計算の理論 最終課題 入力例及び実行ログ

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1 準備

[1]:
```
from typing import List
import simply_typed_lambda_parser as parser
import simply_typed_lambda_calculi as stlc
```

[2]:
```
def prove(proposition: str, typenames: List[str]):
    return parser.parse_type(proposition, typenames=typenames).deduce(env=[],␣
↪hypothesis={}, visit={})
```

2 証明可能な例

2.1 交換則

[3]:
```
prove(r'(A\/B)->(B\/A)', typenames=['A', 'B'])
```

[3]:

$\vdash \lambda t_2 : A \vee B.\, \mathrm{case}\ t_2\ \mathrm{of}\ \mathrm{inl}(t_0) \Rightarrow \mathrm{inr}(t_0) \mid \mathrm{inr}(t_1) \Rightarrow \mathrm{inl}(t_1)$
$: A \vee B \to B \vee A$

[4]:
```
prove(r'(A/\B)->(B/\A)', typenames=['A', 'B'])
```

[4]:

$\vdash \lambda t_0 : A \wedge B.\, (\mathrm{snd}(t_0), \mathrm{fst}(t_0))$
$: A \wedge B \to B \wedge A$

2.2 結合則

[5]:
```
prove(r'(A\/(B\/C))->((A\/B)\/C)', typenames=['A', 'B', 'C'])
```

[5]:

$\vdash \lambda t_4 : A \vee (B \vee C).$
case t_4 of

$\mathrm{inl}(t_0) \Rightarrow \mathrm{inl}(\mathrm{inl}(t_0))$

$|\mathrm{inr}(t_3) \Rightarrow \mathrm{case}\ t_3\ \mathrm{of}$

$\mathrm{inl}(t_1) \Rightarrow \mathrm{inl}(\mathrm{inr}(t_1))$

$|\mathrm{inr}(t_2) \Rightarrow \mathrm{inr}(t_2)$

$: A \lor (B \lor C) \to (A \lor B) \lor C$

[6]:
```
prove(r'(A/\(B/\C))->((A/\B)/\C)', typenames=['A', 'B', 'C'])
```

[6]:

$\vdash \lambda t_0 : A \land \big(B \land C\big)\,.\ \big((\mathrm{fst}(t_0), \mathrm{fst}(\mathrm{snd}(t_0))), \mathrm{snd}(\mathrm{snd}(t_0))\big)$
$: A \land \big(B \land C\big) \to \big(A \land B\big) \land C$

2.3 分配則

[7]:
```
law_of_distribution_1 = r'A/\(B\/C)->(A/\B)\/(A/\C)'
law_of_distribution_2 = r'A\/(B/\C)->(A\/B)/\(A\/C)'
```

[8]:
```
prove(law_of_distribution_1, typenames=['A', 'B', 'C'])
```

[8]:

$\vdash \lambda t_0 : A \land (B \lor C).\, \mathrm{case}\ \mathrm{snd}(t_0)\ \mathrm{of}\ \mathrm{inl}(t_5) \Rightarrow \mathrm{inl}((\mathrm{fst}(t_0), t_5)) \mid \mathrm{inr}(t_6) \Rightarrow \mathrm{inr}((\mathrm{fst}(t_0), t_6))$
$: A \land (B \lor C) \to A \land B \lor A \land C$

[9]:
```
prove(law_of_distribution_2, typenames=['A', 'B', 'C'])
```

[9]:

$\vdash \lambda t_2 : A \lor B \land C.\, \mathrm{case}\ t_2\ \mathrm{of}\ \mathrm{inl}(t_0) \Rightarrow (\mathrm{inl}(t_0), \mathrm{inl}(t_0)) \mid \mathrm{inr}(t_1) \Rightarrow (\mathrm{inr}(\mathrm{fst}(t_1)), \mathrm{inr}(\mathrm{snd}(t_1)))$
$: A \lor B \land C \to (A \lor B) \land (A \lor C)$

2.4 吸収則

[10]:
```
law_of_absorption_1 = r'A/\(A\/B)->A'
law_of_absorption_1_converse = r'A->A/\(A\/B)'
law_of_absorption_2 = r'A\/(A/\B)->A'
law_of_absorption_2_converse = r'A->A\/(A/\B)'
```

[11]:
```
prove(law_of_absorption_1, typenames=['A', 'B'])
```

[11]:

$\vdash \lambda t_0 : A \land (A \lor B).\, \mathrm{fst}(t_0)$
$: A \land (A \lor B) \to A$

[12]:
```
prove(law_of_absorption_1_converse, typenames=['A', 'B'])
```

[12]:

$\vdash \lambda t_0 : A.\, (t_0, \mathrm{inl}(t_0))$
$: A \to A \land (A \lor B)$

[13]:
```
prove(law_of_absorption_2, typenames=['A', 'B'])
```

[13]:

$\vdash \lambda t_2 : A \lor A \land B.\, \mathrm{case}\ t_2\ \mathrm{of}\ \mathrm{inl}(t_0) \Rightarrow t_0 \mid \mathrm{inr}(t_1) \Rightarrow \mathrm{fst}(t_1)$
$: A \lor A \land B \to A$

[14]:
```
prove(law_of_absorption_2_converse, typenames=['A', 'B'])
```

[14]:

$\vdash \lambda t_0 : A.\, \mathrm{inl}(t_0)$
$: A \to A \lor A \land B$

2.5　三重否定の除去

[15]:
```
triple_negation = r'~~~P->~P'
prove(triple_negation, typenames=['P'])
```

[15]:
$\vdash \lambda t_0 : \neg\neg\neg P.\, \lambda t_1 : P.\, t_0\ (\lambda t_2 : \neg P. t_2\ t_1)$
$: \neg\neg\neg P \to (P \to \bot)$

[16]:
```
triple_negation_converse = r'~P->~~~P'
prove(triple_negation_converse, typenames=['P'])
```

[16]:
$\vdash \lambda t_0 : \neg P.\, \lambda t_1 : \neg\neg P.\, t_1\ t_0$
$: \neg P \to (\neg\neg P \to \bot)$

2.6　後半課題 5 のテスト問題

[17]:
```
prove(r'~(A\/B)->~A', typenames=['A', 'B'])
```

[17]:
$\vdash \lambda t_0 : \neg(A \vee B).\, \lambda t_1 : A.\, t_0\ \mathrm{inl}(t_1)$
$: \neg(A \vee B) \to (A \to \bot)$

[18]:
```
prove(r'(A->~A)->~(~A->A)', typenames=['A'])
```

[18]:
$\vdash \lambda t_0 : A \to \neg A.\, \lambda t_1 : \neg A \to A.$
$(\lambda t_2 {:} A.\ t_0\, t_2\, t_2)\ (t_1\ (\lambda t_2 {:} A.\ t_0\, t_2\, t_2))$
$: (A \to \neg A) \to ((\neg A \to A) \to \bot)$

2.7　ド・モルガンの法則

[19]:
```
de_morgan1 = r'~(P\/Q)->(~P/\~Q)'
de_morgan1_converse = r'(~P/\~Q)->~(P\/Q)'
de_morgan2 = r'(P\/Q)->~(~P/\~Q)'
de_morgan3 = r'(P/\Q)->~(~P\/~Q)'
de_morgan4 = r'(~P\/~Q)->~(P/\Q)'
```

[20]:
```
prove(de_morgan1, typenames=['P', 'Q'])
```

[20]:
$\vdash \lambda t_0 : \neg(P \vee Q).\, (\lambda t_1 : P.\, t_0\ \mathrm{inl}(t_1), \lambda t_2 : Q.\, t_0\ \mathrm{inr}(t_2))$
$: \neg(P \vee Q) \to (P \to \bot) \wedge (Q \to \bot)$

[21]:
```
prove(de_morgan1_converse, typenames=['P', 'Q'])
```

[21]:
$$\vdash \lambda t_0 : \neg P \wedge \neg Q.\, \lambda t_3 : P \vee Q.\, \mathrm{case}\ t_3\ \mathrm{of}\ \mathrm{inl}(t_1) \Rightarrow \mathrm{fst}(t_0)\ t_1 \\ \qquad\qquad\qquad\qquad\qquad\qquad \mid \mathrm{inr}(t_2) \Rightarrow \mathrm{snd}(t_0)\ t_2$$
$: \neg P \wedge \neg Q \to (P \vee Q \to \bot)$

[22]:
```
prove(de_morgan2, typenames=['P', 'Q'])
```

[22]:

$\vdash \lambda t_4 : P \vee Q.$
case t_4 of

$\quad \mathrm{inl}(t_0) \Rightarrow \lambda t_1 : \neg P \wedge \neg Q.$
$\qquad \mathrm{fst}(t_1)\ t_0$
$\quad \mid \mathrm{inr}(t_2) \Rightarrow \lambda t_3 : \neg P \wedge \neg Q.$
$\qquad \mathrm{snd}(t_3)\ t_2$

$: P \vee Q \to (\neg P \wedge \neg Q \to \bot)$

[23]:
```
prove(de_morgan3, typenames=['P', 'Q'])
```

[23]:

$\vdash \lambda t_0 : P \wedge Q.$
$\lambda t_3 : \neg P \vee \neg Q.$
case t_3 of $\mathrm{inl}(t_1) \Rightarrow t_1\ \mathrm{fst}(t_0)$
$\quad \mid \mathrm{inr}(t_2) \Rightarrow t_2\ \mathrm{snd}(t_0)$
$: P \wedge Q \to (\neg P \vee \neg Q \to \bot)$

[24]:
```
prove(de_morgan4, typenames=['P', 'Q'])
```

[24]:

$\vdash \lambda t_4 : \neg P \vee \neg Q.$
case t_4 of

$\quad \mathrm{inl}(t_0) \Rightarrow \lambda t_1 : P \wedge Q.$
$\qquad t_0\ \mathrm{fst}(t_1)$
$\quad \mid \mathrm{inr}(t_2) \Rightarrow \lambda t_3 : P \wedge Q.$
$\qquad t_2\ \mathrm{snd}(t_3)$

$: \neg P \vee \neg Q \to (P \wedge Q \to \bot)$

2.8　記号論理学 岡本賢吾先生の証明問題集から

[25]:
```
prove(r'(P->~P) -> (~~Q->~Q) -> ~(P\/Q)', typenames=['P', 'Q'])
```

[25]:

$\vdash \lambda t_0 : P \to \neg P.$
$\lambda t_1 : \neg\neg Q \to \neg Q.$
$\lambda t_6 : P \vee Q.$
case t_6 of

$\quad \mathrm{inl}(t_2) \Rightarrow t_0\ t_2\ t_2$
$\quad \mid \mathrm{inr}(t_4) \Rightarrow t_1\ (\lambda t_5 : \neg Q. t_5\ t_4)\ t_4$

$: (P \to \neg P) \to ((\neg\neg Q \to \neg Q) \to (P \vee Q \to \bot))$

[26]:
```
prove(r'(⊥->P) -> ((P->Q)->(X\/(Q->P))) -> (X->(P\/~Q)) -> Q -> P',␣
↪typenames=['P', 'Q', 'X'])
```

[26]:

$\vdash \lambda t_0 : \bot \to P.$
$\lambda t_1 : (P \to Q) \to X \vee (Q \to P).$

$\lambda t_2 : X \to P \vee \neg Q.$
$\lambda t_3 : Q.$
case t_1 $(\lambda t_4 : P.t_3)$ of
    $\mathrm{inl}(t_{21}) \Rightarrow$ case t_2 t_{21} of
        $\mathrm{inl}(t_{24}) \Rightarrow t_{24}$
        $|\mathrm{inr}(t_{25}) \Rightarrow t_0$ $(t_{25}$ $t_3)$
    $|\mathrm{inr}(t_{26}) \Rightarrow t_{26}$ t_3
$: (\bot \to P) \to (((P \to Q) \to X \vee (Q \to P)) \to ((X \to P \vee \neg Q) \to (Q \to P)))$

[27]:
```
prove(r'((P->X)->~Y) -> ((Q->X)->~~Y) -> ~((P\/Q)->X)', typenames=['P', 'Q', 'X',
    'Y'])
```

[27]:
$\vdash \lambda t_0 : (P \to X) \to \neg Y.$
$\lambda t_1 : (Q \to X) \to \neg\neg Y.$
$\lambda t_2 : P \vee Q \to X.$
t_1 $(\lambda t_8 : Q.\, t_2$ $\mathrm{inr}(t_8))$ $(t_0$ $(\lambda t_3 : P.\, t_2$ $\mathrm{inl}(t_3)))$
$: ((P \to X) \to \neg Y) \to (((Q \to X) \to \neg\neg Y) \to ((P \vee Q \to X) \to \bot))$

[28]:
```
prove(r'(~P\/~Q) -> (P/\Q) -> ~~(P/\Q)', typenames=['P', 'Q'])
```

[28]:
$\vdash \lambda t_6 : \neg P \vee \neg Q.$
case t_6 of
    $\mathrm{inl}(t_0) \Rightarrow \lambda t_1 : P \wedge Q.\, \lambda t_2 : \neg(P \wedge Q).$
        t_0 $\mathrm{fst}(t_1)$
    $|\mathrm{inr}(t_3) \Rightarrow \lambda t_4 : P \wedge Q.\, \lambda t_5 : \neg(P \wedge Q).$
        t_3 $\mathrm{snd}(t_4)$
$: \neg P \vee \neg Q \to (P \wedge Q \to (\neg(P \wedge Q) \to \bot))$

2.9 他の例

[29]:
```
prove(r'Q->P->Q', typenames=['X', 'P', 'Q'])
```

[29]:
$\vdash \lambda t_0 : Q.\, \lambda t_1 : P.\, t_0$
$: Q \to (P \to Q)$

[30]:
```
prove(r'((P\/⊥)->P)->P\/~Q->Q->P', typenames=['X', 'P', 'Q'])
```

[30]:
$\vdash \lambda t_0 : P \vee \bot \to P.$
$\lambda t_5 : P \vee \neg Q.$
case t_5 of
    $\mathrm{inl}(t_1) \Rightarrow \lambda t_2 : Q.$
        t_1
    $|\mathrm{inr}(t_3) \Rightarrow \lambda t_4 : Q.\, t_0$
        $\mathrm{inl}(t_0$ $\mathrm{inr}(t_3$ $t_4))$

$: (P \lor \bot \to P) \to (P \lor \neg Q \to (Q \to P))$

[31]:
```
prove(r'~~(P->Q)->(~~P->~~Q)', typenames=['P', 'Q'])
```

[31]:

$\vdash \lambda t_0 : \neg\neg(P \to Q).$
$\lambda t_1 : \neg\neg P.$
$\lambda t_2 : \neg Q.$
$t_1\ (\lambda t_8 : P.\, t_0\ (\lambda t_3 : P \to Q.\, t_1\ (\lambda t_5 : P.\, t_2\ (t_3\ t_5))))$
$: \neg\neg(P \to Q) \to (\neg\neg P \to (\neg Q \to \bot))$

3 証明不能な例

3.1 二重否定の除去

[32]:
```
double_negation = r'~~P->P'
try:
    prove(double_negation, typenames=['P'])
except stlc.DeductionFailed:
    print('No')
```

No

3.2 パースの法則

[33]:
```
peirce_law = r'((P->Q)->P)->P'
try:
    prove(peirce_law, typenames=['P', 'Q'])
except stlc.DeductionFailed:
    print('No')
```

No

4 Reference

岡本賢吾, *記号論理の証明システム・自然演繹* (Natural Deduction)NJ/NK —(2) *述語論理* (Predicate Logic)