Module-II

* Vector Spaces

In mathematics a vector space is also known as a linear space, it is a collection of objects called vectors that can be added together and multiplied by scalers (numbers), satisfying the specific axioms. Vector spaces provide a frame work for studying and analysing mathematical structure such as geometry, ~~geography~~ algebra and linear transformations.

→ A real vector space V is a non empty set of objects called vectors satisfying the following axioms

① If $u, v \in V$ then $u+v \in V$ i.e V is closed under vector addition.

② If $\alpha \in R$ and $u \in V$ then $\alpha u \in V$ i.e V is closed under scalar multiplication.

→ Properties:-

(I) ① Associative law → $u+(v+w) = (u+v)+w$ for $u, v, w \in V$

② $u+v = v+u \ \forall \ u, v \in V$ (commutative law)

③ $u+0 = 0+u \ \forall \ u \in V$ (Identity law)

④ $u+(-u) = 0 = (-u)+u \ \forall u \in V$ (Inverse law)

(II) There is an external composition in V over R called scalar multiplication i.e $\forall \alpha \in R$ and $u \in V$ then $\alpha u \in V$.

In other words V is closed under scalar multiplication.

(III) The two compositions i.e vector addition and scalar multiplication satisfy the following conditions:

$\forall \ a, b \in R$ and $v, w \in V$

$$\left.\begin{array}{l} ① \ (a+b)v = av+bv \\ ② \ a(v+w) = av+aw \end{array}\right\} \text{(distribution law)}$$

③ $1 \cdot (v) = v$ (multiplicative Identity)

Q. Prove that the set C of all complex numbers i.e (the set of all ordered pairs of real numbers) is a vector space over the field 'R' of all real numbers where vector addition is defined by $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ $\forall (x_1, x_2)(y_1, y_2) \in C$ and scalar multiplication is defined by $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$ $\forall (x_1, x_2) \in C$

Soln:- given $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$

• we need to show that C is closed under vector addition and scalar multiplication

(I) $(C, +)$ is abelian group

(i) Associative property

$\forall (x_1, x_2)(y_1, y_2)(z_1, z_2) \in C$.

$$(x_1, x_2) + [(y_1, y_2) + (z_1, z_2)] = (x_1, x_2) + [(y_1 + z_1), (y_2 + z_2)]$$
$$= [x_1 + y_1 + z_1, x_2 + y_2 + z_2]$$
$$= (x_1 + y_1, x_2 + y_2) + (z_1, z_2)$$
$$= [(x_1, x_2) + (y_1, y_2)] + (z_1, z_2)$$

(ii) Commutativity:

$\forall (x_1, x_2) + (y_1, y_2) \in C$, we have

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$
$$= (y_1 + x_1, y_2 + x_2)$$
$$= (y_1, y_2) + (x_1, x_2)$$

(iii) Existence of Identity:

$\forall (x_1, x_2) \in C$ there exists $(0, 0) \in C$, such that

$$(x_1, x_2) + (0, 0) = (x_1 + 0, x_2 + 0)$$
$$= (x_1, x_2)$$

$$(0, 0) + (x_1, x_2) = (0 + x_1, 0 + x_2) = (x_1, x_2)$$

$$\therefore (x_1, x_2) + (0, 0) = (x_1, x_2) = (0, 0) + (x_1, x_2)$$

(iv) Existence of Inverse: For any $(x_1, x_2) \in C$ there exists $(-x_1, -x_2) \in C$ such that

$$(x_1, x_2) + (-x_1, -x_2) = (0, 0) = (-x_1, -x_2) + (x_1, x_2)$$

Thus, C is an abelian group wrt vector add^n.

$$(x_1 + (-x_1), x_2 + (-x_2))$$
$$= (0, 0)$$

II) Properties of scalar multiplication in C

(i) $\alpha[(x_1, x_2) + (y_1, y_2)] = \alpha(x_1+y_1, x_2+y_2)$

$= (\alpha(x_1+y_1), \alpha(x_2+y_2))$

$= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2)$

$= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2)$

$= \alpha(x_1, x_2) + \alpha(y_1, y_2)$

Thus $\alpha[(x_1,x_2) + (y_1,y_2)] = \alpha(x_1,x_2) + \alpha(y_1,y_2)$

for all $(x_1, x_2)(y_1, y_2) \in C, \alpha \in R$

(ii) $(a+b)(x_1, x_2) = ((a+b)x_1, (a+b)x_2)$

$= (ax_1 + bx_1, ax_2 + bx_2)$

$= (ax_1, ax_2) + (bx_1, bx_2)$

$= a(x_1, x_2) + b(x_1, x_2)$

Thus $(a+b)(x_1, x_2) = a(x_1, x_2) + b(x_1, x_2)$ for all

$(x_1, x_2), (y_1, y_2) \in C$ and $a, b \in R$

(iv) $1.(x_1, x_2) = (x_1, x_2)$ for all $(x_1, x_2) \in C$

Thus the set C satisfies all the properties of vector space. Hence C is a vector space over R.

(iii) $a[(x_1, x_2) + (y_1, y_2)] = a[x_1+y_1, x_2+y_2]$

$= [a(x_1+y_1), a(x_2+y_2)]$

$= [ax_1 + ay_1, ax_2 + ay_2]$

$= [(ax_1, ax_2) + (ay_1, ay_2)]$

$= a(x_1, x_2) + a(y_1, y_2)$

Thus the set C satisfies all the properties of vector space.
Hence, C is a vector space.

Q. Show that the set V of all real value continuous functions of x defined the interval $[0,1]$ is a vector space over the field R of real numbers with respect to vector addition and scalar multiplication is defined by

$$(f_1 + f_2)x = f_1(x) + f_2(x) \quad \forall\ f_1 f_2 \in V$$

$$(af_1)x = af_1(x) \quad \forall\ a \in V,\ f_1 \in V$$

Sol^n: (I) $(V, +)$ is an abelian group

(i) Associativity :

Let $f_1, f_2, f_3 \in V$ be arbitary

$$[(\underbrace{f_1+f_2}_{f_1}) + \underbrace{f_3}_{f_2}](x) = (f_1+f_2)(x) + f_3(x)$$

$$= [f_1(x) + f_2(x)] + f_3(x)$$

$$= f_1(x) + (f_2+f_3)(x)$$

$$= [f_1 + (f_2+f_3)](x)$$

$$\therefore (f_1+f_2)+f_3 = f_1+(f_2+f_3)$$

(ii) Commutativity :

Let $f_1, f_2 \in V$ be arbitary

$$(f_1+f_2)(x) = f_1(x) + f_2(x)$$

$$= f_2(x) + f_1(x)$$

$$= (f_2+f_1)(x), \text{ for all } x$$

$$\therefore f_1+f_2 = f_2+f_1$$

(iii) Existence of Identity :-

Define a function '0' such that $0(x) = 0$ (real Number) $\forall\ x \in [0,1]$

Also, '0' is a continuous function and $\in$ to V

$$(0+f_1)(x) = 0(x) + f_1(x) = 0 + f_1(x) = f_1(x)$$

$$(f_1+0)(x) = f_1(x) + 0(x) = f_1(x)$$

$$0 + f_1 = f_1 + 0 = f_1$$

Thus, the function '0' defined above is an Additive identity element of 'V'.

(iv) Existence of Inverse :

For a function f_1, the function $-f_1$ defined by

$(-f_1)(x) = -f_1(x)$ is called Additive Inverse as $[f_1 + (-f_1)](x) = f_1(x) + (-f_1)(x)$

$$= f_1(x) - f_1(x) = 0 = 0(x)$$

$|||^{ly}$,

$$[-f_1+f_1](x) = 0(x)$$

$\therefore$ Thus, the set 'V' is an abelian group under addition.

II) Properties of scalar Multiplication in V

(i) For $f_1, f_2 \in V$ and $\alpha \in R$, we have

$$[\alpha(f_1+f_2)]x = \alpha[(f_1+f_2)(x)]$$
$$= \alpha[f_1(x) + f_2(x)]$$
$$= \alpha f_1(x) + \alpha f_2(x)$$
$$= (\alpha f_1)x + (\alpha f_2)x$$
$$= (\alpha f_1 + \alpha f_2)x$$
$$\alpha(f_1+f_2) = \alpha f_1 + \alpha f_2$$

(ii) For $f_1 \in V$ and $a, b \in R$

$$[(a+b)f_1](x) = (a+b)f_1(x)$$
$$= af_1(x) + bf_1(x)$$
$$= (af_1)x + (bf_1)x$$
$$= (af_1 + bf_1)(x)$$
$$(a+b)f_1 = af_1 + bf_1$$

(iii) For $f_1 \in V$ and $a, b \in R$

$$[a(bf_1)]x = a[(bf_1)x]$$
$$= a[bf_1(x)]$$
$$= (ab)f_1(x) = [(ab)f_1]x$$
$$a(bf_1) = (ab)f_1$$

(iv) For $f_1 \in V$ and $a, b \in R$

$$(1 \cdot f_1)x = 1 \cdot f_1(x) = f_1(x)$$
$$1 f_1 = f_1$$

$\therefore$ Thus V satisfies all the properties of a vector space and hence V is a vector space.

Q. Show that the set of matrices of the type $m \times n$, where m & n are fixed positive integers is a vector space over R wrt matrix addition & multiplication of matrix by scalar (i.e. a real number).

Sol^n → $m \times n$

(i) Associativity

Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$, $C = [c_{ij}]_{m\times n}$ be three matrices belonging to set M

$(A+B)+C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$= [a_{ij} + b_{ij}] + [c_{ij}]$

$= [(a_{ij} + b_{ij}) + (c_{ij})]$

$= [a_{ij} + (b_{ij} + c_{ij})]$

$= [a_{ij}] + [b_{ij} + c_{ij}]$

$= [a_{ij}] + ([b_{ij} + c_{ij}])$

$= A + (B+C)$

(ii) Commutativity:-

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be the matrix belonging to set M.

$A+B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

$= [b_{ij} + a_{ij}]$

$= [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$

$= B + A$

(iii) Existence of Identity:-

Let $A = [a_{ij}]_{m\times n} \in M$ be arbitary

wkt, zero matrix (null matrix) of type $m\times n$ also belongs to the set M and is denoted by 0.

Now, $A + 0 = [a_{ij}] + [0] = [a_{ij} + 0] = [a_{ij}] = A$

Similarly, $0 + A = A$

(iv) Existence of Inverse:-

If $A = [a_{ij}]_{m\times n}$ belongs to set M, then $-A = [a_{ij}]_{m\times n}$ along belongs to set M.

Now, $A + (-A) = [a_{ij}] + [-a_{ij}] = [a_{ij} - a_{ij}] = [0] = 0$

Similarly, $(-A) + (A) = 0$

Thus, the set M is are abelian group under addition.

II) Properties of Scalar Multiplication in M :-

(i) Let $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ $\in M$ and $\alpha\in R$.

then, $\alpha(A+B) = \alpha([a_{ij}]+[b_{ij}])$

$= \alpha[a_{ij}+b_{ij}]$

$= [\alpha(a_{ij}+b_{ij})]$

$= [\alpha a_{ij} + \alpha b_{ij}]$ [$\because$ Multiplication is distributive in R]

$= [\alpha a_{ij}] + [\alpha b_{ij}]$

$= \alpha[a_{ij}] + \alpha[b_{ij}] = \alpha A + \alpha B$

(ii) Let $A=[a_{ij}]_{m\times n}$ $\in M$ and $a, b\in R$

then, $(a+b)A = (a+b)[a_{ij}]$

$= [(a+b)a_{ij}]$

$= [aa_{ij} + ba_{ij}]$

$= a[a_{ij}] + b[a_{ij}]$

$= aA + bA$

(iii) Let $A=[a_{ij}]_{m\times n}$ $\in M$ and $a, b\in R$

then, $a(bA) = a(b[a_{ij}])$

$= a[(ba_{ij})]$

$= [(ab)a_{ij}]$

$= (ab)[a_{ij}]$

$= (ab)A$

$a(bA) = (ab)A$

(iv) Let $A=[a_{ij}]_{m\times n}$ $\in M$

then, $1.A = 1.[a_{ij}] = [1.a_{ij}] = [a_{ij}] = A$

Thus, M satisfies all the properties of vector space and hence M is a vector space over R.

Q. Prove that the set of all vectors in a plane over the field of real numbers is a vector space with respect to vector addition and scalar multiplication.

Soln:- Let V denote the set of all coplanar vectors and R be the field of real numbers.

$\therefore$ The elements of V are the ordered pairs (x, y) where $x, y \in R$.

$$V = \{(x,y) : x, y \in R\}$$

I) $(V, +)$ is an abelian group

(i) Associativity :-

We know that for all $u, v, w \in V$

$$(u+v)+w = u+(v+w)$$

(ii) Commutativity :-

We know that for all $u, v \in V$

$$u+v = v+u$$

(iii) Existence of additive Identity :-

For every vector $u \in V$ there exists a zero vector $0 \in V$ such that $u+0 = 0+u = u$

(iv) Existence of additive inverse :-

For every vector $u \in V$ there exists a vector $-u \in V$ such that $u+(-u) = (-u)+u = 0$

Thus V is an abelian group with respect to vector addition.

(11) Scalar multiplication in V

(i) For $u, v \in V$ and $\alpha \in R$ we have
$$\alpha(u+v) = \alpha u + \alpha v$$

(ii) For $u \in V$ and $a, b \in R$, we have
$$(a+b)u = au + bu$$

(iii) For $u \in V$ and $a, b \in R$, we have
$$a(bu) = (ab)u$$

(iv) For $u \in V$ and $1 \in R$, we have
$$1u = u$$

Thus set V of coplanar vectors satisfies all the properties of vector addition and scalar multiplication.

$\therefore$ V is a vector space.

H.W

1.) Prove that the set of all vectors in a plane over the field of real numbers is a vector space with respect to vector addition and scalar multiplication.

Sol:- Let V denotes the set of all coplanar vectors and R be the field of real numbers.

$\therefore$ The elements of V are the ordered pairs (x, y) where $x, y \in R$.
$$V = \{(x, y) : x, y \in R\}$$

(I) $(V, +)$ is an abelian group.

(i) Associativity:-
we know that for all $u, v, w \in V$
$$(u+v)+w = u+(v+w)$$

(ii) Commutativity:-
we know that for all $u, v \in V$
$$u+v = v+u$$

(iii) Existence of additive identity:-
for every vector $u \in V$ there exists a zero vector $0 \in V$ such that $u+0 = 0+u = u$.

(iv) Existence of additive inverse:-
For every vector $u \in V$ there exists a vector $-u \in V$ such that $u+(-u) = (-u)+u = 0$.

Thus V is an abelian group with respect to vector addition.

II Scalar multiplication in V.

(i) For $u, v \in V$ and $\alpha \in R$ we have
$$\alpha(u+v) = \alpha u + \alpha v$$

(ii) For $u \in V$ and $a, b \in R$, we have
$$(a+b)u = au + bu$$

(iii) For $u \in V$ and $a, b \in R$, we have
$$a(bu) = (ab)u$$

(iv) For $u \in V$ and $1 \in R$, we have
$$1u = u$$

Thus set V of coplanar vectors satisfies all the properties of vector addition and scalar multiplication

$\therefore$ V is a vector space.

Subspaces:-

A non-empty subset W of a vector space $V(F)$ is said to form a subspace of V if W is also a vector space over F with the same addition and scalar multiplication as for V.

Eg:- Let $W_1 = \{(a,0,0) : a \in R\}$
$W_2 = \{(a,b,0) : a,b \in R\}$

Here W_1 is a subspace of W_2. Also W_1 and W_2 are subspace of R^3.

* **Necessary and Sufficient conditions for a subspace:**
(only statement, no proof)

Theorem 1: W is a subspace of $V(F)$ iff
(i) W is non empty.
(ii) W is closed under vector addition.
i.e $\forall\, w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$
(iii) W is closed under scalar multiplication.
i.e $\forall\, a \in F$ and $w \in W \Rightarrow a.w \in W$.

Theorem 2: W is a subspace of $V(F)$ iff.
(i) W is non empty.
(ii) $\forall\, a,b \in F$ and $v,w \in W \Rightarrow$
$a.v + b.w \in W$.

(1) Show that W is a subspace of $V(R)$ where $W = \{f : f(9) = 0\}$.

Soln→ Since $0 \in W$ as $0(9) = 0$
So W is non empty set.
Let $f, g \in W$.
i.e $f(9) = 0$ and $g(9) = 0$
then $\forall\, a,b \in R$
$$(a.f + b.g)9 = a.f(9) + b.g(9) = a.0 + b.0 = 0$$
Hence $a.f + b.g \in W$.
By theorem(2), W is a subspace of $V(R)$.

② Show that W is a subspace of $V(R)$ where $W = \{ f : f(2) = f(1) \}$

Sol^n → $0 \in W$ since $0(2) = 0 = 0(1)$

hence W is a non empty set.

Let $f, g \in W$ then [redacted] $f(2) = f(1)$ & $g(2) = g(1)$

then $\forall\, a, b \in R$.

$$(a.f + b.g)(2) = a.f(2) + b.g(2) = a.f(1) + b.g(1)$$
$$= (a.f + b.g)(1)$$

hence $a.f + b.g \in W$

So by theorem 2, W is a subspace of $V(R)$.

③ Let $V = R^3$ be the Euclidean 3-space. Let $W = \{ (x, y, z) : ax + by + cz = 0;\ x, y, z \in R \}$, a, b, c being real numbers. Show that W is a subspace of V.

(OR)

Show that any plane passing through the origin is a subspace of R^3.

Sol^n → [redacted]

Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ be any two elements of W where $x_1, x_2, y_1, y_2, z_1, z_2 \in R$.

Then $ax_1 + by_1 + cz_1 = 0$ and $ax_2 + bx_2 + cx_2 = 0$

For $\alpha \in R$, $\alpha u + v = \alpha(x_1, y_1, z_1) + (x_2, y_2, z_2)$

$$\alpha u + v = (\alpha x_1, \alpha y_1, \alpha z_1) + (x_2, y_2, z_2)$$
$$= (\alpha x_1 + x_2,\ \alpha y_1 + y_2,\ \alpha z_1 + z_2) \longrightarrow (1)$$

where $\alpha x_1 + x_2,\ \alpha y_1 + y_2,\ \alpha z_1 + z_2 \in R$.

Now,

$$a(\alpha x_1 + x_2) + b(\alpha y_1 + y_2) + c(\alpha z_1 + z_2) = \alpha(ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2)$$
$$= \alpha(0) + 0 = 0 \longrightarrow (2)$$

From (1) and (2), we have

$\alpha u + v \in W$.

Thus, W is a subspace of V.

(4) Let V be the vector space of all square matrices over R. Determine which of the following are subspaces of V.

(i) $W = \left\{ \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} : x, y, z \in R \right\}$ (ii) $W = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in R \right\}$

(iii) $W = \{ A : A \in V$ and A is singular$\}$ (iv) $W = \{ A : A \in V, A^2 = A \}$

Solⁿ → (i) Let $A = \begin{bmatrix} x_1 & y_1 \\ z_1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & y_2 \\ z_2 & 0 \end{bmatrix}$ be any two elements of W.

If $a, b \in R$ then

$$aA + bB = a\begin{bmatrix} x_1 & y_1 \\ z_1 & 0 \end{bmatrix} + b\begin{bmatrix} x_2 & y_2 \\ z_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 & ay_1 \\ az_1 & 0 \end{bmatrix} + \begin{bmatrix} bx_2 & by_2 \\ bz_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_2 & ay_1 + by_2 \\ az_1 + bz_2 & 0 \end{bmatrix}$$

which is a matrix of the type $\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$ and $ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2 \in R$.

$\therefore aA + bB \in W$

Thus, W is a sub-space of V.

(ii) Let $A = \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}$ be any two elements of W.

If $a, b \in R$, then

$$aA + bB = a\begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix} + b\begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_2 & 0 \\ 0 & ax_2 + by_2 \end{bmatrix}$$

which is a matrix of the type $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ and $ax_1 + bx_2, ay_1 + by_2 \in R$.

$\therefore aA + bB \in W$.

Thus W is a subspace of V.

(iii) Here W is the set of singular matrices.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ be two square matrices.

Since $|A| = 0$ and $|B| = 0$, therefore $A, B \in W$.

If $a, b \in R$ are non zero, then

$$aA + bB = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Also, $|aA_1 + bA_2| = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab \neq 0$, as none of a and b is zero.

$\therefore aA_1 + bA_2 \notin W$.

Thus, W is not a sub-space of V.

(iv) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} : A$

$\therefore A \in W$.

Now, $A + A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$$(A+A)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= A + A$$

$\therefore (A+A)$ is not a member of the set W.

→ W is not closed under addition.

Hence W is not a sub-space of V.

(5.) Let V be the vector space of all real valued continuous functions over R. Show that the set W of solutions of differential equations

$$5\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 2y = 0$$ is a subspace of V.

Sol^n:- Let $y_1, y_2 \in W$. Then y_1, y_2 are solutions of differential equations.

$$5\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 2y = 0 \longrightarrow (1)$$

i.e $5\frac{d^2y_1}{dx^2} - 7\frac{d^2y_1}{dx} + 2y_1 = 0$; $5\frac{d^2y_2}{dx^2} - 7\frac{d^2y_2}{dx^2} + 2y_2 = 0$.

Let $a, b \in R$ be arbitary.

$$5\frac{d^2}{dx^2}(ay_1 + by_2) - 7\frac{d}{dx}(ay_1 + by_2) + 2(ay_1 + by_2)$$

$$= 5\frac{d^2}{dx^2}(ay_1) - 7\frac{d}{dx}(ay_1) + 2(ay_1) +$$

$$5\frac{d^2}{dx^2}(by_2) - 7\frac{d}{dx}(by_2) + 2(by_2).$$

$$= a\left[5\frac{d^2y_1}{dx^2} - 7\frac{dy_1}{dx} + 2y_1\right] + b\left[5\frac{d^2y_2}{dx^2} - 7\frac{dy_2}{dx} + 2y_2\right]$$

$$= a \cdot 0 + b \cdot 0 = 0$$

Thus, $ay_1 + by_2$ is a solution of (1) & so $ay_1 + by_2 \in W$

$\therefore$ W is a subspace of V.

* Linear Combination

Let V be a vector space over a field F and Let $V_1, V_2, V_3 \dots V_n \in V$. Any vector of the form $a_1V_1 + a_2V_2 + \dots + a_nV_n$ in V where $a_i \in F$ is called a linear combination of $V_1, V_2, V_3 \dots V_n$.

* Linear Dependence

Let V be a vector space over the field F. The vectors $V_1, V_2, V_3, \dots V_n$ are said to be linearly dependend over F if there exist a scalar $a_1, a_2 \dots a_n \in F$, not all are zero but linear combination is zero i.e $a_1V_1 + a_2V_2 + \dots a_nV_n = 0$ $\forall\ a_i \neq 0$ where $i \in N$.

* Linearly Independence

Let V be a vector space over the field F the vectors $V_1, V_2, V_3 \dots V_n$ are saided to be linearly independent over F, if there exist a scalars $a_1, a_2, a_3 \dots a_n \in F$ such that $a_1V_1 + a_2V_2 + a_3V_3 + \dots + a_nV_n = 0$ $\forall\ a_i = 0$ where $i \in N$.

Q. Write the vector $v = (1,3,9)$ as a linear combination of the vectors $u_1 = (2,1,3)$, $u_2 = (1,-1,1)$, $u_3 = (3,1,5)$.

Solⁿ: $v = xu_1 + yu_2 + zu_3$

$(1,3,9) = x(2,1,3) + y(1,-1,1) + z(3,1,5)$ —— ①

$(1,3,9) = (2x, x, 3x) + (y, -y, y) + (3z, z, 5z)$

$\Rightarrow 2x + y + 3z = 1$

$x - y + z = 3$

$3x + y + 5z = 9$

$AX = B$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & 5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

P.t.o

The Augmented matrix.

$$[A:B] = \left[\begin{array}{ccc:c} 2 & 1 & 3 & 1 \\ 1 & -1 & 1 & 3 \\ 3 & 1 & 5 & 9 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$= \left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 2 & 1 & 3 & 1 \\ 3 & 1 & 5 & 9 \end{array}\right]$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$= \left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & -5 \\ 0 & 4 & 2 & 0 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_2$

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 0 & 3 & 1 & -5 \\ 0 & 1 & 1 & 5 \end{array}\right]$$

$R_3 \leftrightarrow R_2$

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 0 & 1 & 1 & 5 \\ 0 & 3 & 1 & -5 \end{array}\right]$$

$R_3 \rightarrow R_3 - 3R_2$

$$= \left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -2 & -20 \end{array}\right]$$

$R_3 \rightarrow -\frac{1}{2} \times R_3$

$$= \left[\begin{array}{ccc:c} 1 & -1 & 1 & 3 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 10 \end{array}\right]$$

$\therefore \rho(A) = 3$

$\rho(A:B) = n$

$n = 3$

No. of unknown

$\therefore \rho(A) = \rho(A:B) = n$

$\therefore$ consistent & unique solution

$x - y + z = 3$ — (1)

$y + z = 5$ — (2)

$\boxed{z = 10}$ — (3)

$\therefore \boxed{y = -5}$

$\therefore x - (-5) + 10 = 3$

$\Rightarrow \boxed{x = -12}$

Substitute the values of x, y, z then eqn (1) become.

$(1, 3, 9) = x(2,1,3) + y(1,-1,1) + z(3,1,5)$

$(1,3,9) = -12(2,1,3) - 5(1,-1,1) + 10(3,1,5)$

Ans

Q. Write the vector $v=(4,2,1)$ as a linear combination of the vectors $u_1=(1,-3,1)$, $u_2=(0,1,2)$, $u_3=(5,1,37)$.

Sol^n → $v = xu_1 + yu_2 + zu_3$

$(4,2,1) = x(1,-3,1) + y(0,1,2) + z(5,1,37)$ —①

$(4,2,1) = (x,-3x,x) + (0,y,2y) + (5z,z,37z)$

$\Rightarrow x + 0 + 5z = 4$

$-3x + y + z = 2$

$x + 2y + 37z = 1$

$AX = B$

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -3 & 1 & 1 \\ 1 & 2 & 37 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

The Augmented matrix

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 0 & 5 & 4 \\ -3 & 1 & 1 & 2 \\ 1 & 2 & 37 & 1 \end{array}\right]$$

$R_2 \to R_2 + 3R_1$

$$= \left[\begin{array}{ccc:c} 1 & 0 & 5 & 4 \\ 0 & 1 & 16 & 14 \\ 1 & 2 & 37 & 1 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc:c} 1 & 0 & 5 & 4 \\ 0 & 1 & 16 & 14 \\ 0 & 2 & 32 & -3 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc:c} 1 & 0 & 5 & 4 \\ 0 & 1 & 16 & 14 \\ 0 & 0 & 0 & -31 \end{array}\right]$$

$\therefore \rho(A) = 2$ and $\rho(A:B) = 3$

$\therefore \rho(A) \neq \rho(A:B)$

System of linear eq^n is inconsistent

$\therefore$ sol^n doesn't exist.

$\therefore$ we can't be expressed v as linear combination of the vectors u_1, u_2, u_3.

Q. Determine whether the vectors $V_1 = (1, 2, 3)$, $V_2 = (3, 1, 7)$ & $V_3 = (2, 5, 8)$ are linearly dependent or linearly independent.

Solⁿ → $xV_1 + yV_2 + zV_3 = 0$

$x(1,2,3) + y(3,1,7) + z(2,5,8) = 0$

$x + 3y + 2z = 0$

$2x + y + 5z = 0$

$3x + 7y + 8z = 0$

$AX = 0$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 5 \\ 3 & 7 & 8 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$

$R_3 \to 2R_3$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 1 \\ 0 & -4 & 4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$R_3 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & -5 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 16 \end{bmatrix}$$

$\rho(A) = 3$, $n = 3$ → unique solⁿ

∴ homogeneous system of linear equation posses a unique solution i.e. $x = 0, y = 0, z = 0$

∴ the vectors V_1, V_2, V_3 are linearly independent.

Q. Express the vector $V=(1,-2,5)$ as a linear combination of the vector $V_1=(1,1,1)$ & $V_2=(1,2,3)$ & $V_3=(2,-1,1)$ in the vector space R^3.

Soln → $xV_1+yV_2+zV_3=V$ ①

$\Rightarrow (1,-2,+5)=x(1,1,1)+y(1,2,3)+z(2,-1,1)$

$\Rightarrow (1,-2,+5)=(x,x,x)+(y,2y,3y)+(2z,-z,z)$

$\Rightarrow x+y+2z=1$

$x+2y-z=-2$

$x+3y+z=+5$

$AX=B$

$A=\begin{bmatrix}1&1&2\\1&2&-1\\1&3&1\end{bmatrix}$ $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ $B=\begin{bmatrix}1\\-2\\+5\end{bmatrix}$

The Augmented Matrix

$[A:B]=\left[\begin{array}{ccc:c}1&1&2&1\\1&2&-1&-2\\1&3&1&+5\end{array}\right]$

$R_2\to R_2-R_1$

$\left[\begin{array}{ccc:c}1&1&2&1\\0&1&-3&-3\\1&3&1&+5\end{array}\right]$

$R_3\to R_3-R_1$

$\left[\begin{array}{ccc:c}1&1&2&1\\0&1&-3&-3\\0&2&-1&+4\end{array}\right]$

$R_3\to R_3-2R_2$

$\left[\begin{array}{ccc:c}1&1&2&1\\0&1&-3&-3\\0&0&5&10\end{array}\right]$

$\therefore \rho(A)=3$

$\rho(A:B)=3$

$n=3$

$\therefore \rho(A)=\rho(A:B)=n$

$\therefore$ consistent & unique solue.

$x+y+2z=1$

$y-3z=-3$

$5z=10$

$\boxed{z=2}$

$y-3\times2=-3$

$y-6=-3$

$\cancel{y=-3+6}$

$\boxed{y=3}$

$x+3+2\times2=1$

$x+7=1$

$\boxed{x=-6}$

substitute the values of x,y,z they eqn ① becomes:-

$(1,-2,5)=x(1,1,1)+y(1,2,3)+z(2,-1,1)$

$\Rightarrow (1,-2,5)=-6(1,1,1)+3(1,2,3)+2(2,-1,1)$ Ans

Linear Span

Let S be a subset of vector space V over the field F. The set of all linear combination of vectors in S is called a linear span of S and is denoted by $L(S)$

If $S = \phi$, then $L(S) = 0$

Basis or Base of Vector Space V

Let V be a vector space over the field F. The set of vectors $\{v_1, v_2, v_3 \dots v_n\}$ is called a basis, If ① $v_1, v_2, v_3 \dots v_n$ are linearly independent

② $v_1, v_2, v_3 \dots v_n$ are span of V i.e each vector of V can be uniquely expressed as linear combination of $v_1, v_2, v_3 \dots v_n$.

Dimension of the Vector Space

The number of elements in basis is called the dimension of a vector space and is denoted by $\dim(V) \rightsquigarrow$ (dimension of V).

If V contains a basis with N elements then dimension $V = N$.

Q. Find the value of a, b, c, so that $w = (a,b,c)$ is a linear combination of $u = (1,-3,2)$ & $v = (2,-1,1)$ in R^3 so that w belongs to span of u & v.

Soln → $w = xu + yv$

$(a,b,c) = x(1,-3,2) + y(2,-1,1)$

$(a,b,c) = (x, -3x, 2x) + (2y, -y, y)$

$x + 2y = a$

$-3x - y = b$

$2x + y = c$

The Augmented Matrix

$$[A:B] = \left[\begin{array}{cc:c} 1 & 2 & a \\ -3 & -1 & b \\ 2 & 1 & c \end{array}\right]$$

$R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{cc:c} 1 & 2 & a \\ 0 & 5 & b+3a \\ 0 & -3 & c-2a \end{array}\right]$$

$R_2 \to R_2 \times 1/5$

$$\left[\begin{array}{cc:c} 1 & 2 & a \\ 0 & 1 & b+3a/5 \\ 0 & -3 & c-2a \end{array}\right]$$

$R_3 \to R_3 + 3R_2$

$$\left[\begin{array}{cc:c} 1 & 2 & a \\ 0 & 1 & \frac{b+3a}{5} \\ 0 & 0 & \boxed{c - 2a + 3\frac{(b+3a)}{5}} \end{array}\right]$$

↳ $\frac{5c - a + 3b}{5}$

If $\frac{5c-a+3b}{5}$ it will be equal to 0 then

the system of linear eqn will be consistent

$\therefore \rho(A) = \rho(A:B) = n = 2$

$\therefore$ w is a linear combination of u and v if

$\frac{5c-a+3b}{5} = 0$ or $-a+3b+5c = 0$

90

H.W. **Q.** For what value of k the vector $v=(1,-2,k)$ can be expressed as linear combination of vector $v_1=(3,0,-2)$, $v_2=(2,-1,-5)$ in R^3.

Sol. $v = xv_1 + yv_2$

$(1,-2,k) = x(3,0,-2) + y(2,-1,-5)$

$(1,-2,k) = (3x, 0, -2x) + (2y, -y, -5y)$

$3x + 2y = 1$

$-y = -2$

$-2x - 5y = k$

The Augmented Matrix

$$[A:B] = \left[\begin{array}{cc:c} 3 & 2 & 1 \\ 0 & -1 & -2 \\ -2 & -5 & k \end{array}\right]$$

$R_3 \to 3R_3 + 2R_1$

$$\left[\begin{array}{cc:c} 3 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & -11 & 3k+2 \end{array}\right]$$

$R_3 \to R_3 - 11R_2$

$$\left[\begin{array}{cc:c} 3 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 3k+24 \end{array}\right]$$

If $3k+24$ will be equal to 0 then the system of linear eqn will be consistent.

$\therefore \rho(A) : \rho(A:B) = n = 2$

$\therefore v$ is a linear combination of v_1 & v_2

if $3k+24=0$

$\Rightarrow 3k = -24$

$\Rightarrow \boxed{k=-8}$ Ans.

Q. Express the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in the vector space of 2×2 matrices as a linear combination of $B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

Sol. $A = a_1B + a_2C + a_3D$

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = a_1\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + a_2\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + a_3\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} a_1 & a_1 \\ 0 & -a_1 \end{bmatrix} + \begin{bmatrix} a_2 & a_2 \\ -a_2 & 0 \end{bmatrix} + \begin{bmatrix} a_3 & -a_3 \\ 0 & 0 \end{bmatrix}$$

$a_1 + a_2 + a_3 = 3$

$a_1 + a_2 - a_3 = -1$

$-a_2 = 1 \qquad \boxed{a_2 = -1}$

$-a_1 = -2 \qquad \Rightarrow \boxed{a_1 = 2}$

$a_3 = 3 - a_1 - a_2$

$= 3 - 2 + 1$

$= \boxed{2}$

Subsitute a_1, a_2, a_3 in eq^n (1)

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} - 1\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Ans.

Q. Express the matrix $\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix}$ as linear combination of the matrices

$A = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$

Sol^n

$$A = a_1 A + a_2 B + a_3 C$$

$$\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = a_1 \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + a_3 \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \quad -(1)$$

$$\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 0 & -3a_1 \\ 2a_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2a_2 & a_2 \end{bmatrix} + \begin{bmatrix} 2a_3 & 3a_3 \\ 0 & 5a_3 \end{bmatrix}$$

(1) — $2a_3 = 2 \Rightarrow \boxed{a_3 = 1}$

(2) $\leftarrow$ $-3a_1 + 3a_3 = 0 \Rightarrow -3a_1 = -3a_3$

$\boxed{a_1 = a_3 = 1}$

(3) $\leftarrow$ $2a_1 + 2a_2 = 4$

$2 \times 1 + 2a_2 = 4$

$2a_2 = 2$

$\boxed{a_2 = 1}$

(4) $\leftarrow$ $a_2 + 5a_3 = -5$

$\Rightarrow 1 + 5 \times 1 = -5$

$6 \neq -5$ false

$\therefore a_1 = 1,\ a_2 = 1,\ a_3 = 1$

But not satisfies the eq^n (4) then for the given matrix can't be expressed as linear combination of A, B, C.

Q. Determine whether the vectors $v_1 = (1,4,9)$, $v_2 = (3,1,4)$, $v_3 = (9,3,12)$ are linearly dependent or linearly independent.

Sol^n $v = x v_1 + y v_2 + z v_3$

$$AX = 0$$

$$x v_1 + y v_2 + z v_3 = 0$$

$$x[1,4,9] + y[3,1,4] + z[9,3,12] = [0,0,0]$$

$$x + 3y + 9z = 0$$

$$4x + y + 3z = 0$$

$$9x + 4y + 12z = 0$$

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 4 & 1 & 3 \\ 9 & 4 & 12 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 4 & 1 & 3 \\ 9 & 4 & 12 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & -11 & -33 \\ 9 & 4 & 12 \end{bmatrix}$$

$R_3 \to R_3 - 9R_1$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & -11 & -33 \\ 0 & -23 & -69 \end{bmatrix}$$

$R_3 \to -\frac{1}{23} R_3$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & -11 & -33 \\ 0 & 1 & 3 \end{bmatrix}$$

$R_3 \to 11R_3 + R_2$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & -11 & -33 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$, but $n = 3$

$\therefore$ it is [redacted] inconsistent.

So that the vectors are linearly dependent.

Q. Determine whether or not each of the following forms a basis
$u_1 = (2, 2, 1)$, $u_2 = (1, 3, 7)$,
$u_3 = (1, 2, 2)$ in R^3.

Sol^n $xu_1 + yu_2 + zu_3 = 0$

$x(2,2,1) + y(1,3,7) + z(1,2,2) = 0$

$2x + y + z = 0$

$2x + 3y + 2z = 0$

$x + 7y + 2z = 0$

$AX = 0$

where, $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 7 & 2 \end{bmatrix}$

$R_1 \leftrightarrow R_3$

$\begin{bmatrix} 1 & 7 & 2 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 2R_1$

$\begin{bmatrix} 1 & 7 & 2 \\ 0 & -11 & -2 \\ 0 & -13 & -3 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$\begin{bmatrix} 1 & 7 & 2 \\ 0 & -11 & -2 \\ 0 & -2 & -1 \end{bmatrix}$

$R_3 \to 11R_3 - 2R_2$

$\begin{bmatrix} 1 & 7 & 2 \\ 0 & -11 & -2 \\ 0 & 0 & 7 \end{bmatrix}$

$\therefore x + 7y + 2z = 0$

$-11y - 2z = 0$

$7z = 0$

$\boxed{x = 0, y = 0, z = 0}$

$\rho(A) = 3$

extra

$\therefore \rho(A) = 3$ & number of unknowns $(n) = 3$

$\therefore$ we have unique solution and the solution is $x = 0, y = 0, z = 0$.

Hence, vectors u_1, u_2, u_3 are linearly independent and hence form a basis.

Q. Let W be the subspace of R^5 spaned by $u_1 = (1, 2, -1, 3, 4)$

$u_2 = (2, 4, -2, 6, 8)$

$u_3 = (1, 3, 2, 2, 6)$

$u_4 = (1, 4, 5, 1, 8)$

$u_5 = (2, 7, 3, 3, 9)$

Find a subset of vectors which forms a basis of W.

Sol.: Let $A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - R_1$

$R_5 \to R_5 - 2R_1$

$= \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$

$R_4 \to R_4 - 2R_3$

$R_5 \to R_5 - 3R_3$

$= \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{bmatrix}$

$\Rightarrow R_1, R_3, R_5$ forms a basis

$\therefore$ dimension 3