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Let p and q be distinct positive integers. Suppose p^2 and q^3 are terms of an infinite arithmetic progression whose terms are positive integers. Show that the arithmetic progression contains the sixth power of some integer.

Let the arithmetic progression be $\{a, a+d, a+2d, \dots\}$. The problem statement is equivalent to the following: if a is a quadratic and cubic residue modulo d, then a is a sixth power residue modulo d. We induce on d, with the base case of $d = 1$ being verifiably true.

Suppose $d > 1$ and the statement is true for bases less than d.

First, the problem is easy if $\gcd(a, d) = 1$. Indeed, a^3 and a^2 are both sixth power residues mod d, so their quotient a is too.

Now, suppose $\gcd(a, d) > 1$. Let $x^3 \equiv y^2 \equiv a \pmod{d}$. Let p be a prime dividing $\gcd(a, d)$, with $k = \nu_p(\gcd(a, d)) = \min\{\nu_p(a), \nu_p(d)\}$. We casework on k.

- $k \geq 6$. Then
$$\frac{x^3}{p^6} \equiv \frac{y^2}{p^6} \equiv \frac{a}{p^6} \pmod{\frac{d}{p^6}},$$
so $\frac{a}{p^6}$ is both a quadratic and cubic residue mod $\frac{d}{p^6}$, so the inductive hypothesis implies that $\frac{a}{p^6}$ is a sixth power residue mod $\frac{d}{p^6}$. So we are done by multiplying by p^6.
- $k < 6$. I claim that $\nu_p(d) = k$. Suppose not, then $\nu_p(a) = k < \nu_p(d)$. Then x^3 is a plus a multiple of d. Since $\nu_p(a) < \nu_p(d)$, we have that $\nu_p(x^3) = \nu_p(a)$. Similarly, $\nu_p(y^2) = \nu_p(a)$, so $\nu_p(a)$ is divisible by 6. But $k > 0$, so this is a contradiction.

  So we have $\nu_p(d) = k$ and thus $\gcd(p, \frac{d}{p^k}) = 1$. Then
$$\frac{x^3}{p^6} \equiv \frac{y^2}{p^6} \equiv \frac{a}{p^6} \pmod{\frac{d}{p^k}}.$$
  By the inductive hypothesis, $\frac{a}{p^6}$ is a sixth power residue mod $\frac{d}{p^k}$. Then a is a sixth power mod dp^{6-k} and thus mod d, so we are done.

Thus by induction, any number that is a quadratic and cubic residue modulo a base is a sixth power residue modulo the same base. This implies the problem as desired. ■