2018 ISL A5

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Determine all functions $f : (0, \infty) \to \mathbb{R}$ satisfying

$$\left(x + \frac{1}{x}\right) f(y) = f(xy) + f\left(\frac{y}{x}\right)$$

for all $x, y > 0$.

The solutions are $ax + \frac{b}{x}$ for any constants $a, b \in \mathbb{R}$. These work because $x, \frac{1}{x}$ are solutions and the solution set is a vector space over $\mathbb{R}$.

Observe that

$$\begin{aligned}
\left(x + \frac{1}{x}\right) f(x) &= f(x^2) + f(1) \\
&= \left(\frac{x}{2} + \frac{2}{x}\right) f(2x) - f(4) + f(1) \\
&= \frac{\frac{x}{2} + \frac{2}{x}}{2x + \frac{1}{2x}} \left[f(4x^2) + f(1)\right] - f(4) + f(1) \\
&= \frac{\frac{x}{2} + \frac{2}{x}}{2x + \frac{1}{2x}} \left[\left(4x + \frac{1}{4x}\right) f(x) - f(\frac{1}{4}) + f(1)\right] - f(4) + f(1)
\end{aligned}$$

where we used the substitutions (x, x), $(x/2, 2x)$, $(2x, 2x)$, and $(4x, x)$. So

$$\left(f(\frac{1}{4}) - f(1)\right) \cdot \frac{\frac{x}{2} + \frac{2}{x}}{2x + \frac{1}{2x}} + f(4) - f(1) = \left[\frac{\left(\frac{x}{2} + \frac{2}{x}\right)\left(4x + \frac{1}{4x}\right)}{2x + \frac{1}{2x}} - \left(x + \frac{1}{x}\right)\right] f(x) = \frac{\frac{45}{8}}{2x + \frac{1}{2x}} \cdot f(x)$$

and thus

$$\begin{aligned}
f(x) &= \frac{8}{45}\left(f(\frac{1}{4}) - f(1)\right)\left(\frac{x}{2} + \frac{2}{x}\right) + \frac{8}{45}\left(f(4) - f(1)\right)\left(2x + \frac{1}{2x}\right) \\
&= \frac{4f(\frac{1}{4}) - 20f(1) + 16f(4)}{45} \cdot x + \frac{16f(\frac{1}{4}) - 20f(1) + 4f(4)}{45} \cdot \frac{1}{x}
\end{aligned}$$

as desired. ■