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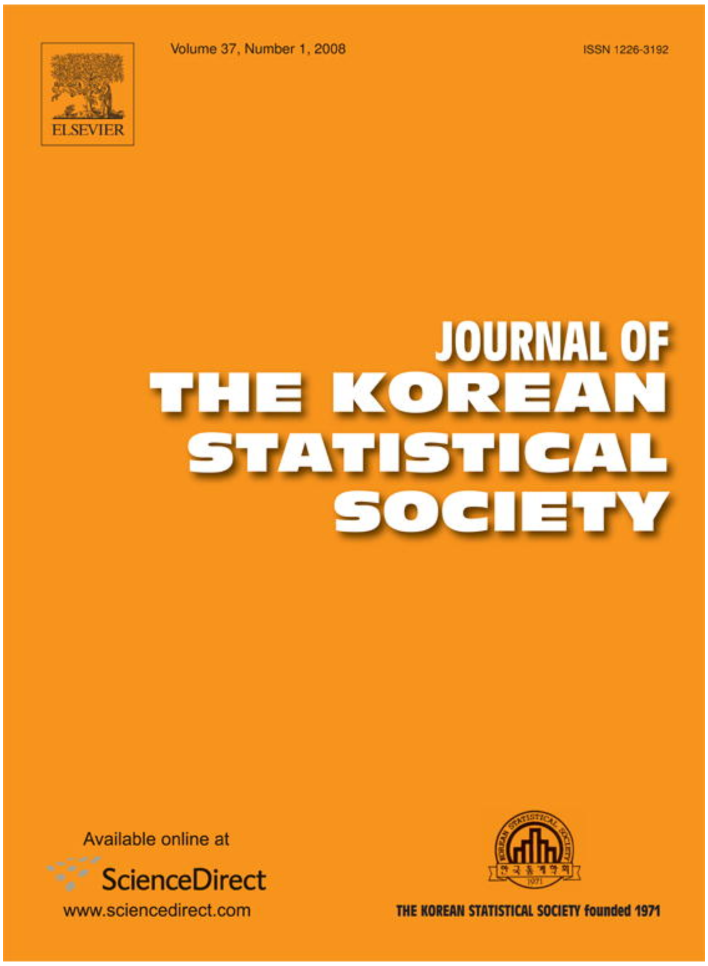
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Moments of truncated Student-t distribution

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Abstract

Truncated moments (TMs) are derived for the generalized Student-t distribution. It is obtained by using a scale mixture of doubly truncated standard normal distribution. Closed form formulae for the TMs are presented up to Nth order under the doubly truncated case (having both upper and lower truncation points). Necessary theories and two applications are provided.

1. Introduction

A random variable X has a doubly truncated generalized Student-t_ν distribution if its probability density function is

$$c\sigma^{-1}\left[\nu+\left(\frac{x-\mu}{\sigma}\right)^2\right]^{-(\nu+1)/2}\left\{\int_\alpha^\beta c\sigma^{-1}\left[\nu+\left(\frac{t-\mu}{\sigma}\right)^2\right]^{-(\nu+1)/2}\mathrm{d}t\right\}^{-1}$$
$$=\sigma^{-1}f_\nu\left(\frac{x-\mu}{\sigma}\right)\left[F_\nu\left(\frac{\beta-\mu}{\sigma}\right)-F_\nu\left(\frac{\alpha-\mu}{\sigma}\right)\right]^{-1},\quad \alpha<x<\beta, \tag{1.1}$$

where $F_\nu(\cdot)$ and $f_\nu(\cdot)$ are the c.d.f. and p.d.f. of the standard Student-t variable with degrees of freedom ν, respectively. $c=\nu^{\nu/2}\Gamma((\nu+1)/2)\{\Gamma(\nu/2)\Gamma(1/2)\}^{-1}$ and Γ is the gamma function. The lower and upper truncation points are α and β, respectively, i.e. the degrees of truncation are $F_\nu((\alpha-\mu)/\sigma)$ (from below) and $1-F_\nu((\beta-\mu)/\sigma)$ (from above).

The effects of truncation on modelling have long been recognized (see, for example, DePriest (1983)) and are commonly referred to in classic statistics literatures such as Johnson, Kotz, and Balakrishnan (1994) and Jawitz (2004). One of the most common techniques for characterizing a truncated distribution is the method of TMs. TMs are useful for providing descriptive information about the distribution. In addition to their utility as general descriptive measures, moments are also employed for estimating the parameters of the distribution.

In the literature, TMs have been derived for several truncated distributions. Especially for the doubly truncated distributions, Shah (1966) considered TMs of a doubly truncated binomial distribution. TMs of a doubly truncated normal were derived by Shah and Jaiswal (1966). Sugiura and Gombi (1985) studied the skewness and kurtosis of a doubly truncated normal and a right truncated Weibull distribution. Recently, Jawitz (2004) has derived several expressions for TMs useful for numerical computation. These include TMs of normal, lognormal, Pearson type III,

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log Pearson type III, and extreme value (Weibull and Gumbel) distributions. However, to the author's knowledge, a general expression for moments of doubly truncated generalized Student-t_ν has not been previously presented. The purpose of this paper is to present an exact expression for the TMs of a doubly truncated generalized Student-t_ν that are important for modelling doubly truncated data sets.

2. Preliminaries

Prior to the derivation of the TMs, we provide lemmas useful for calculating them. For notational convenience, we use $\Phi(\cdot)$ and $\phi(\cdot)$ to indicate respective c.d.f. and p.d.f. of the standard normal variable. $G(c,\ d)$ represents a Gamma distribution with mean cd and variance cd^2. Further denote that $TN_{(a,\,b)}(\mu,\sigma^2)$ and $Tt_{(a,\,b)}(\mu,\sigma^2;\nu)$ are a doubly truncated $N(\mu,\sigma^2)$ and doubly truncated generalized Student-t distribution with location parameter μ and scale parameter σ^2 and degrees of freedom ν, respectively. Here a and b indicate the lower and upper truncation points, respectively. When a distribution is $Tt_{(a,\,b)}(0,1;\nu)$, a doubly truncated standard Student-t distribution, we denote it as $Tt_{(a,\,b)}(\nu)$ for brevity.

Lemma 2.1. *Let $\eta \sim G(\nu/2, 2/\nu)$ and $Z \sim TN_{(\eta^{1/2}a,\ \eta^{1/2}b)}(0,1)$. Then the unconditional (or scale mixture) distribution of $\eta^{-1/2}Z$ is $Tt_{(a,\,b)}(\nu)$.*

Proof. Conditional on η, we see that $\eta^{-1/2}Z \sim TN_{(a,\,b)}(0,\eta^{-1})$. Let $T = \eta^{-1/2}Z$ and $H(\eta)$ be the c.d.f. of η variable. The p.d.f. of T is

$$
\begin{aligned}
f(t) &\propto \int_0^\infty \eta^{-1/2}(2\pi)^{-1/2}\mathrm{e}^{-\eta t^2/2}\mathrm{d}H(\eta),\\
&= \frac{\Gamma(\nu+1)/2\nu^{\nu/2}}{\Gamma(\nu/2)\Gamma(1/2)}(\nu+t^2)^{-(\nu+1)/2}, \quad a < t < b,
\end{aligned}
$$

where the normalizing constant of $f(t)$ is

$$
\int_a^b \frac{\Gamma(\nu+1)/2\nu^{\nu/2}}{\Gamma(\nu/2)\Gamma(1/2)}(\nu+t^2)^{-(\nu+1)/2}\mathrm{d}t = F_\nu(b) - F_\nu(b). \quad \square \tag{2.1}
$$

Lemma 2.2. *If $\eta \sim G(\nu/2, 2/\nu)$,*

$$
E[\Phi(\eta^{1/2}h)] = F_\nu(h) \tag{2.2}
$$

and

$$
E\left[\eta^{-\ell/2}\phi(\eta^{1/2}h)\right] = \frac{\Gamma(\nu-\ell)/2\nu^{\nu/2}}{2^{(\ell+1)/2}\Gamma(\nu/2)\Gamma(1/2)}(\nu+h^2)^{-(\nu-\ell)/2} \tag{2.3}
$$

for $\ell = 0, 1, 2, \ldots, (\nu-1)$.

Proof. (2.2) is immediately obtained from Lemma 2.1. (2.3) is the result of a direct integration using the p.d.f. of η. $\square$

Lemma 2.3. *If $Z \sim TN_{(a,\,b)}(0,1)$, then*

$$
E[(k+1)Z^k] - E[Z^{k+2}] = \frac{b^{k+1}\phi(b) - a^{k+1}\phi(a)}{\Phi(b) - \Phi(a)} \tag{2.4}
$$

for $k = -1, 0, 1, 2, \ldots$.

Proof. We see that

$$
\frac{\mathrm{d}z^{k+1}\phi(z)}{\mathrm{d}z} = (k+1)z^k\phi(z) - z^{k+2}\phi(z) \quad \text{for ; } k = -1, 0, 1, 2, \ldots.
$$

This gives

$$E[(k+1)Z^k] - E[Z^{k+2}] = \int_a^b \left\{(k+1)z^k - z^{k+2}\right\} \phi(z)\mathrm{d}z/(\Phi(b) - \Phi(a))$$
$$= (b^{k+1}\phi(b) - a^{k+1}\phi(a))/(\Phi(b) - \Phi(a))$$

for $k = -1, 0, 1, 2, \ldots$. □

3. The moments

To compute the moments of a random variable $X \sim Tt_{(\alpha\ \beta)}(\mu, \sigma^2; \nu)$, it suffices to compute the moments of $Y = (X - \mu)/\sigma$, where $Y \sim Tt_{(a,\ b)}(\nu)$, $a = (\alpha - \mu)/\sigma$, and $b = (\beta - \mu)/\sigma$. The relationship between them is

$$X = \mu + \sigma Y. \tag{3.1}$$

From Lemma 2.1, we see that Y has density

$$f_Y(y) = [F_\nu(b) - F_\nu(a)]^{-1} \int_0^\infty \phi(z; \eta^{-1})\mathrm{d}H(\eta), \quad a < y < b, \tag{3.2}$$

where $\phi(z; \eta^{-1})$ denotes the density of $N(0, \eta^{-1})$ variable.

The moment generating function of Y is

$$M_Y(t) = \frac{\int_0^\infty \mathrm{e}^{t^2/(2\eta)}\{\Phi(\eta^{1/2}b - \eta^{-1/2}t) - \Phi(\eta^{1/2}a - \eta^{-1/2}t)\}\mathrm{d}H(\eta)}{F_\nu(b) - F_\nu(a)}, \quad t \in \mathbb{R}. \tag{3.3}$$

Naturally, the moments of Y can be obtained by using the moment generating function differentiation. For example:

$$EY = M_Y'(t)|_{t=0} = \frac{\Gamma((\nu-1)/2)\nu^{\nu/2}(A_{(\nu)}^{-(\nu-1)/2} - B_{(\nu)}^{-(\nu-1)/2})}{2[F_\nu(b) - F_\nu(a)]\Gamma(\nu/2)\Gamma(1/2)} \quad \text{for } \nu > 1, \tag{3.4}$$

where $A_{(\nu)} = \nu + a^2$ and $B_{(\nu)} = \nu + b^2$. Unfortunately, for higher moments this rapidly becomes tedious.

An alternative procedure, making use of the following theorem, gives a simple way for calculating the moments.

Theorem 3.1. *For $Y \sim Tt_{(a,\ b)}(\nu)$, the moments of Y are obtained from*

$$E[Y^{k+2}] = E_\eta[\eta^{-(k+2)/2}V^{k+2}], \quad \text{for } k = -1, 0, 1, 2, \ldots, \tag{3.5}$$

where

$$(k+1)V^k - V^{k+2} = \frac{(\eta^{1/2}b)^{k+1}\phi(\eta^{1/2}b) - (\eta^{1/2}a)^{k+1}\phi(\eta^{1/2}a)}{F_\nu(b) - F_\nu(a)},$$

and E_η denoting the expectation is taken with respect to $\eta \sim G(\nu/2, 2/\nu)$ distribution.

Proof. From Lemma 2.1, we see that the distribution Y is equivalent to the unconditional distribution of $\eta^{-1/2}Z$ given $\eta \sim G(\nu/2, 2/\nu)$, where $Z \sim TN_{(\eta^{1/2}a,\ \eta^{1/2}b)}$. This gives $E[Y^{k+2}] = E_\eta E[\eta^{-(k+2)/2}Z^{k+2}\ |\eta]$. Furthermore, when $Z \sim TN_{(\eta^{1/2}a,\ \eta^{1/2}b)}$, the conditional expectation

$$E[(k+1)Z^k \mid \eta] - E[Z^{k+2} \mid \eta] = \frac{(\eta^{1/2}b)^{k+1}\phi(\eta^{1/2}b) - (\eta^{1/2}a)^{k+1}\phi(\eta^{1/2}a)}{\Phi(\eta^{1/2}b) - \Phi(\eta^{1/2}a)}$$

for $k = -1, 0, 1, 2, \ldots$, by Lemma 2.3. Combining these two results along with the fact of Lemma 2.2 that $E_\eta[\Phi(\eta^{1/2}a)] = F_\nu(a)$ and $E_\eta[\Phi(\eta^{1/2}b)] = F_\nu(b)$, we have the result. □

The expectation on the right-hand side of (3.5) can be easily evaluated by using (2.3) of Lemma 2.2. By setting $k = -1, 0, 1, 2$, we obtain four expressions, which may be solved to yield the first four moments of Y. Higher moments could be found similarly.

$$\begin{aligned}
E[Y] &= G_\nu(1)(A_{(\nu)}^{-(\nu-1)/2} - B_{(\nu)}^{-(\nu-1)/2}) \quad \text{for } \nu > 1,\\
E[Y^2] &= \frac{\nu}{\nu-2} + G_\nu(1)(aA_{(\nu)}^{-(\nu-1)/2} - bB_{(\nu)}^{-(\nu-1)/2}) \quad \text{for } \nu > 2,\\
E[Y^3] &= G_\nu(3)(A_{(\nu)}^{-(\nu-3)/2} - B_{(\nu)}^{-(\nu-3)/2}) + G_\nu(1)(a^2A_{(\nu)}^{-(\nu-1)/2} - b^2B_{(\nu)}^{-(\nu-1)/2}) \quad \text{for } \nu > 3,\\
E[Y^4] &= 3\left\{\frac{\nu^2}{(\nu-2)(\nu-4)} + \frac{G_\nu(3)}{2}\left(aA_{(\nu)}^{-(\nu-3)/2} - bB_{(\nu)}^{-(\nu-3)/2}\right)\right\}\\
&\quad + G_\nu(1)(a^3A_{(\nu)}^{-(\nu-1)/2} - b^3B_{(\nu)}^{-(\nu-1)/2}) \quad \text{for } \nu > 4,
\end{aligned}$$

where

$$G_\nu(\ell) = \frac{\Gamma((\nu-\ell)/2)\nu^{\nu/2}}{2[F_\nu(b) - F_\nu(a)]\Gamma(\nu/2)\Gamma(1/2)}, \quad \ell = 1, 3.$$

Denoting EY^ℓ by $\lambda_\ell(a, b\ ; \nu)$, one then has the general formula of the moments of $X \sim Tt_{(\alpha\ \beta)}(\mu, \sigma^2; \nu)$ given by

$$EX^\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} \mu^{\ell-j}\sigma^j\lambda_j(a, b; \nu), \tag{3.6}$$

and $\mathrm{Var}(X) = \sigma^2\mathrm{Var}(Y)$ by (3.1). where $a = (\alpha-\mu)/\sigma$, $b = (\beta-\mu)/\sigma$, and $\mathrm{Var}(Y) = EY^2 - (EY)^2$.

From (3.1), we see that the skewness (the third standardized central moment) and kurtosis of X and Y distributions are the same, and they are

$$\begin{aligned}
\mathrm{Skewness}(Y) &= \mathrm{Var}(Y)^{-3/2}\left\{G_\nu(3)\left(A_{(\nu)}^{-(\nu-3)/2} - B_{(\nu)}^{-(\nu-3)/2}\right) + G_\nu(1)\left(a^2A_{(\nu)}^{-(\nu-1)/2} - b^2B_{(\nu)}^{-(\nu-1)/2}\right)\right.\\
&\quad + 3\left(\frac{\nu}{\nu-2} + G_\nu(1)\left(aA_{(\nu)}^{-(\nu-1)/2} - bB_{(\nu)}^{-(\nu-1)/2}\right)\right)\left(G_\nu(1)\left(B_{(\nu)}^{-(\nu-1)/2} - A_{(\nu)}^{-(\nu-1)/2}\right)\right)\\
&\quad \left. + 2G_\nu(1)^3\left(A_{(\nu)}^{-(\nu-1)/2} - B_{(\nu)}^{-(\nu-1)/2}\right)^3\right\} \quad \text{for } \nu > 3,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Kurtosis}(Y) &= \mathrm{Var}(Y)^{-2}\left\{\frac{3\nu^2}{(\nu-2)(\nu-4)} + \frac{3G_\nu(3)}{2}\left(aA_{(\nu)}^{-(\nu-3)/2} - bB_{(\nu)}^{-(\nu-3)/2}\right)\right.\\
&\quad + G_\nu(1)\left(a^3A_{(\nu)}^{-(\nu-1)/2} - b^3B_{(\nu)}^{-(\nu-1)/2}\right) - 4G(1)\left(A_{(\nu)}^{-(\nu-1)/2} - B_{(\nu)}^{-(\nu-1)/2}\right)\\
&\quad \times \left[G_\nu(3)\left(A_{(\nu)}^{-(\nu-3)/2} - B_{(\nu)}^{-(\nu-3)/2}\right) + G_\nu(1)\left(a^2A_{(\nu)}^{-(\nu-1)/2} - b^2B_{(\nu)}^{-(\nu-1)/2}\right)\right]\\
&\quad + 6G_\nu(1)^2\left(A_{(\nu)}^{-(\nu-1)/2} - B_{(\nu)}^{-(\nu-1)/2}\right)^2\left[\frac{\nu}{\nu-2} + G_\nu(1)\left(aA_{(\nu)}^{-(\nu-1)/2} - bB_{(\nu)}^{-(\nu-1)/2}\right)\right]\\
&\quad \left. - 3G_\nu(1)^4\left(A_{(\nu)}^{-(\nu-1)/2} - B_{(\nu)}^{-(\nu-1)/2}\right)^4\right\} \quad \text{for } \nu > 4.
\end{aligned}$$

As an example, we use the half-t_ν distribution, written by $Tt_{(0,\ \infty)}(\nu)$. If $Y \sim Tt_{(0,\ \infty)}(\nu)$, the p.d.f. in (1.1) reduces to

$$f_Y(y) = 2f_\nu(y), \quad y > 0. \tag{3.7}$$

One finds the moments

$$E[Y] = \frac{\nu^{1/2}\Gamma((\nu-1)/2)}{\Gamma(\nu/2)\Gamma(1/2)} \quad \text{for } \nu > 1,$$

$$\mathrm{Var}(Y) = \frac{\nu}{\nu-2} - \nu\left(\frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)\Gamma(1/2)}\right)^2 \quad \text{for } \nu > 2,$$

$$\mathrm{Skewness}(Y) = \mathrm{Var}(Y)^{-3/2}\left(\frac{\nu^{3/2}\Gamma((\nu-3)/2)}{\Gamma(\nu/2)\Gamma(1/2)}\right)\left\{\frac{1}{\nu-2} + 2\left(\frac{\Gamma((\nu-3)/2)}{\Gamma(\nu/2)\Gamma(1/2)}\right)^2\right\} \quad \text{for } \nu > 3,$$

$$\begin{aligned}\mathrm{Kurtosis}(Y) \;=\; & \mathrm{Var}(Y)^{-2}\left\{\frac{3\nu^2}{(\nu-2)(\nu-4)} + \left(\frac{\nu^2\Gamma((\nu-1)/2)}{[\Gamma(\nu/2)\Gamma(1/2)]^2}\right)\left(6\frac{\Gamma((\nu-1)/2)}{\nu-2}\right.\right.\\ & \left.\left. -\; 4\Gamma((\nu-3)/2) - 3\frac{\Gamma((\nu-1)/2)^2}{[\Gamma(\nu/2)\Gamma(1/2)]^2}\right)\right\} \quad \text{for } \nu > 4.\end{aligned}$$

For the limit points of $b^\ell B_{(\nu)}^{(-\nu-1)/2}$, $\ell = 1, 2, 3$, when $b \to \infty$, we use L'Hopital's rule to get the above results. As expected, we see that $\mathrm{Skewness}(Y) > 0$. For the other example, applying L'Hopital's rule, we consider the moments of $Tt_{(-\infty,\,\infty)}(\nu)$ distribution, the standard Student-t_ν distribution. When $Y \sim Tt_{(-\infty,\,\infty)}(\nu)$, the foregoing results yield $E[Y] = 0$, $\mathrm{Var}(Y) = \nu/(\nu-2)$ for $\nu > 2$, $\mathrm{Skewness}(Y) = 0$, and $\mathrm{Kurtosis}(Y) = 3(\nu-2)/(\nu-4)$ for $\nu > 4$. These values of Y agree with those of the standard Student-t_ν distribution given in Johnson et al. (1994).

4. Applications

4.1. Inequality constrained regression

With the normal linear regression model, we assume that an $n \times 1$ vector of observations $\mathbf{y}$ on our dependent variable satisfies

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{4.1}$$

where $\boldsymbol{\varepsilon} \sim N_n(\mathbf{0}, \sigma^2 I)$, $\mathbf{X} : n \times k$, and $\mathrm{rank}(\mathbf{X}) = k$ for the regression model. Let $\hat{\boldsymbol{\beta}} = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{y}$, $\nu = n - k$, and $s^2 = \nu^{-1}(\mathbf{y} - \mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\beta}})^{\mathrm{T}}(\mathbf{y} - \mathbf{X}^{\mathrm{T}}\hat{\boldsymbol{\beta}})$. Under the above assumptions the joint p.d.f. for the elements of $\mathbf{y}$ given $\mathbf{X}$, $\boldsymbol{\beta}$, and σ, is

$$f(\mathbf{y}|\mathbf{X}, \boldsymbol{\beta}, \sigma) \propto \sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}[\nu s^2 + (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\mathbf{X}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})]\right\}. \tag{4.2}$$

For a possibly improper diffuse prior $p(\boldsymbol{\beta}, \sigma) \propto \sigma^{-1}q(\boldsymbol{\beta})$, the posterior distribution is proportional to

$$\sigma^{-n-1} \exp\left\{-\frac{1}{2\sigma^2}[\nu s^2 + (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\mathbf{X}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})]\right\} q(\boldsymbol{\beta}). \tag{4.3}$$

Integrating over σ, the marginal posterior p.d.f. for the elements of $\boldsymbol{\beta}$ becomes

$$p(\boldsymbol{\beta}|\mathbf{y}, \mathbf{X}) \propto [\nu s^2 + (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\mathbf{X}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})]^{-(\nu+k)/2} q(\boldsymbol{\beta}) \tag{4.4}$$

the product of a multivariate Student-t_ν p.d.f. and $q(\boldsymbol{\beta})$. When, a priori, an inequality constraint for a regression coefficient β_j is given in the form of $\alpha < \beta_j < \beta$. We can define $q(\boldsymbol{\beta})$ as $I(\alpha < \beta_j < \beta)$, where $I(\cdot)$ is an indicator function. By integrating (4.4) with respect to $\beta_1, \beta_2, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_k$, by using the properties of the multivariate Student-t_ν p.d.f., we have the marginal posterior p.d.f. of β_j (see, Lee (1997) for the unconstrained case):

$$p(\beta_j|\mathbf{y}, \mathbf{X}) \propto \left(\nu + \frac{(\beta_j - \hat{\beta}_j)^2}{s^2 h^{jj}}\right)^{-(\nu+1)/2} I(\alpha < \beta_j < \beta), \tag{4.5}$$

a kernel of doubly truncated generalized Student-t_ν p.d.f., where h^{jj} is the (j, j) element of $(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}$. Thus the posterior distribution is

$$\beta_j|\mathbf{y}, \mathbf{X} \sim Tt_{(\alpha,\beta)}(\hat{\beta}_j, s^2 h^{jj}; \nu). \tag{4.6}$$

Under the quadratic loss function, the Bayes estimator of β_j is

$$\hat{\beta}_{j,\text{Bayes}} = \hat{\beta}_j + \frac{s\sqrt{h^{jj}}\Gamma((\nu-1)/2)\nu^{\nu/2}}{2[F_\nu(b) - F_\nu(a)]\Gamma(\nu/2)\Gamma(1/2)}\left((\nu+a^2)^{-(\nu-1)/2} - (\nu+b^2)^{-(\nu-1)/2}\right) \tag{4.7}$$

by (3.6), where $a = (\alpha - \hat{\beta}_j)/(s\sqrt{h^{jj}}\ )$ and $b = (\beta - \hat{\beta}_j)/(s\sqrt{h^{jj}}\ )$. Note that the estimator has the following properties: (i) $\hat{\beta}_{j,\text{Bayes}} \in (\alpha,\ \beta)$; (ii) when $|a| = b$ $(a < b)$, $\hat{\beta}_{j,\text{Bayes}} = \hat{\beta}_j$; (iii) as $\nu \to \infty$, the limit of $\hat{\beta}_{j,\text{Bayes}}$ is

$$\hat{\beta}_j + \frac{s\sqrt{h^{jj}}(\phi(a) - \phi(b))}{\Phi(b) - \Phi(a)}.$$

We see, by Lemma 2.3, that this is equivalent to the Bayes estimator of β_j when the marginal posterior distribution is $\beta_j|\mathbf{y}, \mathbf{X} \sim TN_{(\alpha,\ \beta)}(\hat{\beta}_j, s^2h^{jj})$.

4.2. The sum of a t_{ν_1} and a truncated t_{ν_2} variable

As an example of special problems that can sometimes arise, Weinstein (1964) mentioned the derivation of the distribution of the sum of two independent random variables, one normal and the other truncated normal. When laws of both random variables are generalized Student-t, Weinstein's problem reduces to the following problem.

Suppose $(X - \mu_X)/\sigma_X \sim t_{\nu_1}$ and $Y \sim Tt_{(\alpha,\ \infty)}(\mu_Y, \sigma_Y^2;\ \nu_2)$, and they are independent variables. Let $U = X + Y$, then our problem is to get $\Pr(U \le q)$, $E[U]$ and $\text{Var}(U)$. First, to calculate $\Pr(U \le q)$, make the following transformations:

$$x = (X - \mu_X)/\sigma_X, \tag{4.8}$$
$$y = (Y - \mu_Y)/\sigma_X. \tag{4.9}$$

Then the probability that $x + y = u$ is equal to or less than t is given by the variable

$$\begin{aligned} Q(t) &= \int_{-\infty}^{t}\int_{-\infty}^{u-a(\alpha)} f_{\nu_2}\left(\frac{u-x}{\sigma}\right) f_{\nu_1}(x)\partial x\partial u \Big/ \left[\sigma\left(1 - F_{\nu_2}\left(\frac{a(\alpha)}{\sigma}\right)\right)\right] \\ &= \int_{-\infty}^{t-a(\alpha)}\left[F_{\nu_2}\left(\frac{t-x}{\sigma}\right) - F_{\nu_2}\left(\frac{a(\alpha)}{\sigma}\right)\right] f_{\nu_1}(x)\mathrm{d}x \Big/ \left[\left(1 - F_{\nu_2}\left(\frac{a(\alpha)}{\sigma}\right)\right)\right], \end{aligned}$$

where $a(\alpha) = (\alpha - \mu_Y)/\sigma_X, \sigma = \sigma_Y/\sigma_X$, and $t = (q - \mu_X - \mu_Y)/\sigma_X$. Thus $\Pr(U \le q)$ is equivalent to $Q(t)$. For given values of parameters, a numerical computation (say, using Mathematika) may yield the probability. The expectation and variance of U are directly obtained by using (3.6). They are

$$\begin{aligned} E[U] &= E[X] + E[Y] = \mu_X + \mu_Y + \sigma_Y\left((\nu_2 + a(\alpha)^2)^{-(\nu_2-1)/2}\right)G_{\nu_2}, \\ \text{Var}(U) &= \text{Var}(X) + \text{Var}(Y) = \frac{\nu_1\sigma_X^2}{\nu_1 - 2} + \sigma_Y^2\left\{\frac{\nu_2}{\nu_2 - 2} + G_{\nu_2}\left(a(\alpha)(\nu_2 + a(\alpha)^2)^{-(\nu_2-1)/2}\right)\right. \\ &\quad \left. - \left[G_{\nu_2}\left((\nu_2 + a(\alpha)^2)^{-(\nu_2-1)/2}\right)\right]^2\right\}, \end{aligned}$$

where

$$G_{\nu_2} = \frac{\Gamma((\nu_2 - 1)/2)\nu_2^{\nu_2/2}}{2[1 - F_{\nu_2}(a(\alpha))]\Gamma(\nu_2/2)\Gamma(1/2)}.$$

A particular example considered by Weinstein (1964) is as follows: An item which we make has, among others, two parts which are assembled additively with regard to length. The lengths of both parts are distributed as generalized Student-t distributions. Before assembly, one of the parts is subject to an inspection which removes all individuals below a specified length. As an example, suppose that X comes from a generalized Student-t_4 distribution with $\mu_X = 100$ and a standard deviation of $\sigma_X = 6$, and Y comes from a generalized Student-t_4 distribution with $\mu_Y = 50$

and $\sigma_Y = 3$, but with the restriction that $Y \geq 44$. Our problem is to find the chance that $U = X + Y \leq 138$, $E[U]$, and $\text{Var}(U)$. From the foregoing results, we have

$$\Pr(U \leq 138) = Q(-2) = 1 - F_4(1) - \int_{-\infty}^{-1} F_4\,(4 + 2x)\; f_4(x)\text{d}x / F_4(2) = .06348,$$

$E[U] = 152.640$, and $\text{Var}(U) = 70.0397$. These values can be compared with those of $U^* = X + Y$ obtained from the unrestricted distribution of Y, i.e. $(Y - 50)/3 \sim t_4$. Those values are $\Pr(U^* \leq 138) = .09347$, $E[U^*] = 150$, and $\text{Var}(U^*) = 90$.

5. Concluding remarks

Sometimes, observed data sets are almost exclusively truncated, because of analytical detection limits or spatial and temporal limitations on data collection. Some broadly related proposals and results have appeared in the literature under the concept of the truncated distribution. The present paper has considered the moments of a doubly truncated generalized Student-t distribution, providing descriptive information about the distribution. As given in Section 4, in addition to their utility as general descriptive measures, the moments can be employed for solving statistical problems. However, moment estimators of parameters of the distribution are not available in closed form. A numerical method for the effective estimation of the parameters is worthy of being studied. A study pertaining to this problem is left as a future research topic of interest.

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