MA30287: Mathematics of Planet Earth

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Mathematical Sciences
University of Bath

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Preface

Note 30 Jan 2025

We are currently re-generating the notes for 2025 delivery. If you would like to see the complete notes from prior deliveries there should be a link up on the Moodle site. Hang tight.

Welcome to to the 2023-24 delivery of MA30287 Maths of Planet Earth at the University of Bath. In the below figure, you will see an image that, to some extent, represents the way mathematicians and other learned scientists see the Planet.

Lectures and office hours

Lectures take place at the following times and locations:

- Tuesdays 9:15 in 6W 1.2
- Wednesdays 11:15 in 6W 1.2
- Thursdays 15:15 in 6W 1.2

Office hours: You will be able to find me for an office hour in 4W 2.18 on Thursdays (following the lecture). Typically it is best to set this up, beforehand, by email appointment.

Coursework and examinations

Your final mark will be 25% coursework and 75% final exam.

Details of the coursework will be released in Week 7 and it will be due in Week 10[1].

[1] Subject to confirmation.

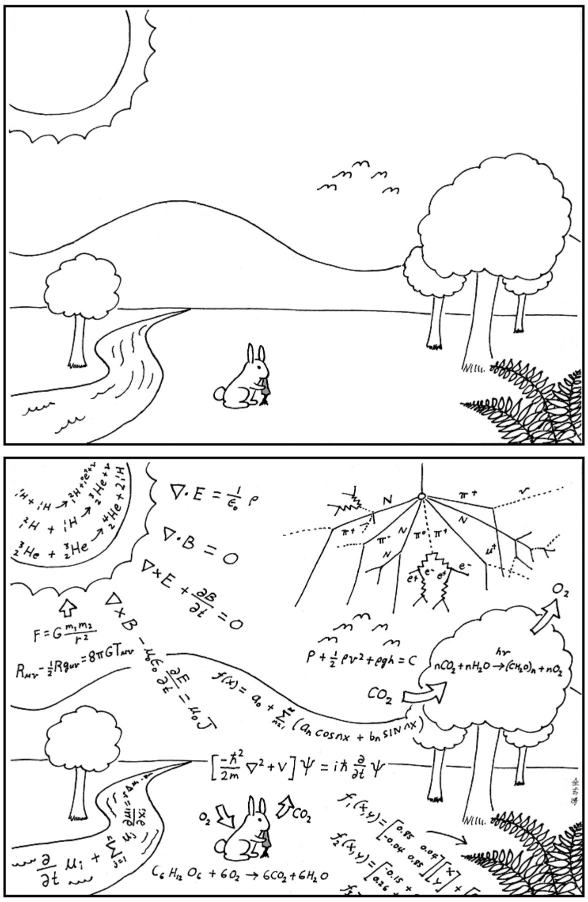

$^1_1H + ^1_1H \rightarrow ^2_1H + ^0_1e^+ + \nu$
$^2_1H + ^1_1H \rightarrow ^3_2He + \gamma$
$^3_2He + ^3_2He \rightarrow ^4_2He + 2^1_1H$
$\nabla \cdot E = \frac{1}{\epsilon_0}\rho$
$\nabla \cdot B = 0$
$\nabla \times E + \frac{\partial B}{\partial t} = 0$
$\nabla \times B - \mu_0\epsilon_0 \frac{\partial E}{\partial t} = \mu_0 J$
$F = G\frac{m_1 m_2}{r^2}$
$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$
$P + \frac{1}{2}\rho v^2 + \rho g h = C$
$nCO_2 + nH_2O \xrightarrow{h\nu} (CH_2O)_n + nO_2$
O_2
CO_2
$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$
$\left[\frac{-\hbar^2}{2m}\nabla^2 + V\right]\Psi = i\hbar\frac{\partial}{\partial t}\Psi$
$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = f_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\Delta u_i$
O_2
CO_2
$C_6H_{12}O_6 + 6O_2 \rightarrow 6CO_2 + 6H_2O$
$f_1(x,y) = \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + $
$f_2(x,y) = \begin{bmatrix} -0.15 & \\ 0.26 & \end{bmatrix}$
N
π^+
π^-
π^0
ν
μ^+
e^+
e^-

Figure 1: This picture was produced by Abstruse Goose and has the title: "this is how scientists see the world".

Resources

In general, you will have access to a few kinds of resources:

1. The Moodle portal will be the main organisation portal.
2. Lecture notes, coursework, and other resources will be found in an online format and will be linked on Moodle.
3. Coding will be done via Noteable, accessed via the Moodle website.

Naturally, because this is a relatively new module at Bath, there will be a fair amount of activity as we settle the material over the semester.

Prerequisites

This course officially requires MA20221 (modeling and dynamical systems) or XX20231 (mathematical and statistical methods for the life sciences). It is designed to be somewhat stand-alone in the sense that applied mathematical techniques learned in other courses will be introduced in some capacity. Such techniques will involve:

- Solutions of ordinary differential equations (MA10230 and MA20220).
- Multivariable calculus, partial differentiation, and multiple integrals (MA10230 and MA10236); some review/introduction of concepts from MA20223.
- Dynamical systems, stability, phase planes (MA20221, MA30060).
- Numerical methods in Python (MA10276).

Whenever possible, I have isolated such reviews/introductions and these can be found in the *Mathematical methods* section of these notes. During the current phase, Bath is undergoing changes to Years 1 and 2, so some of the above content may have shifted; these shifts are typically minor and cosmetic.

Resources

This course is designed around the following sources:

- Sustainable energy – without the hot air by (MacKay, 2009)
- Mathematics & Climate by (Kaper und Engler, 2013)
- Mathematical Geoscience by (Fowler, 2011)
- A gentle introduction to numerical simulations with Python

However, the current MPE course is also quite 'bespoke' at Bath, so there is no single source with all the information. It has been designed to cover a selection of topical subjects and complementary to other materials you might have learned previously in your studies.

Lecture plan

A lecture plan can be found at this link.

Part I

Introduction

CHAPTER 1

What does Mathematics of Planet Earth mean?

Mathematics of Planet Earth seems like an incredibly broad description for a course, but perhaps in order to give a rough idea of what such a course might include, we can consider the following diagram in Figure 1.1, which illustrates different categories and subject areas that are involved in the modelling of a full Earth system.

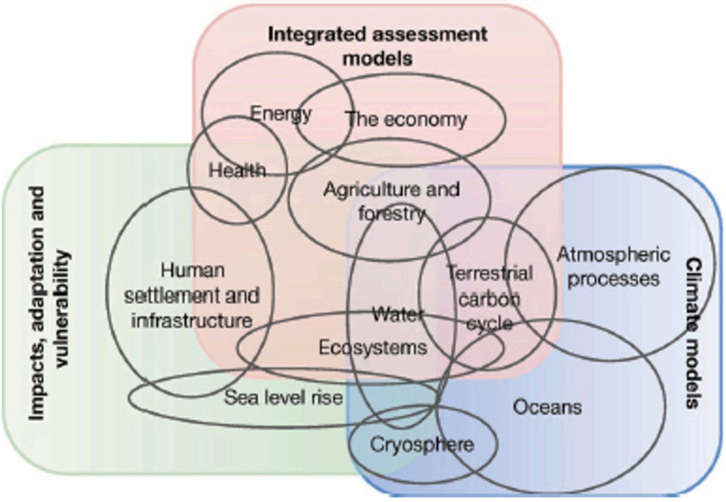


Figure 1.1: The many components of a full Earth System Model

It would be possible to spend a lifetime studying any one aspects of the above categories, and they span many different areas of study, including: (i) engineering (civil, fluids, mechanical, etc.); (ii) physics (geosciences, mechanics); (iii) Earth sciences;

(iv) policy and health; and so forth and so on. As mathematicians, we also have a unique perspective, and applied mathematics plays important roles in many of the above categories.

In essence, this course will include topics and themes are united by aspects of **mathematical modelling** and **mathematical analysis** and this is what distinguishes our style of study from adjacent areas of science and social science.

This course will focus on **models** of Planet Earth. As such, one can divide modelling into (at least) three types: (i) conceptual modelling; (ii) physics-based modelling; and (iii) statistical or data-based modelling. A basic **conceptual model** of the temperature on the planet consists of a differential equation that expresses basic principles of energy conservation, but that significantly coarse-grains the dynamics (to the extent where you would not necessarily need to know very much fluid mechanics or physics to study it). Analysis of such models is done numerically or semi-analytically (similar to the kind of phase-plane analysis of ODEs you would have seen in other modules). In this course, we will also study conceptual models of oceans and floods.

A secondary part of this course will involve more in-depth analysis of the physical models that govern the blue elements of the above figure. This moves us from the toy box (conceptual) models studied above to digging into the underlying physics—this also falls into the category of Mathematical Geoscience or Fluid Mechanics.

Statistical or data-based approaches are not as emphasised in this course (though they remain very important tools). Such approaches include, for example, analyses of time-series of climate variables, or machine-learning approaches for building data-based models on large quantities of data.

CHAPTER 2

Conservation laws

In the first chapter of (Fowler, 2011), there is a concise introduction to the different categories of techniques and approaches that you might use when doing mathematical modelling in the real world. Some of these ideas will be introduced to you in this course.

Here, we provide a brief intro to the highlights, involving the use of conservation laws (and PDEs) and also the concept of non-dimensionalisation (which you would have encountered previously), studied in Chapter 3.

> Partial differential equations
>
> Knowledge partial differential equations (PDEs) is not a prerequisite for this course, but naturally in studying anything related to the physical real world, we must discuss such things. The hope is that the necessary theory for PDEs will be presented to you as this course evolves, so that it can appreciated by both newcomers and experienced readers.

Conservation laws can be expressed as mathematical equations that represent the idea that some quantity is conserved. In processes governing the planet, these might correspond to conservation of heat, of water, of air, of momentum, etc.

In Chapter 4, we will develop the simplest possible model governing the temperature on the surface of the Earth. It is a conservation equation for energy and is zero-dimensional (does not involve time and does not involve spatial variation).

2.1 Derivation of the 1D heat equation

In order to demonstrate some of the basic principles of this course, let us demonstrate the derivation of the heat equation. We are interested in modelling the heat in a volume, V, which, for the sake of concreteness is given by a long cylinder with its axis along $x \in [0, L]$. We assume that the side walls of the cylinder are insulated and the temperature only varies along the x direction.

At any point along this rod, the internal heat is given by $\rho c T(x, t)$, where ρ is the density of the material (kg/m3), c is the specific heat capacity (J/(kg K)), and T is the temperature (K). Therefore, the heat energy along any segment in the rod is calculated from

Internal heat energy

$$\text{heat energy in } [a, b] = \int_a^b \rho c T \, \mathrm{d}x.$$

If the heat changes, then the rate of change of heat energy is given the time derivative of the above quantity. By conservation of energy, any change of the internal energy must be equal to the inflow or outflow of heat at the ends, $x = a$ or $x = b$. We therefore write q for the flux (or flow) of heat.

We need a **constitutive law** that dictates how energy is exchanged at the boundaries. Based on intuition, it is sensible to assume that the flow of heat proceeds from hot to cold. For example, hot air rises towards cool air; or heat from a hot mug of tea flows and diffuses outwards into a cold room. Therefore, we write this as

Fourier's law

Fourier's law in 1D specifies that the heat flux is given by

$$q(x, t) = -k \frac{\partial T}{\partial x}.$$

This is known as Fourier's law. The quantity k is the thermal conductivity, and its units are W/(m K). Because a Watt is a Joule/s, you can also see that the units of k are J/(m K s). The quantity q is the flux, and you can verify that it is given in units of J/(m2 s).

Therefore by energy conservation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b \rho c T \, \mathrm{d}x = q(x = a, t) - q(x = b, t),$$

i.e. the change in internal heat is equal to the flow through the ends. Note that we have added-in the heat flux due to $x = a$ assuming positive q refers to heat moving from left-to-right. Conversely, we subtract it away from $x = b$. We can alternatively write this as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b \rho c T \, \mathrm{d}x = -\int_a^b \frac{\partial q}{\partial x} \, \mathrm{d}x.$$

Substitution Fourier's law, we can then write

$$\int_a^b \rho c \frac{\partial T}{\partial t} \, \mathrm{d}x = \int_a^b k \frac{\partial^2 T}{\partial x^2} \, \mathrm{d}x.$$

Because the above integral identity needs to be true for all possible values of a and b, then it must be true everywhere (this is sometimes referred to as the *du Bois-Reymond lemma* or the *bump lemma*. Therefore we are left with the classic heat equation.

Heat equation

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}. \tag{2.1}$$

In order to produce a sensible physical solution, partial differential equations are typically supplemented by initial conditions and boundary conditions. The **initial condition** prescribes the state of the function at some initial time, typically $t = 0$. **Boundary conditions** prescribe how the function behaves on the boundary of its domain, which in this case is $x = 0$ and $x = L$. An example might be

Initial conditions (IC) and boundary conditions (BC)

$$\begin{aligned} T(x, 0) &= T_0 \\ T(0, 0) &= T_a \\ T(L, 0) &= T_b \end{aligned}$$

which expresses, respectively, that the temperature starts from a constant temperature, T_0, and where the ends of the rod are kept at temperature T_a and T_b.

Steady states and long-time behaviours

When we refer to a **steady-state** solution, we are typically referring to a time-independent solution.

Definition 2.1 (Steady-state solutions). Given an evolving system described by a function, say $f(x, t)$, defined on some spatial domain and with $t > 0$, the steady-state solution refers to time-independent solutions with

$$\frac{\partial f}{\partial t} = 0.$$

One can envisage that as the system evolves with $t \to \infty$, it reaches a state that is independent of time. However, not all systems will approach a steady state. Moreover, not all steady states are stable or attractive (and might never be reachable in a real-life experiment).

For the case of heat flow, such a steady-state solution would be $T(x,t) = T(x)$. In this case,

$$\frac{\partial T}{\partial t} = 0 \Longrightarrow k\frac{\partial^2 T}{\partial x^2} = 0.$$

Therefore, for the heat flow along a segment of length L with left boundary held at T_a and right boundary held at T_b we have

$$T_{\text{steady}}(x) = \left(\frac{T_b - T_a}{L}\right) x + T_a.$$

2.2 Deriving the 1D transport (continuity) equation

Consider the mass transport of some substance with density $\rho(x,t)$, immersed in a fluid, along a one-dimensional line in x. If desired, you may consider the substance as existing in three-dimensional space, and propagating along the x-direction, with its behaviour independent of y and z. Initially, when considered at time t, the mass of the substance between two points, a, and b, is:

$$m_{\text{blob}}(t) = \int_{a(t)}^{b(t)} \rho(x,t)\,\mathrm{d}x.$$

As time increases the particles of the substance will move due to the fluid moving; at the same time, the fluid volume which is initially contained in $x \in [a(0), b(0)]$ will also move. We want to find how the mass of the blob changes in time, and hence consider the quantity

$$\frac{\mathrm{d}m_{\text{blob}}}{\mathrm{d}t} = \int_a^b \frac{\partial \rho}{\partial t}\,\mathrm{d}x + \rho(b,t)\frac{\mathrm{d}b}{\mathrm{d}t} - \rho(a,t)\frac{\mathrm{d}a}{\mathrm{d}t}.$$

The above considers the intrinsic rate of change of the function within the integrand, but then adds the extra mass due to the right boundary shifting rightwards (b) and subtracts the mass due to the left boundary shifting rightwards (a). It is known as the Leibniz integral rule. We can thus write this within the integral as

$$\frac{\mathrm{d}m_{\text{blob}}}{\mathrm{d}t} = \int_a^b \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho(x,t)\frac{\mathrm{d}x}{\mathrm{d}t}\right)\right]\,\mathrm{d}x.$$

However, the quantity

$$\frac{\mathrm{d}x}{\mathrm{d}t} \equiv u(x,t)$$

represents the velocity of the fluid (which for the moment we assume to be a known and provided quantity). Therefore we can write the mass change as

$$\frac{\mathrm{d}m_{\text{blob}}}{\mathrm{d}t} = \int_a^b \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u)\right]\,\mathrm{d}x.$$

This result, which explains how to pass a derivative through an integral express mass of a substance within a flow is known as the *Reynolds Transport Theorem*. We have just derived it in 1D.

Reynolds Transport Theorem

Let $\rho = \rho(x,t)$ be some quantity (such as density) that is advected along a one-dimensional line in x due to a fluid with velocity $u(x,t)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} \rho \,\mathrm{d}x = \int_{a(t)}^{b(t)} \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u)\right] \mathrm{d}x.$$

We may now consider the substance being transported along the x-direction. If there is no interior creation or destruction of the source, then by conservation of mass, it must be the case that

$$\frac{\mathrm{d}m_{\text{blob}}}{\mathrm{d}t} = 0.$$

Thus, again since the above integral identity applies to all possible values of a and b, it must be the case that the integrand is zero. Thus we conclude with the so-called tranport equation.

One-dimensional transport equation

The transport of a substance described by $\rho(x,t)$ advected along a one-dimensional line in x due to a fluid with velocity $u(x,t)$ is given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0.$$

Again, we must consider the above problem in combination with potential initial conditions and boundary conditions. For instance, we might specify that the substance begins from some initial state, say

$$\rho(x,0) = \rho_0(x).$$

The boundary conditions are more subtle. It is not always obvious what the boundary conditions should be on a problem.

Here is an example of a solution of such a problem.

Linear advection equation with constant speed

Solve the problem given by

$$\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = 0,$$

where the velocity $u = c$ is constant. You may assume the initial condition is given by

$$\rho(x, 0) = \rho_0(x) = e^{-x^2}.$$

Solution

The general solution is given by

$$\rho(x, t) = F(x - ct),$$

where F is an arbitrary (differentiable) function. You can verify this via the chain rule, noting that

$$\frac{\partial F}{\partial t} = -cF'(x - ct) \quad \text{and} \quad \frac{\partial F}{\partial x} = F'(x - ct).$$

Applying the initial condition at $t = 0$ gives $F(x) = \rho_0(x)$ and thus

$$\rho(x, t) = e^{-(x-ct)^2}.$$

You may plot this to discover that it is, as perhaps expected, a Gaussian profile moving to the right at constant speed (for $c > 0$).

CHAPTER 3

Dimensional scaling analysis

Putting a mathematical model into non-dimensional form is fundamental (Fowler, 2011). When we refer to a “back-of-the-envelope calculation”, we often mean simple algebraic calculations that still nevertheless provide enormous insight on problems. Such analyses are often based on dimensional analysis, which identifies the relationships between the different quantities involved in the problem.

3.1 Dimensional quantities

Every physical quantity, say Q, can be expressed as a product of a dimensional unit, denoted [Q], and a magnitude, say Q’. Thus we write

$$Q = Q'[Q]$$

For example, if x corresponds to the physical length in a problem, we might select $[x] =$ km or $[x] =$ yards or $[x] =$ m. It is important to choose the dimensionalisation to suit the problem under consideration.

SI Units

The International System (SI) of Base Units sets out a distinct selection of choices for dimensions in certain physical quantities. The seven fundamental dimensional units are

- [Length] = metre, m
- [Time] = seconds, s
- [Mass] = kilogram, kg
- [Temperature] = Kelvin, K,
- [Electric current] = ampere, A

- [Light intensity] = candela, cd
- [Material quantity] = mole, mol

Dimensional units that can be expressed in terms of other fundamental units are known as *derived units*. For example:

- [Speed] = $\mathrm{m\,s^{-1}}$
- [Acceleration] = $\mathrm{m\,s^{-2}}$
- [Force] = $\mathrm{kg\,m\,s^{-2}}$

3.2 Dimensional homogeneity and non-dimensionalisation

All terms in any equation must have the same dimensions. This is the principle of dimensional homogenuity. For example, Newton's second law expresses the fact that

$$F = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2},$$

where m is mass, x is a unit of length, and t is time. We can check, then, that the units do indeed match up on either side. Here, the RHS has units of $[m][x]/[t]^2$ or in SI units, $\mathrm{kg\,m\,s^{-2}}$. This indeed matches our previous given SI unit decomposition for force.

Notice in addition that the input to functions like $\cos\theta$ and e^z must be non-dimensional (or dimensionless).

The process of nondimensionalisation is then as follows. Given an equation, we know that each term must have the same dimension. Therefore, we can scale all the dependent and independent variables by dimensional constants in order to yield a non-dimensional equation.

Why this is an important tool is demonstrated by the below.

3.3 Returning to the heat equation

Exact units are not relevant for dynamics, and it is instead the ratio of units that we care about. To apply this principle, let us non-dimensionalise the equation Equation 2.1. Let us consider the heat equation as given by the system

$$\begin{aligned}\frac{\partial T}{\partial t} &= \kappa\frac{\partial^2 T}{\partial x^2}\\ T(x,0) &= T_{\text{init}}\\ T(0,t) &= T_a\\ T(L,t) &= T_b.\end{aligned}$$

Note that the constant κ is known as the thermal diffusivity, and is given by

$$\kappa = \frac{k}{\rho c},$$

which were the units previously introduced.

We introduce typical scales for each of the variables. For example, we non-dimensionalise the temperature, distance, and time by setting

$$T = [T]T', \qquad x = [x]x', \qquad t = [t]t'.$$

Remember that via the chain rule, we have that

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}x'}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x'} = \frac{1}{[x]}\frac{\mathrm{d}}{\mathrm{d}x'}.$$

Substitution into the equation now yields

$$\begin{aligned} \frac{\partial T'}{\partial t'} &= \frac{\kappa[t]}{[x]^2}\frac{\partial^2 T}{\partial x^2} \\ T'(x', 0) &= \frac{T_{\text{init}}}{[}T] \\ T'(0, t') &= \frac{T_a}{[T]} \\ T'(L/[x], t') &= \frac{T_b}{[T]}. \end{aligned}$$

Be sure to work out the above for yourself. For example, notice that the previous boundary at $x = L$ is now sent to $x' = L/[x]$. Now we see that the final equation is beautifully simple: At this point, we can identify a crucial non-dimensional parameter given by

$$\Pi = \frac{\kappa[t]}{[x]^2} = \frac{k[t]}{\rho c[x]^2}.$$

Although this looks quite complicated, you can test that this parameter is indeed non-dimensional by looking up the units for thermal conductivity, k, density ρ, and specific heat c, and verifying that indeed Π is non-dimensional.

3.4 Choice of units

It is important to remember that there are technically no wrong choices for the scales [x], [T], and [t], as long as they yield consistent balances. However, some choices are better than others for the context of the problem. Let us select:

$$[x] = L \qquad [T] = T_{\text{init}}.$$

This means our unit of length is “one pipe” (if this is indeed a pipe), and our unit of temperature is whatever the initial temperature was set to be. The choice of time is free at the moment, and therefore we can select time as

$$\Pi = \frac{\kappa[t]}{[x]^2} = 1 \implies [t] = \frac{L^2}{k}.$$

Now we see that the final equation is beautifully simple:

$$\begin{aligned} \frac{\partial T'}{\partial t'} &= \frac{\partial^2 T}{\partial x^2} \\ T'(x', 0) &= 1 \\ T'(0, t') &= \frac{T_a}{T_{\text{init}}} \equiv A, \\ T'(1, t') &= \frac{T_b}{T_{\text{init}}} \equiv B. \end{aligned} \tag{3.1}$$

3.5 Interpretation

What is the whole point of the above exercise?

In the original problem, there were five parameters:

$$\kappa, \qquad T_{\text{init}}, \qquad T_a, \qquad T_b, \qquad L.$$

Imagine a situation where you are performing an analysis of heat spread in different pipes of different materials (characterised by κ) and lengths (L), with different initial and boundary temperatures. That is potentially a highly complicated parameter space to search (five-dimensional).

Rather than solving the problem again and again for each change in parameters, what the analysis yielding (Equation 3.1) reveals is that the parameter space only needs to be two-dimensional, for parameters A and B. Therefore, by solving the problem at many different values of A and B, the solution space can be completely mapped.

Here is an example. Consider heat diffusion through common brick, which possesses a thermal diffusivity of

$$\kappa = 5.2 \times 10^{-7} \mathrm{m}^2/\mathrm{s}.$$

Therefore a brick that is about L = 10cm = 0.1m in size has a typical timescale of

$$[t] = \frac{L^2}{\kappa} \approx 1.9 \times 10^4 \mathrm{s} \approx 5 \mathrm{hrs}.$$

The above choice of [t] is representative of the typical timescale you can expect for heat to diffuse through the brick. As you can see it’s very long.

In summary, we have gone from a system where we needed to consider five parameters to a system where we only need to specify two (essentially, the ratio of the initial heat to the boundary values). This is an enormous simplification.

Moreover, your analysis has further identified the key non-dimensional parameters that appear, including:

$$\Pi = \frac{\kappa[t]}{[x]^2}, \qquad A = \frac{T_a}{T_{\text{init}}}, \qquad B = \frac{T_b}{T_{\text{init}}}, \qquad \frac{L}{[x]}$$

In this particular problem, you had the freedom to choose L to remove one parameter (the ratio of lengths), and time could be chosen to set $\Pi = 1$.

CHAPTER 4

Basic energy models

In this chapter, we introduce the simplest climate model to describe the temperature of the Earth. Despite its apparent simplicity, this basic climate model is the foundation of all further discussion. We will find that, according to the most basic model, the surface temperature of the Earth is expected to be much colder than it is actually; we conclude that the likely culprit is greenhouse gases.

Consider the Earth as a single averaged body. Our goal is to obtain an equation for the surface temperature, T, by considering the energy recieved from the sun versus the energy the Earth must emit into space. This kind of balance-equation procedure is used everywhere in mathematical modelling. In essence, we are looking to write down a precise statement of the following:

Basic energy balance

$$\text{Change in internal energy of Earth} = E_{\text{in}} - E_{\text{out}}, \tag{4.1}$$

where E_{in} is the incoming energy from the sun and E_{out} is the energy leaving the Earth. Note that the left hand-side (the internal energy) will depend on the temperature, T.

4.1 The basic energy model

The basic model is derived as follows.

1. By considering the incoming radiation from the sun, obtain an estimate for the incoming energy, E_{in}.
2. Use a constitutive law that indicates how much outgoing energy is produced by an object heated to some temperature.

3. Equate the change in internal energy to be equal to the difference between the above two items.

Note that the primary components of the global energy balance are radiative fluxes: we receive short-wave radiation (UV and visible light) from the Sun, and emit long-wave radiation (infra-red) to space.

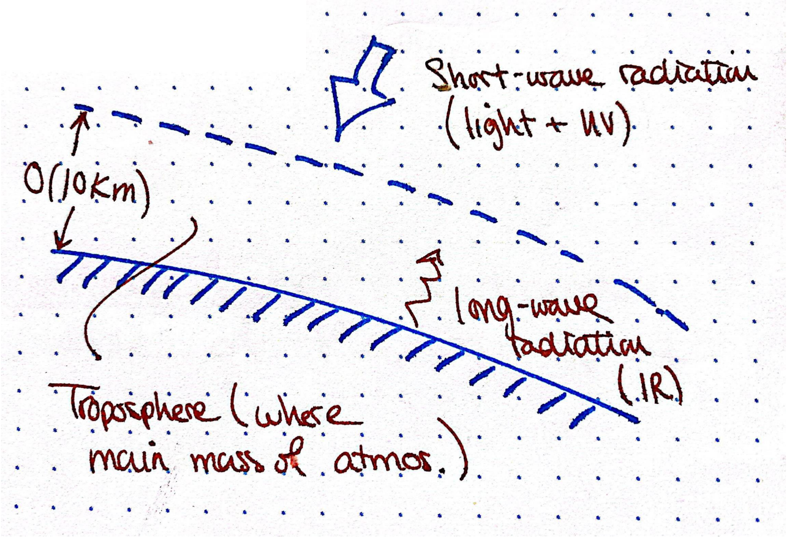


Figure 4.1: Radiation in the atmosphere

First we consider incoming energy.

Energy from the Sun

First, note that the shortwave radiation (UV and radiation) recieved from the sun is $Q \approx 1370\,\mathrm{W{\cdot}m^{-2}}$ (which we consider measured at a point near the planet).
If we consider only that radiation that is absorbed into the Earth, we have

$$E_{\text{in}} = \pi R^2 Q(1-a), \tag{4.2}$$

where R is the Earth's radius.
In the above formula, we have multiplied the flux, Q, with the visible surface area, πR^2. There is an additional multiplication by $(1-a)$ where a is the *planetary albedo*, which characterises amount of energy reflected due to the surface properties. Light surfaces like snow will have high albedo, $a \approx 0.9$, while darker surfaces like the ocean have smaller albedo, $a \approx 0.3$. The global average albedo is $a \approx 0.3$.

Next we consider outgoing energy.

Energy from the Earth

We now wish to characterise the energy, E_{out}, and in the case of Earth, this will correspond to longwave radiation (infra-red) emitted into space. All bodies characterised by a temperature, say T_e, will emit radiation, Q_e. As a model, we can consider Q_e to be given by the Stefan-Boltzmann law, which states that

$$Q_e = \sigma T_e^4, \tag{4.3}$$

where $\sigma \approx 5.67 \times 10^{-8} W m^{-2} K^{-4}$ is the Stefan-Boltzmann constant.
Now although the Earth's surface may emit radiation according to (Equation 4.3), some of this radiation will be absorbed by the atmosphere and reflected back. This is the **greenhouse effect**. As a consequence of the greenhouse gas, the surface temperature of the Earth, T, will be larger than the effective emitting temperature, T_e. For the moment, we model this as

$$T_e = \gamma^{1/4} T_e, \tag{4.4}$$

where $\gamma < 1$ is a greenhouse gas factor, which depends on the properties of the atmosphere.

Now combining the above equations, we have

$$E_{\text{in}} - E_{\text{out}} = \pi R^2 Q(1-a) - 4\pi R^2 \sigma \gamma T^4, \tag{4.5}$$

which gives the incoming energy per unit time.

Internal heat energy

Due to this incoming energy, the Earth will cool or heat in response. We need to know how the internal temperature of an object responds to an input in energy. The general formula is

$$\text{Internal heat energy} = \text{volume} \times (\rho c_p) \times T.$$

The key quantity is the experimentally determined, c_p, which corresponds to the specific heat capacity. It is given in the SI units of $\text{J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1}$, i.e. energy per unit mass per unit temperature. Note that this applied to a shell around the planet of thickness d in the atmosphere, and so the mass is given by

$$\text{mass} = (4\pi R^2) d\rho \tag{4.6}$$

where ρ is the average density of the atmosphere. Let us imagine the increase in temperature, ΔT,during an interval of time, Δt. You can now verify that

$$\Big[(4\pi R^2) d\rho\Big] c_p (\Delta T)$$

returns the units of Joules—i.e. this is the internal energy produced during the time Δt. We then have

$$\Big[(4\pi R^2)d\rho\Big]c_p\Delta T = \Delta t(E_{in} - E_{out}).$$

Putting in (Equation 4.5) and taking the limit of $\Delta t \to 0$, we finally have a heat equation for the Earth's temperature.

Zero-dimensional model for the surface temperature of the Earth

The Earth's temperature, as measured on a layer of thickness d at the troposphere is given by the following ordinary differential equation (ODE) for $T = T(t)$:

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{4}Q(1-a) - \sigma\gamma T^4, \tag{4.7}$$

where we have defined

$$C = \rho c_p d$$

as the heat capacity of the atmosphere. Above, the solar flux, Q is often taken to be $Q = 1370$ W/m2, $\sigma = 5.67 \times 10^{-8}$ W/(m2K4), and $\gamma \leq 1$ is the greenhouse gas factor.

The above equation Equation 4.7 is time-dependent, but we may consider that the surface temperature, either over long-time or in an averaged sense, is dictated by the steady-state (time-independent) solution. Setting $\mathrm{d}T/\mathrm{d}t$ to zero, we see that there is a unique steady-state given by

$$T = \left(\frac{Q(1-a)}{4\sigma\gamma}\right)^{1/4}.$$

If we take $Q \approx 1370\,\mathrm{Wm}^{-2}$, $a \approx 0.3$, $\sigma \approx 5.67 \times 10^{-8}\mathrm{W\,m^{-2}K^{-4}}$, we then get

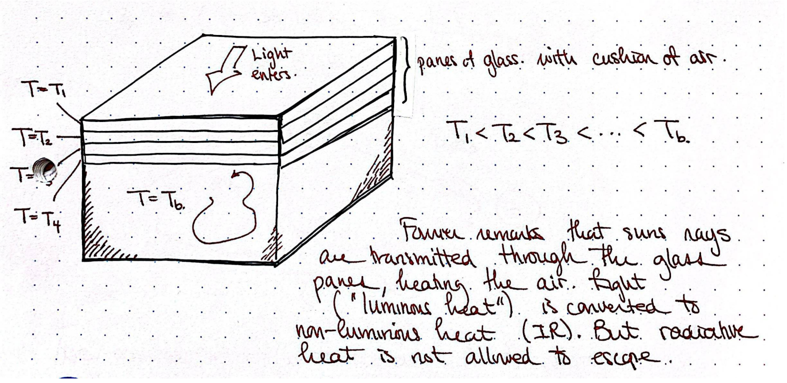
$$T \approx 255\mathrm{K} = -18^\circ\mathrm{C}.$$

under the assumption that $\gamma = 1$. That's pretty cold! The actual average temperature is around $288\mathrm{K} \approx 15^\circ\mathrm{C}$.

The above back-of-the-envelope calculation seems to suggest that the parameter $\gamma < 1$ plays an important role in keeping the Earth warm enough for us to live on, and indeed the value of γ inferred by the above is roughly $\gamma \approx 0.61$. Later on in the course, we will develop a more rigorous model to predict such a γ by studying the properties of the atmosphere.

4.2 The history of global warming

The history of global warming (and hence the estimation of γ) is convoluted, but the origins can be considered as far back as the work of (Fourier, 1827) and (Pouillet, 1838), and is discussed in the work by (Van der Veen, 2000).


Light enters.
panes of glass. with cushion of air.
$T=T_1$
$T=T_2$
$T=$
$T=T_4$
$T=T_b.$
$T_1 < T_2 < T_3 < \cdots < T_b.$

Fourier remarks that suns rays are transmitted through the glass panes, heating the air. Light ("luminous heat") is converted to non-luminous heat (IR). But radiative heat is not allowed to escape.

Figure 4.2: An illustration of Fourier's glass box

4.3 Next in the study of energy balance models?

At this stage, the basic energy balance model studied in this chapter can be made more complex and realistic through many different extensions. For instance:

- Consider the effects of a non-constant and nonlinear albedo. Since the albedo depends on the material property (water vs. ice vs. land), we can incorporate a toy model for spatial variability by allowing the albedo to depend on temperature, $a = a(T)$.
- Consider the time-dependence. What happens if the system starts out of the steady state? Does it tend to relax to its equilibrium profile? Can we analyse the temporal properties of the solutions?
- Can we incorporate latitude (or longitudinal) dependence into the model. For instance, should the albedo and solar radiation be considered as functions of the latitude?
- Should we incorporate the effects of diffusion (heat spreading out) or convection (heat transported with flows)?
- To what extent should we consider the effects of the atmosphere on the absorption and reflection properties of the incoming (and outgoing) energy?

Many of these questions are much more involve, as they require differential or partial differential equations. Therefore in the next few chapters, we will discuss, in a very general way, the collection of applied and numerical mathematics techniques that can be used when studying problems in the physical sciences.

Part II

Practical applied mathematics

As we go deeper into formulating the equations that model or govern aspects of Planet Earth, we will quickly come to the realisation that many such equations, even for the simplest minimal models, are not exactly solvable.

For example, in the analysis of the Ocean models later in the course, we develop the following "simple" model for the temperature in the ocean:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \delta(1-x) - |f(x,y)|x,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - y - |f(x,y)|y,$$

where we have introduced the function,

$$f(x,y;R,\lambda) = \frac{1}{\lambda}(Rx - y),$$

This is quite a difficult problem! This is essentially a set of two nonlinear differential equations for two unknowns and three parameters. What kind of practical applied mathematics can we apply to study such problems?

The intention of this part is to introduce (and in some cases, review) three key concepts:

1. Asymptotic approximations.
2. Numerical solutions of differential equations.
3. Numerical solutions of nonlinear equations (Newton's method).

as well as any additional tools you may need to study the kinds of differential equations involved in Maths of Planet Earth!

CHAPTER 5

Asymptotic approximations I

One powerful set of techniques for approximating solutions to equations is called *asymptotic analysis* or *perturbation theory*. To begin with, in this chapter, we introduce you to these techniques as a means to approximating the solutions to equations like:

$$\epsilon x^2 + x - 1 = 0,$$

when ϵ is a small parameter.

Soon, we apply the techniques to approximating solutions of differential equations.

5.1 A simple quadratic

A singular quadratic

Consider the solution of

$$\epsilon x^2 + x - 1 = 0, \tag{5.1}$$

where ϵ is a fixed and very small positive number, say 0.000001. Forget that we know how to solve a quadratic equation: is it possible to develop a systematic approximation method?

If $\epsilon = 0$, then $x = 1$. Moreover, if we substitute $x = 1$ into the equation, then we see that the error is small and proportional to ϵ. It is natural to seek an approximation in powers of ϵ. We call this an *asymptotic expansion*. We write

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$$

Substitution into the equation yields

$$\epsilon\Big(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots\Big)^2 + \Big(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots\Big) - 1 = 0.$$

Expand and collect terms in powers of ϵ:

$$\Big(x_0 - 1\Big) + \epsilon\Big(x_1 + x_0^2\Big) + \epsilon^2\Big(x_2 + 2x_0x_1\Big) + ... = 0.$$

Now we equate coefficients at each order in ϵ. This gives

$$\begin{aligned} x_0 - 1 = 0 &\Longrightarrow x_0 = 1 \\ x_1 + x_0^2 = 0 &\Longrightarrow x_1 = -1 \\ x_2 + 2x_0x_1 = 0 &\Longrightarrow x_2 = 2 \end{aligned}$$

We therefore have obtained the three-term approximation,

$$x = 1 - \epsilon + 2\epsilon^2 + ...$$

Clearly we could continue this process *ad infinitum* obtaining increasingly accurate approximations to one of the roots.

The singular root

But where has the other quadratic root gone?

The problem is that in considering ϵ to be small, we began by ignoring the leading term, ϵx^2. We effectively assumed that the equation was primarily *balanced* by setting the x term with the -1 term, and the sum of the two terms approximately equalling zero.

But if $|x|$ is large, then clearly our assumption that ϵx^2 being small may not be necessarily true for it depends on how large $|x|$ is compared to ϵ. Note that if $|x|$ is large, then necessarily the last term, -1, is negligible in comparison. Therefore, in order for ϵx^2 to balance x, we see that $|x|$ must be of size $1/\epsilon$.

Therefore this suggests that we should re-scale our solution as follows

$$x = \frac{X}{\epsilon}.$$

Substitution into the original quadratic now yields

$$X^2 + X - \epsilon = 0.$$

Now notice that $\epsilon = 0$ expresses the correct balance in order to detect that missing root. Again we write

$$X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + ...$$

and attempt to solve order by order. Substitution into the equation yields

$$\Big(X_0 + \epsilon X_1 + \epsilon^2 X_2 + ...\Big)^2 + \Big(X_0 + \epsilon X_1 + \epsilon^2 X_2 + ...\Big) - \epsilon = 0.$$

Expand and collect orders of ϵ:

$$X_0^2 + X_0 = 0 \implies X_0 = -1$$
$$2X_0X_1 + X_1 - 1 = 0 \implies X_0 = -1,$$

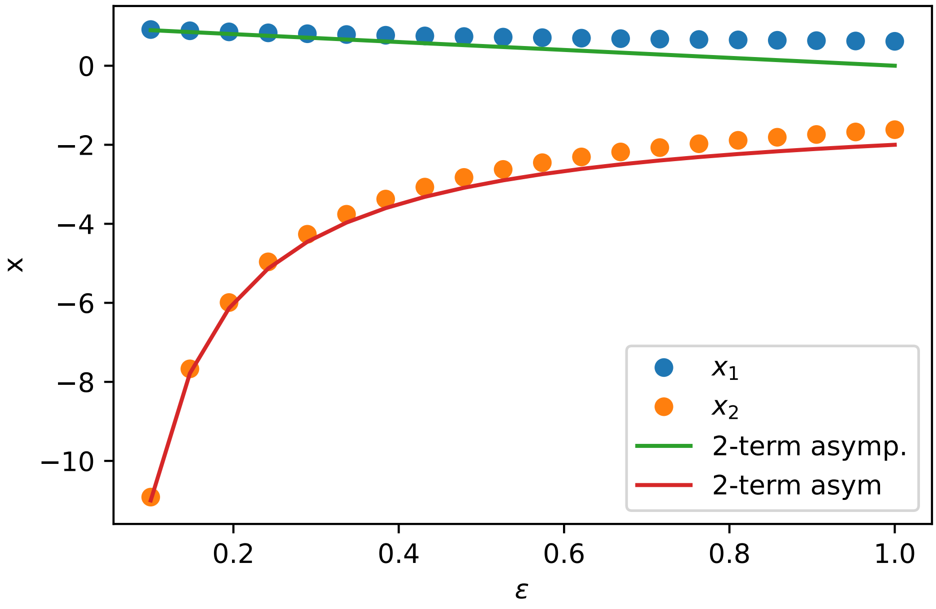
and thus to two orders, we have

$$X = -1 - \epsilon + \ldots \implies x = -\frac{1}{\epsilon} - 1 + \ldots$$

Of course, we have used a very simple example (a solvable quadratic) to illustrate the idea of asymptotic approximations, but you should hopefully see that this method is extensible to much more complicated equations.

```python
import numpy as np
import matplotlib.pyplot as plt

ep = np.linspace(1,0.1, 20)
root1 = (-1 + np.sqrt(1-4*ep*(-1)))/(2*ep)
root2 = (-1 - np.sqrt(1-4*ep*(-1)))/(2*ep)
asym1 = 1 - ep
asym2 = -1/ep - 1
plt.plot(ep, root1, 'o')
plt.plot(ep, root2, 'o')
plt.plot(ep, asym1, '-')
plt.plot(ep, asym2, '-')
plt.legend(['$x_1$', '$x_2$', '2-term asymp.', '2-term asym'])
plt.xlabel('$\epsilon$')
plt.ylabel('x');
```

```
<>:14: SyntaxWarning: invalid escape sequence '\e'
<>:14: SyntaxWarning: invalid escape sequence '\e'
/var/folders/lw/7v5bfgnj6gj4qhr6fykhcp3h0000gn/T/ipykernel_31852/3662636573.py:14: S
  plt.xlabel('$\epsilon$')
```



5.2 Order notation and the tilde sign for asymptotic

We define precisely what we mean when we say that two functions, say f and g, exhibit the same behaviour in some limit, say $\epsilon \to 0$ or $x \to x_0$ or $x \to \infty$ and so forth. For instance, we claim that the graphs of $\sin(x)$ and x look very similar as $x \to 0$. Thus we might write

$$\sin(x) \sim x \quad \text{as } x \to 0. \tag{5.2}$$

This notation of similarity allows us to specify functional behaviours at a deeper level than just limits. As you can see, it is not as useful to specify that

$$\lim_{x\to 0} \sin x = \lim_{x \to 0} x.$$

In contrast, the asymptotic relation is much more prescriptive about the way that the functions are approaching the limit.

Definition of $\sim$, $\gg$, and $\ll$

First, the notation

$$f(x) \ll g(x), \qquad x \to x_0,$$

is read as "$f(x)$ is much smaller than $g(x)$ as $x \to x_0$" and means

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

We may analogously use $g(x) \gg f(x)$ for "much greater than…".

Second, the notation

$$f(x) \sim g(x), \qquad x \to x_0,$$

is read as "$f(x)$ is asymptotic to $g(x)$ as $x \to x_0$", and means that the error between f and g tends to zero as $x \to x_0$, or

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1.$$

We will often say "f is like g" or "f behaves like g",

Here are some examples.

Examples

- $\sin x \sim x \sim \tan x$ as $x \to 0$
- $x^2 + x + 1 \sim \dfrac{x^3 + \sin x}{1 + x}$ as $x \to \infty$
- $\sin x \ll \cos x$ as $x \to 0$

In the examination of limiting processes, often the main issue of consideration is the relative sizes of quantities defined according to their powers. For example, if x is a very small number, with $x = 10^{-5}$, then x^5 is much smaller than x (in terms of our notation, $x^5 \ll x$ as $x \to 0$). On the other hand, we might not care so much about the difference between

$$x^5 \quad \text{vs.} \quad 5x^5$$

The point is that the *order* of x^5 and $5x^5$ is the same as $x \to 0$. The "Big-Oh" notation formalises this distinction.

Definition of Big-Oh

We write $f = O(g)$ as $x \to x_0$ to mean that there exists constants $K > 0$ and $x^* > 0$ such that

$$|f| < K|g| \quad \text{for all } |x - x_0| < x^*.$$

In practice, the use of the order symbol is very natural and you will not need to work with the technical definition. For example, when you derive the terms of the Maclaurin/Taylor series, you are naturally clustering all the terms of the same order (power) together. For us, the O symbol provides a very convenient way of separating terms of different sizes.

Examples

- $2\sin x = O(\tan x)$ as $x \to 0$
- $x^2 + x + 1 = O\left(\dfrac{5x^3 + \sin x}{1 + x}\right)$ as $x \to \infty$

Let us return to the case of the quadratic example (Equation 5.1). Using the O notation, we can write

$$x = \begin{cases} 1 - \epsilon + 2\epsilon^2 + O(\epsilon^3) \\ -\frac{1}{\epsilon} - 1 + O(\epsilon^2) \end{cases}$$

for the two roots. Alternatively, we can truncate the expansions and simply using the $\sim$ symbol:

$$x \sim \begin{cases} 1 - \epsilon \\ -\frac{1}{\epsilon} - 1 \end{cases}$$

CHAPTER 6

Asymptotic approximations II

In the previous chapter, we learned about the technique of asymptotic expansions, whereby the solution of an equation is expressed in terms of a series in powers of a small parameter:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$$

The precise choice of power progression (here integer powers of ϵ will depend on the particular problem. The same idea can be extended to approximating solutions of differential equations. The upshot of this procedure is that at each order of the scheme, a simpler problem can be studied.

Again it is best to demonstrate through examples.

6.1 Returning to the projectile problem

In **?@sec-pclass1** and Chapter 12 you studied the non-dimensionalisation of the projectile problem. Once re-scaled, it takes the following form:

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= -\frac{1}{(1+\epsilon y)^2}, \qquad t > 0 \\ y(0) &= 0, \\ y'(0) &= 1. \end{aligned} \tag{6.1}$$

This is a difficult problem without, in fact, any explicit solutions. However, we can estimate the solution in the limit $\epsilon \to 0$. We expand the solution as

$$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots$$

In order to expand the denominator, you can use Taylor's theorem to expand the function

$$f(x) = (1+x)^\alpha = f(0) + f'(0)x + \dots = 1 + \alpha x + \dots$$

around $x = 0$.

The differential equation now yields

$$y_0'' + \epsilon y_1'' + \epsilon^2 y_2'' + ... = -\left[1 - 2\epsilon\left(y_0 + \epsilon y_1 + ...\right) + ...\right]$$

so grouping terms together order-by-order yields

$$\Big[y_0'' + 1\Big] + \epsilon\Big[y_1'' - 2y_0\Big] + ... = 0.$$

We can similarly substitute the expansion into the initial conditions. Altogether, at leading order, we obtain the following system to solve:

$$\begin{aligned} y_0'' + 1 &= 0, \\ y_0(0) &= 0, \\ y_0'(0) &= 1. \end{aligned}$$

Integrating twice and applying the boundary conditions gives us

$$y_0(t) = -\frac{1}{2}t^2 + t.$$

In fact, this is simply the parabolic motion you would expect from school Physics. The $\epsilon = 0$ solution corresponds to assuming that the mass at the centre of the planet is dominant and then acceleration is constant.

However, we can now proceed to higher order and examine the nonlinear effects. Proceeding to $O(\epsilon)$, we have the following system to solve:

$$\begin{aligned} y_1'' &= 2y_0, \\ y_1(0) &= 0, \\ y_1'(0) &= 0. \end{aligned}$$

Notice the boundary conditions come from the fact there are no ϵ corrections in the original boundary conditions, so $y_n(0) = y_n'(0) = 0$ for all $n > 0$. Again this system is simple to integrate. Integrating the solution for y_0 twice and substitution of the initial conditions yields

$$y_1(t) = -\frac{1}{12}t^4 + \frac{1}{3}t^3.$$

We have thus solved for the asymptotic approximation to two orders. We have

$$y(t) \sim \Big[-\frac{1}{2}t^2 + t\Big] + \epsilon\Big[-\frac{1}{12}t^4 + \frac{1}{3}t^3\Big].$$

This was quite an accomplishment! We have taken a problem that was not easily solvable in explicit form and through fairly simple integrations, obtained an approximation to two orders in ϵ. How good is it? Let us solve the problem numerically and compare with the asymptotic approximation.

6.2 Numerical solutions of IVPs

We first demonstrate how to solve ODEs (initial-value-problems, IVPs) using black-box functions in Python. For starters, most numerical formulations for ODEs will require that the problem be posed in terms of a first-order system of equations. To convert (Equation 6.1) into such a form, create a set of unknowns for the derivatives. Set

$$\mathbf{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

Then we have the following first-order system:

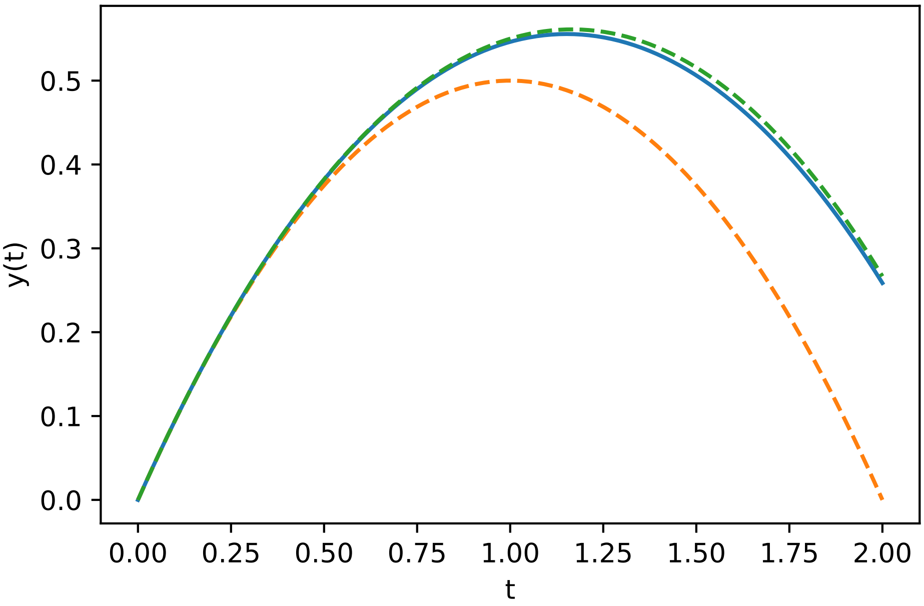
$$\begin{aligned} \mathbf{Y}'(t) &= \mathbf{F}(t, \mathbf{Y}(t)) = \begin{pmatrix} y_1' \\ -\frac{1}{(1+\epsilon y_1)^2} \end{pmatrix} \\ \mathbf{Y}(0) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{6.2}$$

You can find a little guide on using solve_ivp in Python here. Here is the Python code to solve the differential equation.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

ep = 0.2 # epsilon value
tmax = 2 # max time
t = np.linspace(0, tmax, 100) # mesh used for plotting

# Define function for the ODE
def f(t, Y):
    ep = 0.2
    y, yp = Y
    ypp = -1/(1 + ep*y)**2
    return [yp, ypp]

# define the initial condition
Y0 = [0, 1]

sol = solve_ivp(f, [0, tmax], Y0, dense_output=True)

# Prior to plotting, re-interpolate solution on a fine grid
yy = sol.sol(t)
# Asymptotic solutions
```

```
y0 = -1/2*t**2 + t
y1 = -1/12*t**4 + 1/3*t**3

# Plot it all
plt.plot(t, yy[0,])
plt.plot(t, y0, '--')
plt.plot(t, y0 + ep*y1, '--')
plt.xlabel('t')
plt.ylabel('y(t)')
```

```
Text(0, 0.5, 'y(t)')
```



The two-term approximation does beautifully well, even at this moderate value of $\epsilon = 0.2$.

CHAPTER 7

Numerical solutions of IVPs

In the previous section, we used built-in ODE solvers to develop numerical solutions. It is important to gain an understanding how a simple ODE solver works. The simplest scheme is called Euler's method, and this we now explain.

Begin from the system (Equation 6.2). We assume that the solution is represented by a discrete set of points, $\mathbf{Y}_n = \mathbf{Y}(t_n)$ at the times $t_0 = 0$, $t_1 = \Delta t$, $t_2 = 2\Delta t$, and so on. The time derivative is written as a discrete derivative while we approximate the right hand side by its value at the nth time step:

$$\frac{\mathbf{Y}_{n+1} - \mathbf{Y}_n}{\Delta t} = \mathbf{F}(t_n, \mathbf{Y}_n)$$

Rearranging yields a very simple algorithm for solving the ODE:

$$\mathbf{Y}_n = \mathbf{Y}_{n-1} + \mathbf{F}(t_{n-1}, \mathbf{Y}_{n-1})\Delta t$$

for $n = 1, 2, 3, \ldots$

This would be implemented via the following pseudocode:

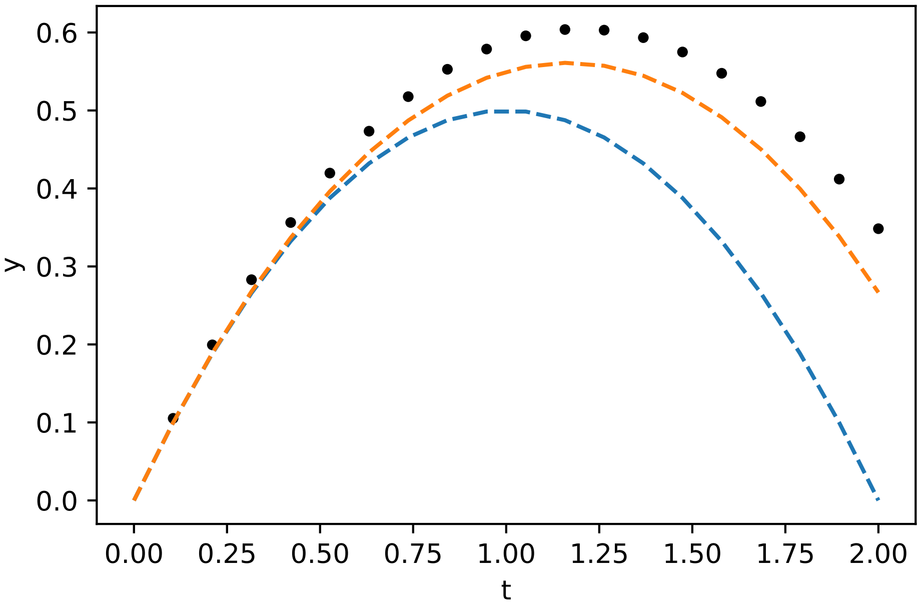
Euler's method

```
1. Input: function f(t, Y)
          time step, dt
          initial condition, Y0

2. Set initial condition Y = Y0

2. Take one Euler step and overwrite previous value
```

```
                  Y = Y + f(t, Y)

3. Increment t by dt and goto 2
```

Euler's method is conceptually simple but quite inaccurate. But in this case, we see that it works fairly well in comparison to the built-in solvers.

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

ep = 0.2            # epsilon value
tmax = 2            # max time
N = 20              # number of steps
t = np.linspace(0, tmax, N) # mesh used for plotting
dt = t[1] - t[0]

# Define function for the ODE
def f(t, Y, ep):
    y, yp = Y
    ypp = -1/(1 + ep*y)**2
    return np.array([yp, ypp])

# define the initial condition
Y = [0.0, 1.0]
ti = 0

# define the solution vector
for i in range(1, N):
    ti = ti + dt  # Increment time
    Y = Y + f(ti, Y, ep)*dt # Euler step
    plt.plot(ti, Y[0], 'k.')

# Asymptotic solutions
y0 = -1/2*t**2 + t
y1 = -1/12*t**4 + 1/3*t**3
plt.plot(t, y0, '--')
plt.plot(t, y0 + ep*y1, '--')
plt.xlabel('t');
plt.ylabel('y');
```


0.6
0.5
0.4
0.3
0.2
0.1
0.0
y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t

CHAPTER 8

Asymptotic approximations III

8.1 Regular vs. singular problems

In Chapter 6, we studied how asymptotic expansions can be used to approximate equations like

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\frac{1}{(1+\epsilon y)^2}$$

by expanding the solution as $y(t) = y_0(t) + \epsilon y_1(t) + ...$ These are known as *regular* problems because a small perturbation, ϵ, does not seem to fundamentally change the $\epsilon = 0$ solution beyond a small perturbation. This is not always the case. In *singular* problems, the situation of $\epsilon \neq 0$ is fundamentally different than the situation from $\epsilon = 0$. You have already seen such an example in Chapter 5. The equation

$$\epsilon x^2 + x - 1 = 0$$

has one root for $\epsilon = 0$ and two roots for non-zero small ϵ—even infinitesimally small values! This is quite interesting. From a wider scientific perspective, you may wonder what other problems in nature possess such singular effects.

The point of this lecture is study a technique known as *matched asymptotics*. These matched asymptotics are often necessary for singularly perturbed differential equations.

8.2 A singular first-order ODE problem

Previously in Chapter 4, we derived a basic equation that governs the temperature on the surface of the planet. This equation had the following form:

$$(\rho c_p V)\frac{\mathrm{d}T}{\mathrm{d}t} = E_{\mathrm{in}}(t, T) - E_{\mathrm{out}}(t, T).$$

For the purpose of this section, let us make up a toy model. We suggest, in non-dimensional form,

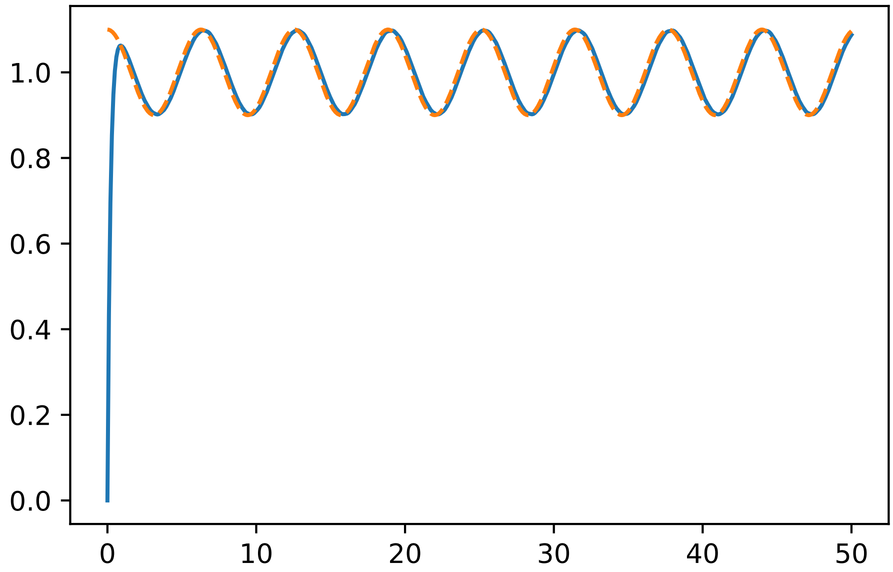
$$\begin{aligned} \epsilon \frac{\mathrm{d}T}{\mathrm{d}t} &= R(t) - T, \quad t \geq 0 \\ T(0) &= T^*, \end{aligned} \tag{8.1}$$

where we consider $\epsilon > 0$ and $\epsilon \ll 1$. You can think of the above model as modelling the temperature on a substance that radiates heat in a fashion proportional to itself ($-T$) and is being subjected to an (incoming) heat source, R. Let us take as an example,

$$R(t) = 1 + A\cos(t).$$

Our choice for R is not so important. This equation is, in fact, solvable in closed form (how?) but let us get additional practice solving numerically.

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

Ts = 0
ep = 0.2
A = 0.1
f = lambda t, T: 1/ep*(1 + A*np.cos(t) - T)

tmax = 50
sol = solve_ivp(f, [0, tmax], [Ts], dense_output=True)

t = np.linspace(0, tmax, 1000)
y = sol.sol(t)
plt.plot(t, y[0])
plt.plot(t, 1 + A*np.cos(t), '--')
```



What do we observe? If $\epsilon = 0$, then we expect the solution $T \sim R(t)$. This is shown with the dashed line. However, this solution does not satisfy the necessary initial condition. We observe that near $t = 0$, the exact solution seems to very rapidly diverge from the approximation in order to satisfy the proper boundary condition. The region in t where this rapid change occurs is called a *boundary layer*.

If we repeat the experiment with an even smaller value of ϵ, we would observe that the size of this boundary layer seems to tends to zero as $\epsilon \to 0$. This numerical experiment thus inspires the following method.

8.3 Boundary layer theory

We seek a method that will allow us to develop a uniformly valid approximation, i.e. an approximation that is good everywhere in the relevant domain, $t \geq 0$. Begin by performing the usual asymptotic approximation:

$$T(t) = T_0(t) + \epsilon T_1(t) + \epsilon^2 T_2(t) + \ldots$$

Substitution into the ODE (Equation 8.1) yields at leading order,

$$0 = R(t) - T_0(t) \implies T_0(t) = R(t) = 1 + A\cos(t).$$

As we have noted, this approximation fails to satisfy the initial condition, $T(0) = T^*$ in general. It is possible to go to higher order but this is not so important at the moment. So for now, we have obtained:

$$T_{\text{outer}} \sim \left[1 + A\cos t\right].$$

We have chosen to refer to this as the *outer* solution for reasons that will be abundantly clear. But rather than satisfying $T(0) = T^*$, this approximation has the limiting behaviour of

$$\lim_{t\to 0} T_{\text{outer}} \sim \Big[1 + A\Big]$$

Above, we have only included the leading term in the limit expression.

The inner scaling

Our intuition follows a very similar logic to the examination of the singular root in Section 5.1. Above, our naive assumption was that $\epsilon T'(t)$ could be ignored since ϵ is a small number. However, this may not be the case if the gradient is very large.

Our intuition further suggests that the boundary layer occurs near $t = 0$ and that it scales in size with ϵ. Therefore, let us set

$$t = \epsilon^{\alpha} s,$$

as a change of coordinates. We expect $\alpha > 0$ (otherwise t is not small), and within this region, we expect the new coordinate, s, to be $O(1)$ (of moderate size). We then transform the unknown function:

$$T(t) = T(\epsilon s) = U(s),$$

and seek a new differential equation for U. By the chain rule,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \epsilon^{-\alpha}\frac{\mathrm{d}U}{\mathrm{d}s}.$$

Before substituting into the equation, we are prudent to examine the behaviour of $R(t)$ near $t = 0$. We know by Taylor's theorem that

$$R(t) = 1 + A\left(1 - \frac{t^2}{2} + \ldots\right).$$

Therefore, under the substitution, we may approximate R by its leading terms:

$$R(\epsilon^{\alpha} s) \sim 1 + A.$$

For now, we will not need more terms than this. Substituting into the ODE now gives

$$\epsilon^{1-\alpha}\frac{\mathrm{d}U}{\mathrm{d}s} \sim \Big[1 + A\Big] - U.$$

Now in order to involve the first term, it is sensible to select

$$1 - \alpha = 0 \Longrightarrow \alpha = 1.$$

The inner equation

Therefore, the correct coordinate re-scaling was the 'obvious' one:

$$t = \epsilon s.$$

Substituting this again in (Equation 8.1):

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}s} &= 1 + A\cos(\epsilon s) - U, \\ U(0) &= T^*. \end{aligned}$$

The procedure is now exactly the same. We expand

$$U(s) = U_0(s) + \epsilon U_1(s) + \epsilon^2 U_2(s) + \ldots$$

At leading order, we get

$$\begin{aligned} U_0' &= 1 + A - U_0 \\ U_0(0) &= T^*. \end{aligned}$$

The above ODE can be solved by integrating factors. Multiplying both sides be e^s, we have

$$(U_0 e^s)' = (1 + A)e^s.$$

Integrate and use the initial condition:

$$U_0(s) = (1 + A) + (T^* - (1 + A))e^{-s}.$$

This is exactly what we expect. Notice that

$$\lim_{s\to\infty} U_0(s) = \lim_{t\to 0} T_{\text{outer}} \tag{8.2}$$

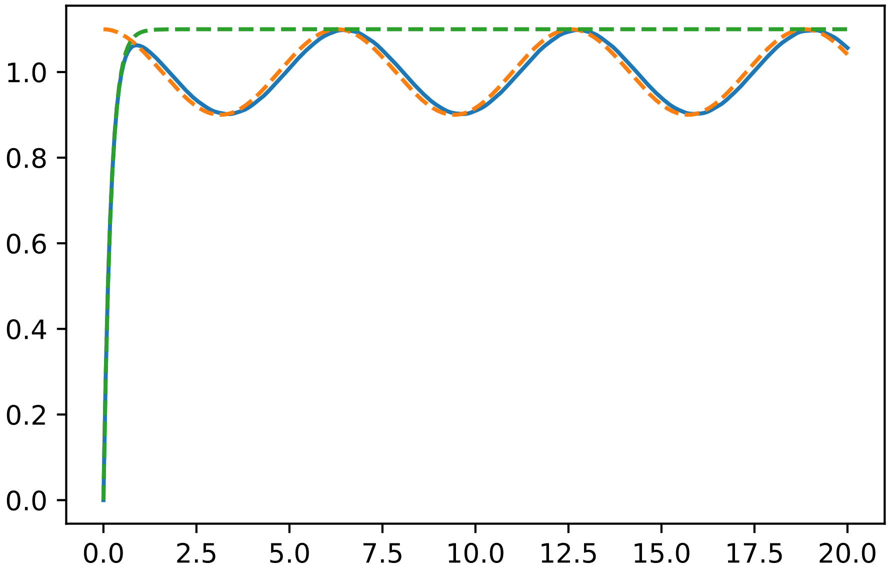
therefore the outer limit of our inner solution matches the inner limit of our outer solution. In terms of outer coordinates, our inner solution is approximated as follows:

$$T_{\text{inner}} \sim (1 + A) + (T^* - (1 + A))e^{-t/\epsilon}.$$

Let's finally plot this with our previous curves:

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

Ts = 0
ep = 0.2
A = 0.1
```

```
f = lambda t, T: 1/ep*(1 + A*np.cos(t) - T)

tmax = 20
sol = solve_ivp(f, [0, tmax], [Ts], dense_output=True)

t = np.linspace(0, tmax, 1000)
y = sol.sol(t)
plt.plot(t, y[0])
plt.plot(t, 1 + A*np.cos(t), '--')
plt.plot(t, (1 + A) + (Ts - (1 + A))*np.exp(-t/ep), '--')
```



It works beautifully!

Summary

Let us summarise the procedure of matched asymptotics.

1. Expand the solution of the differential equation naively in the typical asymptotic expansion (e.g. in powers of ϵ).
2. Notice that the approximation does not satisfy certain boundary conditions.
3. Re-scale the coordinates in the 'inner' regions.
4. Develop an inner solution that satisfies the boundary condition. Ensure it matches the outer solution.

You will get more practice of this procedure in the problem sets.

CHAPTER 9

Nonlinear root finding

Before we move on back to the subject of applications, it is worth providing an introduction to methods for solving nonlinear equations. Many problems you will encounter in applications, including for ordinary or partial differential equations, can be re-formulated as the solution of a nonlinear system of equations.

Newton's method is the most well-known scheme for solving nonlinear equations. Suppose we wish to solve the scalar equation,

$$f(x) = 0,$$

given some initial guess, $x = x_0$, of the root. Suppose the root lies at $x = x^*$. Then by Taylor's theorem,

$$f(x^*) = f(x_0) + f'(x_0)(x^* - x_0) + O(f''(x_0)(x - x_0)^2).$$

If we assume the quadratic terms are negligible then solve for x^* this gives

$$x^* \approx x_0 - \frac{f(x_0)}{f'(x_0)}.$$

There is a geometrical interpretation of the above. Essentially, in order to estimate the root of $f(x) = 0$, we have used the tangent line at the point $x = x_0$, and used the intersection of this tangent line with the axis as the approximation. This procedure can then be iterated.

Thus, provided that the desired root, x^*, is such that $f'(x^*) \neq 0$, and x_0 is sufficiently close to x^*, then the following iterates converge to the root:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

9.1 Demo of Newton's method for scalar equations

Here is a simple demonstration of Newton's method in order to solve for one of the roots of the following:

$$f(x) = x^3 + x - 1.$$

We will start with the initial guess of $x_0 = 0$. We can do this by hand in lectures using a pocket calculator. The solution is $x^* \approx 0.6823278$..

i	xi	f(xi)	f'(xi)	-f(xi)/f'(xi)	error
0					
1					
2					

```python
import numpy as np

x0 = 0
N = 10

def Newton(f, df, x, maxiter=10):
    i = 0
    while (i < maxiter):
        err = f(x)
        x = x - err / df(x)
        print("f(x) = ", np.abs(err), ", x = ", x)
        i = i + 1
    return x, err

f = lambda x: x**3 + x - 1
df = lambda x: 3*x**2 + 1

x, err = Newton(f, df, x0, N)
print("Final approximation = ", x)
```

```
f(x) =  1 , x =  1.0
f(x) =  1.0 , x =  0.75
f(x) =  0.171875 , x =  0.686046511627907
f(x) =  0.008941036638283384 , x =  0.6823395825973142
f(x) =  2.823062168566537e-05 , x =  0.6823278039465127
f(x) =  2.839946056099052e-10 , x =  0.6823278038280194
f(x) =  2.220446049250313e-16 , x =  0.6823278038280193
f(x) =  1.1102230246251565e-16 , x =  0.6823278038280193
```

```
f(x) =  1.1102230246251565e-16 , x =  0.6823278038280193
f(x) =  1.1102230246251565e-16 , x =  0.6823278038280193
Final approximation =  0.6823278038280193
```

It is a good idea to also learn how to do this using built-in packages. The 'fsolve' function provides a Newton-like nonlinear solver. In fact, it can estimate the Jacobian (derivative), so only the function values need to be provided.

```python
import numpy as np
from scipy.optimize import fsolve

f = lambda x: x**3 + x - 1

x0 = 1
x, info, ier, msg = fsolve(f, x0, full_output=True)

print(msg)
print(x)
```

```
The solution converged.
[0.6823278]
```

9.2 Newton's method for systems of nonlinear equations

Newton's method generalises naturally to the case of a system of equations. Suppose we wish to solve for the n unknowns $\mathbf{x} = (x_1, \ldots, x_n)$ via

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ \ldots \\ F_n(\mathbf{x}) \end{pmatrix} = 0.$$

We have, via Taylor's formula,

$$\mathbf{F}(\mathbf{x}_{i+1}) \sim \mathbf{F}(\mathbf{x}_\mathrm{i}) + J(\mathbf{x}_i)(\mathbf{x}_{i+1} - \mathbf{x}_i) + \mathcal{O}(||\mathbf{x}_{i+1} - \mathbf{x}_i||^2),$$

where J is the Jacobian matrix

$$J(\mathbf{x}) = \nabla \mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \ldots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \ldots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}.$$

Therefore, Newton's method forms the iterates of

$$\mathbf{x}_{i+1} = \mathbf{x}_i - J^{-1}(\mathbf{x}_i)\mathbf{F}(\mathbf{x}_i),$$

which takes a very similar form to the scalar case.

However, solution of the inverse of J is typically inefficient, and it is better to instead solve for $\delta_{i+1} = \mathbf{x}_{i+1} - \mathbf{x}_i$ via

$$J(\mathbf{x}_i)\delta_{i+1} = -\mathbf{F}(\mathbf{x}_i),$$

and then calculate $\mathbf{x}_{i+1} = \mathbf{x}_i + \delta_{i+1}$. There are many ways of solving the above matrix problem efficiently using built-in routines that perform, e.g. Gaussian elimination.

9.3 Secant method

In many situations, evaluation of the Jacobian (or derivative) is the most time-consuming or difficult part of a nonlinear solver. Built-in solvers like 'fsolve', in fact, have the ability to approximate the derivative numerically.

The Secant Method is similar to Newton's Method but replaces the derivative by a finite difference. Geometrically, the tangent line is replaced with a line through the two last known guesses. The algorithm goes as follows.

Secant method

1. Develop two initial guesses to the solution, x0 and x1
2. Compute

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}.$$

CHAPTER 10

Numerical solutions of PDEs

We shall begin by introducing the simplest finite-difference routine for solving a PDE using Euler time stepping. Let us consider the solution of the heat equation on a finite interval:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$
$$u(0,t) = 1, \quad u(1,t) = 2, \quad u(x,0) = 2.$$

10.1 Finite difference problem

We consider a discretisation of the spatial domain $x \in [0,1]$ via the $(N+1)$ points

$$x_0 = 0, \quad x_1 = \Delta x, \quad x_2 = 2\Delta x, \quad \ldots, \quad x_{N+1} = 1.$$

Typically Δx is small. We also have time steps of size Δt, and thus going from $t_0 = 0$, $t_1 = \Delta t$, and so forth. We represent the solution by the indexing scheme:

$$u(x_j, t_k) = u_j^j.$$

Like for Euler's method, we consider the approximation of the time derivative as follows:

$$\left(\frac{\partial u}{\partial t}\right)_j^k \approx \frac{u_j^{k+1} - u_j^k}{\Delta t}, \qquad \text{for } 0 \leq j \leq N+1,\, k \geq 0.$$

In order to approximate the second-order derivative, Hwe will use

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j^k \approx \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{(\Delta x)^2}, \qquad 1 \leq j \leq N,\, k \geq 0.$$

Note that the finite difference cannot be applied directly to the first and last points ($j = 0$ and $j = N+1$). Substituting the above finite differences into the heat equation, we see that we now have the prescription:

$$u_j^{k+1} = u_j^k + \frac{\Delta t}{(\Delta x)^2} \left[u_{j+1}^k - 2u_j^k + u_{j-1}^k\right],$$

which applies for $j = 1, 2, \ldots, N$.

10.2 Boundary and initial conditions

The initial conditions are implemented by setting

$$u_j^0 = 2, \qquad \text{for } j = 0, 1, 2, \ldots, N+1.$$

The boundary conditions require

$$u_0^k = 1 \quad \text{and} \quad u_{N+1}^k = 2, \quad \text{for all } k \geq 0.$$

During the lectures, we will examine script `Chap10-SolvingPDEs.ipynb` on the course scripts to see an example implementation. A similar application is applied in `Chap10-winecellar.ipynb` to solve a heat equation.

Part III

Energy balance models

CHAPTER 11

EBM with nonlinear albedo

Recall that we previously introduced the basic energy balance model in Chapter 4. There, we derived the basic heat equation model (Equation 4.7) for the Earth's temperature given by the following ordinary differential equation (ODE) for $T = T(t)$,

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = Q(1-a) - \sigma\gamma T^4, \tag{11.1}$$

where we have defined $C = \rho c_p d$ as the heat capacity of the atmosphere. Above, the solar flux, Q is often taken to be $Q = 1370/4 = 342$ W/m2, $\sigma = 5.67 \times 10^{-8}$ W/(m2K4), and $\gamma \leq 1$ is the greenhouse gas factor.

In this chapter, we discuss the effects of considering a nonlinear albedo, $a = a(T)$, as well as some of the numerical and analytical tools at our disposal for studying the above equation.

11.1 Steady-state analysis

Below, we shall let $T = T_\infty$ be the steady-state solution that is independent of time. Whenever convenient, we will drop the subscript notation and simply refer to the steady state as T.

Previously, we have assumed that the planetary albedo, a, is constant and independent of temperature. In actuality, water can turn to snow and ice and vice versa; since snow and ice have much higher albedo than open water, then we should consider $a = a(T)$.

Let us assume that there are two relevant ranges to consider: T < 150K (cold) and T > 280K (hot). Let us assume that the albedo is, in these two regions:

$$a(T) \approx \begin{cases} 0.7 & \text{if } T < 150\text{K}, \\ 0.3 & \text{if } T > 280\text{K}. \end{cases}$$

The above guarantees that more energy is reflected if temperatures are low. To model this process, we can use a ramp function to specify the albedo over all temperatures:

$$a(T) = A - B\tanh\left(k(T - 265)\right). \tag{11.2}$$

where $A = 0.5$, $B = 0.2$, $k = 0.1$, and $T_0 = 265$K. Recall that the tanh function is given by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Let us further assume that the system is in steady state, so that the temperature is determined by solving the equation

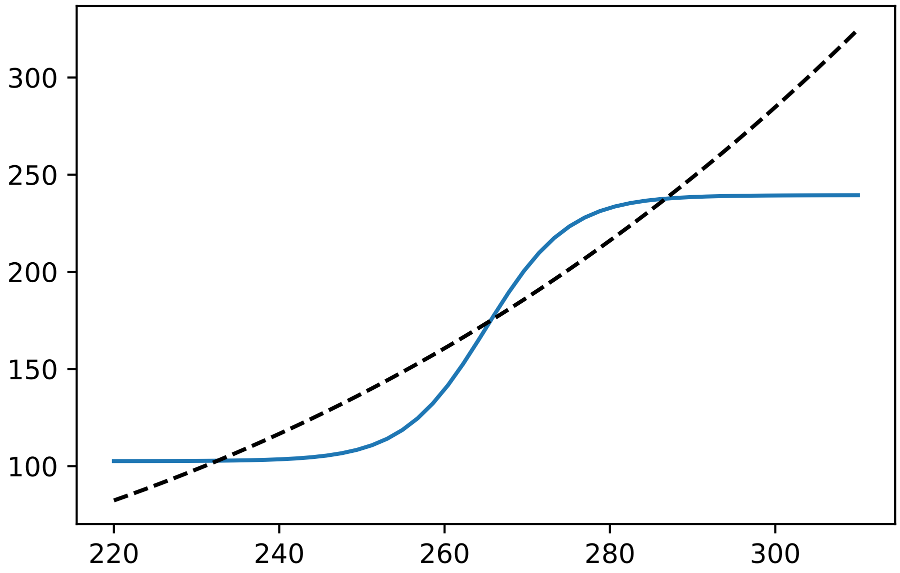
$$f(T) = Q[1 - a(T)] - \sigma\gamma T^4 = 0. \tag{11.3}$$

In the following code, we plot the two terms that make up f, and their intersections indicate roots of $f = 0$. We then use the Python 'fsolve' function to approximate the roots given initial guesses.

```python
import numpy as np
import matplotlib.pyplot as plt
import scipy.optimize as sciopt

Q = 342
sigma = 5.67e-8
gamma = 0.62

TT = np.linspace(220,310,50)

def fun(T):
    a = 0.5 - 0.2*np.tanh((T - 265)/10)
    x = (1-a)*Q
    return x
LHS = fun(TT)

plt.plot(TT, LHS)
plt.plot(TT, gamma*sigma*TT**4, 'k--')

def eq(T):
    x = fun(T) - gamma*sigma*T**4
    return x
T1 = sciopt.fsolve(eq, 230)
T2 = sciopt.fsolve(eq, 265)
T3 = sciopt.fsolve(eq, 290)
```

```
print("T1 = {:.2f}".format(T1[0]))
print("T2 = {:.2f}".format(T2[0]))
print("T3 = {:.2f}".format(T3[0]))
```

```
T1 = 232.55
T2 = 265.56
T3 = 286.74
```



Therefore multiple equilibria are observed.

11.2 Bifurcation diagram

During the lectures, the above script plotting the steady-state solutions will be examined in order to understand the effects of changing Q.

It will be observed that dependent on the solar radiation, there may be one or three steady-state solutions. This leads to the following key bifurcation diagram.

We noted the following:

- The system has three steady states, given by the green, red, and blue curves.
- The middle state is unstable (shown dashed).
- The system exhibits hysteresis. Note that if we decrease Q/Q_0 past the tipping point, marked Q_{T1} in the image, then we would evolve to the lower stable steady state (which is the ice state). However, while we are in the ice stated, if we were to attempt to increase the solar radiation to return to the green branch, we

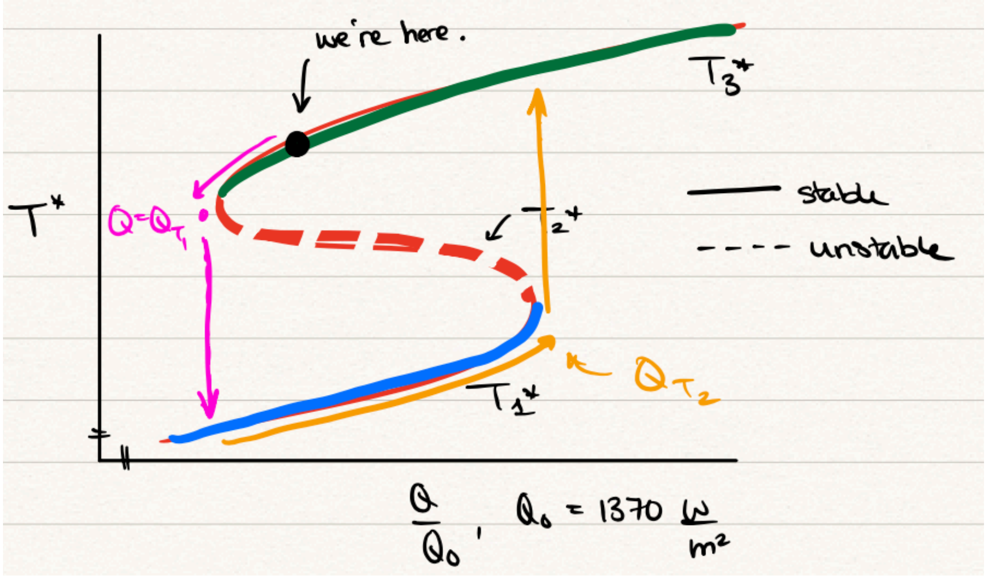


Figure 11.1: Bifurcation diagram of Q/Q_0 vs T_∞

would need to arrive at Q_{T2} to do so; this irreversibility is known as hysteresis. ## Dynamics and phase line solutions

The full time-dependent model is given by

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = f(T),$$

so we may use the positivity or negativity of f in order to sketch the time-dependent behaviour of the system.

To see this, we can perform an asymptotic analysis near the fixed points. Let the initial condition be considered near the fixed point:

$$T(t=0) = T_\infty + \delta,$$

where $\delta \ll 1$. Then we expand the solution into an asymptotic expansion,

$$T(t) = T_\infty + \delta T_1(t) + \delta T^2(t) + \ldots$$

Substitution into the above ODE gives, at $O(\delta)$,

$$C\frac{\mathrm{d}T_1}{\mathrm{d}t} = f'(T_\infty)T_1,$$

and hence, with $T_1(0) = 1$, we have

$$T_1(t) = e^{f'(T_\infty)t/C}.$$

Therefore, depending on the positivity or negativity of the gradient function, the perturbation will either decay or grow as $t \to \infty$.

It can then be verified that the centre equilbria is unstable while the other two are stable. The higher temperature corresponds to the one that the Earth is currently in, but according to this model, there seems to be the possibility of a colder climate (50 degrees colder) where the Earth is entirely covered with snow and ice.

Interestingly, there is some evidence that the Earth's climate may have been in this so-called *Snowball Earth* state up to four times between 750 million and 580 million years ago (Neoproterozoic age). Observations of geological deposits suggest that the Earth has undergone periods of complete global glaciation where there have been very minimal biological activity. During this period, there is a massive build-up of CO_2 in the atmopshere, leading to huge greenhouse effect. As γ decreases in our model, the equilibrium can then shift, suddenly transitioning the Planet into the warm state.

11.3 Re-scaling and Budyko's model

To model the outgoing radiation, we use the quartic Stefan-Boltzmann law. However, over the range of temperatures we are interested-in, it seems that a simpler approximation is sufficient. In your homework, you will investigate the re-scaling and shifting of temperature, such that

$$T = T_0 + [T]\tilde{T},$$

where $T_0 = 265\text{K}$. Then under the assumption that temperatures are not-so-far from T_0, we expand

$$(T_0 + [T]\tilde{T})^4 \sim T_0^4 + 4T_0^3[T]\tilde{T} = C_1 + C_2\tilde{T}.$$

This simplifies the model considerably.

This is, in fact, a theoretical justification of **Budyko's model**, first suggested by Budyko (Budyko, 1969), where instead of the Stefan-Boltmann law that provides the cumbersome quartic power of T, we use instead

$$E_{\text{out}} = \sigma\gamma T^4 \approx A + BT,$$

where A and B will vary with location and climate. For instance, for the Northern Hemisphere, (Kaper und Engler, 2013) gives the values of $A = 203.3\text{Wm}^{-2}$ and $B = 2.09\text{Wm}^{-2}\text{deg}^{-1}$ and where temperature is measured in degrees Celcius. As shown above, one can interpret this as a linear expansion of the Stefan-Boltzmann law (after which similar values of A and B are derived).

The above Budyko model will be a staple of the latitude-dependent model introduced in the next chapter.

Part IV

Exercises

CHAPTER 12

Problem set 1

Note

For feedback, hand in by Friday Week 3.

The intention of this problem set is to practice concepts from material related to conservation laws and non-dimensionalisation. Although these concepts seem quite separate from “Maths of Planet Earth”, actually, they form important pillars of mathematical modelling.

Q1. Bump lemma

Prove the following one-dimensional lemma, which was used in the derivation of the heat equation.

If $\int_a^b g(x)\,\mathrm{d}x = 0$ for all a and $b \in [0,1]$, then $g(x) \equiv 0$ for all $x \in [0,1]$.

Hint: think of a proof by contradiction.

Q2. A source in the heat equation

Consider the same heat experiment discused in Chapter 2 but now consider a bar that has an internal source or sink generating or removing heat, such as the case of a boiler with an internal heating element. By adapting a similar derivation to the one presented, explain why the modified conservation of heat equation is

$$\frac{\partial}{\partial t}\int_a^b \rho c T\,\mathrm{d}x = q(x=a,t) - q(x=b,t) + \int_a^b R(x,t)\,\mathrm{d}x.$$

In addition:

a. By studying the dimensions of the other terms in the above equation, find what the dimensions of R are. What does $R > 0$ mean and $R < 0$?

b. Hence derive the partial differential equation that governs the temperature T.

c. By introducing the appropriate scalings on each of the variables, x, t, and T, non-dimensionalise the PDE and discuss the non-dimensional parameters (there will be two).

Q3. Choice of scalings

Consider the dimensional problem for the motion of a projectile launched from close to the surface of the Earth. The dimensional height of the projectile, $y(t)$, is given by

$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= -\frac{GM}{(R+y)^2},\\ y(0) &= 2\,\mathrm{m},\\ y'(0) &= -V_0\,\mathrm{m/s}.\end{aligned}$$

Assume that the Earth is spherical and of uniform density, with its mass given by $M = (4/3)\pi R^3 \rho$. Non-dimensionalise the height using $y = L\tilde{y}$ and time using $t = T\tilde{t}$. Consider the following cases:

(i) R fixed, $V_0 \to \infty$, ρ fixed;
(ii) R fixed, V_0 fixed, $\rho \to \infty$;
(iii) R fixed, V_0 fixed, $\rho \to 0$;
(iv) $R \to 0$, V_0 fixed, M fixed.

For each case:

a. Explain the physical interpretation of the limits.
b. Choose the scalings L and T to normalise as many terms as possible.
c. Choose the scalins so that the time it takes for the projectile to fall should be finite for the given limit, and for the speed, acceleration, and initial height to be well behaved (finite).
d. Write out the scaled problem and identify all remaining nondimensional parameters.
e. Identify the limiting small parameter for each case. Write out the problem (leading-order problem) when the parameter is set to zero.

Q4. The unique timescale in the heat equation

During our investigation of the heat equation, we found that it was possible to scale time so as to scale out the only non-dimensional parameter that appears in the PDE (Π). This produced (Equation 3.1). The disappearance of all non-dimensional parameters is due to the fact that only a single sensible timescale exists.

By adjusting the boundary conditions, we may create a new problem involving heat flow where a unique 'special' timescale can no longer be chosen.

Consider a system where one side of the rod is heated in some periodic fashion, e.g. set the initial and boundary conditions to be

$$\begin{aligned} T(x,0) &= T_0 \\ T(0,t) &= T_a \cos(\omega t), \\ T(L,t) &= T_b. \end{aligned}$$

a. What must the units of ω be?

b. Non-dimensionalise as usual and, without selecting the timescale, $[t]$, identify the key non-dimensional parameters that remain. Write a brief sentence to describe their physical interpretation.

c. There are two sensible choices for setting the timescale, $[t]$. Identify the two choices and present the reduced set of equations in each case.

Q5. Timescale in the surface energy

Take the basic zero-dimensional energy model studied in (Equation 4.7) for the temperature of the troposphere:

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{4}Q(1-a) - \sigma\gamma T^4.$$

a. Non-dimensionalise the model by choosing $T = [T]T'$, $t = [t]t'$, and $Q = [Q]Q'$. Show that it is possible to select the scalings on the temperature and time so as to completely remove all constants from the problem when Q is assumed to be constant.

b. Thus, show that the analysis of the above equation is equivalent to studying

$$\frac{\mathrm{d}T}{\mathrm{d}t} = 1 - T^4,$$

where we have dropped the primes and assumed that the albedo is such that $1 - a \neq 0$.

c. From your choice of $[t]$, estimate the typical dimensional value using $d \approx 10\text{km}$, $\rho \approx 1\text{kg}\,\text{m}^{-3}$, $c_p \approx 10^3\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$.

Use a pocket calculator to verify your calculations and conclude that this timescale is on the order of a month. What is the relevance of this approximation as it concerns the steady-state solution?

CHAPTER 13

Problem set 2

Note

For feedback, hand in by Friday Week 4.

The intention of this problem set is to practice concepts from Chapter 5 (asymptotic approximations of algebraic equations) to Chapter 7 (Euler's method and numerical solutions of differential equations). These techniques form some of the most powerful techniques at your disposal in applied maths.

13.1 Getting started with Noteable

Use your Moodle course page to access the Noteable Python interface by clicking on the appropriate link in the course materials. Follow the instructions on the Moodle page to add the Git repository to your directory.

1. Navigate to the welcome screen for MA30287 and follow the instructions in `Welcome.ipynb` and `FirstTimeSetup.ipynb` in order to create your own local directory at `/MA30287_workspace/`

2. Navigate to the workspace folder. Click `New -> Notebook`. If asked, select the kernel `Python 3 (ipykernel)`.

3. In the first line of input, select, in toolbar `Code -> Markdown`. This allows you to annotate your notebook with Markdown-style text input.

   In the first line of input, type `# Problem set 2`. Then either type `Shift + Enter` or press the play button which will execute the line(s) of input. Your markdown text should render as a nicely formatted entry.

4. Rename the file to something appropriate, like `ps02_scripts`. You can do this by right-clicking the filename in the file manager and selecting `Rename`.

   Proceed to the next question.

13.2 Testing the solutions of a cubic

In this question, you will develop the numerical solutions for the roots of the cubic equation:

$$\epsilon x^3 - x + 1 = 0, \qquad \epsilon > 0$$

Type the following code into your Jupyter notebook. This code uses a command, which you will learn in a later week, called `fsolve`, in order to solve nonlinear equations. When writing the below code, you may want to separate the import commands into their own input field in the notebook.

```
import numpy as np
from scipy.optimize import fsolve

ep = 0.1        # epsilon value
xguess = 1.1    # Initial guess of root

f = lambda x: ep*x**3 - x + 1
xsol = fsolve(f, xguess)
print("Solved root at x = ", xsol)
```

```
Solved root at x =  [1.15346731]
```

13.3 Analysis of singular cubic equation

Consider the cubic equation

$$\epsilon x^3 - x + 1 = 0,$$

with $\epsilon \ll 1$ and $\epsilon > 0$.

1. Develop the first three terms of an asymptotic expansion about the root by setting

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$$

2. Fill out the following table.

ϵ	x_{exact}	$x_{\text{exact}} - x_0$
0.1	–	–
0.08	–	–
0.06	–	–
0.04	–	–
0.02	–	–

Use your code in Section 13.2 to input numerical approximations to the above entries.

Create a graph **by hand** of the data (it does not have to be extremely accurate), as plotted in the $(\epsilon, x_{\text{exact}} - x_0)$-plane. Fit a line to this graph and estimate the gradient. Is this consistent with what you derived above?

3. By rescaling x appropriately in terms of ϵ, derive the first three terms of the asymptotic approximations of the remaining roots.

13.4 A damped projectile problem

In Chapter 6 you performed the asymptotic analysis for a projectile. The small parameter was ϵ and represented $v_0^2/(gR_E)$ (a parameter that includes the initial velocity, v_0, gravity g, and the radius of the Earth, R_E.

If air resistance is included, then the non-dimensional toy model is instead

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= -\frac{1}{(1+\epsilon y)^2} - \frac{\alpha}{(1+\epsilon y)} \frac{\mathrm{d}y}{\mathrm{d}t}, \\ y(0) &= 0, \\ y'(0) &= 1. \end{aligned} \tag{13.1}$$

where $\alpha \geq 0$ is the parameter that controls air resistance.

1. Begin by assuming that α is a fixed number and consider the limit where $\epsilon \ll 1$. Find a **one-term** asymptotic expansion of the solution for small ϵ.

2. (**Challenging**) Is the effect of the air resistance to increase or decrease the flight time? Justify based on your analytical solution.

13.5 ODE solvers and Euler's method

Return to the setup of the above question.

1. Modify the script shown in Section 6.2 in order to solve the initial-value problem from the previous question at a prescribed value of ϵ and α.

2. Using **a pocket calculator** (or your phone calculator) apply Euler's method with $\Delta t = 0.2$, $\epsilon = 0.2$, and $\alpha = 1$ to determine the position of the projectile at $t = 0.6$.

3. Compare your hand calculation with the result from the Python output, as well as with your asymptotic approximations.

CHAPTER 14

Problem set 3

> Note
>
> For feedback, hand in by Friday Week 5.

In Chapter 8 we discussed how the zero-dimensional energy equation occurs in the form of

$$(\rho c_p V)\frac{\mathrm{d}T}{\mathrm{d}t} = E_{\mathrm{in}}(t, T) - E_{\mathrm{out}}(t, T).$$

This is oversimplistic since in actuality, your temperature function, T, should also depend on space. For instance, if you are interested in modelling T with a depth coordinate, then it would be $T(z, t)$. In this case, we know that temperature would be governed by a second-order differential equation (in space) such as the kind that appears in (Equation 2.1).

In this problem set, we will make up a toy example of a differential equation that might be encountered in heat flow. Consider now (in non-dimensional coordinates):

$$\frac{\partial T}{\partial t} = \epsilon\frac{\partial^2 T}{\partial z^2} + 2\frac{\partial T}{\partial z} + T = 0,$$

which is determined for some function $T(z, t)$. Here, the small parameter $\epsilon > 0$, $\epsilon \ll 1$, is linked to the heat diffusion.

We will study the steady-state version of the problem. So consider only $T = T(z)$. We will make up some boundary conditions as well.

$$\begin{aligned} \epsilon T'' + 2T' + T &= 0, \\ T(0) &= 0, \\ T(1) &= 1. \end{aligned} \tag{14.1}$$

The challenge is to study the above problem for small values of ϵ.

Q1. Conversion to first-order system

By using the procedure reviewed in Section 6.2, convert (Equation 14.1) to a first-order system of equations.

Q2. Numerical solutions

By adapting the code studied in lectures (script `lecture12-SolvingBVPs`), write a numerical code to solve (Equation 14.1) using Python's built-in functions. Use your code to investigate the solution profiles for different values of ϵ.

Q3. Investigation of the boundary layer

Using your code, use the following command in order to investigate the maxima (x_m, T_m) as ϵ varies:

```
ind = np.argmax(Y[0])
print(ep, z[ind], Y[0, ind])
```

You may want to consider, as an example, the values $\epsilon = \{0.05, 0.1, 0.15, 0.2\}$ and fill the following table.

ϵ	x_m	T_m
0.05		
0.1		
0.15		
0.2		

Create a plot of (ϵ, x_m) and discuss the observed trend and its implications.

Q4. Outer asymptotic solutions

Begin by setting

$$T = T_{\text{outer}} = T_0(z) + \epsilon T_1(z) + \epsilon^2 T_2(z) + \ldots$$

Substitute the above expansion into the system and solve for the first two orders.

You may verify that the solution is given by

$$T_0 = e^{1/2} e^{-z/2}, \tag{14.2}$$

$$T_1 = -\frac{1}{8} e^{1/2} e^{-z/2}(z-1). \tag{14.3}$$

Q5. Inner asymptotic solutions

There will be a boundary layer near $z = 0$. Set $z = \epsilon^{\alpha} s$ and $T(z) = U(s)$. Follow the same procedure, as in Chapter 8 in order to determine the correct choice of α for the inner region. This choice should balance the two terms $\epsilon T''$ and $2T'$.

Q6. Matching and comparison

Expanding the inner solution as $U = U_0(s) + \epsilon U_1(s) + ...$, write down the equation and boundary conditions that U_0 must satisfy. You will notice that U_0 is governed by a second-order differential equation and therefore needs two boundary conditions. One boundary condition comes from $z = 0$, i.e.

$$U_0(0) = 0.$$

The other boundary condition is a matching condition:

$$\lim_{s\to\infty} U_0(s) = \lim_{z\to 0} T_0(z),$$

which imposes that the inner solution, as it leaves the boundary layer, matches the outer solution, it tends into the inner region.

Solve for U_0.

CHAPTER 15

Problem set 4

Note

For feedback, hand in by Friday Week 6.

This problem set focuses on EBMs.

Q1. Budyko's model

Previously we derived the energy for outgoing radiation as:

$$E_{\text{out}}(T) = \sigma\gamma T^4,$$

which is estimated via the Stefan-Boltzmann law. Budyko's approximation replaces this law with a linear relationship.

a. Assume that

$$T = T_0 + \bar{T},$$

where $T_0 = 273.15\,\text{K}$, which corresponds to 0° C. We can consider $\bar{T}$ as measured in Celcius (since conversion between Kelvins and Celcius is directly proportional). Substitute the above into E_{out} and linearise, thus assuming $\bar{T}$ is near zero. Place the energy in the form

$$E_{\text{out}} \sim A + B\bar{T}.$$

Using the standard value of $\sigma = 5.67 \times 10^{-8}$ W/m2 K4 and $\gamma = 1$, calculate the values of A and B.

b. Rather than use the formal values derived above, the standard Budyko model uses values of $A = 203.3\text{W/m}^2$ and $B = 2.09\,\text{W/(m}^{2\circ}\text{C)}$ that are found from best fit with observational data.

   Using Jupyter and a short script for numerical root finding, investigate the equilibrium solutions of

$$Q(1 - a(T_0 + \bar{T})) = A + B\bar{T}.$$

   where $Q = 342\,\text{W/m}^2$ and $a(T)$ is given from (Equation 11.2). Where are the equilibrium solutions, $\bar{T}^*$, and how do they compare with those obtained previously using the Stefan-Boltzmann relation?

Q2. Variable Sun output

Satellite data indicates that Q, varies roughly between 341.37 W/m2 and 341.75 W/m2, with a period of about 11 years.

a. Use the simple EBM (Equation 11.3), given by

$$Q(1 - a) = \sigma\gamma T^4,$$

   with a constant albedo, $a = 0.3$ and greenhouse gas factor $\gamma = 0.6$ to estimate the resultant variation (max and min) in the Earth's mean surface temperature T.

b. Similar to (a) but this time, use the Budyko balance equation,

$$Q(1 - a) = A + BT$$

   with $A = 203.3\,\text{Wm}^{-2}$ and $B = 2.09\,\text{W/(m}^2\,{}^\circ\text{C)}$ to estimate the resultant variation in the surface temperature. Use $a = 0.3$.

c. The actual variation in surface temperature is in fact less than what you computed above. Why might this be?

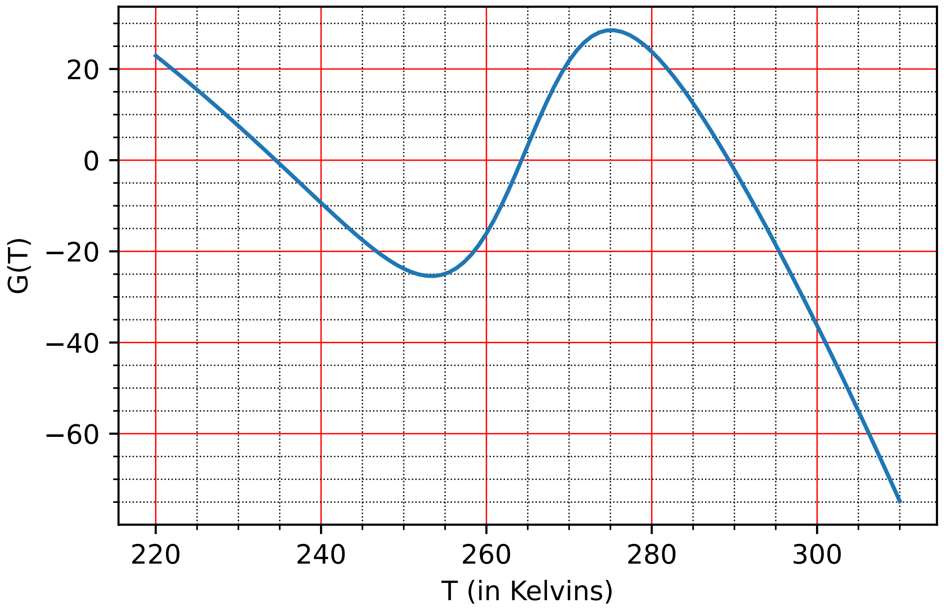
Q3. Phase line analysis

Consider the energy balance equation

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = Q(1 - a(T)) - \sigma\gamma T^4 \equiv G(T). \tag{15.1}$$

with a given by (Equation 11.2). Because the differential equation is autonomous, we can apply phase-line analysis in order to qualitatively understand the evolution. Below is a plot of the function G:

```
import numpy as np
import matplotlib.pyplot as plt

Q = 342; sigma = 5.67e-8; gam = 0.6;
a = lambda T: 0.5 - 0.2*np.tanh(0.1*(T-265));
T = np.linspace(220, 310, 100);
G = Q*(1-a(T)) - sigma*gam*T**4
fig, ax = plt.subplots()
ax.plot(T, G)
ax.grid(); ax.minorticks_on();
# Customize the major grid
ax.grid(which='major', linestyle='-', linewidth='0.5', color='red')
# Customize the minor grid
ax.grid(which='minor', linestyle=':', linewidth='0.5', color='black')
plt.xlabel('T (in Kelvins)'); plt.ylabel('G(T)');
```



a. Sketch the solution $T(t)$ of this equation for $t > 0$ if $T(0) = 230, 240, 260, 270$ and 300.

b. Sketch the solution $T(t)$ of this equation for $t > 0$ if $T(0) = 285$. Then sketch the solution of this equation with the same initial data in the same coordinate system if C is twice as large. Explain your answer.

c. If γ is decreased due to the increased greenhouse effect, the entire curve is shifted upwards. Sketch the solution if $T(0) = 280$. Sketch the solution with the same initial data if γ is decreased. Explain your answer.

Q4. Evolution

Consider again (Equation 15.1). Let T^* be a steady-state solution and set $T = T^* + u(t)$ where $u(t)$ is a small perturbation from the steady state.

a. Show that the perturbation satisfies

$$C\dot{u} = -Du + O(u^2).$$

and hence solve for the general solution of the leading-order perturbation (ignoring quadratic terms). What are the conditions on T^* so that the steady state is linearly stable?

b. Assuming T^* is linearly stable, find the typical response time to a perturbation. For instance, what is the time it takes for the perturbation to reach the value $u(t) = 0$ if $u(0) = 1$? How does this response time change with C? What is the physical interpretation of this regarding the climate?

Q5. Integral of energy over the planet

Ignoring the effects of albedo, the total radiation absorbed over the surface of the planet (per unit time) is given by

$$\iint_{\text{planet}} Qs(y = \sin\varphi)\,\mathrm{d}S.$$

This is what is known as a surface integral (**?@sec-appendix-surfaceintegral**). In the case of the spherical coordinate system, this is calculated by

$$\int_{\theta=0}^{2\pi}\int_{\varphi=-\pi/2}^{\pi/2} Qs(y = \sin\varphi)R_E^2\cos\varphi\,\mathrm{d}\varphi\mathrm{d}\theta.$$

Use the properties of $s(y)$ in (**?@eq-EBM-s**) to conclude that the total radiation absorbed is $4\pi R_E^2 Q$.

Q6. Mean temperature in the latitude-dependent EBM

Consider now the latitude-dependent EBM

$$C\frac{\boxtimes T}{\boxtimes t} = Qs(y)[1 - a(y)] - (A + BT) + k(\bar{T} - T).$$

Recall the albedo is given by $a = a_i$ for $y > y_s$ and $a = a_w$ for $y < y_s$.

a. By integrating the above equation over $y \in [0, 1]$, show that the mean temperature is given by

$$C\frac{\mathrm{d}\bar{T}}{\mathrm{d}t} = Q(1 - \bar{a}) - (A + B\bar{T}), \tag{15.2}$$

where

$$\bar{a} = \int_0^1 s(y)a(y)\,\mathrm{d}y = \alpha_w \int_0^{y_s} s(y)\,\mathrm{d}y + a_i \int_{y_s}^1 s(y)\,\mathrm{d}y.$$

b. In the case that s is given by (**?@eq-EBM-s**), show that

$$\bar{a} = a_i + (a_w - a_i)y_s[1 - 0.241(y_s^2 - 1)]. \tag{15.3}$$

What is $\bar{a}$ in the two situations of a completely ice-covered world and an ice-free world?

c. Using Python, plot a graph of the steady-state temperature, T^*, which satisfies

$$\bar{T}^* = \frac{Q(1 - \bar{a}) - A}{B}, \tag{15.4}$$

as a function of the ice line location, $y_s \in [0, 1]$.

d. In the numerical code designed in lectures `ma30287/notebooks/lecture17-latitudeebm_newton.i` we used numerical quadratic to determine $\bar{T}$. Experiment with a modest number of mesh points and verify how well the integration scheme compares with the exact solution shown above.

Part V

Problem classes

CHAPTER 16

Problem class 1: an introduction to Noteable

During this module, we will perform various computational experiments using Python. In order to make it easy for students to code in Python from a variety of devices (desktops, laptops, tablets, and even a phone), we use a cloud-computing system called Noteable that allows you to code from within a web browser. Students who like to code using their own Python setup on their computer can also do so. The goal of this problem class is to quickly get you up to speed with the basic Noteable system, and how to manipulate outputs. We should cover:

- A walkthrough of setting up with Noteable
- Basic plotting using matplotlib and numpy
- How to export graphics and worksheets

16.1 Getting started with Noteable

Use your Moodle course page to access the Noteable Python interface by clicking on the appropriate link in the course materials. Follow the instructions on the Moodle page to add the Git repository to your directory.

1. Navigate to the welcome screen for MA30287 and follow the instructions in `Welcome.ipynb` and `FirstTimeSetup.ipynb` in order to create your own local directory at `/MA30287_workspace/`

2. Navigate to the workspace folder. Click `New -> Notebook`. If asked, select the kernel `Python 3 (ipykernel)`.

3. In the first line of input, select, in toolbar `Code -> Markdown`. This allows you to annotate your notebook with Markdown-style text input.

   In the first line of input, type `# Problem class 1`. Then either type `Shift + Enter` or press the play button which will execute the line(s) of input. Your markdown text should render as a nicely formatted entry.

4. Rename the file to something appropriate, like `problemclass01`. You can do this by right-clicking the filename in the file manager and selecting `Rename`.

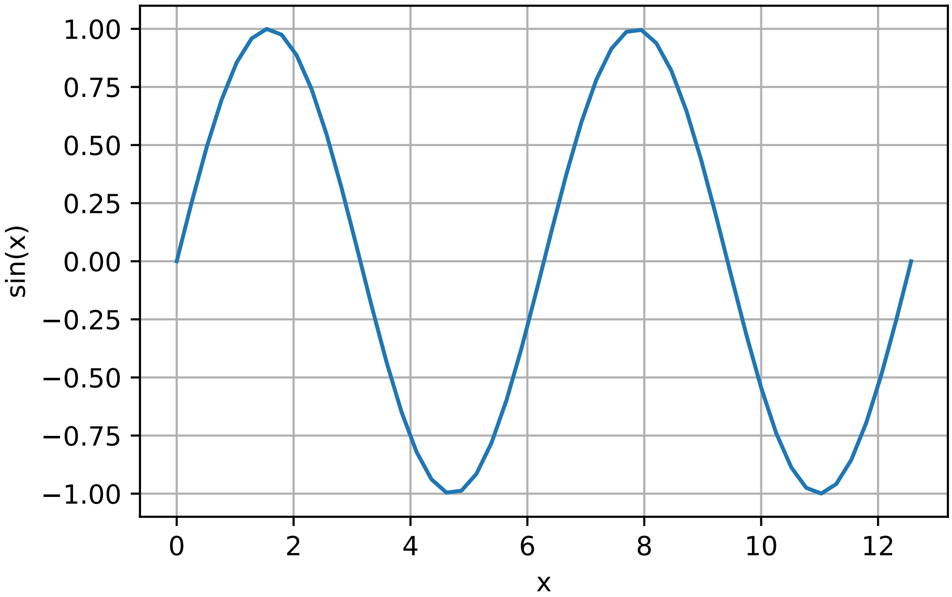
   Proceed to the next question.

16.2 Mathematical plotting

Three key Python packages used throughout this course are numpy, matplotlib, and scipy. `Numpy` provides array functionality, allowing you to naturally manipulate vectors and matrices. `Matplotlib` is a general plotting package for Python, allowing for the creation of (typically) 2D and 3D plots. And `scipy` provides algorithms for differential and integral operations allowing e.g. the solution of differential equations.

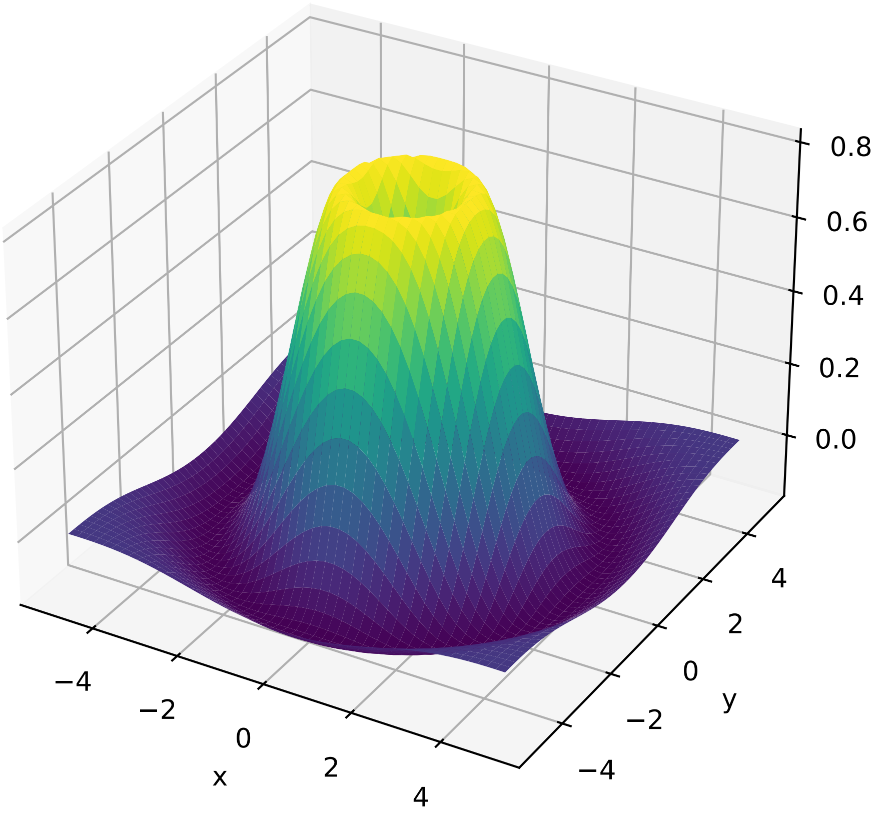
Here is a script to generate a graph of a sine curve.

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(0, 4*np.pi, 50)
y = np.sin(x)
plt.plot(x, y)
plt.xlabel('x');
plt.ylabel('sin(x)');
plt.grid(1)
```



Here is a script to generate a 3D plot.

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Define the function
def f(x, y):
    return np.sin(np.sqrt(x**2 + y**2)) * np.exp(- (x**2 + y**2) / 10)

# Create grid
x = np.linspace(-5, 5, 50)
y = np.linspace(-5, 5, 50)
X, Y = np.meshgrid(x, y)
Z = f(X, Y)

# Plot the surface
fig = plt.figure(figsize=(8, 6))
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(X, Y, Z, cmap='viridis')

# Labels and title
ax.set_xlabel("x")
ax.set_ylabel("y")
ax.set_zlabel("z")
plt.show()
```



16.3 Debugging Jupyter code

It would be good to demonstrate a workflow to understand how errors are diagnosed and studied (e.g. by printing simple variables throughout the script). Also, it would be good to show students how graphs can be zoomed in/out and/or rotated within the browser interface (this may be tricky in Noteable).

16.4 Outputting and saving

During the problem class, we will also show how to:

- Create a zip file of your workspace using the terminal so that you can save a copy of your files onto your local filesystem. This can be done by running the script `ExportMA30287.ipynb` in the root directory of the Noteable system, and then downloading the result.

- Create a copy of your script in PDF form so that it can be sent around as a single document. This can be done by going to File -> Save and Export Notebook As...

16.5 Challenge coding exercise

Create an animation of the 2D or 3D plot, e.g. by making it so the waves ‘ripple’. Using a `for` loop, animate the wave. Then output the animation as a video file.

CHAPTER 17

Problem class 2: dimensional analysis

Abstraction

It may seem strange to study examples from elementary physics in a course that is supposed to be about Planet Earth. But simple examples are the best ways to learn these important techniques. The full climate equations are often very involved. These toy models still nevertheless capture the spirit of what you must do when attacking any scientific problem.

In this problem class, we will practice some concepts about non-dimensionalising in preparation for the Problem Set 1 in Chapter 12. We will cover these two strategies in choosing scalings.

Scaling principle 1

Select the characteristic scales so that as many of the possible non-dimensional numbers, Π_i, $i = 1, 2, 3, \ldots$ are normalised.

Scaling principle 2

Select characteristic scales so that no terms in the model diverge in the physical limit of interest.

Here are the problems we shall do in the problem class.

17.1 Projectile motion

A projectile of mass M (in kg) is launched vertically with initial velocity V_0 (in m/s) from a position Y_0 (in m) above the surface. Thus the mass's position, $Y(t)$ is governed by Newton's second law (applied to the mass and the mass of the Earth) and the set of equations

$$MY_{tt} = -\frac{gR_E^2 M}{(R_E + Y)^2},$$
$$Y(0) = Y_0,$$

where $g = 9.81 m/s^2$ and $R_E = 6.4 \times 10^6 m$ is the radius of the Earth.

1. Non-dimensionalise the equation using arbitary length and time scales.
2. Identify the non-dimensional constants, Π_i.
3. Choose a length scale of $L = Y_0$ and time scale of $T = (L/g)^{1/2}$. Discuss the resultant equation and the interpretation of choosing these scales.
4. Does your above choice allow you to easily study the limit of $R_E \to \infty$? If the limit can be taken, reduce the governing system to a simpler equation.
5. Does your choice in 3. allow you to easily study the limit of $Y_0 \to 0$? If not, choose an alternative choice of length and time scales and in that case, reduce the set of equations.

17.2 Terminal velocity

A ball of radius R (in m) and uniform density ρ (in kg/m3) falls in a viscous fluid. The fluid has density ρ_f (in kg/m3) and viscosity (a measure of friction or resistance) μ (in kg/(m s)). The equation that governs the velocity is

$$\frac{4}{3}\pi R^3 \rho \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{4}{3}\pi R^3 (\rho - \rho_f) g - 6\pi\mu R V,$$
$$V(0) = V_0.$$

1. Choose appropriate velocity and time scales to non-dimensionalise the equation so as to leave only a single non-dimensional number on the drag term (the last term on the right hand-side).
2. Define the non-dimensional parameter expressing a ratio between drag force and gravity force by the Stokes number (St) and confirm that it is

$$St = \frac{9\mu V_0}{2(\rho - \rho_f) g R^3}.$$

3. Comment on the two limits of $St \to 0$ and $St \to \infty$. Can the problem be reduced in these two limits? If so, reduce and solve.

CHAPTER 18

Problem class 3: BVPs

This problem class will involve the completion of the ODE studied in asymptotic approximations III. We will then study the boundary-value problem which forms part of the third problem set, namely:

$$\begin{gathered}\epsilon T'' + 2T' + T = 0, \\ T(0) = 0 \quad \text{and} \quad T(1) = 1.\end{gathered}$$

The associated Python note is in `lectures/lecture12-SolvingBVPs`. You will gain an appreciation for some of the unique elements that are introduced when looking at $\epsilon \to 0$ in boundary-value problems.

CHAPTER 19

Problem class 4: on PDEs

This problem class, we will study an example of a PDE model for heat transport, and then use it to inform ourselves on a typical physical problem.

We are interested in finding the optimal depth for a storage cellar that might be used to house wine. Based on our previous discussions, if we assume that the temperature beneath the ground is given by $T(x, t)$, with $x \geq 0$ below the surface, then the temperature is governed by

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2},$$

for $x \geq 0$. This is a highly averaged model, where we assume there is no transport of heat into the horizontal plane.

We shall also assume that the surface temperature is modelled by an oscillatory function. On the assumption that the temperature has been normalised so that $T = 0$ corresponds to the mean temperature of the surface, then we have

$$T(x = 0, t) = A \cos(\omega t).$$

The wavelength is then $2\pi/\omega$ which we can set to one year. So $\omega = 1/(2\pi)$ yr^{-1}.

In addition, we expect that the temperature should not be singular far beneath (in particular we expect it to get colder as we go further underground), at least on a scale where we are not approaching the centre of the Earth! Thus it is sensible that

$$|T(x, t)| \text{ is bounded as } x \to \infty.$$

You will not require an initial condition of the discussions below.

19.1 Numerical solutions

An example code that uses first-order explicit Euler timestepping is given in `lecture15-winecellar` on the lectures coding folder. A typical numerical output is given here:

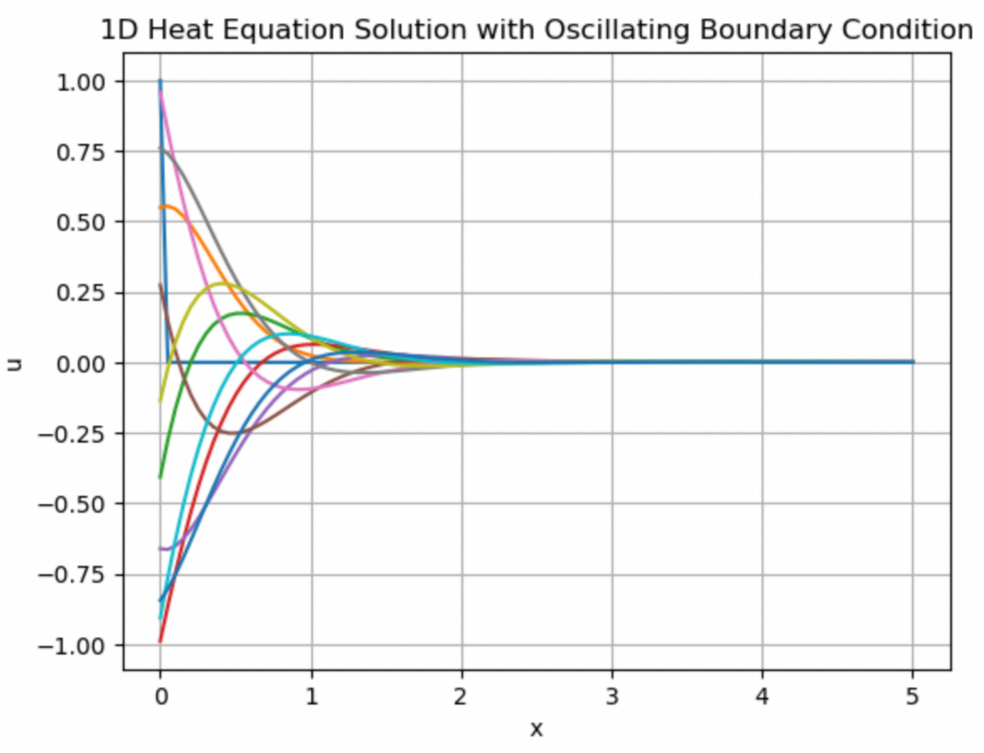


Figure 19.1: Numerical solution of the wine cellar problem

19.2 Analytical solution

Before we begin, it is useful for the purposes of manipulation to write the boundary condition as

$$T(0,t) = A\Re(e^{i\omega t}) = A\Re(\cos(\omega t) + i\sin(\omega t)) = A\cos(\omega t).$$

This has the advantage that we can manipulate exponentials much more easily.

To solve the problem analytically, we attempt a solution method called separation of variables. Let

$$T(x,t) = G(t)H(x).$$

Substitution into the PDE gives

$$G'(t)H(x) = \kappa G(t)H''(x),$$

and we attempt to separate the functions of t to one side and the functions of x to the other.

$$\frac{G'(t)}{G'(t)} = \kappa \frac{H''(x)}{H(x)}$$

Now in order for two completely different functions dependent on two completely different independent variables to be equal in this way, then this implies that both sides must be equal to a constant. So we have

$$\begin{aligned} G' &= \lambda G \\ H'' &= \lambda \kappa H. \end{aligned}$$

Therefore $G = \text{const} \times e^{\lambda t}$. Because of the boundary conditions that would require $T = Ae^{i\omega t}$, you can check that $\lambda = 0$ and λ real are both impossible. The only way to satisfy the boundary conditions is if $\lambda = i\omega$. So we conclude that

$$G(t) = \text{const.} \times e^{i\omega t}.$$

To solve

$$H'' = (i\omega/\kappa)H,$$

you can attempt the substitution $H(x) = e^{rx}$ and solve the resultant equation for the characteristic polynomial in r. You will do this in the problem set, and obtain

$$r = \pm\sqrt{i}\sqrt{\omega/\kappa} = \pm\sqrt{\omega/\kappa}\frac{1+i}{\sqrt{2}}.$$

Therefore

$$H(x) = C_1 e^{\sqrt{\omega/(2\kappa)}x} e^{\sqrt{\omega/(2\kappa)}ix} + C_2 e^{-\sqrt{\omega/(2\kappa)}x} e^{-\sqrt{\omega/(2\kappa)}ix}.$$

The first solution should be discarded since it blows up as $x \to \infty$.

Finally, we conclude that the solution is given by

$$T = e^{i\omega t} e^{-\sqrt{\omega/(2\kappa)}x} e^{-\sqrt{\omega/(2\kappa)}ix}.$$

Move the imaginary arguments together:

$$T = \text{const.} \times e^{-\sqrt{\omega/(2\kappa)}x} e^{i(-\sqrt{\omega/(2\kappa)}x+\omega t)}.$$

Now we take the real part, giving

$$T = Ae^{-\sqrt{\omega/(2\kappa)}x} \cos\left(-\sqrt{\omega/(2\kappa)}x + \omega t\right), \tag{19.1}$$

and we see that having set the constant to A the solution indeed satisfies $T(0,t) = A\cos(\omega t)$.

19.3 Selection of optimal depth

The optimal depth of the wine cellar is selected by taking the depth to be such that the cosine is completely out of phase with the surface solution. This means that

$$-\sqrt{\omega/(2\kappa)}x + \omega t + \pi = \omega t.$$

You can substitute other multiples of π. Thus we have

$$x = \pi\sqrt{\frac{2\kappa}{\omega}}.$$

We can estimate the numbers as follows. Converting from 1 year to seconds yields

$$\omega = \frac{2\pi}{3.15 \times 10^7\mathrm{s}}.$$

Also, the κ for dry soil is around $0.002\mathrm{cm}^2/\mathrm{s}$.

Putting these numbers in yields

$$x \approx 445\mathrm{cm} = 4.45\mathrm{m}.$$

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