BANGALORE INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MATHEMATICS

Transforms Calculus, Fourier Series and Numerical Techniques
(18MAT31)

MODULE - I

LAPLACE TRANSFORMS

Definition

If $f(t)$ is a real valued function defined for all $t \geq 0$ then the Laplace transforms of $f(t)$ is denoted by $L[f(t)]$ and defined as

$$L[f(t)] = \int_{t=0}^{\infty} e^{-st} f(t)\,dt = F(s) = \overline{f}(s)$$

Provided the integral exists. On integration of the definite integral we will be having a function of $'s'$ i.e., $F(s) = \overline{f}(s)$. Where $'S'$ is a real or complex parameter. The symbol $'L'$ is called Laplace transform operator.

Laplace Transform of Elementary Functions

1. $L[a]$ Where $'a'$ is a constant.

$$L[f(t)] = \int_{t=0}^{\infty} e^{-st} f(t)\,dt$$

$$L[a] = \int_{t=0}^{\infty} e^{-st}.a\,dt$$

$$= a\left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \frac{a}{-s}\left[e^{-\infty} - e^{0}\right]$$

$$L[a] = \frac{a}{-s}[0-1] = \frac{a}{s}\ ,\ \ where\ \ s > 0$$

$If\ a = 1\ then$

$$L[1] = \frac{1}{s}, \quad where\ \ s > 0$$

2. $L\left[e^{at}\right]$

$$L\left[f(t)\right] = \int_{t=0}^{\infty} e^{-st} f(t)\,dt$$

$$L\left[e^{at}\right] = \int_{t=0}^{\infty} e^{-st}.e^{at}\ dt$$

$$= \int_{t=0}^{\infty} e^{-(s-a)t}\ dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}$$

$$= \frac{1}{-(s-a)}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{1}{-(s-a)}[0-1]$$

$$\boxed{L\left[e^{at}\right] = \frac{1}{s-a}}\ , \quad where\ \ s > a$$

$$similarly\ \boxed{L\left[e^{-at}\right] = \frac{1}{s+a}}$$

3. $L[\cosh at]$

$$L[\cosh at] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left\{L\left[e^{at}\right] + L\left[e^{-at}\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{1}{s-a}\right] + \left[\frac{1}{s+a}\right]\right\}$$

$$=\frac{1}{2}\left[\frac{(s+a)+(s-a)}{(s+a)(s-a)}\right]$$

$$=\frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]$$

$$\boxed{L[\cosh at]=\frac{s}{s^2-a^2}} \quad where\ \ s>a$$

4. $L[\sinh at]$

$$L[\sinh at]=L\left[\frac{e^{at}-e^{-at}}{2}\right]$$

$$=\frac{1}{2}\left\{L\left[e^{at}\right]-L\left[e^{-at}\right]\right\}$$

$$=\frac{1}{2}\left\{\left[\frac{1}{s-a}\right]-\left[\frac{1}{s+a}\right]\right\}$$

$$=\frac{1}{2}\left[\frac{(s+a)-(s-a)}{(s+a)(s-a)}\right]$$

$$=\frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$$

$$\boxed{L[\sinh at]=\frac{a}{s^2-a^2}} \quad where\ \ s>a$$

5. $L[\sin at]$

$$L[f(t)]=\int_{t=0}^{\infty} e^{-st} f(t)\,dt$$

$$L[\sin at]=\int_{t=0}^{\infty} e^{-st}\sin at\;dt \quad \left[\int e^{at}\sin bt\,dt=\frac{e^{at}}{a^2+b^2}(a\sin bt-b\cos bt)\right]$$

$$=\frac{e^{-st}}{(-s)^2+a^2}\left[-s\sin at-a\cos at\right]_0^{\infty}$$

$$=\frac{1}{s^2+a^2}\left[e^{-\infty}\left(-s\sin at-a\cos at\right)_{t\to\infty}-e^{0}\left(-s\sin 0-a\cos 0\right)\right]$$

$$=\frac{1}{s^2+a^2}\left(0+a\right)$$

$$\boxed{L\left[\sin at\right]=\frac{a}{s^2+a^2}}$$

6. $L\left[\cos at\right]$

$$L\left[f\left(t\right)\right]=\int_{t=0}^{\infty}e^{-st}f\left(t\right)dt$$

$$L\left[\cos at\right]=\int_{t=0}^{\infty}e^{-st}\cos at\;dt\quad\left[\int e^{at}\cos bt\,dt=\frac{e^{at}}{a^2+b^2}\left(a\cos bt+b\sin bt\right)\right]$$

$$=\frac{e^{-st}}{\left(-s\right)^2+a^2}\left[-s\cos at+a\sin at\right]_0^{\infty}$$

$$=\frac{1}{s^2+a^2}\left[e^{-\infty}\left(-s\cos at+a\sin at\right)_{t\to\infty}-e^{0}\left(-s\cos 0+a\sin 0\right)\right]$$

$$=\frac{1}{s^2+a^2}\left(s+0\right)$$

$$\boxed{L\left[\cos at\right]=\frac{s}{s^2+a^2}}$$

7. $L\left[t^n\right]$

$$L\left[f\left(t\right)\right]=\int_{t=0}^{\infty}e^{-st}f\left(t\right)dt$$

$$L\left[t^n\right]=\int_{t=0}^{\infty}e^{-st}.t^n\;dt$$

put $st=x\Rightarrow t=\frac{x}{s}\Rightarrow dt=\frac{dx}{s}$

when : $t=0\Rightarrow x=0$

$t\to\infty\Rightarrow x\to\infty$

$$= \int_{x=0}^{\infty} e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$L\left[t^n\right] = \frac{1}{s^{n+1}} \int_{x=0}^{\infty} e^{-x} x^n \, dx \;, \quad w.k.t \int_{x=0}^{\infty} e^{-x} x^n \, dx = \Gamma(n+1)$$

$$\therefore \quad \boxed{L\left[t^n\right] = \frac{\Gamma(n+1)}{s^{n+1}}}$$

NOTE : $\Gamma(n+1) = n!$ If 'n' is a +ve integer

$\Gamma(n+1) = n\Gamma n$ If 'n' is a +ve real number

$\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$ If 'n' is a -ve fraction

SL. No.	$f(t)$	$L\left[f(t)\right]$
1	1	$\frac{1}{s}$
2	a	$\frac{a}{s}$
3	e^{at}	$\frac{1}{s-a}$
4	e^{-at}	$\frac{1}{s+a}$
5	$\cosh at$	$\frac{s}{s^2-a^2}$
6	$\sinh at$	$\frac{a}{s^2-a^2}$
7	$\cos at$	$\frac{s}{s^2+a^2}$
8	$\sin at$	$\frac{a}{s^2+a^2}$
9	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$

Problems

Find the Laplace transforms of the following functions

1. $L\left[e^{3t}\right]=\dfrac{1}{s-3}$ $\quad w.k.t.\ \boxed{L\left[e^{at}\right]=\dfrac{1}{s-a}}$

2. $L\left[e^{-8t}\right]=\dfrac{1}{s+8}$ $\quad w.k.t.\ \boxed{L\left[e^{-at}\right]=\dfrac{1}{s+a}}$

3. $L[\cosh 4t]=\dfrac{s}{s^2-4^2}=\dfrac{s}{s^2-16}$ $\quad w.k.t\ \boxed{L[\cosh at]=\dfrac{s}{s^2-a^2}}$

4. $L[\sin 2t]=\dfrac{2}{s^2+2^2}=\dfrac{2}{s^2+4}$ $\quad w.k.t.\ \boxed{L[\sin at]=\dfrac{a}{s^2+a^2}}$

5. $L\left[a^t\right]=L\left[e^{(\log a)t}\right]$ $\quad \text{w.k.t.}\ a^t=e^{\log a^t}=e^{t\log a}=e^{(\log a)t}$

$L\left[a^t\right]=\dfrac{1}{s-\log a}$ $\quad \text{w.k.t.}\ \boxed{L\left[e^{at}\right]=\dfrac{1}{s-a}}$

6. $L\left[6^t\right]=L\left[e^{(\log 6)t}\right]$ $\quad \text{w.k.t.}\ 6^t=e^{\log 6^t}=e^{t\log 6}=e^{(\log 6)t}$

$\boxed{L\left[6^t\right]=\dfrac{1}{s-\log 6}}$ $\quad w.k.t.\ \boxed{L\left[a^t\right]=\dfrac{1}{s-\log a}}$

7. $L\left[\left(1+e^t\right)^2\right]$

$$L\left[\left(1+e^t\right)^2\right]=L\left[1+e^{2t}+2e^t\right]$$

$$=L[1]+L\left[e^{2t}\right]+L\left[2e^t\right]$$ $\quad w.k.t.\ \boxed{L\left[e^{at}\right]=\dfrac{1}{s-a}}$

$$\boxed{L\left[\left(1+e^t\right)^2\right]=\frac{1}{s}+\frac{1}{s-2}+2\left(\frac{1}{s-1}\right)}$$

8. $L[\sin 5t\cos 2t]$

$$L[\sin 5t\cos 2t]=L\left[\frac{1}{2}(\sin 7t+\sin 3t)\right] \qquad \left[\because \sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]\right]$$

$$=\frac{1}{2}\{L[\sin 7t]+L[\sin 3t]\} \qquad \text{w.k.t.} \boxed{L[\sin at]=\frac{a}{s^2+a^2}}$$

$$=\frac{1}{2}\left[\frac{7}{s^2+7^2}+\frac{3}{s^2+3^2}\right]$$

$$\boxed{L[\sin 5t\cos 2t]=\frac{1}{2}\left[\frac{7}{s^2+49}+\frac{3}{s^2+9}\right]}$$

9. $L[\sin t\sin 2t\sin 3t]$

$$L[\sin t\sin 2t\sin 3t]=L\left[\frac{1}{2}(\cos t-\cos 3t)\sin 3t\right], \qquad \left[\because \sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]\right]$$

$$=\frac{1}{2}L[\sin 3t\cos t-\sin 3t\cos 3t], \qquad \left[\because \sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]\right]$$

$$=\frac{1}{2}L\left[\frac{1}{2}(\sin 4t+\sin 2t)-\frac{1}{2}(\sin 6t+\sin 0)\right]$$

$$=\frac{1}{4}\{L[\sin 4t]+L[\sin 2t]-L[\sin 6t]-L[0]\} \qquad w.k.t. \boxed{L[\sin at]=\frac{a}{s^2+a^2}}$$

$$=\frac{1}{4}\left[\frac{4}{s^2+4^2}+\frac{2}{s^2+2^2}-\frac{6}{s^2+6^2}-0\right]$$

$$\boxed{L[\sin t\sin 2t\sin 3t]=\frac{1}{4}\left[\frac{4}{s^2+16}+\frac{2}{s^2+4}-\frac{6}{s^2+36}\right]}$$

10. $L[\cos^2 6t]$

$$L[\cos^2 6t]=L\left[\frac{1+\cos 12t}{2}\right] \qquad \boxed{\because \cos^2\theta=\frac{1+\cos 2\theta}{2}}$$

$$=\frac{1}{2}\{L[1]+L[\cos 12t]\}$$

$$=\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+12^2}\right]$$

$$\boxed{L\left[\cos^2 6t\right]=\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+144}\right]}$$

11. $L\left[\sin^3 3t\right]$

$$L\left[\sin^3 3t\right]=\frac{1}{4}L\left[3\sin 3t-\sin 9t\right] \qquad \left[\because \sin^3\theta=\frac{1}{4}\left(3\sin\theta-\sin 3\theta\right)\right]$$

$$=\frac{1}{4}\left\{L\left[3\sin 3t\right]-L\left[\sin 9t\right]\right\}, \qquad w.k.t.\ \boxed{L\left[\sin at\right]=\frac{a}{s^2+a^2}}$$

$$=\frac{1}{4}\left[3\left(\frac{3}{s^2+3^2}\right)-\left(\frac{9}{s^2+9^2}\right)\right]$$

$$\boxed{L\left[\sin^3 3t\right]=\frac{9}{4}\left[\frac{1}{s^2+9}-\frac{1}{s^2+81}\right]}$$

12. $L\left[t^4\right]$

$$\boxed{L\left[t^4\right]=\frac{4!}{s^{4+1}}=\frac{24}{s^5}} \qquad w.k.t.\ L\left[t^n\right]=\frac{n!}{s^{n+1}}\ \text{, If 'n' is a +ve integer}$$

13. $L\left[t^{-7/2}\right]=\dfrac{\Gamma\left(-\frac{7}{2}+1\right)}{s^{-\frac{7}{2}+1}}$ $\qquad w.k.t.\ \boxed{L\left[t^n\right]=\frac{\Gamma(n+1)}{s^{n+1}}}$

$$=\frac{\Gamma\left(-\frac{5}{2}\right)}{s^{-\frac{5}{2}}} \qquad \left[w.k.t.\ \Gamma(n)=\frac{\Gamma(n+1)}{n} \quad \text{If 'n' is a -ve fraction}\right]$$

$$=\frac{1}{s^{-\frac{5}{2}}}\left[\frac{\Gamma\left(-\frac{3}{2}+1\right)}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)}\right]=\frac{1}{s^{-\frac{5}{2}}}\left[\frac{\Gamma\left(-\frac{1}{2}\right)}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)}\right]$$

$$= \frac{1}{s^{-\frac{5}{2}}}\left[\frac{\Gamma\left(-\frac{1}{2}+1\right)}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}\right] = \frac{1}{s^{-\frac{5}{2}}}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}\right]$$

$$= \frac{1}{s^{-\frac{5}{2}}}\left[\frac{\sqrt{\pi}}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}\right]$$

$$\boxed{L\left[t^{-7/2}\right] = \frac{1}{s^{-\frac{5}{2}}}\left[\frac{\sqrt{\pi}}{\left(-\frac{15}{8}\right)}\right] = \frac{-8\sqrt{\pi}}{15s^{-\frac{5}{2}}}}$$

14. $L\left[t^{3/2}\right]$

$$L\left[t^{3/2}\right] = \frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}} = \frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} \qquad w.k.t.\ \boxed{L\left[t^n\right] = \frac{\Gamma(n+1)}{s^{n+1}}}$$

$$= \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{\frac{5}{2}}} \qquad \left[\because \Gamma(n) = (n-1)\Gamma(n-1),\ \text{If } 'n' \text{ is a +ve real number}\right]$$

$$= \frac{\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}}$$

$$\boxed{L\left[t^{3/2}\right] = \frac{\frac{3\sqrt{\pi}}{4}}{s^{\frac{5}{2}}} = \frac{3\sqrt{\pi}}{4s^{\frac{5}{2}}}}$$

15. $L\left[(2t+3)^2\right]$

$$L\left[(2t+3)^2\right] = L\left[4t^2+9+12t\right]$$

$$= 4L\left[t^2\right]+9L[1]+12L[t] \quad \left[\because L\left[t^n\right]=\frac{n!}{s^{(n+1)}} \text{ If}'n'\text{is a possitive integer}\right]$$

$$\boxed{L\left[(2t+3)^2\right]= 4\times\frac{2!}{s^3}+9\times\frac{1}{s}+12\times\frac{1}{s^2}}$$

16. $L\left[\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3\right]$

$$L\left[\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3\right] = L\left[\left(t^{3/2}+t^{-3/2}+3t^{1/2}+3t^{-1/2}\right)\right] \qquad \left[\because [a+b]^3=a^3+b^3+3ab(a+b)\right]$$

$$= L\left[t^{3/2}\right]+L\left[t^{-3/2}\right]+3L\left[t^{1/2}\right]+3L\left[t^{-1/2}\right]$$

$$\left[\because L\left[t^n\right]=\frac{\Gamma(n+1)}{s^{n+1}} \quad \text{If } 'n'\text{is a real number}\right]$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)}{s^{5/2}}+\frac{\Gamma\left(\frac{-1}{2}\right)}{s^{-1/2}}+3\frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}}+3\frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}}$$

$$\left[\begin{array}{ll}\because \Gamma(n)=(n-1)\Gamma(n-1) & \text{If } 'n' \text{ is a possitive real number}\\ \Gamma(n)=\dfrac{\Gamma(n+1)}{n} & \text{If } 'n' \text{ is a negative real number}\\ & \text{but not negative intrger}\end{array}\right]$$

$$= \frac{\frac{3}{2}\times\Gamma\left(\frac{3}{2}\right)}{s^{5/2}}+\frac{\Gamma\left(\frac{1}{2}\right)}{\frac{-1}{2}\times s^{-1/2}}+3\frac{\frac{1}{2}\times\Gamma\left(\frac{1}{2}\right)}{s^{3/2}}+3\frac{\sqrt{\pi}}{s^{1/2}}$$

$$= \frac{\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{5/2}}+\frac{\sqrt{\pi}}{\frac{-1}{2}\times s^{-1/2}}+3\frac{\frac{1}{2}\times\sqrt{\pi}}{s^{3/2}}+3\frac{\sqrt{\pi}}{s^{1/2}}$$

$$\boxed{L\left[\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3\right]=\frac{3\times\sqrt{\pi}}{4\times s^{5/2}}-\frac{2\times\sqrt{\pi}}{s^{-1/2}}+\frac{3\times\sqrt{\pi}}{2\times s^{3/2}}+\frac{3\times\sqrt{\pi}}{s^{1/2}}}$$

17.

$$L\left[2+5t^3+4e^{-3t}+10e^t+\sin 2t\right]=L[2]+L\left[5t^3\right]+L\left[4e^{-3t}\right]+L\left[10e^t\right]+L[\sin 2t]$$

$$=\frac{2}{s}+5\frac{3!}{s^4}+\frac{4}{s+3}+\frac{10}{s-1}+\frac{2}{s^2+2^2}$$

$$\boxed{L\left[2+5t^3+4e^{-3t}+10e^t+\sin 2t\right]=\frac{2}{s}+5\frac{6}{s^4}+\frac{4}{s+3}+\frac{10}{s-1}+\frac{2}{s^2+4}}$$

18. $L\left[4\sin^2 3t+e^{3t+4}+2\cos 4t\cos 2t\right]$

$$=L\left[4\sin^2 3t\right]+L\left[e^{3t+4}\right]+L[2\cos 4t\cos 2t]$$

$$=4L\left[\sin^2 3t\right]+L\left[e^{3t}.e^4\right]+2L[\cos 4t\cos 2t]$$

$$=4L\left[\frac{1-\cos 2(3t)}{2}\right]+e^4L\left[e^{3t}\right]+2L\left[\frac{\cos(4t+2t)+\cos(4t-2t)}{2}\right]$$

$$=\frac{4}{2}\left\{L[1]-L\left[\cos(6t)\right]\right\}+e^4\frac{1}{s-3}+\frac{2}{2}\left\{L[\cos 6t]+L[\cos 2t]\right\}$$

$$=2\left[\frac{1}{s}-\frac{s}{s^2+6^2}\right]+\frac{e^4}{s-3}+\frac{s}{s^2+6^2}+\frac{s}{s^2+2^2}$$

$$=\frac{2}{s}-\frac{2s}{s^2+36}+\frac{e^4}{s-3}+\frac{s}{s^2+36}+\frac{s}{s^2+4}$$

$$=\frac{2}{s}-\frac{s}{s^2+36}+\frac{e^4}{s-3}+\frac{s}{s^2+4}$$

$$\boxed{L\left[4\sin^2 3t+e^{3t+4}+2\cos 4t\cos 2t\right]=\frac{2}{s}-\frac{s}{s^2+36}+\frac{e^4}{s-3}+\frac{s}{s^2+4}}$$

Properties

1. Linearity Property

If $f(t)$ & $g(t)$ are any 2 functions of $'t'$. α & β are constants then

$$L\left[\alpha f(t)+\beta g(t)\right]=\alpha L\left[f(t)\right]+\beta L\left[g(t)\right]$$

2. First Shifting Property

If $L\left[f(t)\right]=F(s)$ then $L\left[e^{at} f(t)\right]=F(s-a)$

Proof: $F(s)=L\left[f(t)\right]$

$$=\int_0^\infty e^{-st} f(t)dt$$

$$F(s-a)=\int_0^\infty e^{-(s-a)t} f(t)dt$$

$$=\int_0^\infty e^{-st}.e^{at} f(t)dt$$

$$=\int_0^\infty e^{-st}\left(e^{at} f(t)\right)dt$$

$$F(s-a)=L\left[e^{at} f(t)\right]$$

$$L\left[e^{at} f(t)\right]=F(s-a)=L\left[f(t)\right]_{s\to(s-a)}$$

Similarly

$$L\left[e^{-at} f(t)\right]=F(s+a)=L\left[f(t)\right]_{s\to(s+a)}$$

Problems

1. $L\left[e^{3t}t^2\right]$

$$L\left[e^{at} f(t)\right] = L\left[f(t)\right]_{s\to(s-a)} = \left[F(s)\right]_{s\to(s-a)} = F(s-a)$$

$$L\left[e^{3t} t^2\right] = F(s-3) = L\left[t^2\right]_{s\to(s-3)} \quad w.k.t.\ L\left[t^n\right] = \frac{n!}{s^{n+1}} \quad \text{, If } 'n' \text{ is a +ve integer}$$

$$= \left(\frac{2!}{s^3}\right)_{s\to(s-3)}$$

$$\boxed{L\left[e^{3t} t^2\right] = \frac{2}{(s-3)^3}}$$

2. $L\left[e^{-3t}\cos^2 t\right]$

$$L\left[e^{-at} f(t)\right] = L\left[f(t)\right]_{s\to(s+a)} = \left[F(s)\right]_{s\to(s+a)} = F(s+a)$$

$$L\left[e^{-3t}\cos^2 t\right] = L\left[\cos^2 t\right]_{s\to(s+3)} \qquad \left[w.k.t. \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$= L\left[\frac{1+\cos 2t}{2}\right]_{s\to(s+3)}$$

$$= \frac{1}{2}\left\{L[1] + L[\cos 2t]\right\}_{s\to(s+3)}$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+2^2}\right]_{s\to(s+3)}$$

$$\boxed{L\left[e^{3t}\cos^2 t\right] = \frac{1}{2}\left[\frac{1}{s+3} + \frac{s+3}{(s+3)^2+4}\right]}$$

3. $L\left[t^5 e^{4t}\cosh 3t\right]$

$$L\left[t^5 e^{4t}\cosh 3t\right] = L\left[t^5 e^{4t}\left(\frac{e^{3t}+e^{-3t}}{2}\right)\right] \qquad \left[\because \cosh\theta = \left(\frac{e^{\theta}+e^{-\theta}}{2}\right)\right]$$

$$=\frac{1}{2}L\left[t^5\left(e^{7t}+e^{t}\right)\right]$$

$$=\frac{1}{2}\left\{L\left[e^{7t}t^5\right]+L\left[e^{t}t^5\right]\right\}$$

$$L\left[e^{at}f(t)\right]=L\left[f(t)\right]_{s\to(s-a)}=\left[F(s)\right]_{s\to(s-a)}=F(s-a)$$

$$L\left[t^5e^{4t}\cosh 3t\right]=\frac{1}{2}\left\{L\left[t^5\right]_{s\to(s-7)}+L\left[t^5\right]_{s\to(s-1)}\right\}$$

$$w.k.t.\ L\left[t^n\right]=\frac{n!}{s^{n+1}}\quad \text{, If 'n' is a +ve integer}$$

$$L\left[t^5e^{4t}\cosh 3t\right]=\frac{1}{2}\left\{\left(\frac{5!}{s^{5+1}}\right)_{s\to(s-7)}+\left(\frac{5!}{s^{5+1}}\right)_{s\to(s-1)}\right\}$$

$$L\left[t^5e^{4t}\cosh 3t\right]=\frac{1}{2}\left\{\left(\frac{120}{(s-7)^6}\right)+\left(\frac{5!}{(s-1)^6}\right)\right\}$$

3. Derivative of the transform Property

If $L\left[f(t)\right]=F(s)$ then

$$L\left[t^n f(t)\right]=(-1)^n\frac{d^n}{ds^n}\left\{L\left[f(t)\right]\right\}=(-1)^n\frac{d^n}{ds^n}\left[F(s)\right]\ \text{, where 'n' is a +ve integer}$$

Problems

1. $L\left[t\cos 3t\right]$

$$L\left[t^n f(t)\right]=(-1)^n\frac{d^n}{ds^n}\left\{L\left[f(t)\right]\right\}=(-1)^n\frac{d^n}{ds^n}\left[F(s)\right]\text{, where 'n' is a +ve integer}$$

$$L\left[t\cos 3t\right]=(-1)^1\frac{d}{ds}\left\{L\left[\cos 3t\right]\right\}$$

$$=(-1)\frac{d}{ds}\left(\frac{s}{s^2+3^2}\right)$$

$$=(-1)\left(\frac{(s^2+3^2)(1)-s(2s)}{(s^2+3^2)^2}\right)$$

$$=(-1)\left(\frac{9-s^2}{(s^2+9)^2}\right)$$

$$\boxed{L[t\cos 3t]=\left(\frac{s^2-9}{(s^2+9)^2}\right)}$$

2. $L\left[t^2 \sin t\right]$

$$L\left[t^n f(t)\right]=(-1)^n \frac{d^n}{ds^n}\left\{L\left[f(t)\right]\right\}=(-1)^n \frac{d^n}{ds^n}\left[F(s)\right], \text{where '}n\text{' is a +ve integer}$$

$$L\left[t^2 \sin t\right]=(-1)^2 \frac{d^2}{ds^2}\left\{L[\sin t]\right\}$$

$$=\frac{d^2}{ds^2}\left[\frac{1}{s^2+1^2}\right]$$

$$=\frac{d}{ds}\left[\frac{-2s}{(s^2+1^2)^2}\right]$$

$$=-2\left[\frac{(s^2+1)^2(1)-s2(s^2+1)(2s)}{(s^2+1)^4}\right]$$

$$=-2\left\{\frac{(s^2+1)\left[(s^2+1)-4s^2\right]}{(s^2+1)^4}\right\}$$

$$= -2\left[\frac{\left[\left(s^2+1\right)-4s^2\right]}{\left(s^2+1^2\right)^3}\right]$$

$$= -2\left[\frac{1-3s^2}{\left(s^2+1^2\right)^3}\right]$$

$$\boxed{L\left[t^2 \sin t\right] = 2\left[\frac{3s^2-1}{\left(s^2+1^2\right)^3}\right]}$$

3. $L\left[t\,e^{-2t}\sin 4t\right]$

$$L\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n}\left\{L\left[f(t)\right]\right\} = (-1)^n \frac{d^n}{ds^n}\left[F(s)\right], \text{where } 'n' \text{ is a +ve integer}$$

$$L\left[t\,e^{-2t}\sin 4t\right] = L\left[e^{-2t}\left(t\sin 4t\right)\right]$$

$$= L\left[t\sin 4t\right]_{s\to(s+2)}$$

$$= (-1)\left\langle \frac{d}{ds}\left\{L\left[\sin 4t\right]\right\}\right\rangle_{s\to(s+2)}$$

$$= (-1)\left\{\frac{d}{ds}\left[\frac{4}{s^2+4^2}\right]\right\}_{s\to(s+2)}$$

$$= -\left[\frac{-4(2s)}{\left(s^2+16\right)^2}\right]_{s\to(s+2)}$$

$$= \left[\frac{8s}{\left(s^2+16\right)^2}\right]_{s\to(s+2)}$$

$$\boxed{L\left[t\,e^{-2t}\sin 4t\right] = \left[\frac{8(s+2)}{\left((s+2)^2+16\right)^2}\right]}$$

4. $L\left[t^5 e^{4t}\cosh 2t\right]$

$$L\left[t^5 e^{4t}\cosh 2t\right] = L\left[t^5 e^{4t}\left(\frac{e^{2t}+e^{-2t}}{2}\right)\right]$$

$$= \frac{1}{2}L\left[t^5\left(e^{6t}+e^{2t}\right)\right]$$

$$= \frac{1}{2}L\left[t^5 e^{6t}+t^5 e^{2t}\right]$$

$$= \frac{1}{2}\left\{L\left[e^{6t}t^5\right]+L\left[e^{2t}t^5\right]\right\}$$

$$= \frac{1}{2}\left\{L\left[t^5\right]_{s\to(s-6)}+L\left[t^5\right]_{s\to(s-2)}\right\}$$

$$= \frac{1}{2}\left[\frac{5!}{s^{5+1}}_{s\to(s-6)}+\frac{5!}{s^{5+1}}_{s\to(s-2)}\right]$$

$$= \frac{1}{2}\left[\frac{120}{s^6}_{s\to(s-6)}+\frac{120}{s^6}_{s\to(s-2)}\right]$$

$$= \frac{120}{2}\left[\frac{1}{(s-6)^6}+\frac{1}{(s-2)^6}\right]$$

$$\boxed{L\left[t^5 e^{4t}\cosh 2t\right] = 60\left[\frac{1}{(s-6)^6}+\frac{1}{(s-2)^6}\right]}$$

5. $L\left[t\left(\sin^3 t-\cos^3 t\right)\right]$

$$L\left[t^n f(t)\right] = (-1)^n\frac{d^n}{ds^n}\left\{L\left[f(t)\right]\right\} = (-1)^n\frac{d^n}{ds^n}\left[F(s)\right], \text{where '}n\text{' is a +ve integer}$$

$$L\left[t\left(\sin^3 t-\cos^3 t\right)\right] = (-1)\frac{d}{ds}L\left[\sin^3 t-\cos^3 t\right]$$

$$= (-1)\frac{d}{ds}L\left[\left(\frac{3\sin t-\sin 3t}{4}\right)-\left(\frac{\cos 3t+3\cos t}{4}\right)\right]$$

$$= \frac{(-1)}{4}\frac{d}{ds}\left\{L[3\sin t]-L[\sin 3t]-L[\cos 3t]-L[3\cos t]\right\}$$

$$= \frac{-1}{4}\frac{d}{ds}\left[3\left(\frac{1}{s^2+1^2}\right)-\left(\frac{3}{s^2+3^2}\right)-\left(\frac{s}{\left(s^2+3^2\right)}\right)-3\left(\frac{s}{s^2+1^2}\right)\right]$$

$$= \frac{-1}{4}\left[\frac{-3(2s)}{\left(s^2+1\right)^2}-\frac{-3(2s)}{\left(s^2+9\right)^2}-\left(\frac{\left(s^2+9\right)(1)-s(2s)}{\left(s^2+9\right)^2}\right)-3\left(\frac{\left(s^2+1\right)(1)-s(2s)}{\left(s^2+1\right)^2}\right)\right]$$

$$\boxed{L\left[t\left(\sin^3 t-\cos^3 t\right)\right]=\frac{-1}{4}\left[\frac{-6s}{\left(s^2+1\right)^2}-\frac{-6s}{\left(s^2+9\right)^2}-\left(\frac{9-s^2}{\left(s^2+9\right)^2}\right)-3\left(\frac{1-s^2}{\left(s^2+1\right)^2}\right)\right]}$$

6. $L\left[e^{-t}\sin 4t+t\cos 2t\right]$

$$L\left[e^{-t}\sin 4t+t\cos 2t\right] = L\left[e^{-t}\sin 4t\right]+L\left[t\cos 2t\right]$$

$$= L\left[\sin 4t\right]_{s\to s+1}+(-1)\frac{d}{ds}L\left[\cos 2t\right]$$

$$= \left[\frac{4}{s^2+4^2}\right]_{s\to s+1}+(-1)\frac{d}{ds}\left(\frac{s}{s^2+2^2}\right)$$

$$= \left[\frac{4}{(s+1)^2+16}\right]+(-1)\left\{\frac{\left(s^2+4\right)1-s(2s)}{\left(s^2+4\right)^2}\right\}$$

$$= \left[\frac{4}{(s+1)^2+16}\right]+(-1)\left[\frac{4-s^2}{\left(s^2+4\right)^2}\right]$$

$$\boxed{L\left[e^{-t}\sin 4t+t\cos 2t\right]=\left[\frac{4}{(s+1)^2+16}\right]+\left[\frac{s^2-4}{\left(s^2+4\right)^2}\right]}$$

4. Division of transform Property

If $L\left[f(t)\right]=F(s)$ then

$$L\left[\frac{f(t)}{t}\right]=\int_s^\infty F(s)ds \quad \text{This property is called division property}$$

1. $L\left[\dfrac{1-e^{-at}}{t}\right]$

$$L\left[\frac{f(t)}{t}\right]=\int_s^\infty L\left(f(t)\right)ds=\int_s^\infty F(s)ds$$

$$L\left[\frac{1-e^{-at}}{t}\right]=\int_s^\infty L\left(1-e^{-at}\right)ds$$

$$=\int_s^\infty\left(\frac{1}{s}-\frac{1}{s+a}\right)ds$$

$$=\left[\log s-\log(s+a)\right]_{s=s}^{\infty}$$

$$=\log\left[\frac{s}{s+a}\right]_{s=s}^{\infty}$$

$$=\log\left[\frac{s}{s\left(1+\frac{a}{s}\right)}\right]_{s=s}^{\infty}$$

$$=\log\left[\frac{1}{\left(1+\frac{a}{s}\right)}\right]_{s=s}^{\infty}$$

$$=\underset{s\to\infty}{lt}\left\{\log\left[\frac{1}{\left(1+\frac{a}{s}\right)}\right]\right\}-\log\left[\frac{1}{\left(1+\frac{a}{s}\right)}\right]$$

$$= \log 1 - \log\left(\frac{1}{\frac{s+a}{s}}\right)$$

$$= -\left[\log 1 - \log\left(\frac{s+a}{s}\right)\right]$$

$$= 0 + \log\left(\frac{s+a}{s}\right)$$

$$\boxed{L\left[\frac{1-e^{-at}}{t}\right] = \log\left(\frac{s+a}{s}\right)}$$

2. $L\left[\frac{\sin^2 t}{t}\right]$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty L(f(t))ds = \int_s^\infty F(s)ds$$

$$L\left[\frac{\sin^2 t}{t}\right] = \int_s^\infty L(\sin^2 t)ds$$

$$= \int_s^\infty L\left(\frac{1-\cos 2t}{2}\right)ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+2^2}\right)ds$$

$$= \left\{\frac{1}{2}\left(\log s - \frac{1}{2}\log\left(s^2+4\right)\right)\right\}_s^\infty$$

$$= \left\{\frac{1}{2}\left(\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right)\right\}_s^\infty$$

$$=\left\{\frac{1}{2}\left(\log\left(\frac{s}{s\sqrt{1+\frac{4}{s^2}}}\right)\right)\right\}_s^{\infty}$$

$$=\lim_{s\to\infty}\frac{1}{2}\left(\log\left(\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right)\right)-\frac{1}{2}\left(\log\left(\frac{1}{\sqrt{1+\frac{4}{s^2}}}\right)\right)$$

$$=\frac{1}{2}\log 1-\frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4}}\right)$$

$$=0+\frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

$$\boxed{L\left[\frac{\sin^2 t}{t}\right]=\frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)}$$

3. $L\left[\dfrac{2\sin t\sin 5t}{t}\right]$

$$L\left[\frac{f(t)}{t}\right]=\int_s^{\infty}L[f(t)]ds=\int_s^{\infty}F(s)ds$$

$$L\left[\frac{2\sin t\sin 5t}{t}\right]=\int_s^{\infty}L(2\sin t\sin 5t)ds \qquad \left[\text{w.k.t}\sin A\sin B=\frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]\right]$$

$$=2\int_s^{\infty}L\left\{\frac{1}{2}[\cos 4t-\cos 6t]\right\}ds$$

$$=\int_s^{\infty}L(\cos 4t-\cos 6t)ds$$

$$=\int_s^{\infty}\left(\frac{s}{s^2+4^2}-\frac{s}{s^2+6^2}\right)ds$$

$$=\left\{\frac{1}{2}\left(\log\left(s^2+4^2\right)-\frac{1}{2}\log\left(s^2+6^2\right)\right)\right\}_s^{\infty}$$

$$=\left\{\frac{1}{2}\left(\log\left(\frac{s^2+16}{s^2+36}\right)\right)\right\}_s^\infty$$

$$=\left\{\frac{1}{2}\left(\log\left(\frac{s^2\left(1+\frac{16}{s^2}\right)}{s^2\left(1+\frac{36}{s^2}\right)}\right)\right)\right\}_s^\infty$$

$$=\underset{s\to\infty}{lt}\ \frac{1}{2}\left(\log\left(\frac{1+\frac{16}{s^2}}{1+\frac{36}{s^2}}\right)\right)-\frac{1}{2}\left(\log\left(\frac{1+\frac{16}{s^2}}{1+\frac{36}{s^2}}\right)\right)$$

$$=\frac{1}{2}\log 1-\frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$$

$$=0+\frac{1}{2}\log\left(\frac{s^2+36}{s^2+16}\right)$$

$$\boxed{L\left[\frac{2\sin t\sin 5t}{t}\right]=\frac{1}{2}\log\left(\frac{s^2+36}{s^2+16}\right)}$$

4. $L\left[\dfrac{e^{-at}-e^{-bt}}{t}\right]$

$$L\left[\frac{f(t)}{t}\right]=\int_s^\infty L\left(f(t)\right)ds=\int_s^\infty F(s)ds$$

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right]=\int_s^\infty L\left(e^{-at}-e^{-bt}\right)ds$$

$$=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$=\left[\log(s+a)-\log(s+b)\right]_{s=s}^\infty$$

$$=\log\left[\frac{s+a}{s+b}\right]_{s=s}^{\infty}$$

$$=\log\left[\frac{s\left(1+\frac{a}{s}\right)}{s\left(1+\frac{b}{s}\right)}\right]_{s=s}^{\infty}$$

$$=\underset{s\to\infty}{lt}\log\left[\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right]-\log\left[\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right]$$

$$=\log 1-\log\left(\frac{s+a}{s+b}\right)$$

$$=0+\log\left(\frac{s+b}{s+a}\right)$$

$$\boxed{L\left[\frac{e^{-at}-e^{-bt}}{t}\right]=\log\left(\frac{s+b}{s+a}\right)}$$

Evaluate the following integral

1. $\int_0^{\infty}\frac{e^{-t}\sin t}{t}dt$

To evaluate the given integral compare with definition of Laplace Transform.

$$\int_0^{\infty}e^{-st}f(t)dt=L[f(t)]$$

leaving the exponential term considering in the given problem $f(t)=\frac{\sin t}{t}$

$$\int_0^{\infty}e^{-st}\frac{\sin t}{t}dt=L\left[\frac{\sin t}{t}\right]$$

w.k.t $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} L\left[f(t)\right] ds$

Consider $L\left[\frac{\sin t}{t}\right] = \int_s^{\infty} L\left[\sin t\right] ds$

$$= \int_s^{\infty} \frac{1}{s^2+1} ds$$

$$= \left[\tan^{-1}(s)\right]_s^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$= \frac{\pi}{2} - \tan^{-1}(s)$$

$$L\left[\frac{\sin t}{t}\right] = \cot^{-1}(s)$$

$$\int_0^{\infty} e^{-st} \frac{\sin t}{t} dt = \cot^{-1}(s)$$

To get the requried integral put $s = 1$ in the above integral

$$\int_0^{\infty} e^{-t} \frac{\sin t}{t} dt = \cot^{-1}(1) = \frac{\pi}{4}$$

$$\boxed{\int_0^{\infty} e^{-t} \frac{\sin t}{t} dt = \frac{\pi}{4}}$$

2. $\int_0^{\infty} e^{-t}\, t \sin^2 3t\, dt$

To evaluate the given integral compare with definition of Laplace Transform.

$$\int_0^\infty e^{-st} f(t)\,dt = L\left[f(t)\right]$$

In the given problem $f(t) = t\sin^2 3t$

$$\int_0^\infty e^{-st} t\sin^2 3t\,dt = L\left[t\sin^2 3t\right]$$

$$L\left[t\sin^2 3t\right] = L\left[t\left(\frac{1-\cos 6t}{2}\right)\right]$$

$$= \frac{1}{2}\left\{L(t) - L(t\cos 6t)\right\}$$

$$\left[\begin{array}{l} \text{w.k.t } \mathrm{L}\left[\mathrm{t}^{\mathrm{n}}\ f(t)\right] = (-1)^n \dfrac{d}{ds^n} L\left[f(t)\right] \\ \qquad\qquad L\left[t^n\right] = \dfrac{n!}{s^{n+1}} \qquad \text{If } 'n' \text{ is possitive integer} \end{array}\right]$$

$$L\left[t\sin^2 3t\right] = \frac{1}{2}\left[\frac{1}{s^2} - (-1)^1 \frac{d}{ds} L\left[\cos 6t\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{d}{ds}\left[\frac{s}{s^2+6^2}\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{\left(s^2+6^2\right)1 - s(2s)}{\left(s^2+6^2\right)^2}\right]$$

$$L\left[t\sin^2 3t\right] = \frac{1}{2}\left[\frac{1}{s^2} + \frac{6^2 - s^2}{\left(s^2+6^2\right)^2}\right]$$

$$\int_0^\infty e^{-st}\, t\sin^2 3t\,dt = \frac{1}{2}\left[\frac{1}{s^2} + \frac{6^2 - s^2}{\left(s^2+6^2\right)^2}\right]$$

To get the requried integral put $s=1$ in the above integral

$$\int_0^\infty e^{-t} t \sin^2 3t\, dt = \frac{1}{2}\left[\frac{1}{1} + \frac{6^2-1}{\left(1+6^2\right)^2}\right]$$

$$\boxed{\int_0^\infty e^{-t} t \sin^2 3t\, dt = \frac{1}{2}\left[1 + \frac{5}{(37)^2}\right] = \frac{702}{1369}}$$

3. $\displaystyle\int_0^\infty \left[\frac{e^{-at}-e^{-bt}}{t}\right] dt$

To evaluate the given integral compare with definition of Laplace Transform.

$$\int_0^\infty e^{-st} f(t)\, dt = L\left[f(t)\right]$$

In the given problem $f(t) = \left[\dfrac{e^{-at}-e^{-bt}}{t}\right]$

$$\int_0^\infty e^{-st}\left[\frac{e^{-at}-e^{-bt}}{t}\right] dt = L\left[\frac{e^{-at}-e^{-bt}}{t}\right]$$

$$\textit{Consider } L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty L\left[e^{-at}-e^{-bt}\right] ds$$

$$= \int_s^\infty \left\{L\left[e^{-at}\right] - L\left[e^{-bt}\right]\right\} ds$$

$$= \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \log\left[\frac{s+a}{s+b}\right]_s^\infty$$

$$= \log\left[\frac{s\left(1+\frac{a}{s}\right)}{s\left(1+\frac{b}{s}\right)}\right]_s^\infty$$

$$= \underset{s\to\infty}{lt} \log\left[\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right] - \log\left[\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right]$$

$$= \log 1 - \log\left[\frac{s+a}{s+b}\right]$$

$$= 0 + \log\left[\frac{s+b}{s+a}\right]$$

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \log\left[\frac{s+b}{s+a}\right]$$

$$\int_0^\infty e^{-st}\left[\frac{e^{-at}-e^{-bt}}{t}\right]dt = \log\left[\frac{s+b}{s+a}\right]$$

To get the requried integral put $s = 0$ in the above equation

$$\int_0^\infty e^{-0t}\left[\frac{e^{-at}-e^{-bt}}{t}\right]dt = \log\left(\frac{b}{a}\right)$$

$$\boxed{\int_0^\infty \left[\frac{e^{-at}-e^{-bt}}{t}\right]dt = \log\left(\frac{b}{a}\right)}$$

4. $\int_0^\infty \left(\frac{\cos 6t - \cos 4t}{t}\right)dt$

To evaluate the given integral compare with definition of Laplace Transform.

$$\int_0^{\infty} e^{-st} f(t)\,dt = L[f(t)]$$

In the given problem $f(t) = \left(\dfrac{\cos 6t - \cos 4t}{t}\right)$

$$consider\ L\left[\left(\frac{\cos 6t - \cos 4t}{t}\right)\right] = \int_s^{\infty} L(\cos 6t - \cos 4t)\,ds$$

$$= \int_s^{\infty} \{L[\cos 6t] - L[\cos 4t]\}\,ds$$

$$= \int_s^{\infty} \left[\frac{s}{s^2+6^2} - \frac{4}{s^2+4^2}\right] ds$$

$$= \left\{\frac{1}{2}\left(\log\left(s^2+6^2\right) - \frac{1}{2}\log\left(s^2+4^2\right)\right)\right\}_s^{\infty}$$

$$= \left\{\frac{1}{2}\left(\log\left(\frac{s^2+36}{s^2+16}\right)\right)\right\}_s^{\infty}$$

$$= \left\{\frac{1}{2}\left(\log\left(\frac{s^2\left(1+\frac{36}{s^2}\right)}{s^2\left(1+\frac{16}{s^2}\right)}\right)\right)\right\}_s^{\infty}$$

$$= \underset{s\to\infty}{lt}\ \frac{1}{2}\left(\log\left(\frac{1+\frac{36}{s^2}}{1+\frac{16}{s^2}}\right)\right) - \frac{1}{2}\left(\log\left(\frac{1+\frac{36}{s^2}}{1+\frac{16}{s^2}}\right)\right)$$

$$= \frac{1}{2}\log 1 - \frac{1}{2}\log\left(\frac{s^2+36}{s^2+16}\right)$$

$$= 0 + \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$$

$$L\left[\left(\frac{\cos 6t - \cos 4t}{t}\right)\right] = \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$$

$$L\left[\left(\frac{\cos 6t-\cos 4t}{t}\right)\right]=\int_0^\infty e^{-st}\left(\frac{\cos 6t-\cos 4t}{t}\right)dt$$

To get the requried integral put $s=0$ in the above equation

$$\int_0^\infty e^{-0t}\left(\frac{\cos 6t-\cos 4t}{t}\right)dt=\frac{1}{2}\log\left(\frac{16}{36}\right)$$

$$\boxed{\int_0^\infty \left(\frac{\cos 6t-\cos 4t}{t}\right)dt=\frac{1}{2}\log\left(\frac{16}{36}\right)}$$

Periodic Function

A function $f(t)$ is said to be periodic function of period $T>0$ if $f(t+nT)=f(t)$

$where\ n=1,2,3,4\cdots\cdots\cdots$

Theorem

If $f(t)$ is a periodic function of period $'T'$. Then $L[f(t)]=\frac{1}{1-e^{-st}}\int_0^T e^{-st}f(t)dt$

Proof: From the definition

$$L[f(t)]=\int_0^\infty e^{-st}f(t)dt$$

$$=\int_0^\infty e^{-su}f(u)du$$

$$=\int_0^T e^{-su}f(u)du+\int_T^{2T} e^{-su}f(u)du+\int_{2T}^{3T} e^{-su}f(u)du+\cdots\cdots\cdots$$

$$=\sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} e^{-su}f(u)du$$

$$\begin{bmatrix} Put\, u = t + nT \Rightarrow du = dt, \\ when\ u = nT \Rightarrow t = 0, u = (n+1)T \Rightarrow t = \pi \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} \int_0^T e^{-s(t+nT)} f(t+nT)\,dt$$

$$= \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t)\,dt$$

$$= \left(1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots\cdots\right) \int_0^T e^{-st} f(t)\,dt$$

$$\boxed{L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt}$$

1. If $f(t) = t^2$, $0 < t < 2\, and\ f(t+2n) = f(t)\ Find\ L[f(t)]$

Solution: Given that $f(t)$ is a periodic function of period 2.

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} t^2\,dt$$

$$= \frac{1}{1-e^{-2s}} \left[t^2 \left(\frac{e^{-st}}{-s} \right) - (2t) \left(\frac{e^{-st}}{-s \times -s} \right) + (2) \left(\frac{e^{-st}}{-s \times -s \times -s} \right) \right]_0^2$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{1}{-s}\left(4e^{-2s} - 0\right) - \frac{2}{s^2}\left(2e^{-2s}\right) - \frac{2}{s^3}\left(e^{-2s} - 1\right) \right]$$

$$= \frac{2}{\left(1-e^{-2s}\right)s^3} \left[-2s^2 e^{-2s} - 2se^{-2s} - e^{-2s} + 1 \right]$$

$$\boxed{L[f(t)] = \frac{2}{\left(1-e^{-2s}\right)s^3} \left[1 - \left(2s^2 + 2s + 1\right) e^{-2s} \right]}$$

2. Find the Laplace transform of the full wave rectifier

$$f(t) = E\sin\omega t \ , \ 0 < t < \frac{\pi}{\omega} \text{ having period } \frac{\pi}{\omega}.$$

Solution: Given that $f(t)$ is a periodic function of period $\frac{\pi}{\omega}$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\pi s/\omega}}\int_0^{\pi/\omega} e^{-st} E\sin\omega t\,dt$$

$$= \frac{E}{1-e^{-\pi s/\omega}}\int_0^{\pi/\omega} e^{-st} \sin\omega t\,dt$$

$$= \frac{E}{1-e^{-\pi s/\omega}}\left[\frac{e^{-st}}{(-s)^2+w^2}(-s\sin\omega t-\omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{E}{\left(1-e^{-\pi s/\omega}\right)\left(s^2+\omega^2\right)}\left[\begin{array}{r} e^{-s\frac{\pi}{\omega}}\left(-s\sin\omega\frac{\pi}{\omega}-\omega\cos\omega\frac{\pi}{\omega}\right) \\ -e^0(-s\sin 0-\varpi\cos 0)\end{array}\right]$$

$$= \frac{E}{\left(1-e^{-\pi s/\omega}\right)\left(s^2+\omega^2\right)}\left[\omega e^{-s\pi/\omega}+\omega\right]$$

$$= \frac{E\omega}{\left(s^2+\omega^2\right)}\left[\frac{1+e^{-\pi s/\omega}}{1-e^{-\pi s/\omega}}\right]$$

Multiply and divide R.H.S by $e^{\pi s/2\omega}$.

$$= \frac{E\omega}{\left(s^2+\omega^2\right)}\left[\frac{\left(1+e^{-\pi s/\omega}\right)e^{\pi s/2\omega}}{\left(1-e^{-\pi s/\omega}\right)e^{\pi s/2\omega}}\right]$$

$$= \frac{E\omega}{\left(s^2+\omega^2\right)}\left[\frac{e^{\pi s/2\omega}+e^{-\pi s/2\omega}}{e^{\pi s/2\omega}-e^{-\pi s/2\omega}}\right]$$

$$= \frac{E\omega}{\left(s^2+\omega^2\right)}\left[\frac{2\cosh\left(\frac{\pi s}{2\omega}\right)}{2\sinh\left(\frac{\pi s}{2\omega}\right)}\right]$$

$$\boxed{L\left[f(t)\right] = \frac{E\omega}{\left(s^2+\omega^2\right)}\coth\left(\frac{\pi s}{2\omega}\right)}$$

3. Given $f(t) = \begin{cases} E & 0 < t < a/2 \\ -E & a/2 < t < a \end{cases}$ *where* $f(t+a) = f(t)$

Show that $L\left[f(t)\right] = \frac{E}{s}\tanh\left(\frac{as}{4}\right)$

Solution: Given that $f(t)$ is a periodic function of period $'a'$.

$$L\left[f(t)\right] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\int_0^{a/2} e^{-st}E\,dt + \int_{a/2}^{a} e^{-st}(-E)\,dt$$

$$= \frac{1}{1-e^{-as}}\left\{\left[(E)\frac{e^{-st}}{-s}\right]_0^{a/2} + \left[(-E)\frac{e^{-st}}{-s}\right]_{a/2}^{a}\right\}$$

$$= \frac{E}{1-e^{-as}}\left[\frac{1}{-s}\left(e^{-as/2}-e^0\right)+\frac{1}{s}\left(e^{-as}-e^{-as/2}\right)\right]$$

$$= \frac{E}{\left(1-e^{-as}\right)s}\left[-e^{-as/2}+1+e^{-as}-e^{-as/2}\right]$$

$$= \frac{E}{\left(1-e^{-as}\right)s}\left[1+e^{-as}-2e^{-as/2}\right]$$

$$= \frac{E}{\left(1-e^{-as}\right)s}\left[1-e^{-as/2}\right]^2$$

$$= \frac{E}{s}\left[\frac{\left(1-e^{-as/2}\right)^2}{\left(1-e^{-as}\right)}\right] = \frac{E}{s}\left[\frac{\left(1-e^{-as/2}\right)^2}{\left(1^2-\left(e^{-as/2}\right)^2\right)}\right]$$

$$= \frac{E}{s}\left[\frac{\left(1-e^{-as/2}\right)^2}{\left(1-e^{-as/2}\right)\left(1+e^{-as/2}\right)}\right]$$

$$= \frac{E}{s}\left[\frac{\left(1-e^{-as/2}\right)}{\left(1+e^{-as/2}\right)}\right]$$

Multiply and divide R.H.S by $e^{as/4}$.

$$= \frac{E}{s}\left[\frac{e^{as/4}\left(1-e^{-as/2}\right)}{e^{as/4}\left(1+e^{-as/2}\right)}\right]$$

$$= \frac{E}{s}\left[\frac{\left(e^{as/4}-e^{-as/4}\right)}{\left(e^{as/4}+e^{-as/4}\right)}\right]$$

$$\boxed{L\left[f(t)\right] = \frac{E}{s}\tanh\left(\frac{as}{4}\right)} \qquad \left[\because \tanh\theta = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}\right]$$

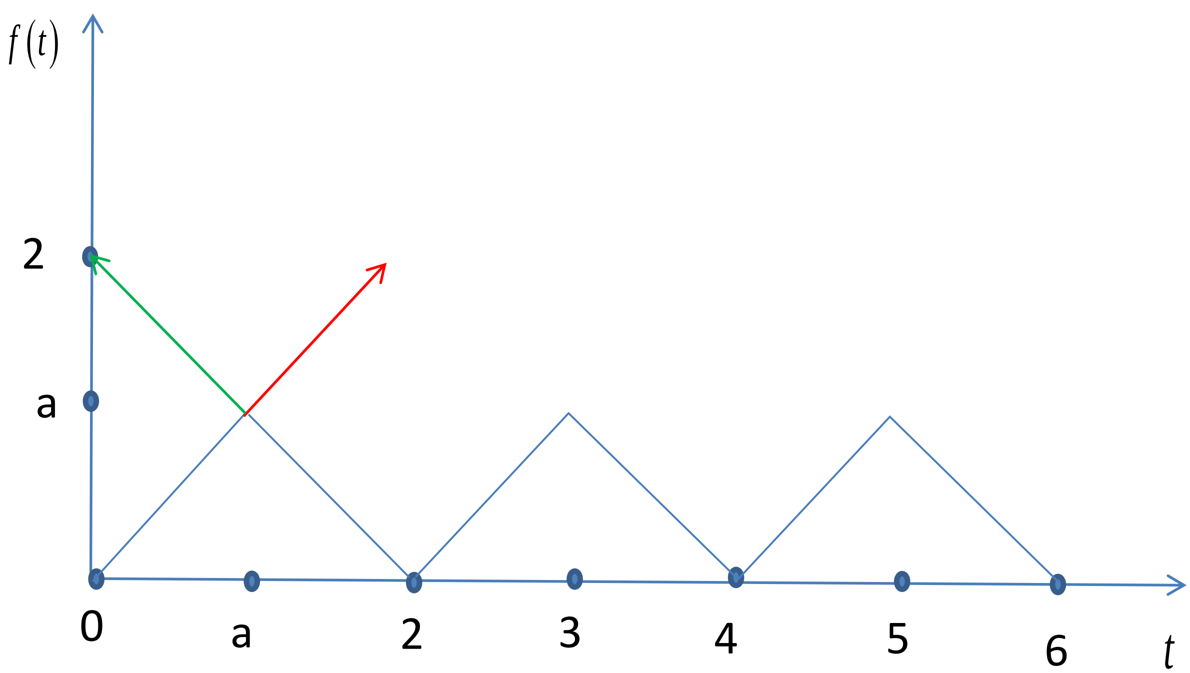
4. Given $f(t) = \begin{cases} t & 0 \le t \le a \\ 2a-t & a \le t \le 2a \end{cases}$ $\quad$ *where* $f(t+2a) = f(t)$

(i) Sketch the graph of $f(t)$ as a periodic function

(ii) Show that $L[f(t)] = \frac{1}{s^2}\tan h\left(\frac{as}{2}\right)$

Solution: $Let\ f(t) = y \Rightarrow y = t$ is a straight line passing through the origin.

$y = 2a - t$ is a straight line passing through the points $(0,2a)\&(2a,0)$



$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}(t)dt + \int_a^{2a} e^{-st}(2a-t)dt\right]$$

$$=\frac{1}{1-e^{-2as}}\left\{\left[(t)\left(\frac{e^{-st}}{-s}\right)-(1)\left(\frac{e^{-st}}{-s\times -s}\right)\right]_0^a+\left[(2a-t)\left(\frac{e^{-st}}{-s}\right)-(-1)\left(\frac{e^{-st}}{-s\times -s}\right)\right]_a^{2a}\right\}$$

$$=\frac{1}{1-e^{-2as}}\left[\frac{1}{-s}\left(ae^{-as}-0\right)-\frac{1}{s^2}\left(e^{-as}-e^{0}\right)+\frac{1}{-s}\left(0-ae^{-as}\right)+\frac{1}{s^2}\left(e^{-2as}-e^{-as}\right)\right]$$

$$=\frac{1}{1-e^{-2as}}\left[-\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}+\frac{ae^{-as}}{s}+\frac{e^{-2as}}{s^2}-\frac{e^{-as}}{s^2}\right]$$

$$=\frac{1}{\left(1-e^{-2as}\right)s^2}\left(1+e^{-2as}-2e^{-as}\right)$$

$$=\frac{\left[1+\left(e^{-as}\right)^2-2e^{-as}\right]}{\left[1-\left(e^{-as}\right)^2\right]s^2}$$

$$=\frac{\left(1-e^{-as}\right)^2}{\left(1-e^{-as}\right)\left(1+e^{-as}\right)s^2}$$

$$L\left[f(t)\right]=\frac{\left(1-e^{-as}\right)}{s^2\left(1+e^{-as}\right)}$$

Multiply and divide R.H.S by $e^{as/2}$.

$$=\frac{1}{s^2}\left[\frac{e^{as/2}\left(1-e^{-as}\right)}{e^{as/2}\left(1+e^{-as}\right)}\right]$$

$$=\frac{1}{s^2}\left[\frac{\left(e^{as/2}-e^{-as/2}\right)}{\left(e^{as/2}+e^{-as/2}\right)}\right]$$

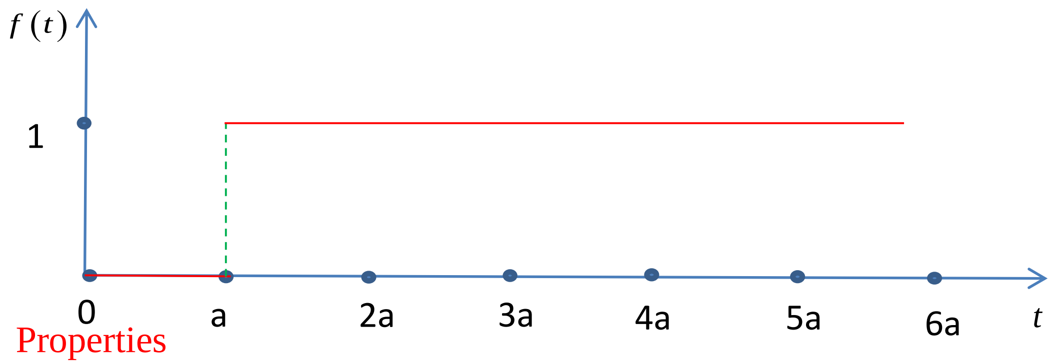
$$\boxed{L\left[f(t)\right]=\frac{1}{s^2}\tanh\left(\frac{as}{2}\right)} \qquad \left[\because \tanh\theta=\frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}\right]$$

Unit Step Function

Definition

The Unit step function $U(t-a)$ or Heaviside function $H(t-a)$ is defined as

$$U(t-a)=H(t-a)=\begin{cases}0 & t\le a\\ 1 & t>a\end{cases} \qquad \text{where 'a' is positive constant}$$



Properties

1. $\boxed{L\left[u(t-a)\right]=\frac{e^{-as}}{s}}$

2. . $\boxed{L\left[f(t-a)u(t-a)\right]=e^{-as}F(s)}$ where $L\left[f(t)\right]=F(s)$

The following two results will be useful in working problems connected with unit step function to find their Laplace Transform.

1.

If $f(t)=\begin{cases}f_1(t) & t\le a\\ f_2(t) & t>a\end{cases}$

Then $f(t)=f_1(t)+\left[f_2(t)-f_1(t)\right]u(t-a)$

consider R.H.S

$$f_1(t)+\left[f_2(t)-f_1(t)\right]u(t-a)=f_1(t)+\left[f_2(t)-f_1(t)\right]\begin{cases}0 & t\le a\\ 1 & t>a\end{cases}$$

$$=f_1(t)+\begin{cases}0 & t\le a\\ f_2(t)-f_1(t) & t>a\end{cases}$$

$$=\begin{cases} f_1(t) & t \le a \\ f_2(t) & t > a \end{cases}$$

$$= f(t) = \text{L.H.S}$$

2.

$$\text{If} \quad f(t) = \begin{cases} f_1(t)\ , & t \le a \\ f_2(t), & a < t \le b \\ f_3(t), & t > b \end{cases}$$

$$\text{Then} \quad f(\text{t}) = f_1(t) + \left[f_2(t) - f_1(t) \right] u(t-a) + \left[f_3(t) - f_2(t) \right] u(t-b)$$

Find the Laplace transform of the following functions.

1. $L\left[\left(e^{t-1} + \sin(t-1) \right) u\left(t-1\right) \right]$

$$w.k.t\, L\left[f(t-a) u\left(t-a\right) \right] = e^{-as} F(s) \text{ where} L\left[f(t) \right] = F(s)$$

$$\text{here } a = 1$$

$$f(t-1) = e^{t-1} + \sin(t-1)$$

$$\text{Change } t \to t+1 \text{ to get } f(t)$$

$$f(t) = e^t + \sin t$$

$$L\left[f(t) \right] = L\left[e^t + \sin t \right]$$

$$L\left[f(t) \right] = \frac{1}{s-1} + \frac{1}{s^2+1} = F(s)$$

$$L\left[\left(e^{t-1} + \sin(t-1) \right) u(t-1) \right] = e^{-s} \left[\frac{1}{s-1} + \frac{1}{s^2+1} \right]$$

2. $L\left[\sin t\, u(t-\pi) \right]$

$$w.k.t\ L\left[f(t-a)\, u(t-a) \right] = e^{-as} L\left[f(t) \right]$$

$$L\left[\sin t\, u(t-\pi) \right] = e^{-\pi s} L\left[f(t) \right]$$

$$f(t-\pi) = \sin t$$

$$\text{Change } t \to t+\pi \text{ to get } f(t)$$

$$f(t) = \sin(t+\pi)$$

$$f(t) = -\sin t$$

$$L\left[\sin t\, u(t-\pi)\right] = e^{-\pi s} L\left[f(t)\right]$$

$$= e^{-\pi s} L\left[-\sin t\right]$$

$$\boxed{L\left[\sin t\, u(t-\pi)\right] = -e^{-\pi s}\left[\frac{1}{s^2+1}\right]}$$

Express the following functions in terms of Heaviside function and hence find their Laplace Transform.

3. If $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$

If $f(t) = \begin{cases} f_1(t) & t \le a \\ f_2(t) & t > a \end{cases}$

Then $f(\mathrm{t}) = f_1(t) + \left[f_2(t) - f_1(t)\right]u(t-a)$

$$L\left[f(t)\right] = L\left[\cos t\right] + L\left\{\left[\sin t - \cos t\right]u(t-\pi)\right\}$$

$$L\left[f(t)\right] = \frac{s}{\left(s^2+1\right)} + L\left\{\left[\sin t - \cos t\right]u(t-\pi)\right\}$$

Consider

$$L\left\{\left[\sin t - \cos t\right]u(t-\pi)\right\}$$

$$L\left[f(t-a)u(t-a)\right] = e^{-as} L\left[f(t)\right]$$

$$L\left\{\left[\sin t - \cos t\right]u(t-\pi)\right\} = e^{-\pi s} L\left[f(t)\right]$$

$$f(t-\pi) = \left[\sin t - \cos t\right]$$

$$f(t) = \left[\sin(t+\pi) - \cos(t+\pi)\right]$$

$$f(t) = -\sin t + \cos t$$

$$L\left\{\left[\sin t - \cos t\right]u(t-\pi)\right\} = e^{-\pi s} L\left[-\sin t + \cos t\right]$$

$$L\left\{\left[\sin t - \cos t\right]u(t-\pi)\right\} = e^{-\pi s}\left[\frac{-1}{s^2+1} + \frac{s}{s^2+1}\right]$$

$$\boxed{L\left[\mathrm{f(t)}\right] = F(s) = \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{s}{s^2+1} - \frac{1}{s^2+1}\right]}$$

4. If $f(t)=\begin{cases}\cos t, & 0<t\le\pi\\ 1, & \pi<t\le 2\pi\\ \sin t, & t>2\pi\end{cases}$

If $f(t)=\begin{cases}f_1(t), & t\le a\\ f_2(t), & a<t\le b\\ f_3(t), & t>b\end{cases}$

Then $f(\text{t})= f_1(t)+\left[f_2(t)-f_1(t)\right]u(t-a)+\left[f_3(t)-f_2(t)\right]u(t-b)$

$$f(\text{t})= f_1(t)+\left[f_2(t)-f_1(t)\right]u(t-\pi)+\left[f_3(t)-f_2(t)\right]u(t-2\pi)$$

$$L\left[\text{f(t)}\right]=L\left[\cos t\right]+L\left\{\left[1-\cos t\right]u(t-\pi)\right\}+L\left\{\left[\sin t-1\right]u(t-2\pi)\right\}$$

1st term $L\left[\cos t\right]=\dfrac{s}{s^2+1}$

2nd term $L\left\{\left[1-\cos t\right]u(t-\pi)\right\}$

$$L\left\{\left[f(t-a)\right]u(t-a)\right\}=e^{-as}L\left[f(t)\right]$$

Here a=π

$$f(t-\pi)=1-\cos t$$

Change $t\to t+\pi$

$$f(t)=1-\cos(t+\pi)$$

$$=1+\cos t$$

$$L\left\{\left[1-\cos t\right]u(t-\pi)\right\}=e^{-\pi s}L\left[1+\cos t\right]$$

$$=e^{-\pi s}\left[\frac{1}{s}+\frac{s}{s^2+1}\right]$$

3rd term $L\left\{\left[\sin t-1\right]u(t-2\pi)\right\}$

$$L\left\{\left[f(t-a)\right]u(t-a)\right\}=e^{-as}L\left[f(t)\right]$$

Here $a=2\pi$

$$f(t-2\pi)=\sin t-1$$

Change $t\to t+2\pi$

$$f(t)=\sin(t+2\pi)-1$$

$$f(t)=\sin t-1$$

$$L\left\{\left[\sin t-1\right]u(t-2\pi)\right\}=e^{-2\pi s}L\left[\sin t-1\right]$$

$$=e^{-2\pi s}\left[\frac{1}{s^2+1}-\frac{1}{s}\right]$$

Equation (1) becomes

$$\boxed{L\left[f(t)\right]=F(s)=\frac{s}{s^2+1}+e^{-\pi s}\left[\frac{1}{s}+\frac{s}{s^2+1}\right]+e^{-2\pi s}\left[\frac{1}{s^2+1}-\frac{1}{s}\right]}$$

5. If $f(t)=\begin{cases}\cos t, & 0<t<\pi\\ \cos 2t, & \pi<t<2\pi\\ \cos 3t, & t>2\pi\end{cases}$

If $f(t)=\begin{cases}f_1(t)\ , & t\le a\\ f_2(t), & a<t\le b\\ f_3(t), & t>b\end{cases}$

Then $f(\mathrm{t})=f_1(t)+\left[f_2(t)-f_1(t)\right]u(t-a)+\left[f_3(t)-f_2(t)\right]u(t-b)$

$$f(\mathrm{t})=f_1(t)+\left[f_2(t)-f_1(t)\right]u(t-\pi)+\left[f_3(t)-f_2(t)\right]u(t-2\pi)$$

$$L\left[f(\mathrm{t})\right]=L\left[\cos t\right]+L\left\{\left[\cos 2t-\cos t\right]u(t-\pi)\right\}+L\left\{\left[\cos 3t-\cos 2t\right]u(t-2\pi)\right\}$$

1st term $\qquad L\left[\cos t\right]=\dfrac{s}{s^2+1}$

2nd term $\quad L\left\{\left[\cos 2t-\cos t\right]u(t-\pi)\right\}$

$$L\left\{\left[f(t-a)\right]u(t-a)\right\}=e^{-as}L\left[f(t)\right]$$

Here $a=\pi$

$$f(t-\pi)=\cos 2t-\cos t$$

Change $t\to t+\pi$

$$f(t)=\cos 2(t+\pi)-\cos(t+\pi)$$

$$=\cos 2t+\cos t$$

$$L\left\{\left[\cos 2t-\cos t\right]u(t-\pi)\right\}=e^{-\pi s}L\left[\cos 2t+\cos t\right]$$

$$=e^{-\pi s}\left[\frac{s}{s^2+2^2}+\frac{s}{s^2+1}\right]$$

Consider 3rd term $L\left\{\left[\cos 3t-\cos 2t\right]u(t-2\pi)\right\}$

$$L\left\{\left[f(t-a)\right]u(t-a)\right\}=e^{-as}L\left[f(t)\right]$$

Here $a=2\pi$

$f(t-2\pi)=\cos 3t-\cos 2t$

Change $t\rightarrow t+2\pi$

$$f(t)=\cos 3(t+2\pi)-\cos 2(t+2\pi)$$
$$f(t)=\cos 3t-\cos 2t$$

$$L\left\{\left[\cos 2t-\cos t\right]u(t-\pi)\right\}=e^{-2\pi s}L\left[\cos 3t-\cos 2t\right]$$
$$=e^{-2\pi s}\left[\frac{s}{s^2+3^2}-\frac{s}{s^2+2^2}\right]$$

Equation (1) becomes

$$\boxed{L\left[f(t)\right]=F(s)=\frac{s}{s^2+1}+e^{-\pi s}\left[\frac{s}{s^2+2^2}+\frac{s}{s^2+1}\right]+e^{-2\pi s}\left[\frac{s}{s^2+3^2}-\frac{s}{s^2+2^2}\right]}$$

6. If $f(t)=\begin{cases}\sin t, & 0\le t<\pi\\ \sin 2t, & \pi\le t<2\pi\\ \sin 3t, & t>2\pi\end{cases}$

$$L\left[f(t)\right]=F(s)=\frac{1}{s^2+1}+e^{-\pi s}\left[\frac{2}{s^2+2^2}+\frac{1}{s^2+1}\right]+e^{-2\pi s}\left[\frac{3}{s^2+3^2}-\frac{2}{s^2+2^2}\right]$$

7. If $f(t)=\begin{cases}1, & 0<t\le 1\\ t, & 1<t\le 2\\ t^2, & t>2\end{cases}$

$$L\left[f(t)\right]=F(s)=\frac{1}{s}+\frac{e^{-s}}{s^2}+e^{-2s}\left[\frac{2}{s^3}+\frac{3}{s^2}+\frac{2}{s}\right]$$

INVERSE LAPLACE TRANSFORM

Definition

If $L\left[f(t)\right]=F(s)$ then $f(t)$ is called the Inverse Laplace transform of $F(s)$ and is denoted by $L^{-1}\left[F(s)\right]$

i.e., $L\left[f(t)\right]=F(s) \Leftrightarrow L^{-1}\left[F(s)\right]=f(t)$

Here L^{-1} is called as Inverse Laplace transform operator.

The inverse Laplace transform is the transformation of a Laplace transform into a function of time.

Note:

1. Inverse Laplace transform holds Linearity property

i.e., $L^{-1}\left[aF(s)\pm bG(s)\right]=aL^{-1}(F(s))\pm bL^{-1}(G(s))$

Inverse Laplace Transform of Elementary Functions

SL. No.	$L^{-1}\left\{F(s)\right\}$	$f(t)$	Example
1	$\frac{1}{s}$	1	$L^{-1}\left[\frac{1}{s}\right]=1$
2	$\frac{a}{s}$	a	$L^{-1}\left[\frac{5}{s}\right]=5,\ L^{-1}\left[\frac{\pi}{s}\right]=\pi$
3	$\frac{1}{s-a}$	e^{at}	$L^{-1}\left[\frac{1}{s-3}\right]=e^{3t}$
4	$\frac{1}{s+a}$	e^{-at}	$L^{-1}\left[\frac{1}{s+5}\right]=e^{-5t}$
5	$\frac{1}{s-\log a}$	a^t	$L^{-1}\left[\frac{1}{s-\log 5}\right]=5^t$

6	$\frac{s}{s^2-a^2}$	$\cosh at$	$L^{-1}\left[\frac{s}{s^2-25}\right]=\cosh 5t$
7	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$	$L^{-1}\left[\frac{1}{s^2-25}\right]=\frac{1}{5}\sinh 5t$
8	$\frac{s}{s^2+a^2}$	$\cos at$	$L^{-1}\left[\frac{s}{s^2+25}\right]=\cos 5t$
9	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$	$L^{-1}\left[\frac{1}{s^2+25}\right]=\frac{1}{5}\sin 5t$
10	$\frac{1}{s^{n+1}}$	$\frac{t^n}{\Gamma(n+1)}$	$L^{-1}\left[\frac{1}{s^2}\right]=\frac{t^1}{\Gamma(2)}=t,$ $L^{-1}\left[\frac{1}{s^4}\right]=\frac{t^3}{\Gamma(4)}=\frac{t^3}{3!}=\frac{t^3}{6},$ $L^{-1}\left[\frac{1}{\sqrt{s}}\right]=\frac{t^{-1/2}}{\Gamma(1/2)}=\frac{1}{\sqrt{t}\sqrt{\pi}}$ $L^{-1}\left[\sqrt{s}\right]=L^{-1}\left[\frac{1}{s^{-1/2}}\right]=\frac{t^{-3/2}}{\Gamma(-(1/2))}=\frac{t^{-3/2}}{\frac{\sqrt{\pi}}{-1/2}}=\frac{-2t^{-3/2}}{\sqrt{\pi}}$
11	$F(s-a)$	$e^{at}f(t)$	$L^{-1}\left[\frac{s-3}{(s-3)^2+25}\right]=e^{3t}L^{-1}\left[\frac{s}{s^2+25}\right]=e^{3t}\cos 5t$
12	$F(s+a)$	$e^{-at}f(t)$	$L^{-1}\left[\frac{s+3}{(s+3)^2+25}\right]=e^{-3t}L^{-1}\left[\frac{s}{s^2+25}\right]=e^{-3t}\cos 5t$
13	$\frac{e^{-as}}{s}$	$u(t-a)$ *or* $H(t-a)$	$L^{-1}\left[\frac{e^{-3s}}{s}\right]=u(t-3)$

Problems

Find the Inverse Laplace Transform of the following functions

1. $\dfrac{1}{s^4}$

$$L^{-1}\left[\frac{1}{s^3}\right]=\frac{t^{3-1}}{\Gamma(3)}=\frac{t^2}{2!}=\frac{t^2}{2} \qquad \left[w.k.t.\ L^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{\Gamma(n+1)}\right]$$

2. $\dfrac{1}{s+2}+\dfrac{3}{2s+5}-\dfrac{4}{3s-2}$

$$L^{-1}\left[\frac{1}{s+2}+\frac{3}{2s+5}-\frac{4}{3s-2}\right]=L^{-1}\left(\frac{1}{s+2}\right)+\frac{3}{2}L^{-1}\left(\frac{1}{s+5/2}\right)-\frac{4}{3}L^{-1}\left(\frac{1}{s-2/3}\right)$$

$$\left[w.k.t.L^{-1}\left[\frac{1}{s-a}\right]=e^{at}\ \&\ L^{-1}\left[\frac{1}{s+a}\right]=e^{-at}\right]$$

$$\boxed{L^{-1}\left[\frac{1}{s+2}+\frac{3}{2s+5}-\frac{4}{3s-2}\right]=e^{-2t}+\frac{3}{2}e^{-(5/2)t}-\frac{4}{3}e^{(2/3)t}}$$

3. $\dfrac{2}{s+3}+\dfrac{5s}{s^2+9}$

$$L^{-1}\left[\frac{2}{s+3}+\frac{5s}{s^2+9}\right]=2L^{-1}\left(\frac{1}{s+3}\right)+5L^{-1}\left(\frac{s}{s^2+3^2}\right)$$

$$\left[w.k.t.L^{-1}\left[\frac{1}{s+a}\right]=e^{-at}\ \&\ L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at\right]$$

$$\boxed{L^{-1}\left[\frac{2}{s+3}+\frac{5s}{s^2+9}\right]=2e^{-3t}+5\cos 3t}$$

4. $\dfrac{1}{3s^2+16}+\dfrac{2s-1}{s^2+8}$

$$L^{-1}\left[\frac{1}{3s^2+16}+\frac{2s-1}{s^2+8}\right]=L^{-1}\left(\frac{1}{3s^2+16}\right)+L^{-1}\left(\frac{2s}{s^2+8}\right)-L^{-1}\left(\frac{1}{s^2+8}\right)$$

$$=\frac{1}{3}L^{-1}\left(\frac{1}{s^2+16/3}\right)+2L^{-1}\left(\frac{s}{s^2+8}\right)-L^{-1}\left(\frac{1}{s^2+8}\right)$$

$$=\frac{1}{3}L^{-1}\left(\frac{1}{s^2+\left(4/\sqrt{3}\right)^2}\right)+2L^{-1}\left(\frac{s}{s^2+\left(\sqrt{8}\right)^2}\right)-L^{-1}\left(\frac{1}{s^2+\left(\sqrt{8}\right)^2}\right)$$

$$\left[w.k.t.\ L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{\sin at}{a}\ \&\ L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at\right]$$

$$=\frac{1}{3}\frac{\sin\left(\frac{4}{\sqrt{3}}t\right)}{\frac{4}{\sqrt{3}}}+2\cos\sqrt{8}\,t-\frac{\sin\sqrt{8}\,t}{\sqrt{8}}$$

$$=\frac{1}{3}\left[\frac{\sqrt{3}}{4}\sin\left(\frac{4}{\sqrt{3}}t\right)\right]+2\cos\sqrt{8}\,t-\frac{\sin\sqrt{8}\,t}{\sqrt{8}}$$

$$\boxed{L^{-1}\left[\frac{1}{3s^2+16}+\frac{2s-1}{s^2+8}\right]=\frac{1}{4\sqrt{3}}\sin\left(\frac{4}{\sqrt{3}}t\right)+2\cos\sqrt{8}\,t-\frac{\sin\sqrt{8}\,t}{\sqrt{8}}}$$

5. $\frac{s+2}{s^2+36}+\frac{4s-1}{s^2+25}$

$$L^{-1}\left[\frac{s+2}{s^2+36}+\frac{4s-1}{s^2+25}\right]=L^{-1}\left(\frac{s}{s^2+36}\right)+2L^{-1}\left(\frac{1}{s^2+36}\right)+4L^{-1}\left(\frac{s}{s^2+25}\right)$$

$$-L^{-1}\left(\frac{1}{s^2+25}\right)$$

$$=\cos 6t+\frac{2}{6}\sin 6t+4\cos 5t-\frac{1}{5}\sin 5t$$

$$\boxed{L^{-1}\left[\frac{s+2}{s^2+36}+\frac{4s-1}{s^2+25}\right]=\cos 6t+\frac{1}{3}\sin 6t+4\cos 5t-\frac{1}{5}\sin 5t}$$

6. $\frac{2}{s\sqrt{s}}+\frac{5}{s^2\sqrt{s}}-\frac{7}{\sqrt{s}}$

$$L^{-1}\left[\frac{2}{s\sqrt{s}}+\frac{5}{s^2\sqrt{s}}-\frac{7}{\sqrt{s}}\right]=L^{-1}\left(\frac{2}{s\sqrt{s}}\right)+L^{-1}\left(\frac{5}{s^2\sqrt{s}}\right)-L^{-1}\left(\frac{7}{\sqrt{s}}\right)$$

$$=2L^{-1}\left(\frac{1}{s^{3/2}}\right)+5L^{-1}\left(\frac{1}{s^{5/2}}\right)-7L^{-1}\left(\frac{1}{s^{1/2}}\right)\quad\left[w.k.t.\ L^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{\Gamma(n+1)}\right]$$

$$=2\frac{t^{3/2-1}}{\Gamma\left(\frac{3}{2}\right)}+5\frac{t^{5/2-1}}{\Gamma\left(\frac{5}{2}\right)}-7\frac{t^{1/2-1}}{\Gamma\left(\frac{1}{2}\right)}\quad\left[w.k.t\,\Gamma(n)=(n-1)\Gamma(n-1)\ \right]$$

$$=2\frac{t^{3/2-1}}{\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)}+5\frac{t^{5/2-1}}{\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)}-7\frac{t^{1/2-1}}{\Gamma\left(\frac{1}{2}\right)}$$

$$=2\frac{t^{1/2}}{\left(\frac{1}{2}\right)\sqrt{\pi}}+5\frac{t^{3/2}}{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}}-7\frac{t^{-1/2}}{\sqrt{\pi}}$$

$$\boxed{L^{-1}\left[\frac{2}{s\sqrt{s}}+\frac{5}{s^2\sqrt{s}}-\frac{7}{\sqrt{s}}\right]=4\frac{\sqrt{t}}{\sqrt{\pi}}+\frac{20}{3}\frac{t^{3/2}}{\sqrt{\pi}}-\frac{7}{\sqrt{\pi}\sqrt{t}}}$$

7. $\dfrac{3\left(s^2-1\right)^2}{2s^5}$

$$L^{-1}\left[\frac{3\left(s^2-1\right)^2}{2s^5}\right]=L^{-1}\left[\frac{3\left(s^4+1-2s^2\right)}{2s^5}\right]$$

$$=L^{-1}\left[\frac{3s^4+3-6s^2}{2s^5}\right]$$

$$=L^{-1}\left[\frac{3}{2s}+\frac{3}{2s^5}-\frac{3}{s^3}\right]$$

$$=\frac{3}{2}L^{-1}\left(\frac{1}{s}\right)+\frac{3}{2}L^{-1}\left(\frac{1}{s^5}\right)-3L^{-1}\left(\frac{1}{s^3}\right)\qquad\left[w.k.t.\ L^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{\Gamma(n+1)}\right]$$

$$= \frac{3}{2} + \frac{3}{2}\left(\frac{t^4}{\Gamma(5)}\right) - 3\left(\frac{t^2}{\Gamma(3)}\right) \qquad \Gamma(n) = (n-1)!$$

$$= \frac{3}{2} + \frac{3}{2}\left(\frac{t^4}{4!}\right) - 3\left(\frac{t^2}{2!}\right)$$

$$= \frac{3}{2} + \frac{3t^4}{48} - \frac{3t^2}{2}$$

$$\boxed{L^{-1}\left[\frac{3(s^2-1)^2}{2s^5}\right] = \frac{3}{2} + \frac{t^4}{16} - \frac{3t^2}{2}}$$

Methods to find the Inverse Laplace transform

I. Inverse of First shifting property

W.K.T $L[f(t)] = F(s)$ then $L[e^{at} f(t)] = F(s-a)$

$$\Rightarrow L^{-1}[F(s-a)] = e^{at} f(t) = e^{at} L^{-1}[F(s)]$$

This is called as shifting rule of Inverse Laplace transform

Problems:

Find the Inverse Laplace transform of the following functions

1. $\dfrac{1}{(s-2)^2}$

$$w.k.t\, L^{-1}[F(s-a)] = e^{at} L^{-1}[F(s)]$$

$$L^{-1}\left[\frac{1}{(s-2)^2}\right] = e^{2t} L^{-1}\left[\frac{1}{s^2}\right] = \frac{e^{2t}t}{\Gamma(2)} = \frac{e^{2t}t}{1} = e^{2t}t$$

2. $\dfrac{1}{(s-2)^2+9}$

$w.k.t\, L^{-1}\left[F(s-a)\right]=e^{at}L^{-1}\left[F(s)\right]$

$$L^{-1}\left[\frac{1}{(s-2)^2+9}\right]=L^{-1}\left[\frac{1}{(s-2)^2+3^2}\right]=e^{2t}L^{-1}\left[\frac{1}{s^2+3^2}\right]=\frac{e^{2t}\sin 3t}{3}$$

3. $\dfrac{s+3}{(s+3)^2+36}$

$w.k.t\, L^{-1}\left[F(s+a)\right]=e^{-at}L^{-1}\left[F(s)\right]$

$$L^{-1}\left[\frac{s+3}{(s+3)^2+36}\right]=L^{-1}\left[\frac{s+3}{(s+3)^2+6^2}\right]=e^{-3t}L^{-1}\left[\frac{s}{s^2+6^2}\right]=e^{-3t}\cos 6t$$

4. $\dfrac{s}{(s+3)^2+49}$

$w.k.t\, L^{-1}\left[F(s+a)\right]=e^{-at}L^{-1}\left[F(s)\right]$

$$L^{-1}\left[\frac{s}{(s+3)^2+49}\right]=L^{-1}\left[\frac{s}{(s+3)^2+7^2}\right]=L^{-1}\left[\frac{s+3-3}{(s+3)^2+7^2}\right]$$

$$=L^{-1}\left[\frac{s+3}{(s+3)^2+7^2}-\frac{3}{(s+3)^2+7^2}\right]$$

$$=L^{-1}\left[\frac{s+3}{(s+3)^2+7^2}\right]-L^{-1}\left[\frac{3}{(s+3)^2+7^2}\right]$$

$$=e^{-3t}L^{-1}\left[\frac{s}{s^2+7^2}\right]-3e^{-3t}L^{-1}\left[\frac{1}{s^2+7^2}\right]$$

$$\boxed{L^{-1}\left[\frac{s}{(s+3)^2+49}\right]=e^{-3t}\cos 7t-\frac{3e^{-3t}\sin 7t}{7}}$$

5. $\dfrac{3s+1}{(s+1)^4}$

$$w.k.t\, L^{-1}\left[F(s+a)\right]=e^{-at}L^{-1}\left[F(s)\right]$$

$$L^{-1}\left[\frac{3s+1}{(s+1)^4}\right]=L^{-1}\left[\frac{3(s+1-1)+1}{(s+1)^4}\right]=L^{-1}\left[\frac{3((s+1)-1)+1}{(s+1)^4}\right]$$

$$=L^{-1}\left[\frac{3((s+1)-1)+1}{(s+1)^4}\right]=L^{-1}\left[\frac{3(s+1)}{(s+1)^4}-\frac{3}{(s+1)^4}+\frac{1}{(s+1)^4}\right]$$

$$=L^{-1}\left[\frac{3}{(s+1)^3}-\frac{2}{(s+1)^4}\right]$$

$$=3L^{-1}\left[\frac{1}{(s+1)^3}\right]-2L^{-1}\left[\frac{1}{(s+1)^4}\right]$$

$$=3e^{-t}L^{-1}\left[\frac{1}{s^3}\right]-2e^{-t}L^{-1}\left[\frac{1}{s^4}\right]$$

$$=3e^{-t}\frac{t^2}{\Gamma(3)}-2e^{-t}\frac{t^3}{\Gamma(4)}$$

$$=3e^{-t}\frac{t^2}{2}-2e^{-t}\frac{t^3}{6}$$

$$=\frac{3}{2}e^{-t}t^2-\frac{1}{3}t^3e^{-t}$$

6. $\dfrac{s^2}{(s-a)^3}$

$$L^{-1}\left[\frac{s^2}{(s-a)^3}\right]=L^{-1}\left[\frac{(s-a+a)^2}{(s-a)^3}\right]=L^{-1}\left[\frac{((s-a)+a)^2}{(s-a)^3}\right]$$

$$=L^{-1}\left[\frac{(s-a)^2+a^2+2(s-a)a}{(s-a)^3}\right]$$

$$= L^{-1}\left[\frac{(s-a)^2}{(s-a)^3} + \frac{a^2}{(s-a)^3} + \frac{2(s-a)a}{(s-a)^3}\right]$$

$$= L^{-1}\left[\frac{1}{(s-a)} + \frac{a^2}{(s-a)^3} + \frac{2a}{(s-a)^2}\right]$$

$$= L^{-1}\left[\frac{1}{(s-a)}\right] + a^2L^{-1}\left[\frac{1}{(s-a)^3}\right] + 2aL^{-1}\left[\frac{1}{(s-a)^2}\right]$$

$$= e^{at}L^{-1}\left[\frac{1}{s}\right] + a^2e^{at}L^{-1}\left[\frac{1}{s^3}\right] + 2a\, e^{at}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{at}(1) + a^2e^{at}\frac{t^2}{\Gamma(3)} + 2a\, e^{at}\frac{t}{\Gamma(2)}$$

$$\boxed{L^{-1}\left[\frac{s^2}{(s-a)^3}\right] = e^{at} + \frac{1}{2}a^2e^{at}t^2 + 2a\,t\,e^{at}}$$

II. Method of Finding Inverse Laplace transform by completing square

Working rule:

- If the given function of s, i.e., $F(s)$ is of the form $F(s)=\dfrac{\phi(s)}{ps^2+qs+r}$, then first express ps^2+qs+r to the form $(s-a)^2 \pm b^2$ and then express $\phi(s)$ in terms of $(s-a)$. Thus the given function reduces to a function of $(s-a)$.
- Using shifting rule of Inverse Laplace transform, i.e., $L^{-1}\left[F(s-a)\right] = e^{at}L^{-1}\left[F(s)\right] = e^{at}f(t)$, the Inverse Laplace transform of the given function can be found.

Problems:

Find the Inverse Laplace transform of the following functions

1. $\dfrac{3s}{s^2+2s-8}$

$$s^2+2s-8=s^2+2(s)(1)+1^2-1^2-8=(s+1)^2-9$$

$$L^{-1}\left[\frac{3s}{s^2+2s-8}\right]=L^{-1}\left[\frac{3s}{(s+1)^2-9}\right]=L^{-1}\left[\frac{3(s+1-1)}{(s+1)^2-9}\right]$$

$$=L^{-1}\left[\frac{3(s+1)-3}{(s+1)^2-9}\right]$$

$$=L^{-1}\left[\frac{3(s+1)}{(s+1)^2-9}-\frac{3}{(s+1)^2-9}\right]$$

$$=3L^{-1}\left[\frac{s+1}{(s+1)^2-9}\right]-3L^{-1}\left[\frac{1}{(s+1)^2-9}\right]$$

$$=3e^{-t}L^{-1}\left[\frac{s}{s^2-3^2}\right]-3e^{-t}L^{-1}\left[\frac{1}{s^2-3^2}\right]$$

$$\boxed{L^{-1}\left[\frac{3s}{s^2+2s-8}\right]=3e^{-t}\cosh 3t-e^{-t}\sinh 3t}$$

2. $\dfrac{2s-1}{s^2+4s+29}$

$$s^2+4s+29=s^2+2(s)(2)+2^2-2^2+29=(s+2)^2+25$$

$$L^{-1}\left[\frac{2s-1}{s^2+4s+29}\right]=L^{-1}\left[\frac{2s-1}{(s+2)^2+25}\right]=L^{-1}\left[\frac{2(s+2-2)-1}{(s+2)^2+5^2}\right]$$

$$=L^{-1}\left[\frac{2((s+2)-2)-1}{(s+2)^2+5^2}\right]$$

$$=L^{-1}\left[\frac{2(s+2)}{(s+2)^2+5^2}-\frac{4}{(s+2)^2+5^2}-\frac{1}{(s+2)^2+5^2}\right]$$

$$= 2L^{-1}\left[\frac{s+2}{(s+2)^2+5^2}\right] - 5L^{-1}\left[\frac{1}{(s+2)^2+5^2}\right]$$

$$= 2e^{-2t}L^{-1}\left[\frac{s}{s^2+5^2}\right] - 5e^{-2t}L^{-1}\left[\frac{1}{s^2+5^2}\right]$$

$$\boxed{L^{-1}\left[\frac{2s-1}{s^2+4s+29}\right] = 2e^{-2t}\cos 5t - e^{-2t}\sin 5t}$$

3. $\dfrac{7s+4}{4s^2+4s+9}$

$$4s^2+4s+9 = 4\left(s^2+s+\frac{9}{4}\right) = 4\left(s^2+2(s)\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+\frac{9}{4}\right)$$

$$= 4\left(\left(s+\frac{1}{2}\right)^2+\frac{8}{4}\right) = 4\left(\left(s+\frac{1}{2}\right)^2+2\right)$$

$$L^{-1}\left[\frac{7s+4}{4s^2+4s+9}\right] = L^{-1}\left[\frac{7s+4}{4\left(\left(s+\frac{1}{2}\right)^2+2\right)}\right] = \frac{1}{4}L^{-1}\left[\frac{7s+4}{\left(\left(s+\frac{1}{2}\right)^2+\left(\sqrt{2}\right)^2\right)}\right]$$

$$= \frac{1}{4}L^{-1}\left[\frac{7\left(s+1/2-1/2\right)+4}{\left(s+\frac{1}{2}\right)^2+\left(\sqrt{2}\right)^2}\right]$$

$$= \frac{1}{4}L^{-1}\left[\frac{7\left(\left(s+1/2\right)-1/2\right)+4}{\left(s+\frac{1}{2}\right)^2+\left(\sqrt{2}\right)^2}\right]$$

$$= \frac{1}{4} L^{-1}\left[\frac{7\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\sqrt{2}\right)^2} - \frac{\frac{7}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\sqrt{2}\right)^2} + \frac{4}{\left(s+\frac{1}{2}\right)^2+\left(\sqrt{2}\right)^2} \right]$$

$$= \frac{1}{4} L^{-1}\left[\frac{7\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\sqrt{2}\right)^2} + \frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\sqrt{2}\right)^2} \right]$$

$$= \frac{1}{4} \left\{ 7L^{-1}\left[\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\sqrt{2}\right)^2} \right] + \frac{1}{2} L^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\sqrt{2}\right)^2} \right] \right\}$$

$$= \frac{1}{4} \left\{ 7e^{-\frac{1}{2}t} L^{-1}\left[\frac{s}{s^2+\left(\sqrt{2}\right)^2} \right] + \frac{1}{2} e^{-\frac{1}{2}t} L^{-1}\left[\frac{1}{s^2+\left(\sqrt{2}\right)^2} \right] \right\}$$

$$\boxed{L^{-1}\left[\frac{7s+4}{4s^2+4s+9} \right] = \frac{1}{4}\left[7e^{-\frac{1}{2}t} \cos\sqrt{2}t + \frac{1}{2\sqrt{2}} e^{-\frac{1}{2}t} \sin\sqrt{2}t \right]}$$

4. $\dfrac{2s+1}{s^2+3s+1}$

$$s^2+3s+1 = s^2+2(s)\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+1$$

$$= \left(s+\frac{3}{2}\right)^2 - \frac{9}{4} + 1 = \left(s+\frac{3}{2}\right)^2 - \frac{5}{4}$$

$$L^{-1}\left[\frac{2s+1}{s^2+3s+1}\right]=L^{-1}\left[\frac{2s+1}{\left(s+\frac{3}{2}\right)^2-\frac{5}{4}}\right]=L^{-1}\left[\frac{2\left(s+\frac{3}{2}-\frac{3}{2}\right)+1}{\left(s+\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\right]$$

$$=L^{-1}\left[\frac{2\left(\left(s+\frac{3}{2}\right)-\frac{3}{2}\right)+1}{\left(s+\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\right]=L^{-1}\left[\frac{2\left(s+\frac{3}{2}\right)-2\left(\frac{3}{2}\right)+1}{\left(s+\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\right]$$

$$=2L^{-1}\left[\frac{s+\frac{3}{2}}{\left(s+\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\right]+L^{-1}\left[\frac{-2}{\left(s+\frac{3}{2}\right)^2-\frac{5}{4\left(\frac{\sqrt{5}}{2}\right)^2}}\right]$$

$$=2e^{-\frac{3}{2}t}L^{-1}\left[\frac{s}{s^2-\left(\frac{\sqrt{5}}{2}\right)^2}\right]-2e^{-\frac{3}{2}t}L^{-1}\left[\frac{1}{s^2-\left(\frac{\sqrt{5}}{2}\right)^2}\right]$$

$$=2e^{-\frac{3}{2}t}\cosh\left(\frac{\sqrt{5}}{2}t\right)-\frac{2e^{-\frac{3}{2}t}\sinh\left(\frac{\sqrt{5}}{2}t\right)}{\frac{\sqrt{5}}{2}}$$

$$\boxed{L^{-1}\left[\frac{2s+1}{s^2+3s+1}\right]=2e^{-\frac{3}{2}t}\cosh\left(\frac{\sqrt{5}}{2}t\right)-\frac{4}{\sqrt{5}}e^{-\frac{3}{2}t}\sinh\left(\frac{\sqrt{5}}{2}t\right)}$$

5. $\dfrac{s+5}{s^2-6s+13}$

$$s^2-6s+13=s^2-2(s)(3)+3^2-3^2+13=(s-3)^2-9+13$$

$$=(s-3)^2+4$$

$$L^{-1}\left[\frac{s+5}{s^2-6s+13}\right]=L^{-1}\left[\frac{s+5}{(s-3)^2+4}\right]=L^{-1}\left[\frac{s-3+3+5}{(s-3)^2+2^2}\right]$$

$$=L^{-1}\left[\frac{(s-3)+3+5}{(s-3)^2+2^2}\right]=L^{-1}\left[\frac{s-3}{(s-3)^2+2^2}\right]+L^{-1}\left[\frac{8}{(s-3)^2+2^2}\right]$$

$$=e^{3t}L^{-1}\left[\frac{s}{s^2+2^2}\right]+8e^{3t}L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$=e^{3t}\cos 2t+\frac{8e^{3t}\sin 2t}{2}$$

$$\boxed{L^{-1}\left[\frac{s+5}{s^2-6s+13}\right]=e^{3t}\cos 2t+4e^{3t}\sin 2t}$$

6. $\dfrac{s}{s^4+4a^4}$

$$s^4+4a^4=\left(s^2+2a^2\right)^2-4s^2a^2 \qquad \left(\because\ a^2+b^2=(a+b)^2-2ab\right)$$

$$s^4+4a^4=\left(s^2+2a^2\right)^2-(2sa)^2$$

$$L^{-1}\left[\frac{s}{s^4+4a^4}\right]=L^{-1}\left[\frac{s}{\left(s^2+2a^2\right)^2-(2sa)^2}\right]$$

$$=L^{-1}\left[\frac{s}{\left(s^2+2a^2+2sa\right)\left(s^2+2a^2-2sa\right)}\right]$$

Expressing Numerator in terms of denominator for further simplication

Consider $\left(s^2+2a^2+2sa\right)-\left(s^2+2a^2-2sa\right)=4sa$

$$\Rightarrow s=\frac{1}{4a}\left[\left(s^2+2a^2+2sa\right)-\left(s^2+2a^2-2sa\right)\right]$$

$$L^{-1}\left[\frac{s}{s^4+4a^4}\right]=L^{-1}\left[\frac{\frac{1}{4a}\left[\left(s^2+2a^2+2sa\right)-\left(s^2+2a^2-2sa\right)\right]}{\left(s^2+2a^2+2sa\right)\left(s^2+2a^2-2sa\right)}\right]$$

$$=\frac{1}{4a}\left[L^{-1}\left[\frac{\left(s^2+2a^2+2sa\right)-\left(s^2+2a^2-2sa\right)}{\left(s^2+2a^2+2sa\right)\left(s^2+2a^2-2sa\right)}\right]\right]$$

$$=\frac{1}{4a}\left[L^{-1}\left[\frac{1}{s^2+2a^2-2sa}-\frac{1}{s^2+2a^2+2sa}\right]\right]$$

$$=\frac{1}{4a}\left[L^{-1}\left[\frac{1}{\left(s^2+a^2-2sa\right)+a^2}-\frac{1}{\left(s^2+a^2+2sa\right)+a^2}\right]\right]$$

$$=\frac{1}{4a}\left[L^{-1}\left[\frac{1}{(s-a)^2+a^2}-\frac{1}{(s+a)^2+a^2}\right]\right]$$

$$=\frac{1}{4a}\left[L^{-1}\left[\frac{1}{(s-a)^2+a^2}\right]-L^{-1}\left[\frac{1}{(s+a)^2+a^2}\right]\right]$$

$$=\frac{1}{4a}\left[e^{at}L^{-1}\left[\frac{1}{s^2+a^2}\right]-e^{-at}L^{-1}\left[\frac{1}{s^2+a^2}\right]\right]$$

$$=\frac{1}{4a}\left[\frac{e^{at}\sin at}{a}-\frac{e^{-at}\sin at}{a}\right]$$

$$=\frac{\sin at}{4a^2}\left[e^{at}-e^{-at}\right]$$

$$=\frac{\sin at\sinh at}{2a^2}$$

7. $\dfrac{s}{s^4+s^2+1}$

$$s^4+s^2+1=\left(s^2+1\right)^2-2s^2+s^2 \qquad \left(\because\ a^2+b^2=(a+b)^2-2ab\right)$$

$$s^4+s^2+1=\left(s^2+1\right)^2-s^2$$

$$L^{-1}\left[\frac{s}{s^4+s^2+1}\right]=L^{-1}\left[\frac{s}{\left(s^2+1\right)^2-s^2}\right]$$

$$=L^{-1}\left[\frac{s}{\left(s^2+1+s\right)\left(s^2+1-s\right)}\right]$$

Expressing Numerator in terms of denominator for further simplication

Consider $\left(s^2+1+s\right)-\left(s^2+1-s\right)=2s$

$$\Rightarrow s=\frac{1}{2}\left[\left(s^2+1+s\right)-\left(s^2+1-s\right)\right]$$

$$L^{-1}\left[\frac{s}{s^4+s^2+1}\right]=L^{-1}\left[\frac{\frac{1}{2}\left[\left(s^2+1+s\right)-\left(s^2+1-s\right)\right]}{\left(s^2+1+s\right)\left(s^2+1-s\right)}\right]$$

$$=\frac{1}{2}L^{-1}\left[\frac{\left(s^2+1+s\right)-\left(s^2+1-s\right)}{\left(s^2+1+s\right)\left(s^2+1-s\right)}\right]$$

$$=\frac{1}{2}L^{-1}\left[\frac{1}{s^2+1-s}-\frac{1}{s^2+1+s}\right]$$

$$=\frac{1}{2}\left[L^{-1}\left[\frac{1}{s^2+1-s}\right]-L^{-1}\left[\frac{1}{s^2+1+s}\right]\right]$$

$$s^2+1-s=s^2-s+1=s^2-s+\left(\tfrac{1}{2}\right)^2-\left(\tfrac{1}{2}\right)^2+1=\left(s-\left(\tfrac{1}{2}\right)\right)^2-\tfrac{1}{4}+1=\left(s-\left(\tfrac{1}{2}\right)\right)^2+\tfrac{3}{4}$$

$$s^2+1+s=s^2+s+1=s^2+s+\left(\tfrac{1}{2}\right)^2-\left(\tfrac{1}{2}\right)^2+1=\left(s+\left(\tfrac{1}{2}\right)\right)^2-\tfrac{1}{4}+1=\left(s+\left(\tfrac{1}{2}\right)\right)^2+\tfrac{3}{4}$$

$$=\frac{1}{2}\left\{L^{-1}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right]-L^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]\right\}$$

$$=\frac{1}{2}\left\{L^{-1}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]-L^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]\right\}$$

$$=\frac{1}{2}\left\{e^{\frac{1}{2}t}L^{-1}\left[\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]-e^{-\frac{1}{2}t}L^{-1}\left[\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]\right\}$$

$$=\frac{1}{2}\left[\frac{e^{\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}}-\frac{e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}}\right]$$

$$=\frac{1}{2}.\frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\left(e^{\frac{1}{2}t}-e^{-\frac{1}{2}t}\right)$$

$$\boxed{L^{-1}\left[\frac{s}{s^4+s^2+1}\right]=\frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\sinh\left(\frac{1}{2}t\right)}$$

III. Method of Finding Inverse Laplace transform by using partial fractions

Working rule:

- If the given function of s, i.e., $F(s)$ is of the form $F(s)=\frac{\phi(s)}{\psi(s)}$ and if the degree of $\phi(s)$ is less than $\psi(s)$ then, use partial fractions to simplify the function as partial fractions converts the algebraic fraction into sum.

- Depending on the nature of $\psi(s)$, use the suitable following partial fraction to simplify
  - $\dfrac{1}{(s+a)(s+b)} = \dfrac{A}{s+a} + \dfrac{B}{s+b}$
  - $\dfrac{1}{(s+a)(s+b)^2} = \dfrac{A}{s+a} + \dfrac{B}{s+b} + \dfrac{C}{(s+b)^2}$
  - $\dfrac{1}{(s^2+a)(s^2+b)} = \dfrac{As+B}{s^2+a} + \dfrac{Cs+D}{s^2+b}$

- On simplifying, the function gets reduced to the standard form, for which Inverse Laplace transform can be transformed

Problems:

Find the Inverse Laplace transform of the following functions

1. $\dfrac{1}{s(s+1)(s+2)}$

By using Partial fractions

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Multiplying by $s(s+1)(s+2)$

$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$

Giving values for s and finding A, B , C

$s = 0 \Rightarrow$

$1 = A(0+1)(0+2) + B(0)(0+2) + C(0)(0+1)$

$1 = A(2) + B(0) + C(0)$

$\Rightarrow \boxed{A = \frac{1}{2}}$

$s = -1 \Rightarrow$

$1 = A(-1+1)(-1+2) + B(-1)(-1+2) + C(-1)(-1+1)$

$1 = A(0) + B(-1) + C(0)$

$\Rightarrow \boxed{B = -1}$

$s = -2 \Rightarrow$

$1 = A(-2+1)(-2+2) + B(-2)(-2+2) + C(-2)(-2+1)$

$1 = A(0) + B(0) + C(2)$

$\Rightarrow \boxed{C = \frac{1}{2}}$

We get $\frac{1}{s(s+1)(s+2)} = \frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{1}{2}(1) - e^{-t} + \frac{1}{2}e^{-2t}$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

2. $\frac{3s+2}{s^2 - s - 2}$

By using Partial fractions

$$\frac{3s+2}{s^2-s-2}=\frac{3s+2}{(s+1)(s-2)}=\frac{A}{s+1}+\frac{B}{s-2}$$

Multiplying by $(s+1)(s-2)$

$$3s+2=A(s-2)+B(s+1)$$

Giving values for s and finding A, B

$$s=-1\Rightarrow$$
$$3(-1)+2=A(-1-2)+B(-1+1)$$
$$-1=A(-3)+B(0)$$
$$\Rightarrow\boxed{A=\frac{1}{3}}$$
$$s=2\Rightarrow$$
$$3(2)+2=A(2-2)+B(2+1)$$
$$8=A(0)+B(3)$$
$$\Rightarrow\boxed{B=\frac{3}{8}}$$

We get $\dfrac{3s+2}{s^2-s-2}=\dfrac{3s+2}{(s+1)(s-2)}=\dfrac{1/3}{s+1}+\dfrac{3/8}{s-2}$

$$L^{-1}\left[\frac{3s+2}{s^2-s-2}\right]=L^{-1}\left[\frac{3s+2}{(s+1)(s-2)}\right]=\frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right]+\frac{3}{8}L^{-1}\left[\frac{1}{s-2}\right]$$
$$=\frac{1}{3}e^{-t}+\frac{3}{8}e^{2t}$$

3. $\dfrac{s+2}{s^2(s+3)}$

By using Partial fractions

$$\frac{s+2}{s^2(s+3)} = \frac{s+2}{s\,s(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

Multiplying by $s^2(s+3)$

$$s+2 = As(s+3) + B(s+3) + Cs^2$$

Giving values for s and finding A, B

$s = 0 \Rightarrow$

$$0+2 = A(0)(0+3) + B(0+3) + C(0)$$

$$2 = A(0) + B(3) + C(0)$$

$$\Rightarrow \boxed{B = \frac{2}{3}}$$

$s = -3 \Rightarrow$

$$-3+2 = A(-3)(-3+3) + B(-3+3) + C\left((-3)^2\right)$$

$$-1 = A(0) + B(0) + C(9)$$

$$\Rightarrow \boxed{C = \frac{-1}{9}}$$

$s = 1 \Rightarrow$

$$1+2 = A(1)(1+3) + B(1+3) + C(1^2)$$

$$3 = A(4) + \frac{2}{3}(4) + C(1)$$

$$3 = 4A + \frac{2}{3}(4) + \left(\frac{-1}{9}\right)(1)$$

$$3 = 4A + \frac{8}{3} - \frac{1}{9}$$

$$3 = 4A + \frac{23}{9}$$

$$A = \frac{1}{4}\left(3 - \frac{23}{9}\right)$$

$$\Rightarrow \boxed{A = \frac{1}{9}}$$

We get

$$\frac{s+2}{s^2(s+3)} = \frac{s+2}{s\,s(s+3)} = \frac{1/9}{s} + \frac{2/3}{s^2} + \frac{-1/9}{s+3}$$

$$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = L^{-1}\left[\frac{s+2}{s\,s(s+3)}\right] = \frac{1}{9}L^{-1}\left[\frac{1}{s}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{9}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{9}(1) + \frac{2}{3}\left(\frac{t}{\Gamma(2)}\right) - \frac{1}{9}e^{-3t}$$

$$= \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$

4. $\dfrac{s+2}{(s-3)(s+1)^2}$

By using Partial fractions

$$\frac{s+2}{(s-3)(s+1)^2} = \frac{A}{s-3} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

Multiplying by $(s-3)(s+1)^2$

$$s+2 = A(s+1)^2 + B(s-3)(s+1) + C(s-3)$$

Giving values for s and finding A, B, C

$s = 3 \Rightarrow$

$$3+2 = A(3+1)^2 + B(3-3)(3+1) + C(3-3)$$

$$5 = A(16) + B(0) + C(0)$$

$$\Rightarrow \boxed{A = \frac{5}{16}}$$

$s=-1 \Rightarrow$

$-1+2=A(-1+1)^2+B(-1-3)(-1+1)+C(-1-3)$

$1=A(0)+B(0)+C(-4)$

$\Rightarrow \boxed{C=-\frac{1}{4}}$

$s=0 \Rightarrow$

$0+2=A(0+1)^2+B(0-3)(0+1)+C(0-3)$

$2=\frac{5}{16}(1)+B(-3)+\left(-\frac{1}{4}\right)(0-3)$

$2=\frac{5}{16}-3B+\frac{3}{4}$

$\Rightarrow \boxed{B=-\frac{5}{16}}$

We get $\frac{s+2}{(s-3)(s+1)^2}=\frac{5/16}{s-3}+\frac{-5/16}{s+1}+\frac{-1/4}{(s+1)^2}$

$$L^{-1}\left[\frac{s+2}{(s-3)(s+1)^2}\right]=\frac{5}{16}L^{-1}\left[\frac{1}{s-3}\right]-\frac{5}{16}L^{-1}\left[\frac{1}{s+1}\right]-\frac{1}{4}L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$=\frac{5}{16}e^{3t}-\frac{5}{16}e^{-t}-\frac{1}{4}e^{-t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$=\frac{5}{16}e^{3t}-\frac{5}{16}e^{-t}-\frac{1}{4}e^{-t}\frac{t}{\Gamma(2)}$$

$$=\frac{5}{16}e^{3t}-\frac{5}{16}e^{-t}-\frac{1}{4}te^{-t}$$

5. $\frac{3s-1}{(s-3)(s^2+4)}$

By using Partial fractions

$$\frac{3s-1}{(s-3)\left(s^2+4\right)}=\frac{A}{s-3}+\frac{Bs+C}{s^2+4}$$

Multiplying by $(s-3)\left(s^2+4\right)$

$$3s-1=A\left(s^2+4\right)+(Bs+C)(s-3)$$

To find the Values of A, B, C

$$3s-1=As^2+4A+Bs^2-3Bs+Cs-3C$$

Equating corresponding coefficients

$$\Rightarrow A+B=0 \qquad -3B+C=3 \qquad 4A-3C=-1$$

$$\Rightarrow A=\frac{8}{13} \qquad B=-\frac{8}{13} \qquad C=\frac{15}{13}$$

We get $\dfrac{3s-1}{(s-3)\left(s^2+4\right)}=\dfrac{8/13}{s-3}+\dfrac{\left(-8/13\right)s+15/13}{s^2+4}$

$$=\frac{8/13}{s-3}+\frac{\left(-8/13\right)s}{s^2+4}+\frac{15/13}{s^2+4}$$

$$L^{-1}\left[\frac{3s-1}{(s-3)\left(s^2+4\right)}\right]=\frac{8}{13}L^{-1}\left[\frac{1}{s-3}\right]-\frac{8}{13}L^{-1}\left[\frac{s}{s^2+4}\right]-\frac{15}{13}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$=\frac{8}{13}e^{3t}-\frac{8}{13}\cos 2t-\frac{15}{26}\sin 2t$$

6. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

By using Partial fractions

$$\frac{5s+3}{(s-1)(s^2+2s+5)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+2s+5}$$

Multiplying by $(s-1)(s^2+2s+5)$

$$5s+3=A(s^2+2s+5)+(Bs+C)(s-1)$$

To find the Values of A, B, C

$$5s+3=As^2+2As+5A+Bs^2-Bs+Cs-C$$

Equating corresponding coefficients

$$\Rightarrow A+B=0 \qquad 2A-B+C=5 \qquad 5A-C=3$$
$$\Rightarrow A=1 \qquad B=-1 \qquad C=2$$

We get $\dfrac{5s+3}{(s-1)(s^2+2s+5)}=\dfrac{1}{s-1}+\dfrac{(-1)s+2}{s^2+2s+5}$

$$=\frac{1}{s-1}+\frac{2-s}{s^2+2s+5}$$

$$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right]=L^{-1}\left[\frac{1}{s-1}\right]+L^{-1}\left[\frac{2-s}{s^2+2s+5}\right]$$

$$=e^t+L^{-1}\left[\frac{2-s}{(s+1)^2+4}\right]$$

$$= e^{t} + L^{-1}\left[\frac{2-(s+1-1)}{(s+1)^2+4}\right]$$

$$= e^{t} + L^{-1}\left[\frac{2-((s+1)-1)}{(s+1)^2+4}\right]$$

$$= e^{t} + L^{-1}\left[\frac{2-(s+1)+1}{(s+1)^2+4}\right]$$

$$= e^{t} + L^{-1}\left[\frac{3}{(s+1)^2+2^2}\right] - L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right]$$

$$= e^{t} + \frac{3}{2}e^{-t}\sin 2t - e^{-t}\cos 2t$$

7. $\dfrac{s^2}{(s^2+1)(s^2+4)}$

Put $s^2 = t$

$$\frac{s^2}{(s^2+1)(s^2+4)} = \frac{t}{(t+1)(t+4)}$$

By using Partial fractions

$$\frac{t}{(t+1)(t+4)} = \frac{A}{t+1} + \frac{B}{t+4}$$

Multiplying by $(t+1)(t+4)$

$$t = A(t+4) + B(t+1)$$

To find the Values of A, B

$$t = At + 4A + Bt + B$$

Equating corresponding coefficients

$$\Rightarrow A+B=1 \qquad 4A+B=0$$

$$\Rightarrow A=-\frac{1}{3} \qquad B=\frac{4}{3}$$

We get $$\frac{s^2}{\left(s^2+1\right)\left(s^2+4\right)}=\frac{t}{(t+1)(t+4)}=\frac{-1/3}{t+1}+\frac{4/3}{t+4}$$

$$\frac{s^2}{\left(s^2+1\right)\left(s^2+4\right)}=-\frac{1}{3}\left[\frac{1}{s^2+1}\right]+\frac{4}{3}\left[\frac{1}{s^2+4}\right]$$

$$L^{-1}\left[\frac{s^2}{\left(s^2+1\right)\left(s^2+4\right)}\right]=-\frac{1}{3}L^{-1}\left[\frac{1}{s^2+1}\right]+\frac{4}{3}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$=-\frac{1}{3}\sin t+\frac{2}{3}\sin 2t$$

IV. Method to Finding Inverse Laplace transform Inverse trig nometric functions like $\tan^{-1}$, $\cot^{-1}$ and Log function

Working rule:

Problems:

Find the Inverse Laplace transform of the following functions

1. $\log\left(\frac{s+a}{s+b}\right)$

Let $F(s)=\log\left(\frac{s+a}{s+b}\right)=\log(s+a)-\log(s+b)$

Differentiating w r t $'s'$

$$F'(s)=\frac{1}{s+a}-\frac{1}{s+b}$$

$$L^{-1}\left[F'(s)\right]=L^{-1}\left[\frac{1}{s+a}\right]-L^{-1}\left[\frac{1}{s+b}\right]$$

$$-t\,f(t)=e^{-at}-e^{-bt}$$

$$f(t)=\frac{e^{-bt}-e^{-at}}{t}$$

2. $\log\left(\dfrac{s^2+1}{s(s+1)}\right)$

Let $\quad F(s)=\log\left(\dfrac{s^2+1}{s(s+1)}\right)=\log(s^2+1)-\log\left(s(s+1)\right)$

$$F(s)=\log(s^2+1)-\left[\log(s)+\log(s+1)\right]$$

$$F(s)=\log(s^2+1)-\log(s)-\log(s+1)$$

Differentiating w r t 's'

$$F'(s)=\frac{2s}{s^2+1}-\frac{1}{s}-\frac{1}{s+1}$$

$$L^{-1}\left[F'(s)\right]=2L^{-1}\left[\frac{s}{s^2+1}\right]-L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{1}{s+1}\right]$$

$$-t\,f(t)=2\cos t-1-e^{-t}$$

$$f(t)=\frac{1+e^{-t}-2\cos t}{t}$$

3. $\log\left(\dfrac{s^2+4}{(s-4)^2}\right)$

Let $\quad F(s)=\log\left(\dfrac{s^2+4}{(s-4)^2}\right)=\log(s^2+4)-\log\left((s-4)^2\right)$

$$F(s) = \log(s^2+4) - 2\log(s-4)$$

Differentiating w r t $'s'$

$$F^{'}(s) = \frac{2s}{s^2+4} - \frac{2}{s-4}$$

$$L^{-1}\left[F^{'}(s)\right] = 2L^{-1}\left[\frac{s}{s^2+4}\right] - 2L^{-1}\left[\frac{1}{s-4}\right]$$

$$-t\, f(t) = 2\cos 2t - 2e^{4t}$$

$$f(t) = \frac{e^{4t} - 2\cos 2t}{t}$$

4. $s\log\left(\frac{s-1}{s+1}\right)$

Let $F(s) = s\log\left(\frac{s-1}{s+1}\right) = s\left[\log(s-1) - \log(s+1)\right]$

$$F(s) = s\log(s-1) - s\log(s+1)$$

Differentiating w r t $'s'$

$$F^{'}(s) = \left[\frac{s}{s-1} + \log(s-1)\right] - \left[\frac{s}{s+1} + \log(s+1)\right]$$

$$F^{'}(s) = \frac{s}{s-1} + \log(s-1) - \frac{s}{s+1} - \log(s+1)$$

Differentiating w r t $'s'$

$$F^{''}(s) = \left[\frac{(s-1)(1) - s(1)}{(s-1)^2}\right] + \frac{1}{s-1} - \left[\frac{(s+1)(1) - s(1)}{(s+1)^2}\right] - \frac{1}{s+1}$$

$$F^{''}(s) = -\frac{1}{(s-1)^2} + \frac{1}{s-1} - \frac{1}{(s+1)^2} - \frac{1}{s+1}$$

$$L^{-1}\left[F''(s)\right] = -L^{-1}\left[\frac{1}{(s-1)^2}\right] + L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{(s+1)^2}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$t^2 f(t) = -e^t L^{-1}\left[\frac{1}{s^2}\right] + e^t - e^{-t} L^{-1}\left[\frac{1}{s^2}\right] - e^{-t}$$

$$t^2 f(t) = -e^t t + e^t - e^{-t} t - e^{-t}$$

$$f(t) = \frac{e^t t - e^t + e^{-t} t + e^{-t}}{t^2}$$

5. $\log\left(\frac{1}{s^2} - 1\right)$

Let $F(s) = \log\left(\frac{1}{s^2} - 1\right) = \log\left(\frac{1-s^2}{s^2}\right)$

$$F(s) = \log\left(1-s^2\right) - \log\left(s^2\right)$$

$$F(s) = \log\left(1-s^2\right) - 2\log(s)$$

Differentiating w r t $'s'$

$$F'(s) = \frac{-2s}{1-s^2} - \frac{2}{s}$$

$$F'(s) = \frac{2s}{s^2-1} - \frac{2}{s}$$

$$L^{-1}\left[F'(s)\right] = 2L^{-1}\left[\frac{s}{s^2-1}\right] - 2L^{-1}\left[\frac{1}{s}\right]$$

$$-t\, f(t) = 2\cosh t - 2$$

$$f(t) = \frac{2 - 2\cosh t}{t}$$

6. $\tan^{-1}\left(\frac{a}{s}\right)$

Let $\quad F(s)=\tan^{-1}\left(\frac{a}{s}\right)$

Differentiating w r t $'s'$

$$F^{'}(s)=\frac{1}{1+\left(a/s\right)^{2}}\left(-\frac{a}{s^{2}}\right)$$

$$F^{'}(s)=\frac{-a}{s^{2}+a^{2}}$$

$$L^{-1}\left[F^{'}(s)\right]=L^{-1}\left[\frac{-a}{s^{2}+a^{2}}\right]$$

$$L^{-1}\left[F^{'}(s)\right]=-L^{-1}\left[\frac{a}{s^{2}+a^{2}}\right]$$

$$-t\,f(t)=-\sin at$$

$$f(t)=\frac{\sin at}{t}$$

7. $\cot^{-1}\left(\frac{s}{2}\right)$

Let $\quad F(s)=\cot^{-1}\left(\frac{s}{2}\right)$

Differentiating w r t $'s'$

$$F^{'}(s)=\frac{-1}{1+\left(s/2\right)^{2}}\left(\frac{1}{2}\right)$$

$$F^{'}(s)=\frac{-2}{s^{2}+4}$$

$$L^{-1}\left[F^{'}(s)\right]=L^{-1}\left[\frac{-2}{s^{2}+4}\right]$$

$$L^{-1}\left[F'(s)\right] = -L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$-t\,f(t) = -\sin 2t$$

$$f(t) = \frac{\sin 2t}{t}$$

8. $\tan^{-1}\left(\frac{2}{s^2}\right)$

Let $F(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$

Differentiating w r t $'s'$

$$F'(s) = \frac{1}{1+\left(\frac{2}{s^2}\right)^2}\left(-\frac{4}{s^3}\right)$$

$$F'(s) = \frac{-4s}{s^4+4}$$

$$L^{-1}\left[F'(s)\right] = L^{-1}\left[\frac{-4s}{s^4+4}\right]$$

Using completing square method

$$s^4+4 = \left(s^2\right)^2 + 2^2 = \left(s^2+2\right)^2 - 2\left(s^2\right)(2)$$

$$s^4+4 = \left(s^2+2\right)^2 - 4s^2$$

$$s^4+4 = \left(s^2+2\right)^2 - (2s)^2$$

$$L^{-1}\left[F'(s)\right] = L^{-1}\left[\frac{-4s}{\left(s^2+2\right)^2-(2s)^2}\right]$$

$$L^{-1}\left[F'(s)\right]=L^{-1}\left[\frac{-4s}{\left(s^2+2+2s\right)\left(s^2+2-2s\right)}\right]$$

$$L^{-1}\left[F'(s)\right]=-L^{-1}\left[\frac{\left(s^2+2+2s\right)-\left(s^2+2-2s\right)}{\left(s^2+2+2s\right)\left(s^2+2-2s\right)}\right]$$

$$L^{-1}\left[F'(s)\right]=-L^{-1}\left[\frac{1}{s^2+2-2s}-\frac{1}{s^2+2+2s}\right]$$

$$L^{-1}\left[F'(s)\right]=L^{-1}\left[\frac{-1}{s^2+2-2s}\right]+L^{-1}\left[\frac{1}{s^2+2+2s}\right]$$

$$L^{-1}\left[F'(s)\right]=L^{-1}\left[\frac{-1}{(s-1)^2+1}\right]+L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$-t\,f(t)=-e^t\sin t+e^{-t}\sin t$$

$$f(t)=\frac{\sin t\left(e^t-e^{-t}\right)}{t}$$

$$f(t)=\frac{\sin t\sinh t}{t}$$

V. Method to Finding Inverse Laplace transform involving Exponential terms

If $L^{-1}\left[F(s)\right]=f(t)$ then

i.e., $L^{-1}\left[e^{-as}F(s)\right]=u(t-a)\,f(t-a)$

This is called as Inverse Heaviside shift theorem (Inverse of second shifting property).

Problems:

Find the Inverse Laplace transform of the following functions

1. $\dfrac{e^{-2s}}{s^2}$

$$L^{-1}\left[\frac{e^{-2s}}{s^2}\right]=L^{-1}\left[e^{-2s}\left(\frac{1}{s^2}\right)\right]$$

$$=u(t-2)\left[L^{-1}\left(\frac{1}{s^2}\right)\right]_{t\to t-2}$$

$$=u(t-2)(t)_{t\to t-2}$$

$$L^{-1}\left[\frac{e^{-2s}}{s^2}\right]==u(t-2)(t-2)$$

2. $\dfrac{e^{-3s}}{s^2+1}+\dfrac{se^{-4s}}{s^2+4}$

$$L^{-1}\left[\frac{e^{-3s}}{s^2+1}+\frac{se^{-4s}}{s^2+4}\right]=L^{-1}\left[e^{-3s}\left(\frac{1}{s^2+1}\right)+e^{-4s}\left(\frac{s}{s^2+4}\right)\right]$$

$$=L^{-1}\left[e^{-3s}\left(\frac{1}{s^2+1}\right)\right]+L^{-1}\left[e^{-4s}\left(\frac{s}{s^2+4}\right)\right]$$

$$=u(t-3)\left[L^{-1}\left(\frac{1}{s^2+1}\right)\right]_{t\to t-3}+u(t-4)\left[L^{-1}\left(\frac{s}{s^2+4}\right)\right]_{t\to t-4}$$

$$=u(t-3)(\sin t)_{t\to t-3}+u(t-4)(\cos 2t)_{t\to t-4}$$

$$L^{-1}\left[\frac{e^{-3s}}{s^2+1}+\frac{se^{-4s}}{s^2+4}\right]=u(t-3)\sin(t-3)+u(t-4)\cos 2(t-4)$$

3. $\dfrac{1+e^{-3s}}{s^2}$

$$L^{-1}\left[\frac{1+e^{-3s}}{s^2}\right]=L^{-1}\left[\frac{1}{s^2}+\frac{e^{-3s}}{s^2}\right]$$

$$=L^{-1}\left[\frac{1}{s^2}\right]+\left[L^{-1}\left(e^{-3s}\frac{1}{s^2}\right)\right]$$

$$=t+u(t-3)\left[L^{-1}\left(\frac{1}{s^2}\right)\right]_{t\to t-3}$$

$$=t+u(t-3)(t)_{t\to t-3}$$

$$L^{-1}\left[\frac{1+e^{-3s}}{s^2}\right]=t+u(t-3)(t-3)$$

4. $\dfrac{e^{-s}}{(s-4)^2}$

$$L^{-1}\left[\frac{e^{-s}}{(s-4)^2}\right]=L^{-1}\left(e^{-s}\frac{1}{(s-4)^2}\right)$$

$$=u(t-1)\left[L^{-1}\left(\frac{1}{(s-4)^2}\right)\right]_{t\to t-1}$$

$$=u(t-1)\left[e^{4t}L^{-1}\left(\frac{1}{s^2}\right)\right]_{t\to t-1}$$

$$=u(t-1)\left[e^{4t}t\right]_{t\to t-1}$$

$$L^{-1}\left[\frac{e^{-s}}{(s-4)^2}\right]=u(t-1)\ e^{4(t-1)}(t-1)$$

5. $\dfrac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$

$$L^{-1}\left[\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right]=L^{-1}\left(\frac{se^{-s/2}}{s^2+\pi^2}+\frac{\pi e^{-s}}{s^2+\pi^2}\right)$$

$$=L^{-1}\left(\frac{se^{-s/2}}{s^2+\pi^2}\right)+L^{-1}\left(\frac{\pi e^{-s}}{s^2+\pi^2}\right)$$

$$=L^{-1}\left(e^{-s/2}\left(\frac{s}{s^2+\pi^2}\right)\right)+L^{-1}\left(e^{-s}\left(\frac{\pi}{s^2+\pi^2}\right)\right)$$

$$=u\left(t-\frac{1}{2}\right)\left[L^{-1}\left(\frac{s}{s^2+\pi^2}\right)\right]_{t\to t-1/2}+u(t-1)\left[L^{-1}\left(\frac{\pi}{s^2+\pi^2}\right)\right]_{t\to t-1}$$

$$=u\left(t-\frac{1}{2}\right)[\cos \pi t]_{t\to t-1/2}+u(t-1)[\sin \pi t]_{t\to t-1}$$

$$L^{-1}\left[\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right]==u\left(t-\frac{1}{2}\right)\cos\pi\left(t-1/2\right)+u(t-1)\sin\pi(t-1)$$

6. $\dfrac{\left(1-e^{-s}\right)\left(2-e^{-2s}\right)}{s^3}$

$$L^{-1}\left[\frac{\left(1-e^{-s}\right)\left(2-e^{-2s}\right)}{s^3}\right]=L^{-1}\left(\frac{2-e^{-2s}-2e^{-s}+e^{-3s}}{s^3}\right)$$

$$=L^{-1}\left(\frac{2}{s^3}\right)-L^{-1}\left(\frac{e^{-2s}}{s^3}\right)-L^{-1}\left(\frac{2e^{-s}}{s^3}\right)+L^{-1}\left(\frac{e^{-3s}}{s^3}\right)$$

$$=2L^{-1}\left(\frac{1}{s^3}\right)-L^{-1}\left(e^{-2s}\left(\frac{1}{s^3}\right)\right)-2L^{-1}\left(e^{-s}\left(\frac{1}{s^3}\right)\right)+L^{-1}\left(e^{-3s}\left(\frac{1}{s^3}\right)\right)$$

$$=2\frac{t^2}{\Gamma(3)}-u(t-2)\left[L^{-1}\left(\frac{1}{s^3}\right)\right]_{t\to t-2}$$
$$-2\,u(t-1)\left[L^{-1}\left(\frac{1}{s^3}\right)\right]_{t\to t-1}+u(t-3)\left[L^{-1}\left(\frac{1}{s^3}\right)\right]_{t\to t-3}$$

$$=2\frac{t^2}{2}-u(t-2)\left[\frac{t^2}{2}\right]_{t\to t-2}-2\,u(t-1)\left[\frac{t^2}{2}\right]_{t\to t-1}+u(t-3)\left[\frac{t^2}{2}\right]_{t\to t-3}$$

$$=t^2-\frac{u(t-2)(t-2)^2}{2}-u(t-1)(t-1)^2+\frac{u(t-3)(t-3)^2}{2}$$

CONVOLUTION THEOREM

Statement

If $L^{-1}\left[F(s)\right]=f(t)$ and $L^{-1}\left[G(s)\right]=g(t)$ then

$$L^{-1}\left[F(s)\,G(s)\right]=\int_0^t f(u)\,g(t-u)\,du=\int_0^t f(t-u)\,g(u)\,du$$

Working rule to find Inverse laplace transform using Convolution theorem

- Consider the given function in the form $F(s)\,G(s)$ i.e., Split the given function into two functions as $F(s)$ and $G(s)$, provided their Inverse exits.
- Find $L^{-1}\left[F(s)\right]=f(t)$ and $L^{-1}\left[G(s)\right]=g(t)$
- Substituting in Convolution theorem $L^{-1}\left[F(s)\,G(s)\right]=\int_0^t f(u)\,g(t-u)\,du$ **or**

  $L^{-1}\left[F(s)\,G(s)\right]=\int_0^t f(t-u)\,g(u)\,du$ and simplifying,

  we get $L^{-1}\left[F(s)\,G(s)\right]$.

Problems

Find the Inverse Laplace transform of the following functions using Convolution theorem

1. $\dfrac{1}{s\left(s^2+4\right)}$

Let $\dfrac{1}{s\left(s^2+4\right)}=F(s)\,G(s)$

$$\Rightarrow F(s)=\frac{1}{s} \quad G(s)=\frac{1}{s^2+4}$$

$$\Rightarrow L^{-1}\left[F(s)\right]=1 \quad L^{-1}\left[G(s)\right]=\frac{1}{2}\sin 2t$$

$$\Rightarrow f(t)=1 \qquad g(t)=\frac{1}{2}\sin 2t$$

By convolution theorem,

$$L^{-1}\left[F(s)\,G(s)\right]=\int_0^t f(t-u)\,g(u)\,du$$

$$L^{-1}\left[\frac{1}{s\left(s^2+4\right)}\right]=\int_0^t (1)\left(\frac{1}{2}\sin 2u\right)du$$

$$=\int_0^t (1)\left(\frac{1}{2}\sin 2u\right)du$$

$$=\frac{1}{2}\int_0^t \sin 2u\,du \;=\frac{1}{2}\left(\frac{-\cos 2u}{2}\right)_{u=0}^{t}$$

$$=-\frac{1}{4}\left(\cos 2t-\cos 0\right)=-\frac{1}{4}\left(\cos 2t-1\right)$$

$$L^{-1}\left[\frac{1}{s\left(s^2+4\right)}\right]=\frac{1}{4}\left(1-\cos 2t\right)$$

2. $\dfrac{1}{s^3\left(s^2+1\right)}$

Let $\dfrac{1}{s^3\left(s^2+1\right)}=F(s)\,G(s)$

$$\Rightarrow F(s)=\frac{1}{s^3} \quad G(s)=\frac{1}{s^2+1}$$

$$\Rightarrow L^{-1}\left[F(s)\right]=\frac{t^2}{2} \qquad L^{-1}\left[G(s)\right]=\sin t$$

$$\Rightarrow f(t)=\frac{t^2}{2} \qquad\qquad g(t)=\sin t$$

By convolution theorem,

$$L^{-1}\left[F(s)\,G(s)\right]=\int_0^t f(t-u)\,g(u)\,du$$

$$L^{-1}\left[\frac{1}{s^3\left(s^2+1\right)}\right]=\int_0^t \frac{(t-u)^2\sin u}{2}\,du$$

Using Bernoulli rule,

$$=\frac{1}{2}\Big[(t-u)^2(-\cos u)-\big(2(t-u)\big)(-\sin u)+(2)(\cos u)\Big]_{u=0}^{t}$$

$$=\frac{1}{2}\Big[\big(0-(t-0)^2(-\cos 0)\big)-(0-0)+(2)(\cos t-\cos 0)\Big]$$

$$=\frac{1}{2}\left[t^2+2\cos t-2\right]$$

3. $\dfrac{1}{s^2(s+1)^2}$

Let $\dfrac{1}{s^2(s+1)^2} = F(s)\,G(s)$

$$\Rightarrow F(s) = \frac{1}{s^2} \quad G(s) = \frac{1}{(s+1)^2}$$

$$\Rightarrow L^{-1}[F(s)] = t \quad L^{-1}[G(s)] = e^{-t}t$$

$$\Rightarrow f(t) = t \quad g(t) = e^{-t}\,t$$

By convolution theorem,

$$L^{-1}[F(s)\,G(s)] = \int_0^t f(t-u)\,g(u)\,du$$

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \int_0^t (t-u)e^{-u}u\ du$$

$$= \int_0^t \left(ut-u^2\right)e^{-u}\ du$$

Using Bernoulli rule,

$$= \left[\left(ut-u^2\right)\left(\frac{e^{-u}}{-1}\right) - (t-2u)\left(\frac{e^{-u}}{(-1)(-1)}\right) + (-2)\left(\frac{e^{-u}}{(-1)(-1)(-1)}\right)\right]_{u=0}^{t}$$

$$= 0 - \left(-te^{-t} - t\right) + 2\left(e^{-t} - 1\right)$$

$$= te^{-t} + t + 2e^{-t} - 2$$

4. $\dfrac{1}{(s+1)\left(s^2+9\right)}$

Let $\dfrac{1}{(s+1)(s^2+9)} = F(s)\,G(s)$

$$\Rightarrow\ F(s) = \frac{1}{s+1} \quad G(s) = \frac{1}{s^2+9}$$

$$\Rightarrow\ L^{-1}\left[F(s)\right] = e^{-t} \qquad L^{-1}\left[G(s)\right] = \frac{1}{3}\sin 3t$$

$$\Rightarrow\ f(t) = e^{-t} \qquad\qquad g(t) = \frac{1}{3}\sin 3t$$

By convolution theorem,

$$L^{-1}\left[F(s)\,G(s)\right] = \int_0^t f(t-u)\,g(u)\,du$$

$$L^{-1}\left[\frac{1}{(s+1)(s^2+9)}\right] = \int_0^t e^{-(t-u)}\,\frac{1}{3}\sin 3u\,du$$

$$= \frac{1}{3}\int_0^t e^{-t+u}\,\sin 3u\,du$$

$$= \frac{1}{3}\int_0^t e^{-t}e^{u}\,\sin 3u\,du$$

$$= \frac{e^{-t}}{3}\int_0^t e^{u}\,\sin 3u\,du$$

We have $\int e^{at}\sin bt\,dt = \dfrac{e^{at}}{a^2+b^2}\left(a\sin bt - b\cos bt\right)$

$$L^{-1}\left[\frac{1}{(s+1)(s^2+9)}\right] = \frac{e^{-t}}{3}\left[\frac{e^{u}}{1+9}\left(\sin 3u - 3\cos 3u\right)\right]_{u=0}^{t}$$

$$= \frac{e^{-t}}{30}\left[e^{t}\left(\sin 3t - 3\cos 3t\right) - \left(e^{0}\left(\sin 0 - 3\cos 0\right)\right)\right]$$

$$= \frac{e^{-t}}{30}\left[e^{t}\left(\sin 3t - 3\cos 3t\right) + 3\right]$$

$$= \frac{\sin 3t - 3\cos 3t + 3e^{-t}}{30}$$

5. $\dfrac{s}{(s-1)\left(s^2+4\right)}$

Let $\dfrac{s}{(s-1)\left(s^2+4\right)} = F(s)\,G(s)$

$$\Rightarrow F(s) = \frac{1}{s-1} \quad G(s) = \frac{s}{s^2+4}$$

$$\Rightarrow L^{-1}\left[F(s)\right] = e^{t} \qquad L^{-1}\left[G(s)\right] = \cos 2t$$

$$\Rightarrow f(t) = e^{t} \qquad\qquad g(t) = \cos 2t$$

By convolution theorem,

$$L^{-1}\left[F(s)\,G(s)\right] = \int_0^t f(t-u)\,g(u)\,du$$

$$L^{-1}\left[\frac{s}{(s-1)\left(s^2+4\right)}\right] = \int_0^t e^{(t-u)}\cos 2u\,du$$

$$= \int_0^t e^{t-u}\cos 2u\,du$$

$$= \int_0^t e^{t}e^{-u}\cos 2u\,du$$

$$= e^t \int_0^t e^{-u} \cos 2u \, du$$

We have $\int e^{at} \cos bt \, dt = \dfrac{e^{at}}{a^2+b^2}\left(a\cos bt + b\sin bt\right)$

$$= e^t \int_0^t e^{-u} \cos 2u \, du$$

$$L^{-1}\left[\frac{s}{(s-1)\left(s^2+4\right)}\right] = e^t\left[\frac{e^{-u}}{1+4}\left(-\cos 2u + 2\sin 2u\right)\right]_{u=0}^{t}$$

$$= \frac{e^t}{5}\left[e^{-t}\left(-\cos 2t + 2\sin 2t\right) - \left(e^0\left(-\cos 0 + 2\sin 0\right)\right)\right]$$

$$= \frac{e^t}{5}\left[e^{-t}\left(-\cos 2t + 2\sin 2t\right) + 1\right]$$

$$= \frac{2\sin 2t - \cos 2t + e^t}{5}$$

6. $\dfrac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}$

Let $\dfrac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)} = F(s)\,G(s)$

$$\Rightarrow F(s) = \frac{s}{s^2+a^2} \qquad G(s) = \frac{s}{s^2+b^2}$$

$$\Rightarrow L^{-1}\left[F(s)\right] = \cos at \qquad L^{-1}\left[G(s)\right] = \cos bt$$

$$\Rightarrow f(t) = \cos at \qquad g(t) = \cos bt$$

By convolution theorem,

$$L^{-1}\left[F(s)\,G(s)\right]=\int_0^t f(u)\,g(t-u)\,du$$

$$L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]=\int_0^t \cos au\ \cos b(t-u)\,du$$

We have $\cos A \cos B \;=\frac{1}{2}\left(\cos(A+B)+\cos(A-B)\right)$

$$L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]=\int_0^t\left[\frac{1}{2}\left(\cos\left(au+bt-bu\right)+\cos\left(au-bt+bu\right)\right)\right]du$$

$$=\int_0^t\left[\frac{1}{2}\left(\cos\left((a-b)u+bt\right)+\cos\left((a+b)u-bt\right)\right)\right]du$$

$$=\frac{1}{2}\left[\frac{\sin\left((a-b)u+bt\right)}{(a-b)}+\frac{\sin\left((a+b)u-bt\right)}{(a+b)}\right]_{u=0}^{t}$$

$$=\frac{1}{2}\left[\frac{\sin at-\sin bt}{(a-b)}+\frac{\sin at-\sin(-bt)}{(a+b)}\right]$$

$$=\frac{1}{2}\left[\frac{(a+b)(\sin at-\sin bt)+(a-b)(\sin at+\sin bt))}{(a-b)(a+b)}\right]$$

$$=\frac{1}{2}\left[\frac{2a\sin at-2b\sin bt}{\left(a^2-b^2\right)}\right]$$

$$=\frac{a\sin at-b\sin bt}{\left(a^2-b^2\right)}$$

7. $\dfrac{s}{\left(s^2+a^2\right)^2}$

Let $\dfrac{s}{\left(s^2+a^2\right)^2}=F(s)\,G(s)$

$$\Rightarrow F(s)=\frac{1}{s^2+a^2} \quad G(s)=\frac{s}{s^2+a^2}$$

$$\Rightarrow L^{-1}\left[F(s)\right]=\frac{1}{a}\sin at \qquad L^{-1}\left[G(s)\right]=\cos at$$

$$\Rightarrow f(t)=\frac{1}{a}\sin at \qquad\qquad g(t)=\cos at$$

By convolution theorem,

$$L^{-1}\left[F(s)\,G(s)\right]=\int_0^t f(u)\,g(t-u)\,du$$

$$L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=\int_0^t \frac{1}{a}\sin au\ \cos a(t-u)\,du$$

We have $\sin A\ \cos B=\frac{1}{2}\left(\sin(A+B)+\sin(A-B)\right)$

$$L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=\frac{1}{a}\int_0^t\left(\frac{1}{2}\left(\sin\left(au+at-au\right)+\sin\left(au-at+au\right)\right)\right)du$$

$$=\frac{1}{2a}\int_0^t\left(\left(\sin at+\sin\left(2au-at\right)\right)\right)du$$

$$=\frac{1}{2a}\left[u\sin at+\frac{-\cos\left(2au-at\right)}{2a}\right]_{u=0}^{t}$$

$$=\frac{1}{2a}\left[(t-0)\sin at-\frac{1}{2a}\left(\cos at-\cos at\right)\right]$$

$$=\frac{t\sin at}{2a}$$

SOLUTION OF DIFFERENTIAL EQUATION USING LAPLACE TRANSFORM

Laplace transform of a Derivatives of function of $'t'$ i.e., $y(t)$

By definition of Laplace transform, $L\left[f(t)\right]=\int_0^\infty e^{-st}f(t)\,dt$

$$L\left[y'(t)\right]=\int_0^\infty e^{-st}y'(t)\,dt$$

$$=\left(e^{-st}y(t)\right)_0^\infty-\int_0^\infty y(t)\left(-s\,e^{-st}\right)dt$$

$$=\left(0-1.y(0)\right)+s\int_0^\infty e^{-st}y(t)\,dt$$

$$\boxed{L\left[y'(t)\right]=s\,L\left[y(t)\right]-y(0)}$$

Similarly, $$L\left[y''(t)\right]=s^2\,L\left[y(t)\right]-s\,y(0)-y'(0)$$

$$L\left[y'''(t)\right]=s^3\,L\left[y(t)\right]-s^2\,y(0)-s\,y'(0)-y''(0)$$

Working rule to find the solution of differential equations using Laplace transform

- Express the given equation in the notations $y'(t)$, $y''(t)$, $y'''(t)$ etc.,

- Take Laplace transform on both sides of the equation.
- Use $L\left[y'(t)\right] = s\,L\left[y(t)\right] - y(0)$

$$L\left[y''(t)\right] = s^2\,L\left[y(t)\right] - s\;y(0) - y'(0)$$

$$L\left[y'''(t)\right] = s^3\,L\left[y(t)\right] - s^2\;y(0) - s\,y'(0) - y''(0) \text{ etc.,}$$

- Substitute the given initial conditions $y(0)$, $y'(0)$, $y''(0)$ etc.,
- Simplify for $L\left[y(t)\right]$, a function of 's'
- Finding the inverse of $L\left[y(t)\right]$, the solution of the given differential equation is obtained.

Problems

Solve the following differential equations using Laplace transform

1. $\dfrac{d^2y}{dx^2} + k^2y = 0$, given that $y(0) = 2 = y'(0)$

Soln: Given $\dfrac{d^2y}{dx^2} + k^2y = 0$

$$y'' + k^2y = 0$$

Take Laplace transform both sides

$$\Rightarrow \quad L\left[y'' + k^2y\right] = L[0]$$

$$\Rightarrow \quad L[y''] + k^2L[y] = L[0]$$

$$\Rightarrow s^2\,L[y] - s\;y(0) - y'(0) + k^2L[y] = 0$$

$$\Rightarrow \quad L[y]\left(s^2 + k^2\right) - 2s - 2 = 0$$

$$\Rightarrow \quad L[y] = \frac{2+2s}{\left(s^2+k^2\right)}$$

Taking L^{-1} both sides

$$L^{-1}\{L[y]\} = L^{-1}\left[\frac{2+2s}{\left(s^2+k^2\right)}\right]$$

$$y(x) = L^{-1}\left[\frac{2}{\left(s^2+k^2\right)}\right] + L^{-1}\left[\frac{2s}{\left(s^2+k^2\right)}\right]$$

$$\boxed{y(x) = 2\frac{\sin kx}{k} + 2\cos kx}$$

2. $y''' + 2y'' - y' - 2y = 0$, given that $y(0) = 0 = y'(0)$, $y''(0) = 6$

Soln: Given $y''' + 2y'' - y' - 2y = 0$

Take Laplace transform both sides

$$L\left[y''' + 2y'' - y' - 2y\right] = L[0]$$

$$L\left[y'''\right] + 2L\left[y''\right] - L\left[y'\right] - 2L[y] = L(0)$$

Using Laplace transform of derivatives

$$\Rightarrow \left(s^3 L[y(t)] - s^2\, y(0) - s\, y'(0) - y''(0)\right) + 2\left(s^2 L[y(t)] - s\, y(0) - y'(0)\right)$$
$$-\left(s L[y(t)] - y(0)\right) - 2[y(t)] = 0$$

$$\Rightarrow \left(s^3 L[y(t)] - 6\right) + 2\left(s^2 L[y(t)]\right) - \left(s L[y(t)]\right) - 2L[y(t)] = 0$$

$$\Rightarrow s^3 L[y(t)] - 6 + 2s^2 L[y(t)] - s L[y(t)] - 2L[y(t)] = 0$$

$$\Rightarrow L[y(t)]\left(s^3 + 2s^2 - s - 2\right) - 6 = 0$$

$$\Rightarrow L[y(t)]\left(s^3 + 2s^2 - s - 2\right) = 6$$

$$\Rightarrow L\left[y(t)\right] = \frac{6}{s^3 + 2s^2 - s - 2}$$

Taking L^{-1} both sides

$$\Rightarrow L^{-1}\left[L\left[y(t)\right]\right] = L^{-1}\left[\frac{6}{s^3 + 2s^2 - s - 2}\right]$$

$$\Rightarrow y(t) = 6\,L^{-1}\left[\frac{1}{s^2(s+2) - 1(s+2)}\right]$$

$$\Rightarrow y(t) = 6\,L^{-1}\left[\frac{1}{(s+2)\left(s^2 - 1\right)}\right]$$

$$\Rightarrow y(t) = 6\,L^{-1}\left[\frac{1}{(s+2)(s+1)(s-1)}\right]$$

Using Partial fractions

$$\frac{1}{(s+2)(s+1)(s-1)} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s-1}$$

Multiplying by $(s+2)(s+1)(s-1)$

$$1 = A(s+1)(s-1) + B(s+2)(s-1) + C(s+1)(s+2)$$

Giving values for s and finding A, B , C

$$s = -2 \Rightarrow$$

$$1 = A(-2+1)(-2-1) + B(-2+2)(-2-1) + C(-2+1)(-2+2)$$

$$1 = A(-1)(-3) + B(0) + C(0)$$

$$\Rightarrow \boxed{A = \frac{1}{3}}$$

$s = -1 \Rightarrow$

$1 = A(-1+1)(-1-1) + B(-1+2)(-1-1) + C(-1+1)(-1+2)$

$1 = A(0) + B(1)(-2) + C(0)$

$\Rightarrow \boxed{B = -\frac{1}{2}}$

$s = 1 \Rightarrow$

$1 = A(1+1)(1-1) + B(1+2)(1-1) + C(1+1)(1+2)$

$1 = A(0) + B(0) + C(2)(3)$

$\Rightarrow \boxed{C = \frac{1}{6}}$

We get $\dfrac{1}{(s+2)(s+1)(s-1)} = \dfrac{1/3}{s+2} + \dfrac{-1/2}{s+1} + \dfrac{1/6}{s-1}$

$$\Rightarrow y(t) = 6L^{-1}\left[\frac{1/3}{s+2} + \frac{-1/2}{s+1} + \frac{1/6}{s-1}\right]$$

$$\Rightarrow y(t) = L^{-1}\left[\frac{2}{s+2} + \frac{-3}{s+1} + \frac{1}{s-1}\right]$$

$$\Rightarrow y(t) = 2L^{-1}\left[\frac{1}{s+2}\right] - 3L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{1}{s-1}\right]$$

$\Rightarrow y(t) = 2e^{-2t} - 3e^{-t} + e^{t}$ is the required solution

3. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = e^{-x}$, with $y(0) = 0 = y'(0)$

Soln: Given $y''(x)+4y'(x)+4y(x)=e^{-x}$

Take Laplace transform both sides

$$L\left[y''(x)+4y'(x)+4y(x)\right]=L\left[e^{-x}\right]$$

$$L\left[y''(x)\right]+4L\left[y'(x)\right]+4L\left[y(x)\right]=L\left[e^{-x}\right]$$

Using Laplace transform of derivatives

$$\Rightarrow\left(s^2L\left[y(x)\right]-s\,y(0)-y'(0)\right)+4\left(sL\left[y(x)\right]-y(0)\right)+4L\left[y(x)\right]=\frac{1}{s+1}$$

$$\Rightarrow s^2L\left[y(x)\right]+4sL\left[y(x)\right]+4L\left[y(x)\right]=\frac{1}{s+1}$$

$$\Rightarrow L\left[y(x)\right]\left(s^2+4s+4\right)=\frac{1}{s+1}$$

$$\Rightarrow L\left[y(x)\right]=\frac{1}{(s+1)\left(s^2+4s+4\right)}$$

$$\Rightarrow L\left[y(x)\right]=\frac{1}{(s+1)(s+2)^2}$$

Taking L^{-1} both sides

$$\Rightarrow L^{-1}\left[L\left[y(x)\right]\right]=L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$

$$\Rightarrow y(x)=L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$

Using Partial fractions

$$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

Multiplying by $(s+1)(s+2)^2$

$1 = A(s+2)^2 + B(s+2)(s+1) + C(s+1)$

Giving values for s and finding A, B , C

$s = -2 \Rightarrow$

$1 = A(-2+2)^2 + B(-2+2)(-2+1) + C(-2+1)$

$1 = A(0) + B(0) + C(-1)$

$\Rightarrow \boxed{C = -1}$

$s = -1 \Rightarrow$

$1 = A(-1+2)^2 + B(-1+2)(-1+1) + C(-1+1)$

$1 = A(1) + B(0) + C(0)$

$\Rightarrow \boxed{A = 1}$

$s = 0 \Rightarrow$

$1 = A(0+2)^2 + B(0+2)(0+1) + C(0+1)$

$1 = A(4) + B(2) + C(1)$

$1 = (1)(4) + B(2) + (-1)(1)$

$1 = 4 + 2B - 1$

$\Rightarrow \boxed{B = -1}$

We get $\dfrac{1}{(s+1)(s+2)^2} = \dfrac{1}{s+1} + \dfrac{-1}{s+2} + \dfrac{-1}{(s+2)^2}$

$$\Rightarrow y(x) = L^{-1}\left[\frac{1}{s+1} + \frac{-1}{s+2} + \frac{-1}{(s+2)^2}\right]$$

$$\Rightarrow y(x) = L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right) - L^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$\Rightarrow y(x) = e^{-x} - e^{-2x} - e^{-2x}L^{-1}\left(\frac{1}{s^2}\right)$$

$$\Rightarrow y(x) = e^{-x} - e^{-2x} - e^{-2x}x \quad \text{is the required solution}$$

4. $y'' + 2y' + 2y = 5\sin t$, with $y(0) = 0 = y'(0)$

Soln: Given $y''(t) + 2y'(t) + 2y(t) = 5\sin t$

Take Laplace transform both sides

$$L\left[y''(t) + 2y'(t) + 2y(t)\right] = L[5\sin t]$$

$$L\left[y''(t)\right] + 2L\left[y'(t)\right] + 2L\left[y(t)\right] = 5L[\sin t]$$

Using Laplace transform of derivatives

$$\left(s^2 L[y(t)] - s\,y(0) - y'(0)\right) + 2\left(sL[y(t)] - y(0)\right) + 2L(y(t)) = \frac{5}{s^2+1}$$

$$s^2 L[y(t)] + 2sL[y(t)] + 2L(y(t)) = \frac{5}{s^2+1}$$

$$L[y(t)]\left(s^2 + 2s + 2\right) = \frac{5}{s^2+1}$$

$$L[y(t)] = \frac{5}{\left(s^2+1\right)\left(s^2+2s+2\right)}$$

Taking L^{-1} both sides

$$L^{-1}\left[L\left[y(t)\right]\right]=L^{-1}\left[\frac{5}{\left(s^2+1\right)\left(s^2+2s+2\right)}\right]$$

$$y(t)= L^{-1}\left[\frac{5}{\left(s^2+1\right)\left(s^2+2s+2\right)}\right]$$

Using Partial fractions

$$\frac{5}{\left(s^2+1\right)\left(s^2+2s+2\right)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+2s+2}$$

Multiplying by $\left(s^2+1\right)\left(s^2+2s+2\right)$

$$5=(As+B)\left(s^2+2s+2\right)+(Cs+D)\left(s^2+1\right)$$

$$5= As^3+2As^2+2As+Bs^2+2Bs+2B+Cs^3+Cs+Ds^2+D$$

$$5=(A+C)s^3+(2A+B+D)s^2+(2A+2B+C)s+2B+D$$

Equating corresponding Coefficients to find *A*, *B* , *C*, *D*

$$5=(A+C)s^3+(2A+B+D)s^2+(2A+2B+C)s+2B+D$$

$$\Rightarrow A+C=0 \qquad 2A+B+D=0 \qquad 2A+2B+C=0 \quad 2B+D=5$$

Solving the equations, we get $A=-2 \qquad B=1 \qquad C=2 \qquad D=3$

$$\Rightarrow \frac{5}{\left(s^2+1\right)\left(s^2+2s+2\right)} = \frac{-2s+1}{s^2+1} + \frac{2s+3}{s^2+2s+2}$$

$$\Rightarrow \frac{5}{\left(s^2+1\right)\left(s^2+2s+2\right)} = \frac{-2s}{s^2+1} + \frac{1}{s^2+1} + \frac{2s}{(s+1)^2+1} + \frac{3}{(s+1)^2+1}$$

$$\Rightarrow y(t) = L^{-1}\left[\frac{-2s}{s^2+1} + \frac{1}{s^2+1} + \frac{2s}{(s+1)^2+1} + \frac{3}{(s+1)^2+1}\right]$$

$$\Rightarrow y(t) = -2L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right] + 2L^{-1}\left[\frac{s}{(s+1)^2+1}\right] + 3L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$\Rightarrow y(t) = -2\cos t + \sin t + 2e^{-t}L^{-1}\left[\frac{s}{s^2+1}\right] + 3e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$\Rightarrow y(t) = -2\cos t + \sin t + 2e^{-t}\cos t + 3e^{-t}\sin t$ is the required solution

5. $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 4x = 3te^{-t}$, given $x=4$, $\dfrac{dx}{dt}=2$ when $t=0$

Soln: Given $x''(t) + 2x'(t) + x(t) = 3te^{-t}$

Take Laplace transform both sides

$$L\left[x''(t) + 2x'(t) + x(t)\right] = L\left[3te^{-t}\right]$$

$$L\left[x''(t)\right] + 2L\left[x'(t)\right] + L\left[x(t)\right] = 3L\left[te^{-t}\right]$$

Using Laplace transform of derivatives

$$\Rightarrow\left(s^2 L\left[x(t)\right]-s\ x(0)-x^{'}(0)\right)+2\left(s L\left[x(t)\right]-x(0)\right)+L\left[x(t)\right]=3\left(L(t)\right)_{s\to s+1}$$

$$\Rightarrow\left(s^2 L\left[x(t)\right]-s(4)-(2)\right)+2\left(s L\left[x(t)\right]-4\right)+L\left[x(t)\right]=3\left(\frac{1}{s^2}\right)_{s\to s+1}$$

$$\Rightarrow\left(L\left[x(t)\right]-s(4)-(2)\right)+2\left(s L\left[x(t)\right]-4\right)+L\left[x(t)\right]=3\left(\frac{1}{(s+1)^2}\right)$$

$$\Rightarrow L\left[x(t)\right]\left(s^2+2s+1\right)-4s-2-8=\frac{3}{(s+1)^2}$$

$$\Rightarrow L\left[x(t)\right](s+1)^2=\frac{3}{(s+1)^2}+4s+10$$

$$\Rightarrow L\left[x(t)\right]=\frac{3}{(s+1)^2}+\frac{4s+10}{(s+1)^2}$$

Taking L^{-1} both sides

$$\Rightarrow L^{-1}\left[L\left[x(t)\right]\right]=L^{-1}\left[\frac{3}{(s+1)^4}+\frac{4s+10}{(s+1)^2}\right]$$

$$\Rightarrow x(t)=3L^{-1}\left[\frac{1}{(s+1)^4}\right]+4L^{-1}\left[\frac{s}{(s+1)^2}\right]+10L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$\Rightarrow x(t)=3e^{-t}L^{-1}\left[\frac{1}{s^4}\right]+4L^{-1}\left[\frac{(s+1)-1}{(s+1)^2}\right]+10e^{-t}L^{-1}\left[\frac{1}{(s)^2}\right]$$

$$\Rightarrow x(t)=3e^{-t}\,\frac{t^3}{\Gamma 4}+4L^{-1}\left[\frac{(s+1)}{(s+1)^2}-\frac{1}{(s+1)^2}\right]+10e^{-t}\,t$$

$$\Rightarrow x(t)=\frac{3t^3}{3!}e^{-t}+10e^{-t}t+4L^{-1}\left[\frac{1}{s+1}-\frac{1}{(s+1)^2}\right]$$

$$\Rightarrow x(t)=\frac{3t^3}{6}e^{-t}+10e^{-t}t+4\left(e^{-t}-e^{-t}t\right)$$

$$\Rightarrow x(t)=\frac{t^3}{2}e^{-t}+6e^{-t}t+4e^{-t}$$

6. $y''(t)+y(t)=H(t-1)$, with $y(0)=0\ \&\ y'(0)=1$

Soln: Given $y''(t)+y(t)=H(t-1)$

Take Laplace transform both sides

$$L\left[y''(t)+y(t)\right]=L\left[H(t-1)\right]$$

$$\Rightarrow L\left[y''(t)\right]+L\left[y(t)\right]=L\left[H(t-1)\right]$$

$$\Rightarrow s^2L\left[y(t)\right]-s\,y(0)-y'(0)+L\left[y(t)\right]=\frac{e^{-s}}{s}$$

$$\Rightarrow L\left[y(t)\right]\left(s^2+1\right)-s(0)-1=\frac{e^{-s}}{s}$$

$$\Rightarrow L\left[y(t)\right]=\frac{1}{\left(s^2+1\right)}\left[\frac{e^{-s}}{s}+1\right]$$

$$\Rightarrow L\left[y(t)\right]=\frac{e^{-s}}{s\left(s^2+1\right)}+\frac{1}{\left(s^2+1\right)}$$

$$\Rightarrow y(t)=L^{-1}\left[e^{-s}\times\frac{1}{s\left(s^2+1\right)}\right]+L^{-1}\left[\frac{1}{\left(s^2+1\right)}\right]$$

$$\left[\text{W.K.T. } L^{-1}\left[e^{-as}F(s)\right]=u(t-a)\left[L^{-1}\left[F(s)\right]\right]_{t\to t-a}\right]$$

$$\Rightarrow y(t)=u(t-1)\left\{L^{-1}\left[\frac{1}{s\left(s^2+1\right)}\right]\right\}_{t\to t-1}+\sin t$$

$$F(s)=\frac{1}{s\left(s^2+1\right)}=\frac{A}{s}+\frac{Bs+C}{\left(s^2+1\right)}$$

$$\Rightarrow\ 1=A\left(s^2+1\right)+(Bs+C)s$$

Put $s=0\Rightarrow\boxed{1=A}$

Consider the co-efficients of $s^2\Rightarrow 0=A+B\Rightarrow B=-A\Rightarrow\boxed{B=-1}$

Consider the co-efficients of $s\ \Rightarrow\boxed{0=C}$

$$\frac{1}{s\left(s^2+1\right)}=\frac{1}{s}+\frac{-s}{\left(s^2+1\right)}$$

$$\Rightarrow\ y(t)=u(t-1)\left\{L^{-1}\left[\frac{1}{s}-\frac{s}{\left(s^2+1\right)}\right]\right\}_{t\to t-1}+\sin t$$

$$\Rightarrow y(t)=u(t-1)\left\{L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{s}{\left(s^2+1\right)}\right]\right\}_{t\to t-1}+\sin t$$

$$\Rightarrow y(t)=u(t-1)\left[1-\cos t\right]_{t\to t-1}+\sin t$$

$$\Rightarrow\boxed{y(t)=u(t-1)\left[1-\cos(t-1)\right]+\sin t}$$