MA30287: Mathematics of Planet Earth

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Preface

Welcome to to the 2023-24 delivery of MA30287 Maths of Planet Earth at the University of Bath.

Here is a picture that represents the course.

Lectures and office hours

Lectures take place at the following times and locations:

- Tuesdays 9:15 in 6W 1.2
- Wednesdays 11:15 in 6W 1.2
- Thursdays 15:15 in 8W 3.22

Office hours: You will be able to find me for an office hour in 4W 2.18 on Thursdays (following the lecture). Typically it is best to set this up, beforehand, by email appointment.

Coursework and examinations

Your final mark will be 25% coursework and 75% final exam.

Details of the coursework will be released in Week 7 and it will be due in Week 10[1].

Resources

In general, you will have access to a few kinds of resources:

1. The Moodle portal will be the main organisation portal.
2. Lecture notes, coursework, and other resources will be found in an online format and will be linked on Moodle.
3. Coding will be done via Noteable, accessed via the Moodle website.

[1] Subject to confirmation.

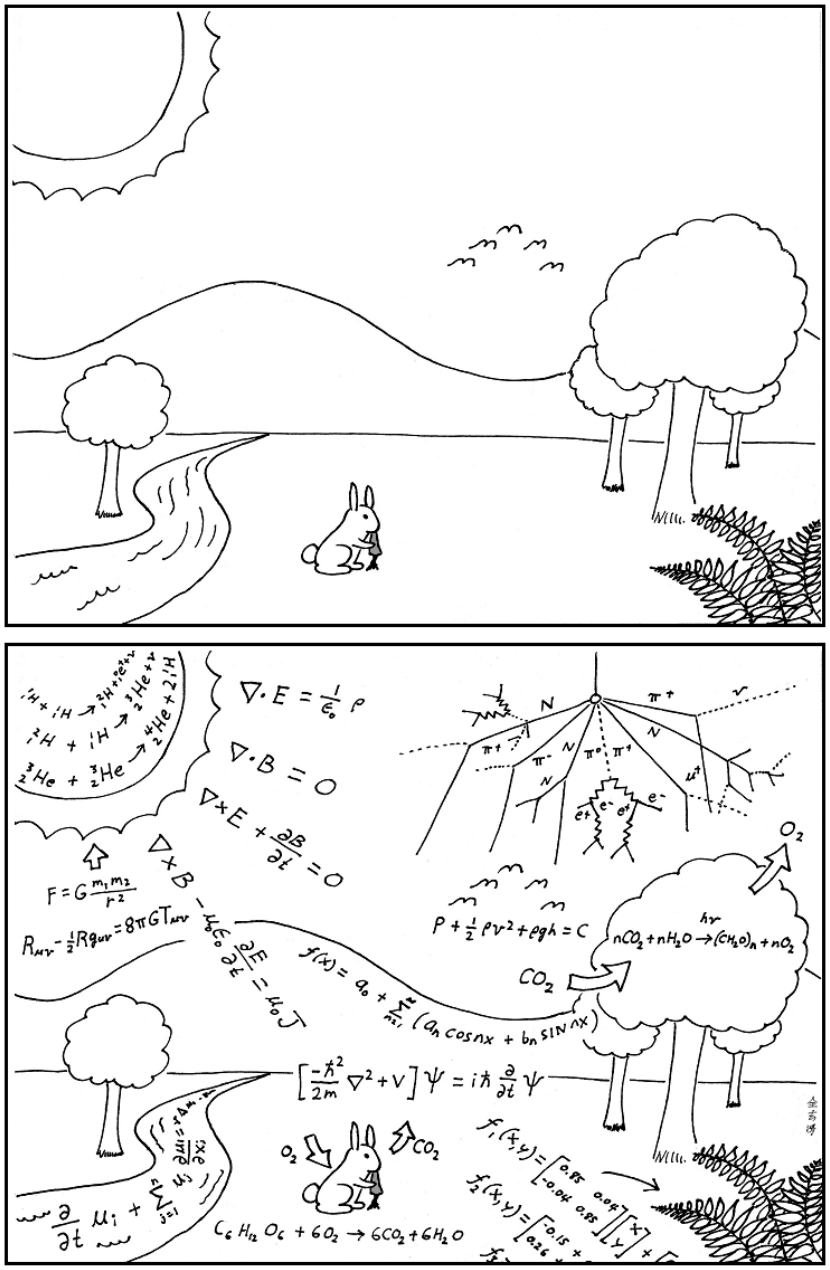


Figure 1: This picture was produced by Abstruse Goose and has the title "This is how scientists see the world".

Naturally, because this is a relatively new module at Bath, there will be a fair amount of activity as we settle the material over the semester. Whenever we complete a lecture note (i.e. a 'chapter'), we will use a box like this to indicate when the material was covered and in which lecture:

> 2023-24 note
>
> The material in this note was covered in Lecture XX.

Hopefully by the time the module ends, every relevant chapter will have such a note. This allows you to judge what material has been 'finalised'.

Prerequisites

This course officially requires MA20221 (modeling and dynamical systems) or XX20231 (mathematical and statistical methods for the life sciences).

It is designed to be somewhat stand-alone in the sense that applied mathematical techniques learned in other courses will be introduced in some capacity. Such techniques will involve:

- Solutions of ordinary differential equations (MA10230 and MA20220).
- Multivariable calculus, partial differentiation, and multiple integrals (MA10230 and MA10236); some review/introduction of concepts from MA20223.
- Dynamical systems, stability, phase planes (MA20221, MA30060).
- Numerical methods in Python (MA10276).

Whenever possible, I have isolated such reviews/introductions and these can be found in the *Mathematical methods* section of these notes.

Resources

This course is designed around the following sources:

- Sustainable energy – without the hot air by (MacKay 2009)
- Mathematics & Climate by (Kaper and Engler 2013)
- Mathematical Geoscience by (Fowler 2011)
- A gentle introduction to numerical simulations with Python

Part I

Introduction

Mathematics of Planet Earth seems like an incredibly broad description for a course, but perhaps in order to give a rough idea of what such a course might include, we can consider the following diagram, which illustrates different categories and subject areas that are involved in the modelling of a full Earth system.

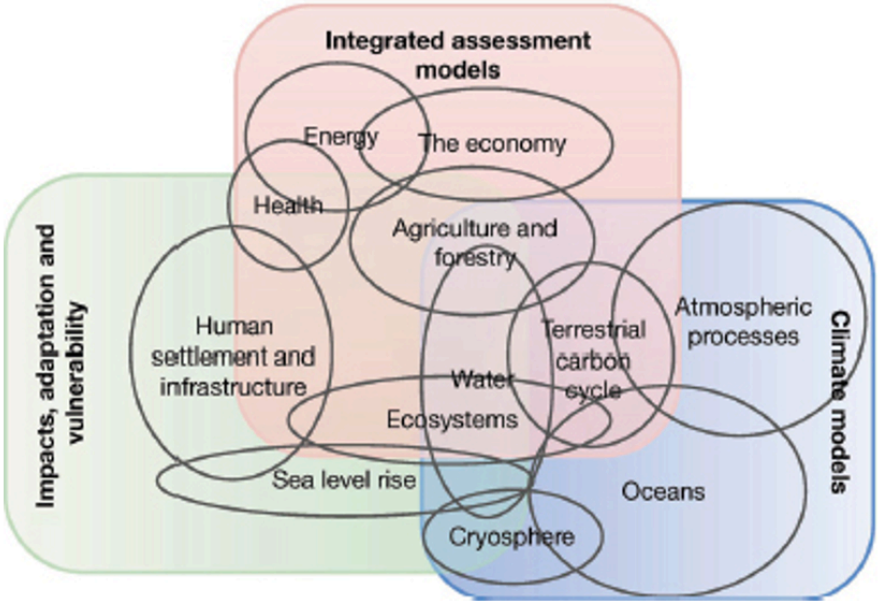


Figure 2: The many components of a full Earth System Model

It would be possible to spend a lifetime studying any one aspects of the above categories, and they span many different areas of study, including: (i) engineering (civil, fluids, mechanical, etc.); (ii) physics (geosciences, mechanics); (iii) Earth sciences; (iv) policy and health; and so forth and so on. As mathematicians, we also have a unique perspective, and applied mathematics plays important roles in many of the above categories.

In essence, this course will include topics and themes are united by aspects of **mathematical modelling** and **mathematical analysis** and this is what distinguishes our style of study from adjacent areas of science and social science.

We will study so-called conceptual or box models of the climate. This involves some of the blue elements of the above figure, and thus we use coarse-grained models of describing the climate. This will involve applying some of the dynamical systems (phase-plane analysis of ODEs) you have learned previously, along with new methods of computation and analysis. The source for this part of the course will be (Kaper and Engler 2013).

A secondary part of this course will involve more in-depth analysis of the physical models that govern the blue elements of the above figure. This moves us from the toy box models studied

above to digging into the underlying physics—this also falls into the category of Mathematical Geoscience. For example, we will use partial differential equations to study the atmosphere and develop a deeper understanding of greenhouse gases. The source for this part of the course will be (Fowler 2011).

1 Conservation laws and constitutive laws

In the first chapter of (Fowler 2011), there is a concise introduction to the different categories of techniques and approaches that you might use when doing mathematical modelling in the real world. Some of these ideas will be introduced to you in this course.

Here, we provide a brief intro to the highlights, involving the use of conservation laws (and PDEs) and also the concept of non-dimensionalisation (which you would have encountered previously), studied in Chapter 2.

> Vectors and PDEs
>
> Vectors and PDEs is not a prerequisite for this course, but naturally in studying anything related to the physical real world, we must discuss partial differential equations. The hope is that the necessary theory for PDEs will be presented to you as this course evolves, so that it can appreciated by both newcomers and experienced readers.

Conservation laws can be expressed as mathematical equations that represent the idea that some quantity is conserved. In processes governing the planet, these might correspond to conservation of heat, of water, of air, of momentum, etc.

In Chapter 3, we will develop the simplest possible model governing the temperature on the surface of the Earth. It is a conservation equation for energy and is zero-dimensional (does not involve time and does not involve spatial variation).

1.1 Derivation of the 1D heat equation

In order to demonstrate some of the basic principles of this course, let us demonstrate the derivation of the heat equation. We are interested in modelling the heat in a volume, V, which, for the sake of concreteness is given by a long cylinder with its axis along $x \in [0, L]$. We assume that the side walls of the cylinder are insulated and the temperature only varies along the x direction.

At any point along this rod, the internal heat is given by $\rho c T(x, t)$, where ρ is the density of the material (kg/m3), c is the specific heat capacity (J/(kg K)), and T is the temperature (K). Therefore, the heat energy along any segment in the rod is calculated from

Internal heat energy

$$\text{heat energy in } [a,b] = \int_a^b \rho c T \, \mathrm{d}x.$$

If the heat changes, then the rate of change of heat energy is given the time derivative of the above quantity. By conservation of energy, any change of the internal energy must be equal to the inflow or outflow of heat at the ends, $x = a$ or $x = b$. We therefore write q for the flux (or flow) of heat.

We need a **constitutive law** that dictates how energy is exchanged at the boundaries. Based on intuition, it is sensible to assume that the flow of heat proceeds from hot to cold. For example, hot air rises towards cool air; or heat from a hot mug of tea flows and diffuses outwards into a cold room. Therefore, we write this as

Fourier's law

Fourier's law in 1D specifies that the heat flux is given by

$$q(x,t) = -k\frac{\partial T}{\partial x}.$$

This is known as Fourier's law. The quantity k is the thermal conductivity, and its units are W/(m K). Because a Watt is a Joule/s, you can also see that the units of k are J/(m K s). The quantity q is the flux, and you can verify that it is given in units of J/(m2 s).

Therefore by energy conservation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho c T \, \mathrm{d}x = q(x=a,t) - q(x=b,t),$$

i.e. the change in internal heat is equal to the flow through the ends. Note that we have added-in the heat flux due to $x = a$ assuming positive q refers to heat moving from left-to-right. Conversely, we subtract it away from $x = b$. We can alternatively write this as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho c T \, \mathrm{d}x = -\int_a^b \frac{\partial q}{\partial x} \, \mathrm{d}x.$$

Substitution Fourier's law, we can then write

$$\int_a^b \rho c \frac{\partial T}{\partial t} \, \mathrm{d}x = \int_a^b k \frac{\partial^2 T}{\partial x^2} \, \mathrm{d}x.$$

Because the above integral identity needs to be true for all possible values of a and b, then it must be true everywhere (this is sometimes referred to as the *du Bois-Reymond lemma* or the *bump lemma*. Therefore we are left with the classic heat equation.

Heat equation

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}. \tag{1.1}$$

In order to produce a sensible physical solution, partial differential equations are typically supplemented by initial conditions and boundary conditions. The **initial condition** prescribes the state of the function at some initial time, typically $t = 0$. **Boundary conditions** prescribe how the function behaves on the boundary of its domain, which in this case is $x = 0$ and $x = L$. An example might be

Initial conditions (IC) and boundary conditions (BC)

$$T(x, 0) = T_0 \tag{1.2}$$
$$T(0, 0) = T_a \tag{1.3}$$
$$T(L, 0) = T_b \tag{1.4}$$

which expresses, respectively, that the temperature starts from a constant temperature, T_0, and where the ends of the rod are kept at temperature T_a and T_b.

1.1.1 Steady states and long-time behaviours

2024 note

This was a new addition in 2024.

When we refer to a **steady-state** solution, we are typically referring to a time-independent solution.

Definition 1.1 (Steady-state solutions)**.** Given an evolving system described by a functio, say $f(x, t)$, defined on some spatial domain and with $t > 0$, the steady-state solution refers to time-independent solutions with

$$\frac{\partial f}{\partial t} = 0.$$

One can envisage that as the system evolves with $t \to \infty$, it reaches a state that is independent of time. However, not all systems will approach a steady state. Moreover, not all steady states are stable or attractive (and might never be reacheable in a real-life experiment).

For the case of heat flow, such a steady-state solution would be $T(x, t) = T(x)$. In this case,

$$\frac{\partial T}{\partial t} = 0 \implies k \frac{\partial^2 T}{\partial x^2} = 0.$$

Therefore, for the heatflow along a segment of length L with left boundary held at T_a and right boundary held at T_b we have

$$T_{\text{steady}}(x) = \left(\frac{T_b - T_a}{L}\right) x + T_a.$$

1.2 Deriving the 1D transport (continuity) equation

> 2024 note
>
> This was a new addition in 2024.

Consider the mass transport of some substance with density $\rho(x,t)$, immersed in a fluid, along a one-dimensional line in x. If desired, you may consider the substance as existing in three-dimensional space, and propagating along the x-direction, with its behaviour independent of y and z. Initially, when considered at time t, the mass of the substance between two points, a, and b, is:

$$m_{\text{blob}}(t) = \int_{a(t)}^{b(t)} \rho(x,t)\,\mathrm{d}x.$$

As time increases the particles of the substance will move due to the fluid moving; at the same time, the fluid volume which is initially contained in $x \in [a(0), b(0)]$ will also move. We want to find how the mass of the blob changes in time, and hence consider the quantity

$$\frac{\mathrm{d}m_{\text{blob}}}{\mathrm{d}t} = \int_a^b \frac{\partial \rho}{\partial t}\,\mathrm{d}x + \rho(b,t)\frac{\mathrm{d}b}{\mathrm{d}t} - \rho(a,t)\frac{\mathrm{d}a}{\mathrm{d}t}.$$

The above considers the intrinsic rate of change of the function within the integrand, but then adds the extra mass due to the right boundary shifting rightwards (b) and subtracts the mass due to the left boundary shifting rightwards (a). It is known as the Leibniz integral rule. We can thus write this within the integral as

$$\frac{\mathrm{d}m_{\text{blob}}}{\mathrm{d}t} = \int_a^b \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho(x,t)\frac{\mathrm{d}x}{\mathrm{d}t}\right)\right]\,\mathrm{d}x.$$

However, the quantity

$$\frac{\mathrm{d}x}{\mathrm{d}t} \equiv u(x,t)$$

represents the velocity of the fluid (which for the moment we assume to be a known and provided quantity). Therefore we can write the mass change as

$$\frac{\mathrm{d}m_{\text{blob}}}{\mathrm{d}t} = \int_a^b \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u)\right] \mathrm{d}x.$$

This result, which explains how to pass a derivative through an integral express mass of a substance within a flow is known as the *Reynolds Transport Theorem*. We have just derived it in 1D.

Reynolds Transport Theorem

Let $\rho = \rho(x,t)$ be some quantity (such as density) that is advected along a one-dimensional line in x due to a fluid with velocity $u(x,t)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} \rho \,\mathrm{d}x = \int_{a(t)}^{b(t)} \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u)\right] \mathrm{d}x.$$

We may now consider the substance being transported along the x-direction. If there is no interior creation or destruction of the source, then by conservation of mass, it must be the case that

$$\frac{\mathrm{d}m_{\text{blob}}}{\mathrm{d}t} = 0.$$

Thus, again since the above integral identity applies to all possible values of a and b, it must be the case that the integrand is zero. Thus we conclude with the so-called tranport equation.

One-dimensional transport equation

The transport of a substance described by $\rho(x,t)$ advected along a one-dimensional line in x due to a fluid with velocity $u(x,t)$ is given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0.$$

Again, we must consider the above problem in combination with potential initial conditions and boundary conditions. For instance, we might specify that the substance begins from some initial state, say

$$\rho(x,0) = \rho_0(x).$$

The boundary conditions are more subtle. It is not always obvious what the boundary conditions should be on a problem.

Here is an example of a solution of such a problem.

Linear advection equation with constant speed

Solve the problem given by

$$\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = 0,$$

where the velocity $u = c$ is constant. You may assume the initial condition is given by

$$\rho(x, 0) = \rho_0(x) = e^{-x^2}.$$

Solution

The general solution is given by

$$\rho(x, t) = F(x - ct),$$

where F is an arbitrary (differentiable) function. You can verify this via the chain rule, noting that

$$\frac{\partial F}{\partial t} = -cF'(x - ct) \quad \text{and} \quad \frac{\partial F}{\partial x} = F'(x - ct).$$

Applying the initial condition at $t = 0$ gives $F(x) = \rho_0(x)$ and thus

$$\rho(x, t) = e^{-(x-ct)^2}.$$

You may plot this to discover that it is, as perhaps expected, a Gaussian profile moving to the right at constant speed (for $c > 0$).

2 Dimensional scaling analysis

Putting a mathematical model into non-dimensional form is fundamental (Fowler 2011). When we refer to a "back-of-the-envelope calculation", we often mean simple algebraic calculations that still nevertheless provide enormous insight on problems. Such analyses are often based on dimensional analysis, which identifies the relationships between the different quantities involved in the problem.

2.1 Dimensional quantities

Every physical quantity, say Q, can be expressed as a product of a dimensional unit, denoted [Q], and a magnitude, say Q'. Thus we write

$$Q = Q'[Q]$$

For example, if x corresponds to the physical length in a problem, we might select $[x] =$ km or $[x] =$ yards or $[x] =$ m. It is important to choose the dimensionalisation to suit the problem under consideration.

2.1.1 SI units

The International System (SI) of Base Units sets out a distinct selection of choices for dimensions in certain physical quantities. The seven fundamental dimensional units are

- [Length] = metre
- [Time] = seconds
- [Mass] = kilogram
- Temperature = Kelvin
- [Electric current] = ampere
- [Light intensity] = candela
- [Material quantity] = mole

Dimensional units that can be expressed in terms of other fundamental units are known as *derived units*. For example:

- [Speed] = metre/second

- [Acceleration] = metre/second2
- [Force] = kilogram . metre/second2

2.2 Dimensional homogeneity and non-dimensionalisation

All terms in any equation must have the same dimensions. This is the principle of dimensional homogenuity. For example, Newton's second law expresses the fact that

$$F = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

We can check, then, that the units do indeed match up on either side. Here, the RHS has units of [m] [x]/[t]2 or in SI units, kg . metres / seconds2. This indeed matches our previous given SI unit decomposition for force.

Notice in addition that the input to functions like $\cos\theta$ and e^z must be non-dimensional (or dimensionless).

The process of nondimensionalisation is then as follows. Given an equation, we know that each term must have the same dimension. Therefore, we can scale all the dependent and independent variables by dimensional constants in order to yield a non-dimensional equation.

Why this is an important tool is demonstrated by the below.

2.3 Returning to the heat equation

> 2023-24 note
>
> The below has been cleaned up to match the notation on the 13 Feb 2023 delivery.

Exact units are not relevant for dynamics, and it is instead the ratio of units that we care about. To apply this principle, let us non-dimensionalise the equation Equation 1.1. Let us consider the heat equation as given by the system

$$\begin{aligned} \frac{\partial T}{\partial t} &= \kappa \frac{\partial^2 T}{\partial x^2} \\ T(x,0) &= T_{\text{init}} \\ T(0,t) &= T_a \\ T(L,t) &= T_b. \end{aligned} \tag{2.1}$$

Note that the constant κ is known as the thermal diffusivity, and is given by

$$\kappa = \frac{k}{\rho c},$$

which were the units previously introduced.

We introduce typical scales for each of the variables. For example, we non-dimensionalise the temperature, distance, and time by setting

$$T = [T]T', \qquad x = [x]x', \qquad t = [t]t'.$$

Substitution into the equation now yields

$$\begin{aligned} \frac{\partial T'}{\partial t'} &= \frac{\kappa[t]}{[x]^2}\frac{\partial^2 T}{\partial x^2} \\ T'(x',0) &= T_{\text{init}} \\ T'(0,t') &= \frac{T_a}{[T]} \\ T'(L/[x],t') &= \frac{T_b}{[T]}. \end{aligned} \tag{2.2}$$

Be sure to work out the above for yourself. For example, notice that the previous boundary at $x = L$ is now sent to $x' = L/[x]$. Now we see that the final equation is beautifully simple: At this point, we can identify a crucial non-dimensional parameter given by

$$\Pi = \frac{\kappa[t]}{[x]^2} = \frac{k[t]}{\rho c[x]^2}.$$

Although this looks quite complicated, you can test that this parameter is indeed non-dimensional by looking up the units for thermal conductivity, k, density ρ, and specific heat c, and verifying that indeed Π is non-dimensional.

2.4 Choice of units

It is important to remember that there are technically no wrong choices for the scales [x], [T], and [t], as long as they yield consistent balances. However, some choices are better than others for the context of the problem. Let us select:

$$[x] = L \qquad [T] = T_{\text{init}}.$$

This means our unit of length is "one pipe" (if this is indeed a pipe), and our unit of temperature is whatever the initial temperature was set to be. The choice of time is free at the moment, and therefore we can select time as

$$\Pi = \frac{\kappa[t]}{[x]^2} = 1 \Longrightarrow [t] = \frac{L^2}{k}.$$

Now we see that the final equation is beautifully simple:

$$\begin{aligned} \frac{\partial T'}{\partial t'} &= \frac{\partial^2 T}{\partial x^2} \\ T'(x', 0) &= 1 \\ T'(0, t') &= \frac{T_a}{T_{\text{init}}} \equiv A, \\ T'(1, t') &= \frac{T_b}{T_{\text{init}}} \equiv B. \end{aligned} \tag{2.3}$$

2.4.1 Interpretation

What is the whole point of the above exercise?

In the original problem, there were five parameters:

$$\kappa, \qquad T_{\text{init}}, \qquad T_a, \qquad T_b, \qquad L.$$

Imagine a situation where you are performing an analysis of heat spread in different pipes of different materials (characterised by κ) and lengths (L), with different initial and boundary temperatures. That is potentially a highly complicated parameter space to search (five-dimensional).

Rather than solving the problem again and again for each change in parameters, what the analysis yielding (Equation 2.3) reveals is that the parameter space only needs to be two-dimensional, for parameters A and B. Therefore, by solving the problem at many different values of A and B, the solution space can be completely mapped.

Here is an example. Consider heat diffusion through common brick, which possesses a thermal diffusivity of

$$\kappa = 5.2 \times 10^{-7} \text{m}^2/\text{s}.$$

Therefore a brick that is about L = 10cm = 0.1m in size has a typical timescale of

$$[t] = \frac{L^2}{\kappa} \approx 1.9 \times 10^4 \text{s} \approx 5\text{hrs}.$$

The above choice of [t] is representative of the typical timescale you can expect for heat to diffuse through the brick. As you can see it's very long.

In summary, we have gone from a system where we needed to consider five parameters to a system where we only need to specify two (essentially, the ratio of the initial heat to the boundary values). This is an enormous simplification.

Moreover, your analysis has further identified the key non-dimensional parameters that appear, including:

$$\Pi = \frac{\kappa[t]}{[x]^2}, \qquad A = \frac{T_a}{T_{\text{init}}}, \qquad B = \frac{T_b}{T_{\text{init}}}, \qquad \frac{L}{[x]}$$

In this particular problem, you had the freedom to choose L to remove one parameter (the ratio of lengths), and time could be chosen to set $\Pi = 1$.

3 Basic energy models

In this chapter, we introduce the simplest climate model to describe the temperature of the Earth. Despite its apparent simplicity, this basic climate model is the foundation of all further discussion. We will find that, according to the most basic model, the surface temperature of the Earth is expected to be much colder than it is actually; we conclude that the likely culprit is greenhouse gases.

Consider the Earth as a single averaged body. Our goal is to obtain an equation for the surface temperature, T, by considering the energy recieved from the sun versus the energy the Earth must emit into space. This kind of balance-equation procedure is used everywhere in mathematical modelling. In essence, we are looking to write down a precise statement of the following:

Basic energy balance

$$\text{Change in internal energy of Earth} = E_{\text{in}} - E_{\text{out}}, \tag{3.1}$$

where E_{in} is the incoming energy from the sun and E_{out} is the energy leaving the Earth. Note that the left hand-side (the internal energy) will depend on the temperature, T.

3.1 The basic energy model

The basic model is derived as follows.

1. By considering the incoming radiation from the sun, obtain an estimate for the incoming energy, E_{in}.
2. Use a constitutive law that indicates how much outgoing energy is produced by an object heated to some temperature.
3. Equate the change in internal energy to be equal to the difference between the above two items.

Note that the primary components of the global energy balance are radiative fluxes: we receive short-wave radiation (UV and visible light) from the Sun, and emit longwave radiation (infra-red) to space.

First we consider incoming energy.

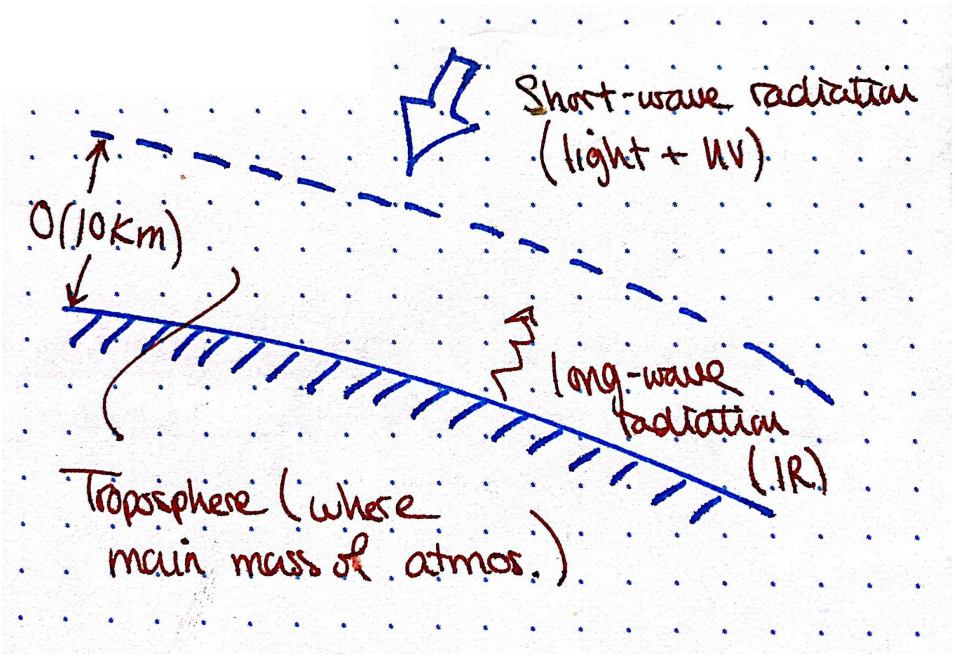


Figure 3.1: Radiation in the atmosphere

Energy from the Sun

First, note that the shortwave radiation (UV and radiation) recieved from the sun is $Q \approx 1370$ W·m^{-2} (which we consider measured at a point near the planet).
If we consider only that radiation that is absorbed into the Earth, we have

$$E_{\text{in}} = \pi R^2 Q(1-a), \tag{3.2}$$

where R is the Earth's radius.
In the above formula, we have multiplied the flux, Q, with the visible surface area, πR^2. There is an additional multiplication by $(1-a)$ where a is the *planetary albedo*, which characterises amount of energy reflected due to the surface properties. Light surfaces like snow will have high albedo, $a \approx 0.9$, while darker surfaces like the ocean have smaller albedo, $a \approx 0.3$. The global average albedo is $a \approx 0.3$.

Next we consider outgoing energy.

Energy from the Earth

We now wish to characterise the energy, E_{out}, and in the case of Earth, this will correspond to longwave radiation (infra-red) emitted into space. All bodies characterised by

a temperature, say T_e, will emit radiation, Q_e. As a model, we can consider Q_e to be given by the Stefan-Boltzmann law, which states that

$$Q_e = \sigma T_e^4, \tag{3.3}$$

where $\sigma \approx 5.67 \times 10^{-8} W m^{-2} K^{-4}$ is the Stefan-Boltzmann constant.
Now although the Earth's surface may emit radiation according to (Equation 3.3), some of this radiation will be absorbed by the atmosphere and reflected back. This is the **greenhouse effect**. As a consequence of the greenhouse gas, the surface temperature of the Earth, T, will be larger than the effective emitting temperature, T_e. For the moment, we model this as

$$T_e = \gamma^{1/4} T_e, \tag{3.4}$$

where $\gamma < 1$ is a greenhouse gas factor, which depends on the properties of the atmosphere.

Now combining the above equations, we have

$$E_{\text{in}} - E_{\text{out}} = \pi R^2 Q(1-a) - 4\pi R^2 \sigma \gamma T^4, \tag{3.5}$$

which gives the incoming energy per unit time.

Internal heat energy

Due to this incoming energy, the Earth will cool or heat in response. We need to know how the internal temperature of an object responds to an input in energy. The general formula is

$$\text{Internal heat energy} = \text{volume} \times (\rho c_p) \times T.$$

The key quantity is the experimentally determined, c_p, which corresponds to the specific heat capacity. It is given in the SI units of $\text{J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1}$, i.e. energy per unit mass per unit temperature. Note that this applied to a shell around the planet of thickness d in the atmosphere, and so the mass is given by

$$\text{mass} = (4\pi R^2) d\rho \tag{3.6}$$

where ρ is the average density of the atmosphere. Let us imagine the increase in temperature, ΔT,during an interval of time, Δt. You can now verify that

$$\Big[(4\pi R^2) d\rho\Big] c_p (\Delta T)$$

returns the units of Joules—i.e. this is the internal energy produced during the time Δt. We then have

$$\Big[(4\pi R^2) d\rho\Big] c_p \Delta T = \Delta t (E_{in} - E_{out}).$$

Putting in (Equation 3.5) and taking the limit of $\Delta t \to 0$, we finally have a heat equation for the Earth's temperature.

Zero-dimensional model for the surface temperature of the Earth

The Earth's temperature, as measured on a layer of thickness d at the troposphere is given by the following ordinary differential equation (ODE) for $T = T(t)$:

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{4}Q(1-a) - \sigma\gamma T^4, \tag{3.7}$$

where we have defined

$$C = \rho c_p d$$

as the heat capacity of the atmosphere. Above, the solar flux, Q is often taken to be $Q = 1370$ W/m2, $\sigma = 5.67 \times 10^{-8}$ W/(m2K4), and $\gamma \leq 1$ is the greenhouse gas factor.

The above equation Equation 3.7 is time-dependent, but we may consider that the surface temperature, either over long-time or in an averaged sense, is dictated by the steady-state (time-independent) solution. Setting $\mathrm{d}T/\mathrm{d}t$ to zero, we see that there is a unique steady-state given by

$$T = \left(\frac{Q(1-a)}{4\sigma\gamma}\right)^{1/4}.$$

If we take $Q \approx 1370\,\mathrm{W m}^{-2}$, $a \approx 0.3$, $\sigma \approx 5.67 \times 10^{-8}\mathrm{W\,m^{-2}K^{-4}}$, we then get

$$T \approx 255\mathrm{K} = -18^{\circ}\mathrm{C}.$$

under the assumption that $\gamma = 1$. That's pretty cold! The actual average temperature is around $288\mathrm{K} \approx 15^{\circ}\mathrm{C}$.

The above back-of-the-envelope calculation seems to suggest that the parameter $\gamma < 1$ plays an important role in keeping the Earth warm enough for us to live on, and indeed the value of γ inferred by the above is roughly $\gamma \approx 0.61$. Later on in the course, we will develop a more rigorous model to predict such a γ by studying the properties of the atmosphere.

3.2 The history of global warming

The history of global warming (and hence the estimation of γ) is convoluted, but the origins can be considered as far back as the work of (Fourier 1827) and (Pouillet 1838), and is discussed in the work by (Van der Veen 2000).

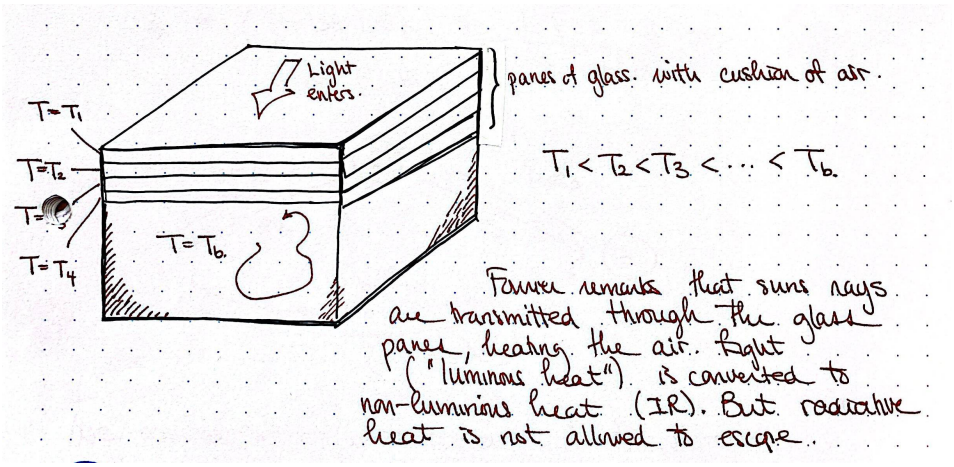


Figure 3.2: An illustration of Fourier's glass box

3.3 Next in the study of energy balance models?

At this stage, the basic energy balance model studied in this chapter can be made more complex and realistic through many different extensions. For instance:

- Consider the effects of a non-constant and nonlinear albedo. Since the albedo depends on the material property (water vs. ice vs. land), we can incorporate a toy model for spatial variability by allowing the albedo to depend on temperature, $a = a(T)$.
- Consider the time-dependence. What happens if the system starts out of the steady state? Does it tend to relax to its equilibrium profile? Can we analyse the temporal properties of the solutions?
- Can we incorporate latitude (or longitudinal) dependence into the model. For instance, should the albedo and solar radiation be considered as functions of the latitude?
- Should we incorporate the effects of diffusion (heat spreading out) or convection (heat transported with flows)?
- To what extent should we consider the effects of the atmosphere on the absorption and reflection properties of the incoming (and outgoing) energy?

Many of these questions are much more involve, as they require differential or partial differential equations. Therefore in the next few chapters, we will discuss, in a very general way, the collection of applied and numerical mathematics techniques that can be used when studying problems in the physical sciences.

4 Problem class 1

Abstraction

It may seem strange to study examples from elementary physics in a course that is supposed to be about Planet Earth. But simple examples are the best ways to learn these important techniques. The full climate equations are often very involved. These toy models still nevertheless capture the spirit of what you must do when attacking any scientific problem.

In this problem class, we will practice some concepts about non-dimensionalising in preparation for the Problem Set 1 in Chapter 14. We will cover these two strategies in choosing scalings.

Scaling principle 1

Select the characteristic scales so that as many of the possible non-dimensional numbers, Π_i, $i = 1, 2, 3, ...$ are normalised.

Scaling principle 2

Select characteristic scales so that no terms in the model diverge in the physical limit of interest.

Here are the problems we shall do in the problem class.

4.1 Projectile motion

A projectile of mass M (in kg) is launched vertically with initial velocity V_0 (in m/s) from a position Y_0 (in m) above the surface. Thus the mass's position, $Y(t)$ is governed by Newton's second law (applied to the mass and the mass of the Earth) and the set of equations

$$MY_{tt} = -\frac{gR_E^2 M}{(R_E + Y)^2}, \tag{4.1}$$

$$Y(0) = Y_0, \tag{4.2}$$

where $g = 9.81 m/s^2$ and $R_E = 6.4 \times 10^6 m$ is the radius of the Earth.

1. Non-dimensionalise the equation using arbitary length and time scales.
2. Identify the non-dimensional constants, Π_i.
3. Choose a length scale of $L = Y_0$ and time scale of $T = (L/g)^{1/2}$. Discuss the resultant equation and the interpretation of choosing these scales.
4. Does your above choice allow you to easily study the limit of $R_E \to \infty$? If the limit can be taken, reduce the governing system to a simpler equation.
5. Does your choice in 3. allow you to easily study the limit of $Y_0 \to 0$? If not, choose an alternative choice of length and time scales and in that case, reduce the set of equations.

4.2 Terminal velocity

A ball of radius R (in m) and uniform density ρ (in kg/m3) falls in a viscous fluid. The fluid has density ρ_f (in kg/m3) and viscosity (a measure of friction or resistance) μ (in kg/(m s)). The equation that governs the velocity is

$$\frac{4}{3}\pi R^3 \rho \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{4}{3}\pi R^3(\rho - \rho_f)g - 6\pi\mu R V, \tag{4.3}$$

$$V(0) = V_0. \tag{4.4}$$

1. Choose appropriate velocity and time scales to non-dimensionalise the equation so as to leave only a single non-dimensional number on the drag term (the last term on the right hand-side).
2. Define the non-dimensional parameter expressing a ratio between drag force and gravity force by the Stokes number (St) and confirm that it is

$$St = \frac{9\mu V_0}{2(\rho - \rho_f)gR^3}.$$

3. Comment on the two limits of $St \to 0$ and $St \to \infty$. Can the problem be reduced in these two limits? If so, reduce and solve.

Part II

Practical applied mathematics

As we go deeper into formulating the equations that model or govern aspects of Planet Earth, we will quickly come to the realisation that many such equations, even for the simplest minimal models, are not exactly solvable.

For example, in the analysis of the Ocean models later in the course, we develop the following "simple" model for the temperature in the ocean:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \delta(1-x) - |f(x,y)|x, \tag{4.5}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - y - |f(x,y)|y, \tag{4.6}$$

where we have introduced the function,

$$f(x,y;R,\lambda) = \frac{1}{\lambda}(Rx - y),$$

This is quite a difficult problem! This is essentially a set of two nonlinear differential equations for two unknowns and three parameters. What kind of practical applied mathematics can we apply to study such problems?

The intention of this part is to introduce (and in some cases, review) three key concepts:

1. Asymptotic approximations.
2. Numerical solutions of differential equations.
3. Numerical solutions of nonlinear equations (Newton's method).

5 Asymptotic approximations

One powerful set of techniques for approximating solutions to equations is called *asymptotic analysis* or *perturbation theory*. To begin with, in this chapter, we introduce you to these techniques as a means to approximating the solutions to equations like:

$$\epsilon x^2 + x - 1 = 0,$$

when ϵ is a small parameter.

Soon, we apply the techniques to approximating solutions of differential equations.

5.1 A simple quadratic

A singular quadratic

Consider the solution of

$$\epsilon x^2 + x - 1 = 0, \tag{5.1}$$

where ϵ is a fixed and very small positive number, say 0.000001. Forget that we know how to solve a quadratic equation: is it possible to develop a systematic approximation method?

If $\epsilon = 0$, then $x = 1$. Moreover, if we substitute $x = 1$ into the equation, then we see that the error is small and proportional to ϵ. It is natural to seek an approximation in powers of ϵ. We call this an *asymptotic expansion*. We write

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...$$

Substitution into the equation yields

$$\epsilon\Big(x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...\Big)^2 + \Big(x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...\Big) - 1 = 0.$$

Expand and collect terms in powers of ϵ:

$$\Big(x_0 - 1\Big) + \epsilon\Big(x_1 + x_0^2\Big) + \epsilon^2\Big(x_2 + 2x_0x_1\Big) + ... = 0.$$

Now we equate coefficients at each order in ϵ. This gives

$$x_0 - 1 = 0 \Longrightarrow x_0 = 1 \tag{5.2}$$
$$x_1 + x_0^2 = 0 \Longrightarrow x_1 = -1 \tag{5.3}$$
$$x_2 + 2x_0x_1 = 0 \Longrightarrow x_2 = 2 \tag{5.4}$$

We therefore have obtained the three-term approximation,

$$x = 1 - \epsilon + 2\epsilon^2 + ...$$

Clearly we could continue this process *ad infinitum* obtaining increasingly accurate approximations to one of the roots.

5.1.1 The singular root

But where has the other quadratic root gone?

The problem is that in considering ϵ to be small, we began by ignoring the leading term, ϵx^2. We effectively assumed that the equation was primarily *balanced* by setting the x term with the -1 term, and the sum of the two terms approximately equalling zero.

But if $|x|$ is large, then clearly our assumption that ϵx^2 being small may not be necessarily true for it depends on how large $|x|$ is compared to ϵ. Note that if $|x|$ is large, then necessarily the last term, -1, is negligible in comparison. Therefore, in order for ϵx^2 to balance x, we see that $|x|$ must be of size $1/\epsilon$.

Therefore this suggests that we should re-scale our solution as follows

$$x = \frac{X}{\epsilon}.$$

Substitution into the original quadratic now yields

$$X^2 + X - \epsilon = 0.$$

Now notice that $\epsilon = 0$ expresses the correct balance in order to detect that missing root. Again we write

$$X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + ...$$

and attempt to solve order by order. Substitution into the equation yields

$$\Big(X_0 + \epsilon X_1 + \epsilon^2 X_2 + ...\Big)^2 + \Big(X_0 + \epsilon X_1 + \epsilon^2 X_2 + ...\Big) - \epsilon = 0.$$

Expand and collect orders of ϵ:

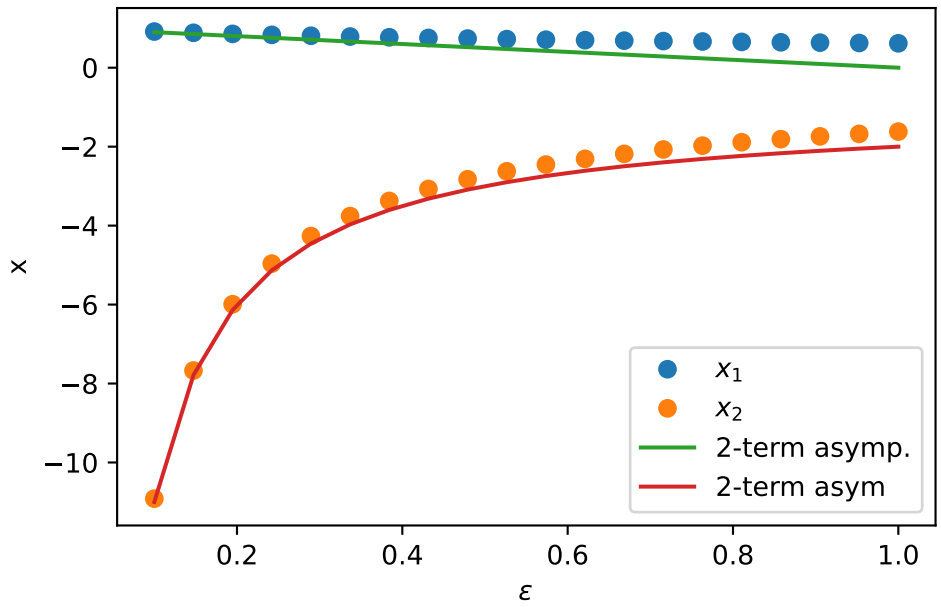
$$X_0^2 + X_0 = 0 \Longrightarrow X_0 = -1 \tag{5.5}$$
$$2X_0X_1 + X_1 - 1 = 0 \Longrightarrow X_0 = -1, \tag{5.6}$$

and thus to two orders, we have

$$X = -1 - \epsilon + ... \implies x = -\frac{1}{\epsilon} - 1 + ...$$

Of course, we have used a very simple example (a solvable quadratic) to illustrate the idea of asymptotic approximations, but you should hopefully see that this method is extensible to much more complicated equations.

```
import numpy as np
import matplotlib.pyplot as plt

ep = np.linspace(1,0.1, 20)
root1 = (-1 + np.sqrt(1-4*ep*(-1)))/(2*ep)
root2 = (-1 - np.sqrt(1-4*ep*(-1)))/(2*ep)
asym1 = 1 - ep
asym2 = -1/ep - 1
plt.plot(ep, root1, 'o')
plt.plot(ep, root2, 'o')
plt.plot(ep, asym1, '-')
plt.plot(ep, asym2, '-')
plt.legend(['$x_1$', '$x_2$', '2-term asymp.', '2-term asym'])
plt.xlabel('$\epsilon$')
plt.ylabel('x');
```



5.2 Order notation and the tilde sign for asymptotic

We define precisely what we mean when we say that two functions, say f and g, exhibit the same behaviour in some limit, say $\epsilon \to 0$ or $x \to x_0$ or $x \to \infty$ and so forth. For instance, we claim that the graphs of $\sin(x)$ and x look very similar as $x \to 0$. Thus we might write

$$\sin(x) \sim x \quad \text{as } x \to 0. \tag{5.7}$$

This notion of *similarity* allows us to specify functional behaviours at a deeper level than just limits. As you can see, it is not as useful to specify that

$$\lim_{x\to 0} \sin x = \lim_{x\to 0} x.$$

In contrast, (Equation 5.7) is much more prescriptive about the way that the functions are approaching the limit.

Definition of $\sim$, $\gg$, and $\ll$

First, the notation

$$f(x) \ll g(x), \qquad x \to x_0,$$

is read as "$f(x)$ is much smaller than $g(x)$ as $x \to x_0$" and means

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0.$$

We may analogously use $g(x) \gg f(x)$ for "much greater than…".
Second, the notation

$$f(x) \sim g(x), \qquad x \to x_0,$$

is read as "$f(x)$ is asymptotic to $g(x)$ as $x \to x_0$", and means that the error between f and g tends to zero as $x \to x_0$, or

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1.$$

We will often say "f is like g" or "f behaves like g",

Here are some examples.

Examples

- $\sin x \sim x \sim \tan x$ as $x \to 0$
- $x^2 + x + 1 \sim \dfrac{x^3 + \sin x}{1 + x}$ as $x \to \infty$
- $\sin x \ll \cos x$ as $x \to 0$

In the examination of limiting processes, often the main issue of consideration is the relative sizes of quantities defined according to their powers. For example, if x is a very small number, with $x = 10^{-5}$, then x^5 is much smaller than x (in terms of our notation, $x^5 \ll x$ as $x \to 0$). On the other hand, we might not care so much about the difference between

$$x^5 \quad \text{vs.} \quad 5x^5$$

The point is that the *order* of x^5 and $5x^5$ is the same as $x \to 0$. The "Big-Oh" notation formalises this distinction.

Definition of Big-Oh

We write $f = O(g)$ as $x \to x_0$ to mean that there exists constants $K > 0$ and $x^* > 0$ such that

$$|f| < K|g| \quad \text{for all } |x - x_0| < x^*.$$

In practice, the use of the order symbol is very natural and you will not need to work with the technical definition. For example, when you derive the terms of the Maclaurin/Taylor series, you are naturally clustering all the terms of the same order (power) together. For us, the O symbol provides a very convenient way of separating terms of different sizes.

Examples

- $2\sin x = O(\tan x)$ as $x \to 0$
- $x^2 + x + 1 = O\left(\dfrac{5x^3 + \sin x}{1 + x}\right)$ as $x \to \infty$

Let us return to the case of the quadratic example (Equation 5.1). Using the O notation, we can write

$$x = \begin{cases} 1 - \epsilon + 2\epsilon^2 + O(\epsilon^3) \\ -\frac{1}{\epsilon} - 1 + O(\epsilon^2) \end{cases}$$

for the two roots. Alternatively, we can truncate the expansions and simply using the $\sim$ symbol:

$$x \sim \begin{cases} 1 - \epsilon \\ -\frac{1}{\epsilon} - 1 \end{cases}$$

6 Asymptotic approximations of ODEs

In the previous chapter, we learned about the technique of asymptotic expansions, whereby the solution of an equation is expressed in terms of a series in powers of a small parameter:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...$$

The precise choice of power progression (here integer powers of ϵ will depend on the particular problem. The same idea can be extended to approximating solutions of differential equations. The upshot of this procedure is that at each order of the scheme, a simpler problem can be studied.

Again it is best to demonstrate through examples.

6.1 Returning to the projectile problem

In Chapter 4 and Chapter 14 you studied the non-dimensionalisation of the projectile problem. Once re-scaled, it takes the following form:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\frac{1}{(1+\epsilon y)^2}, \qquad t > 0 \tag{6.1}$$

$$y(0) = 0, \tag{6.2}$$

$$y'(0) = 1. \tag{6.3}$$

{#eq-asymball} This is a difficult problem without, in fact, any explicit solutions. However, we can estimate the solution in the limit $\epsilon \to 0$. We expand the solution as

$$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + ...$$

In order to expand the denominator, you can use Taylor's theorem to expand the function

$$f(x) = (1+x)^\alpha = f(0) + f'(0)x + ... = 1 + \alpha x + ...$$

around $x = 0$.

The differential equation now yields

$$y_0'' + \epsilon y_1'' + \epsilon^2 y_2'' + ... = -\left[1 - 2\epsilon\left(y_0 + \epsilon y_1 + ...\right) + ...\right]$$

so grouping terms together order-by-order yields

$$\Big[y_0'' + 1\Big] + \epsilon\Big[y_1'' - 2y_0\Big] + ... = 0.$$

We can similarly substitute the expansion into the initial conditions. Altogether, at leading order, we obtain the following system to solve:

$$y_0'' + 1 = 0, \tag{6.4}$$
$$y_0(0) = 0, \tag{6.5}$$
$$y_0'(0) = 1. \tag{6.6}$$

Integrating twice and applying the boundary conditions gives us

$$y_0(t) = -\frac{1}{2}t^2 + t.$$

In fact, this is simply the parabolic motion you would expect from school Physics. The $\epsilon = 0$ solution corresponds to assuming that the mass at the centre of the planet is dominant and then acceleration is constant.

However, we can now proceed to higher order and examine the nonlinear effects. Proceeding to $O(\epsilon)$, we have the following system to solve:

$$y_1'' = 2y_0, \tag{6.7}$$
$$y_1(0) = 0, \tag{6.8}$$
$$y_1'(0) = 0. \tag{6.9}$$

Notice the boundary conditions come from the fact there are no ϵ corrections in the original boundary conditions, so $y_n(0) = y_n'(0) = 0$ for all $n > 0$. Again this system is simple to integrate. Integrating the solution for y_0 twice and substitution of the initial conditions yields

$$y_1(t) = -\frac{1}{12}t^4 + \frac{1}{3}t^3.$$

We have thus solved for the asymptotic approximation to two orders. We have

$$y(t) \sim \Big[-\frac{1}{2}t^2 + t\Big] + \epsilon\Big[-\frac{1}{12}t^4 + \frac{1}{3}t^3\Big].$$

This was quite an accomplishment! We have taken a problem that was not easily solvable in explicit form and through fairly simple integrations, obtained an approximation to two orders in ϵ. How good is it? Let us solve the problem numerically and compare with the asymptotic approximation.

6.2 Numerical solutions of IVPs

We first demonstrate how to solve ODEs (initial-value-problems, IVPs) using black-box functions in Python. For starters, most numerical formulations for ODEs will require that the problem be posed in terms of a first-order system of equations. To convert (**?@eq-asymball**) into such a form, create a set of unknowns for the derivatives. Set

$$\mathbf{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

Then we have the following first-order system:

$$\mathbf{Y}'(t) = \mathbf{F}(t, \mathbf{Y}(t)) = \begin{pmatrix} y_1' \\ -\frac{1}{(1+\epsilon y_1)^2} \end{pmatrix} \tag{6.10}$$

$$\mathbf{Y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.11}$$

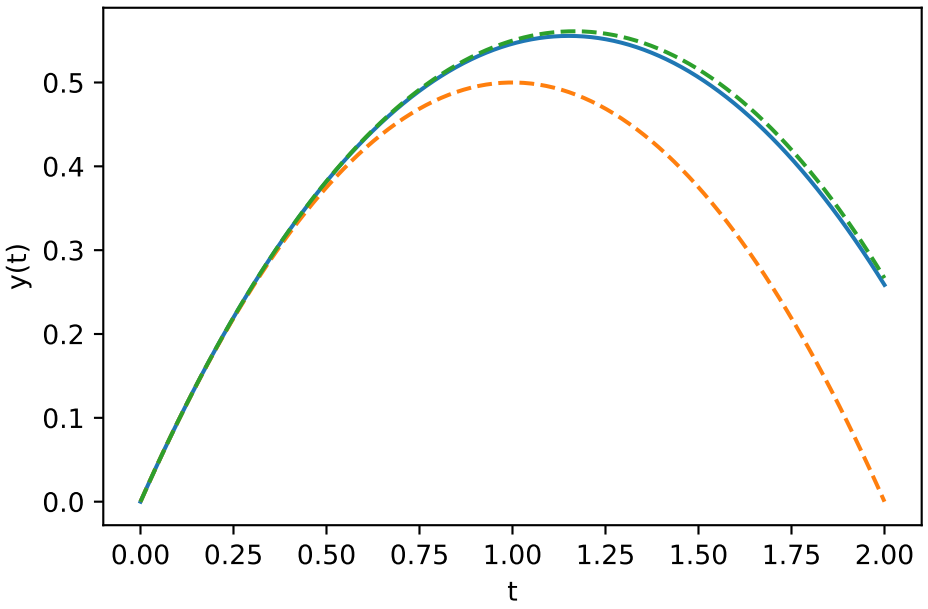
{#eq-asymsys}

You can find a little guide on using solve_ivp in Python here. Here is the Python code to solve the differential equation.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

ep = 0.2 # epsilon value
tmax = 2 # max time
t = np.linspace(0, tmax, 100) # mesh used for plotting

# Define function for the ODE
def f(t, Y):
    ep = 0.2
    y, yp = Y
    ypp = -1/(1 + ep*y)**2
    return [yp, ypp]

# define the initial condition
Y0 = [0, 1]

sol = solve_ivp(f, [0, tmax], Y0, dense_output=True)
```

```
# Prior to plotting, re-interpolate solution on a fine grid
yy = sol.sol(t)
# Asymptotic solutions
y0 = -1/2*t**2 + t
y1 = -1/12*t**4 + 1/3*t**3

# Plot it all
plt.plot(t, yy[0,])
plt.plot(t, y0, '--')
plt.plot(t, y0 + ep*y1, '--')
plt.xlabel('t')
plt.ylabel('y(t)')
```

```
Text(0, 0.5, 'y(t)')
```



The two-term approximation does beautifully well, even at this moderate value of $\epsilon = 0.2$.

7 Euler's method

In the previous section, we used built-in ODE solvers to develop numerical solutions. It is important to gain an understanding how a simple ODE solver works. The simplest scheme is called Euler's method, and this we now explain.

Begin from the system (**?@eq-asymsys**). We assume that the solution is represented by a discrete set of points, $\mathbf{Y}_n = \mathbf{Y}(t_n)$ at the times $t_0 = 0$, $t_1 = \Delta t$, $t_2 = 2\Delta t$, and so on. The time derivative is written as a discrete derivative while we approximate the right hand side by its value at the nth time step:

$$\frac{\mathbf{Y}_{n+1} - \mathbf{Y}_n}{\Delta t} = \mathbf{F}(t_n, \mathbf{Y}_n)$$

Rearranging yields a very simple algorithm for solving the ODE:

$$\mathbf{Y}_n = \mathbf{Y}_{n-1} + \mathbf{F}(t_{n-1}, \mathbf{Y}_{n-1})\Delta t$$

for $n = 1, 2, 3, ...$

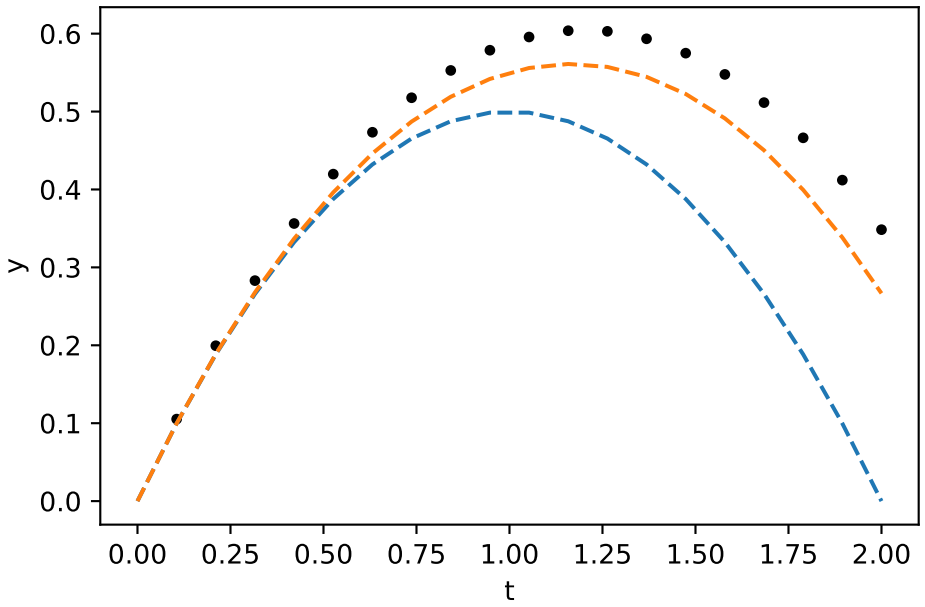
This would be implemented via the following pseudocode:

Euler's method

```
1. Input: function f(t, Y)
            time step, dt
            initial condition, Y0

2. Set initial condition Y = Y0

2. Take one Euler step and overwrite previous value

                  Y = Y + f(t, Y)

3. Increment t by dt and goto 2
```

Euler's method is conceptually simple but quite inaccurate. But in this case, we see that it works fairly well in comparison to the built-in solvers.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

ep = 0.2            # epsilon value
tmax = 2            # max time
N = 20              # number of steps
t = np.linspace(0, tmax, N) # mesh used for plotting
dt = t[1] - t[0]

# Define function for the ODE
def f(t, Y, ep):
    y, yp = Y
    ypp = -1/(1 + ep*y)**2
    return np.array([yp, ypp])

# define the initial condition
Y = [0.0, 1.0]
ti = 0

# define the solution vector
for i in range(1, N):
    ti = ti + dt  # Increment time
    Y = Y + f(ti, Y, ep)*dt # Euler step
    plt.plot(ti, Y[0], 'k.')

# Asymptotic solutions
y0 = -1/2*t**2 + t
y1 = -1/12*t**4 + 1/3*t**3
plt.plot(t, y0, '--')
plt.plot(t, y0 + ep*y1, '--')
plt.xlabel('t');
plt.ylabel('y');
```


0.6
0.5
0.4
0.3
0.2
0.1
0.0
y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
t

8 Matched asymptotics

In Chapter 6, we studied how asymptotic expansions can be used to approximate equations like

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\frac{1}{(1+\epsilon y)^2}$$

by expanding the solution as $y(t) = y_0(t) + \epsilon y_1(t) + ...$ These are known as *regular* problems because a small perturbation, ϵ, does not seem to fundamentally change the $\epsilon = 0$ solution beyond a small perturbation. This is not always the case. In *singular* problems, the situation of $\epsilon \neq 0$ is fundamentally different than the situation from $\epsilon = 0$. You have already seen such an example in Chapter 5. The equation

$$\epsilon x^2 + x - 1 = 0$$

has one root for $\epsilon = 0$ and two roots for non-zero small ϵ—even infinitesimally small values! This is quite interesting. From a wider scientific perspective, you may wonder what other problems in nature possess such singular effects.

The point of this lecture is study a technique known as *matched asymptotics*. These matched asymptotics are often necessary for singularly perturbed differential equations.

Previously in Chapter 3, we derived a basic equation that governs the temperature on the surface of the planet. This equation had the following form:

$$(\rho c_p V)\frac{\mathrm{d}T}{\mathrm{d}t} = E_{\text{in}}(t,T) - E_{\text{out}}(t,T).$$

For the purpose of this section, let us make up a toy model. We suggest, in non-dimensional form,

$$\epsilon\frac{\mathrm{d}T}{\mathrm{d}t} = R(t) - T, \quad t \geq 0 \tag{8.1}$$

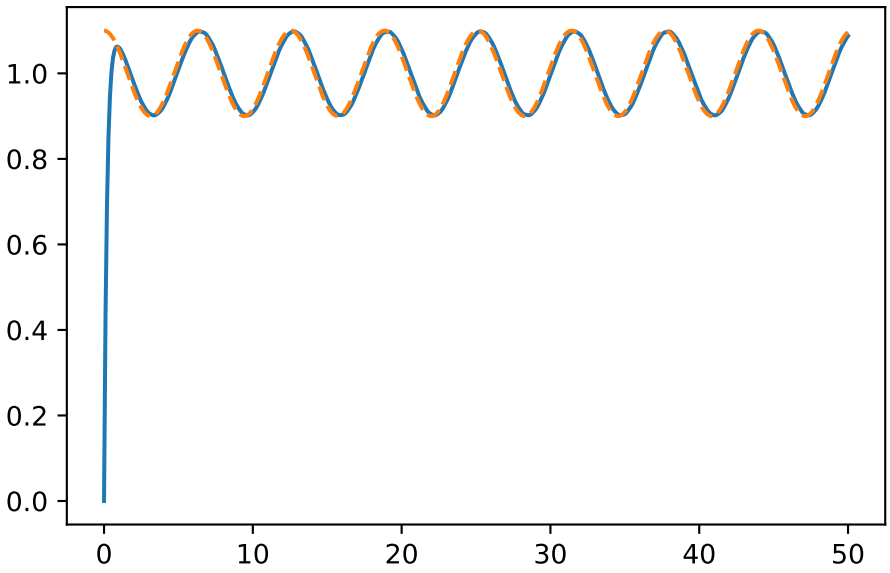
$$T(0) = T^*, \tag{8.2}$$

{#eq-matched-T} where we consider $\epsilon > 0$ and $\epsilon \ll 1$. You can think of the above model as modelling the temperature on a substance that radiates heat in a fashion proportional to itself (-T) and is being subjected to an (incoming) heat source, R. Let us take as an example,

$$R(t) = 1 + A\cos(t).$$

Our choice for R is not so important. This equation is, in fact, solvable in closed form (how?) but let us get additional practice solving numerically.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

Ts = 0
ep = 0.2
A = 0.1
f = lambda t, T: 1/ep*(1 + A*np.cos(t) - T)

tmax = 50
sol = solve_ivp(f, [0, tmax], [Ts], dense_output=True)

t = np.linspace(0, tmax, 1000)
y = sol.sol(t)
plt.plot(t, y[0])
plt.plot(t, 1 + A*np.cos(t), '--')
```



What do we observe? If $\epsilon = 0$, then we expect the solution $T \sim R(t)$. This is shown with the dashed line. However, this solution does not satisfy the necessary initial condition. We observe that near $t = 0$, the exact solution seems to very rapidly diverge from the approximation in order to satisfy the proper boundary condition. The region in t where this rapid change occurs is called a *boundary layer*.

If we repeat the experiment with an even smaller value of ϵ, we would observe that the size of this boundary layer seems to tends to zero as $\epsilon \to 0$. This numerical experiment thus inspires

the following method.

8.1 Boundary layer theory

We seek a method that will allow us to develop a uniformly valid approximation, i.e. an approximation that is good everywhere in the relevant domain, $t \geq 0$. Begin by performing the usual asymptotic approximation:

$$T(t) = T_0(t) + \epsilon T_1(t) + \epsilon^2 T_2(t) + ...$$

Substitution into the ODE (**?@eq-matched-T**) yields at leading order,

$$0 = R(t) - T_0(t) \Longrightarrow T_0(t) = R(t) = 1 + A\cos(t).$$

As we have noted, this approximation fails to satisfy the initial condition, $T(0) = T^*$ in general. It is possible to go to higher order but this is not so important at the moment. So for now, we have obtained:

$$T_{\text{outer}} \sim \Big[1 + A\cos t\Big].$$

We have chosen to refer to this as the `outer` solution for reasons that will be abundantly clear. But rather than satisfying $T(0) = T^*$, this approximation has the limiting behaviour of

$$\lim_{t\to 0} T_{\text{outer}} \sim \Big[1 + A\Big]$$

Above, we have only included the leading term in the limit expression.

8.1.1 The inner scaling

Our intuition follows a very similar logic to the examination of the singular root in Section 5.1.1. Above, our naive assumption was that $\epsilon T'(t)$ could be ignored since ϵ is a small number. However, this may not be the case if the gradient is very large.

Our intuition further suggests that the boundary layer occurs near $t = 0$ and that it scales in size with ϵ. Therefore, let us set

$$t = \epsilon^\alpha s,$$

as a change of coordinates. We expect $\alpha > 0$ (otherwise t is not small), and within this region, we expect the new coordinate, s, to be $O(1)$ (of moderate size). We then transform the unknown function:

$$T(t) = T(\epsilon s) = U(s),$$

and seek a new differential equation for U. By the chain rule,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \epsilon^{-\alpha}\frac{\mathrm{d}U}{\mathrm{d}s}.$$

Before substituting into the equation, we are prudent to examine the behaviour of $R(t)$ near $t = 0$. We know by Taylor's theorem that

$$R(t) = 1 + A\left(1 - \frac{t^2}{2} + ...\right).$$

Therefore, under the substitution, we may approximate R by its leading terms:

$$R(\epsilon^\alpha s) \sim 1 + A.$$

For now, we will not need more terms than this. Substituting into the ODE now gives

$$\epsilon^{1-\alpha}\frac{\mathrm{d}U}{\mathrm{d}s} \sim \Big[1 + A\Big] - U.$$

Now in order to involve the first term, it is sensible to select

$$1 - \alpha = 0 \Longrightarrow \alpha = 1.$$

8.1.2 The inner equation

Therefore, the correct coordinate re-scaling was the 'obvious' one:

$$t = \epsilon s.$$

Substituting this again in (**?@eq-matched-T**):

$$\frac{\mathrm{d}U}{\mathrm{d}s} = 1 + A\cos(\epsilon s) - U, \tag{8.3}$$

$$U(0) = T^*. \tag{8.4}$$

The procedure is now exactly the same. We expand

$$U(s) = U_0(s) + \epsilon U_1(s) + \epsilon^2 U_2(s) + ...$$

At leading order, we get

$$U_0' = 1 + A - U_0 \tag{8.5}$$

$$U_0(0) = T^*. \tag{8.6}$$

The above ODE can be solved by integrating factors. Multiplying both sides be e^s, we have

$$(U_0 e^s)' = (1 + A)e^s.$$

Integrate and use the initial condition:

$$U_0(s) = (1 + A) + (T^* - (1 + A))e^{-s}.$$

This is exactly what we expect. Notice that

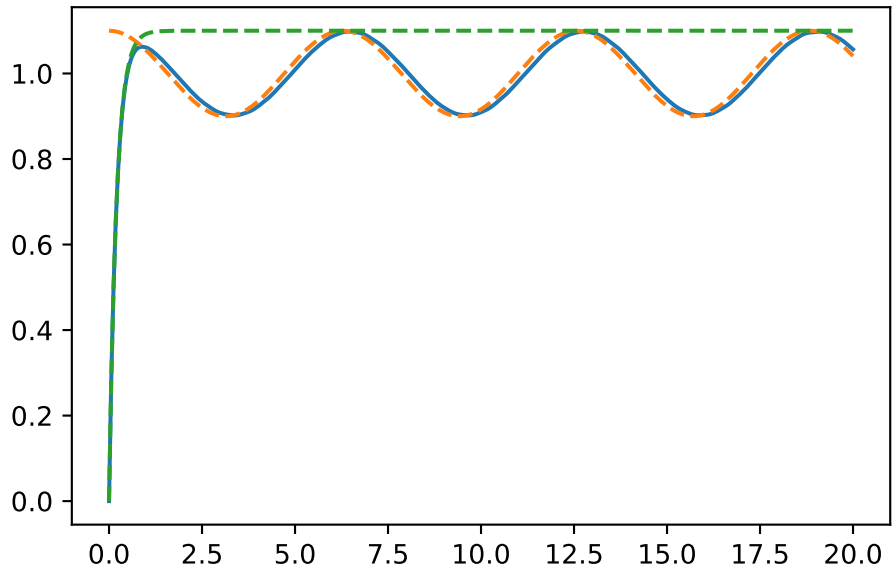
$$\lim_{s\to\infty} U_0(s) = \lim_{t\to 0} T_{\text{outer}} \tag{8.7}$$

therefore the outer limit of our inner solution matches the inner limit of our outer solution. In terms of outer coordinates, our inner solution is approximated as follows:

$$T_{\text{inner}} \sim (1 + A) + (T^* - (1 + A))e^{-t/\epsilon}.$$

Let's finally plot this with our previous curves:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

Ts = 0
ep = 0.2
A = 0.1
f = lambda t, T: 1/ep*(1 + A*np.cos(t) - T)

tmax = 20
sol = solve_ivp(f, [0, tmax], [Ts], dense_output=True)

t = np.linspace(0, tmax, 1000)
y = sol.sol(t)
plt.plot(t, y[0])
plt.plot(t, 1 + A*np.cos(t), '--')
plt.plot(t, (1 + A) + (Ts - (1 + A))*np.exp(-t/ep), '--')
```



It works beautifully!

8.1.3 Summary

Let us summarise the procedure of matched asymptotics.

1. Expand the solution of the differential equation naively in the typical asymptotic expansion (e.g. in powers of ϵ).
2. Notice that the approximation does not satisfy certain boundary conditions.
3. Re-scale the coordinates in the 'inner' regions.
4. Develop an inner solution that satisfies the boundary condition. Ensure it matches the outer solution.

You will get more practice of this procedure in the problem sets.

9 Newton's method

Before we move on back to the subject of applications, it is worth providing an introduction to methods for solving nonlinear equations. Many problems you will encounter in applications, including for ordinary or partial differential equations, can be re-formulated as the solution of a nonlinear system of equations.

Newton's method is the most well-known scheme for solving nonlinear equations. Suppose we wish to solve the scalar equation,

$$f(x) = 0,$$

given some initial guess, $x = x_0$, of the root. Suppose the root lies at $x = x^*$. Then by Taylor's theorem,

$$f(x^*) = f(x_0) + f'(x_0)(x^* - x_0) + O(f''(x_0)(x - x_0)^2).$$

If we assume the quadratic terms are negligible then solve for x^* this gives

$$x^* \approx x_0 - \frac{f(x_0)}{f'(x_0)}.$$

There is a geometrical interpretation of the above. Essentially, in order to estimate the root of $f(x) = 0$, we have used the tangent line at the point $x = x_0$, and used the intersection of this tangent line with the axis as the approximation. This procedure can then be iterated.

Thus, provided that the desired root, x^*, is such that $f'(x^*) \neq 0$, and x_0 is sufficiently close to x^*, then the following iterates converge to the root:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

9.1 Demo of Newton's method for scalar equations

Here is a simple demonstration of Newton's method in order to solve for one of the roots of the following:

$$f(x) = x^3 + x - 1.$$

We will start with the initial guess of $x_0 = 0$. We can do this by hand in lectures using a pocket calculator. The solution is $x^* \approx 0.6823278..$

i	xi	f(xi)	f'(xi)	-f(xi)/f'(xi)	error
0					
1					
2					

```
import numpy as np

x0 = 0
N = 10

def Newton(f, df, x, maxiter=10):
    i = 0
    while (i < maxiter):
        err = f(x)
        x = x - err / df(x)
        print("f(x) = ", np.abs(err), ", x = ", x)
        i = i + 1
    return x, err

f = lambda x: x**3 + x - 1
df = lambda x: 3*x**2 + 1

x, err = Newton(f, df, x0, N)
print("Final approximation = ", x)
```

```
f(x) =  1 , x =  1.0
f(x) =  1.0 , x =  0.75
f(x) =  0.171875 , x =  0.686046511627907
f(x) =  0.008941036638283384 , x =  0.6823395825973142
f(x) =  2.823062168566537e-05 , x =  0.6823278039465127
f(x) =  2.839946056099052e-10 , x =  0.6823278038280194
f(x) =  2.220446049250313e-16 , x =  0.6823278038280193
f(x) =  1.1102230246251565e-16 , x =  0.6823278038280193
f(x) =  1.1102230246251565e-16 , x =  0.6823278038280193
f(x) =  1.1102230246251565e-16 , x =  0.6823278038280193
Final approximation =  0.6823278038280193
```

It is a good idea to also learn how to do this using built-in packages. The 'fsolve' function provides a Newton-like nonlinear solver. In fact, it can estimate the Jacobian (derivative), so only the function values need to be provided.

```python
import numpy as np
from scipy.optimize import fsolve

f = lambda x: x**3 + x - 1

x0 = 1
x, info, ier, msg = fsolve(f, x0, full_output=True)

print(msg)
print(x)
```

```
The solution converged.
[0.6823278]
```

9.2 Newton's method for systems of nonlinear equations

Newton's method generalises naturally to the case of a system of equations. Suppose we wish to solve for the n unknowns $\mathbf{x} = (x_1, \dots, x_n)$ via

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ \dots \\ F_n(\mathbf{x}) \end{pmatrix} = 0.$$

We have, via Taylor's formula,

$$\mathbf{F}(\mathbf{x}_{i+1}) \sim \mathbf{F}(\mathbf{x_i}) + J(\mathbf{x}_i)(\mathbf{x}_{i+1} - \mathbf{x}_i) + \mathcal{O}(||\mathbf{x}_{i+1} - \mathbf{x}_i||^2),$$

where J is the Jacobian matrix

$$J(\mathbf{x}) = \nabla \mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{F_1}{x_1} & \dots & \frac{F_1}{x_n} \\ \vdots & \ddots & \vdots \\ \frac{F_n}{x_1} & \dots & \frac{F_n}{x_n} \end{pmatrix}.$$

Therefore, Newton's method forms the iterates of

$$\mathbf{x}_{i+1} = \mathbf{x}_i - J^{-1}(\mathbf{x}_i)\mathbf{F}(\mathbf{x}_i),$$

which takes a very similar form to the scalar case.

However, solution of the inverse of J is typically inefficient, and it is better to instead solve for $\delta_{i+1} = \mathbf{x}_{i+1} - \mathbf{x}_i$ via

$$J(\mathbf{x}_i)\delta_{i+1} = -\mathbf{F}(\mathbf{x}_i),$$

and then calculate $\mathbf{x}_{i+1} = \mathbf{x}_i + \delta_{i+1}$. There are many ways of solving the above matrix problem efficiently using built-in routines that perform, e.g. Gaussian elimination.

9.3 Secant method

In many situations, evaluation of the Jacobian (or derivative) is the most time-consuming or difficult part of a nonlinear solver. Built-in solvers like 'fsolve', in fact, have the ability to approximate the derivative numerically.

The Secant Method is similar to Newton's Method but replaces the derivative by a finite difference. Geometrically, the tangent line is replaced with a line through the two last known guesses. The algorithm goes as follows.

Secant method

1. Develop two initial guesses to the solution, x0 and x1
2. Compute

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}.$$

Part III

Energy balance models

10 The basic EBM (steady-state)

Recall that we previously introduced the basic energy balance model in Chapter 3. There, we derived the basic heat equation model (Equation 3.7) for the Earth's temperature given by the following ordinary differential equation (ODE) for $T = T(t)$,

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = Q(1-a) - \sigma\gamma T^4, \tag{10.1}$$

where we have defined $C = \rho c_p d$ as the heat capacity of the atmosphere. Above, the solar flux, Q is often taken to be $Q = 1370/4 = 342$ W/m2, $\sigma = 5.67 \times 10^{-8}$ W/(m2K4), and $\gamma \leq 1$ is the greenhouse gas factor.

In this chapter, we discuss some of the numerical and analytical tools at our disposal for studying the above equation.

10.1 Steady-state analysis

Previously, we have assumed that the planetary albedo, a, is constant and independent of temperature. In actuality, water can turn to snow and ice and vice versa; since snow and ice have much higher albedo than open water, then we should consider $a = a(T)$.

Let us assume that there are two relevant ranges to consider: T < 150K (cold) and T > 280K (hot). Let us assume that the albedo is, in these two regions:

$$a(T) \approx \begin{cases} 0.7 & \text{if } T < 150\text{K}, \\ 0.3 & \text{if } T > 280\text{K}. \end{cases}$$

The above guarantees that more energy is reflected if temperatures are low. To model this process, we can use a ramp function to specify the albedo over all temperatures:

$$a(T) = A - B\tanh\left(k(T-265)\right). \tag{10.2}$$

where $A = 0.5$, $B = 0.2$, $k = 0.1$, and $T_0 = 265$K. Recall that the tanh function is given by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

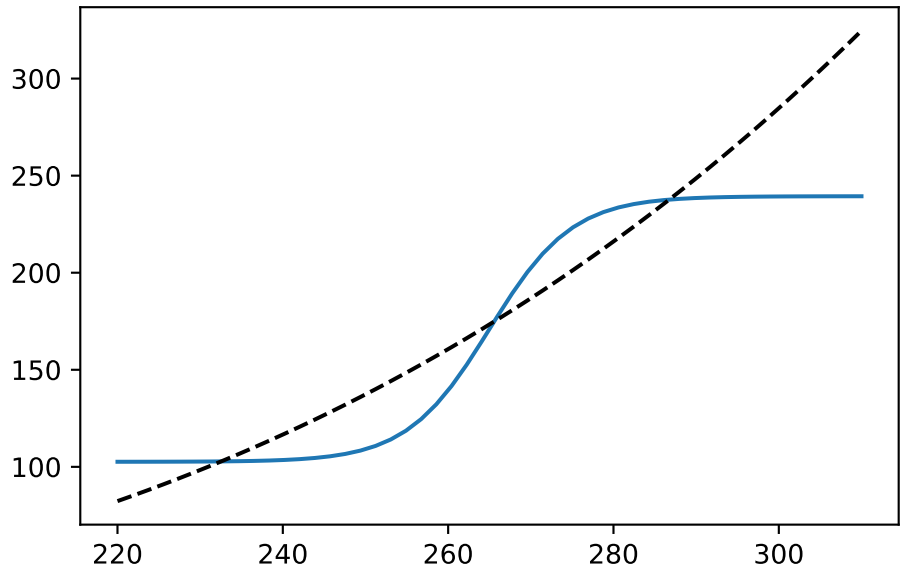
Let us further assume that the system is in steady state, so that the temperature is determined by solving the equation

$$f(T) = Q[1 - a(T)] - \sigma\gamma T^4 = 0. \tag{10.3}$$

In the following code, we plot the two terms that make up f, and their intersections indicate roots of $f = 0$. We then use the Python 'fsolve' function to approximate the roots given initial guesses.

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.optimize as sciopt

Q = 342
sigma = 5.67e-8
gamma = 0.62

TT = np.linspace(220,310,50)

def fun(T):
    a = 0.5 - 0.2*np.tanh((T - 265)/10)
    x = (1-a)*Q
    return x
LHS = fun(TT)

plt.plot(TT, LHS)
plt.plot(TT, gamma*sigma*TT**4, 'k--')

def eq(T):
    x = fun(T) - gamma*sigma*T**4
    return x
T1 = sciopt.fsolve(eq, 230)
T2 = sciopt.fsolve(eq, 265)
T3 = sciopt.fsolve(eq, 290)
print("T1 = {:.2f}".format(T1[0]))
print("T2 = {:.2f}".format(T2[0]))
print("T3 = {:.2f}".format(T3[0]))
```

```
T1 = 232.55
T2 = 265.56
T3 = 286.74
```



Therefore multiple equilibria are observed.

10.2 Dynamics and phase line solutions

The full time-dependent model is given by

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = f(T),$$

so we may use the positivity or negativity of f in order to sketch the time-dependent behaviour of the system.

To see this, we can perform an asymptotic analysis near the fixed points. Let the initial condition be considered near the fixed point:

$$T(t=0) = T^* + \delta,$$

where $\delta \ll 1$. Then we expand the solution into an asymptotic expansion,

$$T(t) = T^* + \delta T_1(t) + \delta T^2(t) + ...$$

Substitution into the above ODE gives, at $O(\delta)$,

$$C\frac{\mathrm{d}T_1}{\mathrm{d}t} = f'(T_0)T_1,$$

and hence, with $T_1(0) = 1$, we have

$$T_1(t) = e^{f'(T_0)t/C}.$$

Therefore, depending on the positivity or negativity of the gradient function, the perturbation will either decay or grow as $t \to \infty$.

It can then be verified that the centre equilbria is unstable while the other two are stable. The higher temperature corresponds to the one that the Earth is currently in, but according to this model, there seems to be the possibility of a colder climate (50 degrees colder) where the Earth is entirely covered with snow and ice.

Interestingly, there is some evidence that the Earth's climate may have been in this so-called *Snowball Earth* state up to four times between 750 million and 580 million years ago (Neoproterozoic age). Observations of geological deposits suggest that the Earth has undergone periods of complete global glaciation where there have been very minimal biological activity. During this period, there is a massive build-up of CO_2 in the atmopshere, leading to huge greenhouse effect. As γ decreases in our model, the equilibrium can then shift, suddenly transitioning the Planet into the warm state.

10.3 Numerical continuation of the steady states

We have done a preliminary steady-state analysis of

$$Q(1 - a(T)) - \sigma\gamma T^4,$$

but we would like to better understand how these steady-states may change depending on the parameters. For example, we would like to understand how the solutions vary of the solar constant related to Q varies; or we would like to understand how solutions vary as γ varies.

More specifically, we would like to design some numerical routines that would allow us to (smartly!) solve for the roots of the above equation as the parameters are varied. Although the above problem (roots of a single equation) is simple enough to do this in a manual way, the methods of *numerical continuation* we learn in this chapter is applicable to much more general set of problems.

Suppose that we are interested in studying how the steady-states (up to three) change as Q changes. Then we are interested in producing a diagram of Q vs. T. The basic idea is to start with an initial solution at some value of Q, increment Q, then solve for the next value using the previous value as a guess. This involves the following pseudocode:

Numerical continuation

```
1. Input guess T0, f, df, parameter Q1

    a. Call Newton's method via Newton(f, df, T0, Q1)
    b. Obtain a preliminary solution (T1, Q1)
```

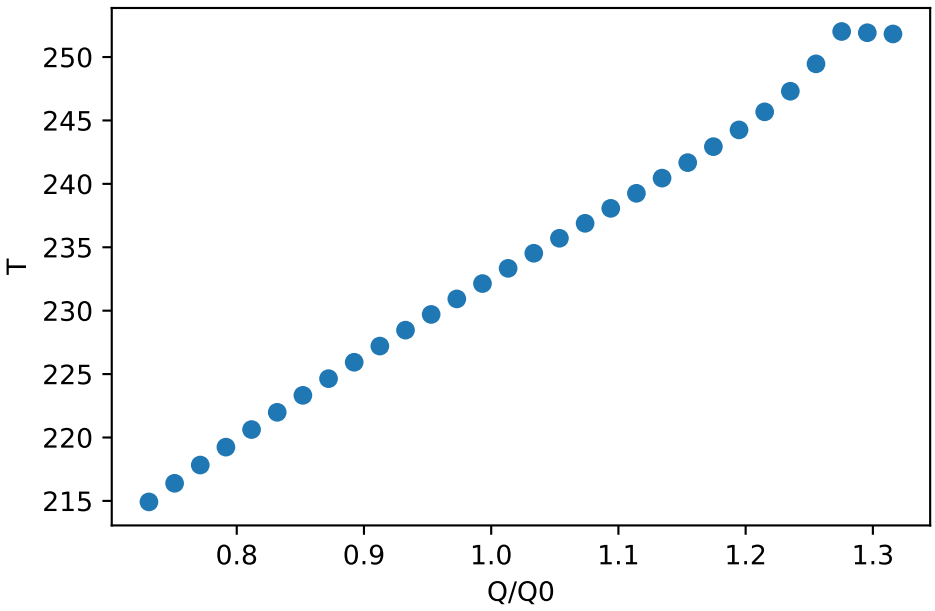
```
2. Increment Q1 = Q1 + dQ

    a. Call Newton's method via Newton(f, df, T1, Q1)
    b. Obtain a new solution (T1, Q1)

3. Repeat 2 until we reach a desired Q value
```

The following code provides continuation for one of the roots.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import fsolve
from scipy.optimize import root

sigma = 5.67e-8
gamma = 0.62
a = lambda T: 0.5 - 0.2*np.tanh((T - 265)/10)

Q0 = 342
Qmat = np.linspace(250, 450, 30)
Tmat = 0*Qmat

# Initial guess
x = 220
for i, Q in enumerate(Qmat):
    f = lambda T: Q*(1-a(T)) - sigma*gamma*T**4
    sol = root(f, x)
    # If solution not found, output error
    if sol.success != 1:
        print("Q/Q0 = ", Q/Q0, ": ", sol.message)
    Tmat[i] = sol.x

plt.plot(Qmat/Q0, Tmat, 'o');

plt.xlabel('Q/Q0');
plt.ylabel('T');
```

```
Q/Q0 =  1.2754587618471467 :  The iteration is not making good progress, as measured by the
  improvement from the last ten iterations.
Q/Q0 =  1.2956241177656787 :  The iteration is not making good progress, as measured by the
  improvement from the last ten iterations.
Q/Q0 =  1.3157894736842106 :  The iteration is not making good progress, as measured by the
```

```
  improvement from the last five Jacobian evaluations.

/var/folders/26/pd267j_136sfv9lw7lc9rcm40000gp/T/ipykernel_4150/2481211673.py:22: Deprecatio
  Tmat[i] = sol.x
```



Above, we have scaled Q with the reference value of $Q_0 = 342$.

In your lecture, problem class, or exercises, you will design continuation on the other branches of solutions, and then study the result and its implications. During the lecture, we studied `MA30287/notebooks/lecture13_EBM.ipynb` which can be found at this link.

10.4 Bifurcation diagram

In reference to the above note, the following diagram was drawn and discussed in Lecture 15.

We noted the following:

- The system has three steady states, given by the green, red, and blue curves.
- The middle state is unstable (shown dashed).
- The system exhibits hysteresis. Note that if we decrease Q/Q_0 past the tipping point, marked Q_{T1} in the image, then we would evolve to the lower stable steady state (which is the ice state). However, while we are in the ice stated, if we were to attempt to increase the solar radiation to return to the green branch, we would need to arrive at Q_{T2} to do so; this irreversibility is known as hysteresis.

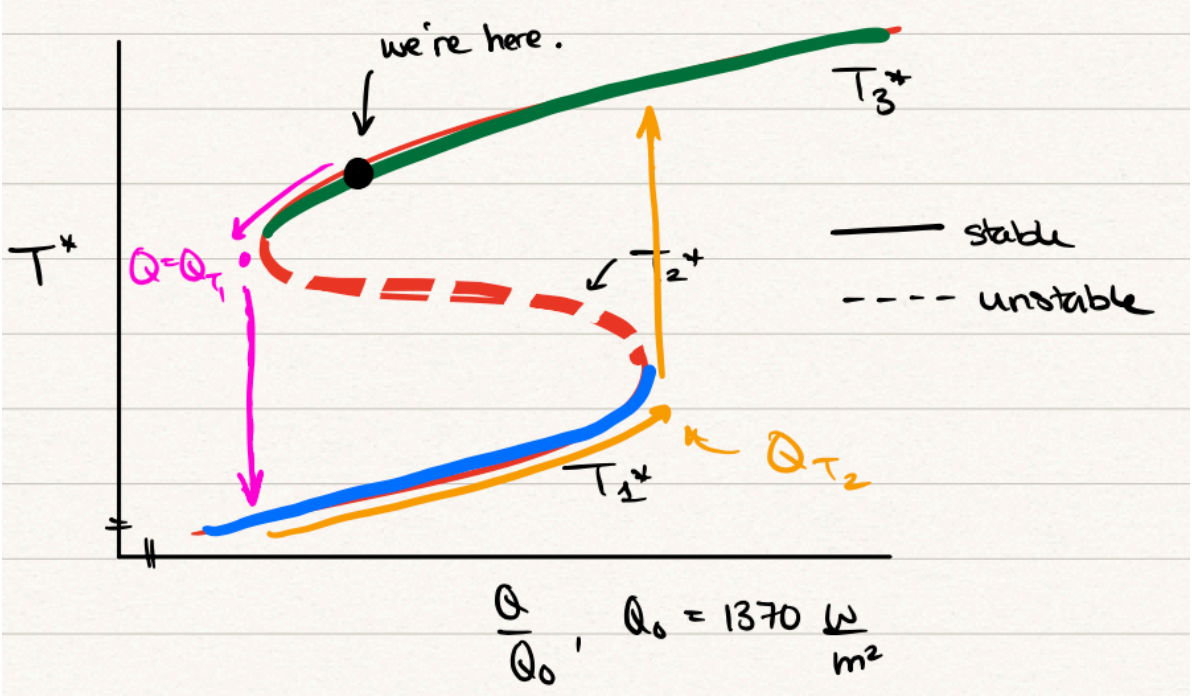


Figure 10.1: Bifurcation diagram of Q/Q_0 vs T^*

10.5 Re-scaling and Budyko's model

To model the outgoing radiation, we use the quartic Stefan-Boltzmann law. However, over the range of temperatures we are interested-in, it seems that a simpler approximation is sufficient. In your homework, you will investigate the re-scaling and shifting of temperature, such that

$$T = T_0 + [T]\tilde{T},$$

where $T_0 = 265$K. Then under the assumption that temperatures are not-so-far from T_0, we expand

$$(T_0 + [T]\tilde{T})^4 \sim T_0^4 + 4T_0^3[T]\tilde{T} = C_1 + C_2\tilde{T}.$$

This simplifies the model considerably.

11 EBM with latitude I

We now introduce a model that allows for two additional effects: an albedo that models ice dynamics and latitude variation; and a transport term, which allows for heat movement (primarily convection).

To begin, let us use a spherical coordinate system where φ is the latitude angle, with $\varphi = 0$ at the equator and $\varphi = \pi/2$ at the North Pole. Note that this differs from the traditional spherical coordinate system where ϕ is the polar angle. It is moreover convenient to use

$$y = \sin\varphi, \quad y \in [-1, 1],\ \varphi \in [-\pi/2, \pi/2].$$

Our conservation equation takes the general form for the temperature, $T(y, t)$:

$$C\frac{\partial T}{\partial t} = E_{\text{in}} - E_{\text{out}} + E_{\text{transport}}, \qquad t > 0 \tag{11.1}$$

where we shall use the following:

$$E_{\text{in}}(y, t) = Qs(y)(1 - \alpha(y)), \tag{11.2}$$
$$E_{\text{out}}(y, t) = A + BT, \tag{11.3}$$
$$E_{\text{transport}}(y, t) = k(\bar{T} - T), \tag{11.4}$$

{#eq-EBMlatcomp} Each of these terms we explain below in turn.

The above equation will be supplemented with initial conditions:

$$T(y, 0) = T_i(y).$$

11.1 Incoming energy

We make the following remarks about E_{in}.

- Notice that now, compared to the previous model, we have weighted the incoming solar constant Q with a term $s(y)$ and also our albedo no longer directly depends on temperature but instead depends on the latitude.
- Points on Earth that are closer to the equator receive more direct sunlight and experience more hours of daylight on average. We thus define a function $s(x)$ to account for this and consider the incoming radiation as weighted by $Qs(y)$.

- Note that the total solar input is calculated by a surface integral:

$$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=-\pi/2}^{\pi/2} Qs(y=\sin\varphi)(a^2\cos\varphi)\,\mathrm{d}\varphi\mathrm{d}\theta = 4\pi a^2 Q\int_0^1 s(y)\,\mathrm{d}y = 4\pi a^2 Q,$$

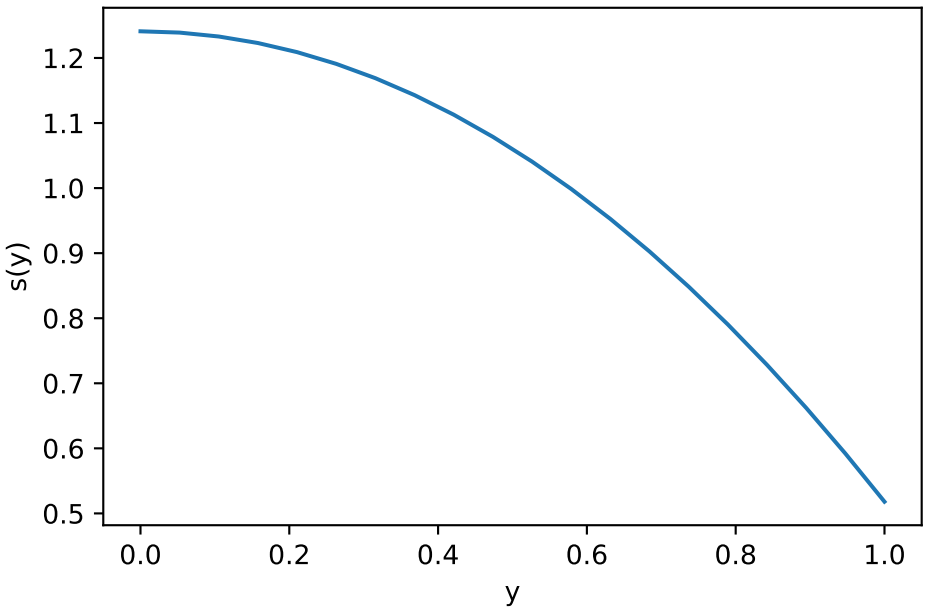
  if the function $s(y)$ has been normalised appropriately so that its integral is equal to one, and furthermore the radiation is assumed symmetric about the equator.

- The above is equal to the solar flux intercepted by an area of the circular disk of the earth seen by the sun, which is $\pi a^2 S$ for $S = 1370\mathrm{W/m}^2$. Therefore $Q = S/4 \approx 342\mathrm{W/m}^2$.

Finally the function $s(y)$ is fitted to data. We can assume it to be given by

$$s(y) = 1 - S_2 P_2(y), \qquad S_2 = 0.482, \quad P_2(y) = (3y^2-1)/2. \tag{11.5}$$

```
import numpy as np
import matplotlib.pyplot as plt

y = np.linspace(0, 1, 20)
plt.plot(y, 1 - 0.482*(3*y**2 - 1)/2)
plt.xlabel('y'); plt.ylabel('s(y)');
```



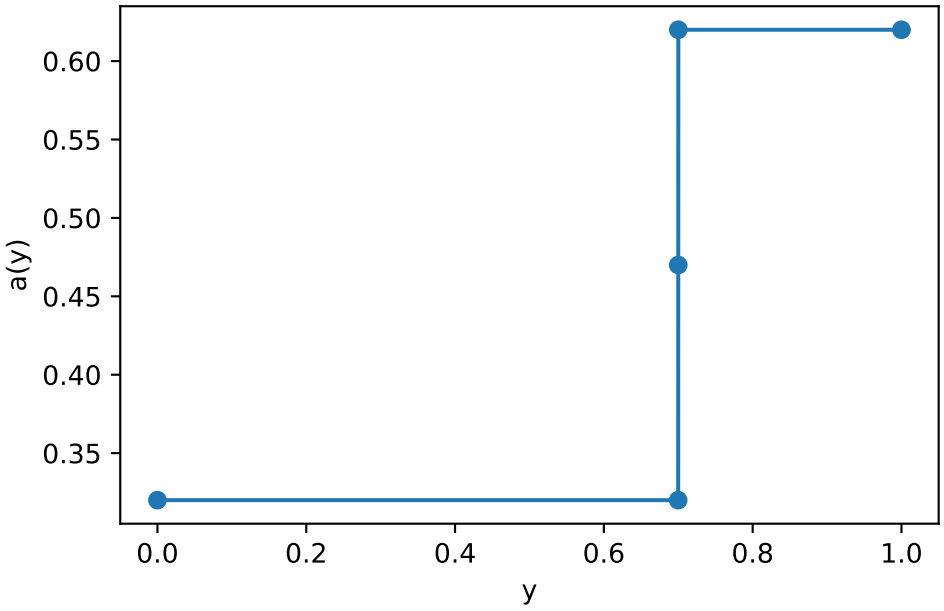
We assume that an ice sheet forms if the temperature is sufficiently low, and $T < T_c = 10^\circ\mathrm{C}$.

The ice forms at an ice line, where $y = y_s$. We then have the following form form the albedo:

$$a(y) = \begin{cases} a_i = 0.62 & y > y_s, \\ a_w = 0.32 & y < y_s, \\ \frac{1}{2}(a_i + a_w) & y = y_s. \end{cases}$$

Therefore we are using a discontinuous model for the albedo. Note that the ice boundary y_s is an unknown to be determined by the model.

```
import numpy as np
import matplotlib.pyplot as plt

ys = 0.7
y = np.array([0, ys-0.0001, ys, ys+0.0001, 1])
ai = 0.62; aw = 0.32;
a = np.piecewise(y, [y < ys, y == ys, y > ys], [aw, 0.5*(aw+ai), ai])
plt.plot(y, a, '-o')
plt.xlabel('y'); plt.ylabel('a(y)');
```



The above forms what is referred to as a free-boundary problem, as the boundary $y = y_s$ is solved as part of the model. Because y_s is essentially determined by where the temperature reaches the critical value, i.e. $T(y_s, t) = T_c$, then the above albedo not only depends on latitude, but also on the current temperature distribution as well.

11.2 Outgoing energy

The outgoing energy is approximated via the linear Budyko approximation discussed in Section 10.5.

$$E_{\text{out}} = A + BT,$$

where $A = 202\text{W/m}^2$ and $B = 1.9\text{W/(m}^2{}^\circ\text{C)}$. These figures are taken from a 1993 paper by Graves, Lee, and North.

11.3 Transport energy

The next term we should consider is the transport. In the case of this simple model, transport is taken to be modelled via a "Newton law of cooling" type expression. We assume that

$$\text{rate of change of heat energy} = k(\bar{T} - T),$$

where it can be verified (exercise) that the global mean temperature is the same as the hemispherically averaged temperature, i.e.

$$\bar{T} = \frac{1}{\text{surface area}} \iint T \,\text{S} = \int_0^1 T(y)\,\text{d}y.$$

Above, we shall use a value of $k = 1.6B$, again estimated based on data and studies.

This completes our review of all the components (**?@eq-EBMlatcomp**) that make up the EBM model (Equation 11.1).

11.4 Steady-state temperature

Let $T^*(y)$ be the steady-state temperature and $\bar{T}^*$ be the corresponding global mean temperature. Then from setting the LHS of (Equation 11.1) to zero, we have the fact that T^* is given by solving the implicit equation:

$$T^*(y) = \Phi(T^*) = \frac{k\bar{T}^* + Qs(y)(1-\alpha(y)) - A}{B + C}.$$

It is important to remember that the albedo, $\alpha(y)$, implicitly depends on T^* because at each location y, we must decide if the temperature is less or greater than the threshold T_c, which is the temperature of the ice line.

The above equation is somewhat tricky because of its implicit nature; it is made even trickier by virtue that multiple equilibria exist at the same parameter values. We will begin studying this problem, but before doing so, let us make a few remarks:

- It is possible to solve for the steady-state global mean, $\bar{T}^*$ as a function of the ice line location, y_s.
- It is possible to solve for the ice line location, y_s, although not entirely explicitly. Up to four solutions are possible.
- Once the ice line is known it is possible to solve for the steady-state.

However:

- A numerical solution can be developed without the need to go through the above procedures.

Part IV

Box models of the ocean

12 Basic models of the ocean

12.1 Terminology and context

The ocean plays a significant role in regulating the Earth's climate, as it acts as a massive heat sink and helps to distribute heat and moisture around the planet. In addition, carbon dioxide is water soluble, and through precipitation and wave motion, is transferred into the oceans. Thus the ocean acts as a sink, absorbing large amounts of this greenhouse gas from the atmosphere.

The thermohaline circulation (THC), also known as the global ocean conveyor belt, is a complex ocean circulation pattern that is driven by differences in water temperature and salinity. It is an important component of the Earth's climate system, as it helps to distribute heat and other properties throughout the planet's oceans.

The thermohaline circulation is driven by the sinking of cold, dense water in the polar regions, which then spreads out and flows towards the equator. As the water warms and becomes less dense, it rises to the surface and returns to the poles, creating a continuous loop of ocean currents. The role of salinity in driving the thermohaline circulation is due to the fact that the dissolved salts in seawater increase its density.

This circulation pattern has a significant impact on global climate, as it helps to regulate the exchange of heat and other properties between the oceans and the atmosphere. Changes in the thermohaline circulation, such as those caused by global warming, can have far-reaching effects on the planet's climate and weather patterns.

12.2 Temperature

> Extra content
>
> The material in this subsection is not examinable in 2022-23 and is given here for further context.

In regards to the temperature, the ocean can be divided into three layers.

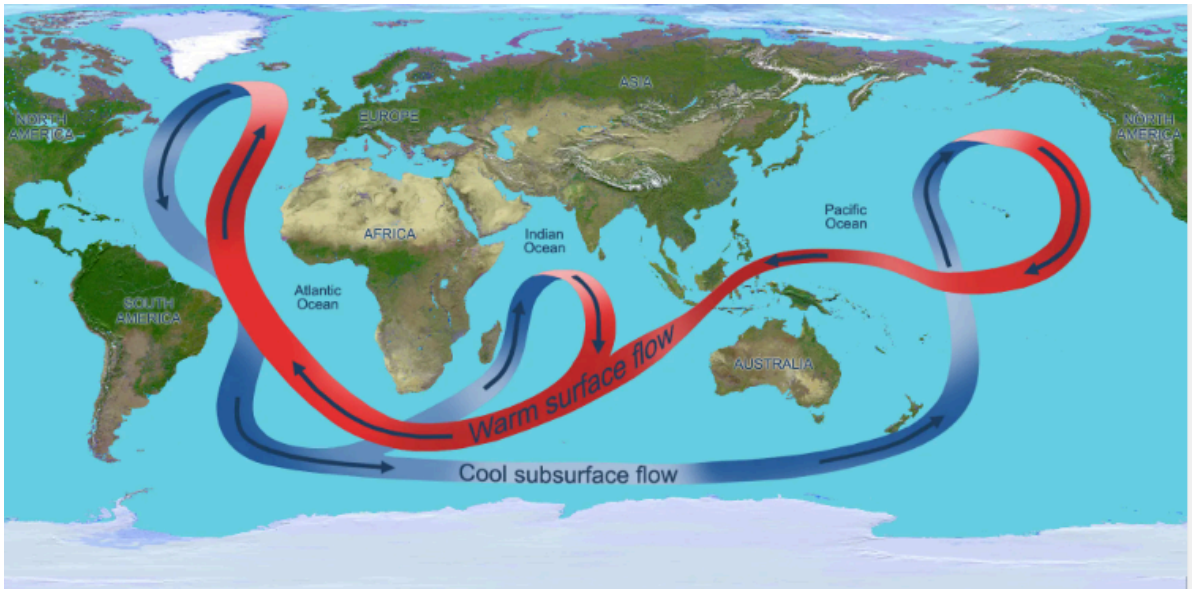


Figure 12.1: The thermohaline circulation. In the Atlantic, the circulation carries warm water (red) north near the surface and cold deep water (blue) south. Image from NASA/JPL.

1. The top layer is thin (on the order of metres) and is heated from the Sun. Mixing is a dominant effect due to wind and waves, and so the temperature in this region is mostly constant.
2. The *thermocline region* is the intermediate layer. Here, the temperature decreaes approximately linearly.
3. The deep *abyssal zone* comprises 98% of the total volume of the oceans. The temperature in this region is mostly constant, and a few degrees above freezing.

Within the intermediate region, the temperature can be modelled by an **advection diffusion** equation,

$$\frac{T}{t} + w\frac{T}{z} = \kappa\frac{^2T}{z^2}, \tag{12.1}$$

where w is the upswelling velocity and κ is the diffusion coefficient of the fluid.

Let us assume that the temperature in this region is near steady state and that the upwards velocity is constant. Then we integrate the ODE to find

$$T(z) = T_0 + T_1 \mathrm{e}^{-z/z^*},$$

where T_0 and T_1 are constants. From (Kaper and Engler 2013), the typical orders for the parameters are $\kappa \sim 10^{-2}\,\mathrm{m}^2\,\mathrm{s}^{-1}$ and $z^* \sim 10^2\,\mathrm{m}$, so $w \sim 10^{-4}\,\mathrm{m\,s}^{-1}$, which is quite slow.

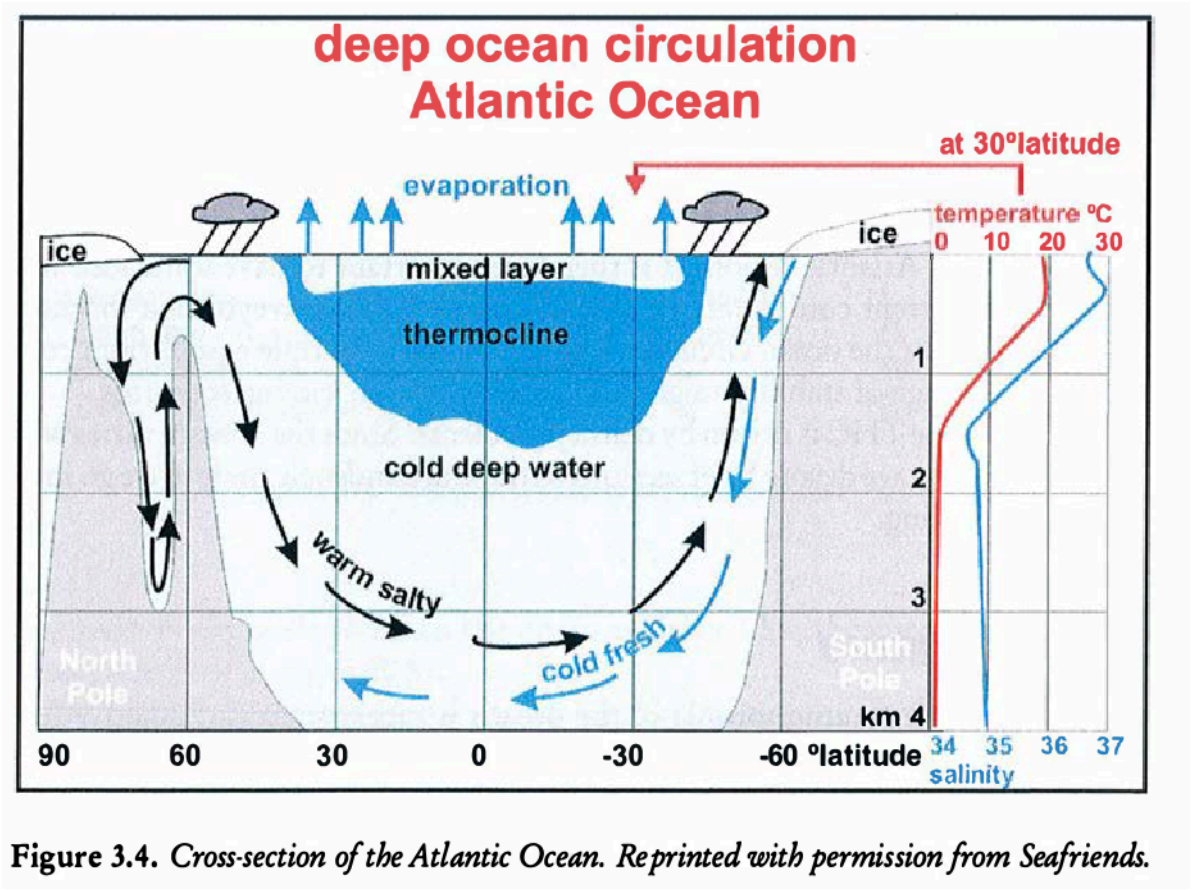


Figure 3.4. *Cross-section of the Atlantic Ocean. Reprinted with permission from Seafriends.*

Figure 12.2: Cross section of the Atlantic Ocean, showing the temperature and salinity profiles on the right. Image from (Kaper and Engler 2013).

12.3 Salinity and density

Salinity is a key component in the oceans since the salts have a large effect on the water density (which consequently drives motion). Salinity is measured in psu or *practical salinity units*, which is a non-dimensional ratio of conductivities. In the mixed layer, the sainity ranges from 31-39 psu, and is about 35 in the abyssal zone. You can inspect the profile in Figure 12.2.

Below in the study of the ocean, we will need a so-called *equation of state*, which connects the density, ρ, with the salinity, say S, and temperature, say T. Here is a typical graph of the relationship.

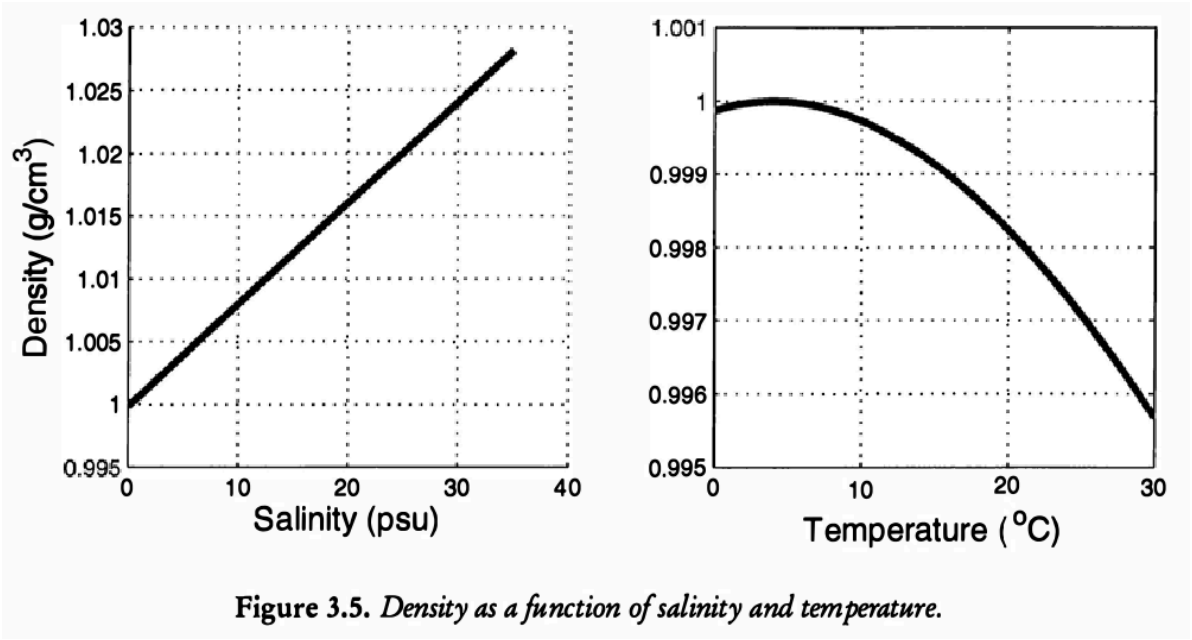


Figure 3.5. *Density as a function of salinity and temperature.*

Figure 12.3: Image from (Kaper and Engler 2013)

12.4 Two-box model of the ocean

Modelling the THC is a challenging task! In principal, this might involve the solution of multiple coupled PDEs for the flows and temperatures, which would then need to be solved on a very complicated domain. In addition, such models require a number of empirical equations of state (connecting density to temperature and salinity).

Toy models can be developed much more easily at the 'systems level' via box models.

12.5 A one-dimensional model (constant temperature)

Construction and assumptions of the box model.

1. We consider two boxes, labeled '1' and '2', respectively via subscripts. Box 1 corresponds to high latitudes (near poles) and Box 2 to low latitudes (near equator). Each box has a corresponding temperature, T_i, and salinity, S_i. In each box, the temperature and salinities are well mixed, so they take different values in either box, in general.

   Differences between the two boxes will drive a flow through the capillary pipe that connects the boxes at the bottom. A compensating flow at the surface ensures that the volume of water in each box does not change. Together, these two exchanges (top and bottom) represent the overturning circulation of the THC.

2. We assume (see above) that the strength of the exchange flow between the boxes is linearly proportional to their differences of temperature and salinity. In particular, let ρ_1 and ρ_2 be the two densities in the corresponding boxes. The flow q in the capillary pipe is driven by the difference

   $$q = k\frac{\rho_1 - \rho_2}{\rho_0}, \tag{12.2}$$

   where k is the hydraulic constant and takes the typical value of $k = 1.5 \cdot 10^{-6}\,\mathrm{s}^{-1}$. The reference density ρ_0 is defined in conjunction with the equation of state.

   In the normal state of the planet, $q > 0$ when the flow through the capillary goes in the direction of the equator as a result of higher densities at high latitudes.

3. We need an equation of state that connects the densities, ρ, with the temperature and salinities. This concerns the figure Figure 12.3. We assume that

   $$\rho = \rho_0(1 - \alpha(T - T_0) + \beta(S - S_0)), \tag{12.3}$$

   where T_0 and S_0 are average values of temperature and salinity; ρ_0 is the density if $T = T_0$ and $S = S_0$; α and β are constants with typical values $\alpha = 1.5 \times 10^{-4}\mathrm{deg}^{-1}$ and $\beta = 8 \times 10^{-4}\mathrm{psu}^{-1}$. Combining (Equation 12.2) with (Equation 12.3) gives

   $$q = k[\alpha(T_2 - T_1) - \beta(S_2 - S_1)]. \tag{12.4}$$

4. External wind forces and Coriolis effects are ignored.

5. We assume that in each box, there is an exchange of heat and salinity to the surrounding environment. For instance, salinity will exchange due to evaporation, precipitation, and runoff. These will have constant of proportionalities of c and d in the equations below.

6. Precipitation, evaporation, and runoff from the continents and atmosphere will also cause the salinity in either box to change. We therefore assume that this is modeled by a flux H.

   Remember that in general, $T_2 > T_1$ (the temperatures at the equator are higher). Consequently, evaporation dominates Box 2; there is a net loss of salt-free moisture in the atmosphere and this causes a compensating virtual flux of salt, $H > 0$, into Box 2. At the same time, there is higher precipitation and runoff into Box 1 (due to lower temperatures), causing a net gain of salt-free moisture and therefore a loss of virtual salt, say $-H < 0$. We assume these two virtual salt fluxes sum to zero.

   Note that in the normal state of affairs, $S_2 > S_1$, and the lower latitude water is saltier than that of high latitudes.

An image of the box model is shown below.

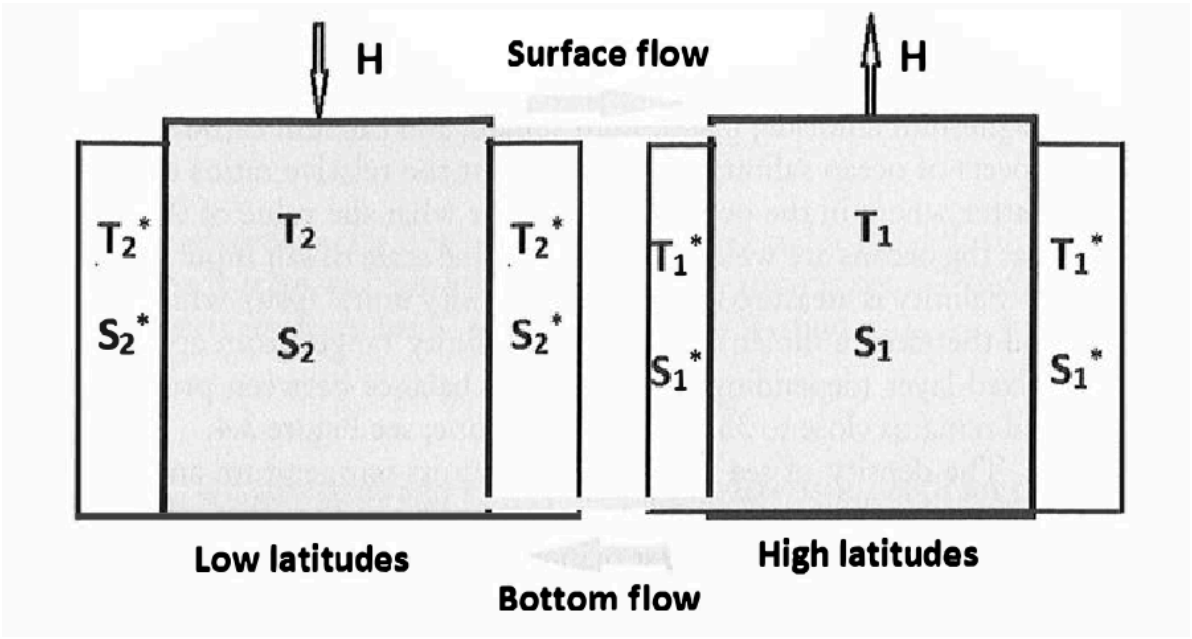


Figure 12.4: Two-box model of the North Atlantic with evaporation and precipitation. Image from (Kaper and Engler 2013)

The equations are given as follows.

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} = c(T_1^* - T_1) + |q|(T_2 - T_1) \tag{12.5}$$
$$\frac{\mathrm{d}T_2}{\mathrm{d}t} = c(T_2^* - T_2) + |q|(T_1 - T_2) \tag{12.6}$$
$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = -H + d(S_1^* - S_1) + |q|(S_2 - S_1) \tag{12.7}$$
$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = H + d(S_2^* - S_2) + |q|(S_1 - S_2) \tag{12.8}$$

{#eq-2box}

Note that in the exchange terms corresponding to the flow q in the capillary pipes, we use an absolute value since it does not matter (flows are exchanged in both lower and upper pipes).

Note also that $q = q(T_1, T_2, S_1, S_2)$ as given in (Equation 12.4).

12.6 Reducing the equations via symmetry

By adding the equations (**?@eq-2box**), it can be verified that in the limit $t \to \infty$, the average temperature in the system, $1/2(T_1 + T_2)$ tends to the average temperature in the surrounding basins $1/2(T_1^* + T_2^*)$. The same conclusion can be drawn for the corresponding salinities.

It is then sensible to write all temperature and salinity in terms of this baseline scenario. So let us write

$$T_1 = \frac{1}{2}m + U_1, \tag{12.9}$$
$$T_2 = \frac{1}{2}m + U_2, \tag{12.10}$$

where $m = T_1^* + T_2^*$. Then the first equation becomes

$$\frac{\mathrm{d}U_1}{\mathrm{d}t} = c\left(T_1^* - U_1 - \frac{1}{2}m\right) + |q|(U_2 - U_1) \tag{12.11}$$
$$= c\left(-T^* - U_1\right) + |q|(U_2 - U_1). \tag{12.12}$$

where $T^* = \frac{1}{2}(T_2^* - T_1^*)$.

The analgous manipulations are done to the quantities for the salinity. In the end, if we

(confusingly) re-write T for U, then we obtain

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} = c(-T^* - T_1) + |q|(T_2 - T_1) \tag{12.13}$$

$$\frac{\mathrm{d}T_2}{\mathrm{d}t} = c(T^* - T_2) + |q|(T_1 - T_2) \tag{12.14}$$

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = -H + d(-S^* - S_1) + |q|(S_2 - S_1) \tag{12.15}$$

$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = H + d(S^* - S_2) + |q|(S_1 - S_2) \tag{12.16}$$

{#eq-2box-again}

Comparing the above to **?@eq-2box**, the main difference is that, in expressing the temperature and salinities with respect to the average values in the basin, we have eliminated two of the constants from the set $(T_1^*, T_1^*, S_1^*, S_2^*)$ now only into two constants (T^*, S^*).

In the situation of zero salt flux. $H = 0$, the above model reduces to Stommel's box model studied in Chapter 13.

12.7 Analysis of a reduced 1D model for the salinity

It is possible to do some analysis on the system (**?@eq-2box-again**), though we may prefer to simply move on to study the more historically well-known reduction called *Stommel's box model* in Chapter 13.

We make the following assumptions:

- We assume that on the timescale of interest in the THC, the temperature of each box equilibrates quickly with the surrounding basin. Therefore the system can be regarded as being in steady state.

- The difference in temperatures between the two boxes is small; together with the top assumption, this implies that $T_1(t) = -T^*$ and $T_2(t) = T^*$.
- Salinity exchanges by negligible amounts with its surrounding basin, i.e. $d = 0$

This leaves us with

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = -H + |q|(S_2 - S_1) \tag{12.17}$$

$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = H + |q|(S_1 - S_2), \tag{12.18}$$

where $q = k(2\alpha T^* - \beta(S_2 - S_1))$.

Now, the formulation for the salinities can be placed into a single equation for $\Delta S = S_2 - S_1$, which satisfies

$$\frac{\mathrm{d}\Delta S}{\mathrm{d}t} = 2H - 2k|\alpha\Delta T - \beta\Delta S|\Delta S. \tag{12.19}$$

This is now a much simpler and elegant formulation for study. Under a suitable non-dimensionalisation and scaling, we can even place the above equation in the final simplified form of

$$\dot{x} = \lambda - |1 - x|x,$$

where $\lambda > 0$ (for temperatures in the usual configuration, with $T_2 - T_1 > 0$, and a suitably defined function $x = x(t)$.

13 Stommel's box model

In addition to the model studied in Section 12.7, there is another kind of reduced model that is examined in this chapter.

In his work (Stommel 1961) did not account for evaporation or precipitation, so there is no virtual salt flux, $H = 0$, but each box does exchange salinity with its surrounding environment, so $d > 0$. The set of four differential equations presented in Chapter 12 (**?@eq-2box-again**) can be reduced two coupled differential equations for the mean temperature and salinity. We notice that if we subtract the pairs of equations, then we can write

$$\begin{aligned} \frac{\mathrm{d}\Delta T}{\mathrm{d}t} &= c(\Delta T^* - \Delta T) - 2|q|\Delta T, \\ \frac{\mathrm{d}\Delta S}{\mathrm{d}t} &= d(\Delta S^* - \Delta S) - 2|q|\Delta S, \end{aligned} \tag{13.1}$$

{#eq-stommelpre}

where $\Delta T^* = 2T^*$, $\Delta S^* = 2S^*$, and we have defined the two new unknowns,

$$\Delta T = T_2 - T_1, \tag{13.2}$$

$$\Delta S = S_2 - S_1, \tag{13.3}$$

and now, the flow q is given by

$$q = k(\alpha \Delta T - \beta \Delta S).$$

13.1 Non-dimensionalisation

We can then nondimensionalise the system by setting

$$x = \frac{\Delta S}{\Delta S^*}, \quad y = \frac{\Delta T}{\Delta T^*}, \quad t' = ct,$$

where t' is nondimensional time. Dropping the primes henceforth, we have the following set of non-dimensional equations to study for the unknowns, $x = x(t)$, and $y = y(t)$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \delta(1-x) - |f(x,y)|x, \tag{13.4}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - y - |f(x,y)|y, \tag{13.5}$$

{#eq-stommelsys} where we have introduced the function,

$$f(x,y;R,\lambda) = \frac{1}{\lambda}(Rx - y), \tag{13.6}$$

where there are now three non-dimensional parameters given by

$$\delta = d/c, \tag{13.7}$$
$$\lambda = c/(2\alpha k \Delta T^*), \tag{13.8}$$
$$R = \beta \Delta S^*/(\alpha \Delta T^*). \tag{13.9}$$

Together **?@eq-stommelsys** and Equation 13.6 form a system of equations for $(x(t), y(t))$, with parameters δ, λ, and R.

Although there are many parameters, it is important to note that the quantity f is crucial since it corresponds to the direction of flow in the bottom pipe of the Ocean model. By convention, the system was set up so that $f > 0$ corresponds to flow through the bottom pipe goes in the direction of the equator (Box 1) as a result of higher densities at the high latitudes. We are subsequently interested in whether it is possible for f to switch sign, which would correspond to a bottom flow going from equator to poles, and the entire system switching direction.

13.2 Equilibrium states

Let the equilibrium states be given by (x^*, y^*) with $f(x^*, y^*) = f^*$. Then setting the right hand sides of (**?@eq-stommelsys**) to zero and solving, we find

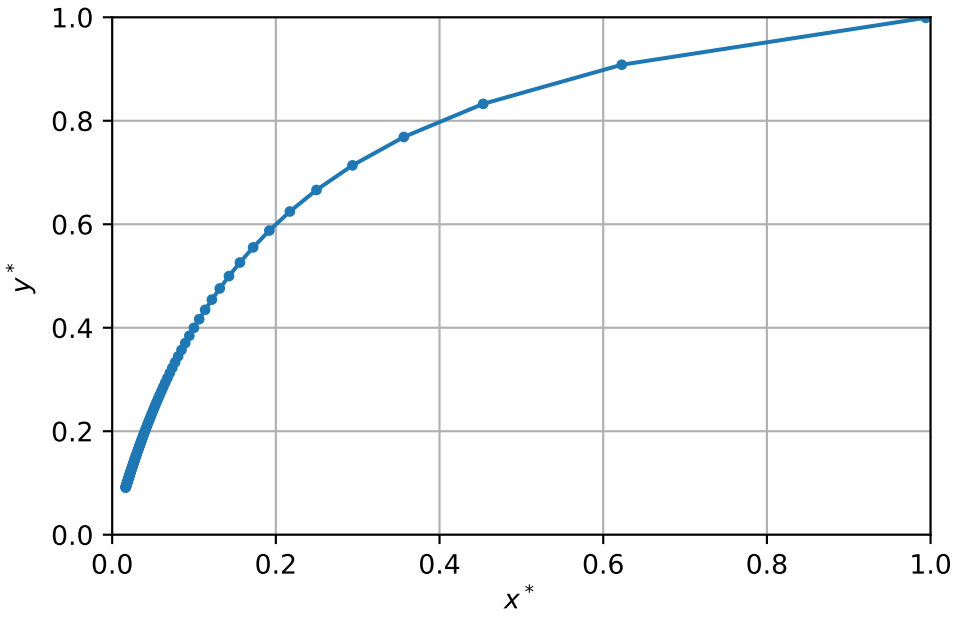
$$x^* = \frac{\delta}{\delta + |f^*|} \quad \text{and} \quad y^* = \frac{1}{1 + |f^*|}. \tag{13.10}$$

We can solve for f^* in both equations and equate the result to one another. This gives

$$\delta \frac{1 - x^*}{x^*} = \frac{1 - y^*}{y^*} = |f^*|.$$

Therefore the equilibrium points lie along the above curve. Let us generate different values of f^* and plot the combination.

```
import numpy as np
import matplotlib.pyplot as plt

delt = 1/6
fmat = np.linspace(1e-3, 10, 101)
xs = delt/(delt + np.abs(fmat))
ys = 1/(1 + np.abs(fmat))
plt.plot(xs, ys, '.-')
plt.xlim((0,1))
plt.ylim((0,1))
plt.xlabel("$x^*$"); plt.ylabel("$y^*$");
plt.grid(1)
```



The above shows the steady-state values at $\delta = 1/6$. Each point refers to a different value of $|f^*|$. Note that as $|f^*| \to \infty$, then $(x^*, y^*) \to (0, 0)$ while as $|f^*| \to 0$, then $(x^*, y^*) \to (1, 1)$.

Now the above does not tell us what equilibrium states will exist at a specific value of δ since it requires information about f^*. We take (Equation 13.6) and combine with (Equation 13.10). We conclude that the equilibrium points must satisfy

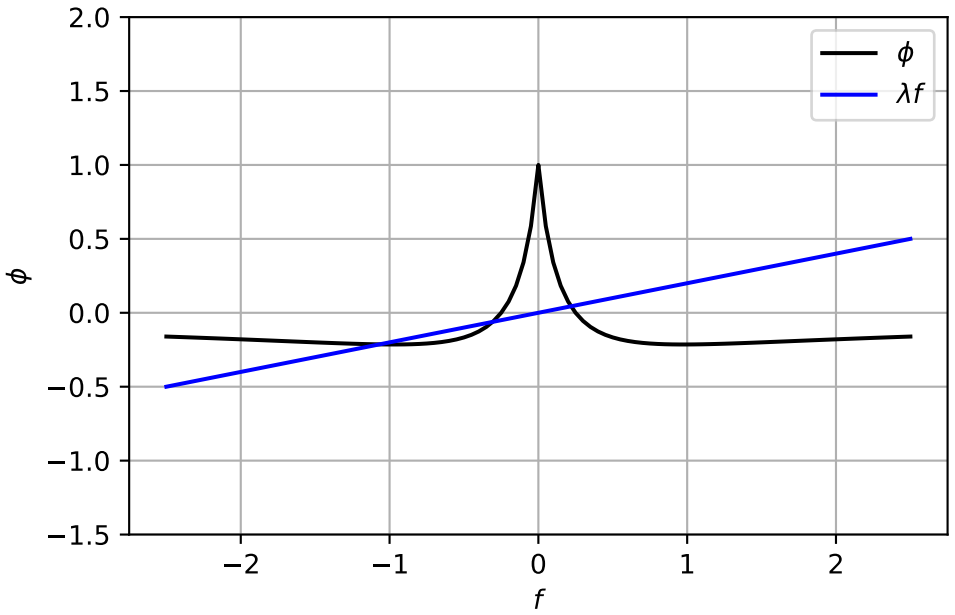
$$\lambda f^* = \phi(f^*),$$

where we have defined the function ϕ according to

$$\phi(f^*) = \frac{\delta R}{\delta + |f^*|} - \frac{1}{1 + |f^*|}. \tag{13.11}$$

Therefore, for each combination of the parameters (λ, δ, R), we must solve $\delta f^* = \phi(f^*)$ to determine f^*. Once the values of f^* are known, then the steady-states (x^*, y^*) are also known. Here is a typical graph showing the potential intersections at the values of $\lambda = 1/5$, $\delta = 1/6$ and $R = 2$.

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.optimize as sciopt

lam = 1/5;
delt = 1/6;
R = 2;

f = np.linspace(-2.5, 2.5, 101)
phi = delt*R/(delt + np.abs(f)) - 1/(1 + np.abs(f))

plt.plot(f, phi, 'k', label='$\phi$')
plt.plot(f, lam*f, 'b', label='$\lambda f$' )
plt.ylim((-1.5,2))
plt.xlabel("$f$")
plt.ylabel("$\phi$")
plt.legend()
plt.grid(1)
```



Although there are many parameters that can alter the shape of the overall graphs, note that if λ is increased from its current value of $\lambda = 1/6$, then two of the left roots will disappear, leaving only a single root.

13.3 Stability

We let $x = x^* + \xi$ and $y = y^* + \eta$ and linearise the system about the fixed points. This gives

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = A \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

where the matrix A is given by

$$A = \begin{pmatrix} -(\delta + |f^*|) \mp \frac{Rx^*}{\lambda} & \pm\frac{x^*}{\lambda} \\ \mp\frac{Ry^*}{\lambda} & -(1 + |f^*|) \pm \frac{y^*}{\lambda} \end{pmatrix}$$

if $f^* \gtrless 0$.

There are two ways or proceeding. We can numerically substitute the fixed points into the above matrix for A and then calculate the eigenvalues (numerically or otherwise). Or we can directly proceed analytically.

If proceeding analytically, we can then calculate the trace and determinant, giving

$$T = -(1 + \delta + 3|f^*|), \tag{13.12}$$

$$D = (\delta + 2|f^*|)(1 + |f^*|) \pm (1 - \delta)\frac{y^*}{\lambda}. \tag{13.13}$$

Using the above, we can analytically calculate the key discriminant expression of $T^2 - 4D$ (or numerically) in order to verify stability. The details of the classification scheme is given in the appendix Chapter 16.

For our purposes, we will mainly proceed using numerics, and bypass the need to study the above results analytically.

In the accompanying lecture script, we will obtain the eigenvalues numerically, and generate the phase plane. At the test point $R = 2$, $\delta = 1/6$, and $\lambda = 1/5$, we generate the following phase plane picture.

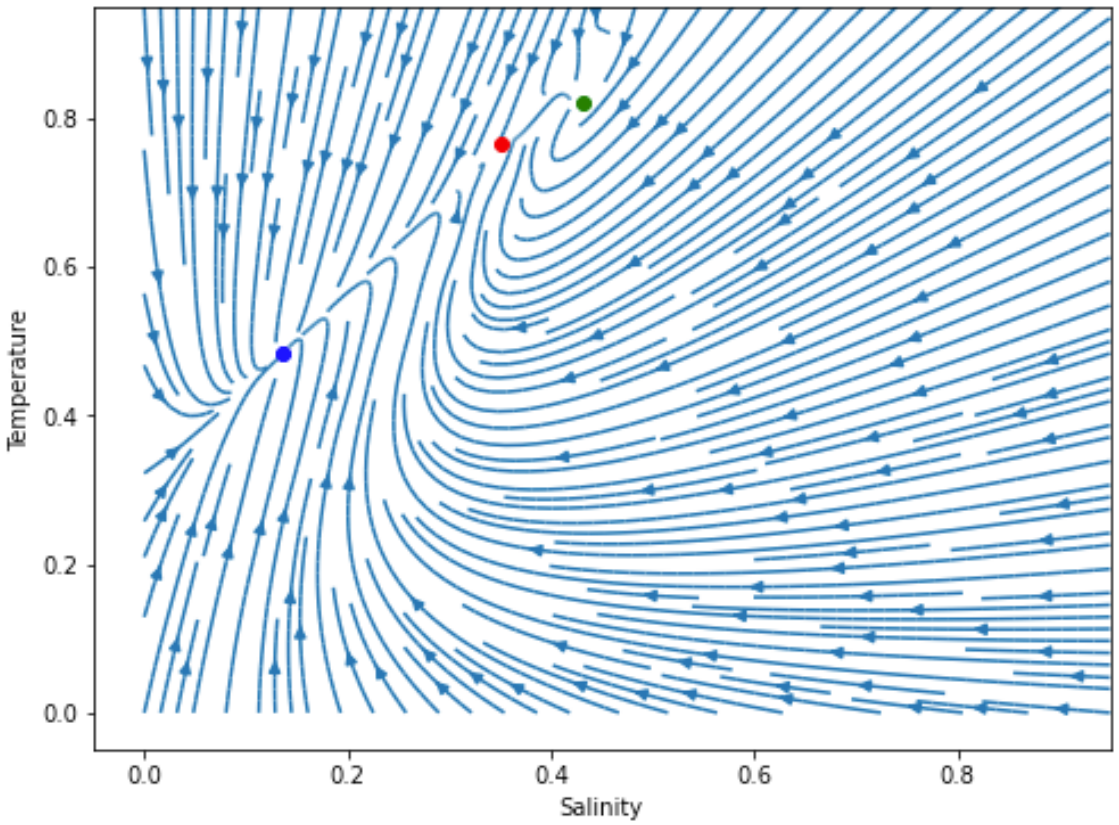


Figure 13.1: Phase plane for $R = 2$, $\lambda = 1/5$ and $\delta = 1/6$. The blue, red, and green fixed points correspond to f_1, f_2, and f_3 in order. The blue point is a stable node, the red point is an unstable saddle, and the green point is a stable spiral.

13.4 Bifurcation diagrams

Once the above stability analysis, it is possible to sketch the final bifurcation diagram.

First, we can note that as λ increases, the graph in Section 13.2 indicates that there may either be three intersections (as shown) or if λ increases then two of the intersections may disappear. We can note that:

f_i^*	$\lambda \to 0^+$	$\lambda \to \infty$
f_1^*	$-\infty$	none
f_2^*	fixed point < 0	none
f_3^*	fixed point > 0	0

We can also ascertain that there should be a point, say $\lambda = \lambda^*$ where the two roots coalesce, $f_1^* = f_2^*$, and thereafter disappear. This allows us to now plot the bifurcation (λ, f^*)-plane:

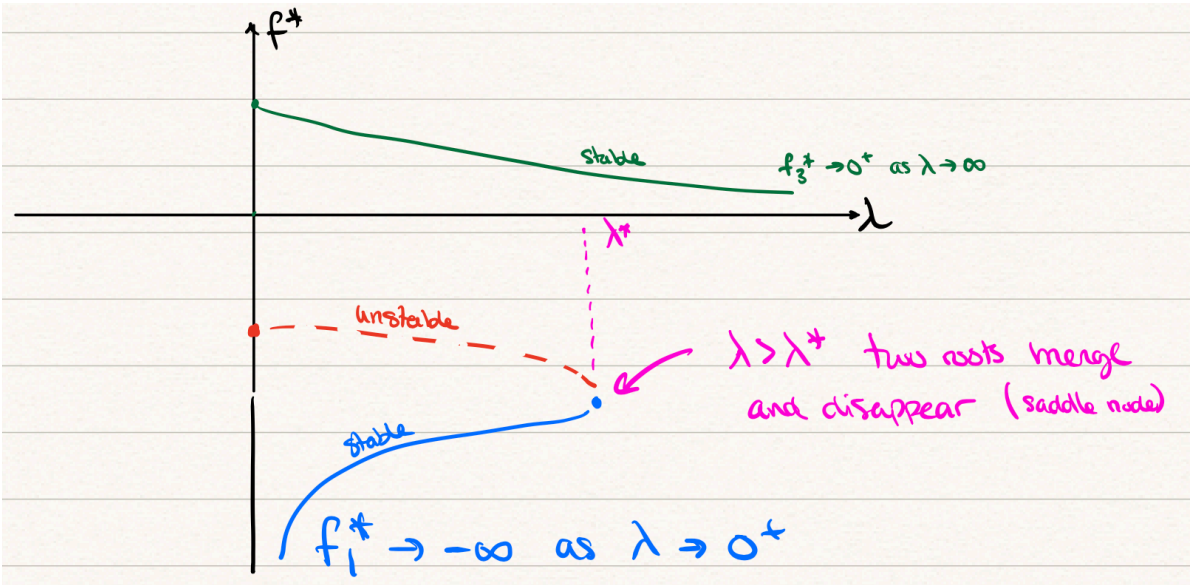


Figure 13.2

13.5 Stommel's conclusion

What is the final takeaway message(s)?

1. It is possible to pose a conceptual model of the ocean and the thermohaline circulation, which governs the exchange of hot-and-cold waters and salty waters from equatorial-to-pole zones.
2. This conceptual model suggests that tipping points and hysteresis are possible in such systems. In particular, it is possible, according to these models, for the flow directions to entirely change!

As noted by [@kaper_book]:

"Clearly, two-box models are only a caricature of the THC. At best, they account for the pole-to-equator circulation in single ocean basin (the North Atlantic). They certainly do not account for the fact that all the Earth's oceans are connected, nor for the fact that the oceans are coupled to the atmosphere and other components of the climate system. Nevertheless, the finding that such simple models predict the possibility of two distinct stable modes of circulation, and that transitions from one more to another can be induced by changing the forcing parameters [–] has had a significant impact in oceanography and climate science."

Part V

Exercises

Exercises and problem sets to be released as the term progresses.

14 Problem set 1

> For feedback, hand in by Friday Week 2.

The intention of this problem set is to practice concepts from material related to conservation laws and non-dimensionalisation. Although these concepts seem quite separate from "Maths of Planet Earth", actually, they form important pillars of mathematical modelling.

14.1 Bump lemma

Prove the following one-dimensional lemma, which was used in the derivation of the heat equation.

If $\int_a^b g(x)\,\mathrm{d}x = 0$ for all a and $b \in [0,1]$, then $g(x) \equiv 0$ for all $x \in [0,1]$.

Hint: think of a proof by contradiction.

14.2 A source in the heat equation

Consider the same heat experiment discused in Chapter 1 but now consider a bar that has an internal source or sink generating or removing heat, such as the case of a boiler with an internal heating element. By adapting a similar derivation to the one presented, explain why the modified conservation of heat equation is

$$\frac{}{t}\int_a^b \rho c T\,\mathrm{d}x = q(x=a,t) - q(x=b,t) + \int_a^b R(x,t)\,\mathrm{d}x.$$

In addition:

a. By studying the dimensions of the other terms in the above equation, find what the dimensions of R are. What does $R > 0$ mean and $R < 0$?

b. Hence derive the partial differential equation that governs the temperature T.

c. By introducing the appropriate scalings on each of the variables, x, t, and T, non-dimensionalise the PDE and discuss the non-dimensional parameters (there will be two).

14.3 Choice of scalings

Consider the dimensional problem for the motion of a projectile launched from close to the surface of the Earth. The dimensional height of the projectile, $y(t)$, is given by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\frac{GM}{(R+y)^2}, \tag{14.1}$$

$$y(0) = 2\,\mathrm{m}, \tag{14.2}$$

$$y'(0) = -V_0\,\mathrm{m/s}. \tag{14.3}$$

Assume that the Earth is spherical and of uniform density, with its mass given by $M = (4/3)\pi R^3 \rho$. Non-dimensionalise the height using $y = L\tilde{y}$ and time using $t = T\tilde{t}$. Consider the following cases:

(i) R fixed, $V_0 \to \infty$, ρ fixed;
(ii) R fixed, V_0 fixed, $\rho \to \infty$;
(iii) R fixed, V_0 fixed, $\rho \to 0$;
(iv) $R \to 0$, V_0 fixed, M fixed.

For each case:

a. Explain the physical interpretation of the limits.
b. Choose the scalings L and T to normalise as many terms as possible.
c. Choose the scalins so that the time it takes for the projectile to fall should be finite for the given limit, and for the speed, acceleration, and initial height to be well behaved (finite).
d. Write out the scaled problem and identify all remaining nondimensional parameters.
e. Identify the limiting small parameter for each case. Write out the problem (leading-order problem) when the parameter is set to zero.

14.4 The unique timescale in the heat equation

During our investigation of the heat equation, we found that it was possible to scale time so as to scale out the only non-dimensional parameter that appears in the PDE (Π). This produced (Equation 2.3). The disappearance of all non-dimensional parameters is due to the fact that only a single sensible timescale exists.

By adjusting the boundary conditions, we may create a new problem involving heat flow where a unique 'special' timescale can no longer be chosen.

Consider a system where one side of the rod is heated in some periodic fashion, e.g. set the initial and boundary conditions to be

$$T(x,0) = T_0 \tag{14.4}$$
$$T(0,t) = T_a \cos(\omega t), \tag{14.5}$$
$$T(L,t) = T_b. \tag{14.6}$$

a. What must the units of ω be?

b. Non-dimensionalise as usual and, without selecting the timescale, $[t]$, identify the key non-dimensional parameters that remain. Write a brief sentence to describe their physical interpretation.

c. There are two sensible choices for setting the timescale, $[t]$. Identify the two choices and present the reduced set of equations in each case.

14.5 Timescale in the surface energy

Take the basic zero-dimensional energy model studied in (Equation 3.7) for the temperature of the troposphere:

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{4}Q(1-a) - \sigma\gamma T^4.$$

a. Non-dimensionalise the model by choosing $T = [T]T'$, $t = [t]t'$, and $Q = [Q]Q'$. Show that it is possible to select the scalings on the temperature and time so as to completely remove all constants from the problem when Q is assumed to be constant.

b. Thus, show that the analysis of the above equation is equivalent to studying

$$\frac{\mathrm{d}T}{\mathrm{d}t} = 1 - T^4,$$

where we have dropped the primes and assumed that the albedo is such that $1 - a \neq 0$.

c. From your choice of $[t]$, estimate the typical dimensional value using $d \approx 10\text{km}$, $\rho \approx 1\text{kg}\,\text{m}^{-3}$, $c_p \approx 10^3\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$.

Use a pocket calculator to verify your calculations and conclude that this time-scale is on the order of a month. What is the relevance of this approximation as it concerns the steady-state solution?

Part VI

Appendices

15 Differential equations

Appendices

These appendices may contain material that is added during the term, dependent on student enquiries and need.

15.1 First-order linear differential equations

First-order linear differential equations are of the form

$$y' + q(x)y = r(x).$$

These are solved by integrating factors. Multiply both sides by

$$e^{\int^x q(x')\,\mathrm{d}x'}$$

so that the equation can be placed in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{\int^x q(x')\,\mathrm{d}x'}\right) = r(x)e^{\int^x q(x')\,\mathrm{d}x'}$$

Then integrate and solve for y.

References

- Paul's online notes

15.2 Second-order constant coefficient ODEs

Second-order linear constant-coefficient ODEs are of the form

$$y'' + ay' + by = f(x).$$

With both forced ($f \neq 0$) and unforced ($f = 0$) varieties studied. For the unforced variant (homogeneous), you will remember that the equation is solved by attempting the ansatz $y = e^{rx}$ and solving for r.

There are many references on this topic since it is the standard second-order theory learned in most initial differential equations courses.

References:

- Paul's online notes

16 Dynamical systems

> Appendices
>
> These appendices may contain material that is added during the term, dependent on student enquiries and need.

We will need some of the theory from your prior ODEs and Modelling and Dynamics courses as related to the solution of 2x2 linear systems of the form

$$\mathbf{Y}'(t) = \mathbf{A}(t)\mathbf{Y},$$

where

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If you need a refresher, refer to your old notes; there is also a good review on these notes and also these notes.

16.1 Stability analysis for 2x2 systems

In the case of 2x2 systems, there is a useful classification and shortcut. The eigenvalues, λ_1 and λ_2 are given by

$$\lambda_{1,2} = \frac{1}{2}(T \pm \sqrt{T^2 - 4D}),$$

where $T = \text{tr}(A) = a + d$ and $D = \det(A) = ad - bc$. Once T and D are known, then the fixed points can be classified based on the following diagram.

Above, the critical parabola is where $T^2 - 4D = 0$.

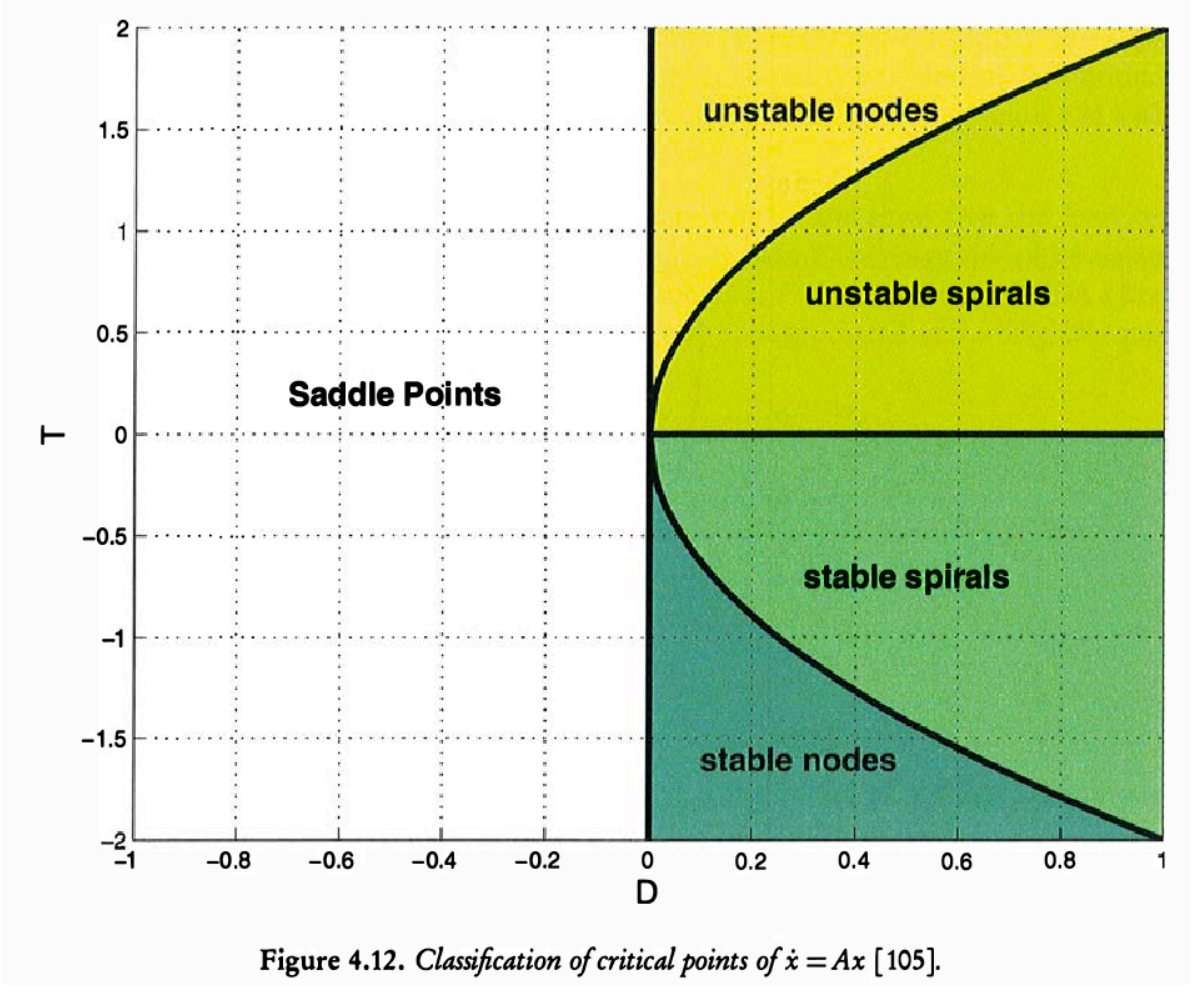


Figure 4.12. *Classification of critical points of* $\dot{x} = Ax$ [105].

Figure 16.1: Classification diagram from (Kaper and Engler 2013)

17 Vector calculus

Appendices

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17.1 Surface integrals

Surface integrals are generalisations of the concept of a multiple integrals to integrals over surfaces. For example, consider a flat plate of area A heated to a certain temperature, and thus emits a certain amount of energy, E, per unit area (per unit time). The total energy emitted per unit time would then be EA.

If we want to consider the same problem, but now posed on a general surface, say the planet S, and where E varies along the surface, then we must consider adding together $E(\mathbf{x})\mathrm{d}S$. We thus chop our surface into smaller pieces, each with area $\mathrm{d}S$, multiply each piece with its corresponding energy, and then sum the result.

The result is the surface integral

$$\iint_S E(\mathbf{x})\,\mathrm{d}S.$$

A tutorial for computing surface integrals can be found here.

You will not need to know how to calculate surface integrals in general in the course.

18 Modelling

Appendices

These appendices may contain material that is added during the term, dependent on student enquiries and need.

19 Finite difference approximations

Appendices

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Many of the numerical methods we need will require using numerical approximations for the derivatives of functions. For instance, using Newton's method to solve for the zeros of a system of equations requires calculating the Jacobian matrix.

19.1 Forwards, backwards, and centred differences

We want to learn how to use finite differences in order to approximate derivatives numerically. We know that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. Therefore, a simple idea to approximating the value of $f'(x)$ is to use a small numerical value for h, and calculate (the gradient of the secant line),

$$f'(x) \approx \frac{f(x+h) - f(x)}{h},$$

where h is a specified small number. The above is known as the **two-point forward-difference formula**. In fact, we can determine exactly the error of such an approximation via Taylor's theorem. If f is twice continuously differentiable, then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(c),$$

for some point $c \in [x, x+h]$. Therefore by re-arrangement, we see the following.

Two-point forward-difference formula

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(c),$$

where $c \in (x, x+h)$.

Notice that the error in using the two-point forward difference approximation is then $\mathcal{O}(h)$, and this error tends to zero as $h \to 0$ (as long as f'' is continuous). We thus call the above formula a first-order finite-difference approximation. If the error is $\mathcal{O}(h^n)$, we call the corresponding formula an nth-order approximation.

Example

Use the two-point forward difference formula with different values of h in order to approximate the derivative of $f(x) = 1/x$ at $x = 2$.

$f(x)$	$f(x+h)$	h	$f'(x)$	Error
...	...	...	...	...
...	...	...	...	...
...	...	...	...	...

Similar formulae can be developed for the backwards difference (send $h \to -h$).

A more accurate formula can be developed via subtracting the Taylor series for $f(x-h)$ from that for $f(x+h)$. This results in:

Three-point centered-difference formula

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f'''(c),$$

where $c \in (x-h, x+h)$.

Thus we see that the centered difference formula is accurate to $O(h^2)$.

19.2 Jacobian matrices

Reference

You will have encountered the Jacobian, firstly in your first-year Methods courses when performing change-of-coordinates in integration formulae, and secondly in your second-year Modelling and Dynamics courses when studying differential equations. It also comes up in the second-year Vectors and PDEs course.

The Jacobian matrix, $\mathbf{J}$, of a vector function, $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$,

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix},$$

is the matrix of all its first-order partial derivatives,

$$\mathbf{J}(\mathbf{x}) = \begin{pmatrix} \frac{f_1}{x_2} & \cdots & \frac{f_1}{x_n} \\ \vdots & \ddots & \vdots \\ \frac{f_m}{x_1} & \cdots & \frac{f_m}{x_n} \end{pmatrix} = \begin{pmatrix} \nabla f_1(\mathbf{x})^T \\ \vdots \\ \nabla f_n(\mathbf{x})^T \end{pmatrix}.$$

We can essentially think of the Jacobian as the multi-dimensional extension of the basic derivative. It encodes all of the (first-order) information about the rate-of-change of the function.

It is interesting and important to consider the numerical evaluation of the Jacobian, in the event that the function $\mathbf{F}$ cannot be easily differentiated exactly. The simplest algorithm is based on approximating each entry of the matrix by a finite difference.

For example, let us consider approximating

$$\frac{f_i}{x_j}(\mathbf{x}_0).$$

We define the jth unit vector by $\mathbf{e}_j = [0\,0 \cdots 1 \cdots 0]$, which the jth entry being one and all entries being zero. Then we can approximate the derivative by a central difference,

$$\frac{f_i}{x_j}(\mathbf{x}_0) \approx \frac{f_i(\mathbf{x}_0 + \mathbf{e}_j h) - f_i(\mathbf{x}_0 - \mathbf{e}_j h)}{2h},$$

where h is a small step size.

Let us test this by calculating the Jacobian for

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{pmatrix}.$$

The pseudocode for this looks like this

Pseudocode for numerical Jacobian

Input: function F (size m), point x (size n), step size h

```
1. Create an (m x n) matrix, J
2. a. Loop through all the rows indexed by i
```

```
      b. Loop through all the columns indexed by j
             Create the ej unit vector
         Calculate the difference of F_i(x + h e_j) - F_i(x - h e_j)
         Divide by 2h
         Assign this value to the (i,j)th value of the Jacobian

Output: the (mxn) Jacobian matrix J
```

Here is a code that puts it into practice.

```
import numpy as np

def jacobian(func,initial,delta=1e-3):
  f = func
  nrow = len(f(initial))
  ncol = len(initial)
  output = np.zeros(nrow*ncol)
  output = output.reshape(nrow,ncol)
  for i in range(nrow):
    for j in range(ncol):
      ej = np.zeros(ncol)
      ej[j] = 1
      dij = (f(initial+ delta * ej)[i] - f(initial- delta * ej)[i])/(2*delta)
      output[i,j] = dij
  return output

def myf(x):
  x1 = x[0]
  x2 = x[1]
  output = np.zeros(3)
  output[0] = x[0]**2
  output[1] = x[1]**2
  output[2] = x[0]*x[1]
  return output

jacobian(myf,[1,2])
```

```
array([[2., 0.],
       [0., 4.],
       [2., 1.]])
```

References

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