Examples and Problems in Analysis

TRISCT

Contents

1 Equalities

1.1 Common Taylor Series

1. $e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots \quad (-\infty < x < +\infty)$
2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots \quad (-\infty < x < +\infty)$
3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots \quad (-\infty < x < +\infty)$
4. $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots \quad (-1 < x < 1)$
5. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots \quad (-1 < x \leqslant x)$
6. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots (-1 \leqslant x \leqslant 1)$
7. $\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots + \frac{(2n-1)!!x^{2n+1}}{(2n)!!(2n+1)} + \cdots (-1 \leqslant x \leqslant 1)$
8. $\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots \quad (-1 < x < 1)$
9. $\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - \cdots + (-1)^n(n+1)x^n + \cdots \quad (-1 < x < 1)$
10. $\sqrt{1+x} = (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots + (-1)^{n-1}\frac{(2n-3)!!}{(2n)!!}x^n + \cdots \quad (-1 \leqslant x \leqslant 1)$
11. $\frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \cdots + (-1)^n\frac{(2n-1)!!}{(2n)!!}x^n + \cdots (-1 < x \leqslant 1)$

1.2 Common Infinite Products

1. $\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) \quad (-\infty < x < +\infty)$

2. $\cos x = \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{(2n-1)^2\pi^2}\right) \quad (-\infty < x < +\infty)$

3. $\frac{\sin x}{x} = \prod_{n=1}^{\infty} \cos \frac{x}{2^n} \quad (x \neq 0)$

4. $\prod_{n=0}^{\infty} (1 + x^{2^n}) = \frac{1}{1-x} \quad (-1 < x < 1)$

1.3 Telescoping (1): $\sum_{n=1}^{N} \cos nx$ and $\sum_{n=1}^{N} \sin nx$

$$\sum_{n=1}^{N} \cos nx = \frac{\sin\left(N + \frac{1}{2}\right)x - \sin\frac{x}{2}}{2\sin\frac{x}{2}} \quad (x \neq 2k\pi)$$

$$\sum_{n=1}^{N} \sin nx = \frac{\cos\left(N + \frac{1}{2}\right)x - \cos\frac{x}{2}}{-2\sin\frac{x}{2}} \quad (x \neq 2k\pi)$$

Note 1.1. *These are obtained from*

$$\begin{aligned} \sin\left(n + \frac{1}{2}\right)x - \sin\left(n - \frac{1}{2}\right)x &= 2\sin\frac{x}{2}\cos nx \\ \cos\left(n + \frac{1}{2}\right)x - \cos\left(n - \frac{1}{2}\right)x &= -2\sin\frac{x}{2}\sin nx \end{aligned}$$

1.4 Telescoping (2): $\prod_{n=1}^{N} \cos \frac{x}{2^n}$

$$\prod_{n=1}^{N} \cos\frac{x}{2^n} = \frac{1}{2^n} \cdot \frac{\sin x}{\sin\frac{x}{2^n}} \quad (x \neq 0)$$

$$\prod_{n=1}^{\infty} \cos\frac{x}{2^n} = \frac{\sin x}{x} \quad (x \neq 0)$$

1.5 Telescoping (3): $\sum_{n=1}^{N} \frac{nx}{(1+x)(1+2x)\cdots(1+nx)}$

$$\begin{aligned}
& \sum_{n=1}^{N} \frac{nx}{(1+x)(1+2x)\cdots(1+nx)} \\
= & \sum_{n=1}^{N} \left(\frac{1}{(1+x)(1+2x)\cdots(1+(n-1)x)} - \frac{1}{(1+x)(1+2x)\cdots(1+nx)} \right) \\
= & \ 1 - \frac{1}{(1+x)(1+2x)\cdots(1+Nx)}
\end{aligned}$$

1.6 Formula (1): $\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}} = 2\cos\frac{\pi}{2^{n+1}}$

Use

$$\cos\frac{x}{2} = \sqrt{\frac{1}{2} + \frac{1}{2}\cos x}$$

1.7 Order Estimate (1): $\frac{p(p+1)\cdots(p+n-1)}{n!}$

$p \neq 0$ is not a negative integer, then

$$\frac{p(p+1)\cdots(p+n-1)}{n!} = O^*\left(\frac{1}{n^{1-p}}\right) \quad (n \to \infty)$$

1.8 Order Estimate (2): $\binom{m}{n}$

The binomial coefficient $\binom{m}{n}$ $\quad (m \notin \mathbb{N})$

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!} = O^*\left(\frac{1}{n^{m+1}}\right)$$

1.9 Order Estimate (3): $\frac{(2n)!!}{(2n-1)!!}$ (Wallis' product)

Wallis's product formula has the following forms

$$\frac{(2n)!!}{(2n-1)!!} \sim \sqrt{\pi n} \quad (n \to \infty)$$

$$\prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right)=\frac{\pi}{2}$$

$$\lim_{n\to\infty}\frac{1}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2=\frac{\pi}{2}$$

1.10 Order Estimate (4): $n!$ (Stirling's formula)

The factorial $n!$ grows at

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad (n\to\infty)$$

1.11 Definite Integral (1): $\int_0^{\pi/2}\sin^n x\mathrm{d}x$

$$\int_0^{\frac{\pi}{2}}\sin^{2n+1}x\mathrm{d}x=\frac{(2n)!!}{(2n+1)!!}$$

$$\int_0^{\frac{\pi}{2}}\sin^{2n}x\mathrm{d}x=\frac{\pi}{2}\cdot\frac{(2n-1)!!}{(2n)!!}$$

1.12 Abel's transformation

The sum $\sum_{i=1}^{m}\alpha_i\beta_i$ can be written as

$$\sum_{i=1}^{m}\alpha_i\beta_i=\alpha_m B_m-\sum_{i=1}^{m-1}(\alpha_{i+1}-\alpha_i)B_i$$

where

$$B_i=\sum_{k=1}^{i}\beta_k$$

If the original summation does not start with $i=1$, one can write

$$\sum_{i=n}^{m}a_ib_i=A_mb_m-A_{n-1}b_n+\sum_{i=n}^{m-1}A_i(b_i-b_{i+1})$$

2 Inequalities

2.1 Bernoulli's Inequality

Bernoulli's Inequality For $x > -1,\ n \in \mathbb{N}^*$,

$$(1+x)^n \geqslant 1 + nx$$

and

$$(1+x)^n = 1 + nx \iff n = 1 \text{ or } x = 0$$

Extensions of Bernoulli's Inequality

$$\begin{aligned} x^\alpha - \alpha x + \alpha - 1 &\leqslant 0 \quad \text{when} \quad 0 < \alpha < 1 \\ x^\alpha - \alpha x + \alpha - 1 &\geqslant 0 \quad \text{when} \quad \alpha < 0 \text{ or } 1 < \alpha \end{aligned}$$

2.2 Hölder's Inequality

Hölder's Inequality (for Sums) Let $x_i, y_i \geqslant 0,\ i = 1, \cdots, n,\ \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned} \sum_{i=1}^n x_i y_i &\leqslant \left(\sum_{i=1}^n x_i^p\right)^{1/p} \left(\sum_{i=1}^n y_i^q\right)^{1/q} \quad , \quad p > 1 \\ \sum_{i=1}^n x_i y_i &\geqslant \left(\sum_{i=1}^n x_i^p\right)^{1/p} \left(\sum_{i=1}^n y_i^q\right)^{1/q} \quad , \quad p < 1,\ p \neq 0 \end{aligned}$$

Hölder's Inequality (for Integrals) Let $f, g \in \mathcal{R}[a,b],\ \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left|\int_a^b (f \cdot g)(x)\mathrm{d}x\right| \leqslant \left(\int_a^b |f|^p (x)\mathrm{d}x\right)^{1/p} \cdot \left(\int_a^b |g|^q (x)\mathrm{d}x\right)^{1/q} \quad , \quad p > 1$$

2.3 Jensen's Inequality

Jensen's Inequality If $f : (a,b) \to \mathbb{R}$ is a convex function, $x_1, \cdots, x_n \in (a,b)$, and $\alpha_1, \cdots, \alpha_n$ are positive numbers such that $\alpha_1 + \cdots + \alpha_n = 1$, then

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \leqslant \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$$

Jensen's Inequality (for Integrals) If f is a continuous convex function on $\mathbb{R}$ and φ an arbitrary continuous function on $\mathbb{R}$, then

$$f\left(\frac{1}{c}\int_0^c \varphi(t)\mathrm{d}t\right) \leqslant \frac{1}{c}\int_0^c f(\varphi(t))\mathrm{d}t$$

2.4 Minkowski's Inequality

Minkowski's Inequality (for Sums) Let $x_i, y_i \geqslant 0,\ i = 1, \cdots, n$. Then

$$\left(\sum_{i=1}^{n}(x_i + y_i)^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p} \quad , \quad p > 1$$

$$\left(\sum_{i=1}^{n}(x_i + y_i)^p\right)^{1/p} \geqslant \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p} \quad , \quad p < 1,\ p \neq 0$$

Minkowski's Inequality (for Integrals) Let $f, g \in \mathcal{R}[a, b]$. Then

$$\left(\int_a^b |f+g|^p\,(x)\mathrm{d}x\right)^{1/p} \leqslant \left(\int_a^b |f|^p\,(x)\mathrm{d}x\right)^{1/p} + \left(\int_a^b |g|^p\,(x)\mathrm{d}x\right)^{1/p} \quad , \quad p \geqslant 1$$

$$\left(\int_a^b |f+g|^p\,(x)\mathrm{d}x\right)^{1/p} \geqslant \left(\int_a^b |f|^p\,(x)\mathrm{d}x\right)^{1/p} + \left(\int_a^b |g|^p\,(x)\mathrm{d}x\right)^{1/p} \quad , \quad 0 < p < 1$$

2.5 Young's Inequality

Young's Inequality If $a > 0,\ b > 0$, then

$$a^{1/p}b^{1/q} \leqslant \frac{1}{p}a + \frac{1}{q}b \quad , \quad p > 1$$

$$a^{1/p}b^{1/q} \geqslant \frac{1}{p}a + \frac{1}{q}b \quad , \quad p < 1 \text{ and } p \neq 0$$

and

$$a^{1/p}b^{1/q} = \frac{1}{p}a + \frac{1}{q}b \iff a = b$$

Or it could be written as

$$xy \leqslant \frac{1}{p}x^p + \frac{1}{q}y^q$$

for $x, y \geqslant 0,\ p, q > 0,\ \frac{1}{p} + \frac{1}{q} = 1$

3 Numerical Series

3.1 Problems

3.1.1 $\sum a_n$ and $\lim na_n$

1. $\lim\limits_{n\to\infty} na_n = a \neq 0 \implies \sum\limits_{n=1}^{\infty} a_n$ diverges.

2. If $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant \cdots \geqslant 0$ and $\sum\limits_{n=1}^{\infty} a_n$ converges, then $\lim\limits_{n\to\infty} na_n = 0$.

   Note 3.1. *An analog in integral is*

$$\int_a^{+\infty} f(x)\mathrm{d}x \text{ converges and } f(x) \text{ is monotonic} \implies f(x) = o\left(\frac{1}{x}\right)$$

3. $\lim\limits_{n\to\infty} na_n$ may not exists for a convergent $\sum\limits_{n=1}^{\infty} a_n$. For example,

$$a_n = \begin{cases} \frac{1}{n} & , \quad n = m^2 \\ \frac{1}{n^2} & , \quad n \neq m^2 \end{cases}$$

4. $\lim\limits_{n\to\infty} na_n = 0$ does not imply the convergence of $\sum\limits_{n=1}^{\infty} a_n$. For example,

$$a_n = \frac{1}{n \ln n}$$

3.1.2 $\sum a_n$, $\sum a_n^2$ and $\sum a_n^3$

1. If $a_n \geqslant 0$ and $\sum\limits_{n=1}^{\infty} a_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n^2$ converges too.

   Note 3.2. *The converse is untrue. An example is* $a_n = \frac{1}{n}$.

2. The convergence of $\sum\limits_{n=1}^{\infty} a_n$ does not imply that of $\sum\limits_{n=1}^{\infty} a_n^3$, for example,

$$a_n = \frac{1}{\sqrt[3]{n}} \cos \frac{2\pi}{3} n$$

3.1.3 $\sum a_n$ and $\sum(a_n + a_{n+1})$

The convergence of $\sum\limits_{n=1}^{\infty} a_n$ implies that of $\sum\limits_{n=1}^{\infty} (a_n + a_{n+1})$. The converse is not true, for example

$$a_n = (-1)^n$$

However, the converse is true if $a_n > 0$.

3.1.4 $\sum a_n$, $\{na_n\}$ and $\sum(a_n - a_{n+1})$

If $\{na_n\}$, $\sum\limits_{n=1}^{\infty} n(a_n - a_{n+1})$ converge, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

3.1.5 $\sum a_n$ and $\sum \sqrt{a_n a_{n+1}}$

If $a_n \geqslant 0$ and $\sum\limits_{n=1}^{\infty} a_n$ converges, then $\sum\limits_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges too. The converse is not true, for example

$$a_n = \begin{cases} \frac{1}{n} & , \quad n = 2k \\ \frac{1}{n^2} & , \quad n = 2k-1 \end{cases}$$

But the converse is true if $\{a_n\}$ is monotonically decreasing.

3.1.6 $\sum a_n$ and $\sum \frac{a_n^\alpha}{n^\beta}$

1. If $a_n > 0$ and $\sum\limits_{n=1}^{\infty} a_n$ converges, and α, $\beta > 0$ are such that $\alpha + \beta > 1$. Then the series

   $$\sum_{n=1}^{\infty} \frac{a_n^\alpha}{n^\beta}$$

   converges.

   Proof 3.1. Hölder's inequality.

2. A special case is that

   $$a_n > 0, \sum_{n=1}^{\infty} a_n < +\infty \implies \forall \delta > 0, \sum_{n=1}^{\infty} \sqrt{\frac{a_n}{n^{1+\delta}}} < +\infty$$

   Note 3.3. *The proposition may not be true for δ, for example,*

   $$\begin{aligned} a_n &= \frac{1}{n(\ln n)^{1+\varepsilon}} \quad (0 < \varepsilon < 1) \\ \implies \sqrt{\frac{a_n}{n}} &= \frac{1}{n(\ln n)^{\frac{1+\varepsilon}{2}}} \quad (0 < \varepsilon < 1) \end{aligned}$$

Note 3.4. *[E14.2-12]*

3.1.7 $\sum a_n$ and $\sum \frac{a_n}{1+a_n}$, $\sum \frac{a_n}{1+n^2 a_n}$, $\sum \frac{a_n}{1+a_n^2}$, $\sum \frac{a_n}{1+n a_n}$

Let $\sum\limits_{n=1}^{\infty} a_n$ be a divergent series with positive terms. Then

1. The series

   $$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$$

   diverges.

Proof 3.2. Consider whether $\limsup\limits_{n\to\infty} a_n = 0$. If so, then a_n is bounded and we can change the denominator to compare with $\sum\limits_{n=1}^{\infty} a_n$; if not, then $\{a_n\}$ does not tend to 0.

2. The series
$$\sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n}$$
converges.

   Proof 3.3. Compare with $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$

3. The series
$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n^2}$$
diverges when $\{a_n\}$ is bounded. When $a_n \to +\infty$, the series has the same convergence of $\sum\limits_{n=1}^{\infty} \frac{1}{a_n}$. Its behavior is uncertain in other situations.

   Note 3.5. *Unlike* $\sum \frac{a_n}{1+a_n}$ *where* $\frac{a_n}{1+a_n} \to 0 \iff a_n \to 0$, *it may happen that* $\frac{a_n}{1+a_n^2} \to 0$ *but* $a_n \not\to 0$.

4. The series
$$\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$$
diverges when $\{na_n\}$ is bounded or tends to $+\infty$. Its behavior is uncertain in other situations.

Note 3.6. *[E14.2-11], [P14.2-2]*

3.1.8 $\sum a_n$ and $\sum \frac{a_n}{S_n^\alpha}$

Let $a_n > 0$ and $S_n = \sum\limits_{k=1}^{n} a_k$.

1. $\alpha > 1 \implies \sum\limits_{n=1}^{\infty} \frac{a_n}{S_n^\alpha}$ converges.

2. $\alpha \leqslant 1$ and $S_n \to +\infty \implies \sum\limits_{n=1}^{\infty} \frac{a_n}{S_n^\alpha}$ diverges.

Note 3.7. *[P14.2-3]*

3.1.9 $\sum a_n \cos nx$ and $\sum a_n \sin nx$

1. If $a_n \geqslant 0$ and tends to 0 monotonically, then

$$\sum_{n=1}^{\infty} a_n \cos nx \text{ converges where } x \neq 2k\pi$$

$$\sum_{n=1}^{\infty} a_n \cos nx \text{ has the same convergence as } \sum_{n=1}^{\infty} a_n \text{ when } x = 2k\pi$$

2. If $a_n \geqslant 0$, then

$$\sum_{n=1}^{\infty} a_n \cos nx \text{ converges uniformly on } \mathbb{R} \iff \sum_{n=1}^{\infty} a_n \text{ converges}$$

3. If $a_n \geqslant 0$ and tends to 0 monotonically, then

$$\sum_{n=1}^{\infty} a_n \sin nx \text{ converges on } \mathbb{R}$$

4. If $a_n \geqslant 0$ and decreases monotonically, then

$$\sum_{n=1}^{\infty} a_n \sin nx \text{ converges uniformly on } \mathbb{R} \iff \lim_{n\to\infty} na_n = 0$$

Note 3.8. *[E14.4-6], [E15.2-8], [P15.2-7]*

3.1.10 $\sum a_n$ and $\sum \left(\frac{a_{n+1}}{a_n} - 1\right)$

If $a_n > 0$ and is monotonically increasing, then

$$\sum_{n=1}^{\infty} \left(\frac{a_{n+1}}{a_n} - 1\right) \text{ converges } \iff \{a_n\} \text{ is bounded}$$

Note 3.9. *[P14.4-3]*

3.1.11 $\sum a_n$ and $\lim \frac{a_1+2a_2+\cdots+na_n}{n}$

If $\sum\limits_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n} = 0$$

Note 3.10. *[E14.4-8]*

3.1.12 $\sum a_n b_n$ and $\lim (a_1 + \cdots + a_n) b_n$

If $\{b_n\}$ $(b_n \geqslant 0)$ tends to 0 monotonically and $\sum\limits_{n=1}^{\infty} a_n b_n$ converges, then

$$\lim_{n\to\infty} (a_1 + \cdots + a_n) b_n = 0$$

Note 3.11. *[P14.4-6]*

3.1.13 $\sum a_n$ and $\lim \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{b_n}$

If $\{b_n\}$ $(b_n > 0)$ and tends to $+\infty$ monotonically and $\sum\limits_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{b_n} = 0$$

Note 3.12. *[P14.4-7]*

Note 3.13. *Go back two problems you will find a special case of this.*

3.1.14 $\sum a_n$ and $\sum \frac{a_1 + 2a_2 + \cdots + n a_n}{n(n+1)}$

If $\sum\limits_{n=1}^{\infty} a_n$ converges, then by setting

$$b_n = \frac{a_1 + 2a_2 + \cdots + n a_n}{n(n+1)}$$

one can obtain a new series such that

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$$

Note 3.14. *[P14.4-8]*

3.1.15 $\sum a_n$ and $\lim (a_1 b_n + a_2 b_{n-1} + a_n b_1) = 0$

If $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely and $\lim\limits_{n\to\infty} b_n \to 0$, then

$$\lim_{n\to\infty} (a_1 b_n + a_2 b_{n-1} + a_n b_1) = 0$$

Note 3.15. *[P14.5-4]*

3.1.16 Other Problems From Chang & Shi

P14.1-4 If $a_n > 0$ and $\{a_n - a_{n+1}\}$ is a strictly decreasing sequence, then

$$\sum_{n=1}^{\infty} a_n < +\infty \implies \lim_{n\to\infty}\left(\frac{1}{a_{n+1}} - \frac{1}{a_n}\right) = +\infty$$

P14.1-5 There exists a constant K such that

$$\sum_{n=1}^{\infty} \frac{n}{a_1 + a_2 + \cdots + a_n} \leqslant K \sum_{n=1}^{\infty} \frac{1}{a_n}$$

holds for all sequences $\{a_n\}$ with positive terms.

E14.2-4 The following series can be tested by comparison using the Cauchy-Hölder inequality.

E14.2-6 The series

$$\sum_{n=2}^{\infty} \frac{1}{\ln n!}$$

diverges.

Proof 3.4. Stirling's formula

E14.2-6 The series

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln \ln n}$$

diverges.

Proof 3.5. Integral test. Notice that

$$\int \frac{1}{x \ln x \ln \ln x} \mathrm{d}x = \ln \ln \ln x + C$$

P14.2-1 The series

$$\sum_{n=1}^{\infty} x^{1+\frac{1}{2}+\cdots+\frac{1}{n}}$$

converges for $0 < x < e^{-1}$ and diverges for $x \geqslant e^{-1}$.

P14.4-1 Let $a_n > 0$. If for $0 < \alpha < 1$,

$$\lim_{n\to\infty} n^\alpha \left(\frac{a_n}{a_{n+1}} - 1\right) = \lambda > 0$$

($\lambda = +\infty$ included), then $\forall k \in \mathbb{N}$, $\sum\limits_{n=1}^{\infty} n^k a_n$ converges.

Note 3.16. *The testing condition resembles that of [E14.4-12], but I don't see much connection here.*

P14.6-1 The Cauchy product of

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^\alpha}, \quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^\beta} \quad (\alpha,\, \beta > 0)$$

converges for $\alpha + \beta > 1$ and diverges for $\alpha + \beta \leqslant 1$.

Proof 3.6. Use Pringsheim's theorem.

4 Series of Functions

4.1 Problems

4.1.1 Uniform Convergence

E15.2-4 Let $u_n(x)$ be monotonic functions on $[a, b]$. If $\sum\limits_{n=1}^{\infty} u_n(a)$, $\sum\limits_{n=1}^{\infty} u_n(b)$ converge absolutely, then $\sum\limits_{n=1}^{\infty} u_n(x)$ converges absolutely and uniformly on $[a, b]$.

E15.2-7 Let $u_n(x)$ be continuous functions on $[a, b]$. If $\sum\limits_{n=1}^{\infty} u_n(x)$ converges at every point of $[a, b)$, but diverges at b, then the convergence of $\sum\limits_{n=1}^{\infty} u_n(x)$ on $[a, b)$ is not uniform.

Note 4.1. *An example is* $\sum\limits_{n=2}^{\infty} \frac{\cos nx}{n \ln n}$.

E15.2-10 Let $[a, b]$ be a finite closed interval and $\{f_n\}$ a sequence of functions, then

$$\forall x \in [a, b],\ \exists \text{ open interval } I_x \ni x,\ f_n|_{I_x} \rightrightarrows f|_{I_x} \implies f_n \rightrightarrows f$$

P15.2-2 (Uniform convergence of product) Let $f_n \rightrightarrows f$, $g_n \rightrightarrows g$ on an interval I. If for each n, f_n, g_n are bounded on I, then $f_n g_n \rightrightarrows fg$.

Note 4.2. *In fact, even if they are not required to be uniformly bounded by hypothesis, it can be proved.*

Note 4.3. *A counterexample if they are not required to be bounded is*

$$f_n(x) = x(1 + \frac{1}{n}), \quad g_n(x) = \begin{cases} \frac{1}{n} & , \quad x = 0,\, x \in \mathbb{R}\backslash\mathbb{Q} \\ q + \frac{1}{n} & , \quad x = \frac{p}{q}(q > 0) \end{cases}$$

P15.2-4 (Bounded derivative series) Let $\sum\limits_{n=1}^{\infty} u_n(x)$ be converge on $[a,b]$. If there exists M, such that for all $x \in [a,b]$ and $n \in \mathbb{N}$, $\left|\sum\limits_{k=1}^{n} u_k'(x)\right| \leqslant M$, then $\sum\limits_{n=1}^{\infty} u_n(x)$ converges uniformly on $[a,b]$.

E15.3-6 (Extension to limit point by uniform convergence) Let $E \subset \mathbb{R}$ be a set and x_0 a limit point of E (it may assume the values $\pm\infty$). If $\sum\limits_{n=1}^{\infty} u_n(x)$ converges uniformly on E and $\lim\limits_{\substack{x \to x_0 \\ x \in E}} u_n(x) = a_n$, then $\sum\limits_{n=1}^{\infty} a_n$ converges and $\lim\limits_{\substack{x \to x_0 \\ x \in E}} \sum\limits_{n=1}^{\infty} u_n(x) = \sum\limits_{n=1}^{\infty} a_n$

P15.3-2 (Application of [E15.3-6]) Let the series $\sum\limits_{n=1}^{\infty} u_n(x)$ satisfy

1. $\int_a^{+\infty} u_n(x)\mathrm{d}x$ converges for all $n \in \mathbb{N}$.
2. $\sum\limits_{n=1}^{\infty} u_n(x)$ converges on $[a,b]$ for all $b > a$.
3. $\sum\limits_{n=1}^{\infty} \int_a^x u_n(t)\mathrm{d}t$ converges uniformly on $[a, +\infty)$

then $\int_a^{+\infty} \sum\limits_{n=1}^{\infty} u_n(x)\mathrm{d}x$ and $\sum\limits_{n=1}^{\infty} \int_a^{+\infty} u_n(x)\mathrm{d}x$ both converges and are equal to each other

$$\int_a^{+\infty} \sum_{n=1}^{\infty} u_n(x)\mathrm{d}x = \sum_{n=1}^{\infty} \int_a^{+\infty} u_n(x)\mathrm{d}x$$

P15.3-3 (Application of [E15.3-6]) Let the series $\sum\limits_{n=1}^{\infty} u_n(x)$ satisfy

1. $\sum\limits_{n=1}^{\infty} u_n(x)$ converges to $f(x)$ on $(x_0 - \delta, x_0 + \delta)$.
2. $u_n(x)$ is differentiable at $x = x_0$ for all $n \in \mathbb{N}$.
3. $\sum\limits_{n=1}^{\infty} \frac{u_n(x)-u_n(x_0)}{x-x_0}$ converges uniformly on $(x_0 - \delta, x_0 + \delta)\backslash x_0$.

then f is differentiable at x_0 and

$$f'(x_0) = \sum_{n=1}^{\infty} u_n'(x_0)$$

P15.3-1 (Bounded convergence and integrability) Let $\sum\limits_{n=1}^{\infty} u_n(x)$ converge on $[a, b]$. If

$$|S_n(x)| = \left|\sum_{k=1}^{n} u_k(x)\right| \leqslant M \quad (a \leqslant x \leqslant b,\ n \in \mathbb{N})$$

then $\sum\limits_{n=1}^{\infty} u_n(x)$ *converges boundedly.* Let $\sum\limits_{n=1}^{\infty} u_n(x)$ converge boundedly on $[a, b]$, and converge uniformly on $[a, c-\delta]$ and $[c+\delta, b]$ for all $\delta > 0$ and some fixed c, then $u_n(x)$ is integrable on $[a, b] \implies \sum\limits_{n=1}^{\infty} u_n(x)$ is integrable on $[a, b]$ and

$$\int_a^b \sum_{n=1}^{\infty} u_n(x)\mathrm{d}x = \sum_{n=1}^{\infty} \int_a^b u_n(x)\mathrm{d}x$$

4.1.2 Convergence Radius

E15.4-3 (Sum/product of coefficients) Let $\sum\limits_{n=0}^{\infty} a_n x^n$ and $\sum\limits_{n=0}^{\infty} b_n x^n$ have convergence radii R_1 and R_2, then

1. $\sum\limits_{n=0}^{\infty} (a_n + b_n)x^n$ has convergence radius $R \geqslant \min(R_1, R_2)$.
2. $\sum\limits_{n=0}^{\infty} (a_n b_n)x^n$ has convergence radius $R \geqslant R_1 R_2$.
3. It may happen that the last two inequalities are strict.

E15.4-6 The convergence radius R of $\sum\limits_{n=0}^{\infty} a_n x^n$ satisfies

$$\liminf_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = l \leqslant R \leqslant L = \limsup_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

P15.4-5 (Limit to the edge) Let $\sum\limits_{n=0}^{\infty} a_n x^n$ have convergence radius R and $a_n \geqslant 0$, then

$$\lim_{x\to R^-}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n$$

4.1.3 Other Problems from Chang & Shi

E15.2-3 $\sum\limits_{n=1}^{\infty} a_n$ converges $\implies \sum\limits_{n=1}^{\infty} a_n e^{-nx}$ converges uniformly on $[0,+\infty)$.

E15.2-10 $\alpha < 1 \iff f_n(x) = xn^{-x}(\ln n)^{\alpha}$ converges uniformly on $[0,+\infty)$.

E15.2-12 Let f_1 be Riemann integrable on $[a,b]$ and define

$$f_{n+1}(x) = \int_a^x f_n(t)\mathrm{d}t$$

, then $f_n \rightrightarrows 0$ on $[a,b]$.

Note 4.4. *Should I try successive approximation of Picard on this?*

E15.2-13 $\alpha > 2 \implies \sum\limits_{n=1}^{\infty} x^{\alpha} e^{-nx^2}$ converges uniformly on $[0,+\infty)$.

P15.2-1 $\sum\limits_{n=1}^{\infty} \frac{nx}{(1+x)(1+2x)\cdots(1+nx)}$ converges nonuniformly on $[0,\delta]$ and uniformly on $[\delta,+\infty)$ for all $\delta > 0$.

P15.2-5 Let $f \in C^2$ in some neighborhood of $x = 0$ and $f(0) = 0$, $0 < f'(0) < 1$. Denote $\underbrace{f\circ f\circ\cdots\circ f}_{n}$ by f_n. The series $\sum\limits_{n=1}^{\infty} f_n(x)$ converges uniformly in a neighborhood of 0.

P15.2-6 $\sum\limits_{n=1}^{\infty} \frac{x^n}{1+x+x^2+\cdots+x^{2n-1}}\cos nx$ converges uniformly on $[0,1]$

P15.3-4 Let $\{a_n\}$ be a sequence in $(0,1)$ and $a_i \neq a_j$ for all $i \neq j$, then

$$f(x) = \sum_{n=1}^{\infty} \frac{|x-a_n|}{2^n}$$

is continuous on $(0,1)$, and is nondifferentiable at each $x = a_n$ and differentiable everywhere else.

P15.3-5 Let $f(x) = \sum\limits_{n=0}^{\infty} \frac{1}{x+2^n} \quad (0 \leqslant x < +\infty)$, then

1. f is continuous on $[0, +\infty)$.
2. $\lim\limits_{x \to +\infty} f(x) = 0$
3. For all $x \in (0, +\infty)$,

$$0 < f(x) - \frac{\ln(1+x)}{x \ln 2} < \frac{1}{1+x}$$

P15.3-6

$$\sum_{n=0}^{\infty} \frac{(n!)^2 2^{n+1}}{(2n+1)!} = \pi$$

P15.3-7 Define a function φ as follows:

$$\begin{aligned}
\varphi(x) &= x(1-x) \quad (0 \leqslant x \leqslant 1) \\
\varphi(-x) &= -\varphi(x) \\
\varphi(x+2) &= \varphi(x)
\end{aligned}$$

then

$$f(x) = \sum_{n=0}^{\infty} \frac{\varphi(n!x)}{(n!)^2}$$

assumes rational values at rational points and irrational values at irrational points.