Final Exam of Computational Neuroscience

Due January 31, 2021

Poisson Spike-Train Statistics

1. Given the Homogeneous Poisson process (mean firing rate is independent of time),

$$P(n) = \frac{(rT)^n}{n!} \exp(-rT). \tag{1}$$

Calculate the mean $\langle n \rangle$ and variance $Var(n)$ of the spike count. Compute the Fano factor $Var(n)/\langle n \rangle$. Calculate the kurtosis of spike count defined as $k = \langle n^4 \rangle - 3\langle n^2 \rangle^2$ in the time interval T.

2. When the firing rate depends on time, we could also extend the homogeneous Poisson process to inhomogeneous Poisson process. When n spikes occurs in an interval T with $0 < t_1 < t_2 < ... < t_n < T$, Prove that the joint probability density is given by

$$p(t_1, t_2, ..., t_n) = \exp\left(-\int_0^T r(t)dt\right) \prod_{i=1}^{n} r(t_i) \tag{2}$$

Then, calculate the probability of seeing n spikes $P(n)$ in the interval T. Check whether it has a similar expression as the homogeneous Poisson process and calculate the Fano Factor.

3. Generate a Poisson spike train with a time-dependent fire rate $r(t) = r_0[1 + cos(2\pi t/\tau)]$ where $r_0 = 100$ Hz and $\tau = 300$ ms. Generate a spike train for 20 s and plot it.

4*. Let's assume that the firing rate of a neuron has the following functional form: $r(t) = r_0 + r_1 \sin(\omega t + \theta)$, where the phase θ is drawn uniformly between 0 and 2π for each trial. Calculate the Fano Factor for the spike count in the time interval T (as a function of T).

Note: Problems with * are optional. However, solving them will give you additional credits.

Sensory Neuron from an Electric Fish

Electric fish can generate and sense electric fields. The response of a electrosensory neuron $R(t)$ is characterized by the firing rate, which is the number of spikes (action potential)occurred within a time window divided by the time bin size Δt. Use the following equation

$$R(t) = R_0 + \int_0^\infty D(\tau)s(t-\tau)dt, \tag{3}$$

with $R_0 = 50$ Hz, and

$$D(\tau) = -\cos(\frac{2\pi(\tau - 20\text{ms})}{140\text{ms}})\exp(-\frac{\tau}{60\text{ms}})\text{Hz/ms}, \tag{4}$$

to predict the response of a neuron of the electrosensory lateral-line lobe to a stimulus. Use an approximate Gaussian white noise stimulus constructed by choosing a stimulus value every 10 ms ($\Delta t = 10$ ms) from a Gaussian distribution with zero mean and variance $\sigma^2/\Delta t$, with $\sigma^2 = 10$. A detailed description of white noise can be found on page 21 of the theoretical neuroscience textbook.

1. Compute the firing rate over a 10 s period.
2. From the results, compute the firing rate-stimulus correlation function $Q_{rs}(\tau)$.
3. Compare $Q_{rs}(-\tau)/\sigma^2$ with the kernel $D(\tau)$ given above.

Winner Take All Circuit

Consider the following recurrent network dynamics of N neurons:

$$\begin{aligned}\frac{dx_i}{dt} &= -x_i + \left[b_i + \alpha x_i - \beta \sum_{j=1, j\neq i}^{N} x_j\right]_+ &(5)\\ &= -x_i + [b_i + (Wx)_i]_+ &(6)\end{aligned}$$

where we have denoted $W = (\alpha + \beta)I - \beta \underline{1}_N$ ($\underline{1}_N$ is an $N \times N$ matrix of ones) and $[x]_+ \equiv max(x, 0)$. $\alpha > 0$ is self-excitation and $\beta > 0$ represents a global inhibition.

The external inputs ($b_i > 0$) are fixed in time and we assume that they are all different from each other. Prove the following:

*A. If $\alpha < 1$, then the network will converge asymptotically to a fixed point from almost all initial conditions.

B. If $\alpha < 1$ and $\beta > 1-\alpha$ then the only possible stable states of the network are fixed points in which only a single neuron is active.

C. Given B, the neurons that can remain active at long time are those for which:

$$b_i \geq \frac{1-\alpha}{\beta} b_{\max} \tag{7}$$

(where $b_{\max} = \max_i b_i$). From this result, derive the conditions which guarantee that, independent of the initial conditions, the network evolves into a state where only the neuron with the largest b_i is active (i.e., the network picks the "winner" neuron).

Instructions for *A: Prove that

$$\begin{aligned} E(x) &= -\sum_i b_i x_i + \frac{1}{2}(1-\alpha)\sum_i x_i^2 + \frac{\beta}{2}\sum_{i \neq j} x_i x_j && (8) \\ &= \frac{1}{2}x^T x - b^T x - \frac{1}{2}x^T W x && (9) \end{aligned}$$

is a Lyapunov function of the system. When calculating $\frac{dE}{dt}$ it is useful to consider separately the contributions from neurons such that $b_i + (Wx)_i > 0$ and those for which $b_i + (Wx)_i < 0$.

Instructions for B: Assume there exists a fixed point with K active neurons. We can arrange the order of the neurons in the system so that: $x_1^*, ...x_K^* > 0$, $x_{K+1}^*, ...x_N^* = 0$ where K denotes the number of active neurons in the fixed point. By linearizing around such a fixed point, prove that if $\alpha < 1$ and $\beta > 1-\alpha$, then the fixed point is stable iff $K = 1$. Note: stability of the inactive neurons is quite straightforward to show. For the active neurons you need to diagonalize a matrix of the form $\underline{1}_K$. If you have difficulty in this part, try at least to analyze the stability of the state with $K = 1$.

Instructions for C: Assume a fixed point with a single active neuron as claimed by **B** and show that consistency requires that the active neuron obey the property Equation **??**.

Note: If you have difficulty with proving A, you can assume A and proceed to prove B and C. Likewise, if you have difficulties proving B, assume that B holds and proceed to prove C.