Module - I

* **Scalar & Vector fields :-** If every point (x, y, z) of a region or in space their corresponds scalar $\phi(x, y, z)$ then ϕ is called scalar point function and scalar field ϕ is defined in R.

ex - $\phi = x^2 + y^2 + z^2$ ex - $\phi = xyz^2$

If every point (x, y, z) of a region or in space their corresponds $\vec{A}(x, y, z)$ then $\vec{A}$ (vector A) is called vector point function and a vector fied $\vec{A}$ is defined in R.

ex - $x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ ex - $\vec{A} = xyz\hat{i} + x^2z\hat{j} + yz\hat{k}$

* **Gradient of scalar function :-** If ϕ is any scalar function then the gradient of ϕ is a vector defined by

$$\boxed{\text{grad } \phi \text{ or } \nabla \cdot \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}}$$

Gradient of scalar function is vector

$$\because \nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

$$\nabla \phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \phi(x, y, z)$$

$$\nabla \phi = \frac{\partial \phi(x, y, z)}{\partial x}\hat{i} + \frac{\partial \phi(x, y, z)}{\partial y}\hat{j} + \frac{\partial \phi(x, y, z)}{\partial z}\hat{k}$$

* **Unit Normal Vector →** If $\phi(x, y, z) = c$ represents a surface then unit normal vector to the surface is defined by

$$\boxed{\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}}$$

* **Directional Derivative →** Directional derivative of ϕ along vector a $(\vec{a})$ is given by

$$\boxed{\text{Directional Derivative (DD)} = \nabla \phi \cdot \hat{a}}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

Maximum Directional Derivative $= |\nabla \phi|$

* Diversions of a vector :- If vector $f\ (\vec{f}) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ then diversions of vector f is scalar defined by

$$\boxed{\text{diversion } \vec{f} \text{ or } \nabla\cdot\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}}$$

Diversion of vector function is scalar

* Solenoidal Vector → A vector $f\ (\vec{f})$ is said to be solenoidal if

$$\boxed{\text{diversion } \vec{f} = 0}$$

* Curl of a Vector function :→ If vector $f(\vec{f}) = f_1 i + f_2 j + f_3 k$, then curl of a vector function is vector defined by

$$\text{curl } \vec{f} \text{ or } \nabla\times\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\boxed{\text{curl } \vec{f} = i\left\{\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right\} - j\left\{\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right\} + k\left\{\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right\}}$$

* Irrotational vector :→ A vector $f(\vec{f})$ is said to be irrotational if $\boxed{\text{curl } \vec{f} = 0}$

* Angle b/w the two vector is given by $\boxed{\cos\theta = \hat{n}_1 \cdot \hat{n}_2}$

Q. If $\phi = x^2y^3z^4$. Find $\nabla\phi$, $|\nabla\phi|$ at $(1,-1,1)$

Sol^n → grad ϕ or $\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

$$= \frac{\partial(x^2y^3z^4)}{\partial x}\hat{i} + \frac{\partial}{\partial y}(x^2y^3z^4)\hat{j} + \frac{\partial}{\partial z}(x^2y^3z^4)\hat{k}$$

$$= y^3z^4(2x)\hat{i} + x^2z^4(3y^2)\hat{j} + x^2y^3(4z^3)\hat{k}$$

$$= 2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k}$$

$$(\nabla\phi)_{(1,-1,1)} = 2(1)(-1)^3(1)^4\hat{i} + 3(1)^2(-1)^2(1)^4\hat{j} + 4(1)^2(-1)^3(1)^3\hat{k}$$

$$= -2\hat{i} + 3\hat{j} - 4\hat{k}$$

$$|\nabla\phi| = \sqrt{(-2)^2 + (3)^2 + (-4)^2}$$

$$= \sqrt{29}$$

Q. Find the unit normal vector to the surface $xy+yz+zx = c$ at $(-1,2,3)$.

Sol^n → $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Let $\phi = xy + yz + zx - c$

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$= (y+z)i + (x+z)j + (y+x)k$$

$$(\nabla\phi)_{(-1,2,3)} = (2+3)i + (-1+3)j + (2-1)k$$

$$= 5i + 2j + k$$

$$|\nabla\phi| = \sqrt{5^2 + 2^2 + 1^2}$$

$$= \sqrt{25+4+1}$$

$$= \sqrt{30}$$

∴ unit normal vector $= \hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\hat{n} = \frac{5}{\sqrt{30}} i + \frac{2}{\sqrt{30}} j + \frac{k}{\sqrt{30}}$$

$$= \frac{5i + 2j + k}{\sqrt{30}}$$

Q. If $\phi = 3x^2y - y^3z^2$, find grad ϕ at $(1,-2,-1)$.

Solⁿ→
$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \frac{\partial(3x^2y - y^3z^2)}{\partial x}i + \frac{\partial(3x^2y - y^3z^2)}{\partial y}j + \frac{\partial(3x^2y - y^3z^2)}{\partial z}k$$

$$= (3y\,2x)i + (3x^2 - z^2 3y^2)j + (-y^3 2z)k$$

$$= (6xy)i + (3x^2 - 3y^2z^2)j - (2y^3z)k$$

$$(\nabla\phi)_{(1,-2,-1)} = [6(1)(-2)]i + [3(1)^2 - 3(-2)^2(-1)^2]j - (2(-2)^3(-1))k$$

$$= -12i + [3 - 12]j - 16k$$

$$= -12i - 9j - 16k$$

Q. Find unit normal vector to the surface $xy^3z^2 = 4$ at $(-1,-1,2)$

Solⁿ→ Let $\phi = xy^3z^2 - 4$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \frac{\partial(xy^3z^2 - 4)}{\partial x}i + \frac{\partial(xy^3z^2 - 4)}{\partial y}j + \frac{\partial(xy^3z^2 - 4)}{\partial z}\hat{k}$$

$$= (y^3z^2)\hat{i} + (3xy^2z^2)\hat{j} + (2xy^3z)\hat{k}$$

$$(\nabla\phi)_{(-1,-1,2)} = [(-1)^3(2)^2]\hat{i} + [3(-1)(-1)^2(2)^2]\hat{j} + [2(-1)(-1)^3(2)]\hat{k}$$

$$= -4\hat{i} - 12\hat{j} + 4\hat{k}$$

$$|\nabla\phi| = \sqrt{(-4)^2 + (-12)^2 + (4)^2}$$

$$= \sqrt{176}$$

$\therefore$ unit normal vector $= \hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{-4i - 12j + 4k}{\sqrt{176}}$$

Q. Find angle b/w the directions of normal to the surface $x^2yz = 1$ at $(-1,1,1)$ and $(1,-1,-1)$.

Sol.→ Let $\phi = x^2yz - 1$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \frac{\partial(x^2yz-1)}{\partial x}\hat{i} + \frac{\partial(x^2yz-1)}{\partial y}\hat{j} + \frac{\partial(x^2yz-1)}{\partial z}\hat{k}$$

$$= 2xyz\hat{i} + x^2z\hat{j} + x^2y\hat{k}$$

$$(\nabla\phi)_{(-1,1,1)} = -2i + j + k$$

$$|\nabla\phi| = \sqrt{(-2)^2 + (1)^2 + (1)^2} = \sqrt{6}$$

$$\therefore \hat{n}_1 = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2i+j+k}{\sqrt{6}}$$

$$(\nabla\phi)_{(1,-1,-1)} = 2i - j - k$$

$$|\nabla\phi| = \sqrt{(2)^2 + (-1)^2 + (-1)^2} = \sqrt{6}$$

$$\therefore \hat{n}_2 = \frac{2i-j-k}{\sqrt{6}}$$

∴ Angle b/w the two vectors is given by $\cos\theta = \hat{n}_1 \cdot \hat{n}_2$.

$$= \left(\frac{-2i+j+k}{\sqrt{6}}\right) \cdot \left(\frac{2i-j-k}{\sqrt{6}}\right)$$

$$= \frac{1}{6}\left[-2(2) + (1)(-1) + (1)(-1)\right]$$

$$= \frac{1}{6}\left[-4-1-1\right]$$

$$= -6/6$$

$$= -1$$

$$\theta = \cos^{-1}(-1)$$

Q. Find the angle b/w the surface $x^2y = 2-z$ & $x\log z = y^2-1$ at $(1,1,1)$.

Soln → Let $\phi_1 = x^2y-2+z$

$$\nabla\phi_1 = \frac{\partial(x^2y-2+z)}{\partial x}\hat{i} + \frac{\partial(x^2y-2+z)}{\partial y}\hat{j} + \frac{\partial(x^2y-2+z)}{\partial z}\hat{k}$$

$$= 2xy\,\hat{i} + x^2\hat{j} + \hat{k}$$

$$(\nabla\phi_1)_{(1,1,1)} = 2i+j+k$$

$$|\nabla\phi_1| = \sqrt{(2)^2+(1)^2+(1)^2} = \sqrt{6}$$

$$\therefore \hat{n}_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{2i+j+k}{\sqrt{6}}$$

Let $\phi_2 = x\log z - y^2+1$

$$\nabla\phi_2 = \log z\,\hat{i} + (-2y)\hat{j} + \frac{x}{z}\hat{k}$$

$$(\nabla\phi_2)_{(1,1,1)} = 0\hat{i} - 2\hat{j} + \hat{k}$$

$$|\nabla\phi_2| = \sqrt{0+4+1} = \sqrt{5}$$

$$\therefore \hat{n}_2 = \frac{0i-2j+k}{\sqrt{5}}$$

∴ The angle b/w the surfaces

$$\cos\theta = \hat{n}_1\cdot\hat{n}_2$$

$$= \left(\frac{2i+j+k}{\sqrt{6}}\right)\left(\frac{0i-2j+k}{\sqrt{5}}\right)$$

$$= \frac{1}{\sqrt{30}}(0-2+1)$$

$$= -\frac{1}{\sqrt{30}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right).$$

Q. Find the angle b/w the surfaces $x^2+y^2-z^2=4$ & $z = x^2+y^2-13$ at $(2,1,2)$

Soln → Let $\phi_1 = x^2+y^2-z^2-4$

$$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$$

$$(\nabla\phi_1)_{(2,1,2)} = 4\hat{i} + 2\hat{j} - 4\hat{k}$$

$$|\nabla\phi_1| = \sqrt{16+4+16} = \sqrt{36} = 6$$

$$\therefore \hat{n}_1 = \frac{4i+2j-4k}{6}$$

Let $\phi_2 = z - x^2 - y^2 + 13$

$$\nabla\phi_2 = -2x\hat{i} - 2y\hat{j} + \hat{k}$$

$$(\nabla\phi_2)_{(2,1,2)} = -4\hat{i} - 2\hat{j} + \hat{k}$$

$$|\nabla\phi_2| = \sqrt{16+4+1} = \sqrt{21}$$

$$\therefore \hat{n}_2 = \frac{-4\hat{i}-2\hat{j}+\hat{k}}{\sqrt{21}}$$

∴ The angle b/w the surfaces

$$\cos\theta = \hat{n}_1\cdot\hat{n}_2$$

$$= \left(\frac{4i+2j-4k}{6}\right)\cdot\left(\frac{-4\hat{i}-2\hat{j}+k}{\sqrt{21}}\right)$$

$$= \frac{1}{6\sqrt{21}}\left(4(-4)+(2)(-2)+(-4)(1)\right)$$

$$= \frac{1}{6\sqrt{21}}(-16-4-4)$$

$$= \frac{1}{\cancel{6}\sqrt{21}}(-\cancel{24}^{\,4})$$

$$= -4/\sqrt{21}$$

$$\therefore \theta = \cos^{-1}\left(-4/\sqrt{21}\right).$$

Q. Find directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ along the direction $\vec{a} = 2i - j - 2k$.

Sol. $D.D = \nabla\phi \cdot \hat{a}$

Let $\phi = x^2yz + 4xz^2$

$$\nabla\phi = \frac{\partial(x^2yz+4xz^2)}{\partial x}\hat{i} + \frac{\partial(x^2yz+4xz^2)}{\partial y}\hat{j} + \frac{\partial(x^2yz+4xz^2)}{\partial z}\hat{k}$$

$$= (2xyz + 4z^2)i + (x^2z)j + (x^2y + 8xz)k$$

$$(\nabla\phi)_{(1,-2,-1)} = (4+4)i + (-1)j + (-2-8)k$$

$$= 8i - j - 10k$$

$\therefore \vec{a} = 2i - j - 2k$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2i-j-2k}{\sqrt{4+1+4}} = \frac{2i-j-2k}{\sqrt{9}}$$

$\therefore D.D = \nabla\phi\cdot\hat{a}$

$$= (8i-j-10k)\cdot\frac{2i-j-2k}{\sqrt{9}}$$

$$= \frac{1}{3}\left(8(2)+(-1)(-1)+(-10)(-2)\right)$$

$$= \frac{1}{3}(16+1+20)$$

$$= 37/3.$$

Q. Find the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ at $(1,1,-1)$ along the direction $2i + j - k$

Sol. $D.D = \nabla\phi \cdot \hat{a}$.

Let $\phi = x^2 - 2y^2 + 4z^2$

$$\nabla\phi = (2x)i - 4yj + 8zk$$

$$(\nabla\phi)_{(1,1,-1)} = 2i - 4j - 8k$$

$\therefore \vec{a} = 2i + j - k$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2i+j-k}{\sqrt{4+1+1}} = \frac{2i+j-k}{\sqrt{6}}$$

$\therefore D.D = \nabla\phi\cdot\hat{a}$

$$= (2i-4j-8k)\cdot\frac{2i+j-k}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}(4-4+8)$$

$$= \frac{1}{\sqrt{6}}\times 8$$

$$= 8/\sqrt{6}$$

Q. find the D.D of xy^3+yz^3 at $(2,-1,1)$ in the direction of the vector $i+2j+2k$.

Sol^n → D.D $= \nabla\phi \cdot \hat{a}$

let $\phi = xy^3 + yz^3$

$$\nabla\phi = (y^3)i + (3xy^2+z^3)j + (3yz^2)k$$

$$(\nabla\phi)_{(2,-1,1)} = (-1^3)i + (3\times2\times(-1)^2+(1)^3)j + (3(-1)(1)^2)k$$

$$= -i + 7j - 3k$$

let $\vec{a} = i+2j+2k$

$$|\vec{a}| = \sqrt{1^2+2^2+2^2} = \sqrt{1+4+4} = \sqrt{9}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{i+2j+2k}{\sqrt{9}}$$

$\therefore$ D.D $= \nabla\phi\cdot\hat{a}$

$$= (-i+7j-3k)\cdot\left(\frac{i+2j+2k}{\sqrt{9}}\right)$$

$$= \frac{1}{3}(-1+14-6)$$

$$= 7/3.$$ Ans.

Q. find the D.D of $4e^{2x-y+z}$ at $(1,1,-1)$ in the direction towards the point $(-3,5,6)$ or $(-3i+5j+6k)$

Sol^n → D.D $= \nabla\phi\cdot\hat{a}$

let $\phi = 4e^{2x-y+z}$

$$\nabla\phi = (4e^{2x-y+z})\times 2i + (4e^{2x-y+z})(-1)j + (4e^{2x-y+z})k$$

$$(\nabla\phi)_{(1,1,-1)} = (8e^0)i - (4e^0)j + (4e^0)k$$

$$= 8i - 4j + 4k$$

let $\vec{a} = -3i+5j+6k$

$$|\vec{a}| = \sqrt{9+25+36} = \sqrt{70}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{-3i+5j+6k}{\sqrt{70}}$$

$\therefore$ D.D $= \nabla\phi\cdot\hat{a}$

$$= (8i-4j+4k)\frac{(-3i+5j+6k)}{\sqrt{70}}$$

$$= \frac{1}{\sqrt{70}}(-24-20+24)$$

$$= \frac{-20}{\sqrt{70}}$$

Q. Show that $3y^4z\,i + 4x^3z^2 j + 3x^2y^2 k$ is solenoidal.

Sol^n Let $\vec{f} = 3y^4z\,i + 4x^3z^2 j + 3x^2y^2 k$
$= f_1 i + f_2 j + f_3 k$

$\nabla\cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$= \frac{\partial(3y^4z)}{\partial x} + \frac{\partial(4x^3z^2)}{\partial y} + \frac{\partial(3x^2y^2)}{\partial z}$

$= 0$

$\therefore \operatorname{div}\vec{f} = 0$

Hence, $\vec{f}$ is solenoidal.

imp: **Q.** If $\vec{f} = (x+y+1)\hat{i} + \hat{j} + (-x-y)\hat{k}$ then show that $\vec{f}\cdot\operatorname{curl}\vec{f} = 0$.

Sol^n Let $\vec{f} = (x+y+1)\hat{i} + \hat{j} + (-x-y)\hat{k}$
$= f_1 \hat{i} + f_2\hat{j} + f_3\hat{k}$

$\operatorname{curl}\vec{f}$ or $\nabla\times\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+1) & 1 & (-x-y) \end{vmatrix}$

$= i\left\{\frac{\partial(-x-y)}{\partial y} - \frac{\partial(1)}{\partial z}\right\} - j\left\{\frac{\partial(-x-y)}{\partial x} - \frac{\partial(x+y+1)}{\partial z}\right\} + k\left\{\frac{\partial(1)}{\partial x} - \frac{\partial(x+y+1)}{\partial y}\right\}$

$= i\{-1-0\} - j\{-1-0\} + k\{0-1\}$
$= -i + j - k$

$\therefore \vec{f}\cdot\operatorname{curl}\vec{f} = ((x+y+1)\hat{i} + \hat{j} + (-x-y)\hat{k})\cdot(-\hat{i}+\hat{j}-\hat{k})$
$= (x+y+1)(-1) + (1)(1) + (-x-y)(-1)$
$= \cancel{-x}\cancel{-y}\cancel{-1}+\cancel{1}+\cancel{x}+\cancel{y}$
$= 0.$

Imp:
Q. If $\vec{f} = (3x^2y - z)\hat{i} + (xz^3+y)\hat{j} - 2x^3z^2\hat{k}$. Find $\operatorname{curl}(\operatorname{curl}\vec{f})$

Sol^n (first curl $\vec{f}$ then answer be $\vec{f}_1$ then find curl $\vec{f}_1$.)

Let $\vec{f} = (3x^2y - z)\hat{i} + (xz^3+y)\hat{j} + 2x^3z^2\hat{k}$

$\operatorname{curl}\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y - z & xz^3+y & -2x^3z^2 \end{vmatrix}$

$= i\left\{\frac{\partial(-2x^3z^2)}{\partial y} - \frac{\partial(xz^3+y)}{\partial z}\right\} - j\left\{\frac{\partial(-2x^3z^2)}{\partial x} - \frac{\partial(3x^2y-z)}{\partial z}\right\} + k\left\{\frac{\partial(xz^3+y)}{\partial x} - \frac{\partial(3x^2y-z)}{\partial y}\right\}$

$= i\{0 - 3xz^2\} - j\{6x^2z^2 - (-1)\} + k\{z^3 - 3x^2\}$
$= (-3xz^2)\hat{i} + (6x^2z^2 - 1)j + (z^3 - 3x^2)k$

Let $\vec{f} = (-3xz^2)\hat{i} + (6x^2z^2-1)\hat{j} + (z^3-3x^2)K$

Now,

$\operatorname{curl}(\operatorname{curl}\vec{f}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3xz^2 & 6x^2z^2-1 & z^3-3x^2 \end{vmatrix}$

$= i\left\{\frac{\partial(z^3-3x^2)}{\partial y} - \frac{\partial(6x^2z^2-1)}{\partial z}\right\} - j\left\{\frac{\partial(z^3-3x^2)}{\partial x} - \frac{\partial(-3xz^2)}{\partial z}\right\} + k\left\{\frac{\partial(6x^2z^2-1)}{\partial x} - \frac{\partial(-3xz^2)}{\partial y}\right\}$

$= i\{0 - 12x^2z\} - j\{-6x + 6xz\} + k\{12xz^2 - 0\}$

$\therefore \operatorname{curl}(\operatorname{curl}\vec{f}) = -12x^2z\hat{i} + (6x - 6xz)\hat{j} + 12xz^2\hat{k}$.

H.W ① Show that $\vec{f} = \frac{xi + yj}{x^2+y^2}$ is both solenoidal & irrotational.

Sol^n → $\vec{f} = \frac{x}{x^2+y^2} i + \frac{y}{x^2+y^2} j + 0\hat{k}$

∴ for solenoidal = division $\vec{f} = 0$

$$\nabla \cdot \vec{f} = \frac{\partial\left(\frac{x}{x^2+y^2}\right)}{\partial x} + \frac{\partial\left(\frac{y}{x^2+y^2}\right)}{\partial y} + \frac{\partial 0}{\partial z}$$

$$= \frac{[(x^2+y^2) - x(2x+0)]}{(x^2+y^2)^2} + \frac{[(x^2+y^2) - y(0+2y)]}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} + 0$$

$$\Rightarrow \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} + 0$$

$$\Rightarrow \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} + 0$$

$\Rightarrow 0.$

∴ div $\vec{f} = 0$ ∴ solenoidal

∴ for irrotational = curl $\vec{f} = 0$

$$\text{curl } \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$= i\{0-0\} - j\{0-0\} + k\left\{\frac{(x^2+y^2)(0) - y(2x+0)}{(x^2+y^2)^2} - \frac{(x^2+y^2)(0) - x(0+2y)}{(x^2+y^2)^2}\right\}$$

$$= 0 - 0 + k\left\{\frac{-2xy + 2xy}{(x^2+y^2)^2}\right\}$$

$= 0.$

∴ irrotational.

Q. Find the maximum D.D of $\phi = xy + yz + zx$ at $(1,1,1)$.

Sol^n $\nabla\phi = (y+z)i + (x+z)j + (y+x)k$

$(\nabla\phi)_{1,1,1} = 2i + 2j + 2k$

Max. D.D $= |\nabla\phi|$

$= \sqrt{2^2+2^2+2^2}$

$= \sqrt{12}$.

Q. If $\vec{A} = 2x^2 i - 3yzj + xz^2\hat{k}$ & $\phi = 2z - x^3 y$. compute

(1) $\vec{A}\cdot\nabla\phi$ and (2) $\vec{A}\times\nabla\phi$ at $(1,-1,1)$.

Sol^n Given, $\phi = 2z - x^3 y$

$\nabla\phi = -3yx^2 i - x^3 j + 2k$

$(\nabla\phi)_{(1,-1,1)} = 3i - j + 2k$

(1) $\vec{A}\cdot\nabla\phi$

$= (2x^2 i - 3yzj + xz^2 k)\cdot(3i - j + 2k)$

$= 6x^2 + 3yz + 2xz^2$

put $(1,-1,1)$

$= 6 - 3 + 2$

$= \underline{5}$.

(2) $\vec{A}\times\nabla\phi$

$(\vec{A})_{(1,-1,1)} = 2i + 3j + k$

$$\vec{A}\times\nabla\phi = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix}$$

$= i(6+1) - j(4-3) + k(-2-9)$

$= 7i - j - 11k$.

Q. Find div $\vec{f}$ and curl $\vec{f}$ where $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$.

Sol^n Let $\phi = x^3+y^3+z^3-3xyz$

$\vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

$\vec{F} = \underbrace{(3x^2-3yz)}_{f_1}i + \underbrace{(3y^2-3xz)}_{f_2}j + \underbrace{(3z^2-3xy)}_{f_3}k$

(1) div $\vec{f}$ or $\nabla\cdot\vec{f}$ =

$$\left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot\left[(3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k\right]$$

$= \frac{\partial(3x^2-3yz)}{\partial x} + \frac{\partial(3y^2-3xz)}{\partial y} + \frac{\partial(3z^2-3xy)}{\partial z}$

$= 6x + 6y + 6z$

(2) curl $\vec{f}$ or $\nabla\times\vec{f}$ $= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

$= i\left\{\frac{\partial(3z^2-3xy)}{\partial y} - \frac{\partial(3y^2-3xz)}{\partial z}\right\} - j\left\{\frac{\partial(3z^2-3xy)}{\partial x} - \frac{\partial(3x^2-3yz)}{\partial z}\right\} + k\left\{\frac{\partial(3y^2-3xz)}{\partial x} - \frac{\partial(3x^2-3yz)}{\partial y}\right\}$

$= i\{\cancel{-3x}+\cancel{3x}\} - j\{\cancel{-3y}+\cancel{3y}\} + k\{\cancel{-3z}+\cancel{3z}\}$

$= 0$.

Q. If $\vec{F} = \nabla(xy^3z^2)$ find
(a) $\text{div}\,\vec{F}$ & (b) $\text{curl}\,\vec{F}$
at $(1,-1,1)$.

Sol^n → let $\phi = xy^3z^2$

$$\vec{F} = \nabla\phi = y^3z^2 i + 3xy^2z^2 j + 2xy^3 z k$$

$$(\vec{F})_{1,-1,1} = -i + 3j - 2k$$

(a) $\text{div}\,\vec{F} =$ [redacted]

$$= \frac{\partial(y^3z^2)}{\partial x} + \frac{\partial(3xy^2z^2)}{\partial y} + \frac{\partial(2xy^3z)}{\partial z}$$

$$= 0 + 6xyz^2 + 2xy^3$$

$$(\text{div}\,\vec{F})_{1,-1,1} = -6 - 2 = -8.$$

(b) $$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$$

$$= i\left\{\frac{\partial(2xy^3z)}{\partial y} - \frac{\partial(3xy^2z^2)}{\partial z}\right\} - j\left\{\frac{\partial(2xy^3z)}{\partial x} - \frac{\partial(y^3z^2)}{\partial z}\right\} + k\left\{\frac{\partial(3xy^2z^2)}{\partial x} - \frac{\partial(y^3z^2)}{\partial y}\right\}$$

$$= i\{6xy^2z - 6xy^2z\} - j\{2y^3z - 2y^3z\} + k\{3y^2z^2 - 3y^2z^2\}$$

$$= 0.$$

Q. If $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$ find
(1) $\text{div}\,\vec{A}$ or $\nabla\cdot\vec{A}$ (3) $\text{div}(\text{curl}\,\vec{A})$ or $\nabla\cdot(\nabla\times\vec{A})$
(2) $\text{curl}\,\vec{A}$ or $\nabla\times\vec{A}$: at $(1,-1,1)$.

Sol^n → (1) $$\nabla\cdot\vec{A} = \frac{\partial(xz^3)}{\partial x} + \frac{\partial(-2x^2yz)}{\partial y} + \frac{\partial(2yz^4)}{\partial z}$$

$$= z^3 - 2x^2z + 8yz^3$$

$$(\nabla\cdot\vec{A})_{(1,-1,1)} = 1 - 2 - 8 = -9.$$

(2) $$\nabla\times\vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= i\left\{\frac{\partial(2yz^4)}{\partial y} - \frac{\partial(-2x^2yz)}{\partial z}\right\} - j\left\{\frac{\partial(2yz^4)}{\partial x} - \frac{\partial(xz^3)}{\partial z}\right\} + k\left\{\frac{\partial(-2x^2yz)}{\partial x} - \frac{\partial(xz^3)}{\partial y}\right\}$$

= [redacted] $i\{2z^4 + 2x^2y\} - j\{0 - 3xz^2\} + k\{-4xyz - 0\}$

$$= (2z^4 + 2x^2y)i + (3xz^2)j - (4xyz)k$$

$$(\nabla\times\vec{A})_{(1,-1,1)} = (2-2)i + 3j - (4\times1\times-1\times1)k = 3j + 4k.$$

(3) $\nabla\cdot(\nabla\times\vec{A}) =$ [redacted]

$$= \frac{\partial(0)}{\partial x} + \frac{\partial(3)}{\partial y} + \frac{\partial(4)}{\partial z}$$

$$= 0.$$

Imp Q. Show that $\vec{f} = (y+z)i + (z+x)j + (x+y)k$ is irrotational. And also find scalar function ϕ such that $\vec{f} = \nabla\phi$.

Sol^n → Let $\vec{f} = \underbrace{(y+z)}_{f_1}i + \underbrace{(z+x)}_{f_2}j + \underbrace{(x+y)}_{f_3}k$

$$\text{curl } \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= i\left\{\frac{\partial(x+y)}{\partial y} - \frac{\partial(z+x)}{\partial z}\right\} - j\left\{\frac{\partial(x+y)}{\partial x} - \frac{\partial(y+z)}{\partial z}\right\} + k\left\{\frac{\partial(z+x)}{\partial x} - \frac{\partial(y+z)}{\partial y}\right\}$$

$$= i[1-1] - j[1-1] + k[1-1]$$

$$= 0$$

∴ $\vec{f}$ is irrotational.

Given, $\vec{F} = \nabla\phi$

$$\Rightarrow (y+z)i + (z+x)j + (x+y)k = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = y+z$$

~~Integrate wrt 'x'~~

$$\int\frac{\partial}{\partial x}\phi = \int(y+z)\,dx + f_1(y,z)$$

$$\phi = xy + zx + f_1(y,z) \quad \text{— (1)}$$

$$\Rightarrow \frac{\partial\phi}{\partial y} = z+x$$

~~Integrate wrt 'y'~~

$$\int\frac{\partial}{\partial y}\phi = \int(z+x)\,dy + f_2(z,x)$$

$$\phi = zy + xy + f_2(z,x) \quad \text{— (2)}$$

$$\Rightarrow \frac{\partial\phi}{\partial z} = x+y$$

~~Integrate wrt 'z'~~

$$\int\frac{\partial}{\partial z}\phi = \int(x+y)\,dz + f_3(x,y)$$

$$\phi = xz + yz + f_3(x,y) \quad \text{— (3)}$$

Choose $f_1(yz) = yz$

$f_2(xz) = zx$

$f_3(xy) = xy$

ish eq^n mai xy nahi hai to ye lenge, similar eq^n (2) mai zx nahi hai to ushme zx and eq^n (1) mai yz

$$\therefore \boxed{\phi = xy + yz + zx} \quad \text{Ans.}$$

Q. Show that $\vec{f} = (z+\sin y)i + (x\cos y - z)j + (x-y)k$ is irrotational and hence find scalar function ϕ such that $\vec{f} = \nabla\phi$.

Sol^n→ Let $\vec{f} = (z+\sin y)i + (x\cos y - z)j + (x-y)k$

$$\nabla\times\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (z+\sin y) & (x\cos y - z) & (x-y) \end{vmatrix}$$

$$= i\left\{\frac{\partial(x-y)}{\partial y} - \frac{\partial(x\cos y - z)}{\partial z}\right\} - j\left\{\frac{\partial(x-y)}{\partial x} - \frac{\partial(z+\sin y)}{\partial z}\right\} + k\left\{\frac{\partial(x\cos y - z)}{\partial x} - \frac{\partial(z+\sin y)}{\partial y}\right\}$$

$$= i\{\cancel{-1}+\cancel{1}\} - j\{\cancel{1}-\cancel{1}\} + k\{\cancel{\cos y}-\cancel{\cos y}\}$$

$= 0.$

∴ f is irrotational.

Given, $\vec{f} = \nabla\phi$

$$\Rightarrow (z+\sin y)i + (x\cos y - z)j + (x-y)k = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = z+\sin y$$

Integrate wrt 'x'

$$\int\frac{\partial}{\partial x}\phi = \int(z+\sin y)\,dx + f_1(z,\sin y)$$

$$\phi = zx + x\sin y + f_1(z,\sin y) \quad -(1)$$

$$\Rightarrow \frac{\partial\phi}{\partial y} = x\cos y - z$$

Integrate wrt 'y'

$$\int\frac{\partial}{\partial y}\phi = \int(x\cos y - z)\,dy + f_2(x\cos y, z)$$

$$\phi = x\sin y - zy + f_2(x\cos y, z) \quad -(2)$$

$$\Rightarrow \frac{\partial\phi}{\partial z} = x-y$$

Integrate wrt 'z'

$$\int\frac{\partial}{\partial z}\phi = \int(x-y)\,dz + f_3(x,y)$$

$$\phi = xz - yz + f_3(x,y) \quad -(3)$$

choose $f_1(z,\sin y) = -zy$
$f_2(x\cos y, z) = zx$
$f_3(x,y) = x\sin y$

$$\therefore \boxed{\phi = x\sin y + zx - zy} \text{ Ans.}$$

Q. Find the value of constant 'a' such that the vector f
$\vec{f} = (axy - z^3)\hat{i} + (a-2)x^2\hat{j} + (1-a)xz^2 k$ is irrotational and hence find scalar function ϕ such that $\vec{f} = \nabla\phi$.

Sol^n → $\text{curl}\,\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix}$

$= i\{0 - 0\} - j\{(1-a)z^2 - (0 - 3z^2)\}$

$+ k\{(a-2)2x - ax\}$

$= i(0) - j[z^2 - az^2 + 3z^2] + k[2ax - 4x - ax]$

$= i[0] - j[4z^2 - az^2] + k[ax - 4x]$

$= i[0] - jz^2(4-a) + kx(a-4)$

$\text{curl}\,\vec{f} = 0i - (4-a)z^2 j + (a-4)xk$

if $\boxed{a = 4}$

∴ $\vec{f}$ is irrotational.

Given, $\nabla\phi = \vec{f}$

$\Rightarrow \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = (4xy - z^3)\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k}$.

$\Rightarrow \frac{\partial\phi}{\partial x} = 4xy - z^3$

Integrate wrt 'x'

$\int \frac{\partial}{\partial x}\phi = \int (4xy - z^3)\,dx + f_1(4xy, z^3)$

$\phi = \frac{4x^2}{2}y - z^3x + f_1(4xy, z^3)$

$\phi = 2x^2y - z^3x + f_1(4xy, z^3)$ — ①

$\Rightarrow \frac{\partial\phi}{\partial y} = 2x^2$

Int wrt 'y'

$\int \frac{\partial}{\partial y}\phi = \int 2x^2\,dy + f_2(2x^2)$

$\phi = 2x^2y + f_2(2x^2)$ — ②

$\Rightarrow \frac{\partial\phi}{\partial z} = -3xz^2$

Int wrt 'z'

$\int \frac{\partial\phi}{\partial z} = \int -3xz^2\,dz + f_3(-3xz^2)$

$\phi = -\frac{3xz^3}{3} + f_3(-3xz^2)$

$\phi = -xz^3 + f_3(-3xz^2)$ — ③

choose, $f_1(4xy, z^3) = 0$.

$f_2(2x^2) = -z^3x$

$f_3(-3xz^2) = 2x^2y$

∴ $\boxed{\phi = 2x^2y - zx^3}$ -Ans.

H.W

Ans $\phi = x^2yz^3$

Q. Show that $\vec{f} = 2xyz^3 i + x^2z^3 j + 3x^2yz^2 k$ is irrotational and find scalar function ϕ such that $\vec{f} = \nabla\phi$.

Sol^n:→

$$\text{curl}\,\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= i\left\{\cancel{3x^2z^2} - \cancel{3x^2z^2}\right\} - j\left\{\cancel{6xyz^2} - \cancel{6xyz^2}\right\} + k\left\{\cancel{2xz^3} - \cancel{2xz^3}\right\}$$

$= 0$

$\therefore \vec{f}$ is irrotational.

Given, $\vec{f} = \nabla\phi$

$\Rightarrow (2xyz^3)i + (x^2z^3)j + (3x^2yz^2)k = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

$\Rightarrow \frac{\partial\phi}{\partial x} = 2xyz^3$

~~Integrate wrt 'x'~~

$\int \frac{\partial\phi}{\partial x} = \int (2xyz^3)dx + f_1(2xyz^3)$

$\phi = x^2yz^3 + f_1(2xyz^3)$ — (1)

$\Rightarrow \frac{\partial\phi}{\partial y} = x^2z^3$

Integrate wrt 'y'

$\int \frac{\partial\phi}{\partial y} = \int (x^2z^3)dy + f_2(x^2z^3)$

$\phi = x^2z^3y + f_2(x^2z^3)$ — (2)

$\Rightarrow \frac{\partial\phi}{\partial z} = 3x^2yz^2$

~~Integrate wrt 'z'~~

$\int \frac{\partial\phi}{\partial z} = \int (3x^2yz^2)dz + f_3(3x^2yz^2)$

$\phi = x^2yz^3 + f_3(3x^2yz^2)$ — (3)

choose $f_1(2xyz^3) = f_2(x^2z^3) = f_3(3x^2yz^2) = x^2yz^3$

$\therefore \boxed{\phi = x^2yz^3}$

$a=6, b=3$

$\phi = 3x^2y + xz^3 - yz$

Q. Find the constants a & b such that $\vec{f} = (axy + z^3)i + (3x^2 - z)j + (bxz^2 - y)k$ is irrotational and also find scalar function ϕ such that $\vec{f} = \nabla\phi$.

Soln:-
$$\text{curl } \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy+z^3) & (3x^2-z) & (bxz^2-y) \end{vmatrix}$$

$= i\{-1-(-1)\} - j\{bz^2 - 3z^2\} + k\{6x - ax\}$

$= i[0] - j[bz^2 - 3z^2] + k[6x - ax]$

$= i[0] - jz^2[b-3] + kx[6-a]$

if $\boxed{b=3}$ & $\boxed{a=6}$

$\therefore \vec{f}$ is irrotational.

Given, $\vec{f} = \nabla\phi$

$\Rightarrow (6xy+z^3)i + (3x^2-z)j + (3xz^2-y)k = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

$\Rightarrow \frac{\partial\phi}{\partial x} = 6xy + z^3$

Integrate wrt 'x'

$\int\frac{\partial}{\partial x}\phi = \int(6xy+z^3)dx + f_1(6xy+z^3)$

$\phi = \frac{6x^2y}{2} + z^3x + f_1(6xy+z^3)$

$\phi = 3x^2y + z^3x + f_1(6xy+z^3)$ — ①

$\Rightarrow \frac{\partial\phi}{\partial y} = 3x^2 - z$

Integrate wrt 'y'

$\int\frac{\partial}{\partial y}\phi = \int(3x^2-z)dy + f_2(3x^2-z)$

$\phi = 3x^2y - zy + f_2(3x^2-z)$ — ②

$\Rightarrow \frac{\partial\phi}{\partial z} = 3xz^2 - y$

Integrate wrt 'z'

$\int\frac{\partial}{\partial z}\phi = \int(3xz^2-y)dz + f_3(3xz^2-y)$

$\phi = xz^3 - yz + f_3(3xz^2-y)$ — ③

Choose $f_1(6xy+z^3) = -zy$

$f_2(3x^2-z) = z^3x$

$f_3(3xz^2-y) = 3x^2y$

$\therefore \boxed{\phi = 3x^2y + xz^3 - yz}$ Ans.

Q. Find a, b, c show that the vector function $\vec{f} = (x+y+az)i + (bx+2y-z)j + (x+cy+2z)k$ is irrotational. Also find scalar function ϕ such that $\vec{f} = \nabla\phi$.

Sol^n curl $\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+az & bx+2y-z & x+cy+2z \end{vmatrix}$

$= i\{c-(-1)\} - j\{a-1\} + k\{b-1\}$

$= i[c+1] - $ [redacted] $+ k[b-1]$

if $\boxed{c=-1}$ and $\boxed{b=1}$ and $\boxed{a=1}$

$\therefore \vec{f}$ is irrotational.

Given, $\vec{f} = \nabla\phi$

$\Rightarrow (x+y+z)i + (x+2y-z)j + (x-y+2z)k = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

$\Rightarrow \frac{\partial\phi}{\partial x} = x+y+z$

Integrate w.r.t 'x'

$\int\frac{\partial\phi}{\partial x} = \int(x+y+z)dx + f_1(x\,y\,z)$

$\phi = \frac{x^2}{2} + yx + zx + f_1(x+y+z)$ — (1)

$\Rightarrow \frac{\partial\phi}{\partial y} = x+2y-z$

Integrate w.r.t 'y'

$\int\frac{\partial\phi}{\partial y} = \int(x+2y-z)dy + f_2(x+2y-z)$

$\phi = xy + y^2 - zy + f_2(x+2y-z)$ — (2)

$\Rightarrow \frac{\partial\phi}{\partial z} = x-y+2z$

Integrate w.r.t 'z'

$\int\frac{\partial\phi}{\partial z} = \int(x-y+2z)dz + f_3(x-y+2z)$

$\phi = xz - yz + z^2 + f_3(x-y+2z)$ — (3)

Choose $f_1(x+y+z) = -zy + y^2 + z^2$

$f_2(x+2y-z) = zx$

$f_3(x-y+2z) = xy$

$\phi = \boxed{\frac{x^2}{2} + xy + xz + y^2 - yz + z^2}$ Ans.

* Vector Integration.

Line Integral: Let $\vec{f}(xyz)$ be any vector field function along the curve c, then the line integral of vector $\vec{f}$ along the curve c is given by $\boxed{\int_c \vec{f}\cdot\vec{dr}}$.

- If c is the closer curve then it is represented by $\boxed{\oint_c \vec{f}\cdot\vec{dr}}$

- If $\vec{f}$ represents force acted upon by a particle in displacing along the curve c then, $\boxed{W = \int_c \vec{F}\cdot\vec{dr}}$ it represents the total work done by the force.

- If $\vec{f}$ is said to be irrotational, $\boxed{\oint_c \vec{F}\cdot\vec{dr} = 0}$

Q. If $\vec{F} = xy\hat{i} + yz\hat{j} + zx\hat{k}$. Evaluate $\int_c \vec{F}\cdot\vec{dr}$ where c is the curve represented, $x=t$ $y=t^2$ $z=t^3$, in $-1 \le t \le 1$

Sol^n→ $\vec{F}\cdot\vec{dr} = xy\,dx + yz\,dy + zx\,dz$

given $x=t$ | $y=t^2$ | $z=t^3$

$dx=dt$ | $dy=2t\,dt$ | $dz=3t^2dt$

$$\vec{F}\cdot\vec{dr} = t\,t^2dt + t^2t^3\,2t\,dt + t^3\,t\,3t^2dt$$
$$= t^3dt + 2t^6dt + 3t^6dt$$
$$= (t^3 + 2t^6 + 3t^6)dt$$
$$= (t^3 + 5t^6)dt$$

on Integrating

$$\int_c \vec{F}\cdot\vec{dr} = \int_{t=-1}^{1} (t^3 + 5t^6)dt$$

$$= \left(\frac{t^4}{4} + \frac{5t^7}{7}\right)_{-1}^{1}$$

$$= \frac{1}{4}\left(\cancel{1^4 - (-1)^4}\right) + \frac{5}{7}\left(1^7 - (-1)^7\right)$$

$$= \frac{5}{7} \times 2$$

$$\therefore \boxed{\int_c \vec{F}\cdot\vec{dr} = \frac{10}{7}} \text{ Ans}$$

$\nearrow -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{5}\hat{k}$

(8) Ans.

Q. If $\vec{f} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$. Evaluate $\int_c \vec{F}\cdot d\vec{r}$

from $(0,0,0)$ to $(1,1,1)$

along the curve given by

$x=t \quad y=t^2 \quad z=t^3$.

Sol$^n \rightarrow$

~~$\vec{F}\cdot d\vec{r} = xy\,dx + yz\,dy + zx\,dz$~~

~~given $x=t$ | $y=t^2$ | $z=t^3$~~

~~$dx=dt$ | $dy=2t\,dt$ | $dz=3t^2dt$~~

~~$\vec{F}\cdot d\vec{r} = 3t^2+6t^2$~~

$\vec{F}\cdot d\vec{r} = (3x^2+6y)dx - (14yz)dy + (20xz^2)dz$

given $x=t$ | $y=t^2$ | $z=t^3$

$dx=dt$ | $dy=2t\,dt$ | $dz=3t^2dt$

$\vec{F}\cdot d\vec{r} = (3t^2+6t^2)dx - (14t^2t^3)dy + 20(t(t^3)^2)dz$

$= (9t^2)dx - (14t^5)dy + (20t^7)dz$

$= \cancel{(9t^2) - (14t^5) + (20t^7)}$

$= 9t^2dt - 14t^5\,2t\,dt + 20t^7\,3t^2dt$

$= (9t^2 - 28t^6 + 60t^9)dt$

on integration

$$\int_c \vec{F}\cdot d\vec{r} = \int_0^1 (9t^2 - 28t^6 + 60t^9)dt$$

$$= \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right]_0^1$$

$= 3(1-0) - 4(1-0) + 6(1-0)$

$3 - 4 + 6$

$= 9 - 4$

$= 5$. Ans.

Q(1) If $\vec{f} = xy\hat{i} - z\hat{j} + x^2\hat{k}$

where c is the curve $x=t^2 \quad y=2t \quad z=t^3$

from $t=0$ to $t=1$. Evaluate $\int_c \vec{F}\times d\vec{r}$.

Sol$^n \rightarrow$ given, $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$

$d\vec{r} = dx\hat{i} - dy\hat{j} + dz\hat{k}$

$$\vec{F}\times d\vec{r} = \begin{vmatrix} i & j & k \\ xy & -z & x^2 \\ dx & dy & dz \end{vmatrix}$$

$= i[-z\,dz - x^2dy] - j[xy\,dz - x^2dx] + k[xy\,dy + z\,dx]$

given, $x=t^2$ | $y=2t$ | $z=t^3$

$dx=2t\,dt$ | $dy=2\,dt$ | $dz=3t^2dt$

$\vec{F}\times d\vec{r} = i[-t^3\,3t^2dt - t^4\,2dt] - j[t^2\,2t\,3t^2dt - t^4\,2t\,dt] + k[t^2\,2t\,2dt + t^3\,2t\,dt]$

$= i[-3t^5dt - 2t^4dt] - j[6t^5dt - 2t^5dt] + k[4t^3dt + 2t^4dt]$

$= -3t^5i\,dt - 2t^4i\,dt - 6t^5j\,dt + 2t^5j\,dt + 4t^3k\,dt + 2t^4k\,dt$.

on integration

$$\int_c \vec{F}\times d\vec{r} = \int_0^1 (-3t^5i - 2t^4i - 6t^5j + 2t^5j + 4t^3k + 2t^4k)dt$$

$$= \int_0^1 (-3t^5i - 2t^4i - 4t^5j + 4t^3k + 2t^4k)dt$$

$$= \left[\frac{-3t^6i}{6}\right]_0^1 - \left[\frac{2t^5i}{5}\right]_0^1 - \left[\frac{4t^6j}{6}\right]_0^1 + \left[\frac{4t^4k}{4}\right]_0^1 + \left[\frac{2t^5k}{5}\right]_0^1$$

$= -\frac{1}{2}(1-0)i - \frac{2}{5}(1-0)i - \frac{2}{3}(1-0)j$

$+ (1-0)K + \frac{2}{5}(1-0)K$

$= -\frac{1}{2}i - \frac{2}{5}i - \frac{2}{3}j + K + \frac{2}{5}K$

$= \left(\frac{-5-4}{10}\right)i - \frac{2}{3}j + \left(\frac{5+2}{5}\right)K$

$= \boxed{\frac{-9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k}}$ Ans.

13/06/2023

(2) $\int_c \phi \cdot \vec{dr}$ when $\phi = 2xyz^2$

w.k.t $\vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$= 2tdt\hat{i} + 2dt\hat{j} + 3t^2dt\hat{k}$

$= (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})dt$

given $\phi = 2xyz^2$

$= 2(t^2)(2t)(t^3)^2$

$= 2t^2 \times 2t \times t^6 = 4t^9$

$\phi \vec{dr} = 4t^9(2ti + 2j + 3t^2k)dt$

$= (8t^{10}\hat{i} + 8t^9\hat{j} + 12t^{11}\hat{k})dt$

Integration

$\int_c \phi \vec{dr} = \int_{t=0}^{1} (8t^{10}\hat{i} + 8t^9\hat{j} + 12t^{11}\hat{k})dt$

$= \left[\frac{8t^{11}}{11}\right]_0^1 \hat{i} + \left[\frac{8t^{10}}{10}\right]_0^1 \hat{j} + \left[\frac{12t^{12}}{12}\right]_0^1 \hat{k}$

$= \boxed{\frac{8}{11}\hat{i} + \frac{8}{10}\hat{j} + \hat{k}}$ Ans.

Q. If $\vec{F} = x^2\hat{i} + xy\hat{j}$, Evaluate $\int_c \vec{F} \cdot \vec{dr}$ from (0,0) to (1,1) along [redacted] the line a) $y = x$ and parabola b) $y = \sqrt{x}$

Sol → $\vec{F} \cdot \vec{dr} = (x^2)dx + (xy)dy$

(1) The line $y = x$

$dy = dx \quad (0 \le x \le 1)$ → yahan y ka jagah x put karenge to x ka limit dalenge.

$\vec{F} \cdot \vec{dr} = x^2 dx + x(x)dx$

$= 2x^2 dx$

on Integrating

$\int_c \vec{F} \cdot \vec{dr} = \int_{x=0}^{1} 2x^2 dx = \left[\frac{2x^3}{3}\right]_0^1 = \boxed{\frac{2}{3}}$ Ans.

(2) a parabola $y = \sqrt{x}$

$y^2 = x$

$x = y^2$ → in place of x we put y so here we take limit of y.

$dx = 2y\,dy \quad (0 \le y \le 1)$

$\vec{F} \cdot \vec{dr} = x^2 dx + xy\,dy$

$= (y^2)^2\, 2y\,dy + (y^2)y\,dy$

$= 2y^5 dy + y^3 dy$

on Integrating

$\int_c \vec{F} \cdot \vec{dr} = \int_{y=0}^{1} 2y^5 dy + \int_{y=0}^{1} y^3 dy$

$= \left[\frac{2y^6}{6_3}\right]_0^1 + \left[\frac{y^4}{4}\right]_0^1$

$= \frac{1}{3} + \frac{1}{4}$

$= \frac{4+3}{12} = \boxed{7/12}$ Ans.

IMP

Q. Find the total work done by the force represented by $\vec{F} = 3xy\hat{i} - y\hat{j} + $ [redacted] $2z\hat{k}$ in moving particle around the circle $x^2+y^2 = 4 = 2 = r$

circle → θ → 0 to 2π

Sol^n → In circle equation [redacted] the parametric form $x = r\cos\theta$, $y = r\sin\theta$

$x = r\cos\theta$	$y = r\sin\theta$	$z = 0$
$x = 2\cos\theta$	$y = 2\sin\theta$	$dz = 0$
$dx = -2\sin\theta\, d\theta$	$dy = 2\cos\theta\, d\theta$	

$\vec{F} = 3xy\hat{i} - y\hat{j} + 2z\hat{k}$

$\vec{F} = 3(2\cos\theta)(2\sin\theta)\hat{i} - 2\sin\theta\hat{j} + \cancel{2\times 0}\hat{k}$

$\vec{F} = 12\sin\theta\cos\theta\,\hat{i} - 2\sin\theta\,\hat{j}$

$\vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$= (-2\sin\theta\, d\theta)\hat{i} + (2\cos\theta\, d\theta)\hat{j} + 0$

$= -2\sin\theta\, d\theta\,\hat{i} + 2\cos\theta\, d\theta\,\hat{j}$

$\vec{F}\cdot\vec{dr} = (12\sin\theta\cos\theta)(-2\sin\theta\, d\theta) + (-2\sin\theta)(2\cos\theta\, d\theta)$

$= -24\sin^2\theta\cos\theta\, d\theta - 4\sin\theta\cos\theta\, d\theta$

on integration

$$\int_C \vec{F}\cdot d\vec{r} = \int_{\theta=0}^{2\pi} -24\sin^2\theta\cos\theta\, d\theta - \int_{\theta=0}^{2\pi} 4\sin\theta\cos\theta\, d\theta$$

= put $t = \sin\theta$
$dt = \cos\theta\, d\theta$

$$= \int_0^{2\pi} -24t^2\, dt - \int_0^{2\pi} 4t\, dt$$

$$= \left[-24^{8}\frac{t^3}{3}\right]_0^{2\pi} - \left[\frac{4^{2}t^2}{2}\right]_0^{2\pi}$$

$= -8\left[\sin^3\theta\right]_0^{2\pi} - 2\left[\sin^2\theta\right]_0^{2\pi}$

$= -8\left[\sin^3(2\pi) - \sin^3(0)\right] - 2\left[\sin^2(2\pi) - \ldots\right.$

$= -8\left[0^3 - 0^3\right] - 2\left[0^2 - 0^2\right]$

$= -8(0) - 2(0)$

$= 0$

$\therefore \boxed{\vec{F}\cdot\vec{dr} = 0}$ — Ans.

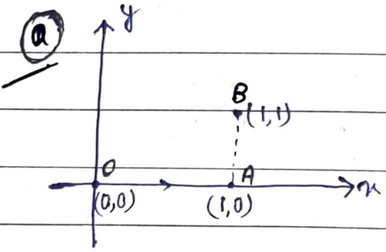
Q. Evaluate $\int_C \vec{F}\cdot\vec{dr}$ where $\vec{F} = xy\hat{i} + (x^2 \ldots$

along ① the path of the straight line from (0,0) to (1,0) and then (1,1).

② the straight line joining the origin (0,0) and (1,2)

Sol^n → [redacted] $\vec{F}\cdot\vec{dr} = xy\,dx + (x^2+y^2)\,dy$

(a)



$$\int_C \vec{F}\cdot\vec{dr} = \int_{OA} \vec{F}\cdot\vec{dr} + \int_{AB} \vec{F}\cdot\vec{dr} \quad \text{→ ①}$$

along OA

$y = 0$, $0 \le x \le 1$
$dy = 0$

$\vec{F}\cdot\vec{dr} = xy\,dx + (x^2+y^2)\,dy$

$= x(0)\,dx + (x^2+0)\,0$

$= 0$

$$\boxed{\int_{OA} \vec{F}\cdot\vec{dr} = 0}$$

along AB

$x = 1 \quad , \quad 0 \le y \le 1$

$dx = 0$

$$\vec{F}\cdot\vec{dr} = xy\,dx + (x^2+y^2)\,dy$$
$$= 1\cdot y\,(0) + (1+y^2)\,dy$$
$$= (1+y^2)\,dy$$

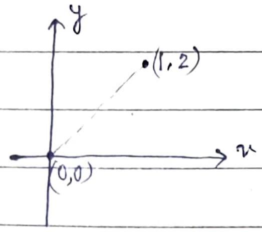
on integration.

$$\int_{AB} \vec{F}\cdot\vec{dr} = \int_{y=0}^{1} (1+y^2)\,dy$$
$$= \left[y + \frac{y^3}{3}\right]_0^1$$
$$= 1 + \frac{1}{3}$$
$$\boxed{= 4/3}$$

put in eqn ①, then

$$\therefore \int_C \vec{F}\cdot\vec{dr} = \int_{OA} \vec{F}\cdot\vec{dr} + \int_{AB} \vec{F}\cdot\vec{dr}$$
$$= 0 + \frac{4}{3}$$
$$\boxed{= \frac{4}{3}} \quad \text{Ans.}$$

(b)



$(x_1, y_1) = (0,0)$

$(x_2, y_2) = (1,2)$

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y-0}{x-0} = \frac{2-0}{1-0}$$

$$\frac{y}{x} = 2$$

$y = 2x$

$dy = 2dx$

$x \to (0,1)$

$$\vec{F}\cdot\vec{dr} = xy\,dx + (x^2+y^2)\,dy$$
$$= x(2x)\,dx + (x^2+4x^2)\,2\,dx$$
$$= 2x^2dx + 10x^2dx$$
$$= 12x^2dx$$

on integrating

$$\int_C \vec{F}\cdot\vec{dr} = \int_{x=0}^{1} 12x^2\,dx$$
$$= \left[\frac{\cancel{12}^{\,4}x^3}{\cancel{3}}\right]_0^1$$
$$\boxed{= 4} \quad \text{Ans.}$$

* Green's Theorem in a plane

If R is a closed region of the xy plane bounded by simple closed curve C and if M and N are two continuous functions of x, y having continuous first order partial derivative in the region R then

$$\boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

Q. Verify Green's Theorem in a plane for $\oint (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where c is the boundary of the region c enclosed by $y = \sqrt{x}$ and $y = x^2$.

Sol^n → $y = \sqrt{x}$ & $y = x^2$

$x^2 = \sqrt{x}$

$= x^4 = x$

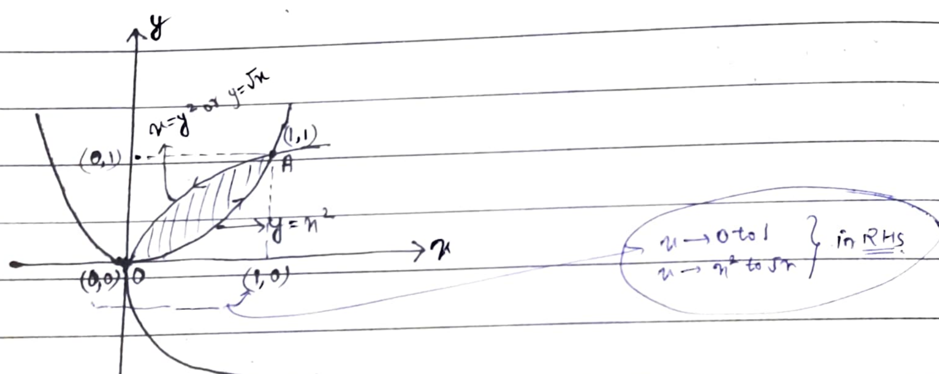
$= x^4 - x = 0$

$= x(x^3 - 1) = 0$

$= x = 0, x = 1$

If $x = 0$ $y = 0$ $(0,0)$

$x = 1$ $y = 1$ $(1,1)$



we have green's theorem

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\left\{ \begin{aligned} M &= 3x^2 - 8y^2 \\ N &= 4y - 6xy \end{aligned} \right\}$$

$\underline{LHS} = \oint_C M\,dx + N\,dy$

$= \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy$

$LHS = I_1 + I_2$

$I_1 = \int_{OA} M\,dx + N\,dy$

$y = x^2 \qquad x \to (0,1)$

$dy = 2x\,dx$

$I_1 = \int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy$

$= \int_{x=0}^{1} (3x^2 - 8x^4)dx + (4x^2 - 6x \times x^2)2x\,dx$

$= \int_{x=0}^{1} (3x^2 - 8x^4 + 8x^3 - 12x^4)dx$

$= \int_{x=0}^{1} (3x^2 + 8x^3 - 20x^4)dx$

$= \left[\frac{\cancel{3}x^3}{\cancel{3}} + \frac{\overset{2}{\cancel{8}}x^4}{\cancel{4}} - \frac{\overset{4}{\cancel{20}}x^5}{\cancel{5}}\right]_0^1$

$= (1 + 2 - 4)$

$\boxed{I_1 = -1}$

$I_2 = \int_{AO} (3x^2 - 8y^2)dx + (4y - 6xy)dy$

$x = y^2 \qquad y \to (1,0)$

$dx = 2y\,dy$

$I_2 = \int_{y=1}^{0} (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)dy$

$= \int_{y=1}^{0} (6y^5 - 16y^3 + 4y - 6y^3)dy$

$= \int_{y=1}^{0} (4y - 22y^3 + 6y^5)dy$

$= \left(\frac{\overset{2}{\cancel{4}}y^2}{\cancel{2}} - \frac{\overset{11}{\cancel{22}}y^4}{\cancel{4}_2} + \frac{\cancel{6}y^6}{\cancel{6}}\right)_1^0$

$= \left(2y^2 - \frac{11}{2}y^4 + y^6\right)_1^0$

$\boxed{I_2 = 5/2}$ $\therefore$ eqⁿ (I) becom

$I = I_1 + I_2 = -1 + \frac{5}{2} = \boxed{I = 3/2}$

RHS $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$M = 3x^2 - 8y^2 \qquad , \qquad N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$

$= \iint_R (-6y + 16y)\,dx\,dy$

$= \iint_R 10y\,dx\,dy$

$= \int_{x=0}^{1} \left[\int_{y=x^2}^{y=\sqrt{x}} 10y\,dy\right] dx$

$= \int_{x=0}^{1} \left[\frac{\cancel{10}y^2}{\cancel{2}}\right]_{x^2}^{\sqrt{x}} dx$

$= \int_{x=0}^{1} 5[x - x^4]\,dx$

$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$

$= 5\left[\left(\frac{1}{2} - \frac{1}{5}\right) - (0-0)\right]$

$= 5\left(\frac{5-2}{10}\right)$

$= 5\left(\frac{3}{10}\right)$

$= \frac{15}{10}$

$= \boxed{3/2}$

$\therefore$ LHS = RHS

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NOTE- Suppose in the problem asked to evaluate the line integral using Green's theorem, we need to do only the RHS part of the theorem for obtaining the desire result.

$I_2 = -1$

$I_1 = 19/20$

$LHS = 1/20$

Q. Verify Green's theorem $\oint_C (xy + y^2)\,dx + x^2\,dy$ where c is the closed curve of the region bounded by $y = x$ & $y = x^2$.

NOTE Suppose in the problem asked to evaluate the line integral using Green's theorem, we need to do only the RHS part of the theorem for obtaining the desire result. (also in Stoke's theorem)

$I_1 = 19/20$ $I_2 = -1$
$LHS = -1/20$

Q. Verify Green's theorem $\oint_C (xy+y^2)\,dx + x^2\,dy$ where c is the closed curve of the region bounded by $y=x$ & $y=x^2$.

Sol^n we have, $y=x$ & $y=x^2$

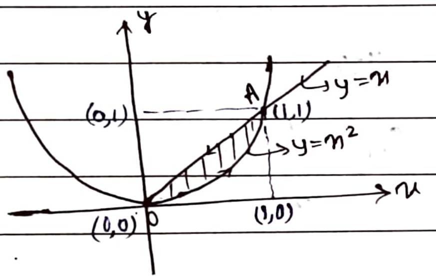
$\therefore x^2 = x$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1)=0$

$\Rightarrow x = 0, 1$

If $x=0 \rightarrow y=0$ $(0,0)$ are points of intersection
$x=1 \rightarrow y=1$ $(1,1)$



we have Green's Theorem

$$\oint M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\underline{LHS} = \oint M\,dx + N\,dy$

$= \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy.$

$= I_1 \quad + I_2$

$I_1 = \int_{OA} M\,dx + N\,dy$

$y = x^2$
$dy = 2x\,dx \quad x \rightarrow (0,1)$

$= \int_{OA} (xy+y^2)\,dx + x^2\,dy$

$$= \int_{x=0}^{1} (x\cdot x^2 + x^4)\,dx + x^2(2x\,dx)$$

$$= \int_{x=0}^{1} (x^3 + x^4 + 2x^3)\,dx$$

$$= \int_{x=0}^{1} (3x^3 + x^4)\,dx$$

$$= \left[\frac{3x^4}{4}\right]_0^1 + \left[\frac{x^5}{5}\right]_0^1$$

$$= 3/4 + 1/5$$

$$= \frac{15+4}{20}$$

$$\boxed{I_1 = 19/20}$$

$$I_2 = \int_{AO} (xy+y^2)\,dx + x^2\,dy$$

$y = x$

$dy = dx$ $\quad y \to (1,0)$

$$= \int_{y=1}^{0} (x\cdot x + x^2)\,dx + x^2(dx)$$

$$= \int_{y=1}^{0} 3x^2\,dx$$

$$= \left[\frac{\cancel{3}x^3}{\cancel{3}}\right]_1^0$$

$$\boxed{I_2 = -1}$$

$\therefore$ $I = I_1 + I_2$

$$= \frac{19}{20} - 1$$

$$= \boxed{\frac{-1}{20}}$$

RHS.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$M = xy + y^2$ $\quad N = x^2$

$\frac{\partial M}{\partial y} = x + 2y$ $\quad \frac{\partial N}{\partial x} = 2x$

$$= \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \iint_R (x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1}\left[\int_{y=x^2}^{y=x} (x-2y)\,dy\right] dx$$

$$= \int_{x=0}^{1} \left[xy - \frac{\cancel{2}y^2}{\cancel{2}}\right]_{y=x^2}^{y=x} dx$$

$$= \int_{x=0}^{1} (x^2 - x^3) - (x^2 - x^4)\,dx$$

$$= \int_{x=0}^{1} (\cancel{x^2} - x^3 - \cancel{x^2} + x^4)\,dx$$

$$= \left[-\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= -1/4 + 1/5$$

$$= \frac{-5+4}{20}$$

$$= \boxed{\frac{-1}{20}}$$

$\therefore$ LHS = RHS

Q. Use Green's Theorem to evaluate $\int_c (x^2+y^2)\,dx + 3x^2y\,dy$ where c is the circle $x^2+y^2=4$. Trace the positive sense.

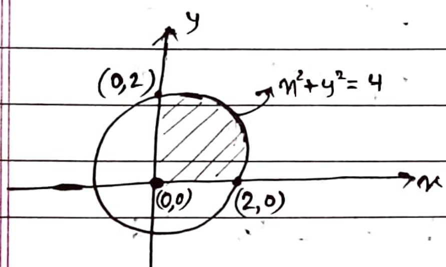
Soln → Here $M = x^2+y^2$ $\qquad N = 3x^2y$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 6xy$$

we have, $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

RHS $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$= \iint_R (6xy - 2y)\,dx\,dy$$



$$= \int_{x=0}^{x=2} \left[\int_{y=0}^{y=\sqrt{4-x^2}} (6xy - 2y)\,dy \right] dx$$

$$= \int_{x=0}^{2} \left[\frac{\cancel{6}^{3} x y^2}{\cancel{2}}\right]_0^{\sqrt{4-x^2}} - \left[\frac{\cancel{2}y^2}{\cancel{2}}\right]_0^{\sqrt{4-x^2}} dx$$

$$= \int_{x=0}^{2} \left[3x[4-x^2] - (4-x^2) \right] dx$$

$$= \int_{x=0}^{2} (12x - 3x^3 - 4 + x^2)\,dx$$

$$= \int_{x=0}^{2} (-4 + 12x + x^2 - 3x^3)\,dx$$

$$= \left[-4x + \frac{\cancel{12}^{6} x^2}{\cancel{2}} + \frac{x^3}{3} - \frac{3x^4}{4} \right]_0^2$$

$$= -8 + 24 + \frac{8}{3} - 12$$

$$= \frac{-24 + 72 + 8 - 36}{3} = \frac{20}{3}$$

$\therefore \oint M\,dx + N\,dy = \boxed{\frac{20}{3}}$ sq. units

$2/3$ sq units

Q. Using Green's theorem in a plane, Evaluate $\int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$ where c is the boundary of the region bounded by $x=0,\ y=0,\ x+y=1$.

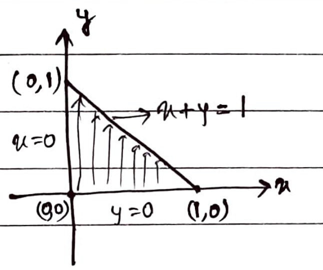
Soln:- Here $M = 2x^2 - y^2 \qquad N = x^2+y^2$

$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = 2x$

we have $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

RHS $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= \iint_R (2x+2y)\,dx\,dy$



$= \int_{x=0}^{1} \left[\int_{y=0}^{y=1-x} (2x+2y)\,dy \right] dx$

$= \int_{x=0}^{1} 2x\,[y]_0^{1-x} + \left[\frac{2y^2}{2}\right]_0^{1-x} dx$

$= \int_{x=0}^{1} 2x(1-x) + (1-x)^2 \, dx$

$= \int_{x=0}^{1} (2x - 2x^2 + 1 + x^2 - 2x)\,dx$

$= \left[\frac{2x^2}{2} - \frac{2x^3}{3} + x + \frac{x^3}{3} - \frac{2x^2}{2}\right]_0^1$

$= \cancel{1} - \frac{2}{3} + 1 + \frac{1}{3} - \cancel{1}$

$= \frac{3-2+1}{3}$

$= 2/3$ sq units

$\therefore \oint M\,dx + N\,dy = 2/3$ sq. units

Q. Evaluate $\int_C (xy+y^2)\,dx + x^2\,dy$. Using Green's theorem when c is the closed curve of the region bounded by $y=x$ & $y=x^2$.

Sol: Here $M = xy+y^2$; $N = x^2$

$$\frac{\partial M}{\partial y} = x+2y \quad ; \quad \frac{\partial N}{\partial x} = 2x$$

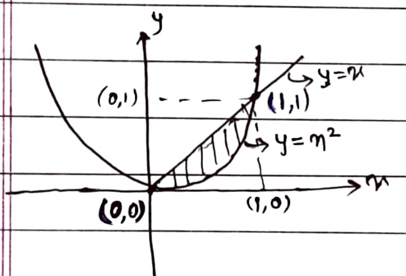
we have;

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

RHS: $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$= \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \iint_R (x-2y)\,dx\,dy$$



$y=x$, $y=x^2$

$y^2 - y = 0$

$y(y-1) = 0$

$y = 0, 1$

~~If $y=0 \to x=0$ (0,0)~~

$y=1 \to x=1$ (1,1)

$$= \int_{x=0}^{1} \left[\int_{y=x^2}^{y=x} (x-2y)\,dy \right] dx$$

$$= \int_{x=0}^{1} \left[xy - \frac{2y^2}{2} \right]_{y=x^2}^{y=x} dx$$

$$= \int_{x=0}^{1} (x^2 - x^3) - (x^2 - x^4)\,dx$$

$$\cancel{= \int_{x=0}^{1} (2x^2 - x^3 - x^4)\,dx}$$

$$\cancel{= \left[\frac{2x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}\right]_0^1}$$

$$\cancel{= \left[\frac{2}{3} - \frac{1}{4} - \frac{1}{5}\right]}$$

$$\cancel{\frac{40-15-12}{60} = \frac{13}{60}}$$

$$= \int_{x=0}^{1} (\cancel{x^2} - x^3 - \cancel{x^2} + x^4)\,dx$$

$$= \left[-\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= -\frac{1}{4} + \frac{1}{5}$$

$$= \frac{-5+4}{20} = \boxed{-\frac{1}{20} \text{ sq. units}}$$ Ans.

* Q. Using Green's theorem to find the area b/w parabolas $x^2=4y$ & $y^2=4x$

Sol^n → If c is a simple closed curve in xy plane then

$$\int_c \frac{x\,dy - y\,dx}{2} \text{ represent area of the enclosed curve}$$

$$\therefore \text{Area} = \int_c \frac{x}{2}dy - \frac{y}{2}dx$$, where c is the area b/w $x^2=4y$ & $y^2=4x$

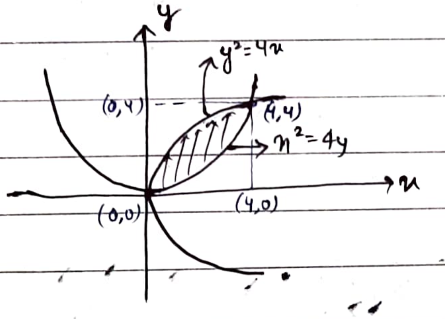
Here, $M = -y/2$ $\quad N = \frac{x}{2}$

$\frac{\partial m}{\partial y} = -\frac{1}{2}$ $\quad \frac{\partial N}{\partial x} = \frac{1}{2}$

we have $\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

RHS $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dx\,dy$



$x^2=4y \quad y^2=4x$

$y^2 = 4\sqrt{4y}$

$y^2 = 4\times 2\sqrt{y}$

sq. both sides

$y^4 = 64y$

$y^4 - 64y = 0$

$y(y^3-64)=0$

$y=0$ & $y^3-64=0$

$y=0$ & $y^3=4^3$

$y=4$

If $y=0 \to x=0$ (0,0)

$y=4 \to x=4$ (4,4)

$= \iint_R dx\,dy$

$= \int_{x=0}^{x=4}\left[\int_{y=\frac{x^2}{4}}^{y=\sqrt{4x}} dy\right] dx$

$= \int_{x=0}^{4} [y]_{x^2/4}^{\sqrt{4x}}\, dx$

$= \int_{x=0}^{4} \left(\sqrt{4x} - \frac{x^2}{4}\right) dx$

$= \left[2 \cdot \frac{x^{3/2}}{3/2}\right]_0^4 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$

$= \frac{4}{3}\left(x^{3/2}\right)_0^4 - \frac{1}{12}\left(x^3\right)_0^4 = \frac{16}{3}$ sq. units $\quad \therefore \oint_c M\,dx + N\,dy = \frac{16}{3}$ sq. units

πab sq. units

Q. Employ Green's Theorem in a plane to show that the area enclosed by a plane curve c is $\int_c \frac{x\,dy - y\,dx}{2}$ and hence find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. ①

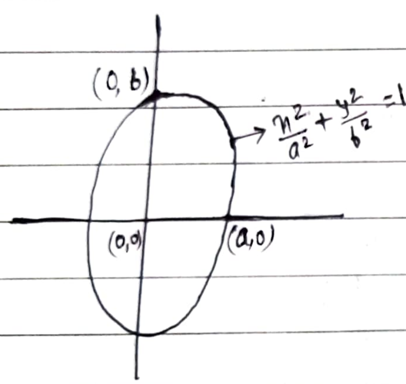
Sol^n $M = -\frac{y}{2}$ $\quad N = \frac{x}{2}$

$\frac{\partial M}{\partial y} = -\frac{1}{2}$ $\quad \frac{\partial N}{\partial x} = \frac{1}{2}$

we have, $\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

RHS $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dx\,dy$



$x = r\cos\theta \qquad y = r\sin\theta$

$dx = -r\sin\theta\,d\theta \qquad dy = r\cos\theta\,d\theta$

put in eq^n ①

→ this is for LHS

[redacted]

$= \iint_R dx\,dy$

$= \int_{x=0}^{a} \int_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx$

$= \int_{x=0}^{a} [y]_0^{b/a\sqrt{a^2-x^2}}\,dx$

$= \int_{x=0}^{a} \frac{b}{a}\sqrt{a^2-x^2}\,dx$

[redacted]

$= \frac{b}{a}\left[\left(\frac{a}{2}\sqrt{a^2-a^2} - \frac{0}{2}\sqrt{a^2-0^2}\right) + \left(\frac{a^2}{2}\sin^{-1}\frac{a}{a} - \frac{a^2}{2}\sin^{-1}\frac{0}{a}\right)\right]$

$= \frac{b}{a}\left[\frac{a^2}{2}\sin^{-1}1 - \frac{a^2}{2}\sin^{-1}0\right]$

$= \frac{b}{a}\left[\frac{a^2}{2} \times \frac{\pi}{2} - \frac{a^2}{2} \times 0\right]$

$= \frac{b a^2 \pi}{4a} = \boxed{\frac{\pi ab}{4} \text{ sq. units}}$ $\Rightarrow 4 \times \frac{\pi ab}{4} = \boxed{\pi ab \text{ sq. units}}$

Ans.

* Stoke's Theorem

If S is a surface bounded by a simple closed curve c and if $\vec{F}$ is any continuously differentiable vector function then

$$\oint_C \vec{F}\cdot\vec{dr} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

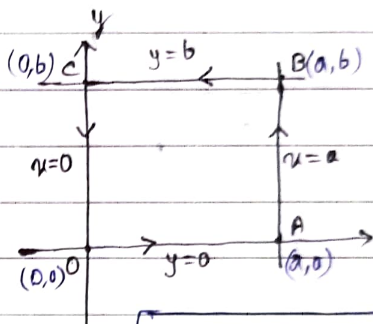
where, $\hat{n}ds = dydz\hat{i} + dzdx\hat{j} + dxdy\hat{k}$

If surface projection is along xy plane then $\vec{ds} = dxdy\hat{k}$

Q. Verify stoke's theorem for the vector $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by $x=0$, $x=a$ & $y=0$, $y=b$

Soln → we have,

$$\oint \vec{F}\cdot\vec{dr} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$



$$LHS = \oint_C \vec{F}\cdot\vec{dr}$$

$$= \int_{OA} \vec{F}\cdot\vec{dr} + \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CO} \vec{F}\cdot\vec{dr}$$

$$LHS = I_1 + I_2 + I_3 + I_4 \quad \text{—} (1)$$

$\iint$ → x → horizontal, y → vertical
↳ $x \to (0,a)$, $y \to (0,b)$
for RHS

$$\vec{F}\cdot\vec{dr} = ((x^2+y^2)\hat{i} - 2xy\hat{j} + 0\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$
$$= (x^2+y^2)dx - 2xy\,dy$$

along OA : $y=0$ $\quad x \to (0,a)$
$dy = 0$

$$I_1 = \int_{OA} \vec{F}\cdot\vec{dr}$$

$$= \int_{x=0}^{x=a} (x^2+y^2)dx - 2xy\,dy$$

$$= \int_{x=0}^{x=a} (x^2+0)dx - 0$$

$$= \int_{x=0}^{x=a} x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \boxed{\frac{a^3}{3}}$$

along AB: $x = a$ $\quad y \to (0, b)$

$dx = 0$

$$I_2 = \int_{AB} \vec{F} \cdot \vec{dr}$$

$$= \int_{y=0}^{y=b} (x^2+y^2)\,dx - 2xy\,dy$$

$$= \int_{y=0}^{y=b} \cancel{(a^2+y^2)(0)} - 2ay\,dy$$

$$= \int_{y=0}^{y=b} -2ay\,dy$$

$$= \left[-\frac{2ay^2}{2} \right]_0^b$$

$$= \boxed{-ab^2}$$

along BC: $y = b$ $\quad x \to (a, 0)$

$dy = 0$

$$I_3 = \int_{x=a}^{x=0} (x^2+b^2)\,dx - 0$$

$$= \left[\frac{x^3}{3} + b^2 x \right]_a^0$$

$$= 0 - \left(\frac{a^3}{3} + ab^2 \right)$$

$$= \boxed{-\frac{a^3}{3} - ab^2}$$

along CO: $x = 0$ $\quad y \to (b, 0)$

$dx = 0$

$$I_4 = \int_{y=b}^{y=0} \cancel{(0+y^2)(0)} - \cancel{2(0)}y\,dy$$

$$= \boxed{0}$$

$\therefore$ eqn ① becomes

$$LHS = I_1 + I_2 + I_3 + I_4$$

$$= \cancel{\frac{a^3}{3}} - ab^2 - \cancel{\frac{a^3}{3}} - ab^2 + 0$$

$$\boxed{LHS = -2ab^2}$$

$$RHS = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\,ds$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= i\,\cancel{[0-0]}^{0} - j\,\cancel{[0-0]}^{0} + k[-2y-2y]$$

$$= -4y\hat{k}$$

$$\text{curl}\,\vec{f} \cdot \hat{n}\,ds = (-4y\hat{k}) \cdot (dy\,dz\,\hat{i} + dz\,dx\,\hat{j} + dx\,dy\,\hat{k})$$

$$= -4y\,dx\,dy$$

$$= \iint_S \text{curl}\,\vec{f} \cdot \hat{n}\,ds$$

$$= \iint_S -4y\,dx\,dy$$

$$= \int_{x=0}^{x=a} \left[\int_{y=0}^{y=b} -4y\,dy \right] dx$$

$$= -\int_{x=0}^{x=a} \left[\frac{4y^2}{2} \right]_0^b dx$$

$$= -2 \int_{x=0}^{a} [y^2]_0^b\,dx$$

$$= -2 \int_{x=0}^{a} (b^2 - 0)\,dx$$

$$= -2b^2 (x)_0^a$$

$$= -2b^2(a-0) = \boxed{-2ab^2} = RHS \quad \therefore LHS = RHS$$

Q. Evaluate $\int_C (xy\,dx + xy^2\,dy)$ by stokes theorem where C is the square in the xy plane with the vertices $(1,0), (-1,0), (0,1), (0,-1)$.

Sol^n: we have;

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$\text{RHS} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}$$

$$= i\{\cancel{0-0}\} - j\{\cancel{0-0}\} + k\{y^2 - x\}$$

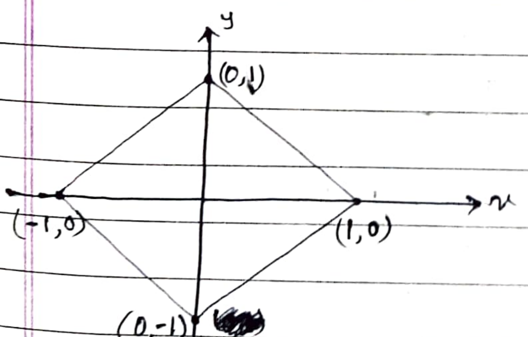
$$= (y^2 - x)\hat{k}$$

$$\text{curl}\,\vec{f}\cdot\hat{n}\,ds = (y^2-x)\hat{k}\cdot(dy\,dz\,\hat{i} + dz\,dx\,\hat{j} + dx\,dy\,\hat{k})$$

$$= (y^2-x)\,dx\,dy$$

$$= \iint_S \text{curl}\,\vec{f}\cdot\hat{n}\,ds$$

$$= \iint (y^2-x)\,dx\,dy$$



$x \to$ horizontal $\to (-1,1)$

$y \to$ vertical $\to (-1,1)$

$$= \int_{x=-1}^{x=1}\left[\int_{y=-1}^{y=1} (y^2 - x)\,dy\right]dx$$

$$= \int_{x=-1}^{1}\left[\frac{y^3}{3} - xy\right]_{-1}^{1}$$

$$= \int_{x=-1}^{1}\left[\frac{1}{3} + \frac{1}{3}\right] - \left[x + x\right]\,dx$$

$$= \int_{x=-1}^{1}\left(\frac{2}{3} - 2x\right)dx$$

$$= \left[\frac{2x}{3}\right]_{-1}^{1} - \left[\frac{2x^2}{2}\right]_{-1}^{1}$$

$$= \left[\frac{2}{3} + \frac{2}{3}\right] - \cancel{\left[1 - 1\right]}^{0}$$

$$= \boxed{\frac{4}{3}} \text{ Ans.}$$

[redacted]

[redacted]

Q. Use Stoke's theorem for $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and c is the boundary of upper half of the sphere $x^2+y^2+z^2=1$.

only x & y

Soln → we have;

$$\oint_C \vec{F}\cdot\vec{dr} = \iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds$$

RHS $= \iint_S \text{curl}\vec{f}\cdot\hat{n}\,ds$

$$\text{curl}\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$= i\{0-1\} - j\{1-0\} + k\{0-1\}$

$= -i - j - k$

$\text{curl}\vec{f}\cdot\hat{n}ds = (-i-j-k)\cdot(dydz\,\hat{i} + dzdx\,\hat{j} + dxdy\,\hat{k})$

$= -dydz - dzdx - dxdy$

$= \iint_S \text{curl}\vec{f}\cdot\hat{n}\,ds$

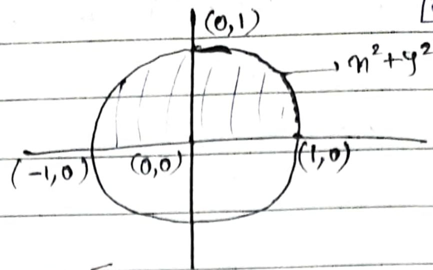
$= \iint_S -dydz - dzdx - dxdy$

*[only consider dx, dy not dz in Stoke's theorem]

we have to find upper half of sphere hence ($z=0$, $dz=0$)

$= \iint_S -dxdy$

$x^2+y^2+z^2=1$, $z \to 0$; $x^2+y^2=1$



$$= \int_{x=-1}^{x=1}\left[\int_{y=0}^{y=\sqrt{1-x^2}} -\,dy\right]dx$$

$$= \int_{x=-1}^{1} -[y]_0^{\sqrt{1-x^2}}\,dx$$

$$= -\int_{x=-1}^{1}\sqrt{1-x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right]_{x=-1}^{x=1}$$

$$= \left(\cancel{\frac{1}{2}\sqrt{1-1^2} - \frac{0}{2}\sqrt{1-1^2}}\right) + \left(\frac{1}{2}\sin^{-1}(1) - \frac{1}{2}\sin^{-1}(-1)\right)$$

$= \cancel{\frac{1}{2}\times\frac{\pi}{2} + \frac{\pi}{2}}$

$= \cancel{\frac{\pi}{4} + \frac{\pi}{2}}$

$= \cancel{\frac{\pi + 2\pi}{4}}$ $\cancel{\frac{2\pi}{4}}$

$= \cancel{\frac{3\pi}{4}}$

$= \frac{1}{2}\times\frac{\pi}{2} + \frac{1}{2}\times\frac{\pi}{2}$

$= \frac{\pi}{4} + \frac{\pi}{4}$

$= 2\pi/4$

$= \pi/2$ $\quad = \frac{\pi}{\cancel{2}} \times \cancel{2}$

$= \pi$ Ans.

$$\boxed{\sqrt{1-x^2} \to \frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}(x)}$$

Q. ~~Verify~~ Evaluate Stoke's theorem for $\vec{F}=(2x-y)\hat{i}-yz^2\hat{j}-y^2z\hat{k}$ where s is the upper half of sphere $x^2+y^2+z^2=1$, c is its boundary:

Sol^n → we have;

$$\oint_c \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{f}\cdot\hat{n}\,ds$$

RHS $= \iint_S \text{curl}\,\vec{f}\cdot\hat{n}\,ds$

$$\text{curl}\,\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

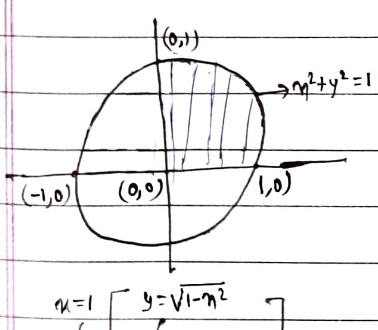
$$= i\{\cancel{-2yz+2yz}\} - j\{\cancel{0-0}\} + k\{0-(-1)\}$$

$= k$

$$\text{curl}\,\vec{f}\cdot\hat{n}\,ds = (k)\cdot(dydz\,\hat{i}+dzdx\,\hat{j}+dxdy\,\hat{k}) = dxdy$$

$$= \iint_S \text{curl}\,\vec{f}\cdot\hat{n}\,ds$$

$$= \iint_S dxdy$$



$$= \int_{x=0}^{x=1}\left[\int_{y=0}^{y=\sqrt{1-x^2}} dy\right]dx$$

$$= \int_{x=0}^{x=1} [y]_0^{\sqrt{1-x^2}}\,dx$$

$$= \int_{x=0}^{x=1} \sqrt{1-x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}(x)\right]_{x=0}^{x=1}$$

$$= \left(\cancel{\frac{1}{2}\sqrt{1-1^2}}^{\,0} - \frac{0}{2}\sqrt{1-0^2}\right) + \left(\frac{1}{2}\sin^{-1}(1) - \frac{1}{2}\sin^{-1}(0)\right)$$

$$= \left(\frac{1}{2}\times\frac{\pi}{2} - \frac{1}{2}\times 0\right)$$

$= \pi/4$

$\therefore$ curl $\vec{f}\cdot\hat{n}\,ds = 4\times\pi/4 = \boxed{\pi}$ Ans:

Condition sirf s nikaln ke liye hai → par answer pure sphere ka nikalna hai.

Repeated Question

Q. If $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$, evaluate $\oint_C \vec{F} \times d\vec{r}$ where C is the curve

$x = t^2, y = 2t, z = t^3$ from $t \to (0,1)$

Soln → given, $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$

$d\vec{r} = dx\hat{i} + dy\hat{j} - dz\hat{k}$

$$\vec{F} \times d\vec{r} = \begin{vmatrix} i & j & k \\ xy & -z & x^2 \\ dx & dy & dz \end{vmatrix}$$

$$= i\left[-z\,dz - x^2 dy\right] - j\left[xy\,dz - x^2 dx\right] + k\left[xy\,dz + z\,dx\right]$$

given, $x = t^2$ | $y = 2t$ | $z = t^3$

$dx = 2t\,dt$ | $dy = 2dt$ | $dz = 3t^2dt$

$$\vec{F} \times d\vec{r} = i\left[-t^3\, 3t^2dt - t^4\, 2dt\right] - j\left[t^2 2t\, 3t^2dt - t^4\, 2t\,dt\right] + k\left[t^2 2t\, 2dt + t^3\, 2t\,dt\right]$$

$$= i\left[-3t^5dt - 2t^4dt\right] - j\left[6t^5dt - 2t^5dt\right] + k\left[4t^3dt + 2t^4dt\right]$$

$$= -3t^5 i\,dt - 2t^4 i\,dt - 6t^5 j\,dt + 2t^5 j\,dt + 4t^3 k\,dt + 2t^4 k\,dt$$

on integration

$$\int_C \vec{F} \times d\vec{r} = \int_0^1 \left(-3t^5 i - 2t^4 i - 6t^5 j + 2t^5 j + 4t^3 k + 2t^4 k\right)dt$$

$$= \int_0^1 \left(-3t^5 i - 2t^4 i - 4t^5 j + 4t^3 k + 2t^4 k\right)dt$$

$$= \left[-\frac{3t^6}{6}\right]_0^1 i - \left[\frac{2t^5}{5}\right]_0^1 i - \left[\frac{4t^6}{6}\right]_0^1 j + \left[\frac{4t^4}{4}\right]_0^1 k + \left[\frac{2t^5}{5}\right]_0^1 k$$

$$= -\frac{1}{2}(1-0)i - \frac{2}{5}(1-0)i - \frac{2}{3}(1-0)j + (1-0)k + \frac{2}{5}(1-0)k$$

$$= -\frac{1}{2}i - \frac{2}{5}i - \frac{2}{3}j + k + \frac{2}{5}k$$

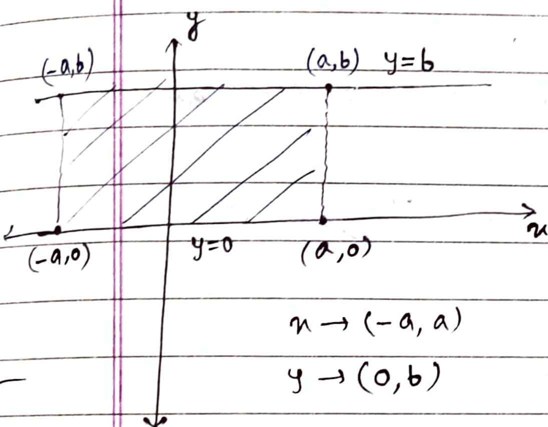
$$= \left(\frac{-5-4}{10}\right)i - \frac{2}{3}j + \left(\frac{5+2}{5}\right)k$$

$$= \boxed{-\frac{9}{10}i - \frac{2}{3}j + \frac{7}{5}k}$$ Ans

Q. Evaluate $\int_c (x^2+y^2)\hat{i} - 2xy\hat{j}$ by Stoke's theorem the rectangle bounded by the line $x = \pm a$, $y=0$ & $y=b$.

Sol^n: we have;

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_C \text{curl}\vec{F}\cdot\hat{n}\,ds$$



$x \to (-a, a)$

$y \to (0, b)$

RHS = $\text{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$

$= \hat{i}[\cancel{0-0}]^{0} - \hat{j}[\cancel{0-0}]^{0} + k[-2y-2y]$

$= -4y\hat{k}$

$\text{curl}\vec{f}\cdot\hat{n}\,ds = (-4y\hat{k})\cdot(dydz\hat{i} + dzdx\hat{j} + dxdy\hat{k})$

$= -4y\,dxdy$

$= \iint_S \text{curl}\vec{f}\cdot\hat{n}\,ds$

$= \iint_S -4y\,dxdy$

$= \int_{x=-a}^{x=a}\left[\int_{y=0}^{y=b} -4y\,dy\right]dx$

$= -\int_{x=-a}^{a}\left[4y^2/2\right]_0^b dx$

$= -2\int_{x=-a}^{a}\left[y^2\right]_0^b dx$

$= -2\int_{x=-a}^{a}(b^2-0)\,dx$

$= -2b^2(x)_{-a}^{a}$

$= -2b^2(a+a)$

$= \boxed{-4ab^2}$ Ans.