PS2: Cobwebbing, bifurcations, and asymptotics

Feedback hand-in: Friday 22 Oct 2021

1. **Basic cobwebbing and bifurcations**

Consider the map $x_{n+1} = \mu \exp(x_n) =: F(x, \mu)$, where μ is a non-zero parameter.

Sketch graphs of $F(x, \mu)$ for both positive and negative values of μ.

Find the values of μ at which the fixed point undergoes (a) a saddle-node bifurcation and (b) a period-doubling bifurcation. Using the graphs for F to infer stability, sketch a bifurcation diagram for the map (using solid lines for stable and dashed for unstable).

2. Basic cobwebbing and bifurcations

Sketch the form of the map $x_{n+1} = \mu - \frac{1}{4}x_n^2$ as μ increases for $\mu \in (-\infty, 5]$.

Find and classify the stability of the invariant sets (in particular all fixed points and period-2 orbits) and bifurcations that occur in the map as μ increases.

Sketch the overall bifurcation diagram in the (μ, x) plane, indicating stability.

3. Taylor series in two variables

Consider the function

$$f(x, y) = y - 2 + (y + 1)x + 3x^2 + x^3.$$

expanded about the point $(x, y) = (-1, 3)$.

(a) By using the explicit Taylor series formula from the notes, show that

$$f(x, y) = -1 + (x + 1) + (x + 1)(y - 3) + O(\text{cubic contributions})$$

(b) Make the substitution $\bar{x} = (x+1)$ and $\bar{y} = y-3$ directly into the form of $f(x, y)$ and expand the expression directly into powers of $\bar{x}$ and $\bar{y}$ using the binomial theorem (or direct expansions).

4. Analysis of a singular quartic equation

Consider solving the roots of

$$f(x) = -\epsilon x^4 + x^2 - x - 2$$

when ϵ is a small number.

(a) Use labels [1], [2], etc. to label each of the terms of the polynomial. By applying the method of dominant balance, develop a one-term approximation to the four roots. You will need to argue two possible balances.

(b) Suppose we argue that the dominant balance should be [1] $\sim$ [3] $\gg$ [2], [4] so that we obtain the approximation

$$-\epsilon x^4 \sim x$$

which suggests that there is at least one solution near $x = -1/\epsilon^{1/3}$. What is wrong with this logic?

(c) Develop the first two terms of an approximation near each of the four roots.