Introduction to Barycentric Coordinates

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"*You always admire what you really don't understand.*"
-Blaise Pascal

The purpose of this document is to give the reader a guide on how to use Barycentric Coordinates to solve geometry problems. This is a very advanced technique, but this document should be approachable by all audiences. It is a rewrite of my original "Barycentric Coordinates Made Easy" handout at https://artofproblemsolving.com/community/c4h2856275 because after four years I realized that it is very outdated and can use a major update. Massive thanks to Evan Chen for making evan.sty. Enjoy!

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§1 The Basics

§1.1 The Reference Triangle and the Coordinate System

Like any other coordinate system (Cartesian, complex, polar...), Barycentric Coordinates have their own reference frames and their own coordinate system.

Definition 1.1. The reference frame when using Barycentric Coordinates is called the *Reference Triangle*.

Typically, for a problem that begins "In triangle ABC..." we tend to let $\triangle ABC$ be the reference triangle, but we can pick any three non-colinear points to form the reference triangle.

Definition 1.2. For a point P, its Barycentric Coordinates are of the form (x, y, z) where $x + y + z = 1$. We say x the $A-$component, y the $B-$component, and z the $C-$component.

Fact 1.3. The vertices of the reference triangle $\triangle ABC$ has coordinates $A = (1, 0, 0), B = (0, 1, 0), C = (0, 0, 1)$.

Let's figure out how to find the coordinates of other points.

§1.2 Finding Coordinates

Lemma 1.4

Let P be a point on side BC of $\triangle ABC$ such that $BP = r$ and $PC = s$ (equivalently, $\frac{BP}{PC} = \frac{r}{s}$). Then the barycentric coordinates of P are $(0, \frac{s}{r+s}, \frac{r}{r+s})$ where lengths are directed.

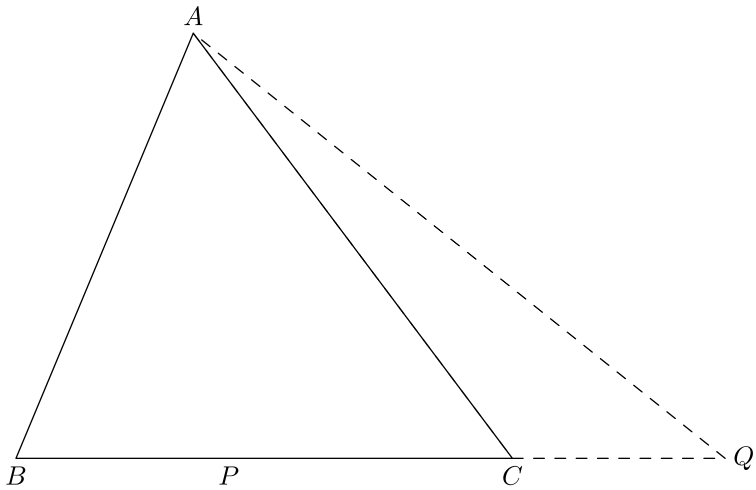

Figure 1: Finding the coordinates of P and Q.

Proof. See Figure 1. We know $\frac{BP}{PC} = \frac{r}{s}$. Now "pretend" that BC is subdivided into $BC = BP + PC = r + s$ pieces each of length $\frac{1}{r+s}$ so that they add to 1.

Now, imagine sliding point B to point C: when this happens, we get from $(0, 1, 0)$ to $(0, 0, 1)$. The $B-$component of point B decreases from 1 to 0 and the $C-$component increases from 0 to 1. Notice that the $A-$component stays constant at 0 since there is absolutely no $A-$ component in the segment BC.

To get from B to P, the $B-$component of point B goes down from 1 to $\frac{s}{r+s}$ while the $C-$coordinate of point B goes up from 0 to $\frac{r}{r+s}$.

Hence, $P = (0, \frac{s}{r+s}, \frac{r}{r+s})$. Notice that the components sum to 1. □

It is much more confusing when the point is outside the reference triangle, such as point Q. Fortunately, our lemma still holds if we use directed lengths.

Exercise 1.5. Let $BQ = t$ and $QC = u$. Find the Barycentric Coordinates of Q.

Proof. Since lengths are directed, we actually have $QC = -u$ as Q is on ray BC; see Figure 1. Then, by our lemma, we have $Q = (0, \frac{-u}{t-u}, \frac{t}{t-u})$. Again, notice that the components sum to 1. □

We actually have a formula for the midpoint of two points using these ideas.

Theorem 1.6 (Midpoint Formula)

For any two points P and Q with Barycentric coordinates $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$, their midpoint M is $M = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$.

Proof. Exercise for the reader. :) □

§1.3 The Area Formula

The power of Barycentric Coordinates is very apparent after seeing the area formula. Let $[ABC]$ denote the area of $\triangle ABC$.

Theorem 1.7

For any three points P, Q, R with coordinates $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$, respectively, we have $|\frac{[PQR]}{[ABC]}| = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$ where $[ABC]$ is the area of the reference triangle.

Now, before you complete freak out because of the determinant, just know that the determinant is just short hand for $x_1(y_2z_3 - z_2y_3) + y_1(z_2x_3 - x_2z_3) + z_1(x_2y_3 - y_2x_3)$.

> **Remark 1.8.** Notice that there are absolute values over the areas. This is because we are using signed areas which can be positive or negative depending on the order of the vertices. We don't have to worry about this as long as we remember to take the absolute value.

Let's see this in practice.

Theorem 1.9 (Medial Triangle)

The area of the medial triangle of $\triangle ABC$ is $\frac{1}{4} \cdot [ABC]$.

Proof. We apply Barycentric Coordinates with respect to $\triangle ABC$. $A = (1, 0, 0), B = (0, 1, 0), C = (0, 0, 1)$. Let the vertices of the medial triangle be P, Q, R. By the Barycentric midpoint formula, we have that $P = (\frac{1}{2}, \frac{1}{2}, 0), Q = (\frac{1}{2}, 0, \frac{1}{2}), R = (0, \frac{1}{2}, \frac{1}{2})$. By the Barycentric area formula, we have $|\frac{[PQR]}{[ABC]}| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4}$ as desired. □

§1.4 Lines

Theorem 1.10 (Barycentric Line)

The Barycentric equation of a line through two points, $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is given by $x(y_1z_2 - z_1y_2) + y(z_1x_2 - x_1z_2) + z(x_1y_2 - y_1x_2) = 0$.

Proof. What is a line? It is basically a triangle with area 0. It is well known that two points define a line. Hence, the Barycentric equation of a line through two points, $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is given by $\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$ where the top row is any point $R = (x, y, z)$ on the line (the parameter). Expanding gives the desired result. □

This equation may look cluncky, but it works very nicely when the line goes through a vertex of the reference triangle.

Exercise 1.11. In triangle ABC, let P_1 be on AB such that $AP_1 : BP_1 = 2 : 1$ and let P_3 is the midpoint of AC. CP_1 and BP_2 meet at P_3. Compute $\frac{[P_1P_2P_3]}{[ABC]}$.

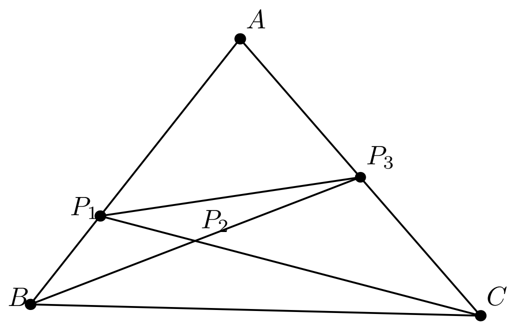

Figure 2: Diagram for Exercise.

Proof. See Figure 2. Let $\triangle ABC$ be the reference triangle so that $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$.

By the ratio of $2 : 1$, $P_1 = (\frac{1}{3}, \frac{2}{3}, 0)$. And P_3 is the midpoint, giving us $P_3 = (\frac{1}{2}, 0, \frac{1}{2})$.

We now use the equations for lines. The equation for BP_3 is $x - z = 0$ (plug B and P_3 in the line formula) and the equation for CP_1 is $2x - y = 0$ (plug in C and P_1). Therefore

$P_2 = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$. This is obtained by solving the system

$$x - z = 0 \tag{1}$$
$$2x - y = 0 \tag{2}$$
$$x + y + z = 1 \tag{3}$$

as P_2 is the intersection of the two lines and we know the coordinates of P_2 must add to 1.

Since, $P_1 = (\frac{1}{3}, \frac{2}{3}, 0)$, $P_2 = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$, and $P_3 = (\frac{1}{2}, 0, \frac{1}{2})$ we can plug them into the area formula to get $\frac{[P_1P_2P_3]}{[ABC]} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{12}$. □

§2 Examples

Let's Barybash a few AMC problems.

§2.1 Example 1 (2019 AMC 8 Problem 24)

Problem 2.1. In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?

Proof. We pick $\triangle ABC$ as our reference triangle. Then $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$. It is quite obvious that $D = (\frac{2}{3}, 0, \frac{1}{3})$ (NOT $D = (\frac{1}{3}, 0, \frac{2}{3})$).

The coordinates of E are easy since it is a midpoint. So $E = (\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

To find the coordinates of F, use the equation of a line. We want to find BC and AE and have them intersect to find F. The equation of a line through two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is $x(y_1z_2 - z_1y_2) + y(z_1x_2 - x_1z_2) + z(x_1y_2 - y_1x_2) = 0$.

Put $B = (x, y, z) = (0, 1, 0)$ and $C = (x, y, z) = (0, 0, 1)$ into the equation to get $x = 0$. (This makes intuitive sense...there is no "A" component in BC.)

Put $A = (1, 0, 0)$ and $E = (\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$ into the equation to get $y(-1 * \frac{1}{6}) + z(1 * \frac{1}{2}) = 0 \Rightarrow \frac{1}{6}y = \frac{1}{2}z \Rightarrow y - 3z = 0$.

To find F, we just need to solve

$$x = 0$$
$$y - 3z = 0$$
$$x + y + z = 1$$

. Solving this system, we find that the coordinates of F will be $\left(0, \frac{3}{4}, \frac{1}{4}\right)$.

In the Barycentric coordinate system, the area formula is

$$[BEF] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \cdot [ABC]$$

Since, $B = (0,1,0)$, $E = (\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$, and $F = \left(0, \frac{3}{4}, \frac{1}{4}\right)$ we can plug them in to have $\frac{[BEF]}{[ABC]} = \begin{vmatrix} 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{vmatrix} \Rightarrow \frac{[BEF]}{360} = \frac{1}{12}$ so $[BEF] = 30$. □

§2.2 Example 2 (2013 AMC 10B Problem 16)

Problem 2.2. In triangle ABC, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?

Proof. We pick $\triangle ABC$ as our reference triangle. Then $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$. Since points D and E are midpoints, we have $D = (0, \frac{1}{2}, \frac{1}{2})$ and $E = (\frac{1}{2}, \frac{1}{2}, 0)$. Since P is the centroid, it has coordinates $P = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ (make sure you see why).

By the Barycentric Area formula, we have $\frac{[EDP]}{[ABC]} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{12}$ (if you got $-\frac{1}{12}$ it doesn't matter since the areas are signed).

By Heron's Formula and the given side lengths, $[\triangle EDP] = \frac{3}{2}$ (or just notice the Pythagorean Triples and it is $\frac{1}{2}$ of the area of a $3-4-5$ triangle). Therefore, $[ABC] = 18$. Drawing the midpoint of side AC and calling it F, it is quite obvious that DEF is a medial triangle. Hence, $[AEDC] = \frac{3}{4} \cdot 18 = 13.5$. □

§2.3 Example 3 (2004 AMC 10B Problem 20)

Problem 2.3. In $\triangle ABC$ points D and E lie on BC and AC, respectively. If AD and BE intersect at T so that $\frac{AT}{DT} = 3$ and $\frac{BT}{ET} = 4$, what is $\frac{CD}{BD}$?

Proof. Let our reference triangle be $\triangle BTD$ and let $B = (1,0,0), T = (0,1,0)$, and $D = (0,0,1)$ (Cool! The reference triangle is not $\triangle ABC$).

From D to T, the x coordinate increases by 1 and the z coordinate decreases by 1. Thus, from T to A, the x coordinate increases by 3 and the z coordinate decreases by 3. Hence, $A = (0, 4, -3)$. (Check that this makes sense.)

From B to T, the x coordinate decreases by 1 and the y coordinate increases by 1. Thus, from T to E, the x coordinate decreases by $\frac{1}{4}$ and the y coordinate increases by $\frac{1}{4}$. Hence, $E = (-\frac{1}{4}, \frac{5}{4}, 0)$.

To find the coordinates of C, we need to find the intersection of lines AE and BD. If a point lies on BD then it satisfies $y = 0$. If a point lies on AE then it must satisfy $x(y_1z_2 - z_1y_2) + y(z_1x_2 - x_1z_2) + z(x_1y_2 - y_1x_2) = 0$ for two points $A = (x_1, y_1, z_1)$ and $E = (x_2, y_2, z_2)$. Plugging in values, we obtain (since we know $y = 0$) $x(-(-3)\frac{5}{4}) + z(-4 * -\frac{1}{4}) \Rightarrow 15x + 4z = 0$. But $x + y + z = 1 \Rightarrow x + z = 1$ since $y = 0$. Solving this system, we find that $x = -\frac{4}{11}$ and $z = \frac{15}{11}$ so $C = (-\frac{4}{11}, 0, \frac{15}{11})$.

We finish with an obvious synthetic observation. Draw segment TC. Since they share an altitude, the ratio $\frac{CD}{BD}$ (what we want) is equal to the ratio of $\frac{[DTC]}{[BTD]}$. Using the

Barycentric Area formula, we have

$$\frac{[DTC]}{[BTD]} = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -\frac{4}{11} & 0 & \frac{15}{11} \end{vmatrix} = \frac{4}{11}.$$

□

§3 Next Steps

This is just an Introduction to Barycentric Coordinates. For a much more detailed and advanced treatment of Barycentric Coordinates, refer to Evan Chen's handout.

To practice using Barycentric Coordinates, just solve as many problems you can. Anything involving ratios of lengths and areas in triangles can usually be solved with this technique. This page has a lot of good problems for you to try. Additionally, try proving famous theorems like Ceva's, Menalaus', Routh's and Stewart's using Barycentric Coordinates.

Thanks for reading! :)