Final Exam of Computational Neuroscience

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Poisson Spike-Train Statistics

1.

$$\begin{aligned}\langle n\rangle = \sum_{n=0}^{\infty} np(n) &= \sum_{n=1}^{\infty} \frac{(rT)^n}{(n-1)!}\exp(-rT) \\ &= \exp(-rT)rT\sum_{n=0}^{\infty}\frac{(rT)^n}{n!} \\ &= \exp(-rT)rT\exp(rT) \\ &= rT\end{aligned} \tag{1}$$

$$\begin{aligned}\langle n^2\rangle = \sum_{n=0}^{\infty} n^2p(n) &= rT\exp(-rT)\sum_{n=0}^{\infty}\frac{(rT)^n}{n!}(n+1) \\ &= rT\exp(-rT)\left[\sum_{n=0}^{\infty} n\frac{(rT)^n}{n!} + \sum_{n=0}^{\infty}\frac{(rT)^n}{n!}\right] \\ &= rT\exp(-rT)\left[rT\exp(rT)+\exp(rT)\right] \\ &= (rT)^2 + rT\end{aligned} \tag{2}$$

$$Var(n) = \langle n^2\rangle - \langle n\rangle^2 = rT \tag{3}$$

$$Fano\ factor : \frac{Var(n)}{\langle n\rangle} = 1 \tag{4}$$

$$\langle n^4\rangle = (rT)^4 + 6(rT)^3 + 7(rT)^2 + rT \tag{5}$$

$$k = \langle n^4\rangle - 3\langle n^2\rangle^2 = -2(rT)^4 + (rT)^2 + rT \tag{6}$$

if we replace $\langle n^4\rangle$ with central moments, the result may look more beautiful:

$$\mu_4 = \langle (n-tT)^4\rangle = 3(rT)^2 + rT \tag{7}$$

$$k = \mu_4 - 3(rT)^2 = rT \tag{8}$$

2. when it comes to inhomogeneous Poisson process:
   we devide time interval $[t_i, t_{i+1}]$ into M time bins of size Δt and setting $M\Delta t = t_{i+1} - t_i$. we will ultimately take limit $\Delta t \to 0$. The firing rate during bin m within this interval is $r(t_i + m\Delta t)$, the firing probability of firing a spike in this bin is $r(t_i + m\Delta t)\Delta t$, the probability of not firing a spike is $1 - r(t_i + m\Delta t)\Delta t$. So, to have no spike during the entire interval, we need string together the whole M bins, because the probability in each bin is independent, we just need to product them together. Denote entire spike numbers between initial time and time t $N(t)$

$$P[N(t_{i+1}) - N(t_i) = 0] = \prod_{m=1}^{M} (1 - r(t_i + m\Delta t)\Delta t) \tag{9}$$

$$\begin{aligned} \ln P[N(t_{i+1}) - N(t_i) = 0] &= \sum_{m=1}^{M} \ln(1 - r(t_i + m\Delta t)\Delta t) \\ &\xlongequal{t\to 0} - \sum_{m=1}^{M} r(t_i + m\Delta t)\Delta t \\ &= -\int_{t_i}^{t_{i+1}} r(t)dt \end{aligned} \tag{10}$$

   the probability density $p(t_1, t_2, ..., t_n)$ is the product of the densities for the individual spikes and the probabilities of not generating spikes during the interspike intervals, between time 0 and the first spike, and between the time of the last spike and the end of the trial period:

$$\begin{aligned} &p(t_1, t_2, ..., t_n) \\ =&P[N(t_1) - N(0) = 0] \cdot P[N(T) - N(t_n) = 0] \cdot \\ &\prod_{i=1}^{n-1} P[N(t_{i+1} - N(t_i)) = 0] \cdot \prod_{i=1}^{n} r(t_i) \\ =&\exp\left(-\int_0^T r(t)dt\right) \prod_{i=1}^{n} r(t_i) \end{aligned} \tag{11}$$

$$\begin{aligned} &p[N(T) - N(0) = n] \\ =&\int_0^T dt_1 \cdots \int_0^T dt_n p(t_1, t_2, ..., t_n)/n! \\ =&\exp\left(-\int_0^T r(t)dt\right) \int_0^T r(t_1)dt_1 \cdots \int_0^T r(t_n)dt_n/n! \\ =&\left(\int_0^T r(t)dt\right)^n \exp\left(-\int_0^T r(t)dt\right)/n! \\ &n = 0, 1, 2 \ldots \end{aligned} \tag{12}$$

it is very similar to homogeneous situation, when r(t) = r(const), this expression returns to homogeneous result

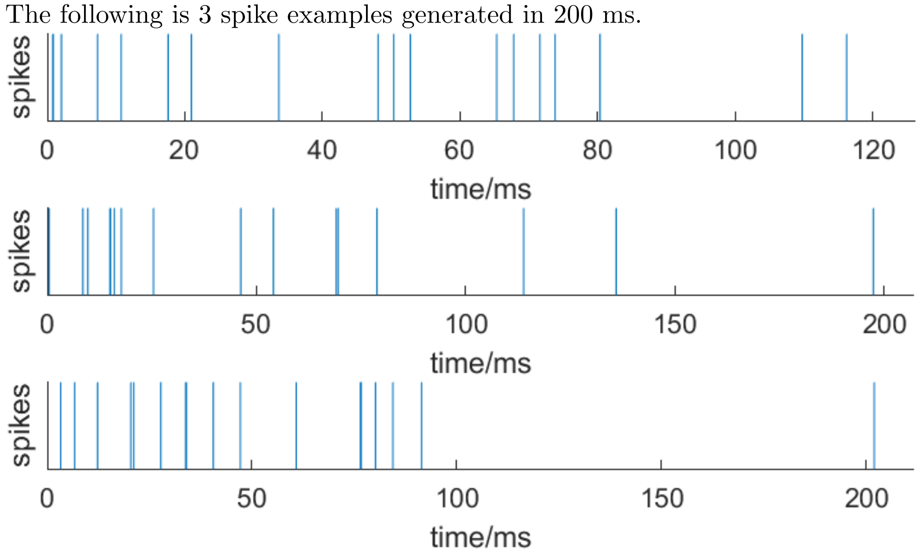
$$\langle n \rangle = \int_0^T r(t)dt \tag{13}$$

$$Var(n) = \int_0^T r(t)dt \tag{14}$$

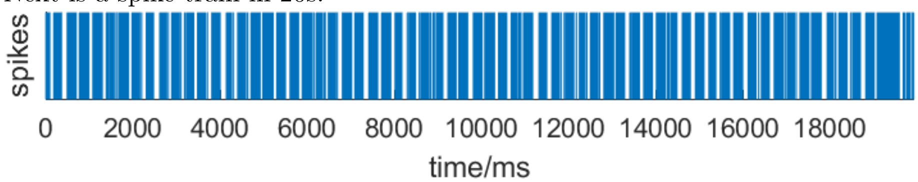
$$Fano\ factor : \frac{Var(n)}{\langle n \rangle} = 1 \tag{15}$$

the result is the same as homogeneous's

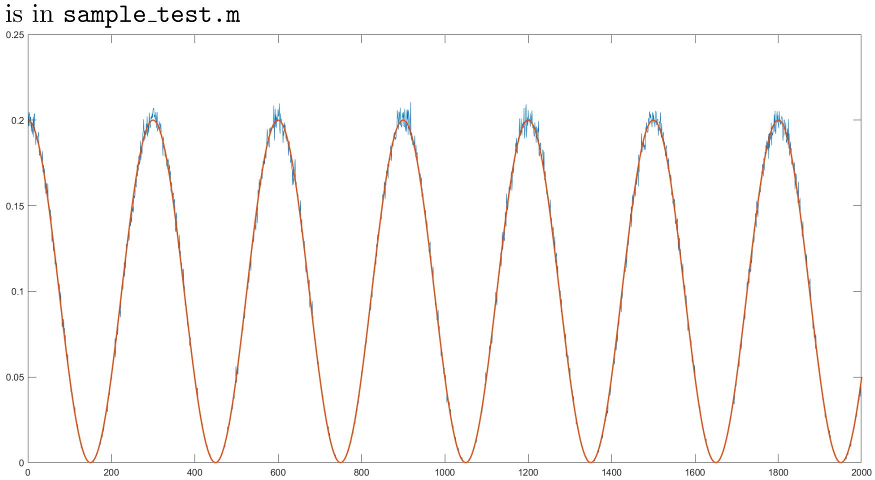
3. Reference the **poisson spike generator** on page 30 of therotical Neuroscience book. First, generate the interspike intervals from an exponential probability density. Then extend it to time-dependent rates by using rejection sampling. Rejection sampling requires a bound r_{max} on the estimated firing rate such that $r_{est}(t) \leq r_{max}$ at all times. We first generate a spike sequence corresponding to the constant rate r_{max} by iterating the rule $t_{i+1} = t_i - \ln(x_{rand})/r_{max}$. The spikes are then thinned by generating another x_{rand} for each i and removing the spike at time t_i from the train if $r_{est}(t_i)/r_{max} < x_{rand}$. If $r_{est}(t_i)/r_{max} \geq x_{rand}$, spike i is retained. Thinning corrects for the difference between the estimated time-dependent rate and the maximum rate.
   main code is in `poisson_spike_generator.m`.
   The following is 3 spike examples generated in 200 ms.



Next is a spike train in 20s:



Then, I did a mento carlo silmulation to test whether the sampling is right. Rerun the code I have used above s times, s is a large number, each s represents a trial. I roughly calculate r(t) using sampling result in a discrete way. r(ith time-bin)$\approx \langle n \rangle$, $\langle \rangle$ is mean from different trial, n is spike numbers in ith time-bin. i = 1,2,3. . . we could see from the following pic, the result is satisfactory, blue line is sample result, red line is real r(t), code is in `sample_test.m`



4. In this problem, we need to use conditional expectation, use formula of total expectation twice

$$
\begin{aligned}
\langle n \rangle &= E[E(n|\theta)] = E\left[\int_0^T (r_0 + r_1 \sin(\omega t + \theta))d\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^T dt(r_0 + r_1 \sin(\omega t + \theta)) \\
&= r_0 T
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&\langle n^2 \rangle = E[E(n^2|\theta)] \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\theta \left[\left(\int_0^T r_0 + r_1 \sin(\omega t + \theta)dt\right)^2 + \int_0^T r_0 + r_1 \sin(\omega t + \theta dt)\right] \\
&= (r_0 T)^2 + \left(\frac{r_1}{w}\right)^2 - \left(\frac{r_1}{w}\right)^2 \cos \omega T + r_0 T
\end{aligned}
\tag{17}
$$

$$Var(n) = \langle n^2 \rangle - \langle n \rangle^2 = r_0 T + 2\left(\frac{r_1}{w}\sin\frac{\omega T}{2}\right)^2 \tag{18}$$

$$Fano\ factor: \frac{Var(n)}{\langle n \rangle} = 1 + \frac{2\left(\frac{r_1}{w}\sin\frac{\omega T}{2}\right)^2}{r_0 T} \tag{19}$$

Sensory Neuron from an Electric Fish

We consider times that are interger multiples of a basic unit of duration Δt, that is times $t = m\Delta t$ for m = 1,2,...,M where $M\Delta t = T$. The function s(t) is then constructed as a discrete sequence of stimulus values. This produce a steplike stimulus waveform, with a constant stimulus value s_m presented during time bin m. In terms of white-noise stimuli definition, each s_m is uncorrelated means

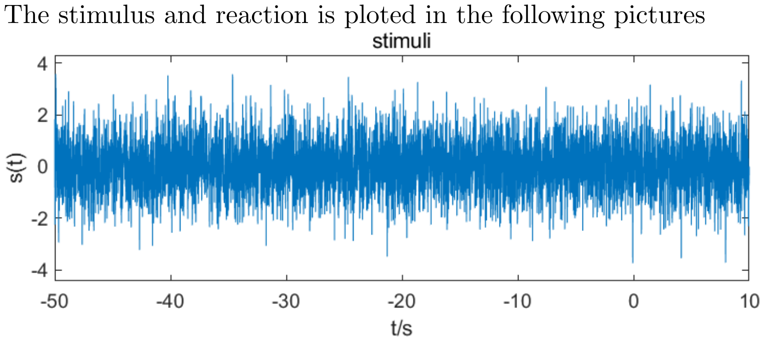
$$\frac{1}{M}\sum_m^M \sum_n s_m s_n = \frac{\sigma_s^2}{\Delta t}\delta_{mn} \tag{20}$$

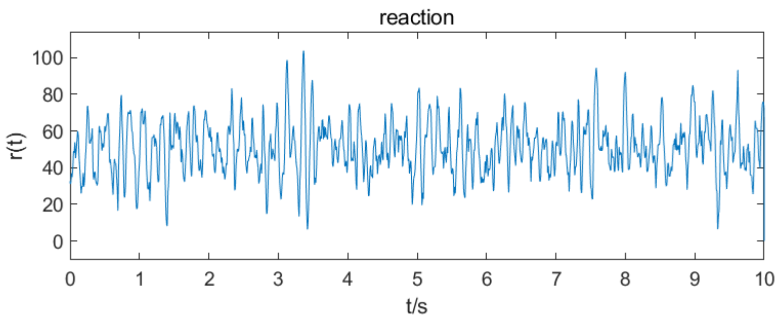
where $\delta_{mn} = 0$,only if $m = n$

1. When $t \in [(n-1)\Delta t, n\Delta t]$, $s = s_n$, during each bin, the value of D could be seen as a constant, for example, when $t - \tau \in [(n-1)\Delta t, n\Delta t]$,$D(\tau) \approx D(t - n\Delta t)$. Then, we can replace integral with limited sum, the following eqaution could be obtained easily

$$\begin{aligned} r(t) &= R_0 + \int_0^\infty D(\tau)s(t-\tau)d\tau \\ &= R_0 + \sum_{n=-\infty}^{[t/\Delta t]} s_n D(t - n\Delta t)\Delta t \end{aligned} \tag{21}$$

with the equation we have aquired above, we realize it in code `fish.m`. The stimulus and reaction is ploted in the following pictures





2. Same as the previous question, replace the integral with limited sum, when $t+\tau \in [(m-1)\Delta t, m\Delta t]$,$D(t-n\Delta t) \approx D(m\Delta t - \tau - n\Delta t)$

$$
\begin{aligned}
Q_{rs}(\tau) &= \frac{1}{T}\int_0^T r(t)s(t+\tau)dt \\
&= \frac{1}{T}\int_0^T (R_0 + \sum_n s_n D(t-n\Delta t)\Delta t)s(t+\tau)dt \\
&= \frac{1}{T}\sum_m (R_0 + \sum_n s_n D(m\Delta t - \tau - n\Delta t)\Delta t)s_m \Delta t \\
&= \frac{1}{T}\sum_m R_0 s_m (\Delta t)^2 + \frac{\Delta t}{T}\sum_m \sum_n s_m s_n D[(m-n)\Delta t - \tau] \\
&= \frac{1}{M}\sum_m \sum_n s_m s_n D[(m-n)\Delta t - \tau]
\end{aligned}
\tag{22}
$$

omit $\dfrac{1}{T}\sum_m R_0 s_m (\Delta t)^2$ because it is higher order infinitesimal. Then, with the definition of white noise stimuli (20)

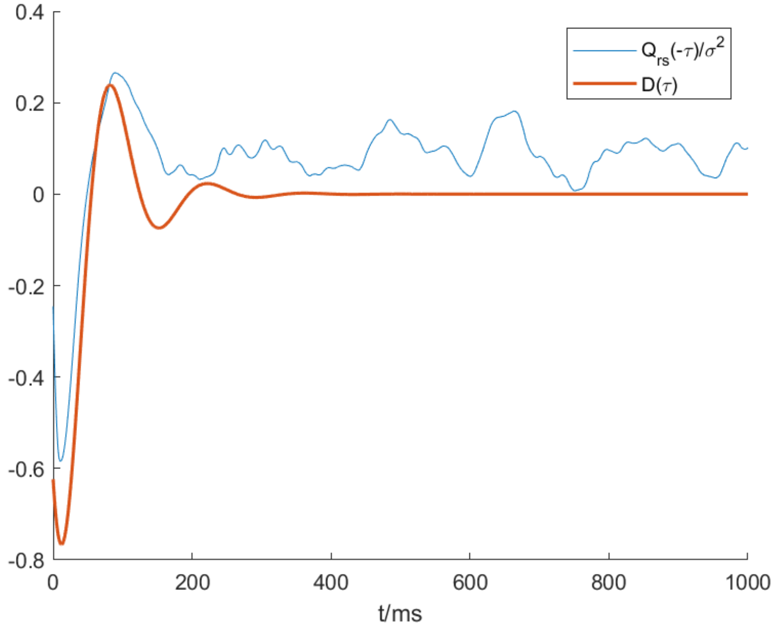
$$Q_{rs}(\tau) = \sigma_s^2 D[(m-n)\Delta t - \tau]\delta_{mn} = \sigma_s^2 D(-\tau) \tag{23}$$

3. From theoretical deduction, it is obvious that

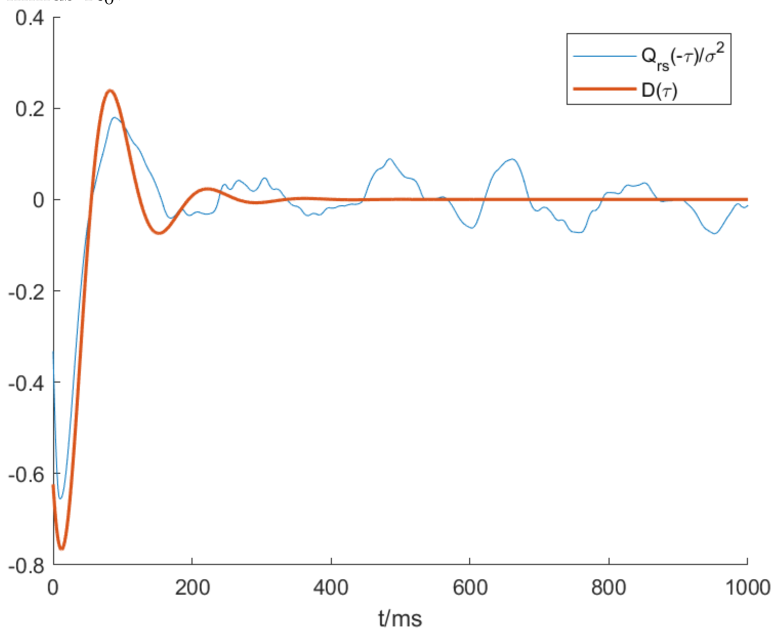
$$Q_{rs}(-\tau)/\sigma^2 = D(\tau) \tag{24}$$

The simulation result could be gotten from code `fish.m` In equation 22, we omit the first term in forth line because when $\Delta t \to 0$, it is a higher order infinitesimal, but in my calculation, I replace integration with discrete sum, so Δt is not infinitesimal, so when we compute Q_{rs}, if we minus R_0 from R, i.e. omit the first term manually, our result would be much more satisfactory.

does not minus R_0:



minus R_0:



Winner Take All Circuit

1. We have known the Lyapunov stability thereom, assume x^* is equilibium point, the system is globally asymptorically stable if following conditions are satisfied

   - $E(x^*) = 0$
   - $E(x) > 0, \forall x \neq x^*$
   - $E(x)$ is radially unbounded
   - $E(x) < 0, \forall x \neq x^*$

   the Lyapunov function is

   $$E(x) = -\sum_i b_i x_i + \frac{1}{2}(1-\alpha)\sum_i x_i^2 + \frac{\beta}{2}\sum_{i\neq j} x_i x_j \tag{25}$$

   Then,

   $$\begin{aligned}\frac{dE}{dt} = \frac{dE}{dx}\dot{x} &= \sum_i(-b_i + (1-\alpha)x_i + \frac{\beta}{2}\sum_{j\neq i} x_j)(-x_i + [b_i + (Wx)_i]_+) \\ &= -\sum_i(-x_i + b_i + (Wx)_i)(-x_i + [b_i + (Wx)_i]_+)\end{aligned} \tag{26}$$

   when $b_i + (Wx)_i > 0$, $\dfrac{dE}{dt} = -\sum_i(-x_i + b_i + (Wx)_i)^2 \leq 0$, the equality holds when x is at equilibium point
   when $b_i + (Wx)_i < 0$,$\dfrac{dE}{dt} = \sum_i x_i(-x_i + b_i + (Wx)_i)$, the discussion about this situation is a little complicated
   let's consider a suitable linear transformation

   $$y_i = b_i + \sum w_{ij} x_j \tag{27}$$

   then we could get that the following two equalities are equivalent

   $$\begin{aligned}\dot{y} + y = b + W[y]_+ &\overset{y=Wx+b}{\Longleftrightarrow} Wx + b + W\dot{x} = b + W[b+Wx]_+ \\ &\overset{W invertible}{\Longleftrightarrow} x + \dot{x} = [b+Wx]_+\end{aligned} \tag{28}$$

   then, E for y is

   $$E = \frac{1}{2}\sum_{kj} F_k(\delta_{kj} - w_{kj})F_j - \sum_k b_k F_k \tag{29}$$

   where $F_i = [y_i]_+$ write E in vector form, $E = \frac{1}{2}F^T(\mathbb{I} - W)F - b^T F$, E is lower bounded if $z^T(\mathbb{I} - W)z$ is copositive, i.e. $z^T(\mathbb{I} - W)z > 0 \forall z \neq 0$

with $z_i \geq 0, \forall i$, so, when it comes to the exact W we have, for simplicity, let the size of W is $(n \times n)$:

$$\begin{pmatrix} F_1 & F_2 \end{pmatrix} \begin{pmatrix} 1-\alpha & \beta \\ \beta & 1-\alpha \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = (1-\alpha)(F_1^2 + F_2^2) + 2\beta F_1 F_2 > 0 \tag{30}$$

since is inequality has to hold for any $x \geq 0$ (and$\beta > 0$), it has to hold that

$$\begin{gathered} 1 - \alpha > 0, \\ i.e. \quad \alpha < 1 \end{gathered} \tag{31}$$

2. Reference the work we done in hw3 about the ring network, replace the **rectified linear network** with **x**, then do calculation and analysis the result.
First try to find the eigenvalue and eigenvector of W. let n = 3

$$\begin{vmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{vmatrix} \to \begin{vmatrix} \alpha-2\beta & -\beta & -\beta \\ \alpha-2\beta & \alpha & -\beta \\ \alpha-2\beta & -\beta & \alpha \end{vmatrix} \to \begin{vmatrix} \alpha-2\beta & -\beta & -\beta \\ 0 & \alpha+\beta & 0 \\ 0 & 0 & \alpha+\beta \end{vmatrix} \tag{32}$$

it is easy to get the eigenvalues are $\{\alpha-(n-1)\beta, \alpha+\beta\}$, the eigenvector for $\alpha-(n-1)\beta$ is $(1, 0, ..., 0)$, for $\alpha+\beta$ are unit vector in the other dimensions, eg. $e_i = (0, ..., 1, ..., 0), i = 2, 3, ..., n$
let $x = \sum_k c_k e_k$, then the equation $\dot{x} = -x + [b + Wx]_+$ could be written as

$$\sum_k \frac{dC_k}{dt} e_k = \sum_k c_k(\lambda_k - 1)e_k + \overrightarrow{b}$$

$$\frac{dC_m}{dt} = C_m(\lambda_m - 1) + \overrightarrow{b} \cdot e_m = C_m(\lambda_m - 1) + b_m \tag{33}$$

the solution is

$$c_m(t) = \frac{b_m}{1-\lambda_m}\{1 - \exp[-t(1-\lambda_m)]\} + c_m(0)\exp(-t(1-\lambda_m)) \tag{34}$$

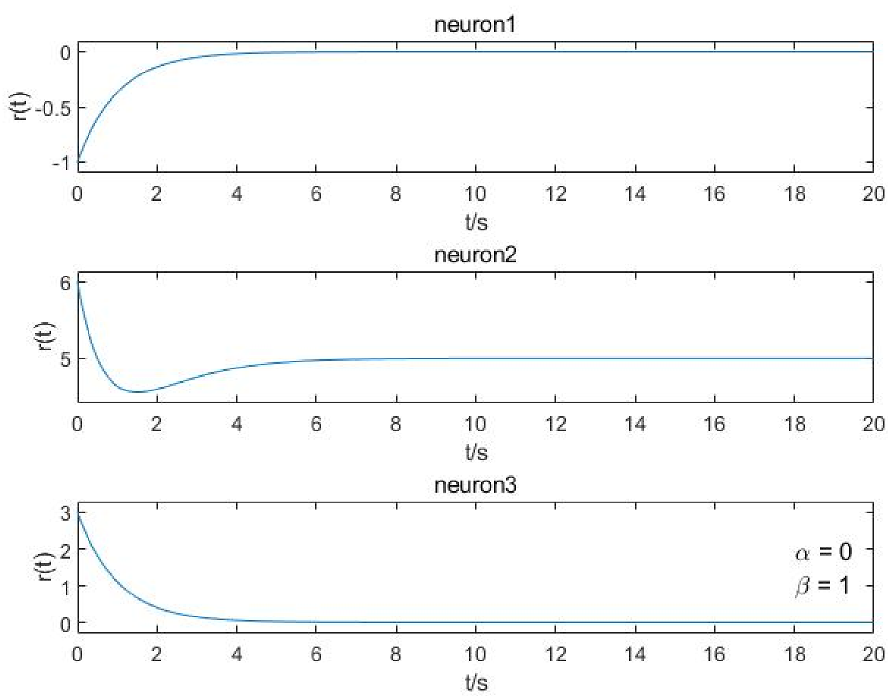
as same, let the exponent term do not explode, we request $\lambda_m < 1$, i.e.

$$\begin{cases} \alpha - (n-1)\beta < 1 \\ \alpha + \beta < 1 \end{cases} \tag{35}$$

but given that $\beta > 1 - \alpha$, i.e. $\alpha + \beta > 1$, the second inequality cannot be satisfied, so when m = 2,3,...,n, the corresponding term will explode.
Then, we replace **x** with **rectified linear network**, the corresponding terms will converge to zero.
Consider the leftover term ,
$\alpha - (n-1)\beta < \alpha - (n-1)(1-\alpha) = 1 + n * (\alpha - 1) < 1$
so this term will converge to a steady state
all in all, only a single neuron is active

3. If just pick the neuron with largest b_{max}, when the system meets the steady state, $x_i^* = [b_i + (Wx)_i]_+ = b_i + \alpha x_i$, let $x_i = max_i b_i$, we could get $\alpha = 0$
   Further, consider the transformation 27 we have taken in the first question. if $x_j = 0$ for all j except $j = argmax(b_j) = m$, this means $y_j = b_j + (Wx)_j = b_j - \beta b_m \leq 0, \forall j \neq m$, thus, $\beta \geq \dfrac{max_{j \neq m} b_j}{b_m} \to 1$
   all in all,
   $$\alpha = 0, \beta = 1 \tag{36}$$
   the following is some simulation results with code `winner_take_all.m` for simplicity, there are only 3 neurons, vector b is set to $\{4, 5, 0.5\}$. We can get more results by changing the initial conditions

   the situation in question 3, network picks the winner neuron:



not only one neuron is exited: