CS 517 Midterm 1

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Problem 1

① Show $(P \to Q) \vee (Q \to P)$ is a tautology.

A tautology is a wff whose truth values are always true

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \vee (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

$(P \to Q) \vee (Q \to P)$ is always true

$\therefore$ it is a tautology

Problem 1

② Prove that $P \to Q$ and $\neg P \vee Q$ are logically equivalent

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$\boxed{\text{The truth tables of } P \to Q \text{ + } \neg P \vee Q \text{ are identical, so logically equivalent}}$

Problem 1

(3) $P = \text{False}$ $\quad P \vee \neg Q = T$

what is Q?

P	Q	$\neg Q$	$P \vee \neg Q$
T	T	F	T ←
T	F	T	T ←
→ F	T	F	F
→ F	F	T	T ←

Due to the truth table if

$$\boxed{P = \text{False} + P \vee \neg Q = T \text{ then } Q = \text{False}}$$

Problem 1

(4) C_H = Calvin is home
B_M = Bonzo is @ movies

Original statement (argument)
$$\neg C_H \vee B_M$$

Negation
$$\neg(\neg C_H \vee B_M)$$

1. $\neg(\neg C_H) \wedge \neg B_M$ De Morgan's Law
2. $C_H \wedge \neg B_M$ negation

$\therefore$ not(Calvin is not home or Bonzo is at the movies) is tautologically equivalent to

$\boxed{\text{Calvin is home and Bonzo is not at the movies}}$

Problem 2 [10+10]

① prove $\sum_{i=\phi}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for $n \geq 1$
n is integer

basis: $n=1$, $n^2=1$

IH: assume: $1^2+2^2+3^2+\dots+k^2 = \frac{k(k+1)(2k+1)}{6}$

IS: show: $1^2+2^2+3^2+\dots+(k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$= 1^2+2^2+3^2+\dots+(k+1)^2$ left side

$= 1^2+2^2+3^2+\dots+k^2+(k+1)^2$ rewrite l.s.

$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$ sub inductive hypothesis

$= (k+1)\left(\frac{k(2k+1)}{6} + (k+1)\right)$ factor out $(k+1)$

$= (k+1)\left(\frac{k(2k+1)}{6} + \frac{6k+6}{6}\right)$ common denominator

$= (k+1)\left(\frac{2k^2+k+6k+6}{6}\right)$ addition + multiplication

$= (k+1)\left(\frac{2k^2+7k+6}{6}\right)$ addition

$= (k+1)\left(\frac{(k+2)(2k+3)}{6}\right)$ factor

$= \frac{(k+1)(k+2)(2k+3)}{6}$ rewrite

$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$ rewrite to match IS

$\therefore$ by induction $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

for $n \in \mathbb{N}$ or $n \geq 1$

and n is integer

Problem 2 [10+10]

② prove $\sum_{i=0}^{n-1} 2^i = 2^n - 1$ for $n \in \mathbb{N}$

Basis: $n = 1$, $2^1 - 1 = 1$

IH, assume: $2^1 + 2^2 + 2^3 + \cdots + 2^{(k-1)} = 2^k - 1$

IS, show: $2^1 + 2^2 + 2^3 + \cdots + 2^k = 2^{(k+1)} - 1$

$= 2^1 + 2^2 + 2^3 + \cdots + 2^k$ left side

$= 2^1 + 2^2 + 2^3 + \cdots + 2^{(k-1)} + 2^k$ rewrite Is

$= 2^k - 1 + 2^k$ sub IH

$= 2(2^k) - 1$ addition

$= 2^{(k+1)} - 1$ law of exponents

$\therefore$ by induction $\sum_{i=\phi}^{n-1} 2^i = 2^n - 1$ for $n \in \mathbb{N}$

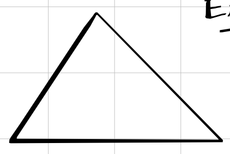
Problem 3

getting a feel for the problem

convex polygon := polygon w/ all ∠'s < 180°

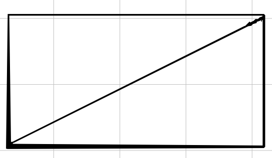
polygon := shape w/ 3 or more sides

Examples

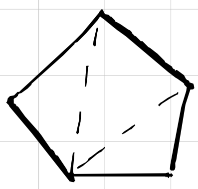


no triangulation needed

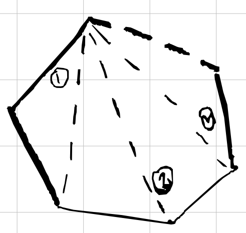
$n = 3$, $n-3 = 3-3 = \emptyset$ lines



$n = 4$, $4-3 = 1$ line



$n = 5$, $5-3 = 2$ lines



$n = k$, $k-3$ lines

1 line for triangulation

remove 2 lines from original n-polygon and adds 1 line back to make $(n-1)$-polygon

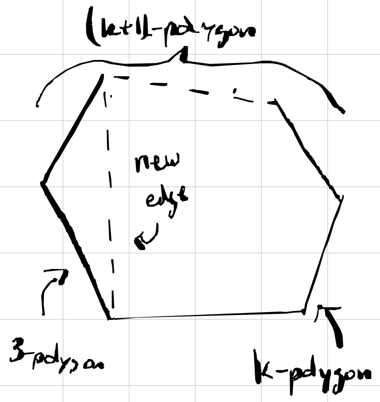
Restatement of problem

triangulation of n-polygon requires $n-3$ lines for n is integer ≥ 3

Basis: 3-polygon , $n = 3$, $n-3 = \emptyset$ lines for triangulation

using 1^{st} Induction: k-polygon requires $k-3$ lines for triangulation

inductive step: show a $(k+1)$-polygon requires $k-2$ lines for triangulation



by adding a single line (+1) we split the $(k+1)$-polygon into a 3-polygon and a k-polygon.

by IH k-polygon requires $k-3$ lines for triangulation

also by IH 3-polygon requires $\emptyset$ lines for triangulation

thus $(k+1)$-polygon requires

3-polygon lines for triangulation
+ k-polygon lines for triangulation
+ 1 line

$k-2$ lines for triangulation

$$\Rightarrow \begin{array}{r} \emptyset \\ + k - 3 \\ + 1 \\ \hline k-2 \end{array} \blacksquare$$

Problem 4

① A = I enter poodle den
B = I have electric poodle prod
C = I have can of mace

$A \to (B \vee C)$

B	C	$B \vee C$	$A \to (B \vee C)$	A
T	F	T	T	T or F

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

if Q is true then
$P \to Q$ must be true

if $P \to Q$ is true the
P may be true or false

∴ this is not a valid argument ∎

or incase you want the whole truth table

A	B	C	$B \vee C$	$A \to B \vee C$	$B \wedge \neg C$
T	T	T	T	T	F
T	T	F	T	T	T
T	F	T	T	T	F
T	F	F	F	F	F
F	T	T	T	T	F
F	T	F	T	T	T
F	F	T	T	T	F
F	F	F	F	T	F

Where $B \wedge \neg C$ is true denotes
the argument from the question. This shows
that the conclusion can be true or false
whereas the argument states A is true
This shows it is an invalid argument ∎

Problem 4

(2) A = I buy new goat
B = I buy used Yugo
C = I need a loan

B	C	$A \wedge B \to C$	$A \wedge B$	A
T	F	T	F	F

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

if C is false and $A \wedge B \to C$ is true then $A \wedge B$ must be false

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

if B is T and $A \wedge B$ is F then A must be F

If A is false I did not buy a new goat ∴ the argument is valid ■

or if you want the whole truth table

A	B	C	$A \wedge B$	$A \wedge B \to C$ (P_2)	$\neg C$	$B \wedge \neg C$ (P_3)	$A \vee B$ (P_1)	$\neg A$
T	T	T	T	T	F	F	T	F
T	T	F	T	F	T	T	T	F
T	F	T	F	T	F	F	T	F
T	F	F	F	T	T	F	T	F
F	T	T	F	T	F	F	T	T
F	T	F	F	T	T	T	T	T
F	F	T	F	T	F	F	F	T
F	F	F	F	T	T	F	F	T

All premises must be true

premise 1 : $A \vee B$ (buy a goat or yugo)

premise 2 : $(A \wedge B) \to C$ (if buy both, I need loan)

premise 3 : $B \wedge \neg C$ (bought yugo + don't need loan)

Where all 3 are true A must be false (or $\neg A$ must be true)

this is the case so the argument is valid ■

Problem 5

$T(1) = 100$

$T(n) = 2T(n/2) + n \quad (\text{for } n \geq 2,\ n = 2^m)$

assume $n = 2^k$

$k = \log_2 n$

Sequence T:

$T(1) = 100$ — Basis

$T(n) = 2T(n/2) + n$ — recurrance relation

$T(n) = 2T(n/2) + n$	expansion 1
$= 2\left(2T\left(\frac{n/2}{2}\right) + \frac{n}{2}\right) + n$	expansion 2
$= 2^2 T\left(\frac{n}{2^2}\right) + \frac{2n}{2} + n$	simplify
$= 2^2 T\left(\frac{n}{2^2}\right) + 2n$	simplify
$= 2^2\left(2T\left(\frac{n/2^2}{2}\right) + \frac{n}{2^2}\right) + 2n$	expansion 3
$= 2^3 T\left(\frac{n}{2^3}\right) + \frac{2^2 n}{2^2} + 2n$	simplify
$= 2^3 T\left(\frac{n}{2^3}\right) + 3n$	simplify
$\dots$	
$= 2^k T\left(\frac{n}{2^k}\right) + kn$	expansion k
$\dots$	
$= n T\left(\frac{n}{n}\right) + n\log_2 n$	expansion $n = 2^k$
$= n\left(T(1) + \log_2 n\right)$	by basis
$= n\left(100 + \log_2 n\right)$	closed form Solution

Theorem

$$T(n) = n(100 + \log_2 n) \quad \text{for } n \geq 1,\ n = 2^m$$

Proof

Basis: $T(1) = 1(100 + \log_2(1))$

$= 100 + 0 = 100$

IH, assume: $T(k) = k(100 + \log_2 k)$

IS, show: $T(2k) = 2k(100 + \log_2 2k)$

$T(2k) = 2T(k) + 2k$	recurrance relation
$= 2\left(k(100 + \log_2 k)\right) + 2k$	inductive hypothesis
$= 2k(100 + \log_2 k) + 2k$	distributive
$= 2k(100 + \log_2 k + 1)$	factor
$= 2k(100 + \log_2 k + \log_2 2)$	$\log_2 2 = 1$
$= 2k(100 + \log_2 2k)$	property of log

$\boxed{\text{thus } T(n) = n(100 + \log_2 n) \text{ for } n \geq 1,\ n = 2^m \text{ where } m \in \mathbb{N}\ \blacksquare}$