Linear Algebra

Trivandrum School on Communication, Coding and Networking 2017

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Linear Algebra

- Vector Spaces
  - Definitions : Fields and Vector Space.
  - Linear Combinations.
  - Linear Independence and Dependence.
  - Subspaces
  - Basis and Dimension.
  - Vectors as tuples.
  - Basis change matrix.
- Linear Transformations.
  - Definition.
  - Linear Transformations as Matrices.
  - Similar matrices.
  - Range and Null Space of Linear Transformations.
  - Rank-Nullity Theorem.
  - Eigen values and vectors of a Linear Operator.

General ideas about Math-based Education and Research

- Math is not hard!
- There are only sets and maps (relations between sets).
- Start from basic axioms.
- Connect simple facts to create bigger facts (not always easy!).
- Imagination and Creativity.

Field - A rough definition

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Need for Linear Algebra in Communications and Coding

- For $\boldsymbol{x} = x(t)$, $\boldsymbol{y} = y(t)$ (complex-valued functions), define

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- To show : If $x(t), y(t)$ are finite-energy, then so is $x(t) + y(t)$.
- Given : $||\boldsymbol{x}|| < \infty$, $||\boldsymbol{y}|| < \infty$, show $||\boldsymbol{x} + \boldsymbol{y}|| < \infty$.

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$$
\begin{aligned}
||\boldsymbol{x}+\boldsymbol{y}||^2 &= ||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2 + <\boldsymbol{x},\boldsymbol{y}> + <\boldsymbol{y},\boldsymbol{x}> \\
&\leq ||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2 + 2|<\boldsymbol{x},\boldsymbol{y}>|. \\
&\leq ||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2 + 2||\boldsymbol{x}||.||\boldsymbol{y}|| \quad (\text{if } \; |<\boldsymbol{x},\boldsymbol{y}>| \leq ||\boldsymbol{x}||.||\boldsymbol{y}||) \\
&< \infty \qquad (\text{as each of the above terms are finite})
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Cauchy-Schwarz inequality

$$|<\boldsymbol{x},\boldsymbol{y}>| \leq ||\boldsymbol{x}||.||\boldsymbol{y}||$$

Proof: Fact: $||\boldsymbol{x}-\lambda\boldsymbol{y}||^2 \geq 0$, for any $\lambda \in \mathbb{C}$. Expand this and substitute $\lambda = \frac{<\boldsymbol{x},\boldsymbol{y}>}{||\boldsymbol{y}||^2}$.

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(Turns out that $<x,y>$ is also an example of a linear algebraic object called inner product)

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Need for Linear Algebra in Communications and Coding

1. Finite-energy signals form a vector space over $\mathbb{C}$.

Field : Formal Definition

Definition: Fields

A *field* $\mathbb{F}$ is a set S with two operations (addition $(+)$ and multiplication(.)) such that

- For any $a, b \in S$, $a + b \in S$ (*closure under addition*)

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- For $a \in S$ there exists an element $b \in S$ such that $a + b = b + a = 0$. (We write this element b as $-a$ and call it the *Additive inverse* of a in S. **Note: Subtraction is just addition with additive inverse.**)

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- For all $a, b \in S$, $a + b = b + a$ (*Addition is Commutative*)

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Definition: Fields (continued)

..such that..

- S is closed under multiplication.
- Multiplication is associative.
- Multiplicative identity exists (denoted by 1).
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Definition: Fields (continued)

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...such that..

- For all $a, b, c \in S$, $a.(b + c) = a.b + a.c$ (Distributivity of multiplication).

It is really over! (I think)

Fields: Informally

Fields

A set where we can add, multiply, subtract (add with additive inverses), and divide (multiply with multiplicative inverses) and things work out *nicely*.

- Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_p$.
- Non-examples: $\mathbb{R}^{m \times k}$ matrices ($m = k \neq 1$).

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- Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_p$.
- Non-examples: $\mathbb{R}^{m\times k}$ matrices $(m = k \neq 1)$.
- Think: What kind of structure exist if $k = m = 1$?, $k = m$?, $k \neq m$?.

Vector Spaces : Formal Definition

A set V is a vector space over $\mathbb{F}$ *(field of scalars)* if the following properties are satisfied :

- V is closed under vector addition, which is commutative and associative. $\forall \boldsymbol{v}, \boldsymbol{w} \in V$, $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v} \in V$.

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- $\forall \boldsymbol{x} \in V$, there exists $\boldsymbol{y} \in V$ such that $\boldsymbol{x} + \boldsymbol{y} = 0$. (Additive inverse exists).

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  $\forall \boldsymbol{x} \in V, \forall \alpha \in \mathbb{F}, \alpha \boldsymbol{x} \in V$.

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- $\forall \boldsymbol{x}, \boldsymbol{y} \in V$ and $\alpha, \beta \in \mathbb{F}$
  1. $1\boldsymbol{x} = \boldsymbol{x}$
  2. $\alpha(\boldsymbol{x} + \boldsymbol{y}) = \alpha \boldsymbol{x} + \alpha \boldsymbol{y}$
  3. $(\alpha\beta)\boldsymbol{x} = \alpha(\beta \boldsymbol{x})$
  4. $(\alpha + \beta)\boldsymbol{x} = \alpha \boldsymbol{x} + \beta \boldsymbol{x}$

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Vector space V over $\mathbb{F}$

A set closed under addition, scalar multiplication (multiplication by scalars from $\mathbb{F}$).

Notation:

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Notation:

- Normal font, (α, β) for scalars.
- Bold fonts $(\boldsymbol{v}, \boldsymbol{w})$ for vectors.
- Caps for Vector spaces (V, W).
- $\mathbb{F}$ for field.

Subspaces

- $W \subseteq V$ is called a subspace if it is a vector space (over $\mathbb{F}$).
- Checking whether a subset is a subspace:
  - For all $\boldsymbol{v}, \boldsymbol{w} \in V$, $\alpha\boldsymbol{v} + \boldsymbol{w} \in V, \forall\alpha$.

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- $V = \mathbb{R}^3$

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Linear Combination of vectors

- A linear combination of a set of vectors $S = \{\boldsymbol{v_i} : i = 1, ..., r\} \subset V$ is

$$\sum_{i=1}^{r} \alpha_i \boldsymbol{v_i},$$

for some $\alpha_i \in \mathbb{F}$.

- Note that if $\alpha_i = 0, \forall i$, then the linear combination gives the $\mathbf{0} \in V$.
- Examples: $S = \{(1\ 0\ 0), (0\ 1\ 0)\}$. Then $(1\ 1\ 0)$ is a linear combination.

Linear Dependence

Linear Dependence of vectors

- Vectors $\{\boldsymbol{v_i} : i = 1, ..., r\}$ are called *linearly dependent*

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for some α_is, at least one of which is non-zero.

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- If $\alpha_j \neq 0$ for some $1 \leq j \leq r$ then

$$\boldsymbol{v_j} = \sum_{i=1, i \neq j}^{r} \beta_i \boldsymbol{v_i},$$

where $\beta_i = \frac{-\alpha_i}{\alpha_j}$.

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Linear Independence

- If $\{\boldsymbol{v_i} : i = 1, ..., r\}$ is not linearly dependent, then they are *linearly independent*.
- Only zero-linear combination gives $\mathbf{0}$.

Examples

- Consider the vectors (from $\mathbb{R}^2$)

$$S = \left\{ \boldsymbol{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \boldsymbol{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \tag{1}$$

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Span of a subset of vectors

Span

The span of a set of vectors $S = \{\boldsymbol{v_i} : i = 1, ..., r\}$ is the set of all linear combinations of the vectors in that set.

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Basis of a Subspace

Basis of a subspace W

A subset B of W is called a basis of W if

1. B is linearly independent set
2. B spans W

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- Any set of k-linearly independent vectors of $\mathbb{F}^k$.

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Any two bases for a subspace contain the same number of vectors

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7. But that means C is dependent (contradiction).

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Basis and Dimension

The following are equivalent:

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Dimension of a Subspace W

$$dim(W) = \text{No. of vectors in any basis of } W.$$

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Theorem

Let V be a finite dimensional vector space and S be a linearly independent subset of vectors from V. Then S can be extended to a basis of V, i.e., there is a basis B for V such that $S \subseteq B$.

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- If $span(S_1) = V$, then we are done. Else find a vector outside $span(S_1)$ and add. ... (repeat).
- We will have a basis for V at the end.

Vectors from n-dimensional V.S as n-tuples

Unique representation of vectors using basis vectors

Let V be a n-dimensional vector space with basis $B = \{\boldsymbol{b_1}, ..., \boldsymbol{b_n}\}$. Then any vector $\boldsymbol{v} \in V$ can be written as a unique linear combination of the basis vectors

$$\boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{b_i}.$$

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$$\boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{b_i}.$$

- In terms of the basis B, we can represent $\boldsymbol{v}$ as the n-tuple,

$$[\boldsymbol{v}]_B = (\alpha_1, \alpha_2, ..., \alpha_n).$$

- This is only a representation, and may change with the basis chosen.

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Vectors as coordinates

- Let $V = \mathbb{R}^2$. Let $B = \{\boldsymbol{b_1} = (1,0), \boldsymbol{b_2} = (0,1)\}$.
- Consider a vector $\boldsymbol{v} = (5,6)$.
- $\boldsymbol{v} = 5\boldsymbol{b}_1 + 6\boldsymbol{b}_2$.
- In terms of B, we have

$$[\boldsymbol{v}]_B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

Change of Basis

How do vector-representations change with change in the basis (from $B = \{\boldsymbol{b_i} : i = 1..n\}$ to $C = \{\boldsymbol{c_i} : i = 1..n\}$) chosen?

$$\text{Given } [\boldsymbol{v}]_B, \text{what is } [\boldsymbol{v}]_C?$$

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- Given $B = \{\boldsymbol{b_i}\}$, we have

$$\boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{b_i},$$

how to get β_is such that

$$\boldsymbol{v} = \sum_{i=1}^{n} \beta_i \boldsymbol{c_i},$$

i.e. what is $[\boldsymbol{v}]_C$?

Change of Basis

Note that

$$
\begin{aligned}
[\boldsymbol{v}]_C &= \sum_{i=1}^{n} \alpha_i [\boldsymbol{b_i}]_C. \\
&= \begin{bmatrix} [\boldsymbol{b_1}]_C & [\boldsymbol{b_2}]_C & \ldots. & [\boldsymbol{b_n}]_C \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \\
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$\begin{bmatrix} [\boldsymbol{b_1}]_C & [\boldsymbol{b_2}]_C & \dots. & [\boldsymbol{b_n}]_C \end{bmatrix}$ is known as the basis change matrix.

Basis change : Example

- Consider the basis $C = \{\boldsymbol{c_1} = (1,0), \boldsymbol{c_2} = (1,1)\}$ for $\mathbb{R}^2$.
- Let $\boldsymbol{v} = (5,6)$. What is $[\boldsymbol{v}]_C$?

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- Let $\boldsymbol{v} = (5,6)$. What is $[\boldsymbol{v}]_C$?
- $$\begin{aligned} [\boldsymbol{v}]_C &= 5[\boldsymbol{b}_1]_C + 6[\boldsymbol{b}_2]_C \\ &= 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 6 \end{bmatrix}. \end{aligned}$$
- Check : $\boldsymbol{v} = -1\boldsymbol{c_1} + 6\boldsymbol{c_2}$.

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Need for Linear Algebra in Communications and Coding

$\mathcal{L}$=Finite energy signals which are also time-limited from $[0, T]$.

Theorem

A basis for $\mathcal{L}$ is

$$f_i(t) = \frac{1}{\sqrt{T}} e^{j2\pi it/T}, \;\; i = 0, \pm 1, \pm 2, \ldots$$

.

Proof:

- Fourier Series expansion.

Need for Linear Algebra in Communications and Coding

1. Finite-energy time-bounded signals form a vector space.

Need for Linear Algebra in Communications and Coding

1. Finite-energy time-bounded signals form a vector space.
2. Span of time-limited sinusoids = Time-limited Finite-Energy signals
   - The sinusoidal basis helps to easily characterize output signal when the signal is passed through 'linear time-invariant' systems.
   - Can think of signals as vectors. Makes Digital Communication possible!

Linear Transformations

- Maps between Vector Spaces (defined over a common field $\mathbb{F}$).
- We like linearity.

Linear Transformation

Let V and W be vector spaces over the field F. A function $T : V \to W$ is a linear transformation if

$$T(c\boldsymbol{v_1} + \boldsymbol{v_2}) = cT(\boldsymbol{v_1}) + T(\boldsymbol{v_2}), \forall \boldsymbol{v_1}, \boldsymbol{v_2} \in V, \text{and}, \forall c \in \mathbb{F}.$$

If $V = W$, then T is called a *linear operator*.

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Linear Tranformation : Examples and Non-Examples

1. T: $R^{2\times 2} \rightarrow R$ where T is defined as
   $T\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right) = x_1 + x_4.$

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4. $T : R^3 \to R^3$ where T is defined as $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ a \end{bmatrix}$.

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   (**No** if $a \neq 0$, **Yes** if $a = 0$)

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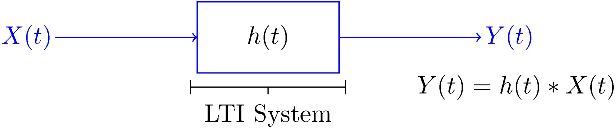
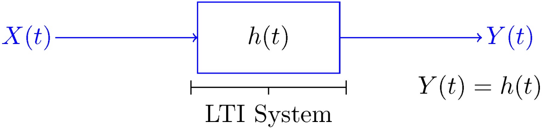
Linear Transformation : Examples and Non-Examples



- $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$.
- Is this is a linear transformation? (What are its domain and codomain?)

Linear Transformation : Examples and Non-Examples



$$Y(t) = h(t) * X(t)$$

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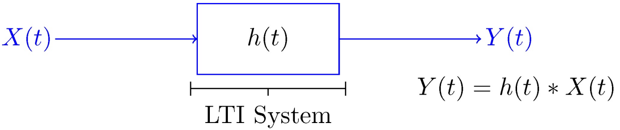

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- Linear Transformation.
- Domain=Codomain=Vector Space of Finite energy signals.

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Need for Linear Algebra in Communications and Coding

1. Finite-energy time-bounded signals form a vector space.
2. Span of time-limited sinusoids $=$ Time-limited Finite-Energy signals.
3. LTI systems are Linear Operators on the Space of Finite Energy Signals.

Sum and Composition of Linear Transformations

- T_1 and T_2 are linear transformations from $V \to W$. Then so is their 'sum' T defined as

$$T(\boldsymbol{v}) = T_1(\boldsymbol{v}) + T_2(\boldsymbol{v}).$$

- So is T' ('composition') defined as

$$T'(\boldsymbol{v}) = T_2(T_1(\boldsymbol{v})).$$

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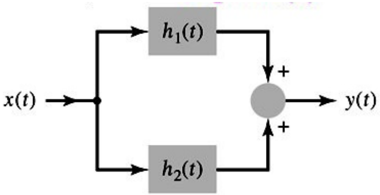
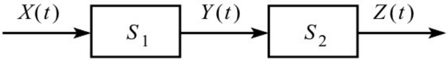
- Series and Parallel LTI systems.





Range and Null Space of a Linear Transformation

Range (Image) and Null-Space (Kernel) of T

- Range (Image):

$$R(T) = \{\boldsymbol{w} \in W : T(\boldsymbol{v}) = \boldsymbol{w}, \text{for some } \boldsymbol{v} \in V\}.$$

- Nullspace (kernel):

$$N(T) = \{\boldsymbol{v} \in V : T(\boldsymbol{v}) = \boldsymbol{0} \in W\}.$$

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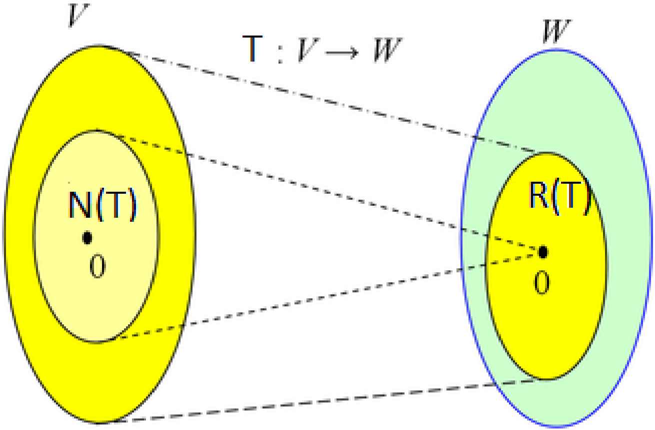
- $R(T)$ is a subspace of W.
- $N(T)$ is a subspace of V.

Range and Null Space



Rank Nullity Theorem

Rank and Nullity

- $Rank(T) = dim(R(T))$.
- $Nullity(T) = dim(N(T))$.

Rank Nullity Theorem

Let V be a finite dimensional vector space and $T : V \to W$ be a L.T. Then

$$dim(V) = Rank(T) + Nullity(T).$$

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Proof of Rank Nullity Theorem

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- It suffices to show that $\{T(\boldsymbol{v_{k+1}}), \ldots, T(\boldsymbol{v_n})\}$ is a basis for $R(T)$.

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- This means $\sum_{i=1}^{n-k} \alpha_i \mathbf{v_{k+i}} \in N(T)$. Thus,

$$\sum_{i=k+1}^{n} \alpha_i \mathbf{v_i} = \sum_{i=1}^{k} \beta_i \mathbf{v_i}.$$

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Proof of Rank Nullity Theorem

- Rearranging,

$$\sum_{i=k+1}^{n} \alpha_i \boldsymbol{v_i} - \sum_{i=1}^{k} \beta_i \boldsymbol{v_i} = \mathbf{0},$$

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for α_is not all zero.
- This is a contradiction as $\{\boldsymbol{v_i} : i = 1, ..., n\}$ is a basis.
- Thus $\{T(\boldsymbol{v_{k+1}}), \ldots, T(\boldsymbol{v_n})\}$ is linearly independent.

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- Apply T on both sides to get the result.

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Example

- Let
$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$
- Consider the linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$ given by $\boldsymbol{x} \to A\boldsymbol{x}$.
- What is the $N(T)$? What is $R(T)$?
- Check if R-N theorem is satisfied.

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Matrix of a Linear Transformation

Characterising linear transformations

Theorem

Let $T : V \to W$ be a L.T. Let $B = \{\mathbf{v_i} : i = 1.., n\}$. Then the action of T on any arbitrary $\mathbf{v} \in V$ is completely specified by its action on the basis vectors $\{\mathbf{v}_i : i = 1, .., n\}$.

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- Choosing a basis B_W for W enables us to write $\boldsymbol{w}$ as a m-tuple $[\boldsymbol{w}]_{B_W}$.
- Fixing B_V and B_W, we have a matrix representation $[T]$ for T.

$$[T][\boldsymbol{v}]_{B_V} = [\boldsymbol{w}]_{B_W}$$

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- $$i^{th} \text{ column of } [T] = [T(\boldsymbol{v_i})]_{B_W}.$$

Example

- Consider the Lin. Operator on the space of real polynomials of degree upto 2, defined as follows.

$$T(a_0+a_1t+a_2t^2) = (a_0+a_2)+(a_1+a_2)t+(a_0+2a_1+3a_2)t^2.$$

- Find its representation under (a) Basis $B = \{1, t, t^2\}$ (b) Basis $C = (1+t, 1+t^2, 1+t+t^2)$.

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    - Embed a low-D subspace in a High-D vector space to a Low-D vector space. (Compression or Source Coding)
    - Embed a low-D vector space as a Low-D subspace of a High-D vector space (Channel Coding).

Eigen values and vectors of a linear operator

Let $T : V \to V$ be a Linear Operator.

Eigen values and vectors

A non-zero $\boldsymbol{v} \in V$ and a constant $\lambda \in \mathbb{F}$ are called the eigen vector and its eigen value of T if

$$T(\boldsymbol{v}) = \lambda \boldsymbol{v}.$$

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- Understanding the I/O relationships of such Lin Operators are easy with such a basis.
- $$\begin{aligned} T(\boldsymbol{v}) &= T\left(\sum \alpha_i \boldsymbol{v}_i\right) \\ &= \sum \alpha_i T(\boldsymbol{v}_i) \\ &= \sum \alpha_i \lambda_i \boldsymbol{v}_i \end{aligned}$$

Example for Eigen vectors and Values

- $\mathcal{L}$=Finite energy signals which are also time-limited from $[0, T]$.
- A basis for $\mathcal{L}$ is

$$f_i(t) = \frac{1}{\sqrt{T}} e^{j2\pi it/T}, \;\; i = 0, \pm 1, \pm 2, \ldots$$

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$$f_i(t) = \frac{1}{\sqrt{T}} e^{j2\pi it/T}, \;\; i = 0, \pm 1, \pm 2, ...$$

- The function $f_i(t)$ are the eigen vectors for any LTI system given by L, with eigen value being the fourier series coefficient of $h(t)$ at $2\pi i/T$.

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Need for Linear Algebra in Communications and Coding

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3. LTI systems are Linear Operators on the Space of Finite Energy Signals.
4. Linear Transformations are heavily used in Coding Theory and Cryptography.
5. Fourier basis are also eigen vectors of LTI systems. So understanding I/O relationships of LTI systems is easy.

Thank You

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