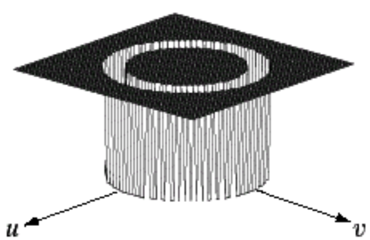
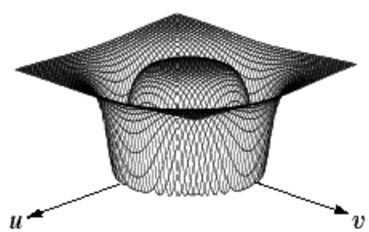
Chapter 5
Image Restoration and Reconstruction

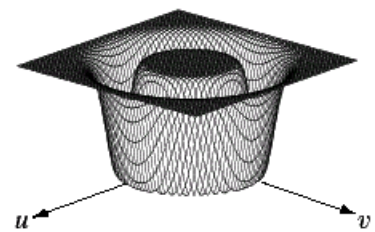
Frequency Domain Filtering for Noise Filtering
Linear, Position-Invariant Degradation
Degradation Function Estimation
Inverse Filtering
Wiener Filtering (Minimum Mean Square Error)

Frequency Domain Filtering for Periodic Noise Reduction

- We assume that the approximate frequency range of the noise is known or can be estimated
- *Band-reject filtering* is appropriate in this case
- E.g. if the periodic noise can be approximated as two-dimensional sinusoidal functions, its Fourier transform will form symmetric impulses





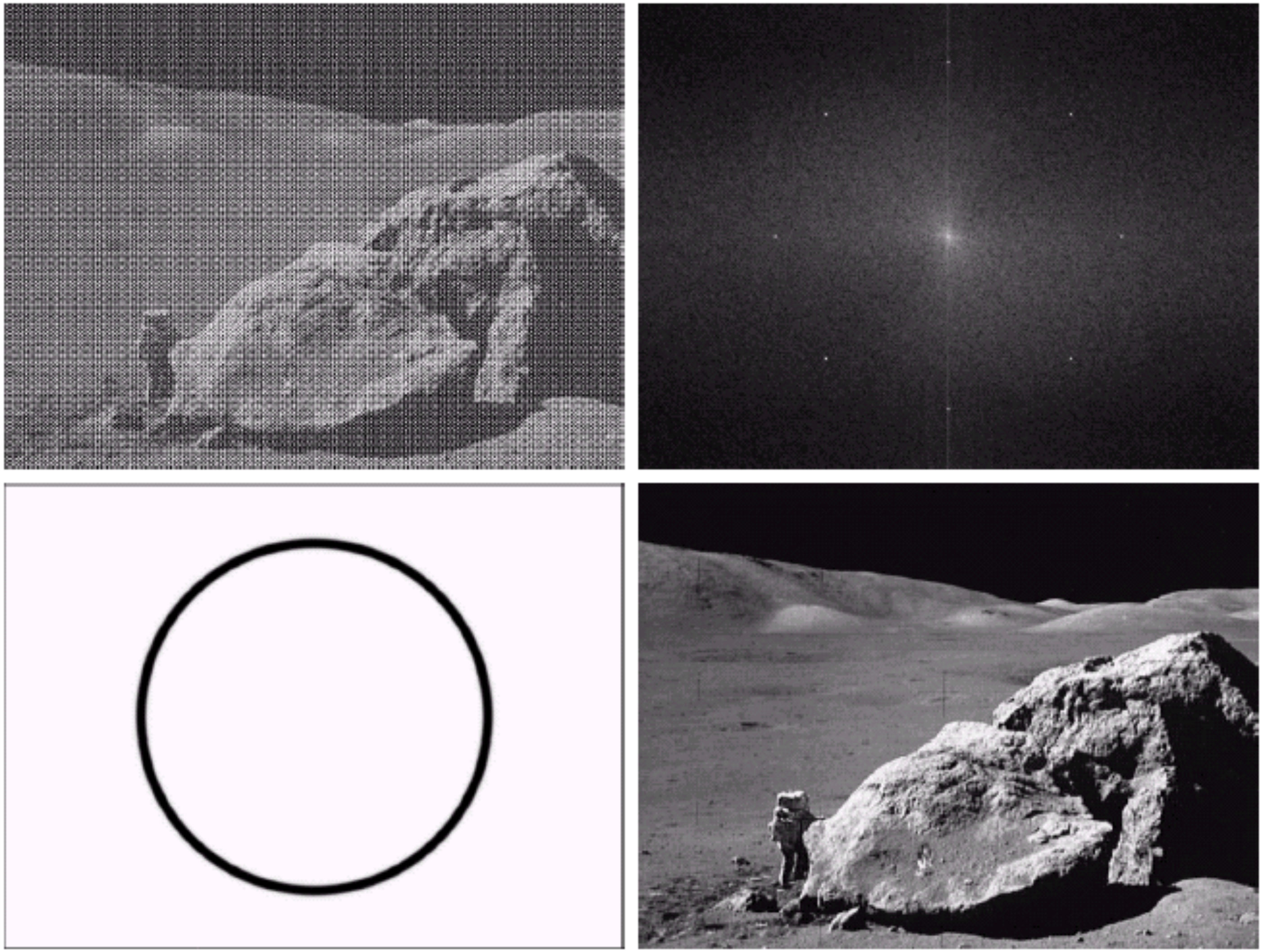


Ideal

Butterworth (1^{st} order)

Gaussian

Frequency Domain Filtering for Periodic Noise Reduction



a b
c d

FIGURE 5.16 (a) Image corrupted by sinusoidal noise. (b) Spectrum of (a). (c) Butterworth bandreject filter (white represents 1). (d) Result of filtering. (Original image courtesy of NASA.)

Frequency Domain Filtering for Periodic Noise Reduction

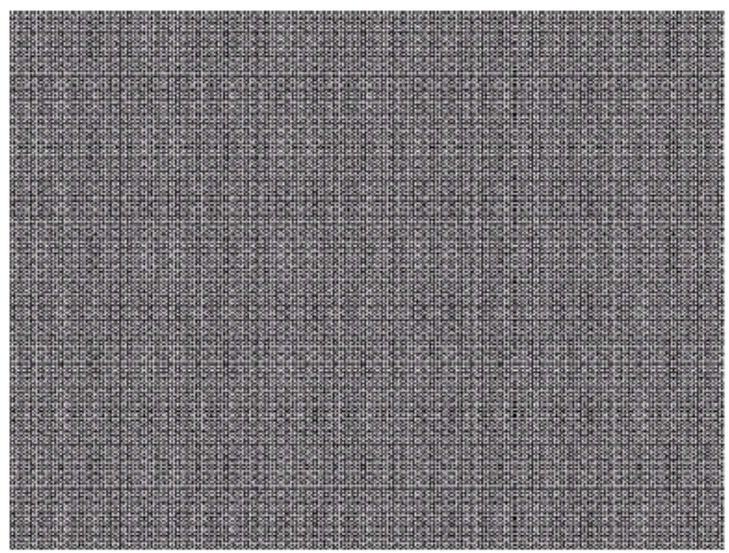
Bandpass Filters

- Can be obtained by inversion of band-reject filters (recall previous slide)

$$H_{BP}(u,v) = 1 - H_{BR}(u,v)$$

- Not commonly used to remove noise, but can isolate it for further study

FIGURE 5.17 Noise pattern of the image in Fig. 5.16(a) obtained by bandpass filtering.



Frequency Domain Filtering for Periodic Noise Reduction

Notch Filters

- Reject or pass frequencies in predefined neighborhoods (not necessarily centered around the origin)
- If not centered at the origin, they appear in symmetrical pairs (due to Fourier transform symmetry)
- *Notch reject* filters and *notch pass* filters are connected via a familiar expression:

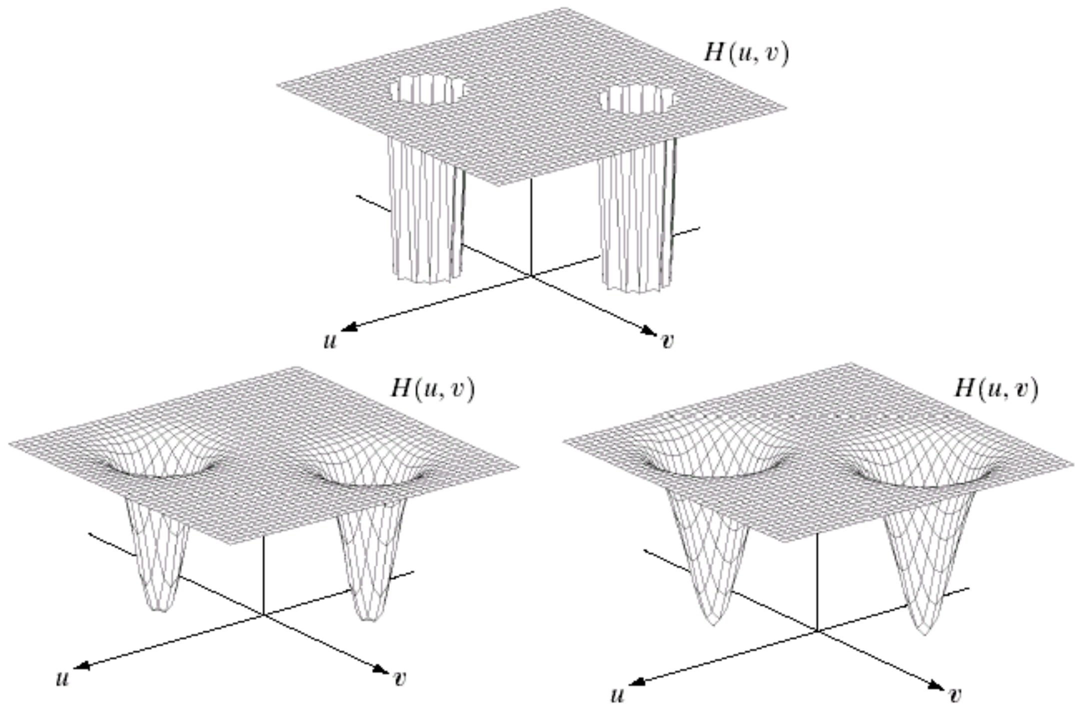
$$H_{NP}(u,v) = 1 - H_{NR}(u,v)$$

Notch Reject Filter Examples

ideal

Butterworth (2nd order)

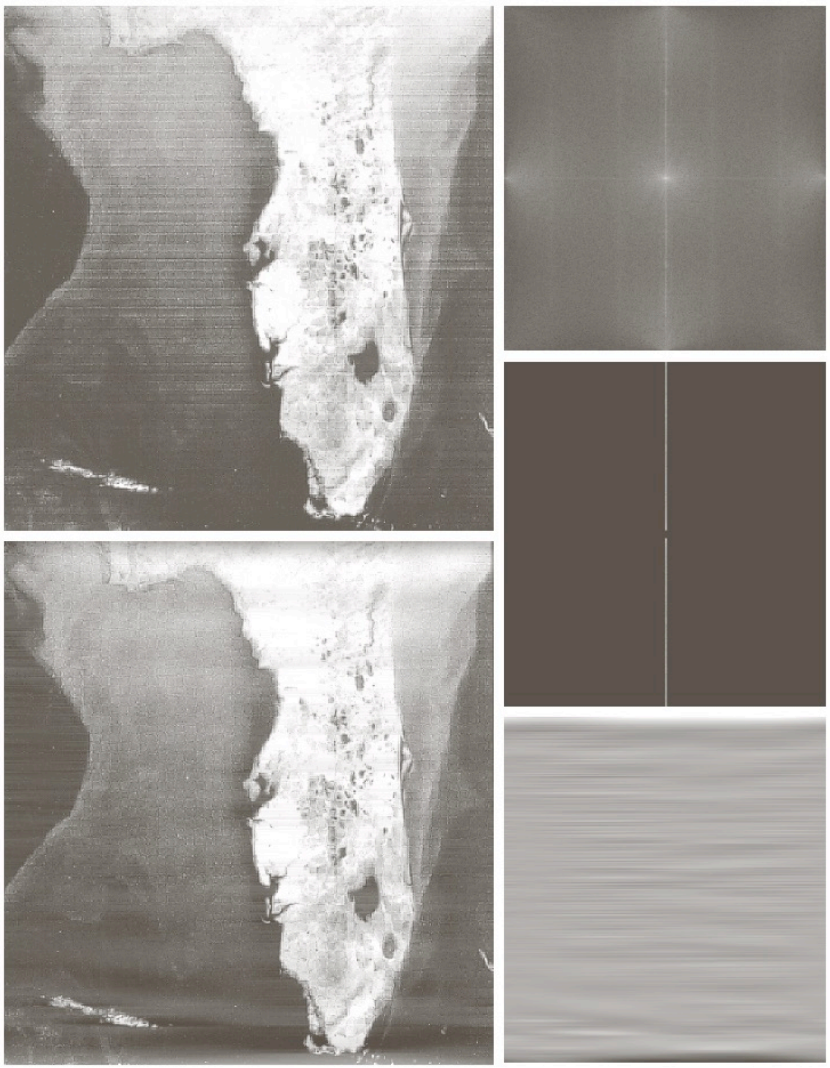
Gaussian



a
b c

FIGURE 5.18 Perspective plots of (a) ideal, (b) Butterworth (of order 2), and (c) Gaussian notch (reject) filters.

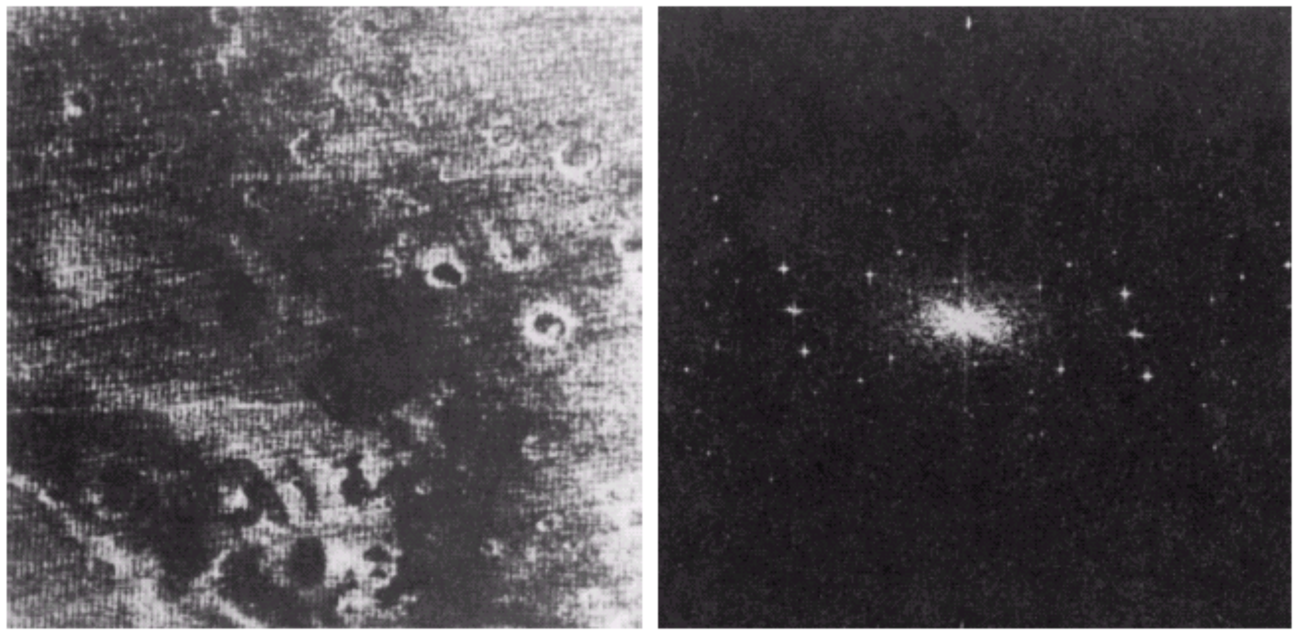
Notch Filtering for Noise Reduction



a b
c
e d

FIGURE 5.19
(a) Satellite image of Florida and the Gulf of Mexico showing horizontal scan lines. (b) Spectrum. (c) Notch pass filter superimposed on (b). (d) Spatial noise pattern. (e) Result of notch reject filtering. (Original image courtesy of NOAA.)

Optimum Notch Filtering



a b

FIGURE 5.20 (a) Image of the Martian terrain taken by *Mariner 6*. (b) Fourier spectrum showing periodic interference. (Courtesy of NASA.)

While the interference patterns are visible in the Fourier domain, they are more subtle and larger in number. Standard notch filtering is not desirable, as it may remove too many details.

Optimum Notch Filtering

- Optimality in the sense of minimizing local variances
- The concept lies in assigning weights to the extracted interference patterns before subtracting them from the image
- Assume that the interference pattern is extracted by some notch pass filter in the frequency domain:

$$N(u,v) = H_{NP}(u,v)G(u,v)$$

  - this filter is generally constructed by visual observation of $G(u, v)$
- The interference pattern in the spatial domain is then:

$$\eta(x,y) = \Im^{-1}\{H_{NP}(u,v)G(u,v)\}$$

- This pattern is approximation of the interference; therefore, simply subtracting it will not yield the best result.

Optimum Notch Filtering

$$\hat{f}(x, y) = g(x, y) - w(x, y)\eta(x, y)$$

- We construct as estimate of the recovered image as specified above, where
- $w(x, y)$ is called a *weighting function* (*modulation function*)
- We can now select this function such that the variance of $\hat{f}(x, y)$ over a specified neighborhood of every point (x, y) is minimized
- Given the neighborhood size of $N = (2a+1)(2b+1)$, the local variance can be determined as:

$$\sigma^2(x, y) = \frac{1}{N}\sum_{s=-a}^{a}\sum_{t=-b}^{b}[\hat{f}(x+s, y+t) - \bar{\hat{f}}(x, y)]^2$$

Optimum Notch Filtering

- In the local variance expression we can substitute $\hat{f}(x,y)$ with our constructed estimate:

$$\sigma^2(x,y) = \frac{1}{N}\sum_{s=-a}^{a}\sum_{t=-b}^{b}\{[g(x+s,y+t) - w(x+s,y+t)\eta(x+s,y+t)] - [\bar{g}(x,y) - \overline{w(x,y)\eta(x,y)}]\}^2$$

- Assume the weights remain constant within the neighborhood:

$$w(x+s,y+t) = w(x,y) \;\Rightarrow\; \overline{w(x,y)\eta(x,y)} = w(x,y)\bar{\eta}(x,y)$$

- Substituting this into the local variance expression and solving for the minimal variance, we obtain:

$$w(x,y) = \frac{\overline{g(x,y)\eta(x,y)} - \bar{g}(x,y)\bar{\eta}(x,y)}{\overline{\eta^2}(x,y) - \bar{\eta}^2(x,y)}$$

- Only one value of $w(x, y)$ per neighborhood is computed!

Optimum Notch Filtering Example

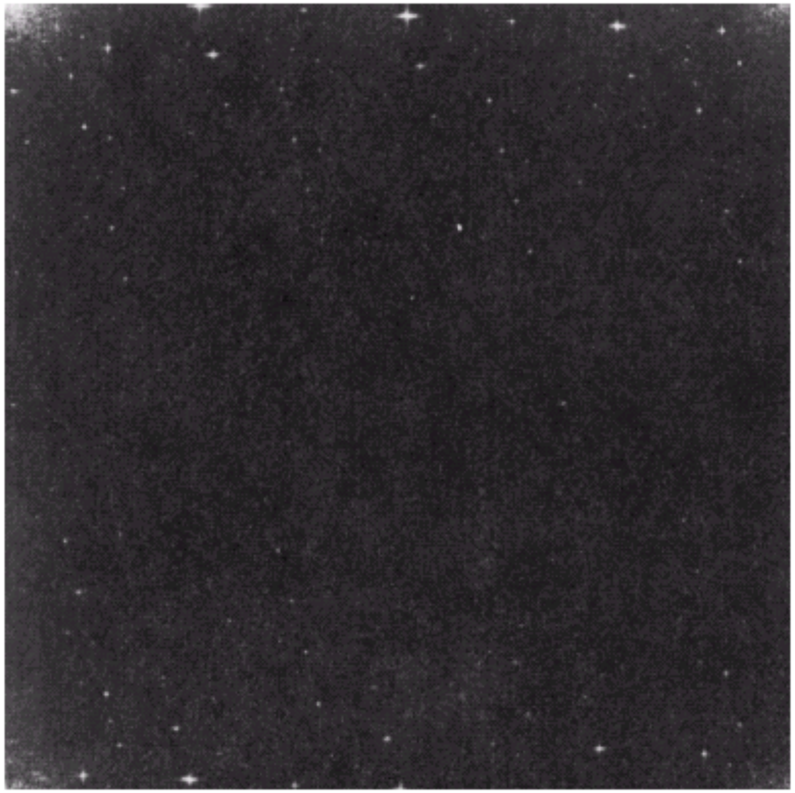
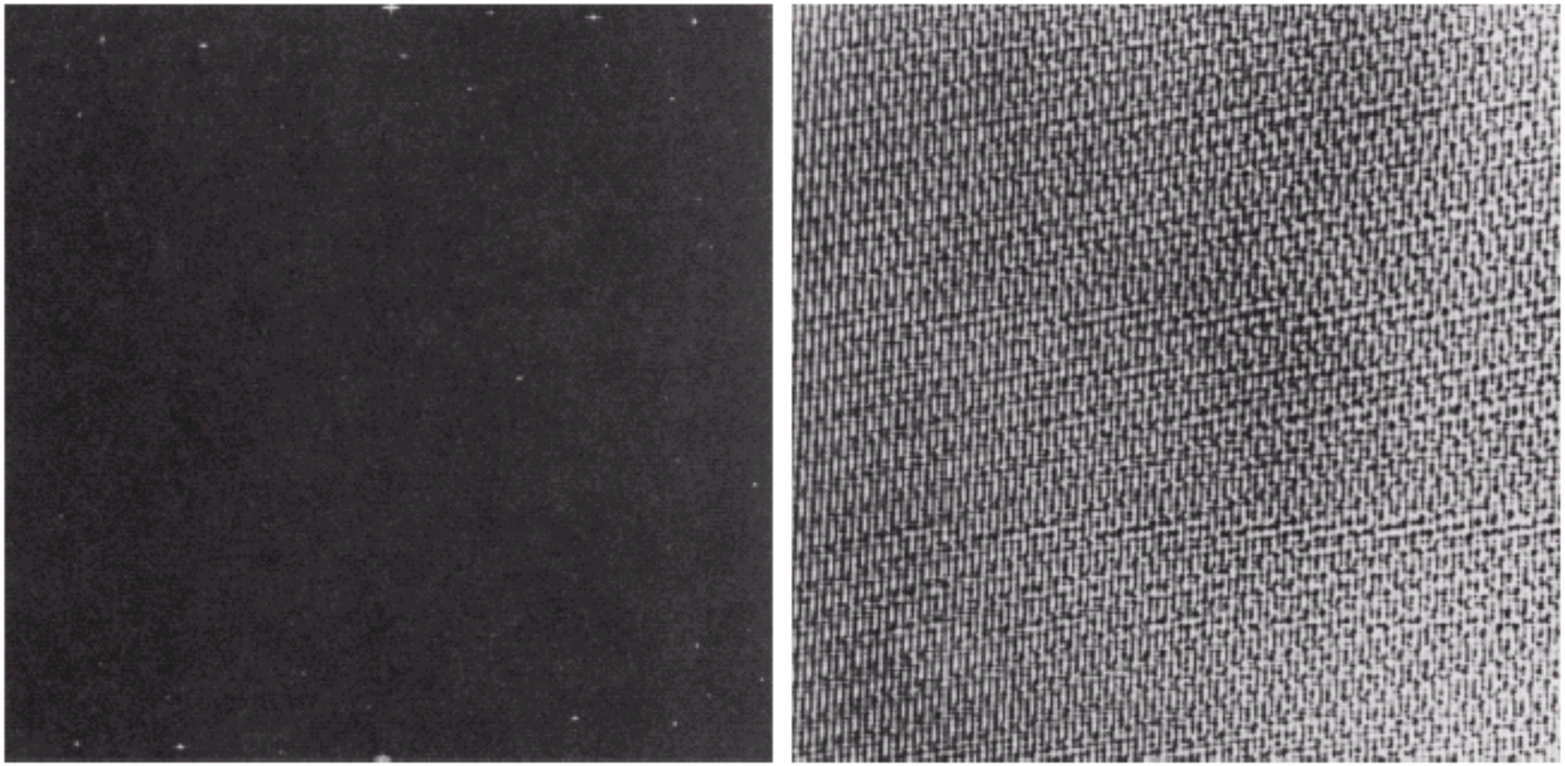


FIGURE 5.21 Fourier spectrum (without shifting) of the image shown in Fig. 5.20(a). (Courtesy of NASA.)

Optimum Notch Filtering Example

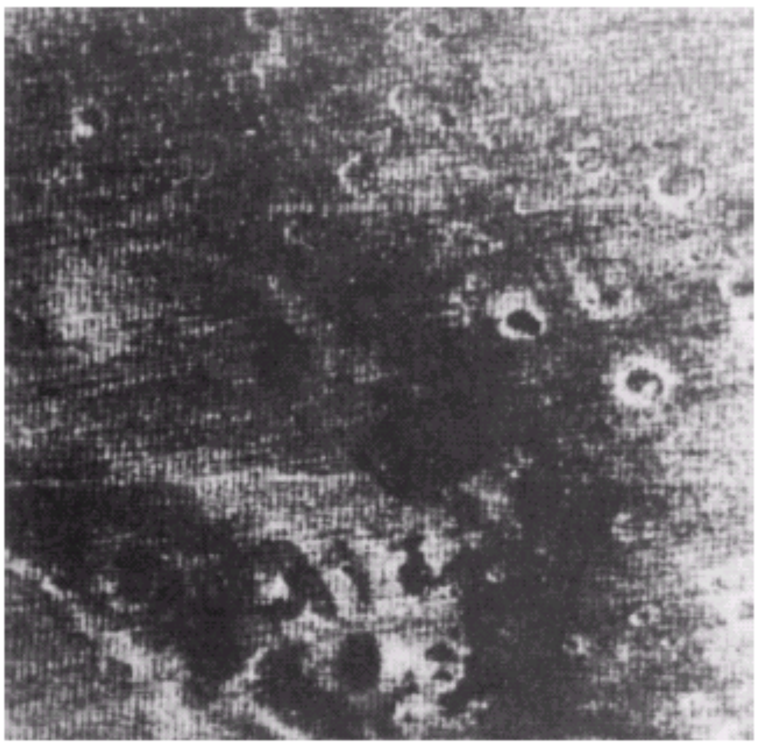


a b

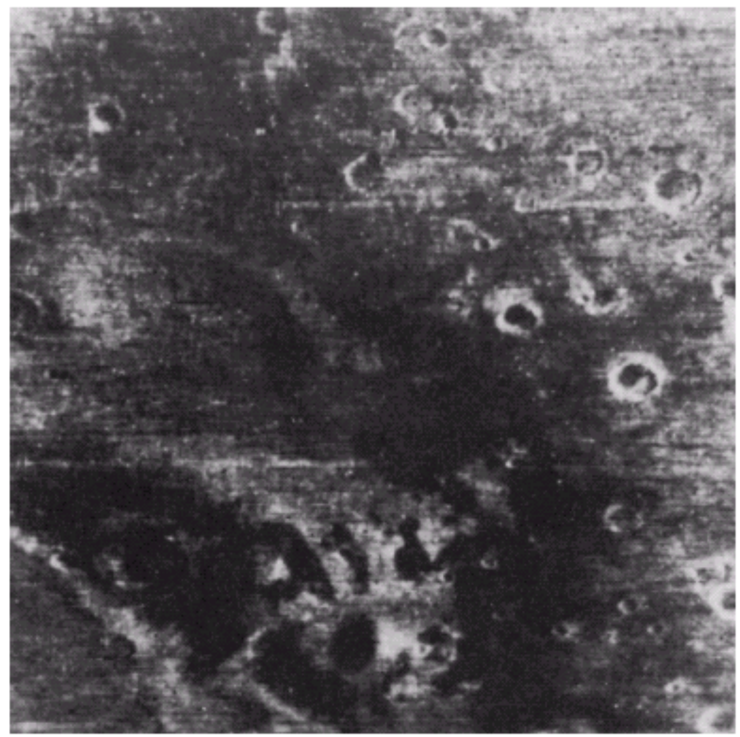
FIGURE 5.22 (a) Fourier spectrum of $N(u, v)$, and (b) corresponding noise interference pattern $\eta(x, y)$. (Courtesy of NASA.)

Optimum Notch Filtering Example

Original



Filtered



General Degradation Model

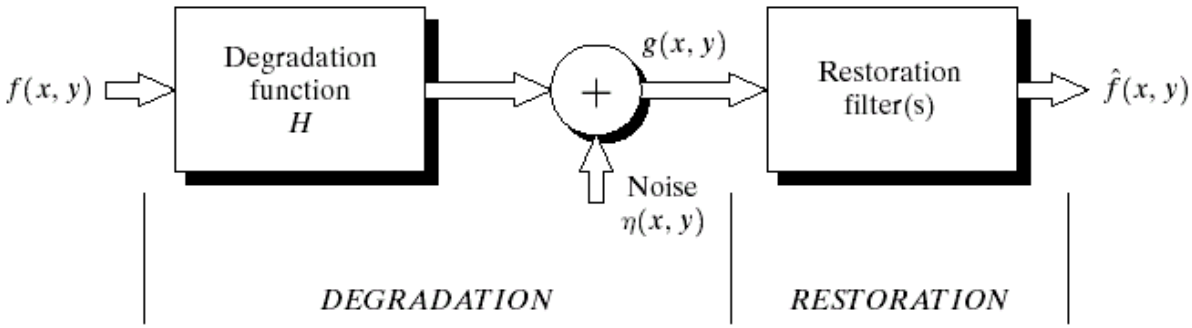


FIGURE 5.1 A model of the image degradation/ restoration process.

<u>Assumption</u>: *H* is linear and position-invariant

Spatial domain: $g(x,y) = h(x,y) * f(x,y) + \eta(x,y)$

Frequency domain: $G(u,v) = H(u,v)F(u,v) + N(u,v)$

<u>Goal</u>: design restoration filter(s) s.t. $\hat{f}(x,y)$ is close to $f(x,y)$

Linear Position-Invariant Degradations

- Assumptions:
  - no noise is present: $\eta(x, y) = 0$
  - degradation function H is linear:

$$H[af_1(x,y)+bf_2(x,y)] = aH[f_1(x,y)]+bH[f_2(x,y)]$$

  - degradation function H is position-invariant:

$$H[f(x,y)] = g(x,y) \Rightarrow H[f(x-\alpha, y-\beta)] = g(x-\alpha, y-\beta)$$

- Recall the *sifting property* of the impulse δ:

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\,\delta(\alpha-x,\beta-y)\,d\alpha\,d\beta$$

- We will take advantage of this representation

Linear Position-Invariant Degradations

- Substitute the last expression into the degradation model:

$$g(x,y) = H[f(x,y)] = H\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\,\delta(x-\alpha,y-\beta)\,d\alpha\,d\beta\right]$$

- Using linearity of H:

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H\left[f(\alpha,\beta)\,\delta(x-\alpha,y-\beta)\right]d\alpha\,d\beta$$

- Note that $f(\alpha, \beta)$ does not depend on (x, y), thus:

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\,H\left[\delta(x-\alpha,y-\beta)\right]d\alpha\,d\beta$$

- The following term is an *impulse response* of H:

$$h(x,\alpha,y,\beta) = H\left[\delta(x-\alpha,y-\beta)\right]$$

Linear Position-Invariant Degradations

- Due to position invariance:

$$H[\delta(x-\alpha, y-\beta)] = h(x-\alpha, y-\beta)$$

- This allows us to obtain the 2-D convolution integral:

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\, h(x-\alpha, y-\beta)\, d\alpha\, d\beta$$

- The <u>impulse response</u> of the linear system is <u>sufficient</u> to compute its response to <u>any input</u>

- Taking the additive noise into account, we obtain:

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\, h(x-\alpha, y-\beta)\, d\alpha\, d\beta + \eta(x,y)$$

$$g(x,y) = h(x,y) * f(x,y) + \eta(x,y)$$

Estimating the Degradation Function

- Assumptions about the nature of the degradation function allow us to utilize tools of linear system theory

- Nonlinear, position-variant degradations can be approximated

- Degradation is modelled as a convolution with the degradation function “image”, which can be estimated

- *Deconvolution* can then be applied to reverse the process

- *Blind deconvolution* refers to the fact that the true degradation is almost never known exactly

Estimating the Degradation Function

1. Estimation by Image Observation

- Find some indicative image regions where the effect of the noise is less significant (e.g. high contrast areas), let this subimage be $g_s(x, y)$
- Process $g_s(x, y)$ such that the result is as good as possible (manual experimentation)
- Estimate the degradation:

$$H_s(u,v) = \frac{G_s(u,v)}{\hat{F}_s(u,v)}$$

- Using $H_s(u, v)$ as a reference, we can construct $H(u,v)$
- This approach is not very practical

Estimating the Degradation Function

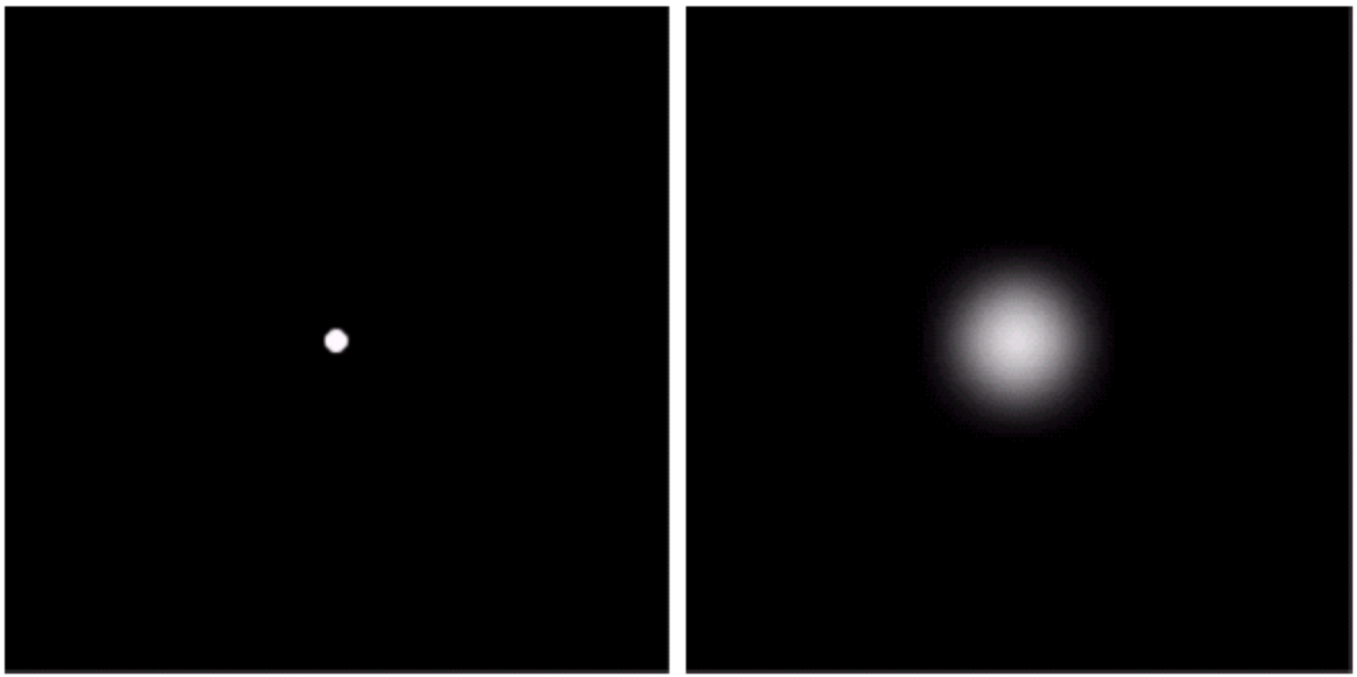
2. Estimation by Experimentation

- Assume we have access to the image generating equipment, which produced the degraded image
- Perform imaging of the impulse signal (small bright dot), using the same system settings
- The impulse should be as strong as possible to minimize noise influence
- Estimate $H(u,v)$ from the degraded impulse image

$$H(u,v) = \frac{G_{imp}(u,v)}{A}$$

where A is the strength of the impulse (since its DFT is constant)

Estimating the Degradation Function by Experimentation: Example



a b

FIGURE 5.24 Degradation estimation by impulse characterization. (a) An impulse of light (shown magnified). (b) Imaged (degraded) impulse.

Estimating the Degradation Function

3. Estimation by Modeling

- Different models of varying complexities have been proposed over the years
- Example: the degradation model by Hufnagel and Stanley (1964), based on atmospheric turbulence:

$$H(u,v) = e^{-k(u^2+v^2)^{5/6}}$$

where k is a constant related to the nature of turbulence

- Compare to the Gaussian low-pass filter:

$$H(u,v) = e^{-D^2(u+v)/2D_0^2}$$

Atmospheric Turbulence Model

FIGURE 5.25 Illustration of the atmospheric turbulence model. (a) Negligible turbulence. (b) Severe turbulence, $k = 0.0025$. (c) Mild turbulence, $k = 0.001$. (d) Low turbulence, $k = 0.00025$. (Original image courtesy of NASA.)

Estimating the Degradation Function

3. Estimation by Modeling (cont'd)

- Models can also be derived from basic principles
- We will consider an example of blurring caused by uniform linear motion between an image and the sensor during acquisition
- Assume that $x_0(t)$ and $y_0(t)$ are the time-varying components of the image motion and T is the motion duration
- With some assumptions on the imaging process, the blurred image can be obtained via integration:

$$g(x,y) = \int_0^T f\left[x - x_0(t), y - y_0(t)\right] dt$$

Estimating the Degradation Function

3. Estimation by Modeling (cont'd)

- The Fourier transform of the blurred image:

$$G(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,e^{-j2\pi(ux+vy)}\,dx\,dy$$

- Substitute $g(x, y)$ and change the order of integration:

$$G(u,v)=\int_{0}^{T}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left[x-x_0(t),y-y_0(t)\right]e^{-j2\pi(ux+vy)}\,dx\,dy\right]dt$$

- The expression under the outermost integral corresponds to the Fourier transform of the displaced function
- By the translation property of Fourier transform:

$$G(u,v)=\int_{0}^{T} F(u,v)\,e^{-j2\pi\left[ux_0(t)+vy_0(t)\right]}\,dt$$

Estimating the Degradation Function

3. Estimation by Modeling (cont'd)

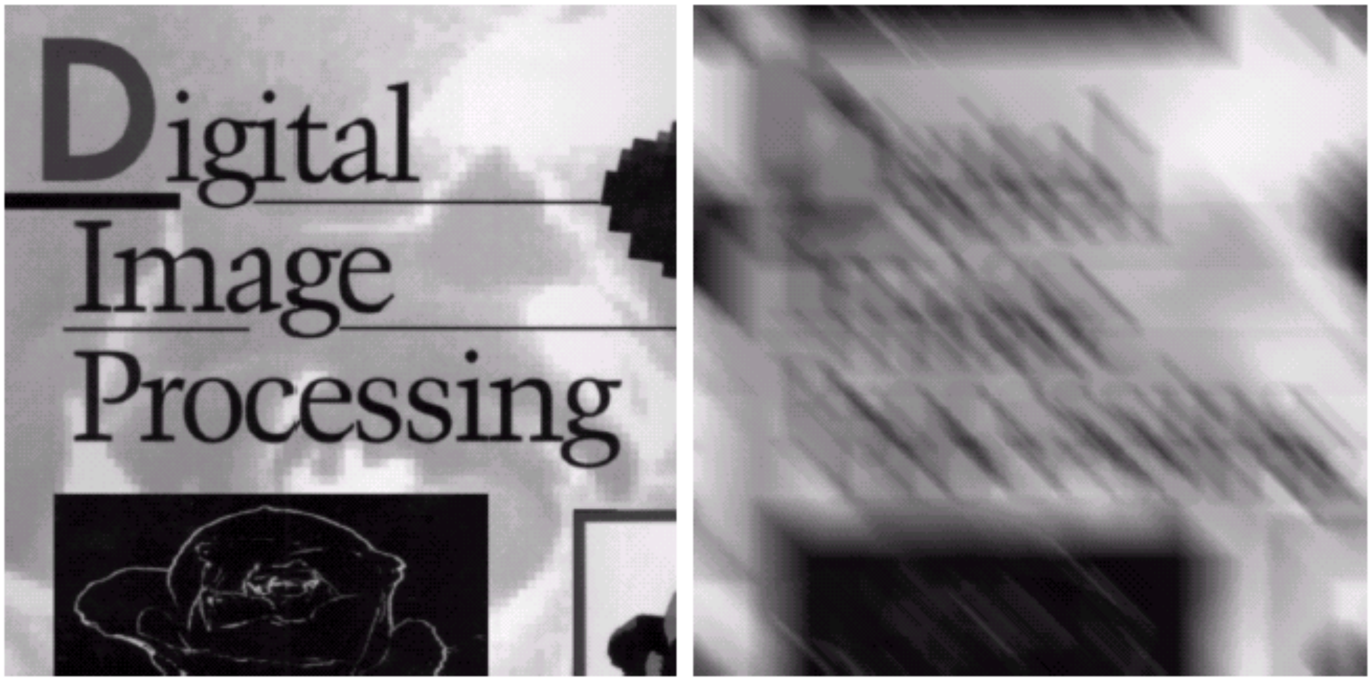
- Since $F(u, v)$ does not depend on t:

$$G(u,v) = F(u,v)\int_0^T e^{-j2\pi\left[ux_0(t)+vy_0(t)\right]}\,dt$$

- Clearly $H(u, v)$ is equal to the above integral
- <u>Example</u>: if we let $x_0(t) = at/T$ and $y_0(t) = bt/T$, the following degradation function is obtained:

$$H(u,v) = \frac{T}{\pi(ua+vb)}\sin\left[\pi(ua+vb)\right]e^{-j\pi(ua+vb)}$$

Motion Blur Modeling Example



a b

FIGURE 5.26 (a) Original image. (b) Result of blurring using the function in Eq. (5.6-11) with $a = b = 0.1$ and $T = 1$.

Inverse Filtering

- Having an estimate of *H*, we can recover the original image:

$$\hat{F}(u,v) = \frac{G(u,v)}{H(u,v)}$$

- *Inverse filtering*: reverses the degrading convolution via the elementwise division of Fourier transforms
- If the noise component is not absent:

$$\hat{F}(u,v) = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

<u>Concerns</u>:

- As noise is unknown, the original image <u>cannot be recovered</u>
- Small values of $H(u, v)$ strongly <u>boost the noise</u>

Pseudo-Inverse Filtering

- <u>Reasonable assumption</u>: often, the largest frequency coefficient values correspond to low frequencies, i.e. the area centered around the origin

- We can apply a low-pass filter on $H(u, v)$ for the purpose of inverse filtering $\Longrightarrow$ *pseudo-inverse filtering*

- The probability of low values in $H(u, v)$ decreases, but small values do not totally disappear $\Longrightarrow$ not completely reliable!

- Besides, standard low-pass filtering limitations do apply, e.g. ringing.

Inverse Filtering Example

a b
c d

FIGURE 5.27 Restoring Fig. 5.25(b) with Eq. (5.7-1). (a) Result of using the full filter. (b) Result with *H* cut off outside a radius of 40; (c) outside a radius of 70; and (d) outside a radius of 85.

Butterworth low pass filtering is used here

Wiener Filtering

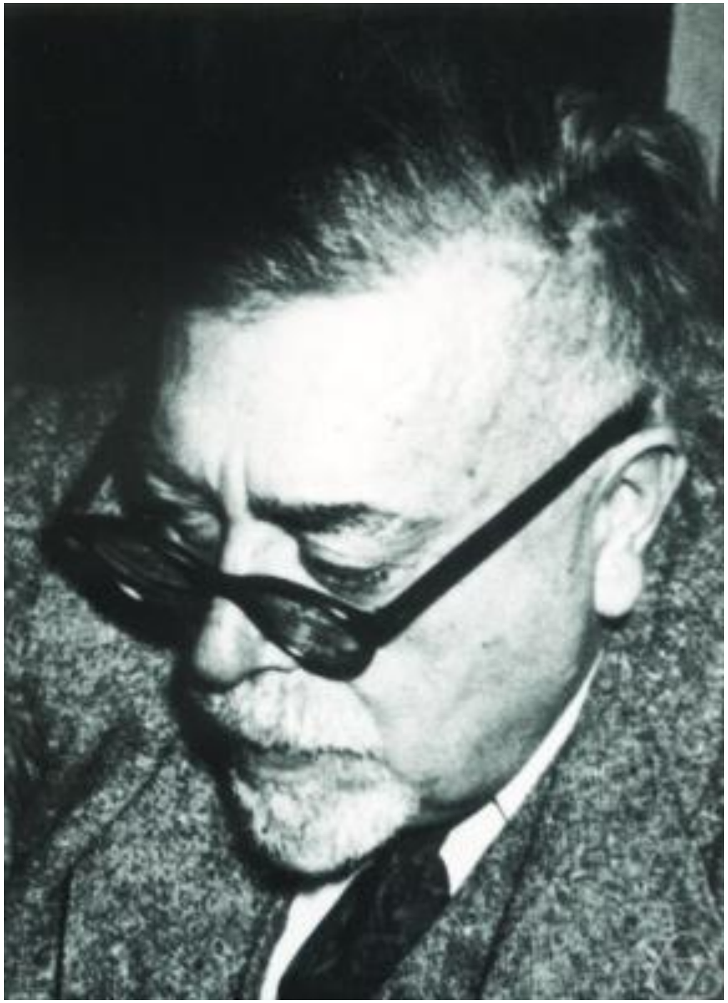
- Noise should be accounted for during the restoration
- We can consider the image and the noise to be random variables and seek to minimize the *mean square error*:

$$e^2 = E\left\{\left(f - \hat{f}\right)^2\right\}$$

- Concept was first proposed by Norbert Wiener in 1942.
- <u>Assumptions</u>:
  - noise and image are not correlated
  - noise <u>or</u> the image have zero mean
  - intensity values of the estimated image $\hat{f}(x, y)$ are a linear function of the degraded image $g(x, y)$

Chapter 5
Image Restoration: Norbert Wiener

- In signal processing, the Wiener filter is a filter proposed by Norbert Wiener during the 1940s and published in 1949. Its purpose is to reduce the amount of noise present in a signal by comparison with an estimation of the desired noiseless signal. The discrete-time equivalent of Wiener's work was derived independently by Kolmogorov and published in 1941. Hence the theory is often called the Wiener-Kolmogorov filtering theory. The Wiener-Kolmogorov was the first statistically designed filter to be proposed and subsequently gave rise to many others including the famous Kalman filter. A Wiener filter is not an adaptive filter because the theory behind this filter assumes that the inputs are stationary.



Wiener Filtering

- The mean square error in the frequency domain:

$$\varepsilon^2 = E\left\{\left|F(u,v) - \hat{F}(u,v)\right|^2\right\}$$

- We will omit the frequency arguments (u, v) in the following derivation for brevity
- The notation is as follows (frequency domain):
  - F – clean image
  - G –degraded image
  - H – degradation function (estimated and ≈ known)
  - W – Wiener filter (our target)
  - N – noise signal
  - S_f – power spectrum of the clean image, $S_f = |F|^2$
  - S_η – power spectrum of the noise, $S_\eta = |N|^2$

Wiener Filtering

- Begin by substituting the estimated image:

$$\varepsilon^2 = E\left\{\left|F-\hat{F}\right|^2\right\} = E\left\{\left|F-WG\right|^2\right\} = E\left\{\left|F-W\left(HF+N\right)\right|^2\right\}$$

- Rearrange by factoring out F: $\varepsilon^2 = E\left\{\left|\left[1-WH\right]F-WN\right|^2\right\}$
- Expand the square:

$$\varepsilon^2 = \left[1-WH\right]\left[1-WH\right]^* E\left\{\left|F\right|^2\right\} + WW^* E\left\{\left|N\right|^2\right\}$$
$$+\left[1-WH\right]W^* E\left\{FN^*\right\} + W\left[1-WH\right]^* E\left\{F^*N\right\}$$

- Recall that F and N are independent; therefore, only the first two elements of the sum are retained:

$$E\left\{FN^*\right\} = E\left\{F^*N\right\} = 0$$

Wiener Filtering

- Using the notation for power spectra, we obtain:

$$\varepsilon^2 = [1 - WH][1 - WH]^* E\{S_f\} + WW^* E\{S_\eta\}$$

- To minimize the error we take the derivative and set it equal to zero, yielding the following equation:

$$\frac{d\varepsilon^2}{dW} = W^* E\{S_\eta\} - H[1 - WH]^* E\{S_f\} = 0$$

- The solution of this equation is the Wiener filter:

$$W(u,v) = \frac{1}{H(u,v)} \left[\frac{|H(u,v)|^2}{|H(u,v)|^2 + S_\eta(u,v)/S_f(u,v)} \right]$$

Wiener Filtering

$$W(u,v) = \frac{1}{H(u,v)} \left[\frac{|H(u,v)|^2}{|H(u,v)|^2 + S_\eta(u,v)/S_f(u,v)} \right]$$

- It follows from the derivation that the Wiener filter is optimal in the mean square error sense
  - given a particular estimate of degradation function!
- In case of *spectrally white noise*, its power spectrum is constant, otherwise it can be estimated
- The power spectrum of the clean image, on the other hand, cannot be easily obtained
- A common approximation is to use some constant value as a *ratio* between the two spectra

Wiener Filtering

- Define *signal-to-noise ratio* as follows:

$$\mathrm{SNR} = \frac{\sum_M \sum_N \left|F(u,v)\right|^2}{\sum_M \sum_N \left|N(u,v)\right|^2}$$

- The Wiener filter can then be specified as:

$$W(u,v) = \frac{1}{H(u,v)} \left[\frac{\left|H(u,v)\right|^2}{\left|H(u,v)\right|^2 + \mathrm{SNR}^{-1}} \right]$$

- Assuming constant SNR, we can use it as a parameter for the Wiener filter and experimentally find appropriate values

Wiener Filtering Interpretation

- Recall $$W(u,v) = \frac{1}{H(u,v)}\left[\frac{|H(u,v)|^2}{|H(u,v)|^2 + S_\eta(u,v)/S_f(u,v)}\right]$$

- No noise is present $\Longrightarrow$ inverse SNR is zero $\Longrightarrow$ the Wiener filter becomes a simple inverse filter

- No degradation $\Longrightarrow$ $H(u, v) = 1$ $\Longrightarrow$ the Wiener filter becomes a smoothing filter, suppressing areas with low SNR:

$$W(u,v)_{H(u,v)=1} = \frac{\text{SNR}}{\text{SNR}+1}$$

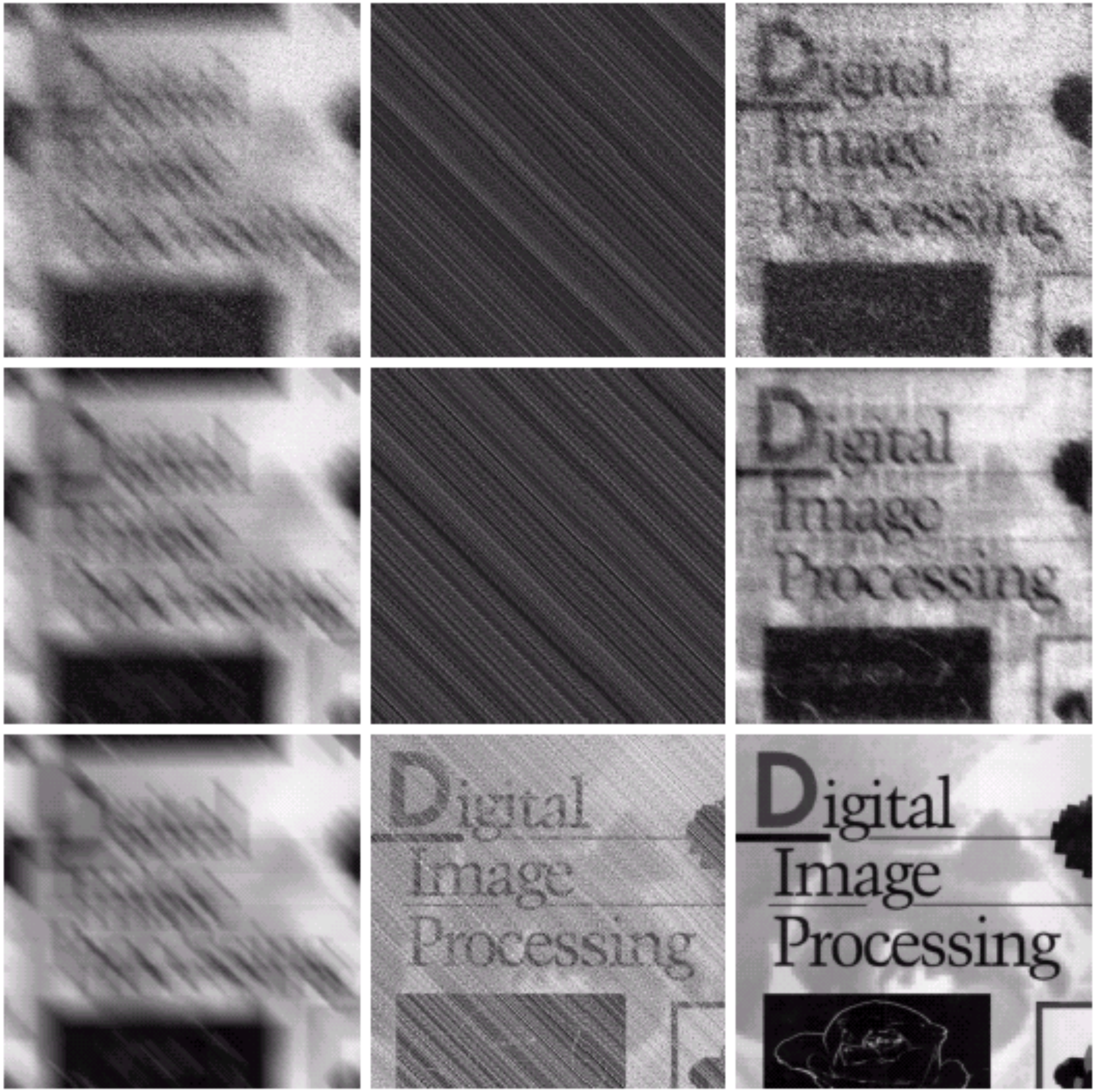
- Otherwise: when both noise and degradation are present, the Wiener filter seeks a compromise between lowpass noise smoothing and high-pass deblurring. The result is a bandpass filter. However, deblurring decreases rapidly as the noise power increases. Experiment with this!

Wiener Filtering Examples

Inverse Pseudo-inverse Wiener

a b c

FIGURE 5.28 Comparison of inverse- and Wiener filtering. (a) Result of full inverse filtering of Fig. 5.25(b). (b) Radially limited inverse filter result. (c) Wiener filter result.



Motion blur + Gaussian noise (variance 650)

Motion blur + Gaussian noise (variance 65)

Motion blur + Gaussian noise (variance 0.065)

a b c
d e f
g h i

Inverse

Wiener

Geometric Mean Filter

$$W_G(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2} \right]^{\alpha} \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \beta \left[S_\eta(u,v) / S_f(u,v) \right]} \right]^{1-\alpha}$$

- Further generalization of the Wiener filter
- Useful for software implementation, as it covers a multitude of possible filters, such as:
  - $\alpha = 1 \Longrightarrow$ inverse filter
  - $\alpha = 0 \Longrightarrow$ parametric Wiener filter (standard if $\beta = 1$)
  - $\alpha = ½ \Longrightarrow$ geometric mean of the two filters
  - $\alpha = ½$ and $\beta = 1 \Longrightarrow$ *spectral equalization filter*