2006 AIME II #15

Tristan Shin

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Given that x, y, and z are real numbers that satisfy:

$$x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}$$
$$y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}$$
$$z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}}$$

and that $x + y + z = \frac{m}{\sqrt{n}}$, where m and n are positive integers and n is not divisible by the square of any prime, find $m + n$.

Consider $\triangle XYZ$ with side lengths x, y, z; the equations imply that the altitudes are $\frac{1}{4}, \frac{1}{5}, \frac{1}{6}$. Then $\triangle XYZ$ is similar to a triangle with side lengths $4, 5, 6$, which has area $\frac{15\sqrt{7}}{4}$. Thus we have that

$$\frac{15\sqrt{7}}{4} \cdot \left(\frac{x}{4}\right)^2 = \frac{1}{2} \cdot x \cdot \frac{1}{4},$$

so $\frac{x}{4} = \frac{2}{15\sqrt{7}}$. Then

$$x + y + z = \frac{2}{15\sqrt{7}}(4 + 5 + 6) = \frac{2}{\sqrt{7}}$$

so the answer is 009. ■