Rec 11-solu

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Rec 11

1. Similar matrices

A,B are similar if $B = PAP^{-1}$

$\Rightarrow$ A B have the same eigenvalues

have the same Jordan forms.

2. Jordan form

Each block $\square = \begin{pmatrix}\lambda & 1 & \\ & \ddots & 1 \\ & & \lambda\end{pmatrix}$

$\#\square$ with eigenvalue $\lambda = \dim(\text{null}(A-\lambda I))$

3. Statistics

Given a set of data $x_1, \cdots, x_n$ in $\mathbb{R}$.

(a) The average $\bar{x} = \frac{1}{n}(x_1 + \cdots + x_n)$

$= \begin{pmatrix}\frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n}\end{pmatrix}\begin{pmatrix}x_1 \\ \vdots \\ x_n\end{pmatrix}$

$\begin{pmatrix}x_1 \\ \vdots \\ x_n\end{pmatrix} - \bar{x}\begin{pmatrix}1 \\ \vdots \\ 1\end{pmatrix} = \underbrace{\left(I - \begin{pmatrix}\frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n}\end{pmatrix}\right)}_{=P}\begin{pmatrix}x_1 \\ \vdots \\ x_n\end{pmatrix}$

$P^T = P$, $P^2 = P$. $\begin{pmatrix}1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0\end{pmatrix}$

(b) The sample variance $\sigma^2 = \frac{\sum \|x_i - \bar{x}\|^2}{n-1}$

$= \frac{\sum (x_i - \bar{x})^T (x_i - \bar{x})}{n-1}$

$= \frac{1}{n-1}(x_1 \cdots x_n) P^T P \begin{pmatrix}x_1 \\ \vdots \\ x_n\end{pmatrix} = \frac{1}{n-1} x^T P x$

4. PCA

A rand(n) + μ ← mean

variance: $A^T A$

$A = U \Sigma V^T$

$\Sigma = \begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & 0\end{pmatrix}$

U_1: Important

$\lambda_1 > \lambda_2 > \cdots$

Problems

1. Suppose A, B are 2×2 similar matrices, say $B = PAP^{-1}$. Let A has two different eigenvalues λ_1, λ_2 and the corresponding eigenvectors are u_1, u_2. Write down the eigenvalues and eigenvectors of B.

$A = U \Sigma U^{-1}$ $\quad U = (u_1, u_2)$ $\quad \Sigma = \begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}$

$B = PAP^{-1} = PU\Sigma U^{-1}P^{-1}$

So B have eigenvalues λ_1, λ_2 and eigenvectors Pu_1, Pu_2.

2. Find the Jordan form of the following matrices

(a) $\begin{pmatrix}1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1\end{pmatrix}$ Eigenvalue 1

$(A - 1I) = \begin{pmatrix}0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{pmatrix}$ null space is 2-dim'l

So the Jordan form is $\begin{pmatrix}1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}$

(b) $\begin{pmatrix}0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1\end{pmatrix}$ $\det(A - \lambda I) = \det\begin{pmatrix}-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 0 & 1-\lambda\end{pmatrix} = (\lambda^2 - 1)(\lambda - 1) = 0$

A has two eigenvalue 1 and one eigenvalue -1.

$(A - 1I) = \begin{pmatrix}-1 & 1 & \\ 1 & -1 & 1 \\ 0 & 0 & 0\end{pmatrix}$ null space is 1'd

$(A - (-1)I) = \begin{pmatrix}1 & 1 & \\ 1 & 1 & 1 \\ 0 & 0 & 2\end{pmatrix}$ null space is 1'd

So Jordan form $\begin{pmatrix}1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{pmatrix}$

3. If A is a symmetric matrix. Can A be similar to $\begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}$? Do not use the fact that symmetric matrices are diagonalizable.

No. If so, $A = P\begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}P^{-1}$

$A = A^T$ $\quad A^T = (P^{-1})^T\begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix}P^T$

$= (P^{-1})^T\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}P^T$

So $(P^{-1})^T\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} = kP$ $\quad k \neq 0$

So $kP^TP = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$ which is impossible due to #3 in Worksheet 10.