ECE657 Tools of Intelligent Systems Design
Assignment 4 Question 5

Huy Trinh

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Question 5: Principal Component Analysis (PCA)

Q5.1

Mean of feature 1:

$$\mu_1 = \frac{8+5+1+17}{4} = 7.75$$

Mean of feature 2:

$$\mu_2 = \frac{12+1+15+4}{4} = 8$$

Center of data

$$X_{\text{centered}} = \begin{pmatrix} 8-7.75 & 12-8 \\ 5-7.75 & 1-8 \\ 1-7.75 & 15-8 \\ 17-7.75 & 4-8 \end{pmatrix} = \begin{pmatrix} 0.25 & 4 \\ -2.75 & -7 \\ -6.75 & 7 \\ 9.25 & -4 \end{pmatrix}$$

We calculate

$$\text{Cov}(X)$$

$$= \frac{1}{3}\begin{pmatrix} 0.25^2 + (-2.75)^2 + (-6.75)^2 + 9.25^2 & 0.25 \cdot 4 + (-2.75)\cdot(-7) + (-6.75)\cdot 7 + 9.25\cdot(-4) \\ 4\cdot 0.25 + (-7)\cdot(-2.75) + 7\cdot(-6.75) + (-4)\cdot 9.25 & 4^2 + (-7)^2 + 7^2 + (-4)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 46.25 & -21.33 \\ -21.33 & 43.33 \end{pmatrix}$$

Q5.2

To find the eigenvalues, we solve the characteristic equation:

$$\det(\text{Cov}(X) - \lambda I) = 0$$

$$\begin{vmatrix} 46.25-\lambda & -21.33 \\ -21.33 & 43.33-\lambda \end{vmatrix} = 0$$

$$(46.25-\lambda)(43.33-\lambda) - (21.33)^2 = 0$$

$$2004.0825 - 46.25\lambda - 43.33\lambda + \lambda^2 - 454.3489 = 0$$

$$\lambda^2 - 89.58\lambda + 1549.7336 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Subtitute into above, we got

$$\lambda = \frac{-(-89.58) \pm \sqrt{(-89.58)^2 - 4 \cdot 1 \cdot 1549.7336}}{2 \cdot 1}$$

$$\lambda = \frac{89.58 \pm \sqrt{8024.4964 - 6198.9344}}{2}$$

$$\lambda = \frac{89.58 \pm \sqrt{1825.562}}{2}$$

$$\lambda = \frac{89.58 \pm 42.73}{2}$$

So, we have two eigenvalues:

$$\lambda_1 = \frac{89.58 + 42.73}{2} = \frac{132.31}{2} = 66.155$$

$$\lambda_2 = \frac{89.58 - 42.73}{2} = \frac{46.85}{2} = 23.425$$

Q5.3: Calculate the Normalized Eigenvectors

Eigenvector v can be found by solving the equation:

$$(\mathrm{Cov}(X) - \lambda I)v = 0$$

For $\lambda_1 = 66.155$:

$$\begin{pmatrix} 46.25 - 66.155 & -21.33 \\ -21.33 & 43.33 - 66.155 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = 0$$

$$\begin{pmatrix} -19.905 & -21.33 \\ -21.33 & -22.825 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = 0$$

Solving for the eigenvector, we get:

$$-19.905v_{11} - 21.33v_{12} = 0$$

$$v_{11} = \frac{-21.33}{19.905}v_{12}$$

Choose v_{12} to be 1 then

$$v_{12} = \frac{-21.33}{19.905} \approx -1.071$$

So unormalized eigen vector is $v_1 = [-1.701, 1]$

Similarly, for $\lambda_2 = 23.425$:

$$\begin{pmatrix} 46.25 - 23.425 & -21.33 \\ -21.33 & 43.33 - 23.425 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = 0$$

$$\begin{pmatrix} 22.825 & -21.33 \\ -21.33 & 19.905 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = 0$$

Solving for the eigenvector, we get:

$$22.825v_{21} - 21.33v_{22} = 0$$

$$v_{21} = \frac{21.33}{22.825}v_{22}$$

To normalize an eigenvector v, divide it by its magnitude mod(v) (say v):

$$v = \sqrt{v_1^2 + v_2^2}$$

Normalizing v_1:

$$mod(v_1) = \sqrt{1^2 + (-0.933)^2} = 1.341$$

$$v_1 = \frac{1}{1.341}[1, -0.933] = [0.746, -0.696]$$

Normalizing v_2:

$$mod(v_2) = \sqrt{1^2 + (1.07)^2} = 1.454 \ v_2 = \frac{1}{1.454}[1, 1.07] = [0.688, 0.736]$$

Therefore, our normalized Eigenvectors are:

$$v_1 = [0.746, -0.696]$$

$$v_2 = [0.688, 0.736]$$

Q5.4: Principal Component with the Highest Eigenvalue

The principal component with the highest eigenvalue is v_1:

$$v_1 = \begin{pmatrix} 0.746 \\ -0.696 \end{pmatrix}$$

Project the original dataset onto v_1:

$$X' = X \cdot v_1 = \begin{pmatrix} 8 & 12 \\ 5 & 1 \\ 1 & 15 \\ 17 & 4 \end{pmatrix} \begin{pmatrix} 0.746 \\ -0.696 \end{pmatrix} = \begin{pmatrix} 8 \cdot 0.746 + 12 \cdot (-0.696) \\ 5 \cdot 0.746 + 1 \cdot (-0.696) \\ 1 \cdot 0.746 + 15 \cdot (-0.696) \\ 17 \cdot 0.746 + 4 \cdot (-0.696) \end{pmatrix} = \begin{pmatrix} 5.968 - 8.352 \\ 3.73 - 0.696 \\ 0.746 - 10.44 \\ 12.682 - 2.784 \end{pmatrix} = \begin{pmatrix} -2.384 \\ 3.034 \\ -9.694 \\ 9.898 \end{pmatrix}$$

This is the reduced dataset in one dimension using the principal component with the highest eigenvalue.