

Functional Analysis

TRISCT

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1 Basic Facts

1.1 The Baire Category Theorem

Theorem 1.1. *In a complete metric space, if there is a countable family of closed sets with no interior, then their union has no interior.*

Corollary 1.1. *If a complete metric space is a countable union of closed sets, then one of them has a nonempty interior.*

1.2 Basic Facts about Normed Vector Spaces

Theorem 1.2. *The following are true about finite dimensional spaces.*

- (i) *All norms are equivalent in a finite dimensional space.*
- (ii) *A finite dimensional normed vector space is complete.*
- (iii) *A finite dimensional subspace of a normed vector space is closed.*
- (iv) *In a finite dimensional normed vector space a closed and bounded set is sequentially compact¹*
- (v) *If in a normed vector space, the unit sphere is compact, then the space is finite dimensional.*

Proof. Let each space in question be denoted by E .

- (i) Let e_1, \dots, e_d be a basis for E . We define for every $\sum_{i=1}^d x_i e_i$ the norm

$$\|x\|_1 = \sum_{i=1}^d |x_i|$$

Then for any other norm $\|\cdot\|$ we have

$$\|x\| = \left\| \sum_{i=1}^d x_i e_i \right\| \leq (\max \|e_j\|) \sum_{i=1}^d |x_i| = (\max \|e_j\|) \|x\|_1$$

So $\|\cdot\|$ is weaker than $\|\cdot\|_1$. On the other hand, if $\|\cdot\|_1$ is not weaker than $\|\cdot\|$, then there is some sequence x_n such that

$$\frac{\|x_n\|}{\|x_n\|_1} \rightarrow 0$$

¹In a general topological space, compact implies sequentially compact but not conversely. However, in a metric space, sequentially compact and compact are equivalent. But we will only use the term “sequentially compact” so as to remind the reader what we are proving.

Of course we may assume $\|x_n\|_1 = 1$. In this case, each coordinate x_i is bounded, and there is a subsequence x_{n_k} converging in coordinates (and hence in the norm $\|\cdot\|_1$) to some x . Since $\|\cdot\|_1$ is stronger than $\|\cdot\|$, x is also the limit of x_{n_k} under $\|\cdot\|$, but $\|x_n\| \rightarrow 0$, hence $x = 0$. But it follows that $\|x_{n_k}\|_1 \rightarrow 0$ as well; a contradiction.

- (ii) Let the space be E with $\dim E = d$. Define for any basis the corresponding 2-norm. Then it is equivalent to the previously given norm and hence induces the same topology as the previous one. Completeness is then equivalent under these two norms. Note that under the 2-norm, there is an isometry $E \cong \mathbb{R}^d$. And the completeness follows from that of \mathbb{R}^d .
- (iii) Let M be the subspace of E and has finite dimension. Restrict the norm to M so that it becomes a Banach space by (ii). Hence any limit point lies in itself and M is therefore closed.
- (iv) Choose a basis and define the corresponding 1-norm. Since it is equivalent to the previously given norm, closedness and boundedness are equivalent under both norms. And under the new norm, each coordinate of a bounded sequence is bounded and hence a subsequence converges in coordinates. It follows that the same subsequence converges with respect to the old norm as well.
- (v) Q

□

1.3 Basic Facts about Linear Functionals

Let E be a vector space over \mathbb{R} . A **functional** f on E is a function $f : E \rightarrow \mathbb{R}$. If f is in addition linear, then it is called a **linear functional**. If instead, f satisfies $f(\alpha x) = \alpha f(x)$ and $f(x + y) \leq f(x) + f(y)$, then f is called a **sublinear** or **Minkowski functional**. If E is normed with $\|\cdot\|$, then we can define the **norm** of a linear functional f on E as follows:

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)| = \sup_{\|x\| \leq 1} |f(x)|$$

The first and the second definitions are clearly equivalent. And obviously

$$\sup_{\|x\| \leq 1} |f(x)| \geq \sup_{\|x\|=1} |f(x)|$$

Conversely we have

$$\sup_{\|x\| \leq 1} |f(x)| \leq \sup_{\substack{\|x\| \leq 1 \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)|$$

Hence all the definitions are equivalent.

Theorem 1.3. *The following are equivalent for a linear functional $f : E \rightarrow \mathbb{R}$ on a normed vector space E .*

- (i) $\|f\| < \infty$;
- (ii) $|f(x)| < M \|x\|$ for some $M < \infty$, such an f is called **bounded**;
- (iii) f is continuous;
- (iv) f is continuous at 0.

Proof. We only prove (iii) \implies (i); others are obvious.² **Hint:** what happens if $\|f\| = \infty$, i.e. there is a sequence x_n with $\|x_n\| = 1$ but $|f(x_n)| \rightarrow \infty$? Consider $f\left(\frac{x_n}{f(x_n)}\right)$ \square

Let E^* be the space of all bounded/continuous/finite-norm linear functionals. E^* is called the **dual space** of E . About the structure of E^* , we have

Theorem 1.4. (i) E^* is a normed vector space;

(ii) E^* is a Banach space.

Proof. (i) Easy;

(ii) One can prove that a Cauchy sequence in E^* has a pointwise limit. One can show this limit is linear and then bounded. \square

Theorem 1.5. *Let $f : E \rightarrow \mathbb{R}$ be a linear functional on a normed vector space (not necessarily continuous). Then $H = [f = \alpha]$ is closed $\iff f$ is continuous.*

Proof. (“ \Leftarrow ”): Obvious. (“ \Rightarrow ”): We can directly show that $f(x)$ is bounded when x belongs to a ball centered at 0. \square

²We cannot use $\|x\| = 1$ is compact because this is not trivial.

1.4 Vector Operations of Subsets in a Normed Vector Space

Theorem 1.6. *Let E be a normed vector space and $A, B \subseteq E$ nonempty subsets.*

- (i) *If one of A, B is open, then $A + B$ is open.*
- (ii) *If A is closed and B is compact, then $A + B$ is closed.*
- (iii) *If both sets are convex, then $A + B$ is convex.*
- (iv) *If A is convex, then $A + A = 2A$.*
- (v) *If A is closed, then for $\lambda > 0$, λA is closed.*
- (vi) *For $\lambda > 0$, $\lambda \overline{A} = \overline{\lambda A}$.*

Proof. (i)

(ii)

(iii)

(iv) Apparently $2A \subseteq A + A$. Conversely, if $x, y \in A$, then $\frac{x+y}{2} \in A$ and $x + y = \frac{x+y}{2} + \frac{x+y}{2} \in 2A$. Hence $A + A \subseteq 2A$.

(v) Let $x_n \rightarrow x$ with $x_n \in \lambda A$. Then $\frac{x_n}{\lambda} \in A \implies \frac{x}{\lambda} \in \overline{A} = A \implies x \in \lambda A$.

(vi) If $x \in \lambda \overline{A}$, then let $x = \lambda y$ with $y \in \overline{A}$. Let $y_n \rightarrow y$ with $y_n \in A$. Then $x_n = \lambda y_n \in \lambda A$ and $x_n \rightarrow x \implies x \in \overline{\lambda A}$. This means

$$\lambda \overline{A} \subseteq \overline{\lambda A}$$

Also, $\lambda A \subseteq \lambda \overline{A}$ and thus

$$\overline{\lambda A} \subseteq \overline{\lambda \overline{A}} = \lambda \overline{A}$$

□

2 The Hahn-Banach Theorems

2.1 Lemmas

Let E be a normed vector space and $C \subseteq E$ an open convex subset with $0 \in C$.³ Then the function is defined on E :

$$p(x) = \inf\{\alpha > 0 : \frac{x}{\alpha} \in C\}$$

³One must require that C contains a neighborhood of 0 for the gauge to be defined. Although convexity is not necessary, without it the definition may be useless.

and is called the **gauge** or the **Minkowski functional** of C .

Lemma 2.1 (Gauge, the Minkowski functional of a set). *Let E be a normed vector space and C an open convex subset of E with $0 \in C$. Then the gauge $p(x)$ of C satisfies*

- (i) p is a Minkowski functional;
- (ii) p is weaker than $\|\cdot\|_E$, in the sense that $0 \leq p(x) \leq M \|x\|_E$ for some M ;
- (iii) $x \in C \iff p(x) < 1$.

Lemma 2.2 (Separation by pivot). *In a normed vector space E , if C is an open and convex subset, then for any $x_0 \notin C$, there exists $f \in E^*$ such that*

$$\forall x \in C, f(x) < f(x_0)$$

Proof. First assume $0 \in C$, then we can define the gauge p of C , which is a Minkowski functional by Lemma 2.1. Then define on $\mathbb{R}x_0$ the linear functional

$$\begin{aligned} f : \mathbb{R}x_0 &\rightarrow \mathbb{R} \\ tx_0 &\mapsto t \end{aligned}$$

and see that it is dominated from above by p because $p(x_0) \geq 1 = f(x_0) \implies f(tx_0) \leq p(tx_0)$ for all $t > 0$ and the case $t < 0$ is trivial. Then f can be extended to $\bar{f} : E \rightarrow \mathbb{R}$ with $f(x) \leq p(x)$, $\forall x \in E$. Then we have $\forall x \in C$, $f(x) \leq p(x) < 1 = f(x_0)$.

For the case in which $0 \notin C$, we choose any $x_1 \in C$, then $0 \in (C - x_1)$ and there exists $f \in E^*$ such that

$$f(C - x_1) < f(x_0 - x_1) \implies f(C) < f(x_0)$$

by linearity. □

2.2 The Theorems

Theorem 2.1 (Helly, Hahn-Banach analytic form). *Let E be a vector space and $G \subseteq E$ a subspace. The following are true.*

- (i) *A linear functional $g : G \rightarrow \mathbb{R}$ dominated from above by a Minkowski functional $p : E \rightarrow \mathbb{R}$ has an extension to E that is still dominated from above by p .*

(ii) A continuous linear functional $g : G \rightarrow \mathbb{R}$ has an extension $f : E \rightarrow \mathbb{R}$ with $\|f\|_{E^*} = \|g\|_{G^*}$.

Proof. First prove that there is an extension to a subspace of at least one more dimension. (The constant on the new dimension needs to be chosen carefully.) Then use Zorn's lemma. \square

Theorem 2.2 (Hahn-Banach, first geometric form). *Let E be a normed vector space. Then for any two disjoint nonempty convex subsets, if one is open then they can be separated by a closed hyperplane.*

Proof. Let A, B be these two sets with A open. Let $C = A - B$ (minus sign denotes vector subtraction here). Then $C = \bigcup_{y \in B} (A - y)$ is open and $0 \notin C$. $A - B$ is also convex, for if $x - y \in A - B$ and $x' - y' \in A - B$ (with $x, x' \in A$, $y, y' \in B$), then

$$(1 - t)(x - y) + t(x' - y') = (1 - t)x + tx' - ((1 - t)y + ty') \in A - B$$

By Lemma 2.2, we can find $f \in E^*$ such that

$$\forall x \in A, y \in B, f(x - y) < 0 \implies f(x) < f(y)$$

\square

Theorem 2.3 (Hahn-Banach, second geometric form). *Let E be a normed vector space. Then for any two disjoint nonempty convex subsets, if one is compact and another is closed, then they can be strictly separated by a closed hyperplane.*

Proof. Let A, B be these sets with A closed and B compact. Let $C = A - B$. Then C is convex (see the proof of Theorem 2.2). C is also closed for if there is a sequence $x_n - y_n$ with $x_n \in A$, $y_n \in B$ that converges to $z \in E$, then a subsequence of y_n converges to some $y \in B$ because B is compact, then the corresponding subsequence of x_n converges to $x = z + y \in A$ because A is closed. Hence $z \in A - B$ and C is closed. Note that $0 \notin C$, hence we can find $B(0, r) \cap C = \emptyset$. By Theorem 2.2 there is some $f \in E^*$ such that

$$\begin{aligned} f(w) &< f(z), \forall w \in B(0, r), z \in C = A - B \\ \implies f(y) + f(w) &< f(x), \forall w \in B(0, r), x \in A, y \in B \end{aligned}$$

$f|_{B(0, r)} \neq 0$ otherwise f is identically 0. Hence we can find $w_0 \in B(0, r)$ such that $f(w_0) \neq 0$. We may assume $f(w_0) = 2\varepsilon > 0$ by replacing w_0 with $-w_0$. Then $f(y) + 2\varepsilon < f(x)$, i.e. $f(y) + \varepsilon < f(x) - \varepsilon$. \square

Theorem 2.4 (Hahn-Banach, finite dimensional case). *Let E be a finite dimensional normed vector space. Then any two disjoint nonempty convex subsets can be separated by a closed hyperplane.*

Proof. **Exercise 1.9.** \square

2.3 Corollaries

Theorem 2.5 (The duality map). *Let E be a normed vector space. Then for every $x \in E$ the set*

$$F(x) = \{f \in E^* : \|f\|_{E^*} = \|x\|_E, \langle f, x \rangle = \|x\|_E^2\}$$

is nonempty. The (multi-valued) map

$$\begin{array}{ccc} E & \rightarrow & E^* \\ x & \mapsto & F(x) \end{array}$$

*is called the **duality map**.*

Theorem 2.6 (Dense subspace). *If a subspace does not admit a nonzero linear functional that annihilates it, then it is dense.*

Proof. We shall prove this using the second form from Note 2.1. Let F be a subspace of E . If $\overline{F} \neq E$, then we find $x_0 \notin \overline{F}$ and use the second geometric form of the Hahn-Banach theorem. We obtain some $f \in E^*$ and some $\varepsilon > 0$ such that

$$f(x) + \varepsilon < f(x_0), \quad \forall x \in \overline{F}$$

$f(x)$ is bounded for $x \in \overline{F}$, which indicates $f(x) = 0, \forall x \in \overline{F}$. But this also implies $f(x_0) = 0$. \square

Note 2.1. There are more ways to state Theorem 2.6.

- (i) For a subspace, if every linear functional that annihilates it must be zero, then it is dense.
- (ii) If a subspace is not dense, then there is a nonzero linear functional that annihilates it.

3 Orthogonality

Let E be a normed vector space. For a subspace $M \subseteq E$, we define the **orthogonal space** (the term “space” is to be justified later) or the **annihilator**⁴ of M to be

$$M^\perp = M^\circ = \{f \in E^* : f(M) = 0\}$$

⁴Sometimes a single element of this set is called an annihilator.

For a subspace $N \subseteq E^*$, we define the **orthogonal space** of N in E or the **annihilator** of N in E to be

$$N^\perp = {}^\circ N = \{x \in E : \langle f, x \rangle = 0, \forall f \in N\}$$

Unless E is reflexive, the **orthogonal space** of N in E^{**} might be different (larger, actually):

$$N^\circ = \{\xi \in E^{**} : \xi(f) = 0, \forall f \in N\}$$

Theorem 3.1. M^\perp, N^\perp are closed subspaces.

Proof. They are clearly subspaces. For closedness, observe that for a given $x \in M$, the set

$$\{f \in E^* : f(x) = 0\}$$

is closed because if $g \in E^*$ is a limit point, then for $f \in \{f \in E^* : f(x) = 0\}$ $\|g(x)\| = \|g(x) - f(x)\| \leq \|g - f\| \|x\|$. Since $\|g - f\|$ can be arbitrarily small we have $g(x) = 0$. Then we have

$$M^\perp = \bigcap_{x \in M} \{f \in E^* : f(x) = 0\}$$

which is therefore closed. The proof that N^\perp is similar. □

Note 3.1. We may also use the following lemma (whose converse is not true) for the proof above.

Lemma 3.1. In E^* , converging in norm implies pointwise convergence.

Proof. Let $f_n \rightarrow f$ in E^* . Then for any $x \in E$, we have

$$\|f_n(x) - f(x)\| \leq \|f - f_n\| \|x\| \rightarrow 0$$

□

Theorem 3.2 (Orthogonal operations). *Let E be a normed vector space. Let $M \subseteq E, N \subseteq E^*$ be subspaces and $G, L \subseteq E$ closed subspaces. The following are true.*

- (i) $\overline{M} = (M^\perp)^\perp$
- (ii) $\overline{N} = (N^\perp)^\perp$
- (iii) $G \cap L = (G^\perp + L^\perp)^\perp$
- (iv) $(G \cap L)^\perp \supseteq \overline{(G^\perp + L^\perp)}$

$$(iv) \quad G^\perp \cap L^\perp = (G + L)^\perp$$

$$(iv) \quad (G^\perp \cap L^\perp)^\perp = \overline{G + L}$$

Theorem 3.3 (Closedness and orthogonality). *Let E be a Banach space. and G, L two closed subspaces of E . The following are equivalent.*

(i) $G + L$ is closed in E

(ii) $G^\perp + L^\perp$ is closed in E^*

(iii) $G + L = (G^\perp + L^\perp)^\perp$

(iv) $G^\perp + L^\perp = (G \cap L)^\perp$

Proof. (i) \iff (iii): Obvious. (ii) \iff (iv): Obvious.

(ii) \Rightarrow (i): By Theorem 5.3, there exists C such that

$$\forall f \in E^*, \quad d(f, G^\perp \cap L^\perp) \leq C(d(f, G^\perp) + d(f, L^\perp))$$

We claim

$$d(f, G^\perp) = \sup_{\substack{x \in G \\ \|x\| \leq 1}} \langle f, x \rangle$$

Indeed, $\forall g \in G^\perp$, then

$$\sup_{\substack{x \in G \\ \|x\| \leq 1}} \langle f, x \rangle = \sup_{\substack{x \in G \\ \|x\| \leq 1}} \langle f - g, x \rangle \leq \sup_{\substack{x \in E \\ \|x\| \leq 1}} \langle f - g, x \rangle = \|f - g\| = d(f, g)$$

By taking the infimum over $g \in G^\perp$,

$$\sup_{\substack{x \in G \\ \|x\| \leq 1}} \langle f, x \rangle \leq d(f, G^\perp)$$

Q: Conversely?

□

4 The Uniform Boundedness Principle

Theorem 4.1 (Banach-Steinhaus, uniform boundedness principle). *If E is a Banach space and F is a normed vector space, then any subset \mathcal{F} of $\mathcal{L}(E, F)$ that is pointwise bounded⁵ is also bounded in norm.*

⁵In the sense that for each given $x \in E$, $\|Tx\|$ is bounded for all $T \in \mathcal{F}$.

Proof. Use the Baire category theorem on the sequence

$$X_n = \{x \in E : \|Tx\| \leq n, \forall T \in \mathcal{F}\}$$

Then one of them has an interior. Consider the values of $T \in \mathcal{F}$ on some ball inside this interior and we are done. \square

Corollary 4.1 (Pointwise convergent sequence of functionals). *Let E be a Banach space and F a normed vector space. Then a pointwise convergent sequence $T_n \in \mathcal{L}(E, F)$ is bounded in norm, and the limit T is also in $\mathcal{L}(E, F)$ with a norm estimate given by*

$$\|T\| \leq \liminf_{n \rightarrow \infty} \|T_n\|$$

Corollary 4.2 (Weak boundedness). *In a Banach space E , weak boundedness is equivalent to strong boundedness. That is to say, if the image of a subset under every continuous linear functional is bounded, then the subset itself is bounded.*

However, for its dual E^* , we don't need E^{**} but merely E for the same argument. I guess this is because the norm of elements in E^* depends solely on E .

Theorem 4.2 (Weak boundedness, dual version). *Let E be a Banach space. Then if the image of a subset of E^* under every $x \in E \subseteq E^{**}$ is bounded, then the subset itself is bounded.*

5 The Open Mapping Theorem and the Closed Graph Theorem

5.1 The Theorems

Theorem 5.1. *Let E, F be Banach spaces. Then*

- (i) *A surjective continuous linear map $E \rightarrow F$ maps $B_E(0, 1)$ to a neighborhood⁶ of $0 \in F$.*
- (ii) *A surjective continuous linear map is an open map.*
- (iii) *A continuous linear map is open in its image, in the sense that it is open if we restrict the codomain to its range.*

⁶The word “neighborhood” here does not necessarily mean an open neighborhood.

Proof. First we use the Baire category theorem on the sequence of closed sets:

$$X_n = \overline{nT(B(0, 1))} = \overline{T(B(0, n))}$$

Since T is surjective, $\bigcup_{n=1}^{\infty} X_n = F$. Then for some n_0 we can find y'_0 , $r > 0$,

$$B(y'_0, r) \subseteq X_{n_0} = \overline{T(B(0, n_0))} \implies B(y'_0/n_0, r/n_0) \subseteq \overline{T(B(0, 1))}$$

Let $y_0 = y'_0/n_0$ and $4c = r/n_0$. Then

$$B(y_0, 4c) \subseteq \overline{T(B(0, 1))}$$

Also, by symmetry,

$$\begin{aligned} -y_0 &\subseteq \overline{T(B(0, 1))} \\ \implies B(0, 4c) &= B(y_0, 4c) - y_0 \subseteq \overline{T(B(0, 1))} + \overline{T(B(0, 1))} = 2\overline{T(B(0, 1))} \end{aligned}$$

by Theorem 1.6. Then

$$B(0, 2c) \subseteq \overline{T(B(0, 1))} \implies B(0, 2c/m) \subseteq \overline{T(B(0, 1/m))} \quad (1)$$

We shall prove now that $B(0, c) \subseteq T(B(0, 1))$. For any $y \in B_F(0, c)$, by (1) we can find $x_1 \in B_E(0, 1)$ with

$$\|y - Tx_1\| < \frac{c}{2}$$

Apply (1) to $y - Tx_1 \in B_F(0, c/2)$ we can find $x_2 \in B_E(0, 1/2)$ such that

$$\|y - Tx_1 - Tx_2\| < \frac{c}{2^2}$$

Suppose we have found x_1, \dots, x_n such that

$$\|y - Tx_1 - \dots - Tx_n\| < \frac{c}{2^n}$$

Then we can find $x_{n+1} \in B_E(0, 1/2^{n+1})$ such that

$$\|y - Tx_1 - \dots - Tx_n - Tx_{n+1}\| < \frac{c}{2^{n+1}}$$

Since $\sum_{i=1}^{\infty} x_i$ is a Cauchy sequence, its limit exists: $x = \sum_{i=1}^{\infty} x_i$ and $\|y - Tx\| = 0$. Hence $y = Tx$ and also $\|x\| < \sum_{i=1}^{\infty} \frac{c}{2^i} = c$. This means $B_F(0, c) \subseteq T(B_E(0, 1))$. \square

Corollary 5.1 (Homeomorphism between Banach spaces, inverse map theorem). *A continuous bijective linear map from a Banach space to another is a homeomorphism.*

Corollary 5.2. *If E is Banach under two norms, then the two norms are equivalent whenever one is weaker than the other.*

Theorem 5.2 (Closed graph theorem). *Let E and F be two Banach spaces. For a linear $T : E \rightarrow F$, if $G(T)$ is closed in $E \times F$, then $T \in \mathcal{L}(E, F)$.*

5.2 Complements and One-sided Inverses

We first introduce something about decomposition.

Theorem 5.3 (Decomposition). *In a Banach space E , if two closed subspaces G, L have a closed sum $G + L$, then is some constant $C > 0$ such that each $z \in G + L$ admits a decomposition of the form*

$$z = x + y, \quad x \in G, \quad y \in L, \quad \|x\|, \|y\| \leq C \|z\|$$

and for all $x \in E$ we have

$$d(x, G \cap L) \leq C(d(x, G) + d(x, L))$$

Let $G \subseteq E$ be a closed subspace of a Banach space E . A subspace $L \subseteq E$ is called a **topological complements** or a **complement** of G if L is closed and $E = G \oplus L$.

Theorem 5.4 (Complemented spaces). *In a Banach space E , the following are true.*

- (i) *Every finite dimensional subspace admits a complement.*
- (ii) *Every closed subspace of finite codimension admits a complement.*
- (iii) *If $N \subseteq E^*$ is a finite dimensional subspace, N^\perp is complemented.*
- (iv) *In a Hilbert space every closed subspace admits a complement.*

Theorem 5.5 (Complements and one-sided inverses). *Let E, F be Banach spaces. The following are true.*

- (i) *For a surjective $T \in \mathcal{L}(E, F)$, we have*

$$T \text{ has a right inverse} \iff \text{Ker}(T) \text{ is complemented}$$

- (ii) *For an injective $T \in \mathcal{L}(E, F)$, we have*

$$T \text{ has a left inverse} \iff \text{Im}(T) \text{ is closed and complemented}$$

6 Adjoint Operators

Let E and F be Banach spaces. An **unbounded linear operator** from E to F is a linear map $A : D(A) \subseteq E \rightarrow F$, where $D(A)$ is a subspace, called the **domain** of A . An unbounded linear operator A is called **bounded** or **continuous** if $D(A) = E$ and $\exists C \geq 0, \forall u \in E, \|Au\| \leq C\|u\|$. We define the **norm** of a bounded operator by

$$\|A\|_{\mathcal{L}(E,F)} = \sup_{u \neq 0} \frac{\|Au\|}{\|u\|}$$

which exists since A is bounded.

A map is called **closed** if its graph is closed.

Now we come to adjoints. Let $A : D(A) \subseteq E \rightarrow F$ be an unbounded linear operator, where $D(A)$ is dense in E . We shall define another linear operator $A^* : D(A^*) \subseteq F^* \rightarrow E^*$. For the domain of A^* , we define

$$D(A^*) = \{v \in F^* : \exists C_v > 0, \forall u \in D(A), |\langle v, Au \rangle| \leq C_v \|u\|, \}$$

Then $D(A^*)$ is a linear subspace of F^* . **Q: Why? A: The constant C is not uniform for all v** We then send each $v \in F^*$ to A^*v , where A^*v is the (unique) extension of

$$\langle v, A(\cdot) \rangle \in (D(A))^*$$

The existence of the extension comes from the Hahn-Banach theorem. The uniqueness comes from $D(A)$ being dense. **Q: Check!** We can check that

$$A^* : D(A^*) \subseteq F^* \rightarrow E^*$$

is linear.

Some properties of A^* are as follows.

Theorem 6.1. *The following are true.*

(i) For any $u \in D(A), v \in D(A^*),$

$$\langle A^*v, u \rangle_{E^*,E} = \langle v, Au \rangle_{F^*,F}$$

(ii) If $\|A\|_{\mathcal{L}(E,F)} < \infty$, then

$$\|A^*\|_{\mathcal{L}(F^*,E^*)} = \|A\|_{\mathcal{L}(E,F)}$$

(iii) A^* is always closed.

(iv) The graphs $G(A^*)$ is isomorphic to $G(A)^\perp$ under the relation:

Proof. (i) By definition.

(ii) First of all $D(A) = E \implies D(A^*) = F^*$. Then we have

$$|\langle A^*v, u \rangle| = |\langle v, Au \rangle| \leq \|v\| \|Au\| \leq \|v\| \|A\| \|u\|$$

$$\implies |\langle A^*v, \cdot \rangle| \leq \|A\| \|v\| \|\cdot\| \implies \|A^*v\| \leq \|A\| \|v\| \implies \|A^*\| \leq \|A\|$$

Hence A^* is bounded. On the other hand,

$$|\langle v, Au \rangle| = |\langle A^*v, u \rangle| \leq \|A^*v\| \|u\| \leq \|A^*\| \|u\| \|v\|$$

$$\implies |\langle \cdot, Au \rangle| \leq \|A^*\| \|u\| \|\cdot\| \implies \|Au\| \leq \|A^*\| \|u\| \implies \|A\| \leq \|A^*\|$$

(iii) Let $(v, f) \in F^* \times E^*$ be a limit point of $G(A^*)$. Then we can find a sequence $v_n \in D(A^*)$ such that

$$v_n \rightarrow v, \quad A^*v_n \rightarrow f$$

We can check $v \in D(A^*)$ and $A^*v = f$.

□

7 Weak Topology

7.1 The Coarsest Topology Associated to a Family of Mappings

Given a set X and a family of mappings $(\varphi_i : X \rightarrow Y_i)_{i \in I}$ from X to a family of topological spaces $(Y_i)_{i \in I}$, we want to find the *coarsest* or *weakest* topology that makes all of (φ_i) continuous.⁷

Theorem 7.1 (Coarsest topology and its basis). *Let X and φ_i be as above. First we consider the family*

$$\mathcal{F} = \{U \subseteq X : U = \varphi_i^{-1}(\omega_i) \text{ for some open } \omega_i \subseteq Y_i\}$$

Let Φ be the family of sets that are finite intersections of members in \mathcal{F} . Let τ be the smallest topology containing \mathcal{F} . Then Φ is a basis for τ .

Proof. By the construction, we see that $X \in \Phi$ and Φ is closed under finite intersections. Hence Φ is eligible for a basis of some topology. Suppose τ_Φ is the topology generated by Φ . Then $\mathcal{F} \subseteq \Phi \implies \tau \subseteq \tau_\Phi$. Conversely, $\mathcal{F} \subseteq \tau_\Phi$ and τ_Φ is a topology, hence $\tau \subseteq \tau_\Phi$. Hence $\tau = \tau_\Phi$ and Φ is a topology for τ . □

⁷The product topology is a special case of this, at least when it is a finite product.

Recall that in any topological space, a **basis of neighborhoods** \mathcal{B}_x of x is a family of neighborhoods of x such that any neighborhood of x contains a neighborhood from the family. In particular, the basis \mathcal{B}_x satisfies

(i) $\forall B \in \mathcal{B}_x, B \neq \emptyset$.

(ii) $\forall B_1, B_2 \in \mathcal{B}_x, \exists B_3 \in \mathcal{B}_x, B_3 \subseteq B_1 \cap B_2$.

So that we can talk about the limit $\lim_{x' \rightarrow x} f(x')$ of a map $f : X \rightarrow Y$. However we won't be needing this now.

Theorem 7.2 (Basis of neighborhoods). *Let $x \in X$, where X is equipped with the coarsest topology associated to $(\varphi_i : X \rightarrow Y_i)$. Then the family*

$$\left\{ \bigcap_{i \in I} \varphi_i^{-1}(V_i) : V_i \ni \varphi_i(x) \text{ is open; only finitely many } V_i \neq Y_i \right\}$$

is a basis of topology for x .

Proof. Each $\bigcap_{i \in I} \varphi_i^{-1}(V_i)$ as above is actually a finite intersection of open sets, and is hence a neighborhood. For any neighborhood U_x of x , since Φ is a basis for the topology, we can write U_x as a union of the basic open sets in Φ :

$$U_x = \bigcup_j B_j, \quad B_j \in \Phi$$

One of the B_j must contain x . Let it be B_0 . Recall that B_0 has the form $B_0 = \bigcap_{i \in \Gamma} \varphi_i^{-1}(V_i)$ where Γ is finite and V_i is open in Y_i . Since $x \in B_0$, we have $\varphi_i(x) \in V_i$ for all $i \in \Gamma$. Also by setting $V_i = Y_i$ for $i \notin \Gamma$, we have $x \in \varphi_i^{-1}(V_i)$ for $i \notin \Gamma$. And we can write $B_0 = \bigcap_{i \in I} \varphi_i^{-1}(V_i)$. Also $B_0 \subseteq U_x$ as desired. \square

Theorem 7.3 (Convergence). *The convergence in the coarsest topology is equivalent to convergence in coordinates⁸*

Proof. If $x_n \rightarrow x$ in this topology, then $\varphi_i(x_n) \rightarrow \varphi_i(x)$ because each φ_i is continuous with respect to this topology. **I haven't thought this through. What is the relation between convergence and continuity?** Conversely, for any neighborhood U of x , we wish to prove $x_n \in U$ for all sufficiently large n . As it turns out it suffices to prove this for some member in the basis of neighborhoods that is contained in U . Let $\bigcap_{i \in \Gamma} \varphi_i^{-1}(V_i)$ be as above, where Γ is finite and $\varphi_i(x) \in V_i$. Since $\varphi_i(x_n) \rightarrow \varphi_i(x)$, for each $i \in \Gamma$, there is some N_i such that $n \geq N_i \implies \varphi_i(x_n) \in V_i$. This means

$$n \geq \max_{i \in \Gamma} (N_i) \implies x_n \in \bigcap_{i \in \Gamma} \varphi_i^{-1}(V_i)$$

The proof is then complete. \square

⁸By the "coordinates" of x I means the family $(\varphi_i(x))$.

Theorem 7.4 (Continuity, of maps to X). *A map into X is continuous \iff its coordinates are continuous.*

Proof. Compare this with the product topology. The proof shall be delayed. \square

7.2 The Weak Topology $\sigma(E, E^*)$.

The **weak topology**⁹ $\sigma(E, E^*)$ on E is the coarsest topology on E associated to E^* . Let τ be the usual (strong) topology (induces by its norm), since each $f \in E^*$ is already continuous, we must have

$$\sigma(E, E^*) \subseteq \tau$$

Theorem 7.5. *Due to the inclusion, the following implications are immediate:*

- (i) O is weakly open $\implies O$ is open.
- (ii) F is weakly closed $\implies F$ is closed.
- (iii) $x_n \rightarrow x \implies x_n \rightharpoonup x$.

Theorem 7.6 (Hausdorff). *The weak topology is Hausdorff.*

Proof. For $x_1 \neq x_2$, by the Hahn-Banach theorem, we find $f \in E^*$ to separate them:

$$f(x_1) < \alpha < f(x_2)$$

Then $x_1 \in f^{-1}((-\infty, \alpha))$, $x_2 \in f^{-1}((\alpha, +\infty))$ are disjoint neighborhoods of them. \square

Theorem 7.7 (Improved basis). *The collection of sets of the form*

$$V(f_1, \dots, f_k; \varepsilon; x_0) = \bigcap_{i=1}^k f_i^{-1}(\langle f_i, x_0 \rangle - \varepsilon, \langle f_i, x_0 \rangle + \varepsilon)$$

with varying $f_i \in E^$, $x_0 \in E$, $\varepsilon > 0$ is a basis for the weak topology.*

Proof. Each set of the form is weakly open. Conversely, it suffices to prove that if O_W is weakly open, then $\forall x_0 \in O_W$, we can find a set of the form that is contained in O_W . \square

The following are some conditions related to weak convergence.

⁹The terms in topology under the weak topology is usually preceded by “weak/weakly”. The weak convergence is denoted by \rightharpoonup .

Theorem 7.8. *The following are true for $\sigma(E, E^*)$.*

- (i) $x_n \rightharpoonup x \iff \forall f \in E^*, f(x_n) \rightarrow f(x)$.
- (ii) *Weakly convergent \implies bounded.*
- (iii) $x_n \rightharpoonup x \implies \|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$.
- (iv) $x_n \rightharpoonup x, f_n \rightarrow f \implies \langle f_n, x_n \rangle \rightarrow \langle f, x \rangle$.

Proof.

- (i) (“ \implies ”): By definition.
 (“ \impliedby ”): Consider the basic open sets.
- (ii) By uniform boundedness principle.
- (iii) For any $f \in E^*$, $|\langle f, x_n \rangle| \leq \|f\| \|x_n\| \implies |\langle f, x \rangle| \leq \|f\| \liminf_n \|x_n\|$.
 Recall that $\|x\| = \sup_{\|f\| \leq 1} |\langle f, x \rangle|$.
- (iv) Easy.

□

7.3 Between the Weak Topology and the Strong Topology

This lemma is to estimate the norm in a finite dimensional Banach space.

Lemma 7.1. *Let E be a Banach space and $n = \dim E < \infty$. If e_1, \dots, e_n is a basis for E and its dual basis is e_1^*, \dots, e_n^* , then for any $x = \sum_{i=1}^n x_i e_i$, we have*

$$\|x\| \leq \sum_{i=1}^n |x_i| \|e_i\| = \sum_{i=1}^n |e_i^*(x)| \|e_i\|$$

Moreover, we can assume $\|e_i\| = 1$ so that

$$\|x\| \leq \sum_{i=1}^n |e_i^*(x)|$$

Theorem 7.9. *Let E be a Banach space.*

- (i) $\dim E < \infty \implies$ *the weak topology on E coincides with the original.*
- (ii) $\dim E = \infty \implies$ *the weak topology on E is strictly coarser than the original.*

Proof. (i) Let $n = \dim E < \infty$. Choose e_1, \dots, e_n as a basis for E and let e_1^*, \dots, e_n^* be its dual basis. Assume $\|e_i\| = 1$. Then for any open set $U \subseteq E$, and any $x_0 \in E$, we have

$$\|x - x_0\| \leq \sum_{i=1}^n |\langle e_i^*, x - x_0 \rangle|$$

Then by Theorem 7.7 we can find a weakly open neighborhood that is contained in U .

(ii) Let E be infinite dimensional. We shall prove that

$$S = \{x \in E : \|x\| = 1\}$$

is not weakly closed, and that

$$B_E(0, 1) = \{x \in E : \|x\| < 1\}$$

is not weakly open. □

Theorem 7.10 (Unit ball). *Let E be a Banach space and $\dim E = \infty$. Then*

the strongly closed unit sphere $\{x \in E : \|x\| \leq 1\}$ is never weakly closed;

the strongly open unit ball $\{x \in E : \|x\| < 1\}$ is never weakly open;

7.4 The Weak* Topology $\sigma(E^*, E)$

The definition is the same.

Theorem 7.11. $\sigma(E^*, E)$ is Hausdorff.

Proof. No need for the Hahn-Banach theorem, but only that $f_1 \neq f_2 \implies$ they don't agree on some point. □

Theorem 7.12. *A basic open set near f_0 is of the form*

$$V(x_1, \dots, x_k, \varepsilon) = \{f \in E^* : |\langle f - f_0, x_i \rangle| < \varepsilon\}$$

Theorem 7.13. *If τ denotes the strong topology on E^* , then*

$$\sigma(E^*, E) \subseteq \sigma(E^*, E^{**}) \subseteq \tau$$

and the following implications are immediate.

(i) *Weakly* open \implies weakly open \implies strongly open.*

(ii) *Strong convergence \implies weak convergence \implies weak* convergence.*

Theorem 7.14 (On the weak* convergence).

(i) $f_n \xrightarrow{*} f \iff \forall x \in E, \langle f_n, x \rangle \rightarrow \langle f, x \rangle.$

(ii) $f_n \xrightarrow{*} f \implies \|f_n\|$ is bounded, $\|f\| \leq \liminf_{n \rightarrow \infty} \|f_n\|.$

(iii) $f_n \xrightarrow{*} f, x_n \rightarrow x \implies \langle f_n, x_n \rangle \rightarrow \langle f, x \rangle.$

We now introduce a representation theorem, with a preceding lemma.

Lemma 7.2 (Linear combination of functionals). *Let X be a vector space and let $\varphi, \varphi_1, \dots, \varphi_k$ be linear functionals on X . If*

$$\bigcap_{i=1}^k \text{Ker} \varphi_i \subseteq \text{Ker} \varphi$$

then φ is a linear combination of φ_i ($1 \leq i \leq k$).

Proof. Consider the embedding

$$\begin{aligned} F : X &\rightarrow \mathbb{R}^{k+1} \\ x &\mapsto (\varphi(x), \varphi_1(x), \dots, \varphi_k(x)) \end{aligned}$$

The assumption says $(1, 0, \dots, 0) \notin \text{Im} F$. Note that $\text{Im} F$ is convex and closed (being a finite dimensional subspace), hence by the Hahn-Banach theorem, we find a closed hyperplane $[G = \alpha]$ in \mathbb{R}^{k+1} to strictly separate $(1, 0, \dots, 0)$ and $\text{Im} F$. Also we assume $G(e_i) = \lambda_{i-1}$ ($1 \leq i \leq k+1$). Then

$$G(1, 0, \dots, 0) = \lambda_0 < \alpha < G(x) = \lambda_0 \varphi(x) + \sum_{i=1}^k \lambda_i \varphi_i(x)$$

Since X is a vector space, then right hand side is 0,

$$\lambda_0 \varphi + \sum_{i=1}^k \lambda_i \varphi_i = 0$$

Moreover, note that $\lambda_0 < \alpha < 0$, so that we divide λ_0 for the result. \square

Theorem 7.15 (Representation theorem for $E_{\sigma(E^*, E)}^{**}$ by E). *Let $E_{\sigma(E^*, E)}^{**}$ denote the set of all linear functionals on E^* that is continuous with respect to $\sigma(E^*, E)$. Then*

$$E_{\sigma(E^*, E)}^{**} = E$$

Proof. (“ \supseteq ”): Obvious. (“ \subseteq ”): Let $\varphi \in E_{\sigma(E^*, E)}^{**}$. The idea is to find a bunch of x_i , such that

$$\bigcap_i \text{Ker } x_i \subseteq \text{Ker } \varphi$$

Naturally the inclusion should happen near 0. Let $U = \{f \in E^*, |\varphi(f)| < 1\}$. Then U is a neighborhood of 0 in $(E^*, \sigma(E^*, E))$. Hence we can find V of the form $V(x_1, \dots, x_n; 0; \varepsilon)$. Then prove that these x_i satisfies the condition in the lemma above. \square

Theorem 7.16 (Representation for closed hyperplanes in $(E^*, \sigma(E^*, E))$). *A closed hyperplanes in $(E^*, \sigma(E^*, E))$ is of the form*

$$\{f \in E^* : \langle f, x_0 \rangle = \alpha\}$$

for some $x_0 \neq 0$, $\alpha \in \mathbb{R}$.

Proof. A hyperplane closed in $\sigma(E^*, E)$ is closed in the strong topology, hence has the form

$$H = \{f \in E^* : \varphi(f) = \alpha\}$$

for some $0 \neq \varphi \in E^{**}$. Now since it is closed in the weak* topology, for any $f_0 \notin H$, we find a weak* neighborhood V of f_0 of the form

$$V = V(x_1, \dots, x_k; f_0; \varepsilon)$$

such that $V \cap H = \emptyset$, i.e. $\forall f \in V, \varphi(f) \neq \alpha$. \square

Theorem 7.17 (Banach-Alaoglu-Bourbaki). *The closed unit ball $\{f \in E^* : \|f\| \leq 1\}$ in E^* is compact in $\sigma(E^*, E)$.*

Note 7.1. Compare with Theorem 7.10.

Proof. First embed the ball $\{f \in E^* : \|f\| \leq 1\}$ into \mathbb{R}^E . Then prove the embedding is a homeomorphism onto its image and prove the image is a compact set by rewriting it. \square

8 Reflexive Spaces and Separable Spaces

Let E be a Banach space. If the canonical injection $E \hookrightarrow E^{**}$ is surjective we say E is **reflexive**. In this case we have $E \cong E^{**}$ as Banach spaces (in the sense that it is a linear isometry).

Q 8.1. Non-Banach reflexive space?

Answer. Impossible, as $E \cong E^{**}$, which is Banach. □

Here are some examples of reflexive spaces:

- E with $\dim E < \infty$.
- L^p for $1 < p < \infty$.
- Hilbert spaces.

There are also nonreflexive spaces:

- L^1 and L^∞ .
- $C(K)$, the space of continuous functions on an infinite compact metric space K .

We will use the following two lemmas to prove an equivalent condition for reflexivity.

Lemma 8.1 (Helly). *Let E be a Banach space and $f_1, \dots, f_k \in E^*$, $\gamma_1, \dots, \gamma_k \in \mathbb{R}$. Then the following are equivalent.*

- (i) $\forall \varepsilon > 0, \exists x_\varepsilon \in E$ such that $\|x_\varepsilon\| \leq 1$, and $\forall i, |\langle f_i, x_\varepsilon \rangle - \gamma_i| < \varepsilon$.
- (ii) $\forall \beta_1, \dots, \beta_k \in \mathbb{R}, \left| \sum_{i=1}^k \beta_i \gamma_i \right| \leq \left\| \sum_{i=1}^k \beta_i f_i \right\|$.

Note 8.1. Another interpretation of (i) is the weak neighborhood

$$\bigcap_{i=1}^k f_i^{-1}(\gamma_i - \varepsilon, \gamma_i + \varepsilon)$$

has a nonempty intersection with $B_E = \{x \in E : \|x\| \leq 1\}$.

Proof. (i) \implies (ii): By directly computing $\left| \sum_{i=1}^k \beta_i (\langle f_i, x \rangle_\varepsilon - \gamma_i) \right|$.

(ii) \implies (i): Consider the map

$$\begin{aligned} \varphi : E &\rightarrow \mathbb{R}^k \\ x &\mapsto (\langle f_1, x \rangle, \dots, \langle f_k, x \rangle) \end{aligned}$$

We see that

$$(i) \iff (\gamma_1, \dots, \gamma_k) \in \overline{\varphi(B_E)}$$

If (i) does not hold then we use a hyperplane in \mathbb{R}^k to strictly separate γ and $\overline{\varphi(B_E)}$. Suppose this hyperplane is determined by $f = \sum_i \beta e_i^*$ where e_i^* is the dual basis of the standard basis. A direct computation yields a contradiction. □

Lemma 8.2 (Goldstine). *Let E be a Banach space. Then the following are true.*

- (i) $J(B_E)$ is dense in $B_{E^{**}}$ with respect to $\sigma(E^{**}, E^*)$.
- (ii) $J(B_E)$ is strongly closed in $B_{E^{**}}$.
- (iii) $J(B_E)$ is not strongly dense in $B_{E^{**}}$ unless they are equal.

Proof. (i) Let $\xi \in B_{E^{**}}$. We wish to prove that in every neighborhood V of $B_{E^{**}}$ there is a point of $J(B_E)$. We assume V to be of the form

$$V = \{\eta \in E^{**} : \forall i : 1 \leq i \leq k, |\langle \eta, f_i \rangle - \langle \xi, f_i \rangle| < \varepsilon\}$$

To show that $J(B_E)$ intersects V is to show that there is some $x \in B_E$ such that we can replace $\langle \eta, f_i \rangle$ as above can be replaced by $\langle f_i, x \rangle$. To use Helly's lemma, we put $\gamma_i = \langle \xi, f_i \rangle$. We can check that in Helly's lemma (ii) holds and hence (i) holds, and hence $J(x_\varepsilon) \in V$ with $x_\varepsilon \in B_E$.

(ii) Q.

(iii) By (ii). □

Corollary 8.1. $J(E)$ is dense in E^{**} with respect to $\sigma(E^{**}, E^*)$.

Theorem 8.1 (Kakutani). *For a Banach space E , E is reflexive $\iff \{x \in E : \|x\| \leq 1\}$ is compact in $\sigma(E, E^*)$.*

The following theorem states the relation between weakly sequentially compact and being reflexive.

Theorem 8.2. *Let E be a Banach space.*

- (i) E is reflexive \implies any bounded sequence has a weakly convergent subsequence.
- (ii) (Eberlein-Šmulian) Every bounded sequence in E has a weakly convergent subsequence $\implies E$ is reflexive.

Proof. The proof of (i) is postponed; that of (ii) is out of reach. □

Theorem 8.3. *A closed subspace of a reflexive space is reflexive.*

Proof. Let M be a closed subspace of a reflexive space E . In view of Kakutani's theorem, we only need to show B_M is closed in $\sigma(M, M^*)$. Step 1 is to prove that $\sigma(M, M^*)$ is the same as the topology induced by $\sigma(E, E^*)$. Since each $f \in E^*$ restricts to $f|_M \in M^*$, we know $\sigma(M, M^*)$ is stronger than the induced topology. On the other hand, the Hahn-Banach theorem says we can extend each $g \in M^*$ to some $\tilde{g} \in E^*$. Hence $\sigma(M, M^*)$ is also weaker.

Q 8.2. Make precise these arguments.

Then we have $B_M = M \cap B_E$. Since M is convex and closed, it is also weakly closed. Also, B_E is weakly compact. Hence B_M is weakly compact in $\sigma(E, E^*)$. Note that since $\sigma(M, M^*)$ is induced by $\sigma(E, E^*)$, any open cover of B_M in the sense of $\sigma(M, M^*)$ lifts to one in the sense of $\sigma(E, E^*)$, and admits a finite subcover. By restricting to M again we have a finite subcover of B_M in the sense of $\sigma(M, M^*)$. \square

Corollary 8.2. E is reflexive $\iff E^*$ is reflexive.

Corollary 8.3. In a reflexive space, bounded, closed, convex \implies weakly compact.

Theorem 8.4. If E, F are Banach spaces, then for an densely defined, closed unbounded operator $A : D(A) \subseteq E \rightarrow F$, it holds that $A^{**} = A$.