2015 ISL N1

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Determine all positive integers M for which the sequence $a_0, a_1, a_2, \ldots,$ defined by $a_0 = \frac{2M+1}{2}$ and $a_{k+1} = a_k \lfloor a_k \rfloor$ for $k = 0, 1, 2, \ldots,$ contains at least one integer term.

The answer is all M besides 1. We instead solve for the complement. Define the sequence $\{b_k\}$ as $b_k = a_k - \frac{1}{2}$; it is clear that $b_{k+1} = b_k^2 + \frac{b_k - 1}{2}$ and we wish to find all b_0 for which $\{b_k\}$ has only integer terms.

The key claim is that if $b_0, \ldots, b_n$ are all integers, then $b_m \equiv 1 \pmod{2^{n-m}}$ for $m = 0, \ldots, n-1$. We induce on n. The base case of $n = 1$ is clear. Now, assume that $b_0, \ldots, b_n, b_{n+1}$ are all integers. Let $b_m = 2^{n-m-1} c_m + 1$ for $m = 0, \ldots, n$; it suffices to prove that c_m is an even integer. Applying the inductive hypothesis to $b_0, \ldots, b_n$ and $b_1, \ldots, b_{n+1}$, we deduce that c_m is always an integer, and an even one for $m = 1, \ldots, n$. So it suffices to show that c_0 is even. But

$$2^{n-2} c_1 + 1 = (2^{n-1} c_0 + 1)^2 + 2^{n-2} c_0 \equiv 2^{n-2} c_0 + 1 \pmod{2^{n-1}}$$

and c_1 is even, so c_0 is even as desired.

Now, the answer is clear: if $\{b_k\}$ has only integer terms, then $b_0 \equiv 1 \pmod{2^n}$ for all positive integers n, so b_0 must be 1. ■