A simple distribution without any moments. (Letter to the editor)

Article *in* Mathematical Scientist · January 2000

A SIMPLE DISTRIBUTION WITHOUT ANY MOMENTS

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This note gives an example of a probability distribution without any finite moments, integer or fractional, which is closely related to the Cauchy distribution.

Keywords: Counterexamples in probability and statistics, fractional moments.

AMS 2010 Mathematics Subject Classification: Primary 60E05, secondary 62E99.

1. INTRODUCTION

The standard "pathological" example of a probability distribution with few existing moments is the Cauchy distribution, with density

$$f(y) \;=\; \frac{1}{\pi}\,\frac{1}{1+y^2} \quad , \quad -\infty < y < \infty\,. \tag{1.1}$$

This distribution does not have any finite absolute moments $E(|Y|^p)$ for *integer* p. However, for $p \in (-1, 1)$ one finds (cf. Stuart and Ord, 1994, p. 104)

$$\begin{aligned} E(|Y|^p) &= \frac{2}{\pi} \int_0^\infty \frac{y^p}{1+y^2}\,dy \\ &= \frac{1}{\pi} \int_0^1 z^{\frac{1+p}{2}-1}\,(1-z)^{\frac{1-p}{2}-1}\,dz \\ &= \frac{1}{\pi}\,B\left(\frac{1+p}{2}, \frac{1-p}{2}\right) \quad < \;\infty\,, \end{aligned}$$

where $B(\cdot,\cdot)$ denotes the beta function. Thus, the Cauchy distribution still has some finite *fractional* absolute moments.

2. THE LOG-CAUCHY DISTRIBUTION

Now consider a log-Cauchy random variable, i.e. a random variable $X = \exp(Y)$, where Y has a distribution with density (1.1). For the density of X one readily obtains

$$f(x) \;=\; \frac{1}{\pi}\,\frac{1}{x\,\{1+(\log x)^2\}} \quad , \quad x > 0\,. \tag{2.1}$$

This is a reprint of a short contribution published in *The Mathematical Scientist* 25 (1), 59–60, 2000.
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This distribution does not have a single finite fractional moment, positive or negative. Indeed, take any $p \neq 0$. The pth moment of X is then

$$\begin{aligned} E(X^p) &= \frac{1}{\pi} \int_0^\infty x^{p-1} \frac{1}{1+(\log x)^2}\, dx \\ &= \frac{1}{\pi} \left(\int_0^1 x^{p-1} \frac{1}{1+(\log x)^2}\, dx + \int_1^\infty x^{p-1} \frac{1}{1+(\log x)^2}\, dx \right) . \end{aligned}$$

For $p > 0$, the first integral in parentheses converges whereas the second diverges; for $p < 0$ the roles are reversed. Consequently, $E(X^p) = \infty$ for all $p \neq 0$.

The density (2.1) is decreasing on its support $\mathbb{R}_+$, just like, e.g., the Pareto densities. However, the heavy-tailed Pareto distributions still have lighter tails than the present example. In fact, the 95% quantile of the log-Cauchy equals 552.25, its 99% quantile equals 6.60×10^{13} – quite impressive for a distribution with a median of one!

3. MORE DISTRIBUTIONS WITHOUT MOMENTS

The log-Cauchy distribution is a close relative of the "somewhat pathological example" of Brown and Tukey (1946, p. 11). Both distributions have essentially the same tail behavior. The main difference is that the Brown-Tukey distribution is supported on the whole real line. However, because of the close relationship with the well-known Cauchy distribution, the present example should be easier to motivate.

Apart from the log-Cauchy distribution, there are many other moment-free distributions. All log-t and many log-stable distributions also have this property. Further examples are easily obtained as follows: take any distribution with regularly varying tails at both plus and minus infinity and consider the exponential transformation of such a random variable. Its distribution will have slowly varying tails at infinity and rapidly varying tails at the origin, hence it will not have any finite moments at all. Moreover, the distribution of any unbounded random variable will be free of positive moments after a sufficient number of exponentiations. For example, the Gumbel (or extreme value) distribution leads to the log-Pareto after three exponentiations. However, as the support of this distribution is bounded away from zero the log-Pareto has negative moments of all orders. A distribution free of all moments can only be obtained from a distribution with regularly varying tails at both ends, the Cauchy distribution being the most familiar example.

REFERENCES

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