Linear Algebra

TRISCT

Contents

1 Determinants

The **determinant** of $A = (a_{ij})$ is defined as $A = \sum_{j_1 \cdots j_n} (-1)^{r(j_1 \cdots j_n)} a_{1j_1} \cdots a_{nj_n}$, where $r(j_1 \cdots j_n)$ is the number of inverse pairs in the arrangement.

Theorem 1.1 (Laplace)**.** *For chosen p rows $i_1, \cdots, i_p$,*

$$\det A = \sum_{1 \leqslant j_1 < \cdots < j_p} \det A \begin{pmatrix} i_1 \cdots i_p \\ j_1 \cdots j_p \end{pmatrix} \det A \begin{pmatrix} i_{p+1} \cdots i_n \\ j_{p+1} \cdots j_n \end{pmatrix} (-1)^{i_1 + \cdots + i_p + j_1 + \cdots + j_p}$$

Theorem 1.2 (Cramer)**.** *Suppose* $\det A \neq 0$. *Then the solution of $Ax = b$ is $x_j = |D_j| \,/\, |A|$, where*

$$D_j = \begin{pmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{pmatrix}$$

and b appears on the j-th column.

The **algebraic complementary minor** of the submatrix $A\begin{pmatrix} i_1 \cdots i_p \\ j_1 \cdots j_p \end{pmatrix}$ is

$$(-1)^{i_1+\cdots+i_p+j_1+\cdots+j_p} A\begin{pmatrix} i_{p+1} \cdots i_n \\ j_{p+1} \cdots j_n \end{pmatrix}$$

The **adjugate matrix** of A is

$$A^* = (A_{ji})$$

where A_{ji} is the algebraic complementary minor of a_{ji}.

Theorem 1.3. *$AA^* = |A|\, I$.*

Theorem 1.4 (Cauchy-Binet)**.** *Let $A = (a_{ij})_{n\times s},\ B = (b_{ij})_{s\times n}$.*

$$\det(AB) = \begin{cases} 0 & , \quad n > s \\ \det A \cdot \det B & , \quad n = s \\ \sum_{1\leqslant k_1<\cdots<k_n\leqslant s} \det A\begin{pmatrix} 1\ 2\cdots n \\ k_1\ k_2\cdots k_n \end{pmatrix} \det B\begin{pmatrix} k_1\ k_2\cdots k_n \\ 1\ 2\cdots n \end{pmatrix} & , \quad n < s \end{cases}$$

2 Equations of Matrices

Theorem 2.1 (Existence of solution)**.** *$Ax = b$ has a solution $\iff r(A) = r(A, b)$.*

3 Matrix Operations

Two matrices are **equivalent** if they can be transformed to each other by elementary multiplcations.

Theorem 3.1. *A is equivalent to $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ where $r = r(A)$.*

Theorem 3.2 (Sylvester)**.** *$r(A) + r(C) \leqslant r(AC) + n$.*

Theorem 3.3 (Frobenius)**.** *$r(AB) + r(BC) \leqslant r(ABC) + r(B)$.*

Theorem 3.4. *$\dim(\mathrm{Ker}(A)) + \dim(\mathrm{Im}(A)) = n$, where $A = (a_{ij})_{m\times n}$.*

Moore-Penrose inverse: TBA.

4 Linear Spaces

Theorem 4.1. *For linear $\psi : V \to W$, $\mathrm{Im}\psi \xrightarrow{\cong} V/\mathrm{Ker}\psi$.*

Theorem 4.2. *$\frac{V+W}{W} \xrightarrow{\cong} \frac{V}{V\cap W}$.*

Theorem 4.3. *$\frac{V}{W} \xrightarrow{\cong} \frac{V/S}{W/S}$.*

All else are omitted.

5 Decompositions

Theorem 5.1 (Cayley-Hamilton). *The characteristic polynomial of a matrix is also its annihilating polynomial.*

Theorem 5.2 (LU). *A has an LU-decomposition $\iff$ all principle minors are nonzero. The decomposition can be obtained by Gaussian elimination and is unique if we require that the diagonal of L be all ones.*

Theorem 5.3 (Cholesky). *If A is symmetric and positive-definite, then there is a lower triangular matrix L, such that $A = LL^T$. This decomposition is unique if we require that the diagonal of L be all positive.*

Theorem 5.4 (Gram-Schmidt orthogonalization and QR decomposition). *For nonsingular A, we can find orthogonal Q and upper triangular R such that $A = QR$.*

Corollary 5.1 (Diagonalization of symmetric matrix). *If G is symmetric and positive-definite, then we can find T such that $T^T GT = I$.*

Theorem 5.5 (Maximal rank). *Let $A = (a_{ij})_{m\times n}$. If A has rank r, then $A = BC$ for $B = (b_{ij})_{m\times r}$ and $C = (c_{ij})_{r\times n}$ for some full-rank B and C.*

Theorem 5.6 (Spectral decomposition). *If A is normal ($A^T A = AA^T$), then there is an orthogonal O such that $O^T AO$ is pseodu-diagonal, which means on its diagonal there is either a 2×2 block representing a dilated rotation, or a real eigenvalue representing a dilation.*

Theorem 5.7 (Spectral decomposition). *If A is symmetric, then there is an orthogonal O such that $O^T AO$ is diagonal with the eigenvalues of A.*

For any A, its **singular values** are defined as the square roots of the nonzero eigenvalues of $A^T A$.

Theorem 5.8 (SVD). *For any A, we can find orthogonal U, V and diagonal Σ such that $A = U\Sigma V$.*

Proof. By the spectral decomposition theorem we can find an orthogonal O such that $O^T AO = \Sigma'$ is diagonal with the eigenvalues of A. By setting $AO = B$ we have $B^T B = \Sigma'$ is diagonal. Hence the columns of B are orthogonal. We then have $A = O^T \Sigma' B$ where B has orthogonal columns, and the last few columns must be 0. Then we can further rewrite B as $B = U\Sigma$. Then $AO = U\Sigma$ as desired. $\square$

Lemma 5.1 (Square root of positive-semidefinite matrix). *Let S be symmetric and positive-semidefinite. Then there exists a unique symmetric positive-semidefinite S' such that $(S')^2 = S$.*

Theorem 5.9 (Polar). *Any A can be written as $A = S\Omega = \Omega' S'$ where Ω, Ω' are orthogonal and S, S' are positive-semidefinite. Furthermore, S and S' are unique.*

6 Positve-definite Symmetirc Matrices

Theorem 6.1 (Positive-definite symmetric matrices)**.** *Let A be symmetric. TFAE.*

(i) $S > 0$.

(ii) *All eigenvalues* > 0.

(iii) $S = (S')^2$ *for some symmetric positive-definite* S'.

(iv) S *has the SVD* $S = \Omega^T \mathrm{diag}(\lambda_1, \cdots, \lambda_n)\Omega$, *where* Ω *is orthogonal.*

(v) $S = P^T P$ *for some invertible* P.

(vi) *All principle minors are positive.*

(vii) *The first* n *principle minors are all positive.*

(viii) *The sums of the principle minors of the same order are positive.*

Theorem 6.2 (Positive-semidefinite symmetric matrices)**.** *Let A be symmetric. TFAE.*

(i) $S \geqslant 0$.

(ii) *All eigenvalues* $\geqslant 0$.

(iii) $S = (S')^2$ *for some symmetric positive-semidefinite* S'.

(iv) S *has the SVD* $S = \Omega^T \mathrm{diag}(\lambda_1, \cdots, \lambda_r, 0, \cdots, 0)\Omega$, *where* Ω *is orthogonal.*

(v) $S = P^T P$ *for some real* P.

(vi) *All principle minors are positive.*

(vii) *The first* n *principle minors are all nonnegative.*

(viii) *The sums of the principle minors of the same order are nonnegative.*

Remark 6.1. Note the counterexample $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$.