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Modeling of an insurance system and its large deviations analysis

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ABSTRACT

We model an insurance system consisting of one insurance company and one reinsurance company as a stochastic process in $\mathbb{R}^2$. The claim sizes $\{X_i\}$ are an iid sequence with light tails. The interarrival times $\{\tau_i\}$ between claims are also iid and exponentially distributed. There is a fixed premium rate c_1 that the customers pay; $c < c_1$ of this rate goes to the reinsurance company. If a claim size is greater than R the reinsurance company pays for the claim. We study the bankruptcy of this system before it is able to handle N number of claims. It is assumed that each company has initial reserves that grow linearly in N and that the reinsurance company has a larger reserve than the insurance company. If c and c_1 are chosen appropriately, the probability of bankruptcy decays exponentially in N. We use large deviations (LD) analysis to compute the exponential decay rate and approximate the bankruptcy probability. We find that the LD analysis of the system decouples: the LD decay rate γ of the system is the minimum of the LD decay rates of the companies when they are considered independently and separately. An analytical and numerical study of γ as a function of (c, R) is carried out.

1. Introduction

A central idea in actuarial risk theory is to model the reserves of an insurance company as a stochastic process and use tools from probability theory and statistics to compute various ruin probabilities and other measures of risk. This idea goes back to Lundberg, and a theory of actuarial risk based on it saw enormous growth in the 20th century [1–3].

Most of the current literature in actuarial risk focuses on the modeling of the risks of a single company. However, all insurance systems around the world involve many interconnected companies. Modeling of these interconnections is crucial for entities which may be interested in probabilities of events that involve interactions between the components of the system. Examples of such entities are reinsurance companies and governmental organizations overseeing financial systems.

The goal of the present paper is the modeling and analysis of the simplest possible insurance system consisting of an insurance company and a reinsurance company. The claims $\{X_i\}$ are assumed to be iid and light tailed, i.e., their common distribution has a moment generating function. If a claim's size is less than R the insurance company handles it, otherwise the reinsurance company does. The interarrival times $\{\tau_i\}$ between claim arrivals is assumed to be iid as well with an exponential distribution with rate λ. The insurance company charges its customers a premium of c_1 per unit time. $c < c_1$ of this premium rate goes to the reinsurance company. It is further assumed that the [re]insurance company has initial reserves $S_0^{(1)}$ [$S_0^{(2)}$], with $S_0^{(2)} > S_0^{(1)}$. Section 2 models the evolution of the reserves of the two companies as a stochastic process S in $\mathbb{R}^2$; the first component $S^{(1)}$ of this process is the reserves of the insurance company, and its second component $S^{(2)}$ the reserves of the reinsurance company.

The rest of the paper is a study of the bankruptcy probability of this system, i.e., the bankruptcy of one of the companies in it, before it can handle a fixed number N of claims. There is no closed form formula for this probability and approximation

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techniques are necessary for its study. If the expected premium received in between claim arrivals plus the initial reserve per claim s_i ($s_i = S_0^{(i)}/N$ is the initial reserve divided by N) of each company is greater than the average claim size received by the same company the bankruptcy probability of the insurance system decays exponentially in N. In Section 3 the theory of large deviations (LD) [4–7] is used to reduce the computation of the decay rate of this probability into a calculus of variations problem. This question further reduces to the minimization of a convex function over the nonconvex region $\{(x_1, x_2) : x_1 \leq -s_1 \text{ or } x_2 \leq -s_2\} \subset \mathbb{R}^2$. The function to be optimized is denoted by L and it is the Fenchel Legendre transform of the log moment generating function of the increment of the reserves process S. In Section 3.1 this optimization problem is written as the minimum of two optimization problems, each over $\mathbb{R}$. In Section 3.2 it is shown that the aforementioned optimization problems over $\mathbb{R}$ correspond to the large deviations analysis of the insurance and the reinsurance companies separately. In this way, we see that the original problem decouples: one can first compute the LD decay rates γ_1 and γ_2 of the probability of bankruptcy for of each of the companies separately. The LD decay rate of the bankruptcy probability of the system is simply the minimum $\gamma_1 \wedge \gamma_2$ of the rates of the LD decay rates of the companies. Section 3.3 uses these results to compute the LD decay rate of an example system whose claim sizes are exponentially distributed.

Section 4 studies the following problem: how to pick the parameters c and R so that the LD decay rate of the ruin probability of the system is maximized (maximization of the decay rate corresponds to the minimization of the ruin probability). We show that for each fixed value of R there is an optimal premium $c^*(R)$ that goes to the reinsurance company that maximizes the LD decay rate of the system. Let $\gamma^*(R)$ denote the decay rate of the ruin probability when c is set optimally. We prove that there are threshold values $R_1^* < R_2^*$ such that $\gamma^*(R) = \gamma_2(c_1, R)$ for $R \leq R_1^*$ and $\gamma(R) = \gamma_1(0, R)$ for $R \geq R_2^*$. The most interesting part of $\gamma^*(R)$ is its behavior on the interval (R_1^*, R_2^*). This range of values of the threshold variable makes a nontrivial sharing of the premium revenue optimal. A rigorous analysis of this behavior appears complicated. Instead, we provide several numerical examples in 4.2.1, which suggest that *no matter how the claim sizes are distributed* $\gamma^*(R)$ *is decreasing on* (R_1^*, R_2^*) if $s_2 > s_1$. This section concludes with a comparison of the insurance system with a single insurance company formed by merging the companies making up the system. Section 5 discusses possible extensions and poses several questions.

2. The model

We first begin with a review of a model for a single company. Once the one dimensional model is set up, it is simple to modify it to a system consisting of two companies.

2.1. The one dimensional model

Typically one models the reserve of an insurance company with

$$s_t \doteq s_0 + ct - \sum_{i=1}^{N_t} X_i.$$

Here $s_0 > 0$ is the initial reserves of the company, N_t is a Poisson process with rate $\lambda > 0$ and it models the arrival times of the claims, X_i is the ith claim size, and finally $c > 0$ is the premium rate that the company charges. X_i are assumed to be iid and independent of N_t.

Under this model, the probability of ruin before time T of the insurance company is given by:

$$P\left(\min_{t \leq T} s_t \leq 0\right). \tag{1}$$

This probability is well studied in the current literature [1–3]. Instead of (1) we will concentrate on the following type of ruin probability:

$$P\left(\min_{n \leq N} s_{T_n} \leq 0\right), \tag{2}$$

where T_n are the arrival times of the claims. (2) is the probability that the insurance company bankrupts before it is able to pay the Nth claim. The difference between (2) and (1) is in how they measure time: (1) measures time in terms of years whereas (2) measures time in terms of the number of claims processed by the company. Both (2) and (1) are useful quantities to measure the risk of a company. The advantage of (2) is that it allows one to consider the process S_t at the times of its jumps. This means that one can setup the model as a simple discrete time random walk as follows.

Let τ_i be the length of time between the ith and the $i+1$th claim. These are the interarrival times of N_t and by definition they are exponentially distributed with common rate λ. Then the reserves of the company at time T_n is the following random walk:

$$\begin{aligned} S_n \doteq s_{T_n} &= cT_n - \sum_{i=1}^{n} X_i \\ &= \sum_{i=1}^{n} (c\tau_i - X_i). \end{aligned} \tag{3}$$

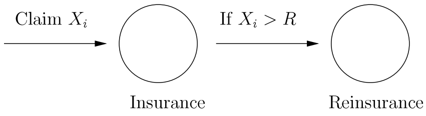

Fig. 1. An insurance system consisting of an insurance and a reinsurance company.

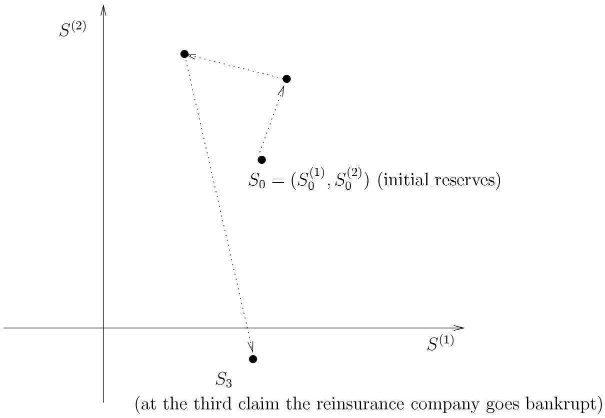

Fig. 2. A sample path of the reserves of the insurance system.

S_n is the reserve of the company at the time when the company received its nth claim. The ruin probability (2) in terms of S_n defined in (3) is

$$p_N = P\left(\min_{n\le N} S_n \le 0\right).$$

2.2. Model for a system of two companies

Now let us extend the previous model to an insurance system consisting of an insurance company and a reinsurance company. The setup is as follows: the two companies agree that if the claim size X_i is greater than a threshold $R > 0$ then this risk is transferred to the reinsurer, see Fig. 1. In exchange the reinsurer takes $c < c_1$ of the premium rate. Let $S_n^{(1)}$ denote the reserves of the insurance company and $S_n^{(2)}$ the reserves of the reinsurance company at the arrival time of the nth claim. The dynamics of these processes is as follows:

$$S_n^{(1)} \doteq S_0^{(1)} + \sum_{i=1}^{n}((c_1 - c)\tau_i - X_i 1_{\{X_i<R\}}), \qquad S_n^{(2)} \doteq S_0^{(2)} + \sum_{i=1}^{n}(c\tau_i - X_i 1_{\{X_i>R\}}).$$

The two dimensional stochastic process $S_n \doteq (S_n^{(1)}, S_n^{(2)})$ models the whole insurance system. A sample path of S is shown in Fig. 2. The reinsurance company depicted in this figure goes bankrupt at the arrival of the third claim. The first two claims are handled by the insurance company. The third claim is a large one and goes to the reinsurer. However, it turns out that the reinsurer doesn't have reserves to meet the claim and goes bankrupt.

A natural probability of interest regarding S is

$$p_N \doteq P\left(\min\left(\min_{k\le N} S_k^{(1)}, \min_{k\le N} S_k^{(2)}\right) \le 0\right). \tag{4}$$

This is the probability that one of the companies goes bankrupt before the processing of the Nth claim. p_N cannot be written in terms of the distributions of the $S^{(i)}$; its computation requires the use of the joint distribution of $S^{(1)}$ and $S^{(2)}$, i.e., the distribution of the process S. Furthermore, it is a probability that depends on the whole sample path of S. For these reasons, the computation of p_N is nontrivial and requires approximation techniques. The rest of this article uses large deviations analysis [4–6] for this purpose.

3. Large deviations analysis of the ruin probability

In order to apply LD theory to the estimation of the ruin probability p_N in (4), the initial reserves $S_0^{(1)}$ and $S_0^{(2)}$ have to grow with N. Therefore, it is assumed that there are two positive real numbers $s_1 < s_2$ and $S_0^{(1)} = Ns_1$ [$S_0^{(2)} = Ns_2$] is the initial reserve of the [re]insurance company. s_1 [s_2] is the reserve of the [re]insurance company per claim to be covered.

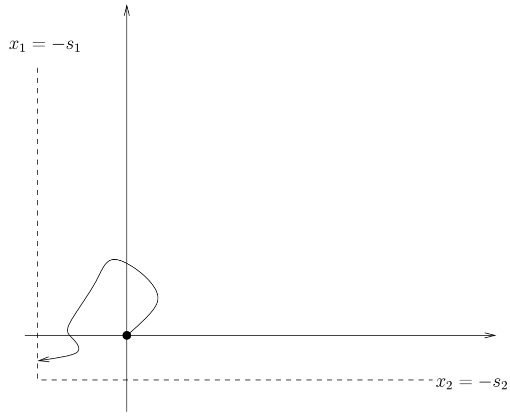

Fig. 3. An example trajectory starting from 0 and hitting ∂_e, optimization in (7) is over such paths.

The increment Y_n of the insurance process is:

$$Y_n \doteq (Y_n^{(1)}, Y_n^{(2)}) \doteq \left((c_1 - c)\tau_n - 1_{\{X_n<R\}}X_n, c\tau_n - 1_{\{X_n>R\}}X_n\right).$$

Note that the first component of Y_n is the amount that goes to the insurance company at the filing of a claim, and the second component is the part that goes to the reinsurer. These components take into account the premiums collected in between the claims ($(c_1 - c)\tau_n$ term for the insurance company and $c\tau_n$ term for the reinsurer.)

Y_n is an iid sequence. In order for an LD analysis to be relevant it must be assumed that the log moment generating function

$$H(\alpha) \doteq \log \mathbb{E}\left[\mathrm{e}^{\langle \alpha, Y\rangle}\right] \tag{5}$$

is finite, at least for $\alpha \in \mathbb{R}^2$ in a neighborhood of 0. It is well known that H is a convex function [4, Lemma 2.2.5, page 27]. The LD result will be in terms of the Fenchel–Legendre transform of H:

$$L(\beta) \doteq \sup_{\alpha\in\mathbb{R}^2} \left[\langle \alpha, \beta\rangle - H(\alpha)\right].$$

Its definition directly implies that L is also a convex function (L is the convex conjugate of H).

Define the scaled exit boundary

$$\partial_e \subset \mathbb{R}^2 \doteq \{(x_1, x_2) : x_1 = -s_1 \text{ or } x_2 = -s_2\}. \tag{6}$$

Under (5), Mogulskii's theorem [4, Theorem 5.1.2, page 176] implies

Proposition 1. *The following limit holds:*

$$\lim_N -\frac{1}{N} \log p_N = \inf_{x\in\mathcal{C}} \int_0^1 L(\dot{x})\mathrm{d}t, \tag{7}$$

where $\mathcal{C}$ is the set of all absolutely continuous functions x from $[0, 1]$ to $\mathbb{R}^2$ such that $x(0) = 0$ and $x(t) \in \partial_e$ for some $t \in [0, 1]$.

The right side of (7) is a calculus of variations problem. Each element of $\mathcal{C}$ can be thought of as an average bankruptcy scenario for the insurance system. Each scenario accumulates an $L(\dot{x})\mathrm{d}t$ amount of cost per step. This cost can be written as a relative entropy and it measures the deviation of the average scenario from the expected path of S. The path with the minimum cost is identified as the most likely bankruptcy scenario.

Note that the cost function L that appears inside the integral is convex and does not depend on t. This and Jensen's inequality imply that it is enough to consider sample paths that are straight lines. For example, consider the path in Fig. 3. The strict convexity of L implies that a cheaper path than the path depicted in this figure is the straight line that connects the end points of this path. This is a well known situation in optimal control and calculus of variations, for a detailed explanation we refer the reader to [8, Chapter 5]. These considerations reduce the calculus of variations problem in (7) to the following two dimensional constrained optimization problem:

$$\lim -\frac{1}{N} \log p_N = \inf_{x\in R} L(x) \doteq \gamma, \tag{8}$$

where

$$R \doteq \{(x_1, x_2) : x_1 \le -s_1, x_2 \le -s_2\}.$$

Note that ∂_e of (6) is the boundary ∂R of R.

Proposition 2. *Suppose the expected earnings of each company per claim it receives plus its initial reserve per claim is positive:*

$$\mathbb{E}[Y_n] = \mathbb{E}\left[\left((c_1 - c)\tau_1 - 1_{\{X_1 < R\}}X_1, c\tau_1 - 1_{\{X_1 > R\}}X_1\right)\right] > -(s_1, s_2), \tag{9}$$

where $>$ denotes component-wise comparison. Then $\gamma > 0$.

Proof. The unique root of L is the average direction in (9). The inequality (9) implies that starting from the origin and moving in the direction of this average will not make it past ∂_e in one unit of time. Proposition 2 follows from this and Proposition 1. □

3.1. Solution of the finite dimensional optimization problem

A solution of (8) proceeds as follows. If the claim size distribution is not constant, H will be a strictly convex function. In any event, let us simply assume that H is strictly convex. It follows that L is also strictly convex. The unique minimizer of L is $\mathbb{E}[Y_n]$. If $\mathbb{E}[Y_n] \in R$ then it is the optimizer of (8) and $\gamma = 0$. Otherwise, because L is strictly convex, $\mathbb{E}[Y_n]$ is the only point of $\mathbb{R}^2$ where L has a zero gradient. Thus, the minimizer of (8) cannot be in the interior of R and therefore the minimum in (8) equals

$$\inf_{x\in\partial R} L(x) = \inf_{x\in\partial_e} L(x) = \inf_{x=(-s_1,x') \text{ or } (x',-s_2), x'\in\mathbb{R}} L(x).$$

Thus, to compute γ, it is enough to compute

$$\gamma_1 \doteq \inf_{x\in\mathbb{R}} L(-s_1, x), \qquad \gamma_2 \doteq \inf_{x\in\mathbb{R}} L(x, -s_2). \tag{10}$$

Let us compute

$$\gamma_1 = \inf_{x\in\mathbb{R}} L(-s_1, x) = \inf_{x\in\mathbb{R}} \sup_{(\alpha_1,\alpha_2)} \left[-\alpha_1 s_1 + \alpha_2 x - H(\alpha_1, \alpha_2)\right],$$

the computation of γ_2 is similar. The log moment generating function H is strictly convex where it is finite, and let us also assume that it is smooth. Then both the sup and the inf have their respective optimizers and one can study them with calculus. The optimizer $(\alpha_1^*(x), \alpha_2^*(x))$ satisfies:

$$\frac{\partial H}{\partial \alpha_1}(\alpha_1^*(x), \alpha_2^*(x)) = -s_1, \qquad \frac{\partial H}{\partial \alpha_2}(\alpha_1^*(x), \alpha_2^*(x)) = x. \tag{11}$$

Then

$$\gamma_1 = \inf_x \left[-\alpha_1^*(x)s_1 + \alpha_2^*(x)x - H(\alpha_1^*(x), \alpha_2^*(x))\right]. \tag{12}$$

Equating the derivative of the expression in brackets with respect to x to 0 yields

$$0 = -\frac{\mathrm{d}}{\mathrm{d}x}\alpha_1^*(x)s_1 + \frac{\mathrm{d}}{\mathrm{d}x}\alpha_2^*(x)x + \alpha_2^*(x) - \frac{\partial H}{\partial \alpha_1}(\alpha_1^*(x), \alpha_2^*(x))\frac{\mathrm{d}}{\mathrm{d}x}\alpha_1^*(x) - \frac{\partial H}{\partial \alpha_2}(\alpha_1^*(x), \alpha_2^*(x))\frac{\mathrm{d}}{\mathrm{d}x}\alpha_2^*(x).$$

Substituting the right sides of the identities in (11) for the partial derivatives of H in the previous display yields $\alpha_2^*(x) = 0$. Substituting this back in (11) gives

$$\frac{\partial H}{\partial \alpha_1}(\alpha_1^*(x), 0) = -s_1, \qquad \frac{\partial H}{\partial \alpha_2}(\alpha_1^*(x), 0) = x. \tag{13}$$

One now solves the first equation to identify α_1^*. With α_1^* identified, the LD decay rate γ_1 is:

$$\gamma_1 = -\alpha_1^* s_1 - H(\alpha_1^*, 0).$$

Note that once α_1^* is known, the second equation in (13) gives the x that optimizes the inf in (12). However, the value of this optimizer is not needed for the computation of γ_1.

Remark 1. It may happen that the range of $\frac{\partial H}{\partial \alpha_1}$ does not contain $-s_1$. This can happen only when it is impossible for the insurance company to go bankrupt in N steps. Such an impossibility can occur because the maximum amount per claim the insurance company has to pay is R and the insurance company may have sufficient reserves to meet all of the first N claims below R. In this case one sets $\alpha_1^* = -\infty$ and $\gamma_1 = \infty$.

3.2. Decoupling

The analysis of the previous subsection implies that one can define γ_1 and γ_2 in (10) as

$$\gamma_1 = \sup_\alpha[-\alpha s_1 - H_1(\alpha)], \qquad \gamma_2 = \sup_\alpha[-\alpha s_2 - H_2(\alpha)],$$

where

$$H_1(\alpha) \doteq \log \mathbb{E}[\exp(\alpha Y^{(1)})], \qquad H_2(\alpha) \doteq \log \mathbb{E}[\exp(\alpha Y^{(2)})].$$

For a moment, let us go back to the model of an insurance company introduced in Section 2.1. An LD analysis can be applied to this model to estimate its bankruptcy probability as well. To that end, let $s = s_0/N$ denote the initial capital of the company per claim, τ the interarrival time of the claims, and X the claim size. If $\mathbb{E}[c\tau - X] > -s$, Mogulskii's theorem implies

$$\lim -\frac{1}{N} \log p_N = \inf_{\alpha}\{-s\alpha - H_0(\alpha)\} \tag{14}$$

where $H_0(\alpha) \doteq \log \mathbb{E}[e^{c\tau - X}]$.

Note that both γ_1 and γ_2 have the same form as the right side of (14). This implies that the LD analysis of the insurance system decouples into an LD analysis of the component companies. That is, to compute the LD decay rate of the system, it is sufficient to carry out a one dimensional LD analysis of each of the component companies separately. The results of these are combined by taking their minimum to yield the LD decay rate of the joint system.

Algorithm to identify the LD decay rate of the system. Let us summarize the foregoing analysis in the form of the following algorithm which computes γ.

1. Find α_1^* such that
$$H_1'(\alpha_1^*) = -s_1, \tag{15}$$
and set
$$\gamma_1 = -\alpha_1^* s_1 - H_1(\alpha_1^*), \tag{16}$$
2. Find α_2^* such that
$$H_2'(\alpha_2^*) = -s_2, \tag{17}$$
and set
$$\gamma_2 = -\alpha_1^* s_2 - H_2(\alpha_2^*), \tag{18}$$
3. $\gamma = \gamma_1 \wedge \gamma_2$.

The following lemma will be useful later on.

Lemma 3.1. *γ_1 [γ_2] in (16) [(18)] is positive iff $\alpha_1^* < 0$ [$\alpha_2^* < 0$].*

Proof. Note that

$$H_1'(\alpha) = \frac{\mathbb{E}[Y_1 e^{\alpha_1 Y_1}]}{\mathbb{E}[e^{\alpha Y_1}]}.$$

This is the expected increment of the reserve of the first company under an exponential change of measure defined by α_1. $H_1'(0)$ is the value of this expectation when there is no change of measure. $\gamma_1 > 0$ iff $H_1'(0)$ greater than $-s_1$. This, the monotonicity of H_1' and (15) imply that $\alpha_1^* < 0$. The argument for α_2^* is analogous. □

Remark 2. Note that the LD analysis of the insurance system is carried out using only the log moment generating functions H_1 and H_2 and it simply consists of solving (15) and (17). These are one dimensional equations involving monotone functions and hence are simple to deal with numerically.

Remark 3. Another output of the LD analysis is the identification of the weakest component in the insurance system and a most likely bankruptcy scenario. The optimizers of (10) identify two bankruptcy scenarios. In the first one the insurance company goes bankrupt and in the second the reinsurance company does. The scenario with the smallest LD decay rate is the more likely one and it is that one which mainly determines the bankruptcy probability of the system.

3.3. An example

Suppose that the claim size X_n is exponentially distribution with rate $a > 0$ and suppose that the interarrival process has intensity $\lambda > 0$.

The log moment generating functions H_1 and H_2 of the companies are

$$\begin{aligned} H_1(\alpha) &= \log \mathbb{E}\left[\exp\left(\alpha Y^{(1)}\right)\right] \\ &= \log \mathbb{E}\left[\exp((c_1 - c)\alpha\tau_1)\right] + \log \mathbb{E}\left[\exp\left(-\alpha X_1 1_{\{X_1 < R\}}\right)\right], \\ H_2(\alpha) &= \log \mathbb{E}\left[\exp\left(\alpha Y^{(2)}\right)\right] \\ &= \log \mathbb{E}\left[\exp(c\alpha\tau_1)\right] + \log \mathbb{E}\left[\exp\left(-\alpha X_1 1_{\{X_1 \geq R\}}\right)\right], \end{aligned}$$

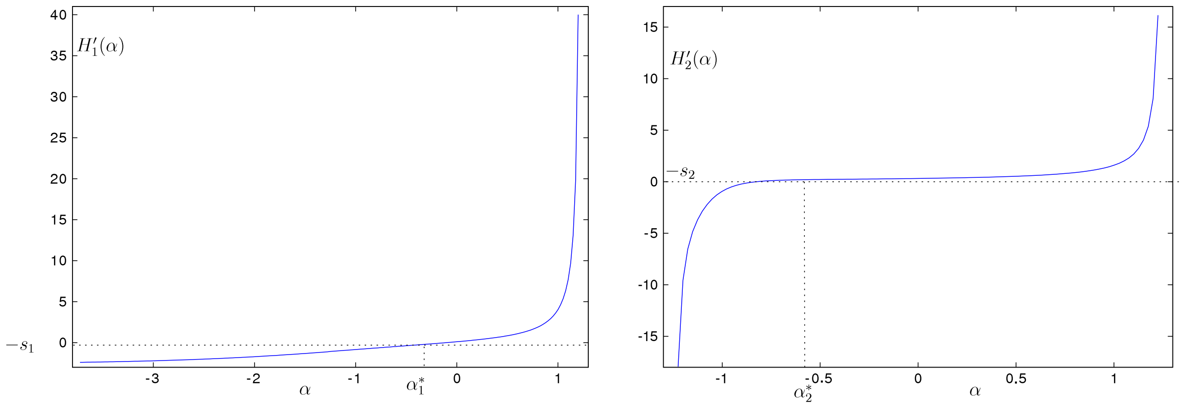

Fig. 4. The graphs of H_1' and H_2' for the parameter values (20).

where we used the assumption that the claim size and the time between claim arrivals are independent of each other. Because τ and X_1 are exponentially distributed these expectations can be calculated explicitly:

$$H_1(\alpha) = -\log(\lambda - (c_1 - c)\alpha) + \log\left(\frac{1 - e^{-(a+\alpha)R}}{a+\alpha} + \frac{e^{-aR}}{a}\right) + \log(a\lambda),$$

$$H_2(\alpha) = -\log(\lambda - c\alpha) + \log\left(\frac{1 - e^{-aR}}{a} + \frac{e^{-(a+\alpha)R}}{a+\alpha}\right) + \log(a\lambda).$$

The derivatives of H_1 and H_2 with respect to α are

$$H_1' = \frac{c_1 - c}{\lambda - ((c_1 - c)\alpha)} + \frac{((a+\alpha)R + 1)e^{-(a+\alpha)R} - 1}{(a+\alpha)^2\left(\frac{1-e^{-(a+\alpha)R}}{a+\alpha} + \frac{e^{-aR}}{a}\right)} \tag{19}$$

and

$$H_2' = \frac{c}{\lambda - (c\alpha)} - \frac{((a+\alpha)R + 1)e^{-(a+\alpha)R}}{(a+\alpha)^2\left(\frac{1-e^{-aR}}{a} + \frac{e^{-(a+\alpha)R}}{a+\alpha}\right)}.$$

3.3.1. A numerical example

To compute γ_1 we need to solve (15). The solution does not seem to have a simple form with H_1' given in (19). Thus we continue with a numerical example. The numerical computations in this work were carried out using Octave [9].

Let us assign the following values to the various parameters:

$$c_1 = 1, \qquad c = 0.2, \qquad R = 3, \qquad a = 5/4, \qquad \lambda = 1. \tag{20}$$

That is, the premium of the insurance company is 1 unit per unit time. 0.2 units of this goes to the reinsurance company. The threshold $R = 3$ units. The claim size is exponentially distributed with mean $1/a = 4/5$ units. And the claims arrive according to a Poisson process with $\lambda = 1$ claim per unit time.

Let the initial reserves of the insurance and the reinsurance company be $S_0^{(1)} = 60$ and $S_0^{(2)} = 100$ respectively. We would like to know how likely it is for this system to be able to cover $N = 200$ claims without bankruptcy.

The s_1 and s_2 values corresponding to the initial reserves $S_0^{(1)}$ and $S_0^{(2)}$ are

$$s_1 = S_0^{(1)}/N = 0.3, \qquad s_2 = S_0^{(2)}/N = 0.5.$$

Computation of γ_1. For these parameter values the graph of H_1' and H_2' are depicted in Fig. 4. A numerical solution of (15) with these parameter values gives

$$\alpha_1^* = -0.39936, \qquad \gamma_1 = -\alpha_1^* s_1 - H_1(\alpha_1^*) = 0.07607.$$

Computation of γ_2. H_2' is depicted in Fig. 4. The solution to (17) and the decay rate γ_2 are

$$\alpha_2^* = -0.50762, \qquad \gamma_2 = -\alpha_2^* s_2 - H_2'(\alpha_2^*) = 0.20376.$$

The LD decay rate of the system is then

$$\gamma = \gamma_1 \wedge \gamma_2 = 0.07607.$$

The approximate ruin probability of the system is $e^{-\gamma N} = 2.47 \times 10^{-7}$. Note that for these parameter values, in the most likely bankruptcy scenario, it is the insurance company that goes bankrupt.

4. γ as a function of c and R

Given the premium rate c_1, how should the companies determine c, the amount of the premium rate that goes to the reinsurance company and the threshold level R? One way to proceed is to choose these parameters so that the insurance system as a whole survives as long as possible. Given the analysis of the previous section, one natural measure of the survival capacity of the insurance system is the LD decay rate γ. The next subsection studies the maximization of γ in the parameters c and R. Throughout we will assume

$$\mathbb{E}[c\tau_1 - X_1] > 0. \tag{21}$$

Under this assumption γ_1 and γ_2 cannot be both zero. Furthermore, γ_1 and γ_2 are continuous in (c, R). These imply that for all of the (c, R) values that will be considered (9) holds and (15)–(18) can be used to compute the LD decay rate of the system.

4.1. Optimization of γ with respect to (c, R)

The LD decay rate of the insurance system is

$$\gamma(c, R) = \gamma_1(c, R) \wedge \gamma_2(c, R). \tag{22}$$

This is the exponential decay rate of the bankruptcy probability of the system and our goal is to maximize γ respect to the variables (c, R).

To carry out this optimization we first compute the derivatives of γ_1 and γ_2 with respect to c and R. Let us rewrite (16), this time indicating the dependence of the various terms on c and R:

$$\begin{aligned} \gamma_1(c, R) &= -\alpha_1(c, R)s_1 - H_1(\alpha_1(c, R), c, R), \\ \gamma_2(c, R) &= -\alpha_2(c, R)s_1 - H_2(\alpha_1(c, R), c, R), \end{aligned} \tag{23}$$

where $\alpha_1(c, R)$ $[\alpha_2(c, R)]$ is the solution α_1^* of (15) [(17)].

Lemma 4.1.

$$\begin{aligned} \frac{\partial \gamma_1}{\partial c}(\alpha_1(c, R), c, R) &= -\frac{\partial H_1}{\partial c}(\alpha_1(c, R), c, R) < 0, \\ \frac{\partial \gamma_1}{\partial R}(\alpha_1(c, R), c, R) &= -\frac{\partial H_1}{\partial R}(\alpha_1(c, R), 0, c, R) \leq 0, \end{aligned}$$

for (c, R) such that $\gamma_1(c, R) \in (0, \infty)$. Similarly,

$$\begin{aligned} \frac{\partial \gamma_2}{\partial c}(\alpha_2(c, R), c, R) &= -\frac{\partial H_2}{\partial R}(0, \alpha_2(c, R), c, R) > 0, \\ \frac{\partial \gamma_2}{\partial R}(\alpha_2(c, R), c, R) &= -\frac{\partial H_2}{\partial R}(0, \alpha_2(c, R), c, R) \geq 0. \end{aligned}$$

for (c, R) such that $\gamma_2(c, R) \in (0, \infty)$.

Proof. We only provide the details for the computation of the first derivative. The computation of the rest is parallel to the one given. It follows from (15) and the implicit function theorem that $\frac{\partial \alpha_1}{\partial c}$ exists when γ_1 is nonzero and finite. Implicit differentiation of (23) gives

$$\frac{\partial \gamma_1}{\partial c} = -\frac{\partial \alpha_1}{\partial c}s_1 - \frac{\partial H_1}{\partial \alpha}\frac{\partial \alpha_1}{\partial c} - \frac{\partial H_1}{\partial c},$$

and (15) implies

$$= -\frac{\partial H_1}{\partial c}(\alpha_1(c, R), c, R).$$

Note that the last derivative equals $\frac{\alpha_1}{\lambda - (c_1 - c)\alpha_1}$. By Lemma 3.1 we know that $\alpha_1(c, R) < 0$. It follows that $\frac{\partial H_1}{\partial c} < 0$. $\square$

Lemma 4.1 implies that γ_1 $[\gamma_2]$ is decreasing [increasing] in the reinsurance premium c and the threshold R.

Let $c^*(R)$ denote the premium rate that maximizes γ of (22) for a fixed R, i.e.,

$$c^*(R) = \mathrm{argmax}_{c \leq c_1} \gamma(c, R),$$

and let $\gamma^*(R)$ denote the maximum:

$$\gamma^*(R) = \max_{c \leq c_1} \gamma(c, R).$$

To keep the exposition brief, we make the following assumption.

Assumption 1. The claim size distribution does not put positive mass on any point.

To compute $c^*(R)$ and $\gamma^*(R)$, let us first consider the case when $R \leq s_1$. In this case, the insurance company has enough funds to pay all of the claims even if all of the premium is collected by the reinsurance company. Therefore,

$$\gamma_1(c, 0) = \infty, \qquad \gamma(c, 0) = \gamma_2(c, 0),$$

for all $c \leq c_1$. Because γ_2 is an increasing function of c, the optimal premium rate for the reinsurance company is

$$c^*(R) = c_1, \text{ if } R \leq s_1, \tag{24}$$

and the maximum LD decay rate for this range of R is

$$\gamma^*(R) = \gamma_2(c_1, R). \tag{25}$$

Assumption 1 implies that $\gamma_1(0, R)$ is continuous at s_1, i.e.,

$$\lim_{R \searrow s_1} \gamma_1(0, R) = \infty. \tag{26}$$

That γ_1 [γ_2] is continuously decreasing [increasing] in R and (26) imply that there is a $R_1^* > s_1$ that satisfies

$$\gamma_1(c_1, R) > \gamma_2(c_1, R), \quad \text{for } R < R_1^*, \qquad \gamma_1(c_1, R^*) = \gamma_2(c_1, R_1^*). \tag{27}$$

Then for $R \leq R_1^*$, (24) and (25) continue to hold. It follows that, because $\gamma_2(c_1, R)$ is increasing in R, $\gamma^*(R)$ is an increasing function of R on $(0, R_1^*)$.

Now consider the case when $R \to \infty$ and $c = 0$. The monotone convergence theorem implies that

$$\lim_{R \to \infty} \mathbb{E}[Y^{(2)}] = 0,$$

which in turn implies

$$\lim_{R \to \infty} \gamma_2(0, R) = \infty.$$

It follows that there is an $R_2^* > R_1^*$ such that

$$\gamma_1(0, R) < \gamma_2(0, R), \quad \text{for } R > R_2^*, \qquad \gamma_1(0, R_2^*) = \gamma_2(0, R_2^*), \quad \text{for } R = R_2^*. \tag{28}$$

This implies

$$c^*(R) = 0 \quad \text{for } R > R_2^*.$$

The corresponding optimal LD rate is

$$\gamma^*(R) = \gamma_1(0, R) \quad \text{for } R > R_2^*.$$

This in particular implies that because $\gamma_1(0, R)$ is decreasing in R, $\gamma^*(R)$ is a decreasing function of R on (R_2^*, ∞).

Now consider the case $R_1^* < R' < R_2^*$. We know that $\gamma_1(c, R)$ is decreasing in R and $\gamma_2(c, R)$ is increasing in R. These and the second part of (27) imply that $\gamma_1(c_1, R') < \gamma_2(c_1, R')$. Monotonicity of γ_1 and γ_2 in R and the second part of (28) imply that $\gamma_1(0, R') > \gamma_2(0, R')$. These and the strict monotonicity of these functions in c imply that there is a unique $0 < c' < c_1$ such that

$$\gamma_1(c', R') = \gamma_2(c', R').$$

This and the monotonicity of γ_i in c imply

$$\max_{0 \leq c \leq c_1} \gamma(c, R') = \max_{0 \leq c \leq c_1} \gamma_1(c, R') \wedge \gamma_2(c, R') = \gamma_1(c', R') = \gamma_2(c', R').$$

Thus for $R_1^* < R' < R_2^*$ we have

$$c^*(R') = c', \qquad \gamma^*(R') = \gamma_1(c', R'),$$

where c' is the unique solution of

$$\gamma_1(c^*, R') = \gamma_2(c^*, R').$$

Again the monotonicity of γ_1 and γ_2 in c and R imply that $c^*(R)$ is decreasing on (R_1^*, R_2^*).

Finally, for $R = 0$ and $c = c_1$, the reinsurance company handles all of the claims and $\gamma^*(0) = \gamma_2(c, 0)$, and for $R = \infty$ and $c = 0$ the insurance company handles them and $\gamma^*(\infty) = \gamma_1(0, \infty)$. Because $s_2 > s_1$, $\gamma_2(c_1, 0) > \gamma_1(0, \infty)$ and therefore $\gamma^*(0) > \gamma^*(\infty)$. These results are summarized in the following proposition.

Proposition 3. *There exists* $0 < R_1^* < R_2^* < \infty$ *such that*

$$\gamma^*(R) = \begin{cases} \gamma_2(c_1, R), & R \leq R_1^*, \\ \gamma_1(0, R), & R \geq R_2^*, \\ \gamma_1(c^*(R), R), & R_1^* < R < R_2^*, \end{cases}$$

where $c^*(R)$ *is the unique solution of*

$$\gamma_1(c, R) = \gamma_2(c, R).$$

$\gamma^*(R)$ *is increasing on* $(0, R_1^*)$, *it is decreasing on* (R_2^*, ∞) *and* $\gamma^*(0) > \gamma^*(\infty)$. $c^*(R)$ *is a decreasing function.*

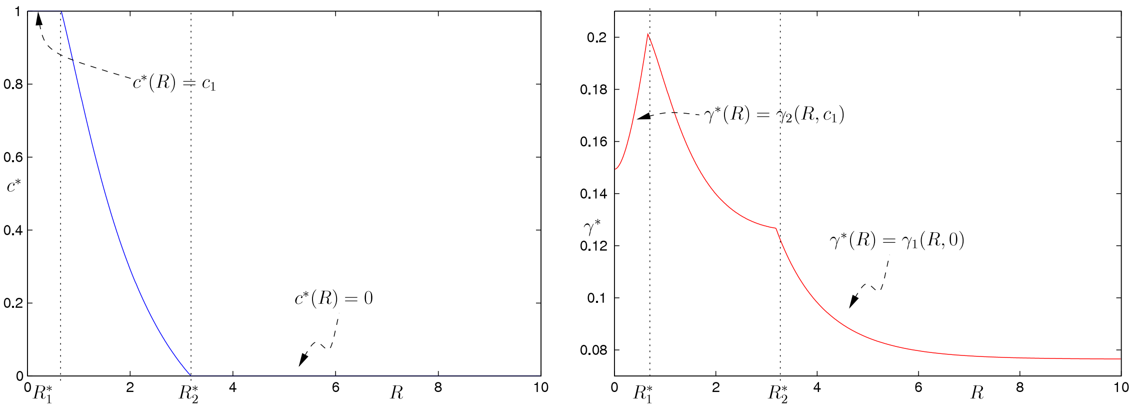

Fig. 5. The functions γ^* and c^* for the example in Section 3.3.1.

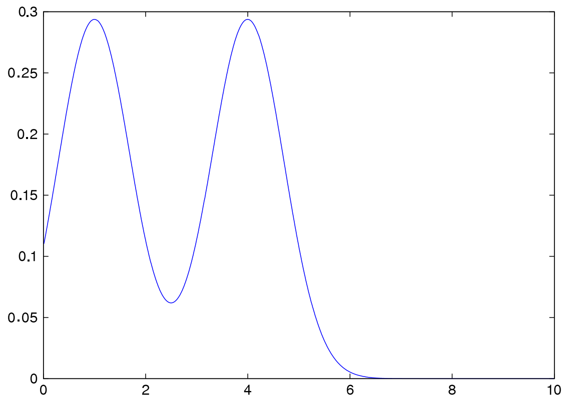

Fig. 6. The density function of claim size for the second numerical example.

4.2. $\gamma^*(R)$ on the interval (R_1^*, R_2^*)

The most interesting part of $\gamma^*(R)$ is its behavior on this interval. This range of values of the threshold variable make a nontrivial sharing of the premium revenue optimal. A rigorous analysis of this behavior appears complicated. We now provide several numerical examples, which suggest that *no matter how the claim sizes are distributed* $\gamma^*(R)$ *is decreasing on* (R_1^*, R_2^*) *if* $s_2 > s_1$.

4.2.1. Numerical examples

We continue with the numerical example of Section 3.3.1. The graph of the function $c^*(R)$ and $\gamma^*(R)$ for the parameter values given in that subsection are depicted in Fig. 5. As is clear, the functions γ^* and c^* behave as described in Proposition 3. Furthermore, γ^* is decreasing on the interval (R_1^*, R_2^*).

A claim size distribution with two extremums. Now take for the distribution of the claim size the following density:

$$f(x) = C\left(\mathrm{e}^{-(x-1)^2} + \mathrm{e}^{-(x-4)^2}\right),$$

where C is chosen so that $\int_{\mathbb{R}^+} f(x)\mathrm{d}x = 1$. The graph of f is depicted in Fig. 6. The rest of the parameter values are assigned as follows:

$$s_1 = 1, \qquad s_2 = 1.5, \qquad \lambda = 1, \qquad c_1 = 2.77. \tag{29}$$

The functions $\gamma^*(R)$ and $c^*(R)$ for these parameter values are depicted in Fig. 7. Their behavior agrees with what is stated in Proposition 3. We further observe that $\gamma^*(R)$ is decreasing on the interval (R_1^*, R_2^*).

A random distribution of claim size. Let us now take as the claim size distribution a random distribution on the interval $[0, 10]$ and the following values for the rest of the parameters:

$$s_1 = 1, \qquad s_2 = 2, \qquad \lambda = 1, \qquad c_1 = 5.2. \tag{30}$$

The randomly generated distribution function for the claim values is depicted in Fig. 8. The functions $\gamma^*(R)$ and $c^*(R)$ for these parameter values are depicted in Fig. 9. We observe the same behavior as before, $\gamma^*(R)$ and $c^*(R)$ behave as described in Proposition 3 and $\gamma^*(R)$ is decreasing on (R_1^*, R_2^*).

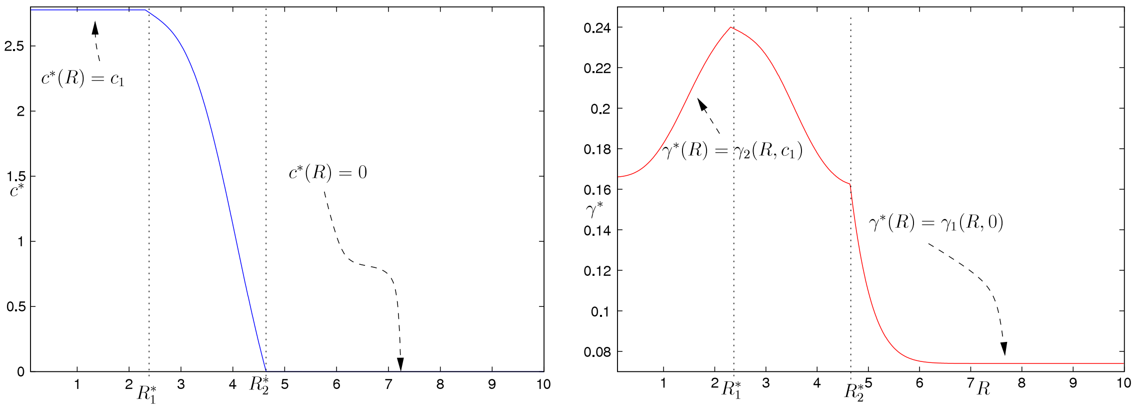

Fig. 7. The functions γ^* and c^* for the parameter values (29).

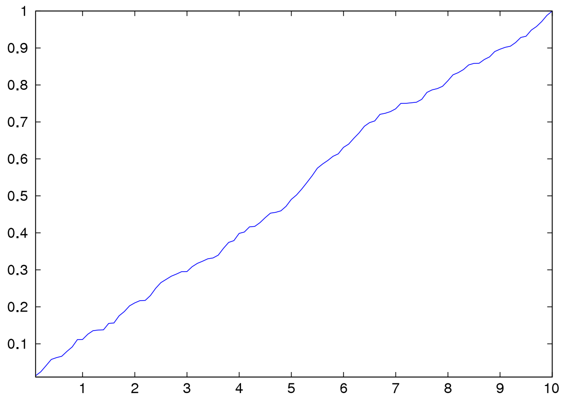

Fig. 8. The distribution function of claim size for the second numerical example.

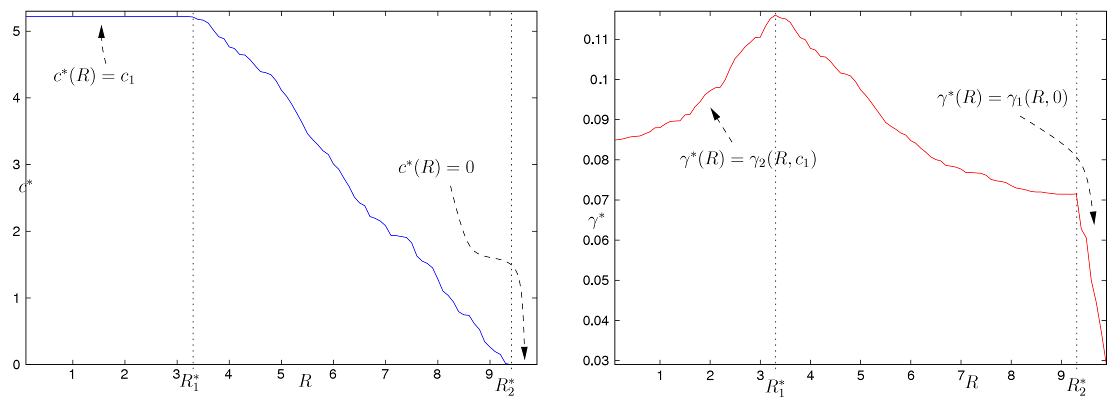

Fig. 9. The functions γ^* and c^* for the parameter values (30).

4.3. Discussion

The numerical examples suggest that $\gamma(c, R)$ takes its maximum value at the point (c_1, R_1^*). With $c = c_1$, all of the premium goes to the reinsurance company. R_1^* is set at a level such that the insurance company can handle its responsibilities with its initial funds with a very high probability. In the meantime, all of the premium is added to the reserves of the reinsurance company. This is almost like merging of the two companies (the funds of the company with the lower initial reserve is effectively being transferred to the company with the greater reserve).

The numerical examples also suggest that $\gamma^*(R)$ is decreasing on the interval (R_1^*, R_2^*). If this is accurate, the situation looks as follows from the point of view of the insurance company. The insurance company has an LD decay rate $\gamma' = \gamma_1(0, \infty)$ of its bankruptcy probability on its own. It can increase this decay rate by getting into a reinsurance agreement with a company with greater reserves. It can set itself a target LD decay rate $\gamma_t > \gamma_0$ and use this target to determine the threshold R_t of the reinsurance contract by solving $\gamma^*(R_t) = \gamma_t$. The higher γ_t, the smaller R_t will be. If the corresponding $c^*(R_t)$ is agreeable to both sides, a reinsurance contract can be made. In this way the insurance company will have improved its LD decay rate from γ' to γ_t in a way that also tries to minimize the probability of bankruptcy of the system formed by the two companies.

It is interesting to compare the best performance of the system under the reinsurance contract to that of a single company I formed by merging the companies in the system. Every sequence of claims that bankrupts I will also bankrupt the insurance system. This implies that the LD decay rate of I is always larger than $\gamma(c, R)$. To see how large the difference can be, let us look, for example, at the numerical example from 3.3.1. R_1^* for this system is 0.66 and $\gamma(c_1, R_1^*) = 0.20119$, which corresponds to an approximate bankruptcy probability of $\exp(-\gamma N) = 3.3 \times 10^{-18}$. The initial reserves of I is $60 + 100$. This is equal to $s = 160/200 = 0.8$ units of money per claim to be covered. The LD decay rate of I is computed using (15) and (16) with $s_1 = 0.8$ and $c = c_1 = 1$, $R = \infty$ and is $\gamma_0 = 0.2837$. The corresponding approximate bankruptcy probability is $\exp(-\gamma_0 N) = 2.2 \times 10^{-25}$, which is 10^{-7} times smaller than the approximate bankruptcy probability of the insurance system.

5. Conclusion

One straightforward extension of the current model is by increasing the number of insurance companies that work with the reinsurance company. It appears that the analysis of such a setup will be similar to the two dimensional case which we treated in this article.

Higher dimensional and more realistic models bring with themselves interesting practical considerations. Suppose, there are a number of insurance companies and one reinsurer. If a bankruptcy indeed does happen, what happens next? The answer depends on the laws regulating the insurance system and on the contracts made between the parties involved. If a model of the insurance system which takes into account these kind of considerations can be built, the following type of problems can be studied: "what is the most likely way for the insurance system to collapse?"

A possibility is the design and analysis of dynamic reinsurance contracts that change in time as the reserves and the parameters of the system evolve in time. The design of such contracts can also aim at the minimization of bankruptcy probability of the insurance system.

Another important direction of generalization is the study of the same problem with heavy tails, because many real life claim sizes are modeled with heavy tailed distributions. The assumption of light tails, and in particular, the existence of the moment generating function in a neighborhood of 0 is required for the type of LD analysis carried out in this work. Thus, for the heavy tailed case different methods will be needed.

An interesting mathematical problem is the identification of the family of distributions of claim sizes for which the function $\gamma^*(R)$ is decreasing on the interval (R_1^*, R_2^*). $\gamma^*(R)$ was decreasing for all of the examples studied in the present work.

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