3.2 Geometric Brownian Motion

A stochastic process $S(t)$ is a *geometric Brownian motion* if $\log S(t)$ is a Brownian motion with initial value $\log S(0)$; in other words, a geometric Brownian motion is simply an exponentiated Brownian motion. Accordingly, all methods for simulating Brownian motion become methods for simulating geometric Brownian motion through exponentiation. This section therefore focuses more on modeling than on algorithmic issues.

Geometric Brownian motion is the most fundamental model of the value of a financial asset. In his pioneering thesis of 1900, Louis Bachelier developed a model of stock prices that in retrospect we describe as ordinary Brownian motion, though the mathematics of Brownian motion had not yet been developed. The use of geometric Brownian motion as a model in finance is due primarily to work of Paul Samuelson in the 1960s. Whereas ordinary Brownian motion can take negative values — an undesirable feature in a model of the price of a stock or any other limited liability asset — geometric Brownian motion is always positive because the exponential function takes only positive values. More fundamentally, for geometric Brownian motion the *percentage* changes

$$\frac{S(t_2)-S(t_1)}{S(t_1)}, \frac{S(t_3)-S(t_2)}{S(t_2)}, \ldots, \frac{S(t_n)-S(t_{n-1})}{S(t_{n-1})} \tag{3.16}$$

are independent for $t_1 < t_2 < \cdots < t_n$, rather than the absolute changes $S(t_{i+1}) - S(t_i)$. These properties explain the centrality of geometric rather than ordinary Brownian motion in modeling asset prices.

3.2.1 Basic Properties

Suppose W is a standard Brownian motion and X satisfies

$$dX(t) = \mu\, dt + \sigma\, dW(t),$$

so that $X \sim \mathrm{BM}(\mu, \sigma^2)$. If we set $S(t) = S(0)\exp(X(t)) \equiv f(X(t))$, then an application of Itô's formula shows that

$$\begin{aligned} dS(t) &= f'(X(t))\, dX(t) + \tfrac{1}{2}\sigma^2 f''(X(t))\, dt \\ &= S(0)\exp(X(t))[\mu\, dt + \sigma\, dW(t)] + \tfrac{1}{2}\sigma^2 S(0)\exp(X(t))\, dt \\ &= S(t)(\mu + \tfrac{1}{2}\sigma^2)\, dt + S(t)\sigma\, dW(t). \end{aligned} \tag{3.17}$$

In contrast, a geometric Brownian motion process is often specified through an SDE of the form

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dW(t), \tag{3.18}$$

an expression suggesting a Brownian model of the "instantaneous returns" $dS(t)/S(t)$. Comparison of (3.17) and (3.18) indicates that the models are inconsistent and reveals an ambiguity in the role of "μ." In (3.17), μ is the drift of the Brownian motion we exponentiated to define $S(t)$ — the drift of $\log S(t)$. In (3.18), $S(t)$ has drift $\mu S(t)$ and (3.18) implies

$$d \log S(t) = (\mu - \tfrac{1}{2}\sigma^2)\, dt + \sigma\, dW(t), \tag{3.19}$$

as can be verified through Itô's formula or comparison with (3.17).

We will use the notation $S \sim \mathrm{GBM}(\mu, \sigma^2)$ to indicate that S is a process of the type in (3.18). We will refer to μ in (3.18) as the *drift parameter* though it is not the drift of either $S(t)$ or $\log S(t)$. We refer to σ in (3.18) as the *volatility parameter* of $S(t)$; the diffusion coefficient of $S(t)$ is $\sigma^2 S^2(t)$.

From (3.19) we see that if $S \sim \mathrm{GBM}(\mu, \sigma^2)$ and if S has initial value $S(0)$, then

$$S(t) = S(0) \exp\left([\mu - \tfrac{1}{2}\sigma^2]t + \sigma W(t)\right). \tag{3.20}$$

A bit more generally, if $u < t$ then

$$S(t) = S(u) \exp\left([\mu - \tfrac{1}{2}\sigma^2](t-u) + \sigma(W(t) - W(u))\right), \tag{3.21}$$

from which the claimed independence of the returns in (3.16) becomes evident. Moreover, since the increments of W are independent and normally distributed, this provides a simple recursive procedure for simulating values of S at $0 = t_0 < t_1 < \cdots < t_n$:

$$S(t_{i+1}) = S(t_i) \exp\left([\mu - \tfrac{1}{2}\sigma^2](t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i} Z_{i+1}\right), \tag{3.22}$$
$$i = 0, 1, \ldots, n-1,$$

with $Z_1, Z_2, \ldots, Z_n$ independent standard normals. In fact, (3.22) is equivalent to exponentiating both sides of (3.3) with μ replaced by $\mu - \frac{1}{2}\sigma^2$. This method is *exact* in the sense that the $(S(t_1), \ldots, S(t_n))$ it produces has the joint distribution of the process $S \sim \mathrm{GBM}(\mu, \sigma^2)$ at $t_1, \ldots, t_n$ — the method involves no discretization error. Time-dependent parameters can be incorporated by exponentiating both sides of (3.4).

Lognormal Distribution

From (3.20) we see that if $S \sim \mathrm{GBM}(\mu, \sigma^2)$, then the marginal distribution of $S(t)$ is that of the exponential of a normal random variable, which is called a lognormal distribution. We write $Y \sim LN(\mu, \sigma^2)$ if the random variable Y has the distribution of $\exp(\mu + \sigma Z)$, $Z \sim N(0,1)$. This distribution is thus given by

$$\begin{aligned} P(Y \le y) &= P(Z \le [\log(y) - \mu]/\sigma) \\ &= \Phi\left(\frac{\log(y) - \mu}{\sigma}\right) \end{aligned}$$

and its density by

$$\frac{1}{y\sigma}\phi\left(\frac{\log(y)-\mu}{\sigma}\right). \tag{3.23}$$

Moments of a lognormal random variable can be calculated using the basic identity

$$\mathsf{E}[e^{aZ}] = e^{\frac{1}{2}a^2}$$

for the moment generating function of a standard normal. From this it follows that $Y \sim LN(\mu, \sigma^2)$ has

$$\mathsf{E}[Y] = e^{\mu+\frac{1}{2}\sigma^2}, \quad \mathsf{Var}[Y] = e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right);$$

in particular, the notation $Y \sim LN(\mu, \sigma^2)$ does not imply that μ and σ^2 are the mean and variance of Y. From

$$P(Y \le e^{\mu}) = P(Z \le 0) = \tfrac{1}{2}$$

we see that e^{μ} is the median of Y. The mean of Y is thus larger than the median, reflecting the positive skew of the lognormal distribution.

Applying these observations to (3.20), we find that if $S \sim \mathrm{GBM}(\mu, \sigma^2)$ then $(S(t)/S(0)) \sim LN([\mu - \frac{1}{2}\sigma^2]t, \sigma^2 t)$ and

$$\mathsf{E}[S(t)] = e^{\mu t}S(0), \quad \mathsf{Var}[S(t)] = e^{2\mu t}S^2(0)\left(e^{\sigma^2 t}-1\right).$$

In fact, we have

$$\mathsf{E}[S(t)|S(\tau), 0 \le \tau \le u] = \mathsf{E}[S(t)|S(u)] = e^{\mu(t-u)}S(u), \quad u < t, \tag{3.24}$$

and an analogous expression for the conditional variance. The first equality in (3.24) is the Markov property (which follows from the fact that S is a one-to-one transformation of a Brownian motion, itself a Markov process) and the second follows from (3.21).

Equation (3.24) indicates that μ acts as an average growth rate for S, a sort of average continuously compounded rate of return. Along a single sample path of S the picture is different. For a standard Brownian motion W, we have $t^{-1}W(t) \to 0$ with probability 1. For $S \sim \mathrm{GBM}(\mu, \sigma^2)$, we therefore find that

$$\frac{1}{t}\log S(t) \to \mu - \tfrac{1}{2}\sigma^2,$$

with probability 1, so $\mu - \frac{1}{2}\sigma^2$ serves as the growth rate along each path. If this expression is positive, $S(t) \to \infty$ as $t \to \infty$; if it is negative, then $S(t) \to 0$. In a model with $\mu > 0 > \mu - \frac{1}{2}\sigma^2$, we find from (3.24) that $\mathsf{E}[S(t)]$ grows exponentially although $S(t)$ converges to 0. This seemingly pathological behavior is explained by the increasing skew in the distribution of $S(t)$: although $S(t) \to 0$, rare but very large values of $S(t)$ are sufficiently likely to produce an increasing mean.

3.2.2 Path-Dependent Options

Our interest in simulating paths of geometric Brownian motion lies primarily in pricing options, particularly those whose payoffs depend on the path of an underlying asset S and not simply its value $S(T)$ at a fixed exercise date T. Through the principles of option pricing developed in Chapter 1, the price of an option may be represented as an expected discounted payoff. This price is estimated through simulation by generating paths of the underlying asset, evaluating the discounted payoff on each path, and averaging over paths.

Risk-Neutral Dynamics

The one subtlety in this framework is the probability measure with respect to which the expectation is taken and the nearly equivalent question of how the payoff should be discounted. This bears on how the paths of the underlying asset ought to be generated and more specifically in the case of geometric Brownian motion, how the drift parameter μ should be chosen.

We start by assuming the existence of a constant continuously compounded interest rate r for riskless borrowing and lending. A dollar invested at this rate at time 0 grows to a value of

$$\beta(t) = e^{rt}$$

at time t. Similarly, a contract paying one dollar at a future time t (a zero-coupon bond) has a value at time 0 of e^{-rt}. In pricing under the risk-neutral measure, we discount a payoff to be received at time t back to time 0 by dividing by $\beta(t)$; i.e., β is the numeraire asset.

Suppose the asset S pays no dividends; then, under the risk-neutral measure, the discounted price process $S(t)/\beta(t)$ is a martingale:

$$\frac{S(u)}{\beta(u)} = \mathsf{E}\left[\frac{S(t)}{\beta(t)}|\{S(\tau), 0 \le \tau \le u\}\right]. \tag{3.25}$$

Comparison with (3.24) shows that if S is a geometric Brownian motion under the risk-neutral measure, then it must have $\mu = r$; i.e.,

$$\frac{dS(t)}{S(t)} = r\,dt + \sigma\,dW(t). \tag{3.26}$$

As discussed in Section 1.2.2, this equation helps explain the name "risk-neutral." In a world of risk-neutral investors, all assets would have the same average rate of return — investors would not demand a higher rate of return for holding risky assets. In a risk-neutral world, the drift parameter for $S(t)$ would therefore equal the risk-free rate r.

In the case of an asset that pays dividends, we know from Section 1.2.2 that the martingale property (3.25) continues to hold but with S replaced by the sum of S, any dividends paid by S, and any interest earned from investing the dividends at the risk-free rate r. Thus, let $D(t)$ be the value of any dividends paid over $[0, t]$ and interest earned on those dividends. Suppose the asset pays a *continuous dividend yield* of δ, meaning that it pays dividends at rate $\delta S(t)$ at time t. Then D grows at rate

$$\frac{dD(t)}{dt} = \delta S(t) + rD(t),$$

the first term on the right reflecting the influx of new dividends and the second term reflecting interest earned on dividends already accumulated. If $S \sim \text{GBM}(\mu, \sigma^2)$, then the drift in $(S(t) + D(t))$ is

$$\mu S(t) + \delta S(t) + rD(t).$$

The martingale property (3.25), now applied to the combined process $(S(t) + D(t))$, requires that this drift equal $r(S(t) + D(t))$. We must therefore have $\mu + \delta = r$; i.e., $\mu = r - \delta$. The net effect of a dividend yield is to reduce the growth rate by δ.

We discuss some specific settings in which this formulation is commonly used:

- *Equity Indices.* In pricing index options, the level of the index is often modeled as geometric Brownian motion. An index is not an asset and it does not pay dividends, but the individual stocks that make up an index may pay dividends and this affects the level of the index. Because an index may contain many stocks paying a wide range of dividends on different dates, the combined effect is often approximated by a continuous dividend yield δ.
- *Exchange Rates.* In pricing currency options, the relevant underlying variable is an exchange rate. We may think of an exchange rate S (quoted as the number of units of domestic currency per unit of foreign currency) as the price of the foreign currency. A unit of foreign currency earns interest at some risk-free rate r_f, and this interest may be viewed as a dividend stream. Thus, in modeling an exchange rate using geometric Brownian motion, we set $\mu = r - r_f$.
- *Commodities.* A physical commodity like gold or oil may in some cases behave like an asset that pays *negative* dividends because of the cost of storing the commodity. This is easily accommodated in the setting above by taking $\delta < 0$. There may, however, be some value in holding a physical commodity; for example, a party storing oil implicitly holds an option to sell or consume the oil in case of a shortage. This type of benefit is sometimes approximated through a hypothetical *convenience yield* that accrues from physical storage. The net dividend yield in this case is the difference between the convenience yield and the cost rate for storage.
- *Futures Contracts.* A futures contract commits the holder to buying an underlying asset or commodity at a fixed price at a fixed date in the future. The *futures price* is the price specifed in a futures contract at which both

the buyer and the seller would willingly enter into the contract without either party paying the other. A futures price is thus not the price of an asset but rather a price agreed upon for a transaction in the future.

Let $S(t)$ denote the price of the underlying asset (the spot price) and let $F(t,T)$ denote the futures prices at time t for a contract to be settled at a fixed time T in the future. Entering into a futures contract at time t to buy the underlying asset at time $T > t$ is equivalent to committing to exchange a known amount $F(t,T)$ for an uncertain amount $S(T)$. For this contract to have zero value at the inception time t entails

$$0 = e^{-r(T-t)}\mathsf{E}[(S(T) - F(t,T))|\mathcal{F}_t], \tag{3.27}$$

where $\mathcal{F}_t$ is the history of market prices up to time t. At $t = T$ the spot and futures prices must agree, so $S(T) = F(T,T)$ and we may rewrite this condition as

$$F(t,T) = \mathsf{E}[F(T,T)|\mathcal{F}_t].$$

Thus, the futures price is a martingale (in its first argument) under the risk-neutral measure. It follows that if we choose to model a futures price (for fixed maturity T) using geometric Brownian motion, we should set its drift parameter to zero:

$$\frac{dF(t,T)}{F(t,T)} = \sigma\, dW(t).$$

Comparison of (3.27) and (3.25) reveals that

$$F(t,T) = e^{(r-\delta)(T-t)}S(t),$$

with δ the net dividend yield for S. If either process is a geometric Brownian motion under the risk-neutral measure then the other is as well and they have the same volatility σ.

This discussion blurs the distinction between futures and forward contracts. The argument leading to (3.27) applies more specifically to a forward price because a forward contract involves no intermediate cashflows. The holder of a futures contract typically makes or receives payments each day through a margin account; the discussion above ignores these cashflows. In a world with deterministic interest rates, futures and forward prices must be equal to preclude arbitrage so the conclusion in (3.27) is valid for both. With stochastic interest rates, it turns out that futures prices continue to be martingales under the risk-neutral measure but forward prices do not. The theoretical relation between futures and forward prices is investigated in Cox, Ingersoll, and Ross [90]; it is also discussed in many texts on derivative securities (e.g., Hull [189]).

Path-Dependent Payoffs

We turn now to some examples of path-dependent payoffs frequently encountered in option pricing. We focus primarily on cases in which the payoff depends on the values $S(t_1), \ldots, S(t_n)$ at a fixed set of dates $t_1, \ldots, t_n$; for these it is usually possible to produce an unbiased simulation estimate of the option price. An option payoff could in principle depend on the complete path $\{S(t), 0 \leq t \leq T\}$ over an interval $[0, T]$; pricing such an option by simulation will often entail some discretization bias. In the examples that follow, we distinguish between discrete and continuous monitoring of the underlying asset.

- *Asian option: discrete monitoring.* An Asian option is an option on a time average of the underlying asset. Asian calls and puts have payoffs $(\bar{S} - K)^+$ and $(K - \bar{S})^+$ respectively, where the strike price K is a constant and

$$\bar{S} = \frac{1}{n} \sum_{i=1}^{n} S(t_i) \tag{3.28}$$

  is the average price of the underlying asset over the discrete set of monitoring dates $t_1, \ldots, t_n$. Other examples have payoffs $(S(T) - \bar{S})^+$ and $(\bar{S} - S(T))^+$. There are no exact formulas for the prices of these options, largely because the distribution of $\bar{S}$ is intractable.
- *Asian option: continuous monitoring.* The continuous counterparts of the discrete Asian options replace the discrete average above with the continuous average

$$\bar{S} = \frac{1}{t - u} \int_u^t S(\tau)\, d\tau$$

  over an interval $[u, t]$. Though more difficult to simulate, some instances of continuous-average Asian options allow pricing through the transform analysis of Geman and Yor [135] and the eigenfunction expansion of Linetsky [237].
- *Geometric average option.* Replacing the arithmetic average $\bar{S}$ in (3.28) with

$$\left(\prod_{i=1}^{n} S(t_i) \right)^{1/n}$$

  produces an option on the geometric average of the underlying asset price. Such options are seldom if ever found in practice, but they are useful as test cases for computational procedures and as a basis for approximating ordinary Asian options. They are mathematically convenient to work with because the geometric average of (jointly) lognormal random variables is itself lognormal. From (3.20) we find (with μ replaced by r) that

$$\left(\prod_{i=1}^{n} S(t_i) \right)^{1/n} = S(0) \exp\left([r - \tfrac{1}{2}\sigma^2] \frac{1}{n} \sum_{i=1}^{n} t_i + \frac{\sigma}{n} \sum_{i=1}^{n} W(t_i) \right).$$

From the Linear Transformation Property (2.23) and the covariance matrix (3.6), we find that

$$\sum_{i=1}^{n} W(t_i) \sim N\left(0, \sum_{i=1}^{n}(2i-1)t_{n+1-i}\right).$$

It follows that the geometric average of $S(t_1), \ldots, S(t_n)$ has the same distribution as the value at time T of a process $\mathrm{GBM}(r-\delta, \bar{\sigma}^2)$ with

$$T = \frac{1}{n}\sum_{i=1}^{n} t_i, \quad \bar{\sigma}^2 = \frac{\sigma^2}{n^2 T}\sum_{i=1}^{n}(2i-1)t_{n+1-i}, \quad \delta = \tfrac{1}{2}\sigma^2 - \tfrac{1}{2}\bar{\sigma}^2.$$

An option on the geometric average may thus be valued using the Black-Scholes formula (1.44) for an asset paying a continuous dividend yield. The expression

$$\exp\left(\int_u^t \log S(\tau)\, d\tau\right)$$

is a continuously monitored version of the geometric average and is also lognormally distributed. Options on a continuous geometric average can similarly be priced in closed form.

- *Barrier options.* A typical example of a barrier option is one that gets "knocked out" if the underlying asset crosses a prespecified level. For instance, a *down-and-out call* with barrier b, strike K, and expiration T has payoff

$$\mathbf{1}\{\tau(b) > T\}(S(T)-K)^+,$$

where

$$\tau(b) = \inf\{t_i : S(t_i) < b\}$$

is the first time in $\{t_1, \ldots, t_n\}$ the price of the underlying asset drops below b (understood to be ∞ if $S(t_i) > b$ for all i) and $\mathbf{1}\{\ \}$ denotes the indicator of the event in braces. A down-and-*in* call has payoff $\mathbf{1}\{\tau(b) \leq T\}(S(T)-K)^+$: it gets "knocked in" only when the underlying asset crosses the barrier. Up-and-out and up-and-in calls and puts are defined analogously. Some knock-out options pay a rebate if the underlying asset crosses the barrier, with the rebate possibly paid either at the time of the barrier crossing or at the expiration of the option.

These examples of discretely monitored barrier options are easily priced by simulation through sampling of $S(t_1), \ldots, S(t_n), S(T)$. A continuously monitored barrier option is knocked in or out the instant the underlying asset crosses the barrier; in other words, it replaces $\tau(b)$ as defined above with

$$\tilde{\tau}(b) = \inf\{t \geq 0 : S(t) \leq b\}.$$

Both discretely monitored and continuously monitored barrier options are found in practice. Many continuously monitored barrier options can be priced in closed form; Merton [261] provides what is probably the first such formula and many other cases can be found in, e.g., Briys et al. [62]. Discretely monitored barrier options generally do not admit pricing formulas and hence require computational procedures.

- *Lookback options.* Like barrier options, lookback options depend on extremal values of the underlying asset price. Lookback puts and calls expiring at t_n have payoffs

$$(\max_{i=1,\ldots,n} S(t_i) - S(t_n)) \quad \text{and} \quad (S(t_n) - \min_{i=1,\ldots,n} S(t_i))$$

respectively. A lookback call, for example, may be viewed as the profit from buying at the lowest price over $t_1, \ldots, t_n$ and selling at the final price $S(t_n)$. Continuously monitored versions of these options are defined by taking the maximum or minimum over an interval rather than a finite set of points.

Incorporating a Term Structure

Thus far, we have assumed that the risk-free interest rate r is constant. This implies that the time-t price of a zero-coupon bond maturing (and paying 1) at time $T > t$ is

$$B(t,T) = e^{-r(T-t)}. \tag{3.29}$$

Suppose however that at time 0 we observe a collection of bond prices $B(0,T)$, indexed by maturity T, incompatible with (3.29). To price an option on an underlying asset price S consistent with the observed term structure of bond prices, we can introduce a deterministic but time-varying risk-free rate $r(u)$ by setting

$$r(u) = -\frac{\partial}{\partial T} \log B(0,T)\bigg|_{T=u}.$$

Clearly, then,

$$B(0,T) = \exp\left(-\int_0^T r(u)\,du\right).$$

With a deterministic, time-varying risk-free rate $r(u)$, the dynamics of an asset price $S(t)$ under the risk-neutral measure (assuming no dividends) are described by the SDE

$$\frac{dS(t)}{S(t)} = r(t)\,dt + \sigma\,dW(t)$$

with solution

$$S(t) = S(0)\exp\left(\int_0^t r(u)\,du - \tfrac{1}{2}\sigma^2 t + \sigma W(t)\right).$$

This process can be simulated over $0 = t_0 < t_1 < \cdots < t_n$ by setting

$$S(t_{i+1}) = S(t_i)\exp\left(\int_{t_i}^{t_{i+1}} r(u)\,du - \tfrac{1}{2}\sigma^2(t_{i+1}-t_i) + \sigma\sqrt{t_{i+1}-t_i}Z_{i+1}\right),$$

with $Z_1, \ldots, Z_n$ independent $N(0,1)$ random variables.

If in fact we are interested only in values of $S(t)$ at $t_1, \ldots, t_n$, the simulation can be simplified, making it unnecessary to introduce a short rate $r(u)$ at all. If we observe bond prices $B(0,t_1), \ldots, B(0,t_n)$ (either directly or through interpolation from other observed prices), then since

$$\frac{B(0,t_i)}{B(0,t_{i+1})} = \exp\left(\int_{t_i}^{t_{i+1}} r(u)\,du\right),$$

we may simulate $S(t)$ using

$$S(t_{i+1}) = S(t_i)\frac{B(0,t_i)}{B(0,t_{i+1})}\exp\left(-\tfrac{1}{2}\sigma^2(t_{i+1}-t_i) + \sigma\sqrt{t_{i+1}-t_i}Z_{i+1}\right). \quad (3.30)$$

Simulating Off a Forward Curve

For some types of underlying assets, particularly commodities, we may observe not just a spot price $S(0)$ but also a collection of forward prices $F(0,T)$. Here, $F(0,T)$ denotes the price specified in a contract at time 0 to be paid at time T for the underlying asset. Under the risk-neutral measure, $F(0,T) = \mathsf{E}[S(T)]$; in particular, the forward prices reflect the risk-free interest rate and any dividend yield (positive or negative) on the underlying asset. In pricing options, we clearly want to simulate price paths of the underlying asset consistent with the forward prices observed in the market.

The equality $F(0,T) = \mathsf{E}[S(T)]$ implies

$$S(T) = F(0,T)\exp\left(-\tfrac{1}{2}\sigma^2 T + \sigma W(T)\right).$$

Given forward prices $F(0,t_1), \ldots, F(0,t_n)$, we can simulate using

$$S(t_{i+1}) = S(t_i)\frac{F(0,t_{i+1})}{F(0,t_i)}\exp\left(-\tfrac{1}{2}\sigma^2(t_{i+1}-t_i) + \sigma\sqrt{t_{i+1}-t_i}Z_{i+1}\right).$$

This generalizes (3.30) because in the absence of dividends we have $F(0,T) = S(0)/B(0,T)$. Alternatively, we may define $M(0) = 1$,

$$M(t_{i+1}) = M(t_i)\exp\left(-\tfrac{1}{2}\sigma^2(t_{i+1}-t_i) + \sigma\sqrt{t_{i+1}-t_i}Z_{i+1}\right), \quad i = 0, \ldots, n-1,$$

and set $S(t_i) = F(0,t_i)M(t_i)$, $i = 1, \ldots, n$.

Deterministic Volatility Functions

Although geometric Brownian motion remains an important benchmark, it has been widely observed across many markets that option prices are incompatible with a GBM model for the underlying asset. This has fueled research into alternative specifications of the dynamics of asset prices.

Consider a market in which several options with various strikes and maturities are traded simultaneously on the same underlying asset. Suppose the market is sufficiently liquid that we may effectively observe prices of the options without error. If the assumptions underlying the Black-Scholes formula held exactly, all of these option prices would result from using the same volatility parameter σ in the formula. In practice, one usually finds that this implied volatility actually varies with strike and maturity. It is therefore natural to seek a minimal modification of the Black-Scholes model capable of reproducing market prices.

Consider the extreme case in which we observe the prices $C(K,T)$ of call options on a single underlying asset for a continuum of strikes K and maturities T. Dupire [107] shows that, subject only to smoothness conditions on C as a function of K and T, it is possible to find a function $\sigma(S,t)$ such that the model

$$\frac{dS(t)}{S(t)} = r\,dt + \sigma(S(t),t)\,dW(t)$$

reproduces the given option prices, in the sense that

$$e^{-rT}\mathsf{E}[(S(T)-K)^+] = C(K,T)$$

for all K and T. This is sometimes called a *deterministic volatility function* to emphasize that it extends geometric Brownian motion by allowing σ to be a deterministic function of the current level of the underlying asset. This feature is important because it ensures that options can still be hedged through a position in the underlying asset, which would not be the case in a stochastic volatility model.

In practice, we observe only a finite set of option prices and this leaves a great deal of flexibility in specifying $\sigma(S,t)$ while reproducing market prices. We may, for example, impose smoothness constraints on the choice of volatility function. This function will typically be the result of a numerical optimization procedure and may never be given explicitly.

Once $\sigma(S,t)$ has been chosen to match a set of actively traded options, simulation may still be necessary to compute the prices of less liquid path-dependent options. In general, there is no exact simulation procedure for these models and it is necessary to use an Euler scheme of the form

$$S(t_{i+1}) = S(t_i)\left(1 + r(t_{i+1}-t_i) + \sigma(S(t_i),t_i)\sqrt{t_{i+1}-t_i}Z_{i+1}\right),$$

with $Z_1, Z_2, \ldots$ independent standard normals, or

$$S(t_{i+1}) =$$
$$S(t_i)\exp\left([r - \tfrac{1}{2}\sigma^2(S(t_i),t_i)](t_{i+1}-t_i) + \sigma(S(t_i),t_i)\sqrt{t_{i+1}-t_i}Z_{i+1}\right),$$

which is equivalent to an Euler scheme for $\log S(t)$.

3.2.3 Multiple Dimensions

A multidimensional geometric Brownian motion can be specified through a system of SDEs of the form

$$\frac{dS_i(t)}{S_i(t)} = \mu_i \, dt + \sigma_i \, dX_i(t), \quad i = 1, \ldots, d, \tag{3.31}$$

where each X_i is a standard one-dimensional Brownian motion and $X_i(t)$ and $X_j(t)$ have correlation ρ_{ij}. If we define a $d \times d$ matrix Σ by setting $\Sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$, then $(\sigma_1 X_1, \ldots, \sigma_d X_d) \sim \mathrm{BM}(0, \Sigma)$. In this case we abbreviate the process $S = (S_1, \ldots, S_d)$ as $\mathrm{GBM}(\mu, \Sigma)$ with $\mu = (\mu_1, \ldots, \mu_d)$. In a convenient abuse of terminology, we refer to μ as the drift vector of S, to Σ as its covariance matrix and to the matrix with entries ρ_{ij} as its correlation matrix; the actual drift vector is $(\mu_1 S_1(t), \ldots, \mu_d S_d(t))$ and the covariances are given by

$$\mathsf{Cov}[S_i(t), S_j(t)] = S_i(0) S_j(0) e^{(\mu_i + \mu_j)t} \left(e^{\rho_{ij}\sigma_i\sigma_j} - 1 \right).$$

This follows from the representation

$$S_i(t) = S_i(0) e^{(\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i X_i(t)}, \quad i = 1, \ldots, d.$$

Recall that a Brownian motion $\mathrm{BM}(0, \Sigma)$ can be represented as $AW(t)$ with W a standard Brownian motion $\mathrm{BM}(0, I)$ and A any matrix for which $AA^\top = \Sigma$. We may apply this to $(\sigma_1 X_1, \ldots, \sigma_d X_d)$ and rewrite (3.31) as

$$\frac{dS_i(t)}{S_i(t)} = \mu_i \, dt + a_i \, dW(t), \quad i = 1, \ldots, d, \tag{3.32}$$

with a_i the ith row of A. A bit more explicitly, this is

$$\frac{dS_i(t)}{S_i(t)} = \mu_i \, dt + \sum_{j=1}^{d} A_{ij} \, dW_j(t), \quad i = 1, \ldots, d.$$

This representation leads to a simple algorithm for simulating $\mathrm{GBM}(\mu, \Sigma)$ at times $0 = t_0 < t_1 < \cdots < t_n$:

$$S_i(t_{k+1}) = S_i(t_k) e^{(\mu_i - \frac{1}{2}\sigma_i^2)(t_{k+1} - t_k) + \sqrt{t_{k+1} - t_k} \sum_{j=1}^{d} A_{ij} Z_{k+1,j}}, \quad i = 1, \ldots, d, \tag{3.33}$$

$k = 0, \ldots, n-1$, where $Z_k = (Z_{k1}, \ldots, Z_{kd}) \sim N(0, I)$ and $Z_1, Z_2, \ldots, Z_n$ are independent. As usual, choosing A to be the Cholesky factor of Σ can reduce the number of multiplications and additions required at each step. Notice that (3.33) is essentially equivalent to exponentiating both sides of the recursion (3.15); indeed, all methods for simulating $\mathrm{BM}(\mu, \Sigma)$ provide methods for simulating $\mathrm{GBM}(\mu, \Sigma)$ (after replacement of μ_i by $\mu_i - \frac{1}{2}\sigma_i^2$).

The discussion of the choice of the drift parameter μ in Section 3.2.2 applies equally well to each μ_i in pricing options on multiple underlying assets. Often,

$\mu_i = r - \delta_i$ where r is the risk-free interest rate and δ_i is the dividend yield on the ith asset S_i.

We list a few examples of option payoffs depending on multiple assets:

- *Spread option.* A call option on the spread between two assets S_1, S_2 has payoff
$$([S_1(T) - S_2(T)] - K)^+$$
with K a strike price. For example, *crack spread* options traded on the New York Mercantile Exchange are options on the spread between heating oil and crude oil futures.
- *Basket option.* A basket option is an option on a portfolio of underlying assets and has a payoff of, e.g.,
$$([c_1 S_1(T) + c_2 S_2(T) + \cdots + c_d S_d(T)] - K)^+.$$
Typical examples would be options on a portfolio of related assets — bank stocks or Asian currencies, for instance.
- *Outperformance option.* These are options on the maximum or minimum of multiple assets and have payoffs of, e.g., the form
$$(\max\{c_1 S_1(T), c_2 S_2(T), \cdots, c_d S_d(T)\} - K)^+.$$
- *Barrier options.* A two-asset barrier option may have a payoff of the form
$$\mathbf{1}\{\min_{i=1,\ldots,n} S_2(t_i) < b\}(K - S_1(T))^+;$$
This is a down-and-in put on S_1 that knocks in when S_2 drops below a barrier at b. Many variations on this basic structure are possible. In this example, one may think of S_1 as an individual stock and S_2 as the level of an equity index: the put on the stock is knocked in only if the market drops.
- *Quantos.* Quantos are options sensitive both to a stock price and an exchange rate. For example, consider an option to buy a stock denominated in a foreign currency with the strike price fixed in the foreign currency but the payoff of the option to be made in the domestic currency. Let S_1 denote the stock price and S_2 the exchange rate, expressed as the quantity of domestic currency required per unit of foreign currency. Then the payoff of the option in the domestic currency is given by
$$S_2(T)(S_1(T) - K)^+. \tag{3.34}$$
The payoff
$$\left(S_1(T) - \frac{K}{S_2(T)}\right)^+$$
corresponds to a quanto in which the level of the strike is fixed in the domestic currency and the payoff of the option is made in the foreign currency.

Change of Numeraire

The pricing of an option on two or more underlying assets can sometimes be transformed to a problem with one less underlying asset (and thus to a lower-dimensional problem) by choosing one of the assets to be the numeraire. Consider, for example, an option to exchange a basket of assets for another asset with payoff

$$\left(\sum_{i=1}^{d-1} c_i S_i(T) - c_d S_d(T)\right)^+,$$

for some constants c_i. The price of the option is given by

$$e^{-rT}\mathsf{E}\left[\left(\sum_{i=1}^{d-1} c_i S_i(T) - c_d S_d(T)\right)^+\right], \tag{3.35}$$

the expectation taken under the risk-neutral measure. Recall that this is the measure associated with the numeraire asset $\beta(t) = e^{rt}$ and is characterized by the property that the processes $S_i(t)/\beta(t)$, $i = 1, \ldots, d$, are martingales under this measure.

As explained in Section 1.2.3, choosing a different asset as numeraire — say S_d — means switching to a probability measure under which the processes $S_i(t)/S_d(t)$, $i = 1, \ldots, d-1$, and $\beta(t)/S_d(t)$ are martingales. More precisely, if we let P_β denote the risk-neutral measure, the new measure P_{S_d} is defined by the likelihood ratio process (cf. Appendix B.4)

$$\left(\frac{dP_{S_d}}{dP_\beta}\right)_t = \frac{S_d(t)}{\beta(t)}\frac{\beta(0)}{S_d(0)}. \tag{3.36}$$

Through this change of measure , the option price (3.35) can be expressed as

$$\begin{aligned}
&e^{-rT}\mathsf{E}_{S_d}\left[\left(\sum_{i=1}^{d-1} c_i S_i(T) - c_d S_d(T)\right)^+ \left(\frac{dP_\beta}{dP_{S_d}}\right)_T\right] \\
&= e^{-rT}\mathsf{E}_{S_d}\left[\left(\sum_{i=1}^{d-1} c_i S_i(T) - c_d S_d(T)\right)^+ \left(\frac{\beta(T)S_d(0)}{S_d(T)\beta(0)}\right)\right] \\
&= S_d(0)\mathsf{E}_{S_d}\left[\left(\sum_{i=1}^{d-1} c_i \frac{S_i(T)}{S_d(T)} - c_d\right)^+\right],
\end{aligned}$$

with E_{S_d} denoting expectation under P_{S_d}. From this representation it becomes clear that only the $d-1$ ratios $S_i(T)/S_d(T)$ (and the constant $S_d(0)$) are needed to price this option under the new measure. We thus need to determine the dynamics of these ratios under the new measure.

Using (3.32) and (3.36), we find that

$$\left(\frac{dP_{S_d}}{dP_\beta}\right)_t = \exp\left(-\tfrac{1}{2}\sigma_d^2 + a_d W(t)\right).$$

Girsanov's Theorem (see Appendix B.4) now implies that the process

$$W^d(t) = W(t) - a_d^\top t$$

is a standard Brownian motion under P_{S_d}. Thus, the effect of changing numeraire is to add a drift $a^\top$ to W. The ratio $S_i(t)/S_d(t)$ is given by

$$\begin{aligned}\frac{S_i(t)}{S_d(t)} &= \frac{S_i(0)}{S_d(0)} \exp\left(-\tfrac{1}{2}\sigma_i^2 t + \tfrac{1}{2}\sigma_d^2 t + (a_i - a_d)W(t)\right) \\ &= \frac{S_i(0)}{S_d(0)} \exp\left(-\tfrac{1}{2}\sigma_i^2 t + \tfrac{1}{2}\sigma_d^2 t + (a_i - a_d)(W^d(t) + a_d^\top)\right) \\ &= \frac{S_i(0)}{S_d(0)} \exp\left(-\tfrac{1}{2}(a_i - a_d)(a_i - a_d)^\top + (a_i - a_d)W^d(t)\right),\end{aligned}$$

using the identities $a_j a_j^\top = \sigma_j^2$, $j = 1, \ldots, d$, from the definition of the a_j in (3.32). Under P_{S_d}, the scalar process $(a_i - a_d)W^d(t)$ is a Brownian motion with drift 0 and diffusion coefficient $(a_i - a_d)(a_i - a_d)^\top$. This verifies that the ratios S_i/S_d are martingales under P_{S_d} and also that $(S_1/S_d, \ldots, S_{d-1}/S_d)$ remains a multivariate geometric Brownian motion under the new measure. It is thus possible to price the option by simulating just this $(d-1)$-dimensional process of ratios rather than the original d-dimensional process of asset prices.

This device would not have been effective in the example above if the payoff in (3.35) had instead been

$$\left(\sum_{i=1}^{d} c_i S_i(T) - K\right)^+$$

with K a constant. In this case, dividing through by $S_d(T)$ would have produced a term $K/S_d(T)$ and would thus have required simulating this ratio as well as S_i/S_d, $i = 1, \ldots, d-1$. What, then, is the scope of this method? If the payoff of an option is given by $g(S_1(T), \ldots, S_d(T))$, then the property we need is that g be homogeneous of degree 1, meaning that

$$g(\alpha x_1, \ldots, \alpha x_d) = \alpha g(x_1, \ldots, x_d)$$

for all scalars α and all $x_1, \ldots, x_d$. For in this case we have

$$\frac{g(S_1(T), \ldots, S_d(T))}{S_d(T)} = g(S_1(T)/S_d(T), \ldots, S_{d-1}(T)/S_d(T), 1)$$

and taking one of the underlying assets as numeraire does indeed reduce by one the relevant number of underlying stochastic variables. See Jamshidian [197] for a more general development of this observation.