RESEARCH PROPOSAL

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May 5, 2022

Contents

Chapter 1

Basic definitions

1.1 Group, ring and field

A set A equipped with a binary operation $+$,denoted as $(A, +)$ is a group if:

- $a + b \in A \; \forall a, b \in A$
- $a + (b + c) = (a + b) + c \quad \forall a, b, c \in A$
- $\exists! 0 \in A \; so \; that \; a + 0 = 0 + a = a \quad \forall a \in A$

$(A, +)$ is called an Abelian group if it is a group and $a + b = b + a \quad \forall a, b \in A$

A set A equipped with 2 binary operations $+$ and . is a ring if:

- $(A, +)$ is an Abelian group
- $a(b + c) = ab + ac \quad \forall a, b, c \in A$
- $a(bc) = (ab)c \quad \forall a, b, c \in A$

A ring $(A, +, .)$ is a commutative ring if $a.b = b.a \quad \forall a, b \in A$ $(A, +, .)$ is a field if:

$(A, +, .)$ is a commutative ring $\exists! 1 \in A$ such that $a.1 = 1.a = a \quad \forall a \in A$
For each $a \in A, \; \exists! a^{-1} \in A$ such that $a.a^{-1} = a^{-1}.a = 1$

$\mathbb{F}$ is a finite field if it is a field with finite cardinality. F_q is a finite field then $q = p^r$ for some odd prime number p and positive integer r

1.2 Discrete Fourier Analysis

Let $(G, +)$ be an Abelian group with a finite number of element, $\chi : G \to \mathbb{C}^{\times}$ is called an additive character on G if:

- $\chi(0) = 1$
- $\chi(a + b) = \chi(a) \cdot \chi(b) \quad \forall a, b \in G$
- $|\chi(a)| = 1 \quad \forall a \in G$

The set of all characters χ on G is denoted as $\widehat{G}$ Let the scalar product of f and g $G \to \mathbb{C}$ be:

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}$$

The Fourier transform of function $f : G \to \mathbb{C}$ is called $\hat{f} : \widehat{G} \to \mathbb{C}$

$$\hat{f}(\chi) = \langle f, \chi \rangle = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{\chi(x)}$$

The trivial character on G is denoted as χ_0 that:

$$\chi_0(x) = 1 \quad \forall x \in G$$

Theorem 1.1. *Let χ be a character on G, the following holds:*

$$\frac{1}{|G|} \sum_{x \in G} \chi(x) = \begin{cases} 0 \ if \ \chi \neq \chi_0 \\ 1 \ if \ \chi = \chi_0 \end{cases}$$

From which we derive a corollary
Corollary 1.2: If χ_1 and χ_2 are 2 distinct characters on G, then

$$\langle \chi_1, \chi_2 \rangle = 0$$

Theorem 1.3: (Parseval's equality) Let $\hat{f}, \hat{g}$ be the Fourier transform of f, g respectively, then:

$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$$

Lemma 1.4: (Fourier inversion formula) Let f and $\hat{f}$ be functions introduced above. then we have the equality:

$$f(x) = \sum_{\chi \in \widehat{G}} \langle f, \chi \rangle \chi(x) = \sum_{\chi \in \widehat{G}} \hat{f}(\chi)\chi(x)$$

Proposition 1.5: Let $\widehat{G}$ is the character set of G, then $|\widehat{G}| = |G|$
Lemma 1.5: (The Plancherel formula)

$$\sum_{\chi \in \widehat{G}} |\hat{f}(\chi)|^2 = \frac{1}{|G|} \sum_{x \in G} |f(x)|^2$$

Hint: Consider $|f(x)|^2 = f(x)\overline{f(x)}$, use the expression in **Lemma 1.4**, and use Theorem 1.1

1.3 Geometric incidence

On R^2, given a set of point P with p elements and a set of lines L with l elements, an incidence is generated whenever a line in L passes through a point in P.

Let $I(P.L)$ denote the number of incidences.

Lemma 1.8 (Trotter-Szemeredi theorem)

$$I(P, L) \lesssim l^{\frac{2}{3}} p^{\frac{2}{3}} + l + p$$

The number of edge intersections in a multigraph G is denoted as $cr(G)$

Lemma 1.9 Given a graph G with n vertices and e edges, then:

$$cr(G) \geq e - 3n + 6$$

Lemma 1.10 (Crossing number inequality) Let G be a graph with n vertices and e edges, let $e \geq 4n$, then:

$$cr(G) \gtrsim \frac{e^3}{n^2}$$

Chapter 2

Erdos Distance Problem in $\mathbb{R}^d$

2.1 Introduction

In 1946, Hungarian mathematician Paul Erdos proposed a question on the distinct distances on a set of points in a subspace P of $\mathbb{R}^d$
Let the set

$$\Delta(P) = \{||a-b|| \ ; \ a, b \in P\} \text{ and } \Delta(n) = \min_{P \subseteq \mathbb{R}^d, \ |P|=n} |\Delta(P)|$$

Erdos wished to find the upper and lower bounds for $\Delta(n)$

2.2 For $\mathbb{R}^2$

2.2.1 $n^{1/2}$ theory and the upper bound

Erdos proved in 1946 that $\Delta(n) \gtrsim n^{\frac{1}{2}}$ and $\Delta(n) \lesssim \dfrac{n}{\sqrt{\log(n)}}$.

2.2.2 $n^{4/5}$ theory

László A. Székely proved in 1993 that $\Delta(n) \gtrsim n^{\frac{4}{5}}$
Hint: Use Lemma 1.8, 1.10 and 1.11

2.2.3 Latest result

Katz and Guth in 2015 proved the best lower bound by far:

$$|\Delta(n)| \gtrsim \frac{n}{\log(n)}$$

2.3 For $\mathbb{R}^d$; $d \geq 3$

For this scenerio, Erdos conjectured that:

$$|\Delta(n)| \gtrsim n^{\frac{2}{d}}$$

This has not yet been shown, however a weaker bound was derived by applying induction on the proof of $n^{\frac{1}{2}}$

$$|\Delta(n)| \gtrsim n^{\frac{1}{d}}$$

The best estimation was proven by Solymosi and Vu in $\mathbb{R}^d$:

$$|\Delta(n)| \gtrsim n^{\frac{2}{d}-\frac{1}{d(d+1)}}$$

Chapter 3

Erdos Distance Problem in finite field

3.1 Erdos-Falconer distance problem

A set $\mathbb{F}_q^d$ is called a vector space over finite field $\mathbb{F}_q$ of cardinality q if it is defined as:

$$\mathbb{F}_q^d = \left\{x = (x_1, x_2, ..., x_d)\ ;\ x_i \in \mathbb{F}_q,\ i = \overline{1, d}\right\}$$

Distance between $x, y \in \mathbb{F}_q^d$ is:

$$d(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_d - y_d)^2$$

Properties of $d(x, y)$:

1. $d(x + z, y + z) = d(x, y) \quad \forall z \in \mathbb{F}_q^d$
2. For any orthogonal matrix A, let $\cdot$ denotes the matrix multiplication, assume $x \in \mathbb{F}_q^d$ is a vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$. Then $d(A \cdot x, A \cdot y) = d(x, y)$

(A is an orthogonal matrix if $A \cdot A^T = I$)
Let $P \subseteq \mathbb{F}_q^d$, we defined $\Delta_{\mathbb{F}_q}(P) = \{d(x, y)\ ;\ x, y \in P\}$
Theorem 3.1: Let $P \subset \mathbb{F}_q^2$ such that $1 < |P| < p^2$, then:

$$|\Delta_{\mathbb{F}_q}(P)| \geq |P|^{1/2}$$

Erdos-Falconer distance problem: What is the smallest $\alpha > 0$ such that for every $P \subset \mathbb{F}_q^d$ and $|P| \gg q^\alpha$, then $\lim\limits_{q\to\infty} \dfrac{\Delta_{\mathbb{F}_q}(P)}{q^d} > 0$

Falconer conjecture: If $A \subset \mathbb{R}^d$ is a compact set with Hausdorff dimension greater than $\dfrac{d}{2}$, then $|\Delta(A)|$ has a positive Lesbeague measure

Latest results:

- (Du, Iosevich, Ou, Wang and Zhang in 2020) For $d \geq 4$ and even: $\alpha = \dfrac{d}{2} + \dfrac{1}{4}$
- (Du and Zhang in 2018) For $d \geq 5$ and odd: $\alpha = \dfrac{d}{2} + \dfrac{d}{4d-2}$

3.2 General results

Let $E \subset \mathbb{F}_q^d$ we define the function $E : \mathbb{F}_q^d \to \mathbb{F}_q$:

$$E(x) = \begin{cases} 0 \; if \; x \notin E \\ 1 \; if \; x \in E \end{cases}$$

The Fourier transform of E, called $\widehat{E} : \mathbb{F}_q^d \to \mathbb{C}$ is:

$$\widehat{E}(m) = q^{-d} \sum_{x \in \mathbb{F}_q^d} e^{-\frac{2\pi i x m}{q}} E(x)$$

The Fourier inversion formula:

$$E(x) = \sum_{m \in \mathbb{F}_q^d} e^{\frac{2\pi i x m}{q}} \widehat{E}(m)$$

The Plancherel formula:

$$\sum_{m \in \mathbb{F}_q^d} |\widehat{E}(m)|^2 = q^{-d} \sum_{x \in \mathbb{F}_q^d} |E(x)|^2$$

Corollary 3.4: Suppose that $|E| \gtrsim q^{\frac{d+1}{2}}$. Then $|\Delta(E)| \gtrsim q$

Similar to $E(x)$, we define $S_r(x) : \mathbb{F}_q^d \to \mathbb{C}$:

$$S_r(x) = \begin{cases} 1 \; if \; ||x|| = r \\ 0 \; if \; ||x|| \neq r \end{cases}$$

Lemma 3.5: q is a prime number, then:

$$\left| \sum_{j \in \mathbb{F}_q / \{0\}} e^{-\frac{2\pi i}{q}(jr + j^{-1}r)} \right| \lesssim \sqrt{q} \quad \forall r, r' \in \mathbb{F}_q$$

Lemma 3.5: The sphere S_r satisfies:

$$|\widehat{S}_r(m)| \lesssim q^{-d} q^{\frac{d-1}{2}} \quad \forall m \in \mathbb{F}_q^d$$

Theorem 3.6: Let $E \subset \mathbb{F}_q^d$, then:

$$|\Delta(E)| \gtrsim \min \left\{ q, \frac{|E|}{q^{\frac{d-1}{2}}} \right\}$$

Reference

[1] Julia Garibaldi, Alex Iosevich, and Steven Senger, *The Erdos Distance Problem*, Student Mathematical Library, vol.56 (2011).

[2] Thang Pham, *Erdös-Falconer distance conjecture over finite fields*,MoCCA (2020)