# Global Attractor for a Semilinear Parabolic System

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**Abstract.** The aim of this paper is to prove existence of a global attractor of the semigroup generated by the first initial boundary value problem for a semilinear parabolic system in the potential form in an arbitrary (bounded or unbounded) domain.

**Keywords:** Semilinear parabolic system; Global solution; Global attractor; Lyapunov functional.

## 1. Introduction

The understanding of the asymptotic behaviour of dynamical systems is one of the most important problems of morden mathematical physics. One way to attack the problem for disspative dynamical system is to consider its global attractor. This is an invariant set that attracts all the trajectories of the system. The existence of the global attractor has been derived for a large class of PDEs (see e.g. [3,4,8] and references therein). One of the most studied gradient partial differential equations is the reaction-diffusion equations and systems, which model several physical phenonmena like heat conduction, population dynamics, etc (see [1-10]).

In the survey article [8], the author consider the following problem in a bounded domain $\Omega \subset \mathbb{R}^N$,

$$\begin{aligned} u_t - \triangle u + f(u) + g(x) = 0,\ x \in \Omega, t > 0, \\ u(x,t) = 0, x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ x \in \Omega, \end{aligned}$$

where $u_0 \in H_0^1(\Omega)$ given, $g \in L^2(\Omega)$. Under some conditions of the nonlinear term $f(u)$, the author prove the existence of a global attractor in $H_0^1(\Omega)$. The proof of the main results relies on the existence of a Lyapunov functional and the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, since $\Omega$ is bounded. In this paper, we will extend these results in two directions: firstly, we consider a semilinear parabolic system in the potential form; secondly, we let $\Omega$ be an arbitrary (bounded or unbounded) domain in $\mathbb{R}^N$ $(N \geqslant 3)$.

More precisely, in this paper we are interested in the following problem: Consider the semilinear parabolic system in the potential form

$$U_t - \triangle U + \nabla F(U) + G(x) = 0, \tag{1.1}$$

with boundary conditions

$$U = 0 \ \text{ on } \ \partial\Omega, \tag{1.2}$$

and initial conditions

$$U(x,0) = U_0(x), \tag{1.3}$$

where $U = (u, v)$ is the unknown function, $U_0 = (u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ given, $G = (g_1, g_2) \in L^2(\Omega) \times L^2(\Omega)$, and $F : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following conditions.

If $\Omega$ is a bounded domain, we assume that

$$\max\Big\{\Big|\frac{\partial^2 F}{\partial u^2}\Big|, \Big|\frac{\partial^2 F}{\partial u \partial v}\Big|, \Big|\frac{\partial^2 F}{\partial v^2}\Big|\Big\} \leqslant c_0\Big(|u|^{\frac{2}{N-2}} + |v|^{\frac{2}{N-2}} + 1\Big) \tag{1.4}$$

$$F(u,v) \geqslant -\frac{\mu}{2}(u^2+v^2) - C_1 \tag{1.5}$$

$$U\nabla F(U) = u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} \geqslant -\mu(u^2+v^2) - C_2, \tag{1.6}$$

where $\mu < \lambda_1$, $\lambda_1$ is the first eigenvalue of $-\triangle$ in $\Omega$ with homogeneous Dirichlet condition, $C_1$ and $C_2$ are the nonegative constants.

In the case $\Omega$ is an unbouded domain, instead of the conditions (1.4) – (1.6), we suppose that

$$\max\Big\{\Big|\frac{\partial^2 F}{\partial u^2}\Big|, \Big|\frac{\partial^2 F}{\partial u \partial v}\Big|, \Big|\frac{\partial^2 F}{\partial v^2}\Big|\Big\} \leqslant c_0\Big(|u|^{\frac{2}{N-2}} + |v|^{\frac{2}{N-2}} + 1\Big) \tag{1.4'}$$

$$\lambda_1 = \inf\{\|\nabla u\|^2 : u \in H_0^1(\Omega), \|u\| = 1\} > 0 \tag{1.5'}$$

$$F(u,v) \geqslant -\frac{\mu}{2}(u^2+v^2), \ \mu < \lambda_1 \tag{1.5''}$$

$$F(0,0) = \frac{\partial F}{\partial u}(0,0) = \frac{\partial F}{\partial v}(0,0) = 0 \tag{1.5'''}$$

$$U\nabla F(U) = u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} \geqslant -\mu(u^2+v^2). \tag{1.6'}$$

Denote by $V$ the space $H_0^1(\Omega) \times H_0^1(\Omega)$, with the norm

$$\|U\|_V = \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\frac{1}{2}}, \ U = (u, v) \in V,$$

and by $H$ the space $L^2(\Omega) \times L^2(\Omega)$, with the norm

$$\|F\|_H = \left(\|f\|^2 + \|g\|^2\right)^{\frac{1}{2}}, \ F = (f, g) \in H.$$

Hereafter $\|.\|_p$ denotes the norm $\|.\|_{L^p(\Omega)}$ and in the case $p = 2$ we may omit the index. We introduce the operator $A = \big(-\triangle_D, -\triangle_D\big)$ with domain

$$D(A) = \{U \in V |\ AU \in H\} = (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)).$$

It is well-known that $A$ is a sectorial operator on $H$ with the fractional power spaces $X^\alpha$, in particular $X^0 = H, X^{1/2} = V$. Then problem (1.1) – (1.3) can be rewritten as an abstract evolutionary equation in $V$:

$$\frac{dU}{dt}(t) + A(U(t)) = -\nabla F(U) - G, \ \ U(0) = U_0 \in V. \tag{1.7}$$

Putting

$$\Phi(U) = \frac{1}{2}\|\nabla U\|^2 + \int\limits_\Omega F(U)dx + (G, U)_H. \tag{1.8}$$

The first purpose of this paper is to prove the solutions $U$ of problem (1.7) exists uniquely, globally and satisfies

$$\frac{d}{dt}\Phi(U(t)) = -\|U_t(t)\|^2. \tag{1.9}$$

Then if for each $U_0 = (u_0, v_0) \in V$, putting $S(t)U_0 = U(t)$, the unique solution with the initial data $U_0 \in V$, we obtain a strongly continuous semigroup $S(t), t \geqslant 0$, on $V$. The second purpose of this paper is to prove the existence of a compact connected global attractor $\mathcal{A}$ of the semigroup $S(t)$.

Let us describe organization and methods used in this paper. In Section 2, we recall some definitions and results of theory of infinite dimensional dissipative systems which we will use. Section 3 presents results on the existence of a classical solution $U(t)$ on $[0, +\infty)$ and the existence of a global attractor of (1.7) in a bounded domain with the hypotheses (1.4) - (1.6). Under the assumption (1.4), one can check that $\nabla F : V \to H$ is a locally Lipschitzian map. This guarantees the existence and uniqueness of a local solution. Then, by using condition (1.5) and the remarkable fact that the equation admits the natural Lyapunov functional (1.8), we are able to prove that the solution exists globally in time and satisfies the energy equation (1.9). Besides that, we also show that orbits of bounded sets are bounded. Finally, by proving the asymptotically compact property of the semigroup $S(t)$ and using the disspativeness condition (1.6) for proving the boundedness of the set $E$ of equilibrium points, we obtain the existence of a global attractor $\mathcal{A}$. In Section 4, we prove a similar result in the case of an unbounded domain with the hypotheses (1.4') - (1.6'). The significant difference between the case of a bounded domain and the case of an unbounded domain is that the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is no longer compact if $\Omega$ is unbounded, and this poses some additional difficulties for the proof of asymptotic compactness of $S(t)$. Thus, new ideas are needed to obtain the asymptotic compactness property.

One way to overcome these difficulties is using weighted Sobolev spaces. For example, Babin and Vishik [2] used this technique to prove the existence of a global attractor for the equation of the form

$$u_t + u - \triangle u = f(u) + g(x),\ x \in \mathbb{R}^N, t > 0,$$
$$u(0) = u_0, u_0 \in L^2(\mathbb{R}^N),$$

with $f$ satisfying $f(0) = 0, f(u)u \leqslant 0, f'(u) \leqslant l$, and the growth condition $|f'(u)| \leqslant c(1+|u|^\alpha), \alpha \geqslant 0$ if $N \leqslant 2, \alpha \leqslant \min\{2/(N-2), 4/N\}$ if $N \geqslant 3$. The choice of weighted spaces, however, imposes some severe conditions on the forcing term $g$ and on the initial data $u_0$. In this paper, we use the method of tail-estimates, introduced by Wang in [10], to establish the asymptotic compactness of the semigroup $S(t)$ in the space $H = L^2(\Omega) \times L^2(\Omega)$ (see Lemmas 4.1 - 4.3 below). Then we show that the solutions are actually asymptotically compact in the natural energy space $V = H_0^1(\Omega) \times H_0^1(\Omega)$ (see Lemmas 3.2 - 3.3). Finally, repeating the arguments as in the case of a bounded domain, we obtain the existence of a global attractor of $S(t)$ in $V$ in the case $\Omega$ is unbounded.

## 2. Preliminary Results

For convenience of the readers, we begin by summarizing some definitions and results in [3,4,8] which we will use.

### 2.1. *Existence of Global Attractors*

Let $X$ be a metric space (not necessarily complete) with metric $d$. If $C \subset X$ and $b \in X$ we set $\rho(b, C) := \inf_{c \in C} d(b, c)$, and if $B \subset X, C \subset X$ we set dist $(B, C) := \sup_{b \in B} \rho(b, C)$. Let $S(t)$ be a *continuous semigroup* on the metric space $X$.

A set $A \subset X$ is *positively invariant* if $S(t)A \subset A$, for any $t \geqslant 0$. The set $A$ is *invariant* if $S(t)A = A$, for any $t \geqslant 0$.

The positive orbit of $x \in X$ is the set $\gamma^+(x) = \{S(t)x | t \geqslant 0\}$. If $B \subset X$, the *positive orbit* of $B$ is the set

$$\gamma^+(B) = \cup_{t \geqslant 0} S(t)B = \cup_{z \in B} \gamma^+(z).$$

More generally, for $\tau \geqslant 0$, we define the orbit after the time $\tau$ of $B$ by

$$\gamma_\tau^+(B) = \gamma^+(S(\tau)B).$$

Let $\emptyset \neq A \subset X$. We define the $\omega-$ *limit set* $\omega(A)$ of $A$ as

$$\omega(A) = \cup_{s \geqslant 0} \overline{\gamma^+(S(s)A)}^X.$$

The subset $A \subset X$ *attracts* a set $B$ if dist $(S(t)B, A) \to 0$ as $t \to +\infty$.

The subset $A$ is a *global attractor* if $A$ is closed, bounded, invariant, and attracts all bounded sets.

The semigroup $S(t)$ is *asymptotically compact* if, for any bounded subset $B$ of $X$ such that $\gamma_\tau^+(B)$ is bounded for some $\tau \geqslant 0$, every set of the form $\{S(t_n)z_n\}$, with $z_n \in B$ and $t_n \geqslant \tau, t_n \to +\infty$ as $n \to +\infty$, is relatively compact.

A continuous semigroup $S(t)$ is a *continuous gradient system* if there exists a function $\Phi \in C^0(X, \mathbb{R})$ such that $\Phi(S(t)u) \leqslant \Phi(u), \forall t \geqslant 0, \forall u \in X$, and $\Phi(S(t)u) = \Phi(u), \forall t \geqslant 0$ implies that $u$ is an equilibrium point, i.e. $S(t)u = u \ \forall t \geqslant 0$. The function $\Phi$ is called a strict Lyapunov functional.

Let $E$ be the set of equilibrium points for the semigroup $S(t)$. We give the definition of the unstable set of $E$ by

$$W^u(E) = \{y \in X : S(-t)y \text{ is defined for } t \geqslant 0 \text{ and } S(-t)y \to E \text{ as } t \to \infty\}.$$

From Proposition 2.19 and Theorem 4.6 in [8], we have

**Theorem 2.1.** Let $S(t), t \geqslant 0$, be an asymptotically compact gradient system, which has the property that, for any bounded set $B \subset X$, there exist $\tau \geqslant 0$ such that $\gamma_\tau^+(B)$ is bounded. If the set of equilibrium points $E$

is bounded, then $S(t)$ has a compact global attractor $\mathcal{A}$ and $\mathcal{A} = W^u(E)$. Moreover, if $X$ is a Banach space, then $\mathcal{A}$ is connected.

If the global attractor $\mathcal{A}$ exists, then (see [3, pg.21]) it contains a *global minimal attractor* $\mathcal{M}$ which is defined as a minimal closed positively invariant set possesing the property

$$\lim_{t\to+\infty} \text{dist } (S(t)y, \mathcal{M}) = 0 \text{ for every } y \in X.$$

Moreover, if $\mathcal{M}$ is compact then it is invariant and $\mathcal{M} = \cup_{z\in V}\omega(z)$.

**2.2.** *Sectorial Evolutionary Equations*

Assume that $A$ is a sectorial operator on $X$ and there is an $\alpha \in [0,1)$ such that $f : X^\alpha \to X$ is locally Lipschitz continuous. Consider the equation

$$\frac{du}{dt} + Au = f(u), \quad t > 0, \quad u(0) = u_0 \in X^\alpha. \tag{2.1}$$

A solution of (2.1) on $[0,\tau)$ is a continuous function $u : [0,\tau) \to X^\alpha, u(0) = u_0$, such that $f(u(.)) : [0,\tau) \to X$ is a continuous function, $u(t) \in D(A)$ and $u$ satisfies (2.1) on $(0,\tau)$. One can show that the solutions of (2.1) coincide with those solutions of the integral equation

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(u(s))ds, 0 \leqslant t < \tau, \tag{2.2}$$

for which $u : [0,\tau) \to X^\alpha$ is continuous and $f(u(.)) : [0,\tau) \to X$ is continuous. We have the following ([4, Theorem 4.2.1])

**Theorem 2.2.** Under the above hypotheses on $A, f$, there is a unique classical solution $u \in C^0([0,t_{\max}), X^\alpha) \cap C^1((0,t_{\max}), X) \cap C^0((0,t_{\max})), D(A))$ of (2.1) on a maximal interval of existence $[0, t_{\max}(u_0))$. If $t_{\max}(u_0) < \infty$, then there is a sequence $t_n \to t^-_{\max}(u_0)$ such that $\|u(t_n)\|_\alpha \to \infty$.

## 3. Existence of Global Attractors in Bounded Domains

**Theorem 3.1.** Assume that $F$ satisfies the conditions $(1.4), (1.5)$. Then for any $U_0 = (u_0, v_0) \in V$, the problem (1.7) has a unique classical solution $U \in C^0([0,\infty), V) \cap C^1((0,\infty), H) \cap C^0((0,\infty), D(A))$. Moreover, for the classical solution $U$, $\Phi(U(t)) \in C^1((0,\infty))$ with

$$\frac{d}{dt}\Phi(U(t)) = -\|U_t(t)\|^2 = -\|u_t\|^2 - \|v_t\|^2, \quad t \in (0,\infty).$$

*Proof.* We divide the proof into three steps.

*Step 1*: We prove that $\nabla F$ is a map from $V$ into $H$, that is for any $(u,v) \in V$ then $\frac{\partial F}{\partial u}(u,v), \frac{\partial F}{\partial v}(u,v) \in L^2(\Omega)$. Indeed, we have

$$|\frac{\partial F}{\partial u}(u,v) - \frac{\partial F}{\partial v}(u,v)| = |\int\limits_0^u \frac{\partial^2 F}{\partial u^2}(t,v)dt| \leqslant c_0 \int\limits_0^{|u|} \Big(|t|^{\frac{2}{N-2}} + |v|^{\frac{2}{N-2}} + 1\Big)dt$$
$$\leqslant c_0\Big(|u|^{\frac{N}{N-2}} + |v|^{\frac{2}{N-2}}|u| + |u|\Big).$$

It implies that

$$|\frac{\partial F}{\partial u}(u,v)| \leqslant |\frac{\partial F}{\partial u}(0,v)| + c_0\Big(|u|^{\frac{N}{N-2}} + |v|^{\frac{2}{N-2}}|u| + |u|\Big).$$

Repeating the above argument, we obtain that

$$|\frac{\partial F}{\partial u}(0,v)| \leqslant |\frac{\partial F}{\partial u}(0,0)| + c_0 \int\limits_0^{|v|} \Big(|t|^{\frac{2}{N-2}} + 1\Big)dt$$
$$\leqslant |\frac{\partial F}{\partial u}(0,0)| + c_0\Big(|v|^{\frac{N}{N-2}} + |v|\Big).$$

Therefore, we have

$$|\frac{\partial F}{\partial u}(u,v)| \leqslant |\frac{\partial F}{\partial u}(0,0)| + c_0\Big(|u|^{\frac{N}{N-2}} + |v|^{\frac{N}{N-2}} + |v|^{\frac{2}{N-2}}|u| + |u| + |v|\Big).$$

Using the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$, we get

$$|\frac{\partial F}{\partial u}(u,v)|^2 \leqslant 2|\frac{\partial F}{\partial u}(0,0)|^2 + 50c_0^2\Big(|u|^{\frac{2N}{N-2}} + |v|^{\frac{2N}{N-2}} + |v|^{\frac{4}{N-2}}|u|^2 + |u|^2 + |v|^2\Big).$$

In order to prove $\frac{\partial F}{\partial u}(u,v) \in L^2(\Omega)$, it is sufficient to prove that

$$\int\limits_\Omega |v|^{\frac{4}{N-2}}|u|^2 dx < \infty.$$

Applying Holder's inequality, we get

$$\int\limits_\Omega |v|^{\frac{4}{N-2}}|u|^2 dx \leqslant \|v\|^{\frac{4}{N-2}}_{\frac{2N}{N-2}} \|u\|^2_{\frac{2N}{N-2}} < \infty$$

because $H_0^1(\Omega)$ is continuously embedded into $L^{\frac{2N}{N-2}}(\Omega)$. Analogously, we can prove that $\frac{\partial F}{\partial v}(u,v) \in L^2(\Omega)$. Thus we can rewrite (1.1) - (1.3) as the abstract evolutionary equation (1.7) in $V$.

*Step 2*: We prove that the map $\nabla F : V \to H$ is Lipschitzian on bounded sets of $V$, that is

$$\|\nabla F(u_1, v_1) - \nabla F(u_2, v_2)\|_H \leqslant M(\rho)\|(u_1, v_1) - (u_2, v_2)\|_V, \tag{3.1}$$

for all $u = (u_1, u_2), v = (v_1, v_2) \in V$ provided $\|\nabla u_i\|, \|\nabla v_i\| \leqslant \rho,\ i = 1, 2$. Then, by Theorem 2.2, there exists $t_{\max} = t_{\max}(r) > 0$ such that, for any $U_0 \in V$, with $\|U_0\|_V \leqslant r$, equation (1.7) has a unique classical solution $U \in C^0([0, t_{\max}), V) \cap C^1((0, t_{\max}), H) \cap C^0((0, t_{\max}), D(A))$.

Firstly, note that

$$\begin{aligned}&\|\nabla F(u_1, v_1) - \nabla F(u_2, v_2)\|_H^2 \\ &= \|\frac{\partial F}{\partial u}(u_1, v_1) - \frac{\partial F}{\partial u}(u_2, v_2)\|^2 + \|\frac{\partial F}{\partial v}(u_1, v_1) - \frac{\partial F}{\partial v}(u_2, v_2)\|^2.\end{aligned}$$

Using Lagrange formula, we have

$$\begin{aligned}&|\frac{\partial F}{\partial u}(u_2, v_2) - \frac{\partial F}{\partial u}(u_1, v_1)| = |(u_2 - u_1)\frac{\partial^2 F}{\partial u^2}(u_1 + \theta(u_2 - u_1), v_1 + \theta(v_2 - v_1)) \\ &+ (v_2 - v_1)\frac{\partial^2 F}{\partial u \partial v}(u_1 + \theta(u_2 - u_1), v_1 + \theta(v_2 - v_1))|,\ 0 < \theta < 1 \\ &\leqslant c_0(|u_2 - u_1| + |v_2 - v_1|)\big[1 + |u_1 + \theta(u_2 - u_1)|^{\frac{2}{N-2}} + |v_1 + \theta(v_2 - v_1)|^{\frac{2}{N-2}}\big] \\ &\leqslant C(|u_1 - u_2| + |v_1 - v_2|)(1 + |u_1|^{\frac{2}{N-2}} + |u_2|^{\frac{2}{N-2}} + |v_1|^{\frac{2}{N-2}} + |v_2|^{\frac{2}{N-2}}).\end{aligned}$$

This implies that

$$\begin{aligned}&|\frac{\partial F}{\partial u}(u_1, v_1) - \frac{\partial F}{\partial u}(u_2, v_2)|^2 \\ &\leqslant C(|u_1 - u_2|^2 + |v_1 - v_2|^2)(1 + |u_1|^{\frac{4}{N-2}} + |u_2|^{\frac{4}{N-2}} + |v_1|^{\frac{4}{N-2}} + |v_2|^{\frac{4}{N-2}}).\end{aligned}$$

Therefore

$$\begin{aligned}&\|\frac{\partial F}{\partial u}(u_1, v_1) - \frac{\partial F}{\partial u}(u_2, v_2)\|^2 \\ &\leqslant C\Big[\|u - v\|_V^2 + \int_\Omega (|u_1|^{\frac{4}{N-2}} + |u_2|^{\frac{4}{N-2}})(|u_1 - u_2|^2 + |v_1 - v_2|^2)dx \\ &\quad + \int_\Omega (|v_1|^{\frac{4}{N-2}} + |v_2|^{\frac{4}{N-2}})(|u_1 - u_2|^2 + |v_1 - v_2|^2)dx\Big].\end{aligned}$$

Applying Holder's inequality, we obtain

$$\begin{aligned}&\int_\Omega (|u_1|^{\frac{4}{N-2}} + |u_2|^{\frac{4}{N-2}})(|u_1 - u_2|^2 + |v_1 - v_2|^2)dx \\ &\leqslant (\|u_2 - u_1\|^2_{\frac{2N}{N-2}} + \|v_2 - v_1\|^2_{\frac{2N}{N-2}})(\|u_1\|^{\frac{4}{N-2}}_{\frac{2N}{N-2}} + \|u_2\|^{\frac{4}{N-2}}_{\frac{2N}{N-2}}),\end{aligned}$$

and

$$\int_\Omega (|v_1|^{\frac{4}{N-2}} + |v_2|^{\frac{4}{N-2}})(|u_1-u_2|^2 + |v_1-v_2|^2)dx$$
$$\leqslant (\|u_2-u_1\|^2_{\frac{2N}{N-2}} + \|v_2-v_1\|^2_{\frac{2N}{N-2}})(\|v_1\|^{\frac{4}{N-2}}_{\frac{2N}{N-2}} + \|v_2\|^{\frac{4}{N-2}}_{\frac{2N}{N-2}}).$$

Since $H_0^1(\Omega)$ is continuously embedded into $L^{\frac{2N}{N-2}}(\Omega)$, we get

$$\|\frac{\partial F}{\partial u}(u_1,v_1) - \frac{\partial F}{\partial u}(u_2,v_2)\|^2 \leqslant M_1(\rho)\|u-v\|_V^2 \quad \text{if } \|\nabla u_i\|, \|\nabla v_i\| \leqslant \rho.$$

Analogously, we have

$$\|\frac{\partial F}{\partial v}(u_1,v_1) - \frac{\partial F}{\partial v}(u_2,v_2)\|^2 \leqslant M_1(\rho)\|u-v\|_V^2 \quad \text{if } \|\nabla u_i\|, \|\nabla v_i\| \leqslant \rho.$$

Thus, (3.1) is proved.

*Step 3:* We prove that $t_{\max} = +\infty$. Putting

$$\Phi(U) = \Phi(u,v) = \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\nabla v\|^2 + \int_\Omega F(u,v)dx + (G,U)_H.$$

By computing directly, one can check that $\Phi(U(t)) \in C^1((0,\infty))$ and

$$\frac{d}{dt}\Phi(U(t)) = -\|u_t\|^2 - \|v_t\|^2, \;\; t\in[0,t_{\max}).$$

Using the hypothesis (1.5), we get

$$\Phi(U(t)) \geqslant \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\nabla v\|^2 - \frac{\mu}{2}(\|u\|^2 + \|v\|^2) + (g_1,u) + (g_2,v) - C_1|\Omega|.$$

Furthermore, from inequalities

$$\|w\|^2 \leqslant \frac{1}{\lambda_1}\|\nabla w\|^2, \;\; \forall w\in H_0^1(\Omega),$$

$$\frac{\epsilon}{2}\|w\|^2 + \frac{1}{2\epsilon}\|g\|^2 \geqslant (g,w), \quad \forall w,g\in L^2(\Omega), \epsilon>0,$$

we have

$$\begin{aligned}\Phi(U_0) \geqslant \Phi(U(t)) &\geqslant \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\nabla v\|^2 - \frac{\mu}{2\lambda_1}\Big(\|\nabla u\|^2 + \|\nabla v\|^2\Big) \\ &\quad + (g_1,u) + (g_2,v) - C_1|\Omega| \\ &\geqslant \frac{1}{2}(1-\frac{\mu+\epsilon}{\lambda_1})\Big(\|\nabla u\|^2 + \|\nabla v\|^2\Big) - \frac{1}{2\epsilon}\|G\|_H^2 - C_1|\Omega|.\end{aligned}$$

Choose $\epsilon > 0$ being small enough, we get

$$\|U(t)\|_V = \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{\frac{1}{2}} \leqslant M, \ \ \forall t \in [0, t_{\max}),$$

where $M$ is a constant independent of $t$. This implies that $t_{\max} = +\infty$. Indeed, let $t_{\max} < +\infty$ and $\limsup_{t \to t_{\max}^-} \|U(t)\|_V < +\infty$. Then there exists a sequence $(t_n)_{n \geqslant 1}$ and a constant $K$ such that $t_n \to t_{\max}^-$, as $n \to +\infty$, and $\|U(t_n)\|_V < K,\ n = 1, 2, \ldots.$ As we have already shown above, for each $n \in \mathbb{N}$ there exists a unique solution of the problem (1.7) with initial data $U(t_n)$ on $[t_n, t_n + T^*]$, where $T^* > 0$ depending on $K$ and independent of $n \in \mathbb{N}$. Thus, we can get $t_{\max} < t_n + T^*$, for $n \in \mathbb{N}$ large enough. This contradicts the maximality of $t_{\max}$ and the proof of Theorem 3.1 is completed.

**Remark 3.1.** From the proof of Theorem 3.1, we see that the function $\Phi$ is a strict Lyapunov functional and is bounded below. Moreover, using Taylor fomular

$$\begin{aligned} F(u,v) &= F(0,0) + u\frac{\partial F}{\partial u}(0,0) + v\frac{\partial F}{\partial v}(0,0) \\ &+ \frac{1}{2}\Big[u^2\frac{\partial^2 F}{\partial u^2}(\theta u, \theta v) + 2uv\frac{\partial^2 F}{\partial u \partial v}(\theta u, \theta v) + v^2\frac{\partial^2 F}{\partial v^2}(\theta u, \theta v)\Big],\ 0 < \theta < 1, \end{aligned}$$

we easily prove that: For all $U_0$, with $\|U_0\|_V \leqslant R$, there exists a number $M > 0$ only depending on $R$ such that $\|U(t)\|_V \leqslant M,\ \forall t \geqslant 0$. In other words, orbits of bounded sets are bounded.

In order to prove the asymptotic compactness of the semigroup $S(t)$, we first note that $A$ is a sectorial operator in the space $X = H$ with the fractional power spaces $X^\alpha$. From the properties of a sectorial operator and the fact that $X^{1/2} = V,\ X^0 = H$ (see e.g. [5]), we have the analytic semigroup $e^{-tA}$ generated by the operator $A$ satisfying the following estimates

$$\begin{aligned} &\|e^{-At}u\| \leqslant Me^{at}\|u\|, \text{ for all } \ u \in H \text{ and all } t > 0, \\ &\|e^{-At}u\|_V \leqslant Me^{at}t^{-1/2}\|u\|, \text{ for all } \ u \in H \text{ and all } t > 0, \end{aligned}$$

where $M$ and $a$ are two positive constants. Furthermore, we need the following lemma (see the book by Henry [5], Chapter 7).

**Lemma 3.1.** Assume that $\varphi(t)$ is a continuous nonegative function on the interval $(0, T)$ such that

$$\varphi(t) \leqslant c_0 t^{-\gamma_0} + c_1 \int_0^t (t-s)^{-\gamma_1} \varphi(s) ds, \ t \in (0, T),$$

where $c_0, c_1 \geqslant 0$ and $0 \leqslant \gamma_0, \gamma_1 < 1$. Then there exists a constant $K = K(\gamma_1, c_1, T)$ such that

$$\varphi(t) \leqslant \frac{c_0}{1-\gamma_0} t^{-\gamma_0} K(\gamma_1, c_1, T), \ t \in (0, T).$$

**Lemma 3.2.** Let $(U_n)_{n\in\mathbb{N}}$ be a sequence in $V$, let $U \in V$, and assume that $U_n \to U$ in $H$. Then $S(t)U_n \to S(t)U$ in $V$, uniformly on $[t_0, t_1]$, for all $t_1 > t_0 > 0$.

*Proof.* Let $t_1 > 0$ be fixed. Since the set $\{U_n | n \in \mathbb{N}\} \cup \{U\}$ is bounded in $V$, there exists a positive number $R$ such that $\|S(t)U_n\|_V \leqslant R$ and $\|S(t)U\|_V \leqslant R$, for all $t \in [0, t_1]$ and for all $n \in \mathbb{N}$. Let $L$ be the Lipschitz constant for $\nabla F$ on the ball of radius $R$ in $V$. Write $U_n(t) := S(t)U_n$ and $U(t) := S(t)U$. Set $f := -\nabla F - G$. For $t \in [0, t_1]$, we have

$$U_n(t) = e^{-At} U_n + \int_0^t e^{-A(t-s)} f(U_n(s)) ds,$$

$$U(t) = e^{-At} U + \int_0^t e^{-A(t-s)} f(U(s)) ds.$$

It follows that, for all $t \in (0, t_1]$,

$$\|U_n(t) - U(t)\|_V \leqslant M e^{at_1} t^{-\frac{1}{2}} \|U_n - U\|_H + MLe^{at_1} \int_0^t (t-s)^{-\frac{1}{2}} \|U_n(s) - U(s)\|_V ds.$$

By the singular Gronwall's inequality (see Lemma 3.1), there is a constant $C_1 = C_1(L, M, a, t_1)$ such that for all $t \in (0, t_1]$,

$$\|U_n(t) - U(t)\|_V \leqslant 2Me^{at_1} t^{-1/2} C_1 \|U_n - U\|_H.$$

It implies that $S(t)U_n \to S(t)U$ in $V$ as $n \to \infty$, and uniformly on $[t_0, t_1]$ for all $t_1 > t_0 > 0$. This completes the proof.

The next lemma shows that, thanks to Lemma 3.2, the asymptotic compactness of $S(t)$ in $V$.

**Lemma 3.3.** Let $(U_n)_{n\in\mathbb{N}}$ be a bounded sequence in $V$ and let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers, $t_n \to +\infty$ as $n \to \infty$. Then there exists a strictly increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ and a function $U \in V$ such that $S(t_{n_k})U_{n_k} \to U$ in $V$ as $k \to \infty$. In other words, $S(t)$ is asymptotically compact in the strong $V$ topology.

*Proof.* Fix any positive $T$. Since $t_n \to \infty$ as $n \to \infty$, we have $t_n > T$ for all $n \geqslant n_0$. Because $(U_n)$ is bounded in $V$, $\{S(t_n - T)U_n\}_{n\geqslant n_0}$ is bounded in $V$. Then there exists a strictly increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}, n_k \geqslant n_0$ for all $k \in \mathbb{N}$, and a function $v \in V$ such that $S(t_{n_k} - T)U_{n_k} \rightharpoonup v$ in $V$ as $k \to \infty$. On the other hand, since $\Omega$ is bounded, the embedding $V \hookrightarrow H$ is compact. Thus, $\{S(t_n - T)U_n\}$ is relatively compact in $H$. We can therefore choose the sequence $(n_k)_{k\in\mathbb{N}}$ such that $S(t_{n_k} - T)U_{n_k} \to v$ in $H$. By Lemma 3.2, we have

$$S(t_{n_k})U_{n_k} = S(T) \circ S(t_{n_k} - T)U_{n_k} \to S(T)v = U \text{ in } V \text{ as } k \to \infty.$$

The proof is complete.

We can now state one of the main results of this paper.

**Theorem 3.2.** Under the conditions (1.4) - (1.6), the semigroup $S(t), t \geqslant 0$, generated by (1.7) has a compact connected global attractor $\mathcal{A} = W^u(E)$ in the space $V = H_0^1(\Omega) \times H_0^1(\Omega)$.

*Proof.* Firstly, from Remark 3.1 we see that for any bounded set $B \subset X$, $\gamma^+(B)$ is bounded, and the function $\Phi$ defined by (1.8) is a strict Lyapunov funcional. We now prove that the set $E$ of the equilibrium points of (1.7) is bounded. Notice that

$$E = \{z = (u, v) \in D(A) | Az + \nabla F(z) + G = 0\},$$

so if $z = (u, v)$ is an equilibrium point, we have

$$(-\Delta z, z)_H + (\nabla F(z), z)_H + (G, z)_H = 0.$$

Hence and from (1.6) it follows that

$$\begin{aligned}
0 &= \|\nabla z\|_H^2 + (\nabla F(z), z)_H + (G, z)_H \\
&\geqslant \|\nabla z\|_H^2 - \mu\|z\|_H^2 - C_2|\Omega| - \frac{\epsilon}{2}\|z\|_H^2 - \frac{1}{2\epsilon}\|G\|_H^2 \\
&\geqslant (1 - \frac{\mu + \epsilon/2}{\lambda_1})\|\nabla z\|_H^2 - \frac{1}{2\epsilon}\|G\|_H^2 - C_2|\Omega|,
\end{aligned}$$

where $\epsilon$ is chosen such that $1 > \frac{\mu+\epsilon/2}{\lambda_1}$. This implies that the set $E$ of the equilibrium points of (1.7) is bounded in $V$. On the other hand, by Lemma 3.3, $S(t)$ is asymptotically compact. Applying Theorem 2.1, one obtains the conclusion of the theorem.

The following proposition describes the asymptotic behavior of solutions of (1.7) as $t \to \infty$.

**Propositions 3.1.** Under the conditions (1.4) - (1.6), the semigroup $S(t), t \geqslant 0$, generated by (1.7) has a global minimal attractor $\mathcal{M}$, given by $\mathcal{M} = E$, in the space $V = H_0^1(\Omega) \times H_0^1(\Omega)$. Consequently, we have

$$\lim_{t\to+\infty} \operatorname{dist}(S(t)y, E) = 0 \text{ for every } y \in V.$$

*Proof.* The existence of the global minimal attractor $\mathcal{M}$ follows immediately from the fact that the semigroup $S(t)$ has a compact global attractor (see Sec. 2.1). We will show that $\mathcal{M} = E$.

It is obvious that $E \subset \mathcal{M}$. We now prove that $\mathcal{M} \subset E$. Indeed, since $\mathcal{M} = \cup_{z\in V}\omega(z)$, it suffices to show that $\omega(z) \subset E$, for any $z \in V$. Taking $a \in \omega(z)$ arbitrarily, by the Characterization Lemma (see e.g. [8, pg.893]), there exists a real sequence $(t_n), t_n \to +\infty$, such that $S(t_n)z = u(t_n) \to a$. Since the Lyapunov functional $\Phi$ is bounded below, it implies that

$$\Phi(a) = \lim_{t\to+\infty} \Phi(u(t_n)) = \inf\{\Phi(S(t)z) = \Phi(u(t))|t \geqslant 0\},$$

i.e. $\Phi$ =const on $\omega(z)$. Therefore, by the nonincreasing property of Lyapunov function along the orbit $S(t)z$ and the positively invariant property of $\omega(z)$, we conclude that $a \in E$. This completes the proof.

## 4. Existence of Global Attractors in Unbounded Domains

Using the conditions (1.4') - (1.5"') and repeating the arguments used in the proof of Theorem 3.1, we obtain the existence and uniqueness of the global solution for problem (1.1) - (1.3) in an unbounded domain.

**Theorem 4.1.** Assume that $F$ satisfies the conditions (1.4') - (1.5"'). Then for any $U_0 = (u_0, v_0) \in V$, the problem (1.7) has a unique classical solution $U \in C^0([0,\infty), V) \cap C^1((0,\infty), H) \cap C^0((0,\infty), D(A))$. Moreover, for the classical solution $U, \Phi(U(t)) \in C^1((0,\infty))$ with

$$\frac{d}{dt}\Phi(U(t)) = -\|U_t(t)\|^2 = -\|u_t\|^2 - \|v_t\|^2, \quad t \in (0,\infty).$$

If $\Omega$ is an unbounded domain, we still have the Lyapunov functional $\Phi(U(t))$, likely in the case of a bounded domain. This combining with the conditions (1.5') - (1.6') implies that orbits of bounded sets are bounded, and the set $E$ of the equilibrium points is bounded in $V$. Thus, in order to prove the existence of global attractor $\mathcal{A}$, we only have to prove that the semigroup $S(t)$ is asymptotically compact in $V$.

Firstly, in Lemma 4.1 below, we prove a tail-estimate of solutions. Then, using this estimate, we prove the asymptotic compactness of $S(t)$ in $H$. Finally, using Lemmas 3.2 - 3.3 in Section 3, we obtain the asymptotic compactness of $S(t)$ in $V$.

**Lemma 4.1.** Let $\bar{\vartheta} : \mathbb{R} \longrightarrow [0,1]$ be a function of class $C^1$ with $\bar{\vartheta}(s) = 0$ for $s \in (-\infty, 1]$ and $\bar{\vartheta}(s) = 1$ for $s \in [2, \infty)$. Putting $\vartheta = \bar{\vartheta}^2$. For $k \in \mathbb{N}$, define functions $\bar{\vartheta}_k : \mathbb{R}^N \longrightarrow \mathbb{R}$ and $\vartheta_k : \mathbb{R}^N \longrightarrow \mathbb{R}$ by

$$\bar{\vartheta}_k(x) = \bar{\vartheta}(\frac{|x|^2)}{k^2}) \text{ and } \vartheta_k(x) = \vartheta(\frac{|x|^2}{k^2}).$$

Set $C_\vartheta = 2\sqrt{2}\sup_{y\in\mathbb{R}}|\vartheta'(y)|$ and $C_{\bar{\vartheta}} = 2\sqrt{2}\sup_{y\in\mathbb{R}}|\bar{\vartheta}'(y)|$. Let $q \in (0, \lambda_1 - \mu)$ be arbitrary, $\mu$ is the constant in $(1.5'')$ and $(1.6')$. For each $k \in \mathbb{N}$, denote $b_k = \max(C_{\bar{\vartheta}}^2 k^{-2},\ 2C_{\bar{\vartheta}}k^{-1}, C_\vartheta k^{-1})$, $c_k = \frac{1}{2\epsilon}\|\bar{\vartheta}_k G\|_H^2$ $(0 < \epsilon < 2(\lambda_1 - \mu - q))$. Then for any $R \geqslant 0, \tau \in (0,\infty)$, and $U : [0,\infty) \longrightarrow V$ is a positive semi-trajectory such that $\|U(t)\|_V \leqslant R, \forall t \in [0,\tau]$, we have

$$\int_\Omega \vartheta_k(x)|U(t,x)|^2 dx \leqslant R^2 e^{-2qt} + (Cb_k R^2 + c_k)/q, \quad t \in [0,\tau]. \tag{4.1}$$

*Proof.* Since $\nabla\vartheta_k(x) = \frac{2}{k^2}\vartheta'(\frac{|x|^2}{k^2})x$ and $\nabla\bar{\vartheta}_k(x) = \frac{2}{k^2}\bar{\vartheta}'(\frac{|x|^2}{k^2})x$, we have

$$\sup_{x\in\Omega}|\nabla\vartheta_k(x)| \leqslant C_\vartheta/k \quad \text{and} \quad \sup_{x\in\Omega}|\nabla\bar{\vartheta}_k(x)| \leqslant C_{\bar{\vartheta}}/k.$$

Firstly, we will prove that for all $v = (v_1, v_2) \in V,\ q \in (0, \lambda_1)$,

$$\begin{aligned} &-(v, \vartheta_k v)_V + q\int_\Omega \vartheta_k(x)|v(x)|^2 - (G, \vartheta_k v)_H \\ &\leqslant b_k \int_\Omega \{|v_1|^2 + |v_2|^2 + 2|v_1||\nabla v_1| + 2|v_2||\nabla v_2|\}dx + \frac{1}{2\epsilon}\|\bar{\vartheta}_k G\|_H^2. \end{aligned} \tag{4.2}$$

Since

$$\begin{aligned} &-(v, \vartheta_k v)_V + q\int_\Omega \vartheta_k(x)|v(x)|^2 - (G, \vartheta_k v)_H \\ =&-(v_1, \vartheta_k v_1)_{H_0^1} - (v_2, \vartheta_k v_2)_{H_0^1} + q\int_\Omega \vartheta_k(x)|v_1|^2 dx + q\int_\Omega \vartheta_k(x)|v_2|^2 dx \\ &-(g_1, \vartheta_k v_1) - (g_2, \vartheta_k v_2), \end{aligned}$$

we now prove that

$$-(v_1, \vartheta_k v_1)_{H_0^1} + q\int_\Omega \vartheta_k(x)|v_1|^2 dx \leqslant b_k \int_\Omega \{|v_1|^2 + 2|v_1||\nabla v_1|\}dx + \frac{1}{2\epsilon}\|\bar{\vartheta}_k g_1\|^2. \tag{4.3}$$

Indeed, we have

$$\begin{aligned}
&-(v_1, \vartheta_k v_1)_{H_0^1} + q\int_\Omega \vartheta_k v_1^2 dx - (g_1, \vartheta_k v_1)\\
=& -\int_\Omega \nabla v_1 \nabla(\vartheta_k v_1)dx + q\int_\Omega \vartheta_k v_1^2 dx - (g_1, \vartheta_k v_1)\\
=& -\int_\Omega |\nabla(\bar{\vartheta}_k v_1)|^2 dx + q\int_\Omega (\bar{\vartheta}_k v_1)^2 dx - (g_1, \vartheta_k v_1) + \int_\Omega \big[v_1^2(\nabla\bar{\vartheta}_k)(\nabla\bar{\vartheta}_k)\\
&\quad + 2v_1\bar{\vartheta}_k(\nabla\bar{\vartheta}_k)(\nabla v_1) - v_1(\nabla\vartheta_k)\nabla v_1\big]dx\\
\leqslant& -\|\nabla(\bar{\vartheta}_k v_1)\|^2 + \frac{q}{\lambda_1}\|\nabla(\bar{\vartheta}_k v_1)\|^2 + \frac{\epsilon}{2}\|\bar{\vartheta}_k v_1\|^2 + \frac{1}{2\epsilon}\|\bar{\vartheta}_k g_1\|^2\\
&+ \int_\Omega \{(C_{\bar{\vartheta}}^2 k^{-2})|v_1|^2 + (2C_{\bar{\vartheta}}k^{-1})|v_1||\nabla v_1| + (C_\vartheta k^{-1})|v_1||\nabla v_1|)\}dx.\\
\leqslant& b_k\int_\Omega \{|v_1|^2 + 2|v_1||\nabla v_1|\}dx - (1 - \frac{q+\epsilon/2}{\lambda_1})\|\nabla(\bar{\vartheta}_k v_1)\|^2 + \frac{1}{2\epsilon}\|\bar{\vartheta}_k g_1\|^2\\
\leqslant& b_k\int_\Omega \{|v_1|^2 + 2|v_1||\nabla v_1|\}dx + \frac{1}{2\epsilon}\|\bar{\vartheta}_k g_1\|^2,
\end{aligned}$$

where $0 < \epsilon < 2(\lambda_1 - q)$. Therefore (4.2) is proved. Similarly, we have

$$-(v_2, \vartheta_k v_2)_{H_0^1} + q\int_\Omega \vartheta_k(x)|v_2|^2 dx \leqslant b_k \int_\Omega \{|v_2|^2 + 2|v_2||\nabla v_2|\}dx + \frac{1}{2\epsilon}\|\bar{\vartheta}_k g_2\|^2. \tag{4.4}$$

From (4.3) and (4.4) we get (4.2). Since $\|v\|_H{}^2 \leqslant \frac{1}{\lambda_1^2}\|v\|_V^2$, $(v, \nabla v)_H \leqslant \frac{1}{\lambda_1}\|v\|_V^2$, and (4.2), we have

$$-(v, \vartheta_k v)_V + q\int_\Omega \vartheta_k(x)|v(x)|^2 \leqslant C b_k R^2 + \frac{1}{2\epsilon}\|\bar{\vartheta}_k G\|^2, \tag{4.5}$$

for all $0 < q < \lambda_1, 0 < \epsilon < 2(\lambda_1 - q)$. Here $C$ is a constant only depending on $\Omega$.

Let $V_k : V \longrightarrow \mathbb{R}$ be a function defined by

$$V_k(U) = \frac{1}{2}\int_\Omega \vartheta_k(x)|U(x)|^2 dx, \quad U \in V.$$

Then $V_k$ is Fréchet differentiable and

$$(V_k(U))'(v) = \int_\Omega \vartheta_k(x)U(x)v(x)dx, \quad U, v \in V.$$

Assume that $U : [0,\infty) \longrightarrow V$ is a positive semi-trajectory of the semigroup $S(t)$. It is easy to check that

$$(V_k \circ U)'(t) = \int_\Omega \vartheta_k(x) U(t,x)(U'(t))(x)dx = (U_t(t), \vartheta_k U(t))_H.$$

Let $q \in (0, \lambda_1 - \mu)$ and $\epsilon > 0$ be chosen such that $q + \mu + \epsilon/2 < \lambda_1$. Set $c_k = \frac{1}{2\epsilon}\|\bar{\vartheta}_k G\|_H^2$, we have

$$\begin{aligned}
&(V_k \circ U)'(t) + 2q(V_k \circ U)(t) \\
=\,& (U_t(t), \vartheta_k U(t))_H + 2q(V_k \circ U)(t) \\
=\,& -(U(t), \vartheta_k U(t))_V + q\int_\Omega \vartheta_k(x)|U(t)|^2 dx - (G, \vartheta_k U(t))_H \\
&- \int_\Omega \nabla F(U(t))(x)\vartheta_k(x)U(t,x)dx \\
\leqslant\,& -(U(t), \vartheta_k U(t))_V + (q+\mu)\int_\Omega \vartheta_k(x)|U(t)|^2 dx - (G, \vartheta_k U(t))_H \\
&\quad (\text{since } U\nabla F(U) \geqslant -\mu U^2) \\
\leqslant\,& C b_k R^2 + c_k.
\end{aligned}$$

Hence it follows that

$$(V_k \circ U)'(t) + 2q(V_k \circ U)(t) \leqslant C b_k R^2 + c_k, \quad \forall t \in [0,\tau].$$

Therefore

$$V_k(U(t)) \leqslant e^{-2qt} V_k(U(0)) + \frac{C b_k R^2 + c_k}{2q},\ t \in [0,\tau].$$

Lemma 4.1 is proved.

**Lemma 4.2.** If $a \in C_0^1(\mathbb{R}^N)$, then the map $h : H_0^1(\Omega) \longrightarrow L^2(\Omega), u \longmapsto a|_\Omega.u$, is defined, linear and compact.

*Proof.* Since $a \in C_0^1(\mathbb{R}^N)$, there exists a ball $U$ such that $\operatorname{supp} a \subset U$. We write $h = h_5 \circ h_4 \circ h_3 \circ h_2 \circ h_1$, where
$h_1 : H_0^1(\Omega) \longrightarrow H_0^1(\mathbb{R}^N), u \longmapsto \bar{u}$; $\quad h_2 : H_0^1(\mathbb{R}^N) \longrightarrow H_0^1(U), v \longmapsto (av)|_U$;
$h_3 : H_0^1(U) \longrightarrow L^2(U), v \longmapsto v$; $\quad h_4 : L^2(U) \longrightarrow L^2(\mathbb{R}^N), v \longmapsto \bar{v}$;
$h_5 : L^2(\mathbb{R}^N) \longrightarrow L^2(\Omega), v \longmapsto v|_\Omega$.
It is clear that the maps $h_1, h_4$ and $h_5$ are defined, linear and bounded. We can easily prove $h_2$ is also defined, linear and bounded. Since $H_0^1(U)$ is compactly embedded into $L^2(U)$, $h_3$ is defined, linear, and compact. This implies that $h$ is a compact linear map.

**Lemma 4.3.** Let $(U_n)$ be bounded in V, $(t_n)$ be a number sequence such that $\lim_{n\to\infty} t_n = +\infty$. Then there exists a strictly increase sequence $(n_k)$ and $U \in V$ such that $\lim_{n\to\infty} S(t_{n_k})U_{n_k} = U$ in $H$. In other words, $S(t)$ is asymptotically compact in the strong $H$ topology.

*Proof.* Since orbits of bounded sets are bounded, we have

$$\|S(t)U_n\|_V \leqslant R, \ \forall t \geqslant 0, \ \forall n \geqslant 1.$$

Put $\alpha$ is the Kuratowski measure of non-compactness on $H = L^2(\Omega) \times L^2(\Omega)$

$$\alpha(B) = \inf\{l > 0 : B \text{ admits a finite cover by sets of diameter } \leqslant l\},$$

where $B$ is a bounded subset of $H$. Using properties of this measure (see e.g. [8, pg. 910-911]), we have for every $k, n_0$,

$$\begin{aligned}\alpha\{U_n(t_n)|n \in \mathbb{N}\} &\leqslant \alpha\{(1-\vartheta_k)U_n(t_n)|n \in \mathbb{N}\} + \alpha\{\vartheta_k U_n(t_n)|n \in \mathbb{N}\} \\ &= \alpha\{(1-\vartheta_k)U_n(t_n)|n \in \mathbb{N}\} + \alpha\{\vartheta_k U_n(t_n)|n \geqslant n_0\},\end{aligned}$$

where $U_n(t_n) := S(t_n)U_n$. Using (4.1) with noting that $t_n \to \infty$ as $n \to \infty$ and $b_k \to 0$ as $k \to \infty$, we have for any $\epsilon > 0$, there exist $k, n_0$ such that $\|\vartheta_k U_n(t_n)\|_H < \epsilon, \forall n \geqslant n_0$. Thus

$$\alpha\{U_n(t_n)|n \in \mathbb{N}\} \leqslant \alpha\{(1-\vartheta_k)U_n(t_n)|n \in \mathbb{N}\} + \epsilon.$$

Since $1-\vartheta_k \in C_0^1(\mathbb{R}^N)$, using Lemma 4.2, we have the map $U \longmapsto (1-\vartheta_k)U$ is compact from $V = H_0^1(\Omega) \times H_0^1(\Omega)$ to $H = L^2(\Omega) \times L^2(\Omega)$. Hence

$$\alpha\{(1-\vartheta_k)U_n(t_n)|n \in \mathbb{N}\} = 0.$$

We therefore obtain that $\alpha\{U_n(t_n)|n \in \mathbb{N}\} = 0$, i.e., $\{U_n(t_n)\}$ is precompact on $H$. Lemma 4.3 is proved.

From Lemma 4.3 and Lemmas 3.2 - 3.3 in Section 3, we obtain the asymptotic compactness of $S(t)$ in $V$. After that, repeating the proofs of Theorem 3.2 and Proposition 3.1, we obtain

**Theorem 4.2.** Under the conditions (1.4') - (1.6'), the semigroup $S(t), t \geqslant 0$, generated by (1.7) has a compact connected global attractor $\mathcal{A} = W^u(E)$ in the space $V = H_0^1(\Omega) \times H_0^1(\Omega)$.

**Proposition 4.1.** Under the conditions (1.4') - (1.6'), the semigroup $S(t), t \geqslant 0$, generated by (1.7) has a global minimal attractor $\mathcal{M}$, given by $\mathcal{M} = E$, in the space $V = H_0^1(\Omega) \times H_0^1(\Omega)$. Consequently, we have

$$\lim_{t \to +\infty} \mathrm{dist}(S(t)y, E) = 0 \ \text{ for every } \ y \in V.$$

## 5. Some Remarks

**5.1.** All the results of this paper remain true if we replace Dirichlet conditions (1.2) by homogeneous Neumann conditions, or by mixed boundary conditions (Robin conditions). Furthermore, we can replace the Laplacian operator by any second order operator $\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) + a_0(x)$, where $a_{ij}, a_0$ are smooth enough functions of $x$ and the matrix $[a_{ij}]_{i,j}$ is symmetric, positive definite, for all $x \in \overline{\Omega}$.

**5.2.** In this paper we are restricted in the case $N \geqslant 3$ for only the clarity of the presentation. The results of this paper remain true (even in the case $N = 1$ or $N = 2$) if replacing the condition (1.4) (or (1.4')) by the following condition

$$\max\Big\{\Big|\frac{\partial^2 F}{\partial u^2}\Big|, \Big|\frac{\partial^2 F}{\partial u \partial v}\Big|, \Big|\frac{\partial^2 F}{\partial v^2}\Big|\Big\} \leqslant c_0\Big(|u|^\alpha + |v|^\alpha + 1\Big),$$

where $0 < \alpha \leqslant \frac{2}{N-2}$ if $N \geqslant 3$, and $\alpha > 0$ is arbitrary if $N = 1, 2$.

**5.3.** It is easy to extend the above results, both in a bounded domain and in an unbounded domain, for a system in the following form

$$U_t - D\triangle U + \nabla F(U) + G(x) = 0,$$

where $U = (u_1, ..., u_m), G(x) = (G_1(x), ...., G_m(x)) \in [L_2(\Omega)]^m$, and $D$ is a diagonal real matrix $D = \mathrm{diag}(d_1, ...., d_m),\ d_i > 0, \forall i = 1, ..., m$.

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