Definitions in Topology and Geometry

TRISCT

Contents

Part I
General Topology

1 Topological Spaces

1.1 Definitions

Topology See topological space.

Topological space Let X be a set and $\mathcal{T} \subset 2^X$. If $\mathcal{T}$ satisfies the following axioms:

- $\varnothing \in \mathcal{T}$, $X \in \mathcal{T}$;
- for any finitely many sets $\{U_i\}_{i=1}^n$ in $\mathcal{T}$, $\bigcap_{i=1}^n U_i \in \mathcal{T}$;
- for arbitrarily many sets $\{U_i\}_{i \in I}$ in $\mathcal{T}$, $\bigcup_{i \in I} U_i \in \mathcal{T}$,

then we say $\mathcal{T}$ is a **topology on** X and that $(X, \mathcal{T})$ (or simply X) is a **topological space**.

Open set In a topological space $(X, \mathcal{T})$, a subset U of X is called **open** if $U \in \mathcal{T}$.

Interior Let A be a set in a topological space $(X, \mathcal{T})$. The **interior** A° of A is the union of all open sets contained in A.

Closure Let A be a set in a topological space $(X, \mathcal{T})$. The **closure** $\overline{A}$ of A is the intersection of all closed sets containing A.

Neighborhood Let A be a set and x a point in a topological space $(X, \mathcal{T})$. A is said to be a **neighborhood** of x if there exists an open set U such that $x \in U \subset A$.

Limit point Let A be a set in a topological space $(X, \mathcal{T})$. $x \in X$ is called a **limit point** of A if any of these equivalent conditions hold:

- each neighborhood of x contains infinitely many points of A;
- each neighborhood of x contains a point of A other than itself.

A limit point is also called a **cluster point** or **accumulation point**.

Derived set Let A be a set in a topological space $(X, \mathcal{T})$. The **derived set** A' of A is the set of all limit points of A.

Boundary point Let A be a set in a topological space $(X, \mathcal{T})$. A boundary point of A is a point of which each neighborhood intersects with both the interior and complement of A.

Boundary Let A be a set in a topological space $(X, \mathcal{T})$. The **boundary** ∂A of A is defined by any one of the following equivalent conditions:

- $\partial A =$the set of all boundary points of A;
- $\partial A = \overline{A} \cap \overline{X \backslash A}$.

Exterior Let A be a set in a topological space $(X, \mathcal{T})$. The **exterior** of A is $X \backslash (A^\circ \cup \partial A)$.

Dense set A set A is said to be **dense** in B, if each point of B is either in A or a limit point of A, or equivalently, any neighborhood of any point in B contains one point of A.

Isolated point A point of A is **isolated** if some neighborhood of it contains only itself.

Perfect set A **perfect set** is a closed set without any isolated point.

Connected topological space A topological space is **connected** if it is not the disjoint union of nonempty open subsets.

Connected set A set E in a topological space X is said to be **connected** if any one of these equivalent conditions is satisfied.

- If $E = A \cup B$ is a disjoint decomposition, and $A \cap \bar{B} = \varnothing$, $\bar{A} \cap B = \varnothing$, then either $E \subset A$ or $E \subset B$;
- If $E = A \cup B$ is a disjoint decomposition, and E intersect both A and B, then at least one of $A \cap \bar{B}$, $\bar{A} \cap B$ is nonempty;
- E is cannot be covered by two disjoint open sets while intersecting both of them;

Not connected set A set E in a topological space X is said to be **not connected** if any one of the equivalent conditions is satisfied.

- E is the union of two nonempty sets A and B such that

$$\bar{A} \cap B = \varnothing = A \cap \bar{B}$$

- E can be covered by two disjoint open sets, and E intersect each of them.

Note 1.1. *Connectedness can be described by decomposition into open sets or any sets, but in the latter case, one needs to describe the disconnectedness by considering*

$$A \cap \bar{B}, \ \bar{A} \cap B$$

Note 1.2. *An important application of this notion is to divide a connected set into two open parts satisfying contradictory properties, and then by contradiction one concludes that one property holds in the whole set.*

Base Let X be a set and $\mathcal{F} \subset 2^X$. A subfamily $\mathcal{B}$ of $\mathcal{F}$ is said to be a **base** of $\mathcal{F}$ if $x \in F \in \mathcal{F} \implies \exists B \in \mathcal{B}, \ x \in B \subset F$.

Basis Let X be a set. A family $\mathcal{B} \subset 2^X$ is called a **basis** if the following hold:

- $X = \bigcup_{B \in \mathcal{B}} B$;
- $B_1, B_2 \in \mathcal{B}, \ x \in B_1 \cap B_2 \implies \exists B \in \mathcal{B}, \ x \in B \subset B_1 \cap B_2$.

Note 1.3. *Actually they are the same.*

Covering A family $\mathcal{C} \subset 2^X$ is said to be a **covering** of X if $\forall x \in X, \ \exists E \in \mathcal{C}, \ x \in E$. And $\mathcal{C}$ is called **finite** (resp. **countable**) if it contains finite (resp. countable) members. A covering is **locally finite** if for every $x \in X$, there exists a neighborhood of x that intersects only a finite number of this covering. A covering consisting of open sets is called an **open covering**. For two coverings $\mathcal{C}$ and $\mathcal{D}$, if $\mathcal{D}$ is a subset of $\mathcal{C}$, then $\mathcal{D}$ is said to be a **subcovering** of $\mathcal{C}$ and that $\mathcal{C}$ is **reducible** to $\mathcal{D}$; if each member of $\mathcal{D}$ is contained in some member of $\mathcal{C}$, then $\mathcal{D}$ is called a **refinement** of $\mathcal{C}$.

Lindelöf topological space A topological space $(X, \mathcal{T})$ is called **Lindelöf** if every open covering is reducible to a countable open subcovering.

Compact topological space A topological space is called **compact** if every open covering is reducible to a finite open subcovering.

Paracompact topological space A topological space is called **paracompact** if every open covering has a refinement which is a locally finite covering.

First countable space A topological space $(X, \mathcal{T})$ is said to be **first countable** if for each $x \in X$, the system of neighborhoods $\{U_x\}$ of x has a countable base.

Second countable A topological space $(X, \mathcal{T})$ is said to be **second countable** if $\mathcal{T}$ has a countable base.

Note 1.4. *Second countable* $\implies$ *Lindelöf*

2 Maps

2.1 Definitions

Continuous Mapping Let (E, u) and (F, v) be two topological spaces. A mapping $f : E \to F$ is said to be **continuous** if $\forall V \in v,\ f^{-1}(V) \in u$.

Continuous (at a point) Let (E, u) and (F, v) be two topological spaces. A mapping $f : E \to F$ is said to be **continuous** at $x_0 \in E$ if for each neighborhood V of $f(x_0)$, $f^{-1}(V)$ is a neighborhood of x_0.

Open mapping Let (E, u) and (F, v) be two topological spaces. A mapping $f : E \to F$ is said to be **open** if $\forall U \in u,\ f(U) \in v$.

Closed mapping Let (E, u) and (F, v) be two topological spaces. A mapping $f : E \to F$ is said to be **closed** if f maps closed sets to closed sets.

Homeomorphism Let (E, u) and (F, v) be two topological spaces. A mapping $f : E \to F$ is called a **homeomorphism** if it is bijective, continuous and open (or equivalently, f^{-1} is continuous).

Curve Let X be a topological space. A **curve** or **path** in X is a continuous mapping $\gamma : [\alpha, \beta] \to X$ where $[\alpha, \beta]$ is an interval in $\mathbb{R}$ called the **parameter interval**. If $\gamma(\alpha) = \gamma(\beta)$, then γ is called a **closed curve**.

3 Separation Axioms

3.1 Definitions

Separation axioms The following are the separation axioms.

- $\mathbf{A}_0$ For any two distinct points of a topological space, at least one of them has an open neighborhood which does not contain the other point.
- $\mathbf{A}_1$ For any two distinct points of a topological space, each has an open neighborhood not containing the other point.
- $\mathbf{A}_2$ For any two distinct points of a topological space, each has an open neighborhood which does not intersect the other point.
- $\mathbf{A}_3$ For any point x and any closed set F such that $x \notin F$ in a topological space there exist disjoint open sets $U_i,\ i = 1, 2$ such that $x \in U_1$ and $F \subset U_2$.

A_4 For any closed subset F of (E,u) and any element $x \in E \backslash F$, there exists a continuous function from E into the closed unit interval $[0,1]$ such that $f(x)=0$ and $f(y)=1$ for all $y \in F$, or $f(x)=0$ and $f(F)=1$.

A_5 Let C_1 and C_2 be any two disjoint closed subsets of a topological space (E,u). Then there exist two disjoint open subsets such that $C_i \subset U_i,\ i=1,2$.

A_6 Let M_1 and M_2 be any two subsets of a topological space (E,u) such that $(M_1 \cap \overline{M}_2) \cup (\overline{M}_1 \cap M_2) = \varnothing$. Then there exist disjoint open subsets U_i such that $M_i \subset U_i,\ i=1,2$.

T_i-space A topological space satisfying A_i for $i=0,1,2,3,5,6$ is called a T_i-space.

Hausdorff space A T_2-space.

Separated space A Hausdorff space.

Part II
Differential Geometry

4 Manifolds

4.1 Definitions

Topological manifold An n-dimensional **topological manifold** is a set M such that:

- M is a paracompact Hausdorff topological space;
- $\forall p \in M,\ \exists$ open $U_p \ni p$, U_p is homeomorphic to some open set $U \subset \mathbb{R}^n$.

Dimension of a topological manifold The n mentioned above.

Closed Euclidean half-space The n**-dimensional closed Euclidean half-space** is defined as $\mathbb{H}^n = \{x=(x_1,\cdots,x_n) \in \mathbb{R}^n | a_n \geqslant 0\}$. The **boundary** of $\mathbb{H}^n$ is $\partial\mathbb{H}^n = \{x=(x_1,\cdots,x_n) \in \mathbb{R}^n | a_n = 0\}$. The **interior** of $\mathbb{H}^n$ is $\mathbb{H}^n \backslash \partial\mathbb{H}^n$.

Chart Let M be a set. A **chart** on M is a bijection of a subset $U \subset M$ onto an open subset of some $\mathbb{R}^d$, in which case we call these charts $\mathbb{R}^d$-valued.

Note 4.1. *Some authors define charts as homeomorphism from the coordinate space to the manifold and some define the converse. Either is okay because the inverse of a homeomorphism is still a homeomorphism.*

Atlas Let M be a set. An **atlas** of class C^r on M is a collection of pairs $\{(U_i, \varphi_i)\}_{i\in I}$ of subsets U_i in M and $\mathbb{R}^d$-valued charts φ_i from these subsets, satisfying the following conditions:

1. $\bigcup_{i\in I} U_i = X$.
2. Sets of the form $\varphi_i(U_i \cap U_j)$ are open in $\mathbb{R}^d$.
3. The coordinate transfer map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is a C^r-diffeomorphism whenever $U_i \cap U_j \neq \varnothing$.

Note 4.2. $\mathbb{R}^d$ *above above may be replaced by any Banach space (complete normed vector space), in which dimension and differential operation are defined.*

Equivalent atlases Two C^r atlases are said to be **equivalent** if their union is also a C^r atlas.

Structure A **structure** is an equivalence class of atlases.

Differentiable structure A C^r **differentiable structure** on M is an equivalence class of C^r atlases.

Smooth structure A **smooth structure** is a C^∞ differentiable structure.

Differentiable manifold A **differentiable manifold** of class C^r is a set M together with a specified C^r structure on M such that the topology induced by the structure is Hausdorff and paracompact. If the charts are $\mathbb{R}^d$-valued, then we say the manifold has **dimension** d, denoted by $\dim(M)$.

Tangent vector Let M be a differentiable manifold of class C^r. Let x be a point of M. We consider triples (U_x, φ, v) where (U_x, φ) is a chart at x and v is an element of the vector space in which $\varphi(U_x)$ lies. We say that two such triples (U_x, φ, v) and (V_x, ϕ, w) are **equivalent** if the derivative of $\phi \circ \varphi^{-1}$ at $\varphi(x)$ maps v to w. The formula reads:

$$[d(\phi \circ \varphi^{-1})(\varphi(x))]v = w$$

An equivalence class of such triples is called a **tangent vector** of M at x.

Tangent space The set of all tangent vectors of a manifold M at a point x is a linear space called the **tangent space of** M **at** x and is denoted by T_xM.

Tangent space [Arnol'd] The set of velocity vectors of motions leaving point x of a domain M is a vector space attached to x. Its dimension is the dimension of M. This space is called the **tangent space** to the domain M at the point x and is denoted T_xM.

Tangent bundle Let M be a differentiable manifold. The disjoint union of all tangent spaces

$$TM = \coprod_{x \in M} T_xM$$

is called the **tangent bundle** of M. We have the following **natural projection**:

$$\begin{array}{rcrcl} \pi & : & TM & \to & M \\ & & v & \mapsto & x \ (v \in T_xM) \end{array}$$

The mapping is surjective and has a right inverse $X : M \to TM,\ x \mapsto X(x) \in T_xM$, such that the diagram commutes:

$$\begin{tikzcd} M \arrow[r, "X"] \arrow[dr, "\mathrm{id}_M"'] & TM \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

Tangent vector field A mapping $X : M \to TM,\ x \mapsto X(x) \in T_xM$ is called a **tangent vector field**.

Differential

5 *Manifolds in $\mathbb{R}^n$

5.1 Definitions

These definitions are essentially the restrictions of more general definitions to sets in $\mathbb{R}^n$.

Topological manifold in $\mathbb{R}^n$ A **topological manifold** of dimension d in $\mathbb{R}^n$ is a subset $S \subset \mathbb{R}^n$ each point of which has a neighborhood in S homeomorphic to $\mathbb{R}^d$. The local homeomorphism is called a **local chart**. If a manifold can be defined by a single chart, it is called a **elementary manifold**. A family of charts that covers the manifold is called an **atlas** of the manifold.

Smooth manifold in $\mathbb{R}^n$ A **d-dimensional smooth manifold of class C^r** in $\mathbb{R}^n$ is a subset $S \in \mathbb{R}^n$ that is a topological manifold, and has an atlas whose charts are C^r-diffeomorphisms and have d at each point.

Orientation of a manifold Two charts are call **consistent** if either they do not intersect, or the transition mapping has a positive Jacobian. An **orienting atlas** of a manifold is an atlas consisting of pairwise consistent charts. If such an atlas exists, the manifold is called **orientable**, otherwise **nonorientable**. Two atlases are called **equivalent** if their union is also an orienting atlas. An equivalence class of orienting atlases is called an **orientation class** or simply an **orientation** of the manifold. An **oriented manifold** is a manifold fixed with an orientation.

Manifold with boundary in $\mathbb{R}$ A **topological manifold of dimension d with boundary** in $\mathbb{R}^n$ is a subset $S \subset \mathbb{R}^n$ each point of which has a neighborhood in S homeomorphic either to $\mathbb{R}^d$ or $\mathbb{H}^d$. A point in S whose neighborhood in S is homeomorphic to $\mathbb{H}^d$ is called a **boundary point** of S. The set of all boundary points is called the **boundary** of S. If the manifold has a smooth atlas whose charts are C^r-diffeomorphisms, then it is called a **smooth manifold of class C^r**.

***NOT A DEFINITION** **Consistent orientation** If the Jacobian of transition from the frame $(n, \xi_2, \cdots, \xi_d)$ to $(\xi_1, \xi_2, \cdots, \xi_d)$ is positive, where n is the exterior normal vector on the boundary, we say the boundary has a **consistent orientation** with the original manifold.

Tangent space If a d-dimensional surface $S \subset \mathbb{R}^n$, is defined parametrically in a neighborhood of $x_0 \in S$ by means of a smooth mapping $(t_1, \cdots, t_d) = t \mapsto x = (x_1, \cdots, x_n)$ such that $x_0 = x(0)$ and the matrix $x'(0)$ has rank d, then the d-dimensional surface in $\mathbb{R}^n$ defined parametrically by the matrix equality

$$x - x_0 = x'(0)t$$

is called the **tangent plane** or **tangent space** to the surface S at $x_0 \in S$. The coordinate form is

$$\begin{aligned}
x_1 - x_1^0 &= \frac{\partial x_1}{\partial t_1}(0)t_1 + \cdots + \frac{\partial x_1}{\partial t_d}(0)t_d \\
&\vdots \\
x_n - x_n^0 &= \frac{\partial x_n}{\partial t_1}(0)t_1 + \cdots + \frac{\partial x_n}{\partial t_d}(0)t_d
\end{aligned}$$

6 Differential Forms

6.1 Definitions

Differential form A **real-valued differential p-form** ω on $D \subset \mathbb{R}^n$ is a family of skew-symmetric forms $\{\omega(x) : (T_x D)^p \to \mathbb{R}\}_{x \in D}$. p is called the **degree** or **order** of ω, and a p-form is usually denoted ω^p.

Exterior differential Let $\omega(x) = a_{i_1 \cdots i_p}(x) \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$ has differentiable coefficients, then its **exterior differential** is

$$\mathrm{d}\omega(x) = \mathrm{d}a_{i_1 \cdots i_p}(x) \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}$$

7 Integration of a Differential Form

7.1 Theorems

The General Stokes Formula Let S be an oriented piecewise smooth d-dimensional compact surface with boundary ∂S in $G \subset \mathbb{R}^n$, in which a smooth $(d-1)$-form ω is defined. Then

$$\int_S \mathrm{d}\omega = \int_{\partial S} \omega$$

Special cases are as follows:

(i) (**Green's theorem, in** $\mathbb{R}^2$) Let $\overline{D} \subset \mathbb{R}^2$ be a domain that satisfy the condition above and P, Q smooth functions in $\overline{D}$. Then we have

$$\boxed{\begin{aligned} &\omega = P\mathrm{d}x + Q\mathrm{d}y \\ &\quad \Longrightarrow \\ &\int_{\partial \overline{D}} P\mathrm{d}x + Q\mathrm{d}y = \int_{\overline{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y \end{aligned}}$$

(ii) (**Gauss-Ostrogradskii's Formula, in** $\mathbb{R}^3$) Let $\overline{D} \subset \mathbb{R}^3$ be a domain that satisfy the condition above and P, Q, R smooth functions in $\overline{D}$. Then we have

$$\boxed{\begin{aligned} &\omega = P\mathrm{d}y \wedge \mathrm{d}z + Q\mathrm{d}z \wedge \mathrm{d}x + R\mathrm{d}x \wedge \mathrm{d}y \\ &\quad \Longrightarrow \\ &\int_{\partial \overline{D}} P\mathrm{d}y \wedge \mathrm{d}z + Q\mathrm{d}z \wedge \mathrm{d}x + R\mathrm{d}x \wedge \mathrm{d}y \\ &= \int_{\overline{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \end{aligned}}$$

(iii) (**Stokes' Formula, in** $\mathbb{R}^3$) Let $S \subset \mathbb{R}^3$ be a 2-dimensional manifold that satisfy the condition above and P, Q, R smooth functions on S. Then we have

$$\boxed{\begin{aligned} &\omega = P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z \\ &\quad \Longrightarrow \\ &\int_{\partial S} P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \int_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z \\ &+ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y \end{aligned}}$$