Vapnik-Chervonenkis dimension

We need to prove exactly two things

(i.) $\tilde{C}_{(N+1,N)} = 2^{N+1}$

Proof:

$$\begin{aligned}
2^{N+1} &= 2 \cdot (1+1)^N \\
&\overset{\dagger}{=} 2 \cdot \sum_{k=0}^{N} \binom{N}{k} 1^{N-k} 1^k \\
&= 2 \cdot \sum_{k=0}^{N} \binom{N}{k} \\
&= \tilde{C}_{(N+1,N)}
\end{aligned}$$

q.e.d

(ii.) $\tilde{C}_{(N+2,N)} < 2^{N+2}$

Proof:

$$\begin{aligned}
2^{N+2} &= 2 \cdot (1+1)^{N+1} \\
&\overset{\dagger}{=} 2 \cdot \left[\binom{N+1}{N+1} + \sum_{k=0}^{N} \binom{N+1}{k} \right] 1^{N+1-k} 1^k \\
&= 2 \cdot \left[1 + \sum_{k=0}^{N} \binom{N+1}{k} \right] 1^{N+1-k} 1^k \\
&= 2 \cdot \sum_{k=0}^{N} \binom{N+1}{k} + 2 \\
&= \tilde{C}_{(N+2,N)} + 2
\end{aligned}$$

q.e.d

$\dagger$:= here we use the binomial formula