Topology

TRISCT

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1 Topological Spaces

1.1 The Definition of a Topological Space

A **topological space** is a nonempty set X, together with a family τ of subsets of X, satisfying

(i) $\varnothing, X \in \tau$;

(ii) τ is closed under finite intersections;

(iii) τ is closed under arbitrary unions.

The subsets in τ are called **open sets**. A **closed set** is one whose complement is open. τ itself is called a **topology** on X. Any subset Y of a topological space X inherits naturally a topology. In this case, Y is called a **subspace** of X.

1.2 Limit Points

Let X be a topological space. Given a subset $A \subseteq X$, a **limit point** of the subset A is a point $p \in X$ such that any neighborhood U of x contains at least one point in A distinct from p, i.e. $U \cap (A \backslash \{p\}) = U \backslash \{p\} \cap A \neq \varnothing$. For a subset A, let A' be the set of all the limit points of A. This A' is called the **derived set** of A. $\overline{A} = A \cup A'$ is called the **closure** of A. A subset $A \subseteq X$ is said to be **dense** in X if $\overline{A} = X$.

Let $A \subseteq X$. An **interior point** of A is a point for which there exists a neighborhood of it contained in A. The **interior** of A is the collection of all the interior points of A. A **boundary point** of A is a point that is neither in the interior of A, nor in that of $X \backslash A$.

Using limit points, we can characterize these concepts in a different manner. In the following theorems, let A be a subset of a topological space X.

Theorem 1.1 (Equivalent conditions for an open set). *TFAE.*

(i) A is open;

(ii) Every point in A has a neighborhood contained in A.

Proof.
(i) $\implies$ (ii): A itself is the neighborhood.
(ii) $\implies$ (i): Let $x \in A$. Then there is an open set U_x with $x \in U_x \subseteq A$. Hence $A = \bigcup_{x \in A} U_x$ is open. $\square$

Theorem 1.2 (Equivalent conditions for a closed set). *TFAE.*

(i) A is closed, i.e. has an open complement;

(ii) $A' \subseteq A$, i.e. all limit points of A lie in A;

(iii) $A = \overline{A}$.

Proof.
(i) $\implies$ (ii): The complement $X \backslash A$ is open, hence contains no limit points of A.
(ii) $\implies$ (i): Any point in $X \backslash A$ is not a limit point of A, hence has a deleted neighborhood that does not intersect A, but the point itself is not in A either. It

follows that this neighborhood is contained in $X \backslash A$.
(ii) $\implies$ (iii): $A \subseteq \overline{A}$ is obvious. $\overline{A} \subseteq A$ since $A' \subseteq A$.
(iii) $\implies$ (ii): Trivial. $\square$

Theorem 1.3 (Equivalent conditions for the closure). *TFAE.*

(i) $K = \overline{A}$, *i.e.* $K = A \cup A'$;

(ii) $K = \bigcap_{\substack{A \subseteq F \\ F \text{ is closed}}} F$;

(iii) $K \supseteq A$ *is closed and is such that for any closed set* $F \supseteq A$*, it holds* $K \subseteq F$.

Proof.
(i) $\iff$ (ii): We first prove that $\overline{A}$ is closed. This is easily done by showing that $X \backslash \overline{A}$ is open. Hence

$$\bigcap_{\substack{A \subseteq F \\ F \text{ is closed}}} F \subseteq \overline{A}$$

Conversely if $A \subseteq F$ and F is closed, then every limit point of A is also one of F, and hence lies in $F \implies A' \subseteq F \implies \overline{A} \subseteq F$. It follows that

$$\overline{A} \subseteq \bigcap_{\substack{A \subseteq F \\ F \text{ is closed}}} F$$

(ii) $\implies$ (iii): Obvious.
(iii) $\implies$ (ii): Obviously $K \subseteq \bigcap_{\substack{A \subseteq F \\ F \text{ is closed}}} F$ if K satisfies (iii). However, K itself appears in the intersection, hence the incluson is actually equality. $\square$

1.3 Topological Bases

Let X be a topological space equipped with the topology τ. A **topological basis** for τ is a subset $\beta \subseteq \tau$ (i.e. a subcollection of open sets) for which every element $O \in \tau$ can be written in the form

$$O = \bigcup_{B_i \in \beta,\ i \in I} B_i$$

The open sets in β are called **basic open sets**. A **topological subbasis** for τ is a subset $\sigma \subseteq \tau$ such that τ is the smallest topology on X containing σ.

Note 1.1. Note that we cannot necessarily write an open set in τ in the form of a union of members in a subbasis.

Note 1.2. You might want to compare this with the weak topology.

Now we want to determine for a collection β of subsets in a set X (with no topology previously specified), whether the topology τ generated by β has β as its basis. We have the following theorem.

Theorem 1.4. *Let X be a set. Let β be a nonempty collection of subsets in X. Let τ_β be the topology generated by β. Suppose β satisfies:*

(i) $X = \bigcup_{A\in\beta} A$.

(ii) For any finitely many members $A_1, \cdots, A_k \in \beta$ and any $x \in \bigcap_{i=1}^k A_i$, there exists $A \in \beta$ with $x \in A \subseteq \bigcup_{i=1}^k A_i$.

Then β is a basis for τ_β.[1]

Proof. We shall show the family F_β (as in the footnote) is a topology on X. First of all $\varnothing \in \mathcal{F}_\beta$ by assumption and $X = \bigcup_{A\in\beta} A$ by property (i). Then let $\{O_\alpha\}_{\alpha\in\Lambda}$ be such a family that each O_α has the form

$$O_\alpha = \bigcup_{i\in I_\alpha} A_i, \text{ with } A_i \in \beta$$

Suppose Λ is an arbitrary index set, we have

$$\bigcup_{\alpha\in\Lambda} O_\alpha = \bigcup_{\alpha\in\Lambda}\bigcup_{i\in I_\alpha} A_i$$

which is a union of members in β, and lies in $\mathcal{F}_\beta$ by the definition of $\mathcal{F}_\beta$. This proves $\mathcal{F}_\beta$ is closed under arbitrary unions. Now suppose on the other hand that $\Lambda = \{\alpha_1, \alpha_2\}$ has two indexes, then

$$\bigcap_{\alpha\in\Lambda} O_\alpha = \left(\bigcup_{i\in I_{\alpha_1}} A_i\right) \cap \left(\bigcup_{j\in I_{\alpha_2}} A_j\right)$$

Let $x \in \left(\bigcup_{i\in I_{\alpha_2}} A_i\right) \cap \left(\bigcup_{j\in I_{\alpha_1}} A_j\right)$. Then there exists A_{i_x} with $i_x \in I_{\alpha_1}$ and A_{j_x} with $j_x \in I_{\alpha_2}$ such that $x \in A_{i_x} \cap A_{j_x}$. By (ii), there exists $A_x \in \beta$ such that

$$x \in A_x \subseteq A_{i_x} \cap A_{j_x} \subseteq \left(\bigcup_{i\in I_{\alpha_2}} A_i\right) \cap \left(\bigcup_{j\in I_{\alpha_1}} A_j\right)$$

[1] Another equivalent statement is: if β satisfies (i) and (ii), then the family $\mathcal{F}_\beta$ of subsets, obtained by adjoining all arbitrary unions of members of β and the empty set, is a topology.

Then taking the union over $x \in \left(\bigcup_{i\in I_{\alpha_2}} A_i\right) \cap \left(\bigcup_{j\in I_{\alpha_1}} A_j\right)$ we get

$$\begin{aligned} \left(\bigcup_{i\in I_{\alpha_2}} A_i\right) \cap \left(\bigcup_{j\in I_{\alpha_1}} A_j\right) &= \bigcup_x \{x\} \\ \subseteq \bigcup_x A_x \subseteq & \left(\bigcup_{i\in I_{\alpha_2}} A_i\right) \cap \left(\bigcup_{j\in I_{\alpha_1}} A_j\right) \\ \implies \left(\bigcup_{i\in I_{\alpha_2}} A_i\right) \cap \left(\bigcup_{j\in I_{\alpha_1}} A_j\right) &= \bigcup_x A_x \in \mathcal{F}_\beta \end{aligned}$$

By induction we can conclude that $\mathcal{F}_\beta$ is closed under finite intersections. It follows that $\mathcal{F}_\beta$ is a topology. Apparently any topology containing β must contain $\mathcal{F}_\beta$ as well. Hence $\tau_\beta = \mathcal{F}_\beta$ and β is a basis for τ_β. $\square$

2 Maps

2.1 Continuity

Let $f : X \to Y$ be a map between topological spaces. We have the following equivalent conditions for the continuity of f.

Theorem 2.1 (Equivalent conditions for continuity)**.** *The following are equivalent*

(i) *f is continuous, i.e. the preimage of an open set in Y under f is open in X.*

(ii) *The preimage of a closed set in Y under f is closed in Y.*

(iii) *Given a basis for the topology in Y, the preimage of any basic open set in Y under f is open in X.*

(iv) *$f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.*

(v) *$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all $B \subseteq Y$.*

(vi) *For an arbitrary open cover of X, f is continuous on every covering member*[2]*.*

(vii) *For a finite closed cover of X, f is continuous on every covering member.*

[2] In the sense that the restriction of f to that set is continuous.

2.2 Basis for a Set and Limits over a Basis

Let X be a set and β a collection of subsets of X. Then β is a **basis** for X if the two conditions are met:

(i) $\varnothing \notin \beta$

(ii) $\forall B_1, B_2 \in \beta,\ \exists B_3 \in \beta,\ B_3 \subseteq B_1 \cap B_2$.

Let X be a set with a basis β as above and Y a topological space. If f is a map $X \to Y$, then **limit of f over β** is defined to be some point $y \in Y$ satisfying: for any neighborhood U_y of y there exists a member $B \in \beta$ such that $f(B) \subseteq U_y$. We can prove that in a Hausdorff space the limit is unique, and we shall denote it by $y = \lim_\beta f(x)$.

2.3 Extension of Maps in Metric Spaces

Let X be a metric space and $A \subseteq X$. We define the **distance** of a point $x \in X$ to A to be

$$d(x, A) = \inf_{y \in A} d(x, A)$$

Lemma 2.1 (Distance to a set is continuous)**.** *$d(x, A)$ as above is continuous in x.*

Theorem 2.2 (Tietze, extension theorem)**.** *A continuous function defined on a closed subset of a metric space has a continuous extension to the whole space.*

Proof. If f is bounded, we find a uniformly convergent series with continuous terms defined on the whole space to approach f on C. If f is not bounded, we use compose f with arctan first. $\square$

3 Compact Spaces and the Hausdorff Separation Axiom

3.1 Definitions

A space is **compact** is any open cover admits a finite subcover. Note we can also define compactness for a subset using an open cover with sets open in the whole space. A space is **Hausdorff** if any two distinct points have disjoint neighborhood.

3.2 Properties of Compact Sets

Theorem 3.1. *The image of a compact set under a continuous map is compact.*

Theorem 3.2. *A closed subset of a compact set is compact.*

Theorem 3.3 (Compact implies sequentially compact)**.** *An infinite subset of a compact set has a limit point in the set. Equivalently, a sequence in a compact set has a convergent subsequence.*

3.3 Properties of a Hausdorff space

Theorem 3.4. *A limit in a Hausdorff space is unique.*

Theorem 3.5. *A compact set in a Hausdorff space is closed.*

Theorem 3.6 (Inverse theorem for continuous functions)**.** *If $f : X \to Y$ is continuous and bijective, where X is compact and Y is Hausdorff, then f is closed*[3] *and hence f^{-1} is continuous. Hence f is a homeomorphism.*

Theorem 3.7 (Pullback of the Hausdorff property)**.** *If $f : X \to Y$ is continuous and injective, then Y is Hausdorff $\implies$ X is Hausdorff.*

Proof. Let $x \neq x' \in X$. Then $f(x) \neq f(x') \in Y$. Since Y is Hausdorff, we can find disjoint neighborhoods V, V' of $f(x), f(x')$ respectively. Hence $f^{-1}(V)$ and $f^{-1}(V')$ are respectively disjoint neighborhoods of x and x'. □

3.4 Compact Sets in $\mathbb{R}^n$

Theorem 3.8 (Lebesgue number)**.** *Let $\{U_\alpha\}$ be an open cover of a compact metric space. Then there exists a number $\delta > 0$ subordinate to $\{U_\alpha\}$ such that for any set $A : \operatorname{diam}(A) < \delta$, A can be covered by a single member of $\{U_\alpha\}$.*

4 Product Topology and More on Compactness

5 Identification Spaces

Let X be topological space, a **partition** of X is a decomposition of X into a disjoint union of nonempty subsets:

$$X = \bigcup_i P_i$$

[3]In the sense that f maps a closed set to a closed set.

We put on X the equivalence relation $\sim$ that two points are equivalent if they are in the same set of the partition, and let $Y = X/\sim$. Let $\pi : X \to Y = X/\sim$ be the canonical projection. We define a set A in Y to be open if $\pi^{-1}(A)$ is open in X. We see that the gives a topology on Y, called the **identification topology**, and Y is made into a topological space, called the **identification space** (associated to the partition $X = \bigcup_i P_i$).

The following result provides a criterion for the continuity of a function from an identification space.

Theorem 5.1 (Continuity test, from an identification space)**.** *Let Y be an identification space of X, with $\pi : X \to Y$ the canonical projection. Then for a map $f : Y \to Z$ where Z is a topological space, f is continuous $\iff$ $f \circ \pi$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "f\circ\pi"'] \arrow[dr, phantom, bend left, "\overset{\text{continuity}}{\Leftrightarrow}"] & Y \arrow[d, "f"] \\ & Z \end{tikzcd}$$

Proof. Let O be an open subset of Z, $f^{-1}(O)$ is open in Y $\iff$ $\pi^{-1}f^{-1}(O) = (f \circ \pi)^{-1}(O)$ is open in X. $\square$

Now we generalize the concept of identification spaces by introducing its category definition. A map $f : X \to Y$ is called an **identification map** if it is surjective, continuous, and satisfies O is open in Y $\iff$ $f^{-1}(O)$ is open in X.[4] This definition is to be justified as follows.

Theorem 5.2 (Category definition, identification map)**.** *Let $f : X \to Y$ be an identification map in the category sense. Then f induce on X an equivalence relation that $x \sim x'$ if and only if $f(x) = f(x')$. Let $Y_* = X/\sim$ and $\pi_* : X \to Y_* = X/\sim$ the canonical projection. Then the map*[5]

$$\begin{aligned} \overline{f} : Y_* &\to Y \\ [x] &\mapsto f(x) \end{aligned}$$

is well-defined and is a homeomorphism. Moreover, if Z is another topological

[4] Or equivalently, if f is surjective and Y is equipped with the largest topology for which f is continuous.

[5] $[x]$ denotes the equivalence class x is in.

space, then $g : Y \to Z$ is continuous $\iff$ $g \circ f$ is continuous.

$$
\begin{tikzcd}
& X \arrow[rr, "g \circ f"'] \arrow[dl, "\pi_*"'] \arrow[dr, "f"] & & Z \\
Y_* \arrow[rr, "\overline{f}"] & & Y \arrow[ur, "g"'] \arrow[ur, phantom, shift left=4, "\textit{continuity}\ \Updownarrow"]
\end{tikzcd}
$$

Proof. The definition of the equivalence on X shows that $\overline{f}$ is well-defined and injective. $\overline{f}$ is surjective because f is. Then it suffices to show f, f^{-1} are continuous. Let $A \subseteq Y$. Then A is open in $Y \iff f^{-1}(A) = (\overline{f} \circ \pi_*)^{-1}(A) = \pi_*^{-1} \circ \overline{f}^{-1}(A)$ is open in $X \iff \overline{f}^{-1}(A)$ is open in Y_*. The proof is then complete. $\square$