A Guessing Game: When Angle A = 60 Degrees

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Sometimes figuring out what to prove is harder than actually proving it!

Those of you who are familiar with Evan Chen's handouts will notice a striking resemblance between this handout and his Mixtilinear Incircles Guessing Game. Indeed, I liked his idea of a Guessing Game very much that I wanted to make my own Guess Game on a different configuration. As a result, I pretty much copied the idea, structure, style sheet, etc... (everything but the Geometry) of this handout from Evan's original Guessing Game. Huge credit goes to Evan Chen: without his work, this handout would not be possible.

In the pages that follow, I will describe a rather simple geometric configuration. It is your job to discover as many "coincidences" as you can: nontrivial collinearities, congruent shapes, symmetric shapes, and whatever else you can find. Can you prove your claims?

Contents

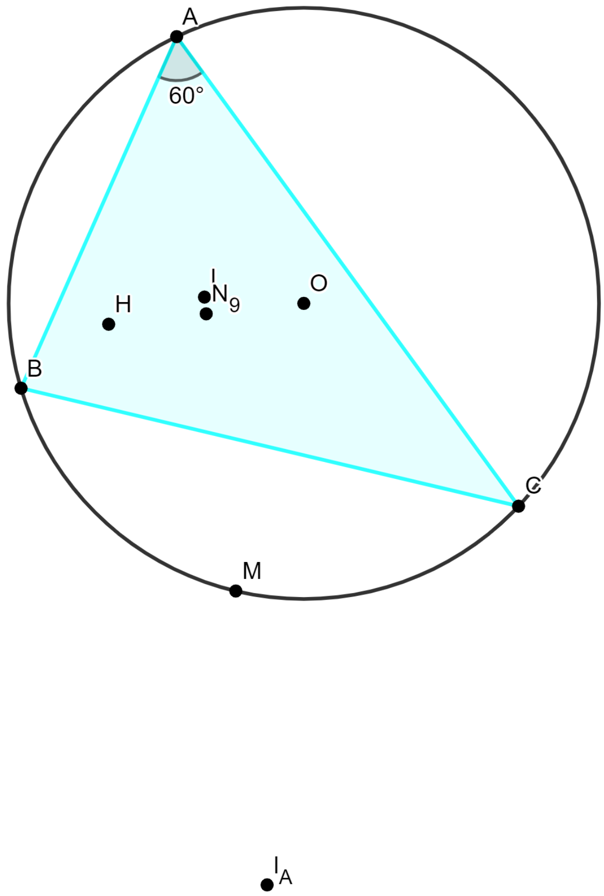
1 The Configuration

Let ABC be an acute triangle with $\angle A = 60°$. Let I, O, H, N_9, I_A denote the incenter, circumcenter, orthocenter, center of the nine-point circle, and A-excenter of $\triangle ABC$, respectively. Furthermore, let M be the midpoint of $\overline{II_A}$.

The next page contains a diagram with all these points and nothing more. There are also some hints to get you started. However, before seeing this, **try to find as many things as you can using your own ruler and compass.** During a competition, one does not have access to computer-drawn diagrams!

2 Some Hints

Having examined the configuration with your hand-drawn diagram, try once more with a computer-generated diagram. What more can you find?

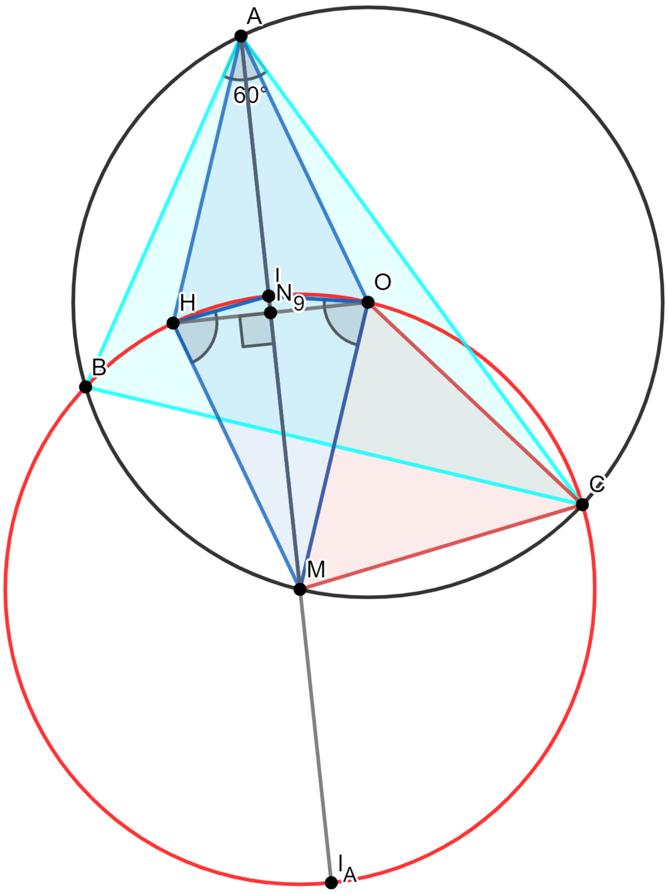


Here are some possible hints for things you could look for:

- A pair of congruent circles.
- An equilateral triangle.
- Two pairs of congruent triangles.
- A cyclic hexagon.
- A rhombus.
- A hidden perpendicular bisector.
- A (perhaps?) surprising colinearity.

My list of properties has eight items. When you want to see my answers, turn the page.

3 Answers



1. (ABC) and (IBC) are reflections over BC.
2. $\triangle COM$ is equilateral.
3. $\triangle AHI \cong \triangle AOI$.
4. $CHIOBI_A$ is cyclic.
5. $\triangle IMH \cong \triangle IMO$.
6. $AOMH$ is a rhombus.
7. Line AIM is the perpendicular bisector of $\overline{OH}$.
8. N_9 lies on the line $\overline{AIM}$.

This list is by no means exhaustive — there are probably more properties that I just haven't mentioned.

4 Solutions

These solutions freely use previously established facts to prove newer ones.

1. Denote (ABC) and (IBC) as ω_1 and ω_2 respectively. Clearly $H \in \omega_2$ and $H' \in \omega_1$ where H' is the reflection of H over BC. This is true by the Reflecting the Orthocenter Lemma. Now, for a point P on ω_1, we have $\measuredangle BPC = \measuredangle BAC = 60^\circ$ and $\measuredangle BP'C = -\measuredangle BPC = 120^\circ = \measuredangle BIC$ as needed.

2. Clearly, IM bisects $\angle BIC = 120^\circ$, so $\triangle MIC$ is isosceles with $\angle MIC = 60^\circ$, implying the result.

3. It is well-known that $AH = 2R\cos A$. Now note that $AH = 2R\cos A = 2R \cdot \frac{1}{2} = R = AO$. Also, $AI = AI$ and $\measuredangle HAI = \measuredangle IAO$ since H and O are isogonal conjugates. Thus, $\triangle AHI \cong \triangle AOI$.

4. Observe the following angle equalities:

$$\angle BHC = 180^\circ - \angle A = 120^\circ$$
$$\angle BIC = 90^\circ + \frac{\angle A}{2} = 120^\circ$$
$$\angle BOC = 2\angle A = 120^\circ$$

   so hexagon $CHIOBI_A$ is cyclic.

5. We have $MH = MO$ and $MI = MI$. Also note that $\measuredangle HMI = \measuredangle IMO$ since H and O are isogonal conjugates and thus reflections across the line $\overline{AIM}$. Thus $\triangle IMH \cong \triangle IMO$.

6. Observe that $AH \parallel OM$. From the above, we have $MH = MO = AO = AH$, finishing. (Actually, the parallel condition was not even needed.)

7. This follows from $AO = AH$ (or $MH = MO$) combined with the O and H isogonality.

8. N_9 is the midpoint of $\overline{OH}$ which does the trick.

5 Problems

Now that you are an expert at this configuration, try some problems!

1. In $\triangle ABC$, points D and E are constructed on sides BC and CA such that AD and BE are angle bisectors. Prove that $\angle ACB = 60^\circ$ if and only if $AE + BD = AB$. (2016 Kyiv City MO and 2017 Saint Petersburg State University School Olympiad)

2. In $\triangle ABC$, let $\angle B = 60^\circ$, O denote the circumcenter, and L denote the foot of the B-angle bisector. The circumcirle of triangle BOL meets the circumcircle of ABC at point $D \neq B$. Prove that $BD \perp AC$. (2019 Saudi Arabia IMO Training Test 3.1)

3. In $\triangle ABC$, $AB > AC$. Let O and I be the circumcenter and incenter respectively. Prove that if $\angle AIO = 30^\circ$, then $\angle ABC = 60^\circ$. (2020 China North Mathematical Olympiad Advanced Level P2)

4. In $\triangle ABC$, let O denote the circumcenter and H denote the orthocenter. Prove that $\angle A = 60^\circ$ if (a) $AO = AH$ or if (b) quadrilateral $BOCH$ is cyclic. (AoPS Forums)

5. In $\triangle ABC$, angle $\angle B = 60^\circ$. Points D and E are constructed on sides BC and CA such that AD and BE are angle bisectors. If O is the incenter, prove that $OD = OE$. (AoPS Forums)

6. In non-isosceles $\triangle ABC$ with $\angle ABC = 60$, a point T in its interior is selected such that $\angle ATC = \angle BTC = \angle BTA = 120^\circ$. Let M be the intersection point of the medians in $\triangle ABC$ and let TM intersect (ATC) at K. Find $\frac{TM}{MK}$. (2021 ARO 9.6)

7. Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC . Prove that $2\angle AHI = 3\angle ABC$. (2007 APMO)

8. Suppose $\triangle ABC$ has orthocenter F, incenter E and circumcenter D. Find all angles $\angle B$ such that $AFED$ is a cyclic quadrilateral. (AoPS Forums)

9. Let $\triangle ABC$ be acute with $\angle A = 60^\circ$. Prove that $IH = IO$, where I, H, O denote the incenter, orthocenter, and circumcenter, respectively. (AoPS Forums)

10. Triangle ABC has $\angle BAC = 60^\circ$, $\angle CBA \leq 90^\circ$, $BC = 1$, and $AC \geq AB$. Let H, I, and O be the orthocenter, incenter, and circumcenter of $\triangle ABC$, respectively. Assume that the area of the pentagon $BCOIH$ is the maximum possible. What is $\angle CBA$?(2011 AMC 12A Problem 25)

11. Let ABC be a scalene triangle with $\angle A = 60^\circ$. Let E and F be the feet of the angle bisectors of $\angle ABC$ and $\angle ACB$, respectively, and let I be the incenter of $\triangle ABC$. Let P, Q be distinct points such that $\triangle PEF$ and $\triangle QEF$ are equilateral. If O is the circumcenter of of $\triangle APQ$, show that $\overline{OI} \perp \overline{BC}$. (2017 ELMO SL G2)