Examples in Analysis

TRISCT

Contents

1 Examples in Analysis

1.1 Common Sense—I Guess

1. $x \neq k\pi,\ k \in \mathbb{Z} \implies \lim\limits_{n\to\infty} \sin nx \neq 0$.

1.2 Common Series Used to Compare With

1. $\sum\limits_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges for $\alpha > 1$ and diverges for $\alpha \leqslant 1$.

   Proof 1.1. Integral test.

2. $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^\alpha}$ converges for $\alpha > 1$ and diverges for $\alpha \leqslant 1$.

   Proof 1.2. Integral test. Notice that

$$\int \frac{1}{x(\ln x)^\alpha}\mathrm{d}x = \int \frac{\mathrm{d}(\ln x)}{(\ln x)^\alpha} = \begin{cases} \frac{(\ln x)^{1-\alpha}}{1-\alpha} + C & , \quad \alpha \neq 1 \\ \ln\ln x + C & , \quad \alpha = 1 \end{cases}$$

   Note 1.1. *Ermakov's test is an easier way to avoid the integral.*

3. The convergence of $\sum\limits_{n=3}^{\infty} \frac{1}{n(\ln n)^p(\ln\ln n)^q}$ is as follows

   - $p > 1 \implies$ it converges for all q.
   - $p = 1 \implies$ it converges for $q > 1$.

1.3 So Classic That They Have Names

Hypergeometric series The hypergeometric series is defined as

$$F(\alpha,\beta,\gamma,x)=1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)}\frac{x^n}{n!}$$

It is well-defined for $|x|<1$ and satisfies the hypergeometric equation

$$x(x-1)y''-[\gamma-(\alpha+\beta-1)x]y'+\alpha\beta y=0$$

1.4 What Convergence Test Should I Apply

Cauchy's test

1. $\sum\limits_{n=2}^{\infty}\frac{1}{(\ln n)^n}$
2. $\sum\limits_{n=1}^{\infty}\left(\frac{x}{n}\right)^n \quad (x>0)$
3. $\sum\limits_{n=1}^{\infty}\left(\frac{x}{a_n}\right)^n \quad (x>0,\ a_n>0,\ \lim\limits_{n\to\infty}a_n=a)$
4. $\sum\limits_{n=1}^{\infty}\tau(n)x^n \quad (\tau(n)$ is the number of factors of $n)$
5. $1+a+ab+a^2b+a^2b^2+\cdots+a^nb^{n-1}+a^nb^n+\cdots \quad (a,b>0,\ a\neq b)$

d'Alembert's test

1. $1+\sum\limits_{n=1}^{\infty}\frac{x^n}{n!} \quad (x>0)$
2. $\sum\limits_{n=1}^{\infty}nx^{n-1} \quad (x>0)$
3. $\sum\limits_{n=1}^{\infty}\frac{x^n}{n^s} \quad (x>0,s>0)$
4. $\sum\limits_{n=1}^{\infty}n!\left(\frac{x}{n}\right)^n \quad (x>0)$

   Note 1.2. *You have to use the nonlimit form of d'Alembert's test on this one*
5. $1+a+ab+a^2b+a^2b^2+\cdots+a^nb^{n-1}+a^nb^n+\cdots \quad (a,b>0,\ a\neq b)$

   Note 1.3. *Not a good choice*

Raabe's test

1. $1+\sum\limits_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}$
2. $\sum\limits_{n=1}^{\infty}\frac{n!}{(x+1)\cdots(x+n)} \quad (x>0)$
3. $\sum\limits_{n=1}^{\infty}\frac{n!x^n}{(x+a_1)(2x+a_2)\cdots(nx+a_n)} \quad (x>0,\ a_n>0,\ \lim\limits_{n\to\infty}a_n=a)$
4. $\sum\limits_{n=1}^{\infty}\frac{1}{n!}\left(\frac{n}{e}\right)^n$

   Note 1.4. *The limit of Raabe sequence of this series should be obtained by applying L'Hospital's rule to a corresponding function*

Gauss's test

1. The hypergeometric series.
2. $1+\left(\frac{1}{2}\right)^p+\left(\frac{1\cdot 3}{2\cdot 4}\right)^p+\cdots+\left(\frac{(2n-1)!!}{(2n)!!}\right)^p+\cdots \quad (p>0)$

Cauchy's integral test

1. $\sum\limits_{n=2}^{\infty}\frac{1}{n\cdot\ln^{1+\sigma}n} \quad (\sigma>0)$
2. $\sum\limits_{n=3}^{\infty}\frac{1}{n\cdot\ln n\cdot\ln\ln n}$
3. $\sum\limits_{n=3}^{\infty}\frac{1}{n\cdot\ln n\cdot(\ln\ln n)^{1+\sigma}} \quad (\sigma>0)$

2 Counterexamples in Analysis

2.1 Nondecreasing alternating series that do not converge

The following alternating series do not converge

1. $\frac{1}{\sqrt{2}-1}-\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}+\cdots+\frac{1}{\sqrt{n}-1}-\frac{1}{\sqrt{n}+1}+\cdots$
2. $\sum\limits_{n=1}^{\infty}\left[\frac{(-1)^{n-1}}{\sqrt{n}}+\frac{1}{n}\right]$
3. $\sum\limits_{n=1}^{\infty}(-1)^n\cdot\frac{2+(-1)^n}{n}$

2.2 $\sum a_n^2$ converges but $\sum a_n$ does not

$\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges but $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ does not.

2.3 Case where comparison in the limit form fails

The series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{(-1)^{n-1}}{\sqrt{n}} + \frac{1}{n} \right)$$

are such that

$$\lim_{n \to \infty} \frac{b_n}{a_n} = 1$$

but the latter does not converge.