STATEMENT OF PURPOSE: RESEARCH

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1. Introduction

I received my Bachelors degree from the University of Oregon in 2008 with the intention of studying number theory in graduate school. I had done a research project as an undergraduate on the Kronecker-Weber theorem, and thought I would enjoy pursuing similar topics as a graduate student. My first semester at the University of Hawaii at Manoa I took a course in lattice theory from my future adviser J.B. Nation. Afterward I found myself increasingly interested in problems involving lattices and universal algebra.

My studies during and after graduate school, beyond the requisite work in algebra and analysis, have centered on universal algebra and lattice theory. This includes enough logic and model theory to enable my studies in these areas. Universal algebra, and the lattice theory so essential for it, seems a highly specialized field when one first encounters it, but it has numerous connections to other fields of mathematics and computer science. Most of this is at a theoretical level. However, there are applications in computing and data analysis.

I have been fortunate to have the opportunity to dabble in other areas of mathematics as well. I have worked during the summer on the topic of modern cryptology with a resident number theorist at UH Manoa, Michelle Manes, and have taken a course in information theory offered by the school of engineering. I am hopeful that the generalized viewpoint of universal algebra and lattice theory will enable me to make contributions to these or other fields. Any opportunity to find practical applications for my areas of research interest would be most welcome. The majority of my graduate and post graduate career has been spent building the toolbox needed to approach hard problems. I look forward to applying these tools to problems in mathematical research.

2. Research Background

A *lattice* is an ordered set $\langle L, \leq \rangle$ such that each pair of elements of L has both a greatest lower bound and a least upper bound. We use the terms *join* and least upper bound interchangeably. Similarly the terms *meet* and greatest lower bound are used interchangeably. We use the notation $x \vee y$ to mean the join of elements x and y, and $x \wedge y$ to mean the meet of these two elements. An equivalence relation θ on $\mathcal{L}$ is called a *congruence* if $x \ \theta \ y$ implies $(x \wedge z) \ \theta \ (y \wedge z)$ and $(x \vee z) \ \theta \ (y \vee z)$.

An element p of a lattice $\mathcal{L}$ is *join irreducible* if $p \leq a \vee b$ implies that $a \leq p$ or $b \leq p$. A *meet irreducible* element of a lattice $\mathcal{L}$ is defined dually by reversing all inequalities and replacing least upper bounds with greatest lower bounds. If a lattice is finite every join irreducible element p has a unique lower cover denoted

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p_*, that is to say for all $x \in \mathcal{L}$ if $x < p$ then $x \leq p_*$. For a meet irreducible element m in a finite lattice its unique upper cover is denoted m^*. We use the notation $JI(\mathcal{L})$ and $MI(\mathcal{L})$ to refer to the set of all join irreducible and, respectively, meet irreducible elements of a lattice $\mathcal{L}$.

A subset C of a lattice $\mathcal{L}$ is *convex* if whenever a and c are in C and $a \leq b \leq c$ then $b \in C$. For any $x, z \in \mathcal{L}$ define $z/x = \{y \in \mathcal{L} \colon x \leq y \leq z\}$. Such a set is called an *interval* in $\mathcal{L}$. Intervals are by definition convex sets. Other examples of convex sets are *lower pseudo-intervals*, which are finite unions of intervals that share the same least element. An *upper pseudo-interval* is the dual concept.

Let C be a convex subset of a lattice $\mathcal{L}$ and let $L[C]$ denote the disjoint union $(L \setminus C) \cup (C \times 2)$. Order $L[C]$ by $x \leq y$ if one of the following conditions holds:

$$\begin{aligned}
&(1)\ x,\ y \in L \setminus C \text{ and } x \leq_{\mathcal{L}} y,\\
&(2)\ (x,i),\ (y,j) \in C \times 2, \text{ and } x \leq_{C\times 2} y,\\
&(3)\ x \in L \setminus C,\ (y,j) \in C \times 2, \text{ and } x \leq_{\mathcal{L}} y,\\
&(4)\ (x,i) \in C \times 2,\ y \in L \setminus C, \text{ and } x \leq_{\mathcal{L}} y.
\end{aligned}$$

Let $\mathcal{K}$ and $\mathcal{L}$ be lattices. A homomorphism $h \colon \mathcal{K} \to \mathcal{L}$ is called *lower bounded* if for every $a \in \mathcal{L}$, the set $h^{-1}(\{x \in \mathcal{L} \colon x \geq a\})$ is either empty or has a least element. An *upper bounded* homomorphism is defined dually, and a homomorphism is called *bounded* if it is both upper and lower bounded. A lattice $\mathcal{L}$ is said to be *lower bounded* if for any lattice $\mathcal{K}$ and homomorphism $f \colon \mathcal{K} \to \mathcal{L}$ it follows that f is a lower bounded homomorphism. The notion of an *upper bounded* lattice is defined dually, and a lattice is called *bounded* if it is both lower and upper bounded.

Theorem 2.1 (Day). *A finite lattice $\mathcal{L}$ is lower bounded if and only if $\mathcal{L}$ can be constructed from the one-element lattice by doubling a sequence of lower pseudo-intervals.*

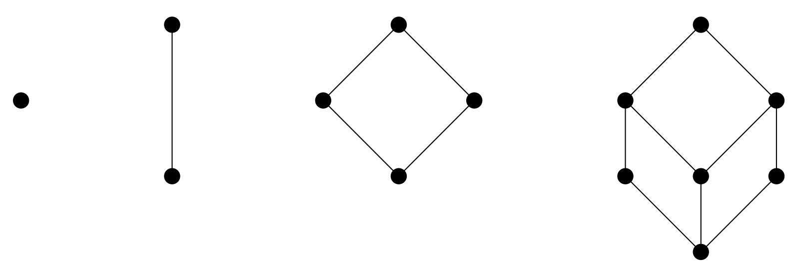

FIGURE 1. Doublings of lower pseudo-intervals

The dual of this result, and the combination of both characterizations gives characterizations of finite upper bounded lattices and finite bounded lattices, respectively. Alan Day's proof of this theorem hinges on the fact that when doubling a lower pseudo-interval, the only change in the congruences of $\mathcal{L}[C]$ from those of $\mathcal{L}$ is the introduction of a single minimal nontrivial congruence, e.g., the kernel of the canonical epimorphism $h \colon \mathcal{L}[C] \to \mathcal{L}$.

In the definition of the doubling construction presented earlier we only insisted that C be a convex set. Indeed, one can easily use doubling to obtain a lattice that is neither lower nor upper bounded (see below). Alan Day was able to classify lattices obtained by doubling arbitrary convex sets shortly before his death in [2].

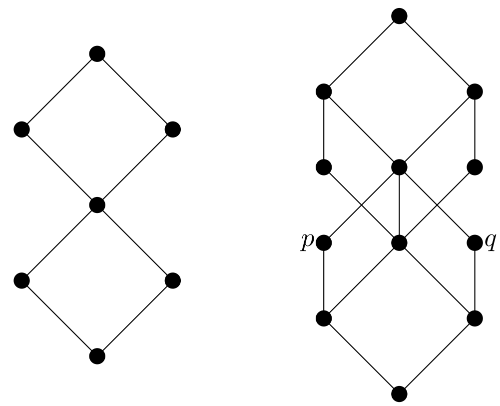

FIGURE 2. $\mathcal{L}[C]$ is neither lower nor upper bounded.

As in the case of lower bounded lattices, the key component of the proof is to show that the only congruence added when doubling a single connected (in the sense of considering the ordered set $\mathcal{L}$ to be a directed graph) convex set is a minimal one. The result then follows as one can, without loss of generality, assume that every stage of the construction of a given lattice that the convex set doubled is connected.

3. Dissertation Research

My most recent published work, [6], takes the foundation of doubling and expands the scope of prior constructive techniques. In his recent dissertation [13], Heiko Reppe demonstrated that convexity is not a necessary condition for a subset of a lattice to be doubled and still yield a lattice. His canonical example is reproduced below. Reppe goes on in his dissertation to classify those subsets of a lattice that can be doubled and yield a lattice, he calls these sets *municipal*.

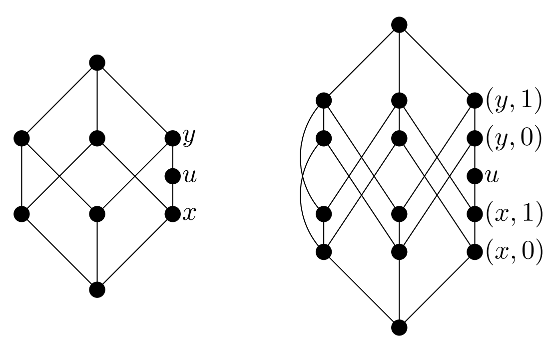

FIGURE 3. Reppe's lattice.

Investigating those lattices constructible by doubling municipal sets led J.B. Nation and I to realize that in examples such as the one above, elements such as u are in fact unnecessary. One can see in Figure 4 that by simply ignoring part of the order relation on $\mathcal{L}$, one can still double and produce a lattice. This generic idea, when formalized, led us to view doubling in a new way. Whereas prior to now doubling had been thought of as acting on elements of a lattice, we took a new approach and thought of it as acting on the edges of the Hasse Diagram, i.e., the covering relations in the partial order of $\mathcal{L}$.

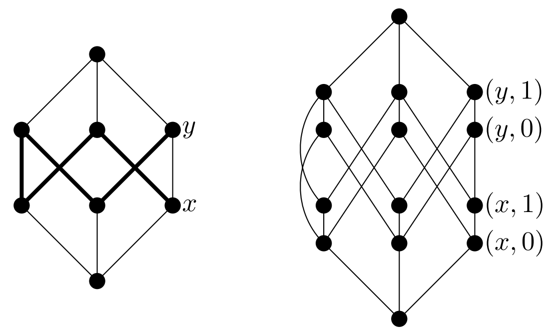

FIGURE 4. A, so far, unexplained doubling.

Let $\mathcal{L}$ and $\mathcal{K}$ be ordered sets. Let E be subset of the partial order on $\mathcal{L}$ that is transitive, i.e., if $(x,y) \in E$ and $(y,z) \in E$ then $(x,z) \in E$. One can define a new partially ordered set $\mathcal{L} \star_E \mathcal{K}$ by "inflating" E as follows. Let E' denote the set of elements of $\mathcal{L}$ that are included in the relations of E. The universe of the poset $\mathcal{L} \star_E \mathcal{K}$ is $(L \setminus E') \cup (E' \times \mathcal{K})$. Let $x, y \in \mathcal{L}$ and $a, b \in \mathcal{K}$. We define a binary relation $\sqsubseteq$ on $\mathcal{L} \star_E \mathcal{K}$ by

- **i:** $x \sqsubseteq y$ if $x \leq_{\mathcal{L}} y$,
- **ii:** $(x,a) \sqsubseteq y$ if $x \leq_{\mathcal{L}} y$,
- **iii:** $x \sqsubseteq (y,b)$ if $x \leq_{\mathcal{L}} y$,
- **iv:** $(x,a) \sqsubseteq (y,b)$ if $x \leq_{\mathcal{L}} y$ and $a \leq_{\mathcal{K}} b$,
- **v:** $(x,a) \sqsubseteq (y,b)$ if $x \leq_{\mathcal{L}} y$, and there exists $(u,v) \notin E$ such that $x \leq u < v \leq y$.

We prove in [6] that the relation $\sqsubseteq$ is a partial order. Note the fundamental difference between this new inflation and the older concepts of doubling. Day's construction never permitted a situation such as $(x,1) \leq (y,0)$ in Figure 4. While such a scenario is possible by doubling using Reppe's municipal sets, as in Figure 3, this only occurs if there exists $u \in \mathcal{L}$ such that $u \notin E'$ and $x < u < y$ for $x, y \in E'$. Our new approach to inflation allows all of the previous kinds of constructions, as well as additional constructions such as in Figure 4.

Note also that in this new formulation it is possible to inflate by ordered sets other than the two element lattice, so we are not just restricted to doubling. This has been done before when inflating convex sets, such as in [4], and so in [6] we present the construction is in the fullest possible generality. The most important result, however, comes from restricting $\mathcal{K}$ to be the two element lattice as in Day's doubling constructions.

With some additional assumptions, we can ensure that $\mathcal{L} \star_E \mathcal{K}$ is a lattice. A subset $E \subseteq \leq_{\mathcal{L}}$ is called *all-or-nothing* if the existence of a chain

$$\{(x_0,x_1),(x_1,x_2),(x_3,x_4),\dots,(x_{n-1},x_n)\} \subseteq E$$

implies that if u, v are such that $x_0 \leq u < v \leq x_n$ then $(u, v) \in E$. This is a stronger condition than transitivity. Indeed, let $x < t < y$ and $x < v < y$ in $\mathcal{L}$. Then the set $F := \{(x,t),(t,y),(x,y)\}$ is transitive, but is not all or nothing as $(x,v),(v,y) \notin F$.

Theorem 3.1. *Given a lattice $\mathcal{L}$, a subset $E \subseteq \leq_{\mathcal{L}}$, and a lattice $\mathcal{K}$ with minimum and maximum, the ordered set $\mathcal{L} \star_E \mathcal{K}$ is a lattice if and only if E is all or nothing.*

The proof of the "if" direction of the theorem is apparent from Figure 5, as the elements a and b do not have a least upper bound. Should E fail to be all-or-nothing, a similar situation will occur in any lattice constructed by inflation. The "only if" direction requires the somewhat tedious definition and verification of the new join and meet operations on $\mathcal{L} \star_E \mathcal{K}$.

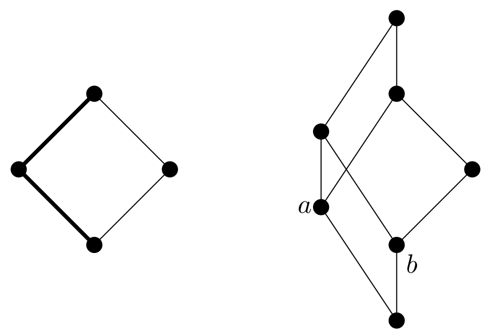

FIGURE 5. A doubling that fails to produce a lattice.

The question immediately arises: can we classify those finite lattices constructible using this new inflation technique when $\mathcal{K}$ is the two element lattice? Inspiration for the form of this classification came from [12].

Theorem 3.2. *Let $\mathcal{L}$ be a finite lattice. For a congruence θ of $\mathcal{L}$ define $\|\theta\|$ to be the size of largest block of the equivalence relation θ. The lattice $\mathcal{L}$ is constructible by a sequence of inflations of all or nothing sets by the two element lattice if and only if there exists a chain of congruences of $\mathcal{L}$*

$$\theta_0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_k$$

such that θ_0 is the identity relation on $\mathcal{L}$, θ_k is the universal relation on $\mathcal{L}$, and $\|\theta_i/\theta_{i-1}\| \leq 2$ for all i such that $1 \leq i \leq k$.

While the details of the proof include a number of technicalities, the basic principle is easy to describe. Similar to past examples of doubling one may, without loss of generality, assume that the all-or-nothing sets use in the construction are connected. Because of this, the only new congruence in $\mathcal{L} \star_E \mathcal{K}$ is a minimal nontrivial congruence. Those elements related by this new congruence are those that arose from doubling. As such, the size of the largest block of the new congruence is 2, which preserves codability. Conversely, one can prove by induction that if $\mathcal{L}$ is binary cut through codable and θ is an appropriately chosen minimal nontrivial congruence of $\mathcal{L}$, then one can construct $\mathcal{L}$ from $\mathcal{L}/\theta$ by inflating an all or nothing set E by the two element lattice.

The next question to consider is whether there exist lattices that generate varieties (a class of lattices that is closed with respect to homomorphic images, sublattices, and direct products) whose finite members are all binary cut-through codable. Constructing examples of such lattices turns out to be a tricky proposition, as it is not enough to assume the variety is generated by binary cut-through codable lattices. The lattice pictured in Figure 6 is binary cut-through codable, as are all of its sublattices and homomorphic images. However, it contains a sublattice, the interval y/x, that is binary-cut through codable but has a homomorphic image that is not binary-cut through codable. Thus, the variety generated by this lattice is not binary cut-through codable.

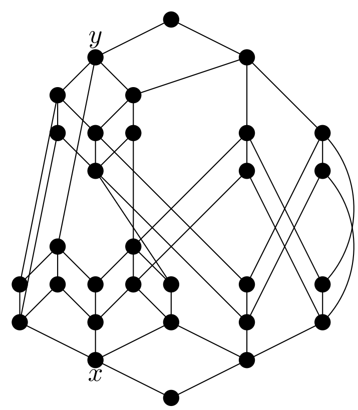

FIGURE 6. A finite lattice $\mathcal{L}$ that generates a variety that is not binary cut-through codable.

Theorem 3.3. *Let $n \geq 3$ and $\mathcal{B}(n)$ be the boolean algebra on n elements, i.e., the lattice of subsets of an n element set ordered by inclusion. There exists $E \subseteq \leq_{\mathcal{B}(n)}$ such that $\mathcal{B}(n) \star_E \mathbf{2}$ generates a variety that is binary cut-through codable.*

These new lattices $\mathcal{B}(n) \star_E \mathbf{2}$ are all constructed similarly to that shown in Figure 4. One takes all elements of the lattice corresponding to subsets of cardinality m and $m+1$ for $1 \leq m \leq n-2$. The set of edges E consists of all edges connecting these lattice elements except for a single arbitrarily chosen edge. This simple construction creates a lattice that satisfies the hypotheses of a lemma that ensures such pathologies as seen in Figure 6 are excluded in any finite lattice in the family of sublattices of $\mathcal{B}(n) \star_E \mathbf{2}$.

Constructing the sets E in Theorem 3.3 in this way also ensures that $\mathcal{B}(n) \star_E \mathbf{2}$ is *subdirectly irreducible.* A lattice $\mathcal{A}$ is called a *subdirect product* of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ if and only if $\mathcal{A}$ is a sublattice of the direct product $\mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_k$, and the projection homomorphism $\pi_i \colon \mathcal{A}_i \to \mathcal{A}$ is onto for every i such that $1 \leq i \leq k$. We say $\mathcal{A}$ is subdirectly irreducible if and only if whenever $\mathcal{A}$ can be written as a subdirect product of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$, then $\mathcal{A} \cong \mathcal{A}_i$ for some i such that $1 \leq i \leq k$.

A lattice variety is determined entirely by the subdirectly irreducible lattices it contains. One of the consequences of Jónsson's Lemma is that if $\mathcal{L}$ is a sudirectly irreducible lattice, then the only other subdirectly irreducible lattices in the variety generated by $\mathcal{L}$ are homomorphic images of sublattices of $\mathcal{L}$. As the integer n in

Theorem 3.3 is arbitrary, this means that there exist infinitely many varieties that are binary cut-through codable.

4. Future Projects

My dissertation presents a classification of those lattices obtainable from our newly generalized form of doubling, but the classification is not as strong as some of those in earlier examples. In the case of the various families of bounded lattices it is known that a finite lattice $\mathcal{L}$ is lower bounded if and only if it satisfies a listable set of identities [8, Theorem 5.3, pg. 541]. This result then easily extends to upper bounded and bounded lattices. At present we do not have an analogous result for binary cut-through codable lattices.

Project 4.1. *Determine a set of identities that classify binary cut through codable lattices.*

Sun and Li in [12] also ask whether there is an exclusion result similar to that for the well known class of semidistributive lattices [12, Problem 2, pg. 277].

Project 4.2. *Determine whether there exist finitely many minimal varieties that contain non binary cut-through codable lattices. Determine whether there is an exclusion result for binary cut-through codable lattices.*

While my current research centers around order and lattices, my training at UH Manoa includes a firm foundation in universal algebra. In the Summer of 2011, I was fortunate to be able to attend conferences in both Toronto and New York City and mingle with researchers in fields related to my areas of study. In Toronto the topic of the conference at the Fields Institute was current research on constraint satisfaction problems.

A constraint satisfaction problem consists of a set of variables, a set of possible values that those variables may be assigned, and a set of relations which the assigned values must satisfy. Universal algebra studies the constraint satisfaction dichotomy conjecture: that a given constraint satisfaction problem is solvable in polynomial time if the universal algebra generated by the problem omits type one minimal algebras, and NP-hard otherwise.

I am currently participating in a seminar at UH Manoa that is working on a similar problem that is much smaller in scale. Ralph Freese and Matt Valeriote showed in [5] that there exists a polynomial time algorithm to decide if the variety generated by an idempotent finite algebra omits various kinds of minimal algebras. An interesting class of finite algebras was classified by Keith Kearnes in [7] in terms of omitting a different set of minimal algebras. The seminar is currently attempting to decide if the proof of the Freese and Valeriote's result can be adapted to this class of algebras classified by Kearnes.

Project 4.3. *Determine if there exists a polynomial time algorithm to decide if a variety generated by a finite idempotent algebra $\mathcal{A}$ has a difference term.*

The second conference I attended that summer at Yeshiva University brought together diverse researchers from numerous fields that relate to universal algebra. I had the pleasure of presenting an introductory talk on the subject of permutohedra, a class of lattices that naturally arise when considering the weak Bruhat order on Coxeter groups. At the time Professor Kira Adaricheva and I were looking into addressing open problems brought up in the work of Santocanale and Wehrung. Before we had made much progress Santocanale and Wehrung published a follow up paper [11] that generalized the permutohedron and answered several of the questions we had been working on. This newer paper presents several new questions which may be tractable.

Project 4.4. *Investigate whether every finite bounded lattice can be embedded into a generalized permutohedron. Investigate whether there is a nontrivial lattice identity that holds in every generalized permutohedron.*

At a conference organized by J.B. Nation and Monique Chyba at UH Manoa I became familiar with the research of Professor Jinfang Wang of Chiba University. Wang's research relates the study of conditional probabability to universal algebra and lattice theory by formalizing known properties of conditional probability functions to construct what is called a Cain Algebra. This formalization allows for more elegant proofs of known results and the possibility of proving new results by working strictly with the formal algebraic properties of the Cain Algebra rather than directly with conditional probability functions.

The construction of the Cain Algebra begins with a (possibly infinite) boolean lattice $\mathcal{L}$. The elements of the lattice are analogues of random variables ordered by their dependence. Wang then constructs objects he refers to as *coins*, from the direct product $\mathcal{L}\otimes\mathcal{L}$. The set of all possible coins is the universe of the Cain Algebra. The structure of the Cain Algebra is constructed to allow common relationships among conditionally independent random variables to be easily formalized.

What is not obvious to Wang is whether the Cain Algebra is sufficient to formalize all desirable theories of conditional independence. In terms of universal algebra, this problem can be interpereted as follows.

Project 4.5. *Determine if the quasi-varieties generated by Cain Algebras model the implications necessary to formalize complicated theories of conditional independence.*

References

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