Theorems in Mathematical Analysis

TRISCT

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Part I
Preliminaries

1 Lemmas

1.1 Techniques

Abel's transformation The sum $\sum_{i=1}^{m} \alpha_i\beta_i$ can be written as

$$\sum_{i=1}^{m} \alpha_i\beta_i = \alpha_m B_m - \sum_{i=1}^{m-1}(\alpha_{i+1} - \alpha_i)B_i$$

where

$$B_i = \sum_{k=1}^{i} \beta_k$$

If the original summation does not start with $i = 1$, one can write

$$\sum_{i=n}^{m} a_i b_i = A_m b_m - A_{n-1}b_n + \sum_{i=n}^{m-1} A_i(b_i - b_{i+1})$$

1.2 Order Estimate

1. If $p \neq 0$ is not a negative integer, then

$$\frac{p(p+1)\cdots(p+n-1)}{n!} = O^*\left(\frac{1}{n^{1-p}}\right) \quad (n \to \infty)$$

2. The binomial coefficient

$$\binom{m}{n} = O^*\left(\frac{1}{n^{m+1}}\right)$$

1.3 Lemmas With Names

Abel's Lemma If the finite sequence $\{\alpha_i\}_{i=1}^m$ is nonincreasing or nondecreasing and $B_i = \sum\limits_{k=1}^{i} \beta_k$ is such that $|B_i| \leqslant L$ for $i = 1, 2, \cdots, m$, then

$$\left|\sum_{i=1}^{m} \alpha_i \beta_i\right| \leqslant L \cdot (|\alpha_1| + 2\,|\alpha_m|)$$

Hadamard's Lemma Let $f : U \to \mathbb{R}$ be a function of class $C^{(p)}(U; \mathbb{R})$, $p \geqslant 1$, defined in a convex neighborhood U of the point $0 = (0, \cdots, 0) \in \mathbb{R}^m$ and such that $f(0) = 0$. Then there exist function $g_i \in C^{(p-1)}(U; \mathbb{R})$, $(i = 1, \cdots, m)$ such that the equality

$$f(x_1, \cdots, x_m) = \sum_{i=1}^{m} x_i g_i(x_1, \cdots, x_m)$$

holds in U, and $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Morse's Lemma If $f : G \to \mathbb{R}$ is a function of class $C^{(3)}(G; \mathbb{R})$ defined on an open set $G \subset \mathbb{R}^m$ and $x_0 \in G$ is a nondegenerate critical point of that function, then there exists a diffeomorphism $g : V \to U$ of some neighborhood of the origin 0 in $\mathbb{R}^m$ onto a neighborhood U of x_0 such that

$$(f \circ g)(y) = f(x_0) - [(y_1)^2 + \cdots + (y_k)^2] + [(y_{k+1})^2 + \cdots + (y_m)^2]$$

for all $y \in V$.

Riemann-Lebesgue Lemma Let $f(x)$ be integrable and absolutely integrable on $[a, b]$[1]. It holds that

$$\lim_{p\to\infty} \int_a^b f(x) \sin px \mathrm{d}x = 0, \quad \lim_{p\to\infty} \int_a^b f(x) \cos px \mathrm{d}x = 0$$

Generalized Riemann-Lebesgue Lemma Let $f \in \mathcal{R}[a, b]$, $\varphi \in \mathcal{R}[0, T]$, $\varphi(x + T) = x$, then

$$\lim_{n\to\infty} \int_a^b f(x)\varphi(nx)\mathrm{d}x = \frac{1}{T} \int_0^T \varphi(x)\mathrm{d}x \int_a^b f(x)\mathrm{d}x$$

[1] a or b may be infinity

2 Equalities and Inequalities

2.1 Common Taylor Series

1. $e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots \quad (-\infty < x < +\infty)$
2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots \quad (-\infty < x < +\infty)$
3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \quad (-\infty < x < +\infty)$
4. $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots \quad (-1 < x < 1)$
5. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots \quad (-1 < x \leqslant x)$
6. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots (-1 \leqslant x \leqslant 1)$
7. $\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots + \frac{(2n-1)!!x^{2n+1}}{(2n)!!(2n+1)} + \cdots (-1 \leqslant x \leqslant 1)$
8. $\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots \quad (-1 < x < 1)$
9. $\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - \cdots + (-1)^n (n+1) x^n + \cdots \quad (-1 < x < 1)$
10. $\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots + (-1)^{n-1}\frac{(2n-3)!!}{(2n)!!}x^n + \cdots \quad (-1 \leqslant x \leqslant 1)$
11. $\frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \cdots + (-1)^n \frac{(2n-1)!!}{(2n)!!}x^n + \cdots (-1 < x \leqslant 1)$

2.2 Common Infinite Products

1. $\sin x = x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right) \quad (-\infty < x < +\infty)$
2. $\cos x = \prod_{n=1}^{\infty}\left(1 - \frac{4x^2}{(2n-1)^2\pi^2}\right) \quad (-\infty < x < +\infty)$
3. $\frac{\sin x}{x} = \prod_{n=1}^{\infty}\cos\frac{x}{2^n} \quad (x \neq 0)$
4. $\prod_{n=0}^{\infty}(1 + x^{2^n}) = \frac{1}{1-x} \quad (-1 < x < 1)$

2.3 Pythagoras's theorem

Let X be a real inner product space, the following are true.

1. If $\{l_i\}$ is an orthogonal system, then

$$\left\| \sum_i l_i \right\|^2 = \sum_i \|l_i\|^2$$

2. If $\{e_i\}$ is an orthonormal system, then

$$\left\| \sum_i x_i e_i \right\|^2 = \sum_i \|x_i e_i\|^2 = \sum_i |x_i|^2$$

2.4 Bernoulli's Inequality

Bernoulli's Inequality For $x > -1,\ n \in \mathbb{N}^*$,

$$(1+x)^n \geqslant 1 + nx$$

and

$$(1+x)^n = 1 + nx \iff n = 1 \text{ or } x = 0$$

Extensions of Bernoulli's Inequality

$$\begin{aligned} x^\alpha - \alpha x + \alpha - 1 \leqslant 0 \quad &\text{when} \quad 0 < \alpha < 1 \\ x^\alpha - \alpha x + \alpha - 1 \geqslant 0 \quad &\text{when} \quad \alpha < 0 \text{ or } 1 < \alpha \end{aligned}$$

2.5 Hölder's Inequality

Hölder's Inequality (for Sums) Let $x_i, y_i \geqslant 0,\ i = 1, \cdots, n,\ \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned} \sum_{i=1}^n x_i y_i \leqslant \left(\sum_{i=1}^n x_i^p \right)^{1/p} \left(\sum_{i=1}^n y_i^q \right)^{1/q} \quad &, \quad p > 1 \\ \sum_{i=1}^n x_i y_i \geqslant \left(\sum_{i=1}^n x_i^p \right)^{1/p} \left(\sum_{i=1}^n y_i^q \right)^{1/q} \quad &, \quad p < 1,\ p \neq 0 \end{aligned}$$

Hölder's Inequality (for Integrals) Let $f, g \in \mathcal{R}[a, b],\ \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left| \int_a^b (f \cdot g)(x) \mathrm{d}x \right| \leqslant \left(\int_a^b |f|^p (x) \mathrm{d}x \right)^{1/p} \cdot \left(\int_a^b |g|^q (x) \mathrm{d}x \right)^{1/q} \quad , \quad p > 1$$

2.6 Jensen's Inequality

Jensen's Inequality If $f : (a, b) \to \mathbb{R}$ is a convex function, $x_1, \cdots, x_n \in (a, b)$, and $\alpha_1, \cdots, \alpha_n$ are positive numbers such that $\alpha_1 + \cdots + \alpha_n = 1$, then

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \leqslant \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$$

Jensen's Inequality (for Integrals) If f is a continuous convex function on $\mathbb{R}$ and φ an arbitrary continuous function on $\mathbb{R}$, then

$$f\left(\frac{1}{c}\int_0^c \varphi(t)\mathrm{d}t\right) \leqslant \frac{1}{c}\int_0^c f(\varphi(t))\mathrm{d}t$$

2.7 Minkowski's Inequality

Minkowski's Inequality (for Sums) Let $x_i, y_i \geqslant 0,\ i = 1, \cdots, n$. Then

$$\left(\sum_{i=1}^n (x_i + y_i)^p\right)^{1/p} \leqslant \left(\sum_{i=1}^n x_i^p\right)^{1/p} + \left(\sum_{i=1}^n y_i^p\right)^{1/p} \quad , \quad p > 1$$

$$\left(\sum_{i=1}^n (x_i + y_i)^p\right)^{1/p} \geqslant \left(\sum_{i=1}^n x_i^p\right)^{1/p} + \left(\sum_{i=1}^n y_i^p\right)^{1/p} \quad , \quad p < 1,\ p \neq 0$$

Minkowski's Inequality (for Integrals) Let $f, g \in \mathcal{R}[a, b]$. Then

$$\left(\int_a^b |f+g|^p(x)\mathrm{d}x\right)^{1/p} \leqslant \left(\int_a^b |f|^p(x)\mathrm{d}x\right)^{1/p} + \left(\int_a^b |g|^p(x)\mathrm{d}x\right)^{1/p} \quad , \quad p \geqslant 1$$

$$\left(\int_a^b |f+g|^p(x)\mathrm{d}x\right)^{1/p} \geqslant \left(\int_a^b |f|^p(x)\mathrm{d}x\right)^{1/p} + \left(\int_a^b |g|^p(x)\mathrm{d}x\right)^{1/p} \quad , \quad 0 < p < 1$$

2.8 Young's Inequality

Young's Inequality If $a > 0,\ b > 0$, then

$$a^{1/p}b^{1/q} \leqslant \frac{1}{p}a + \frac{1}{q}b \quad , \quad p > 1$$

$$a^{1/p}b^{1/q} \geqslant \frac{1}{p}a + \frac{1}{q}b \quad , \quad p < 1 \text{ and } p \neq 0$$

and

$$a^{1/p}b^{1/q} = \frac{1}{p}a + \frac{1}{q}b \iff a = b$$

Or it could be written as

$$xy \leqslant \frac{1}{p}x^p + \frac{1}{q}y^q$$

for $x, y \geqslant 0,\ p, q > 0,\ \frac{1}{p} + \frac{1}{q} = 1$

Part II
Basic Theorems

3 Theorems in Analysis

3.1 Contraction Mapping Principle

Picard-Banach Fixed-point Principle A contraction mapping $f : X \to X$ of a complete metric space (X, d) into itself has a unique fixed point a. Moreover, for any point $x_0 \in X$ the recursively defined sequence $x_0,\ x_1 = f(x_0), \cdots,\ x_{n+1} = f(x_n), \cdots$ converges to a. The rate of convergence is given by the estimate

$$d(a, x_n) \leqslant \frac{q^n}{1-q} d(x_1, x_0)$$

Stability of the Fixed Point Let (X, d) be a complete metric space and (Ω, τ) a topological space that will play the role of a parameter space in what follows. Suppose to each value of the parameter $t \in \Omega$ there corresponds a contraction mapping $f_t : X \to X$ and that the following conditions hold.

(a) The family $\{f_t : t \in \Omega\}$ is uniformly contracting, that is, there exists $q,\ 0 < q < 1$, such that each mapping f_t is a q-contraction

(b) For each $x \in X$ the mapping $f_t(x) : \Omega \to X$ is continuous as a function of t at some point $t_0 \in \Omega$, that is $\lim\limits_{t \to t_0} f_t(x) = f_{t_0}(x)$

Then the solution $a(t) \in X$ of the equation $x = f_t(x)$ depends continuously on t at t_0, that is, $\lim\limits_{t \to t_0} a(t) = a(t_0)$

3.2 Differential Calculus

Mean-value Theorem Let $f : G \to \mathbb{R}$ be a real-valued function defined in a region $G \subset \mathbb{R}^m$, and let the closed line segment $[x, x+h]$ be contained in G.

If the function f is continuous on $[x, x+h]$ and differentiable on $(x, x+h)$, then there exists a point $\xi \in (x, x+h)$ such that

$$f(x+h) - f(x) = f'(\xi)h$$

Finite-increment Theorem Let $f : U \to Y$ be a continuous mapping of an open set U of a normed space X into a normed space Y. If the closed interval $[x, x+h] = \{\xi \in X : \xi = x + \theta h,\ 0 \leqslant \theta \leqslant 1\}$ is contained in U and the mapping f is differentiable at all points of the open interval $(x, x+h) = \{\xi \in X : \xi = x + \theta h,\ 0 < \theta < 1\}$, then the following estimate holds:

$$\|f(x+h) - f(x)\| \leqslant \sup_{\xi \in (x, x+h)} \|f'(\xi)\| \cdot \|h\|$$

Taylor's Formula If $f : U(x) \to \mathbb{R}$ is defined and belongs to class $C^{(n)}(U(x); \mathbb{R})$ in a neighborhood $U(x) \subset \mathbb{R}^m$ of $x \in \mathbb{R}^m$, and the closed interval $[x, x+h]$ is completely contained in $U(x)$, then the following equality holds

$$f(x+h) - f(x) = \sum_{k=1}^{n-1} \frac{1}{k!}(h_1\partial_1 + \cdots + h_m\partial_m)^k f(x) + r_{n-1}(x; h)$$

where

$$\begin{aligned} r_{n-1}(x; h) &= \int_0^1 \frac{(1-t)^{n-1}}{(n-1)!}(h_1\partial_1 + \cdots + h_m\partial_m)^n f(x+th)\mathrm{d}t \\ &= \frac{1}{n!}(h_1\partial_1 + \cdots + h_m\partial_m)^n f(x+\theta h) \\ &= \frac{1}{n!}(h_1\partial_1 + \cdots + h_m\partial_m)^n f(x) + o(\|h\|^n) \end{aligned}$$

Taylor's Formula for Mappings If a mapping $f : U \to Y$ from a neighborhood $U = U(x)$ of x in a normed space X into a normed space Y has derivatives up to order $n-1$ inclusive in U and has an n-th order derivative $f^{(n)}(x)$ at x, then

$$f(x+h) = f(x) + f'(x)h + \cdots + \frac{1}{n!}f^{(n)}(x)h^n + o(\|h\|^n), \quad h \to 0$$

3.3 Implicit Function Theorem

Implicit function theorem If $F : U \to \mathbb{R}^n$ defined in a neighborhood U of $(x_0, y_0) \in \mathbb{R}^{m+n}$ is such that

- $F \in C^{(p)}(U; \mathbb{R}^n)$
- $F(x_0, y_0) = 0$
- $F'_y(x_0, y_0)$ is an invertible matrix

then there exists an $(m+n)$-dimensional interval $I = I_x^m \times I_y^n \subset U$, where

$$I_x^m = \{x \in \mathbb{R}^m : |x - x_0| < \alpha\}, \quad I_y^n = \{y \in \mathbb{R}^n : |y - y_0| < \beta$$

and a mapping $f \in C^{(p)}(I_x^m; I_y^n)$ such that

$$F(x, y) = 0 \iff y = f(x)$$

for any point $(x, y) \in (I_x^m \times I_y^n)$ and

$$f'(x) = -[F'_y(x, f(x))]^{-1}[F'_x(x, f(x))]$$

General Implicit Function Theorem Let X, Y, Z be normed spaces, Y being a complete space. Let $W = \{(x, y) \in X \times Y : |x - x_0| < \alpha,\ |y - y_0| < \beta\}$ be a neighborhood of (x_0, y_0). Suppose that the mapping $F : W \to Z$ satisfies the following conditions

- $F(x_0, y_0) = 0$
- $F(x, y)$ is continuous at (x_0, y_0)
- $F'(x, y)$ is defined in W and continuous at (x_0, y_0)
- $F'_y(x_0, y_0)$ is an invertible transformation

Then there exists a neighborhood U of $x_0 \in X$, a neighborhood V of $y_0 \in Y$, and a mapping $f : U \to V$ such that

- $U \times V \subset W$
- If $(x, y) \in U \times V$, then $F(x, y) = 0 \iff y = f(x)$
- $y_0 = f(x_0)$
- f is continuous at x_0

Inverse Function Theorem If $f : G \to \mathbb{R}^m$ of a domain $G \subset \mathbb{R}^m$ is such that

- $f \in C^{(p)}(G; \mathbb{R}^m)$
- $y_0 = f(x_0)$
- $f'(x_0)$ is invertible

then there exists a neighborhood $U(x_0) \subset G$ and a neighborhood $V(y_0)$ such that $f : U(x_0) \to V(y_0)$ is a $C^{(p)}$-diffeomorphism. Moreover, if $x \in U(x_0),\ y = f(x)$, then

$$(f^{-1})'(y) = (f'(x))^{-1}$$

Rank Theorem Let $f : U \to \mathbb{R}^n$ be a mapping defined in a neighborhood $U \subset \mathbb{R}^m$ of $x_0 \in \mathbb{R}^m$. If $f \in C^{(p)}(U; \mathbb{R}^n)$ and f has the same rank k everywhere in U, then there exists neighborhoods $O(x_0),\ O(y_0),\ y_0 = f(x_0)$ and $C^{(p)}$-diffeomorphisms $u = \varphi(x), v = \psi(y)$ of $O(x_0),\ O(y_0)$, such that $v = \psi \circ f \circ \varphi^{-1}(u)$ has the coordinate representation

$$\begin{aligned} v &= (v_1, \cdots, v_n) \\ &= \psi \circ f \circ \varphi^{-1}(u) \\ &= \psi \circ f \circ \varphi^{-1}(u_1, \cdots, u_k, \cdots, u_m) \\ &= (u_1, \cdots, u_k, 0, \cdots, 0) \end{aligned}$$

in $O(u_0) = \psi(O(x_0)),\ u_0 = \psi(x_0)$

4 Integral Calculus

4.1 Basic theorems

First mean-value theorem

Second mean-value theorem Let f be integrable on $[a, b]$. Then

1. If g is nonnegative and monotonically increasing on $[a, b]$, then there exists $\xi \in [a, b]$,

$$\int_a^b f(x)g(x)\mathrm{d}x = g(b) \int_\xi^b f(x)\mathrm{d}x$$

2. If g is nonnegative and monotonically decreasing on $[a, b]$, then there exists $\xi \in [a, b]$,

$$\int_a^b f(x)g(x)\mathrm{d}x = g(a) \int_a^\xi f(x)\mathrm{d}x$$

3. If g is monotonic $[a, b]$, then there exists $\xi \in [a, b]$,

$$\int_a^b f(x)g(x)\mathrm{d}x = g(a) \int_a^\xi f(x)\mathrm{d}x + g(b) \int_\xi^b f(x)\mathrm{d}x$$

4.2 Improper integrals

Comparison test (inequality)

Comparison test (limit form)

Dirichlet's test (IBFZ) Let f, g be such that

1. $\int_a^A f(x)\mathrm{d}x$ is bounded.
2. g monotonically tends to 0.

Then
$$\int_a^{+\infty} f(x)\mathrm{d}x$$
converges.

Abel's test (ICFB) Let f, g be such that

1. $\int_a^{+\infty} f(x)\mathrm{d}x$ converges.
2. g is monotonic and bounded.

Then
$$\int_a^{+\infty} f(x)\mathrm{d}x$$
converges.

Absolute convergence implies convergence (unbounded domain)

$$\int_a^{+\infty} |f(x)|\, \mathrm{d}x \text{ converges} \implies \int_a^{+\infty} f(x)\mathrm{d}x \text{ converges}$$

4.3 Improper Multiple Integral

Absolutely convergence and convergence imply each other (unbounded domain)
Let $D \subset \mathbb{R}^2$ be unbounded, then

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y \text{ converges} \iff \iint_D |f(x,y)|\, \mathrm{d}x\mathrm{d}y \text{ converges}$$

5 Integrabilities

Bounded domain, bounded function Integrable $\implies$ absolutely integrable

Bounded domain, unbounded function Absolutely integrable $\implies$ integrable

Unbounded domain, bounded function Absolutely integrable $\implies$ integrable

Bounded domain, bounded function Integrable $\implies$ square-integrable

Bounded domain, unbounded function Square-integrable $\implies$ absolutely integrable $\implies$ integrable

Part III
Family of Functions

6 Family of Functions Depending on a Parameter

6.1 Convergence of a Family of Functions Depending on a Parameter

Cauchy Criterion for Uniform Convergence Let $\{f_t : t \in T\}$ be a family of functions depending on a parameter, and $\mathcal{B}$ a base in T. A necessary and sufficient condition for $\{f_t : t \in T\}$ to converge uniformly on $E \subset X$ over $\mathcal{B}$ is that for every $\varepsilon > 0$ there exists $B \in \mathcal{B}$ such that $|f_{t_1}(x) - f_{t_2}(x)| < \varepsilon$ for every $t_1, t_2 \in B$ and every $x \in E$. In formal language one can state it as follows:

$$\exists f,\, f_t \underset{\mathcal{B}}{\rightrightarrows} f \iff \forall \varepsilon > 0,\, \exists B \in \mathcal{B},\, \forall t_1, t_2 \in V,\, \forall x \in E,\, |f_{t_1}(x) - f_{t_2}(x)| < \varepsilon$$

6.2 Functional Properties of a Limit Function

A sufficient condition for two limiting passages to commute Let $\{F_t : t \in T\}$ be a family of functions $F_t : X \to \mathbb{C}$ depending on a parameter $t \in T$; let $\mathcal{B}_X$ be a base in X and $\mathcal{B}_T$ a base in T. If the family converges uniformly on X over $\mathcal{B}_T$ to a function $F : X \to \mathbb{C}$ and $\lim\limits_{\mathcal{B}_X} F_t(x) = A_t$ exists for each

$t \in T$, then both repeated limits $\lim\limits_{\mathcal{B}_X} \left(\lim\limits_{\mathcal{B}_T} F_t(x) \right)$ and $\lim\limits_{\mathcal{B}_T} \left(\lim\limits_{\mathcal{B}_X} F_t(x) \right)$ exist and the equality

$$\lim_{\mathcal{B}_X} \left(\lim_{\mathcal{B}_T} F_t(x) \right) = \lim_{\mathcal{B}_T} \left(\lim_{\mathcal{B}_X} F_t(x) \right)$$

holds. The diagram for this theorem is as follows:

$$\begin{tikzcd}
F_t(x) \arrow[r, Rightarrow, "\mathcal{B}_T"] \arrow[d, "\mathcal{B}_X"'] & F(x) \arrow[d, "\exists \mathcal{B}_X"'] \\
A_t \arrow[r, "\exists \mathcal{B}_T"'] \arrow[ur, dash, dashed] & A
\end{tikzcd}$$

in which the hypotheses are written above the diagonal and the consequences below it.

Continuity and passages to the limit Let $\{F_t : t \in T\}$ be a family of functions $F_t : X \to \mathbb{C}$ depending on a parameter $t \in T$; let $\mathcal{B}_T$ be a base in T. If $f_t \underset{\mathcal{B}}{\rightrightarrows} f$ on X and the functions f_t are continuous at $x_0 \in X$, then the function $f : X \to \mathbb{C}$ is also continuous at that point.

$$\begin{tikzcd}
f_t(x) \arrow[r, Rightarrow, "\mathcal{B}_T"] \arrow[d, "x \to x_0"'] & f(x) \arrow[d, "x \to x_0"] \\
f_t(x_0) \arrow[r, "\mathcal{B}_T"'] \arrow[ur, dash, dashed] & f(x_0)
\end{tikzcd}$$

in which the hypotheses are written above the diagonal and the consequences below it.

Corollary 1. If a sequence of functions that are continuous on a set converges uniformly on that set, then the limit function is continuous on the set.

Corollary 2. If a series of functions that are continuous on a set converges uniformly on that set, then the sum of the series is continuous on the set.

Dini's theorem If a sequence of continuous functions on a compact set converges monotonically to a continuous function, then the convergence is uniform.

Proof 6.1. Extracting a finite covering should do the job.

Corollary 3. If the terms of the series $\sum_{n=1}^{\infty} a_n(x)$ are nonnegative functions $a_n : K \to \mathbb{R}$ that are continuous on a compact set K and the series converges to a continuous function on K, then it converges uniformly on K.

Integration and passage to limit Let $\{f_t : t \in T\}$ be a family of functions $f_t : [a,b] \to \mathbb{C}$ depending on the parameter $t \in T$, and let $\mathcal{B}$ be a base in T. If the functions of the family are integrable on $[a,b]$ and $f_t \rightrightarrows f$ on $[a,b]$ over $\mathcal{B}$, then the limit function $f : [a,b] \to \mathbb{C}$ is also integrable on $[a,b]$ and

$$\int_a^b f(x)\mathrm{d}x = \lim_{\mathcal{B}} \int_a^b f_t(x)\mathrm{d}x$$

The diagram for this theorem is as follows:

$$\begin{tikzcd}
F_t(p) \arrow[rr, Rightarrow] \arrow[d, "\lambda(P)\to 0"'] & & F(p) \arrow[d, "\exists\lambda(P)\to 0"] \arrow[dll, dashed, no head] \\
A_t \arrow[rr] & & A
\end{tikzcd}$$

The notations are defined as:

$$\begin{aligned}
p &= (P,\xi) \text{ is a partition with distinguished points.} \\
F_t(p) &= \sum_{i=1}^{n} f_t(\xi_i)\Delta x_i \\
F(p) &= \sum_{i=1}^{n} f(\xi_i)\Delta i \\
A_t &= \int_a^b f_t(x)\mathrm{d}x \\
A &= \int_a^b f(x)\mathrm{d}x
\end{aligned}$$

Proof 6.2. We can use the fact that $|f(x) - f_t(x)|$ can be arbitrarily small to estimate the difference between $|F(p) - F_t(p)|$. The latter can be considered a function defined on the topological space $\mathcal{P} = \{(P,\xi)\}$ of all partitions, by applying the theorem for the commutativity of limiting passages we obtain the desired result.

Corollary 4. If the series $\sum\limits_{n=1}^{\infty} f_n(x)$ consisting of integrable functions on a closed interval $[a,b] \subset \mathbb{R}$ converges uniformly on that closed interval, then its sum is also integrable on $[a,b]$ and

$$\int_a^b \left(\sum_{n=1}^{\infty} f_n(x) \right) \mathrm{d}x = \sum_{n=1}^{\infty} \int_a^b f_n(x) \mathrm{d}x$$

Differentiation and passage to the limit Let $\{f_t : t \in T\}$ be a family of functions $f_t : [a,b] \to \mathbb{C}$ defined on a convex bounded set X (in a normed space, one-dimensional I think) and depending on the parameter $t \in T$, and let $\mathcal{B}$ be a base in T. If the functions of the family are differentiable on X, the family of derivatives $\{f_t' : t \in T\}$ converges uniformly on X to a function $\varphi : X \to \mathbb{C}$, and the original family $\{f_t : t \in T\}$ converges at even one point $x_0 \in X$, then it converges uniformly on the entire set X to a differentiable function $f : X \to \mathbb{C}$, and $f' = \varphi$.

Corollary 5. If the series $\sum\limits_{n=1}^{\infty} f_n(x)$ of functions $f_n : X \to \mathbb{C}$ that are differentiable on a bounded convex subset $X \subset \mathbb{R}, \mathbb{C}$ (or other normed space) converges at even one point $x \in X$ and the series $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on X, then $\sum\limits_{n=1}^{\infty} f_n(x)$ also converges uniformly on X, its sum is differentiable on X, and

$$\left(\sum_{n=1}^{\infty} f_n(x) \right)'(x) = \sum_{n=1}^{\infty} f_n'(x)$$

7 Integrals Depending on a Parameter

7.1 Proper Integrals

Continuous dependence on the parameter If $f(x,u)$ is continuous on $[a,b] \times [\alpha,\beta]$, then the integral

$$\int_a^b f(x,u) \mathrm{d}x$$

depends continuously on u.

Smooth dependence on the parameter If $f(x,u)$ and $\frac{\partial f}{\partial u}(x,u)$ are continuous on $[a,b] \times [\alpha,\beta]$, then the integral

$$\int_a^b f(x,u) \mathrm{d}x$$

depends smoothly (C^1) on u, and

$$\frac{\mathrm{d}}{\mathrm{d}u}\int_a^b f(x,u)\mathrm{d}x = \int_a^b \frac{\partial f}{\partial u}(x,u)\mathrm{d}x$$

Interchangeable double integral If f is continuous on $[a,b]\times[\alpha,\beta]$, then

$$\int_\alpha^\beta \int_a^b f(x,u)\mathrm{d}x\mathrm{d}u = \int_a^b \int_\alpha^\beta f(x,u)\mathrm{d}u\mathrm{d}x$$

Continuous dependence on the domain and the parameter If $f(x,u)$ is continuous on $[a,b]\times[\alpha,\beta]$, and $p(u), q(u)$ are continuous on $[\alpha,\beta]$ and are bounded on $[\alpha,\beta]$ such that $a \leqslant p(u), q(u) \leqslant b$, then

$$\psi(u) = \int_{p(u)}^{q(u)} f(x,u)\mathrm{d}x$$

depends continuously on u.

Smooth dependence on the domain and the parameter If $f(x,u)$ and $\frac{\partial f}{\partial u}(x,u)$ are continuous on $[a,b]\times[\alpha,\beta]$, and $p(u), q(u)$ are differentiable on $[\alpha,\beta]$ and are bounded on $[\alpha,\beta]$ such that $a \leqslant p(u), q(u) \leqslant b$, then

$$\psi(u) = \int_{p(u)}^{q(u)} f(x,u)\mathrm{d}x$$

depends smoothly (C^1) on u, and

$$\psi'(u) = \int_{p(u)}^{q(u)} \frac{\partial f}{\partial u}(x,u)\mathrm{d}x + f(q(u),u)q'(u) - f(p(u),u)p'(u)$$

7.2 Improper Integrals

Equivalent conditions for uniform convergence The following are equivalent.

1. $\int_a^{+\infty} f(x,u)\mathrm{d}x$ converges uniformly.
2. The remainder $\left|\int_A^{+\infty} f(x,u)\mathrm{d}x\right|$ tends to 0 as $A \to +\infty$, irrespective of the choice of u.
3. The oscillation $\left|\int_{A'}^{A''} f(x,u)\mathrm{d}x\right|$ tends to 0 ($A', A'' > A$) as $A \to +\infty$, irrespective of the choice of u.

4. For any monotonically increasing sequence $\{A_n\} \to +\infty$ ($A_1 = a$), the series
$$\sum_{n=1}^{\infty} \int_{A_n}^{A_{n+1}} f(x,u)\mathrm{d}x$$
converges uniformly on $[\alpha, \beta]$.

Weierstrass's test Let $f(x,u)$ be continuous on $[a,+\infty)$. If there exists a continuous function $F(x)$ on $[a,+\infty)$ such that

1. $\int_a^{+\infty} F(x)\mathrm{d}x$ converges.
2. For all sufficiently large x and every u, $|f(x,u)| \leqslant F(x)$.

then $\int_a^{+\infty} f(x,u)\mathrm{d}x$ converges uniformly.

Proof 7.1. Cauchy criterion.

Dirichlet's test (IBFZ) Let $f(x,u), g(x,u)$ be such that

1. $\int_a^{A} f(x,u)\mathrm{d}x$ is uniformly bounded.
2. $g(x,u)$ monotonically and uniformly tends to 0.

Then
$$\int_a^{+\infty} f(x,u)g(x,u)\mathrm{d}x$$
converges uniformly.

Abel's test (ICFB) Let $f(x,u), g(x,u)$ be such that

1. $\int_a^{+\infty} f(x,u)\mathrm{d}x$ converges uniformly.
2. $g(x,u)$ is monotonic and uniformly bounded.

Then
$$\int_a^{+\infty} f(x,u)g(x,u)\mathrm{d}x$$
converges uniformly.

Note 7.1. *IBFZ stands for "integral bounded, function tends to 0". ICFB stands for "integral convergent, function bounded". In both cases the function needs to be monotonic.*

Dini's theorem Let $f(x,u)$ be continuous and nonnegative on $[a,+\infty) \times [\alpha,\beta]$. If $\varphi(u) = \int_a^{+\infty} f(x,u)\mathrm{d}x$ is continuous on $[\alpha,\beta]$, then

$$\int_a^{+\infty} f(x,u)\mathrm{d}x$$

converges uniformly on $[\alpha,\beta]$.

Interchangeable limits Let $f(x,u)$ be such that

1. $f(x,u) \underset{u\to u_0}{\rightrightarrows} g(x)$
2. $\int_a^{+\infty} f(x,u)\mathrm{d}x$ converges uniformly.

Then

$$\lim_{u\to u_0} \int_a^{+\infty} f(x,u)\mathrm{d}x = \int_a^{+\infty} \lim_{u\to u_0} f(x,u)\mathrm{d}x$$

Continuous dependence on the parameter Let $f(x,u)$ be continuous on $[a,+\infty)\times [\alpha,\beta]$ and let the integral $\int_a^{+\infty} f(x,u)\mathrm{d}x$ be uniformly convergent on $[\alpha,\beta]$, then

$$\varphi(u) = \int_a^{+\infty} f(x,u)\mathrm{d}x$$

is continuous on α,β.

Smooth dependence on the parameter Let $f(x,u)$ and $\frac{\partial f}{\partial u}(x,u)$ both be continuous on $[a,+\infty)\times[\alpha,\beta]$ and let the integral $\int_a^{+\infty} \frac{\partial f}{\partial u}(x,u)\mathrm{d}x$ be uniformly convergent on $[\alpha,\beta]$, then

$$\int_a^{+\infty} f(x,u)\mathrm{d}x$$

is differentiable on $[\alpha,\beta]$ and

$$\frac{\mathrm{d}}{\mathrm{d}u}\int_a^{+\infty} f(x,u)\mathrm{d}x = \int_a^{+\infty} \frac{\partial f}{\partial u}(x,u)\mathrm{d}x$$

Interchangeable integrals (finite×infinite) Let $f(x,u)$ be continuous on $[a,+\infty)\times [\alpha,\beta]$ and let the integral $\int_a^{+\infty} f(x,u)\mathrm{d}x$ be uniformly convergent on $[\alpha,\beta]$, then $\varphi(u) = \int_a^{+\infty} f(x,u)\mathrm{d}x$ is integrable and

$$\int_\alpha^\beta \varphi(u)\mathrm{d}u = \int_\alpha^\beta\int_a^{+\infty} f(x,u)\mathrm{d}x\mathrm{d}u = \int_a^{+\infty}\int_\alpha^\beta f(x,u)\mathrm{d}u\mathrm{d}x$$

Interchangeable integrals (infinite×infinite) Let $f(x,u)$ be such that

1. f is continuous on $[a, +\infty) \times [\alpha + \infty)$.
2. $\int_a^{+\infty} f(x, u)\mathrm{d}x$ converges uniformly with respect to $u \in [\alpha, \beta]$ for all $[\alpha, \beta] \subset [\alpha, +\infty)$.
3. $\int_\alpha^{+\infty} f(x, u)\mathrm{d}u$ converges uniformly with respect to $x \in [a, b]$ for all $[a, b] \subset [a, +\infty)$.

Then the convergence of either of the integrals

$$\int_a^{+\infty} \int_\alpha^{+\infty} |f(x, u)|\, \mathrm{d}u\mathrm{d}x \quad \text{and} \quad \int_\alpha^{+\infty} \int_a^{+\infty} |f(x, u)|\, \mathrm{d}x\mathrm{d}u$$

implies the convergence of the other and the equality between them.

Interchangeable integrals (infinite×infinite, nonnegative) Let $f(x, u)$ be such that

1. f is continuous and nonnegative on $[a, +\infty) \times [\alpha + \infty)$.
2. $\int_a^{+\infty} f(x, u)\mathrm{d}x$ is continuous on $[\alpha, +\infty)$.
3. $\int_\alpha^{+\infty} f(x, u)\mathrm{d}u$ is continuous on $[a, +\infty)$.

Then the convergence of either of the integrals

$$\int_a^{+\infty} \int_\alpha^{+\infty} f(x, u)\mathrm{d}u\mathrm{d}x \quad \text{and} \quad \int_\alpha^{+\infty} \int_a^{+\infty} f(x, u)\mathrm{d}x\mathrm{d}u$$

implies the convergence of the other and the equality between them.

Part IV
Series

8 Numerical Series

8.1 Convergence of Numerical Series with Nonnegative Terms

Comparison theorem Let $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ be series with nonnegative terms. If there exists $N \in \mathbb{N}$ such that $a_n \leqslant b_n$ for all $n > N$, then

$$\sum_{n=1}^{\infty} b_n < +\infty \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n < +\infty$$
$$\sum_{n=1}^{\infty} a_n = +\infty \quad \Longrightarrow \quad \sum_{n=1}^{\infty} b_n = +\infty$$

Comparison by inequalities Please refer to the Cauchy-Hölder inequality. For examples, see Sect. *E14.2-4, Other Problems From Chang & Shi, Numerical Series.*

Comparison by squeeze If $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ are convergent and $a_n \leqslant c_n \leqslant b_n$, then $\sum\limits_{n=1}^{\infty} c_n$ converges.

Comparison theorem in limit form Let $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ be series with nonnegative terms and

$$l = \lim_{n\to\infty} \frac{a_n}{b_n}$$

(a) If $l < +\infty$, then $\sum\limits_{n=1}^{\infty} b_n < +\infty \Longrightarrow \sum\limits_{n=1}^{\infty} a_n < +\infty$

(b) If $l > 0$, then $\sum\limits_{n=1}^{\infty} a_n < +\infty \Longrightarrow \sum\limits_{n=1}^{\infty} b_n < +\infty$

Note 8.1. *This may fail if the series contain negative terms. See counterexamples.*

Comparison theorem in quotient form Let $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ be series with non-negative terms. If there exists $N \in \mathbb{N}$ such that for all $n > N$

$$\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$$

then $\sum\limits_{n=1}^{\infty} b_n < +\infty \implies \sum\limits_{n=1}^{\infty} a_n < +\infty$

Cauchy's test Let $\sum\limits_{n=1}^{\infty} a_n$ be a series and

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{a_n}$$

Then the following are true:

(a) if $\alpha < 1$, $\sum\limits_{n=1}^{\infty} a_n$ converges;

(b) if $\alpha > 1$, $\sum\limits_{n=1}^{\infty} a_n$ diverges;

(c) there exist both convergent and divergent series for which $\alpha = 1$.

Cauchy's proposition If $\{a_n\}$ is a decreasing sequence with nonnegative terms, that is, $a_1 \geqslant a_2 \geqslant \cdots \geqslant 0$, then the series $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converges.

Kummer's test Let $\sum\limits_{n=1}^{\infty} a_n$ be a series with positive terms and $\{c_n\}$ be any sequence of positive numbers. Suppose the following limit exists.

$$\alpha = \lim_{n\to\infty} \left(c_n \frac{a_n}{a_{n+1}} - c_{n+1} \right)$$

Then

(a) if $\alpha > 0$, the series $\sum\limits_{n=1}^{\infty} a_n$ converges;

(b) if $\alpha < 0$, and $\sum\limits_{n=1}^{\infty} \frac{1}{c_n}$ diverges, the series $\sum\limits_{n=1}^{\infty} a_n$ diverges;

(c) there exist both convergent and divergent series for which $\alpha = 1$.

d'Alembert's test Suppose the following limit exists for $\sum_{n=1}^{\infty} a_n$

$$\alpha = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

Then

(a) if $\alpha < 1$, $\sum_{n=1}^{\infty} a_n$ converges;

(b) if $\alpha > 1$, $\sum_{n=1}^{\infty} a_n$ diverges;

(c) there exist both convergent and divergent series for which $\alpha = 1$.

Note 8.2. *This is obtained by setting $c_n = 1$ in Kummer's test.*

Raabe's test Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and

$$\alpha = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right)$$

or equivalently

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\alpha}{n} + o\left(\frac{1}{n}\right) \quad (n \to \infty)$$

Then

(a) if $\alpha > 1$, $\sum_{n=1}^{\infty} a_n$ converges;

(b) if $\alpha < 1$, $\sum_{n=1}^{\infty} a_n$ diverges;

(c) there exist both convergent and divergent series for which $\alpha = 1$.

Note 8.3. *This is obtained by setting $c_n = n$ in Kummer's test.*

Bertrand's test Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and suppose the following limit exists.

$$\alpha = \lim_{n\to\infty} \ln n \left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right]$$

Then

(a) if $\alpha > 1$, the series $\sum\limits_{n=1}^{\infty} a_n$ converges;

(b) if $\alpha < 1$, the series $\sum\limits_{n=1}^{\infty} a_n$ diverges;

(c) there exist both convergent and divergent series for which $\alpha = 1$.

Note 8.4. *This is obtained by setting $c_n = n \ln n$ in Kummer's test.*

Gauss's test Let $\sum\limits_{n=1}^{\infty} a_n$ be a series with positive terms and

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\alpha}{n \ln n} + o\left(\frac{1}{n \ln n}\right) \quad (n \to \infty)$$

Then

(a) if $\alpha > 1$, $\sum\limits_{n=1}^{\infty} a_n$ converges;

(b) if $\alpha < 1$, $\sum\limits_{n=1}^{\infty} a_n$ diverges;

(c) there exist both convergent and divergent series for which $\alpha = 1$.

Unknown name's test Let $\sum\limits_{n=1}^{\infty} a_n$ be a series with positive terms. If for all $n > n_0$,

$$(1 - \sqrt[n]{a_n})\frac{n}{\ln n} \geqslant p > 1$$

then it converges. If for all $n > n_0$,

$$(1 - \sqrt[n]{a_n})\frac{n}{\ln n} \leqslant 1$$

then it diverges.

Lobachevsky's test Let $\sum\limits_{n=1}^{\infty} a_n$ be a series with positive terms. If $\{a_n\}$ tends to 0 monotonically, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} p_m \cdot 2^{-m} \text{ converges}$$

where p_m is the maximum number that satisfies

$$a_n \geqslant 2^{-m}$$

Logarithm test Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms. If for all $n > n_0$

$$\frac{\ln(1/a_n)}{\ln n} \geqslant p > 1$$

then it converges. If for all $n > n_0$

$$\frac{\ln(1/a_n)}{\ln n} \leqslant 1$$

then it diverges.

Maclaurin-Cauchy integral test Suppose $f : [1, +\infty) \to \mathbb{R}$ is a decreasing function assuming only nonnegative values. If the sequence $\{a_n\}$ is such that

$$a_n = f(n)$$

then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{+\infty} f(x)\mathrm{d}x$ converges.

Ermakov's test Suppose $f : [1, +\infty) \to \mathbb{R}$ is a positive and decreasing function. If there exists $x_0 \geqslant 1$, such that for all $x \geqslant x_0$

(a) $\frac{f(e^x)\cdot e^x}{f(x)} \leqslant q < 1$, then $\sum_{n=1}^{\infty} f(n)$ converges;

(b) $\frac{f(e^x)\cdot e^x}{f(x)} \geqslant 1$, then $\sum_{n=1}^{\infty} f(n)$ diverges.

Abel-Dini theorem Let $\sum_{n=1}^{\infty} d_n$ be a series with positive terms and D_n the partial sum of it. If $\sum_{n=1}^{\infty} d_n$ diverges, then so does $\sum_{n=1}^{\infty} \frac{d_n}{D_n}$. However, $\sum_{n=1}^{\infty} \frac{d_n}{D_n^{1+\sigma}}$ converges for all $\sigma > 0$.

Proof 8.1. Use the Cauchy criterion for the divergence of $\sum_{n=1}^{\infty} d_n$ and the finite increment theorem on $\int \frac{\mathrm{d}x}{x^{1+\sigma}} = -\frac{1}{\sigma}\frac{1}{x^{\sigma}}$ for the convergence of $\sum_{n=1}^{\infty} \frac{d_n}{D_n^{1+\sigma}}$

Dini's theorem Let $\sum_{n=1}^{\infty} c_n$ be a series with positive terms and γ_n the nth remainder of it. If $\sum_{n=1}^{\infty} c_n$ is convergent, then $\sum_{n=1}^{\infty} \frac{c_n}{\gamma_{n-1}}$ diverges. However, $\sum_{n=1}^{\infty} \frac{c_n}{\gamma_{n-1}^{1-\sigma}}$ converges for $0 < \sigma < 1$.

Existence of a slower convergent series For every convergent series $\sum_{n=1}^{\infty} c_n$, there exists a slower convergent series

$$\sum_{n=1}^{\infty}(\sqrt{\gamma_{n-1}} - \sqrt{\gamma_n})$$

where γ_n is the remainder.

Existence of a slower divergent series For every divergent series $\sum_{n=1}^{\infty} d_n$, there exists a slower divergent series

$$\sum_{n=1}^{\infty}(\sqrt{D_n} - \sqrt{D_{n-1}})$$

where D_n is the partial sum and $D_0 = 0$.

8.2 Convergence of Numerical Series with Arbitrary Terms

The Cauchy criterion The series $\sum_{n=1}^{\infty}$ converges if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that for all $m \geqslant n > N$,

$$|a_n + \cdots + a_m| < \varepsilon$$

Comparison theorem (for nonnegative series) Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be series with nonnegative terms. If there exists $N \in \mathbb{N}$ such that $a_n \leqslant b_n$ for all $n > N$, then

$$\sum_{n=1}^{\infty} b_n < +\infty \implies \sum_{n=1}^{\infty} a_n < +\infty$$
$$\sum_{n=1}^{\infty} a_n = +\infty \implies \sum_{n=1}^{\infty} b_n = +\infty$$

Cauchy's test Let $\sum_{n=1}^{\infty} a_n$ be a series and

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

Then the following are true:

(a) if $\alpha < 1$, $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely;

(b) if $\alpha > 1$, $\sum\limits_{n=1}^{\infty} a_n$ diverges;

(c) there exist both absolutely convergent and divergent series for which $\alpha = 1$.

d'Alembert's test Suppose the following limit exists for $\sum\limits_{n=1}^{\infty} a_n$

$$\alpha = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then

(a) if $\alpha < 1$, $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely;

(b) if $\alpha > 1$, $\sum\limits_{n=1}^{\infty} a_n$ diverges;

(c) there exist both absolutely convergent and divergent series for which $\alpha = 1$.

Alternating series test (1): Leibniz series If the series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$ is such that for some $N \in \mathbb{N}$,

$$0 \leqslant a_{n+1} < a_n, \quad \forall n > N$$

and

$$a_n \to 0 \quad (n \to \infty)$$

then $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Alternating series test (2) Let $a_n > 0$. If

$$\lim_{n\to\infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right) = \lambda > 0$$

then the alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.

Property of Leibniz series If $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$ is a Leibniz series, then

$$\left| \sum_{n=N}^{N+p} (-1)^{n-1} a_n \right| \leqslant a_N$$

Abel's test Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. If $\sum\limits_{n=1}^{\infty} b_n$ is convergent and $\{|a_n|\}$ is monotonic and bounded, then

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

Dirichlet's test Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. If $\sum\limits_{n=1}^{m} b_n$ is bounded and $\{a_n\}$ monotonically tends to 0, then

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

8.3 Operations on Numerical Series

Absolute convergence Absolute convergence implies convergence.

Equivalent condition for absolute convergence

$$\sum_{n=1}^{\infty} |a_n| < +\infty \iff \sum_{n=1}^{\infty} a_n^+, \sum_{n=1}^{\infty} a_n^- < +\infty$$

Sufficient condition for absolute convergence If for all sequences $\{x_n\}$ that tends to 0, the series $\sum\limits_{n=1}^{\infty} a_n x_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely. The condition "tends to" cannot be weakened to "monotonically tends to", for example, $\sum\limits_{n=1}^{\infty} (-1)^{n-1} x_n$.

Change of order in absolutely convergent series Any change of order of the terms in an absolutely convergent series does not affect its convergence and the value it converges to.

Necessary condition for conditional convergence

$$\sum_{n=1}^{\infty} a_n \text{ converges conditionally} \implies \sum_{n=1}^{\infty} a_n^{=} \sum_{n=1}^{\infty} a_n^{-} = +\infty$$

The converse is not true, for example, $a_n = (-1)^{n-1}$.

Note 8.5. *Conditional convergence also implies that*

$$\lim_{n\to\infty} \frac{S_n^+}{S_n^-} = 1$$

Riemann's theorem for conditionally convergent series If a series is conditionally convergent, then by reordering the terms one can obtain a new series converging to any preassigned number, including $\pm\infty$.

Cauchy's theorem for the product of series If the series $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ converge absolutely to A, B respectively, then the sum

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_j$$

where the summation can happen in any order of the terms, converges absolutely to AB.

Mertens' theorem for Cauchy product If the series $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ converge to A, B respectively, and at least one of them converges absolutely, then the Cauchy product of the two series converges to AB.

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} a_k b_{n+1-k}\right) = AB$$

Abel's theorem for Cauchy product If the series $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ converge to A, B respectively, and the Cauchy product of them converges, it converges to AB.

Pringsheim's theorem for Cauchy product Let $\{a_n\}$ and $\{b_n\}$ be sequences that tends to 0 monotonically and denote the Cauchy product of the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}a_n = A$ and $\sum\limits_{n=1}^{\infty}(-1)^{n-1}b_n = B$ by $\sum\limits_{n=1}^{\infty}(-1)^{n-1}c_n$, where $c_n = \sum\limits_{k=1}^{n}(-1)^{n-1}a_k b_{n+1-k}$. The following conditions are equivalent:

1. $\sum\limits_{n=1}^{\infty}(-1)^{n-1}c_n$ is convergent.
2. $\lim\limits_{n\to\infty} c_n = 0$
3. $\lim\limits_{n\to\infty} a_n(b_1 + \cdots + b_n) = 0$ and $\lim\limits_{n\to\infty} b_n(a_1 + \cdots + a_n) = 0$.

9 Infinite Products

9.1 Convergence of Infinite products

A sufficient condition for convergence A sufficient condition for the convergence of $\prod_{n=1}^{\infty}(1+a_n)$ is that $\sum_{n=1}^{\infty}\ln(1+a_n)$ converges.

Theorem If $a_n > 0$ (resp. $a_n < 0$) for every sufficiently large n, then $\prod_{n=1}^{\infty}(1+a_n)$ and $\sum_{n=1}^{\infty} a_n$ converges and diverges simultaneously.

Theorem If $\sum_{n=1}^{\infty} a_n^2$ converges, then $\prod_{n=1}^{\infty}(1+a_n)$ and $\sum_{n=1}^{\infty} a_n$ converges and diverges simultaneously.

Theorem If $-1 < a_n < 0$, then the divergence of $\sum_{n=1}^{\infty} a_n$ implies the divergence of $\prod_{n=1}^{\infty}(1+a_n)$ to 0.

Theorem If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} a_n^2$ diverges, then $\prod_{n=1}^{\infty}\ln(1+a_n)$ diverges to 0.

9.2 Operations on Infinite Products

Absolute convergence Absolute convergence implies convergence.

Change of order in absolutely convergent series Any change of order of the terms in an absolutely convergent infinite product does not affect its convergence and the value it converges to.

Riemann's theorem for conditionally convergent infinite products If an infinite product is conditionally convergent, then by reordering the terms one can obtain a new infinite product converging to any preassigned positive number, or diverging to $+\infty$ or 0.

10 Series of Functions

10.1 Convergence of Series of Functions

Cauchy Criterion for Uniform Convergence The series $\sum_{n=1}^{\infty} a_n(x)$ converges uniformly on E if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n(x) + \cdots + a_m(x)| < \varepsilon$$

for all natural numbers m, n satisfying $m \geqslant n > N$ and every $x \in E$.

Comparison theorem If the series $\sum_{n=1}^{\infty} a_n(x)$ and $\sum_{n=1}^{\infty} b_n(x)$ are such that $|a_n(x)| \leqslant b_n(x)$ for every $x \in E$ and for all sufficiently large indices $n \in \mathbb{N}$, then the uniform convergence of the series $\sum_{n=1}^{\infty} b_n(x)$ on E implies the absolute and uniform convergence of $\sum_{n=1}^{\infty} a_n(x)$ on E.

Weierstrass M-test for uniform convergence If for $\sum_{n=1}^{\infty} a_n(x)$ one can exhibit a convergent numerical series $\sum_{n=1}^{\infty} M_n$ such that $\sup_{x \in E} |a_n(x)| \leqslant M_n$ for all sufficiently large $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n(x)$ converges absolutely and uniformly on E.

The Abel-Dirichlet test for uniform convergence A sufficient condition for uniform convergence on E of $\sum_{n=1}^{\infty} a_n(x) b_n(x)$ where $a_n : X \to \mathbb{C}$ are complex-valued functions and $b_n : X \to \mathbb{R}$ are real-valued functions is that either pair of the following be satisfied:

1. (Dirichlet)
   - (α_1) the partial sums $s_k(x) = \sum_{n=1}^{k} a_n(x)$ are uniformly bounded on E;
   - (β_1) $b_n(x)$ tends monotonically and uniformly to 0 on E;
2. (Abel)
   - (α_1) $\sum_{n=1}^{\infty} a_n(x)$ converges uniformly on E;
   - (β_1) $b_n(x)$ is monotonic and uniformly bounded on E.

10.2 Convergence of Power Series

Uniqueness of power series If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ in a neighborhood of $x = 0$, then $a_n = b_n$.

Proposition 1. If a power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ converges at a point $\zeta \neq z_0$, then it converges absolutely and uniformly in any disk $K_q = \{z \in \mathbb{C} : |z - z_0| < q\,|\zeta - z_0|\}$, where $0 < q < 1$.

Nature of convergence of a power series (Cauchy-Hadamard) A power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ converges in the disk $K = \{z \in \mathbb{C} : |z - z_0| < R\}$ whose radius of convergence is determined by the Cauchy-Hadamard formula $R = \left(\limsup_{n\to\infty} \sqrt[n]{|c_n|}\right)^{-1}$. Outside the disk the series diverges. On any closed disk contained in the interior of the disk K of convergence of the series, a power series converges absolutely and uniformly.

Proof 10.1. Use Weierstrass M-test.

Convergence at the endpoint (1) If $\sum_{n=0}^{\infty} a_n x^n$ diverges at the endpoint $x = R$ of its disk of convergence, then the series does not converge uniformly on $[0, R]$.

Convergence at the endpoint (2) If $\sum_{n=0}^{\infty} a_n x^n$ converges at the endpoint $x = R$ of its disk of convergence, then the series converges uniformly on $[0, R]$.

So-called second Abel theorem on power series If a power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ converges at $\zeta \in \mathbb{C}$, then it converges uniformly on the closed interval with endpoints $[z_0, \zeta]$.

Proposition 2. If a power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ converges at ζ, it converges uniformly on the closed interval $[z_0, \zeta]$ from z_0 to ζ, and the sum of the series is continuous on that interval.

Abel summation Abel's method of summing $\sum_{n=0}^{\infty} c_n$ is to define the sum as

$$\sum_{n=0}^{\infty} c_n = \lim_{x\to 1^-} \sum_{n=0}^{\infty} c_n x^n$$

If the left-hand side exists, then it is consistent with the conventional case; if not, it may happen that the right-hand side exists while the left-hand side does not, and thus assign to the divergent series a new meaning.

Proposition 3. Let $K \subset \mathbb{C}$ be the convergence disk for $\sum\limits_{n=0}^{\infty} c_n(z - z_0)^n$. If K contains more than just the point z_0, then the sum of the series $f(z)$ is differentiable inside K and

$$f'(z) = \sum_{n=1}^{\infty} nc_n(z - z_0)^{n-1}$$

Moreover, the function $f(z) : K \to \mathbb{C}$ can be integrated over any path $\gamma : [0,1] \to K$, and if $[0,1] \ni t \xrightarrow{\gamma} z(t) \in K$, $z(0) = z_0$, and $z(1) = z$, then

$$\int_\gamma f(z)\mathrm{d}z = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(z - z_0)^{n+1}$$

Tauber's theorem (1) If $\sum\limits_{n=1}^{\infty} a_n = A(c,1)$ and $a_n = \left(\frac{1}{n}\right)$, then the series $\sum\limits_{n=1}^{\infty} a_n$ converges in the ordinary sense to the same sum.

Tauber's theorem (1′) If the radius of convergence of $\sum\limits_{n=0}^{\infty} a_n x^n$ is $R = 1$ and $\lim\limits_{x\to 1^-} \sum\limits_{n=1}^{\infty} a_n x^n = A$, then

$$a_n = o\left(\frac{1}{n}\right) \implies \sum_{n=0}^{\infty} a_n = A$$

Tauber's theorem (2) If the radius of convergence of $\sum\limits_{n=0}^{\infty} a_n x^n$ is $R = 1$ and $\lim\limits_{x\to 1^-} \sum\limits_{n=1}^{\infty} a_n x^n = A$, then

$$a_n \geqslant 0\,(\forall n \in \mathbb{N}) \implies \sum_{n=0}^{\infty} a_n = A$$

Tauber's theorem (3) Suppose $\sum\limits_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$ and $\lim\limits_{x\to 1^-} \sum\limits_{n=0}^{\infty} a_n x^n = A$. If $\lim\limits_{n\to\infty} \frac{a_1+2a_2+\cdots+na_n}{n} = 0$, then the series $\sum\limits_{n=0}^{\infty} a_n$ converges to A in the ordinary sense.

Expansion into Taylor series A sufficient condition for a $C^\infty(x_0 - R, x_0 + R)$ function f to be Taylor expandable at x_0 is that its derivative of every order is uniformly bounded on $(x_0 - R, x_0 + R)$.

10.3 Operations on Power Series

Integral of a power series A power series $f(x) = \sum_{n=1}^{\infty} a_n x^n$ can be integrated over the interval from 0 to x, where $0 < |x| < R$, and the integral is

$$\int_0^x f(x)\mathrm{d}x = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_{n-1}}{n}x^n + \cdots$$

Derivative of a power series A power series $f(x) = \sum_{n=1}^{\infty} a_n x^n$ is differentiable inside its convergence disk, and the derivative is

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

Product of power series If $\sum_{n=1}^{\infty} a_n x^n$, $\sum_{n=1}^{\infty} b_n x^n$ both have convergence radius R, then for all $x \in (-R, R)$,

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \left(\sum_{n=0}^{\infty} c_n x^n\right)$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Substituting a power series into another one If $\varphi(y)$ can be expanded into the power series on $(-\rho, \rho)$

$$\varphi(y) = \sum_{n=0}^{\infty} h_m y^m$$

while $y = f(x)$ can be expanded on $(-R, R)$ as follows

$$y = f(x) = \sum_{n=0}^{\infty} a_n x^n$$

in such a way that $|a_0| = |f(0)| < \rho$, then for sufficiently small x, $|f(x)| < \rho$, therefore the composite $\varphi(f(x))$ exists and can be expanded into a power series.

Proof 10.2. See [Fichtenholz, pp.408, term 446].

Inverse of a power series If $f(x)$ can be expanded into a power series $\sum_{n=1}^{\infty} a_n x^n$ in a neighborhood of $x = 0$ and $a_0 \neq 0$, then $\frac{1}{f(x)}$ can also be expanded into a power series near $x = 0$.

10.4 Taylor/Maclaurin Series

Sufficient condition for a function to be Taylor expandable If all derivatives of f are uniformly bounded on $(x_0 - \delta, x_0 + \delta)$, that is,

$$\exists M \in \mathbb{R}, \forall n \in \mathbb{N}, \forall x \in (x_0 - \delta, x_0 + \delta), \left|f^{(n)}(x)\right| \leqslant M$$

then f can be expanded into its Taylor series on $(x_0 - \delta, x_0 + \delta)$.

11 Sum of Divergent Series

11.1 Abel

12 Fourier Series

12.1 Fourier Series in General

Orthogonal complement Let $\{l_k\}$ be a finite or countable orthogonal system in X, and suppose the Fourier series x_l of x converges to $x_l \in X$. Then $h = x - x_l$ is orthogonal to x_l, to the entire space generated by $\{l_k\}$, and to the closure of that.

Length of orthogonal complement Since $x = h + x_l$ is a decomposition into orthogonal vectors, their lengths satisfy:

$$\begin{aligned} \|x\|^2 &= \|h\|^2 + \|x_l\|^2 \\ &= \|h\|^2 + \sum_k \frac{|\langle x, l_k\rangle|^2}{\langle l_k, l_k\rangle} \end{aligned}$$

Also,

$$\begin{aligned} \|x - x_l\|^2 &= \|x\|^2 - \|x_l\|^2 \\ &= \|x\|^2 - \sum_k \frac{|\langle x, l_k\rangle|^2}{\langle l_k, l_k\rangle} \\ &\geqslant 0 \end{aligned}$$

Bessel's inequality $\|x\|^2 \geqslant \|x_l\|^2$ is the *Bessel's equality.* It can be written in terms of the Fourier coefficients.

$$\begin{aligned}
\|x\|^2 &\geqslant \sum_k \frac{|\langle x, l_k\rangle|^2}{\langle l_k, l_k\rangle} \quad \text{(Orthogonal system)} \\
\|x\|^2 &\geqslant \sum_k |\langle x, e_k\rangle|^2 \quad \text{(Orthonormal system)}
\end{aligned}$$

Extremal property The Fourier series x_l (if convergent) of a vector x in an orthonormal system $\{e_k\}$ give the best approximation in $L = \langle\!\langle \{e_k\} \rangle\!\rangle$.

$$\forall y \in L,\ \|x - x_l\| \leqslant \|x - y\|$$

and

$$\|x - x_l\| = \|x - y\| \iff y = x_l$$

Parseval's equality For $x \in X$ and an orthogonal system $\{l_k\}$ in X, the equality

$$\|x\|^2 = \sum_k \frac{|\langle x, l_k\rangle|^2}{\langle l_k, l_k\rangle}$$

is called **Parseval's equality**. It holds if the Fourier expansion of x equals itself.

Convergence conditions In a complete normed vector space X, given an orthogonal system $\{l_k\}$, the following are equivalent:

1. x can be approximated with arbitrary accuracy by vectors in $\{l_k\}$.
2. Fourier expansion holds for x with respect ro $\{l_k\}$.
3. Parseval's equality holds for x and $\{l_k\}$.

Note 12.1. *In other words, the approximating sequence converges to its Fourier expansion.*

Convergence in a complete space The Fourier series of any vector is convergent if the space is complete.

Completeness conditions Let there be an inner product space X and $\{l_k\}$ a finite or countable orthogonal system in X. The following are equivalent.

1. $\{l_k\}$ is complete with respect to $E \subset X$.

2. Every $x \in E \subset X$ can be approximated with arbitrary accuracy by finite linear combinations of $\{l_k\}$.
3. Fourier expansion holds for all $x \in E \subset X$.

$$x = \sum_k \frac{\langle x, l_k \rangle}{\langle l_k, l_k \rangle} l_k$$

4. Parseval's equality holds for all $x \in E \subset X$.

$$\|x\|^2 = \sum_k \frac{|\langle x, l_k \rangle|^2}{\langle l_k, l_k \rangle}$$

Properties of a complete system The following are true.

1. A vector that is orthogonal to a complete system has norm 0.
2. Expansion into a complete system is unique, that is, if x, y has the same Fourier series with respect to a complete system, then $\|x - y\| = 0$.

Trigonometric system is complete The trigonometric system

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \dots, \frac{\cos kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}}, \dots \right\}$$

is complete in the space of all ISI[2] functions on $[-\pi, \pi]$. That is, every ISI function on $[-\pi, \pi]$ can be approximated with arbitrary accuracy in the sense of the norm

$$\|f\| = \sqrt{\int_{-\pi}^{\pi} f^2(x) \mathrm{d}x}$$

Moreover, by the extremal property of the Fourier coefficients, the approximating sequence can be chosen exactly as the partial sums of its Fourier series.

12.2 Trigonometric Series

In this part of the subsection, f is assumed to be IAI[3] on $[-\pi, \pi]$ and has period 2π unless otherwise specified, and $T(f)$ denotes the trigonometric series of f (the partial sum is denoted $T_n(f)$).

Properties of coefficients The following are true.

[2] Integrable and square-integrable, which will be denoted by $\mathcal{R}_2$

[3] Integrable and absolutely integrable

1. $\lim_{n\to\infty} a_n(f) = \lim_{n\to\infty} b_n(f) = 0$
2. If f' is IAI on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$, then
$$a_n = o\left(\frac{1}{n}\right), \ b_n = o\left(\frac{1}{n}\right)$$
3. If f is monotonic on $(-\pi, \pi)$, then
$$a_n = O\left(\frac{1}{n}\right), \ b_n = O\left(\frac{1}{n}\right)$$
4. $f(x+\pi) = f(x) \implies a_{2n-1} = b_{2n-1} = 0$
   $f(x+\pi) = -f(x) \implies a_{2n} = b_{2n} = 0$
5. f is monotonically increasing on $(0, 2\pi) \implies b_n \geqslant 0$
6. f is monotonically decreasing on $(0, 2\pi) \implies b_n \leqslant 0$
7. f is bounded, has period 2π, and satisfies the Lipschitz condition of order α, that is,
$$|f(x) - f(y)| \leqslant L\,|x-y|^{\alpha}$$
   then
$$a_n = O\left(\frac{1}{n^{\alpha}}\right), \quad b_n = O\left(\frac{1}{n^{\alpha}}\right)$$

Localization The convergence of the Fourier series of f at x is only determined by the behavior of f near x.

Dini's test

$$\exists\, s \in \mathbb{R}, \exists\, \delta > 0, \ \frac{f(x+t) + f(x-t) - 2s}{t} \text{ is IAI on } [0, \delta] \implies T(f)(x) = s$$

Test of convergence by Lipschitz

$$\exists\, \alpha \in (0, 1], \ f \in \mathrm{Lip}^{\alpha}(x) \implies T(f)(x) = \frac{f(x^+) + f(x^-)}{2}$$

Note 12.2. *$f \in \mathrm{Lip}^{\alpha}(x)$ means f satisfies the Lipschitz condition of order α near x.*

Test of convergence by derivative

$$\exists \lim_{t\to 0^+} \frac{f(x+t) - f(x^+)}{t}, \ \exists \lim_{t\to 0^+} \frac{f(x-t) - f(x^-)}{-t}$$
$$\implies T(f)(x) = \frac{f(x^+) + f(x^-)}{2}$$

Test of convergence by piecewise differentiability

$$f \text{ is piecewise differentiable on } [-\pi,\pi] \implies T(f)(x) = \frac{f(x^+)+f(x^-)}{2}$$

Fejér's theorem

$$\exists\, f(x^-),\, \exists\, f(x^+) \implies T(f)(x) \xlongequal{\text{Cesàro}} \frac{f(x^+)+f(x^-)}{2}$$

Corollary

$$\exists\, f(x^-),\, \exists\, f(x^+),\, \exists\, T(f)(x) \implies T(f)(x) = \frac{f(x^+)+f(x^-)}{2}$$

Fejér's theorem

$$f \in C(\mathbb{R}),\; f \text{ has period } 2\pi \implies \forall x \in \mathbb{R},\, T_n(f)(x) \overset{\text{Cesàro}}{\rightrightarrows} f(x)$$

Weierstrass's theorem $f \in C[-\pi,\pi]$, $f(-\pi,\pi) \implies f$ can be uniformly approximated by trigonometric polynomials.

In the following part of this subsection f is assumed to be ISI on $[-\pi,\pi]$.

Trigonometric system is complete The trigonometric system

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \dots, \frac{\cos kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}}, \dots\right\}$$

is complete in the space of all ISI functions on $[-\pi,\pi]$. That is, every ISI function on $[-\pi,\pi]$ can be approximated with arbitrary accuracy in the sense of the norm

$$\|f\| = \sqrt{\int_{-\pi}^{\pi} f^2(x)\mathrm{d}x}$$

Moreover, by the extremal property of the Fourier coefficients, the approximating sequence can be chosen exactly as the partial sums of its Fourier series.

Coefficients If f has the Fourier series

$$T(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

then the Fourier coefficients with respect to the system

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \cdots, \frac{\cos kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}}, \cdots\right\}$$

is

$$\begin{aligned} c_0 &= \frac{\sqrt{\pi}a_0}{\sqrt{2}} \\ c_{2k-1} &= \sqrt{\pi}a_k \\ c_{2k} &= \sqrt{\pi}b_k \end{aligned}$$

Convergence of Fourier series If $f \in \mathcal{R}_2[-\pi, \pi]$,then the Fourier series of f converges to f in the sense of the norm $\langle\cdot,\cdot\rangle = \int_{-\pi}^{\pi}(\cdot,\cdot)\mathrm{d}x$.

Bessel's inequality If $f \in \mathcal{R}_2[-\pi, \pi]$ has the Fourier series $T(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$, then Bessel's inequality gives

$$\int_{-\pi}^{\pi} f^2(x)\mathrm{d}x \geqslant \sum_{k=0}^{2n} c_k^2 = \pi\left(\frac{a_0}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2)\right)$$

Parseval's equality If $f \in \mathcal{R}_2[-\pi, \pi]$ has the Fourier series $T(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$, then Parseval's equality gives

$$\int_{-\pi}^{\pi} f^2(x)\mathrm{d}x = \sum_{k=0}^{\infty} c_k^2 = \pi\left(\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)\right)$$

Parseval's equality of an inner product If $f, g \in \mathcal{R}_2[-\pi, \pi]$ have the Fourier series $T(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$, $T(g)(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}(\alpha_k \cos kx + \beta_k \sin kx)$ respectively, then the Parseval's equality for their inner product is:

$$\int_{-\pi}^{\pi} f(x)g(x)\mathrm{d}x = \pi\left(\frac{a_0\alpha_0}{2} + \sum_{k=1}^{\infty}((a_k + \alpha_k)^2 + (b_k + \beta_k)^2)\right)$$

Termwise integration If $f \in \mathcal{R}_2[-\pi, \pi]$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

then the integral of f over $[a, b] \subset [-\pi, \pi]$ is

$$\int_a^b f(x)\mathrm{d}x = \int_a^b \frac{a_0}{2}\mathrm{d}x + \sum_{k=1}^{\infty} \int_a^b (a_k \cos kx + b_k \sin kx)\mathrm{d}x$$

12.3 The Fourier Transform

In this subsection, f is assumed to be IAI on $\mathbb{R}$.

Coefficients If f is IAI on $\mathbb{R}$, then the coefficients

$$a(u) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\cos ut\mathrm{d}t, \quad b(u) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\sin ut\mathrm{d}t$$

that appear in the Fourier integral are uniformly continuous on $(-\infty, +\infty)$.

The partial Fourier integral (1) The partial Fourier integral of f is

$$\begin{aligned}
S(\lambda, x) &= \int_0^{\lambda} (a(u)\cos ux + b(u)\sin ux)\mathrm{d}u \\
&= \frac{1}{\pi}\int_0^{\lambda} \mathrm{d}u \left(\cos ux \int_{-\infty}^{+\infty} f(t)\cos ut\mathrm{d}t + \sin ux \int_{-\infty}^{+\infty} f(t)\sin ut\mathrm{d}t\right) \\
&= \frac{1}{\pi}\int_0^{\lambda} \mathrm{d}u \int_{-\infty}^{+\infty} f(t)(\cos ux \cos ut + \sin ux \sin ut)\mathrm{d}t \\
&= \frac{1}{\pi}\int_0^{\lambda} \mathrm{d}u \int_{-\infty}^{+\infty} f(t)\cos(u(x-t))\mathrm{d}t
\end{aligned}$$

The partial Fourier integral (2) Let f be IAI on $\mathbb{R}$, then the partial Fourier integral above may be integrated in a different order, which then gives:

$$\begin{aligned}
S(\lambda, x) &= \frac{1}{\pi}\int_0^{\lambda} \mathrm{d}u \int_{-\infty}^{+\infty} f(t)\cos(u(x-t))\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\mathrm{d}t \int_0^{\lambda} \cos(u(x-t))\mathrm{d}u \\
&= \frac{1}{\pi}\int_0^{+\infty} (f(x+t) + f(x-t))\frac{\sin \lambda t}{t}\mathrm{d}t
\end{aligned}$$

Localization theorem The convergence of the Fourier integral $S(+\infty, x)$ of f is determined only by the local behavior of f near x.

Dini's theorem Let f be IAI over $\mathbb{R}$. If there exists s for a fixed x, such that $\frac{f(x+t)+f(x-t)-2s}{t}$ is IAI on some interval $[0, \delta]$, then the Fourier integral of f converges to s at x.

Sufficient condition for convergence Let f be IAI on $\mathbb{R}$ and have generalized left and right derivatives at x. Then the Fourier integral of f at x converges to $\frac{f(x^+)+f(x^-)}{2}$

Fourier cosine integral Suppose the Fourier integral of f converges to f and let f be even. Then the Fourier integral gives:

$$f(x) = \frac{2}{\pi} \int_0^{+\infty} \cos ux \mathrm{d}u \int_0^{+\infty} f(t) \cos ut \mathrm{d}t$$

Fourier cosine transform If we write the formula above as

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \cos ux \mathrm{d}u \left(\sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(t) \cos ut \mathrm{d}t \right)$$

then by setting $g(u) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(t) \cos ut \mathrm{d}t$ it is obtained that

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} g(u) \cos ux \mathrm{d}u \\ g(u) &= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(t) \cos ut \mathrm{d}t \end{aligned}$$

We say f, g are the **Fourier cosine transforms** of each other.

Fourier sine integral

$$f(x) = \frac{2}{\pi} \int_0^{+\infty} \sin ux \mathrm{d}u \int_0^{+\infty} f(t) \sin ut \mathrm{d}t$$

Fourier sine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \sin ux \mathrm{d}u \left(\sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(t) \sin ut \mathrm{d}t \right)$$

$$
\begin{aligned}
f(x) &= \sqrt{\frac{2}{\pi}}\int_0^{+\infty} g(u)\sin ux \mathrm{d}u \\
g(u) &= \sqrt{\frac{2}{\pi}}\int_0^{+\infty} f(t)\sin ut \mathrm{d}t
\end{aligned}
$$

We say f, g are the **Fourier sine transforms** of each other.

Part V
Spaces

13 Continuous Multilinear Transformations $\mathcal{L}$

Throughout this section, $X_1, \cdots, X_n, Y$ will be normed vector spaces and $\mathcal{L}(X_1, \cdots, X_n; Y)$ will denote the space of all *continuous multilinear transformations* from $X_1 \times \cdots \times X_n$ to Y.

13.1 Multilinear Transformations

Continuity The following are equivalent for a multilinear mapping $A : X_1 \times \cdots \times X_n \to Y$.

1. $A \in \mathcal{L}(X_1, \cdots, X_n; Y)$
2. $\|A\| < +\infty$
3. A is bounded.
4. A is continuous at $(0, \cdots, 0) \in X_1 \times \cdots \times X_n$.

$\mathcal{L}$ as a normed space The space $\mathcal{L}(X_1, \cdots, X_n; Y)$ is a normed vector space.

Norm of a composite Let X, Y, Z be normed spaces and $A \in \mathcal{L}(X; Y)$, $B \in \mathcal{L}(X; Z)$, then

$$\|AB\| \leqslant \|A\| \cdot \|B\|$$

Completeness Y is complete $\implies$ $\mathcal{L}(X; Y)$ is complete.

13.2 Linear Transformations on $\mathbb{R}^n$

In this subsection let $\mathcal{L} = \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$.

Continuity The following are true for a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$.

1. $A \in \mathcal{L}$
2. $\|A\| < +\infty$
3. A is bounded.

Dimension $\dim \mathcal{L} = n^2$

Completeness The space $\mathcal{L}$ is a complete normed vector space.

Properties of the norm The following are true for A, $B \in \mathcal{L}$.

1. $\|\lambda A\| = |\lambda| \, \|A\|$
2. $\|A + B\| \leqslant \|A\| + \|B\|$
3. $\|AB\| \leqslant \|A\| \, \|B\|$
4. $\max\limits_{1 \leqslant j \leqslant n} \sum\limits_{j=1}^{n} a_{ij}^2 \leqslant \|A\|^2 \leqslant \sum\limits_{1 \leqslant i,j \leqslant n} a_{ij}^2$

Norm of a composite Let X, Y, Z be normed spaces and $A \in \mathcal{L}(X;Y)$, $B \in \mathcal{L}(X;Z)$, then

$$\|AB\| \leqslant \|A\| \cdot \|B\|$$