These notes are based on the Oxford Numerical Methods course taught by David Marshall (2023 for the NERC DTP) with additional context from LeVeque [1].

Contents

1 Root finding

Consider a function $f(x)$. If f is a quadratic polynomial, we may find the zeros of f using the quadratic formula, but for degrees 5 or larger, there exists no general formula for the zeros (Abel–Ruffini theorem). In general, finding an x^* such that $f(x^*) = 0$ cannot be computed exactly. Instead one must employ numerical root finding algorithms. Common methods include the bisection method and Newton's method.

1.1 Bisection method

1.2 Newton's method

A quicker alternative to bisection is Newton's method (also known as Newton-Raphson). Given an initial guess x_0 and a sufficiently nice derivative f', we may estimate the zero x^* of f.

Consider the Taylor expansion of f around x_n (see Appendix A.1 for the big O notation and Appendix A.2 for Taylor):

$$f(x) = f(x_n) + (x - x_n)f'(x_n) + O(x^2).$$

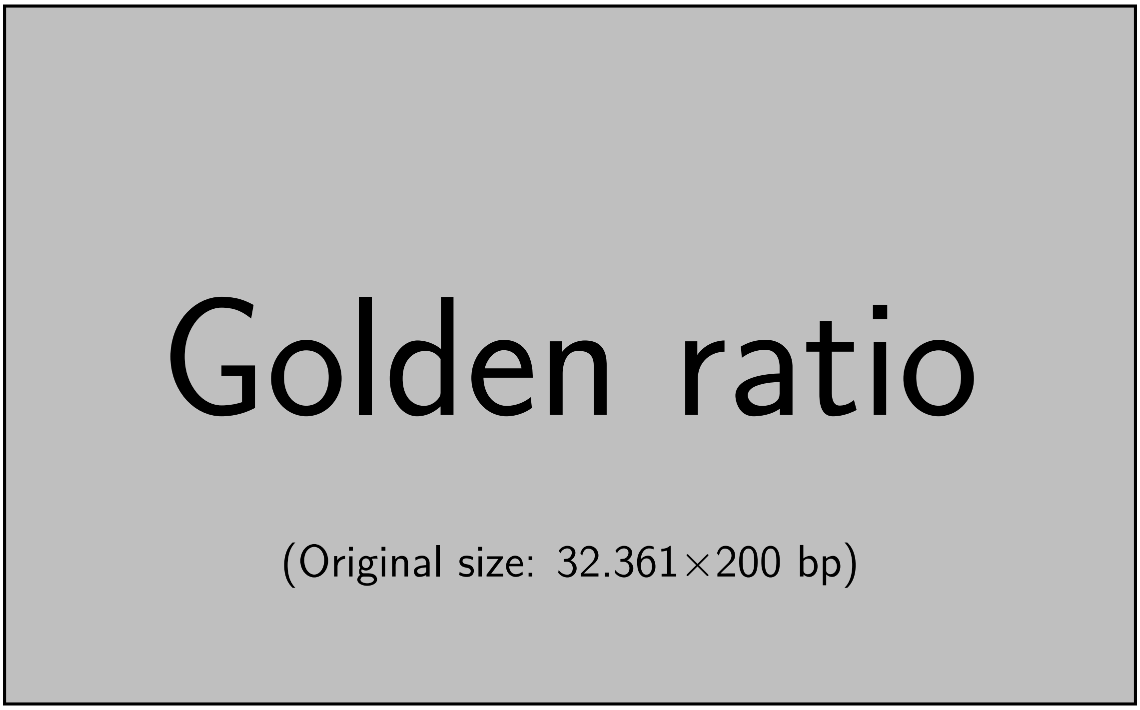

Figure 1: Bisection method

If we suppose that x_n is close to the root x^*, the zero of the linear approximation x_{n+1} is a good approximation for x^*

$$f(x_{n+1}) \approx 0 = f(x_n) + (x_{n+1} - x_n) f'(x_n).$$

Rearranging, we arrive at the iterative formula for Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

1.3 Higher dimensions

Consider the system of m equations in n variables $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ given by

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) = 0, \\ f_2(x_1, x_2, \ldots, x_n) = 0, \\ \qquad\qquad \vdots \\ f_m(x_1, x_2, \ldots, x_n) = 0. \end{cases}$$

2 Finite difference

3 Von Neumann analysis

4 Numerical linear algebra

A Background theory

A.1 Big O notation

A.2 Taylor expansions

Theorem A.1 (Taylor)

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots$$

References

[1] R. J. LeVeque. *Finite difference methods for ordinary and partial differential equations: steady-state and time-dependent problems.* Society for Industrial and Applied Mathematics, Philadelphia, PA, 2007.