)(x^2 + bx + c)$$

where x_- and x_+ are the fixed points of F. Then expand the polynomial and match powers of x in order to solve for b and c. Show that the 2-cycle that emerges from x^* exists for $\mu > 3$ but not in $\mu < 3$.

(d) Consider the 'second iterate map' $G(x) := F^2(x) \equiv F(F(x))$. Write the derivative of G in terms of derivatives of F and hence show that the derivative of G is constant on the 2-cycle. Hence show that the 2-cycle is stable over the range $3 < \mu < 1+\sqrt{6}$.

(e) Explain what happens at $\mu = 1+\sqrt{6}$ and sketch a bifurcation diagram in the (μ, x) plane for $0 < \mu < 1+\sqrt{6}$ to summarise the existence and stability of the invariant sets discussed in this question.

2. **(N-cycles)**

Generalise the first part of question 1(d) above to the case of an N-cycle; i.e. for a general map $F(x)$ define $G := F^N(x)$ and suppose that F has an N-cycle. Deduce an expression for $G'(x)$ in terms of F' and hence show that G' takes the same value at each point on the N-cycle.

3. **(Stability)**

By considering the graph of $F(x)$ for each of the maps below, decide whether the fixed point at the origin is Lyapunov stable, quasi-asymptotically stable, both, or neither. Explain and justify your answers. Draw 'cobweb' diagrams to help describe the dynamics near the origin in each case.

(a) $F(x) = -x$,

(b) $F(x) = x + x^2$,

(c) $F(x) = x - x^3$,

(d) $F(x) = x + x^3$.