2020 USOMO #1

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19 June 2020

Let ABC be a fixed acute triangle inscribed in a circle ω with center O. A variable point X is chosen on minor arc AB of ω, and segments CX and AB meet at D. Denote by O_1 and O_2 the circumcenters of triangles ADX and BDX, respectively. Determine all points X for which the area of triangle OO_1O_2 is minimized.

Note that $O_1O \perp AX$ and $O_2O \perp BX$. Using directed angles modulo π,

$$\measuredangle O_1OO_2 = \measuredangle(O_1O, O_2O) = \measuredangle(AX, BX) = \measuredangle ACB.$$

Let P_1, P_2 be the feet from D to AX, BX; by a similar argument, $\measuredangle P_1DP_2 = \measuredangle ACB$.

From $\measuredangle O_1OO_2 = \measuredangle ACB$, it suffices to minimize $OO_1 \cdot OO_2$. Let ρ denote unsigned power.

Claim

$$OO_1 = \frac{\rho(D,\omega)}{2DP_1} \quad \text{and} \quad OO_2 = \frac{\rho(D,\omega)}{2DP_2}$$

Proof. We proceed using complex numbers with ω as the unit circle. The circumcenter of $\triangle ADX$ is at

$$\frac{\begin{vmatrix} a & 1 & 1 \\ d & |d|^2 & 1 \\ x & 1 & 1 \end{vmatrix}}{\begin{vmatrix} a & a^{-1} & 1 \\ d & d^{-1} & 1 \\ x & x^{-1} & 1 \end{vmatrix}} = \frac{\begin{vmatrix} a & 0 & 1 \\ d & |d|^2 - 1 & 1 \\ x & 0 & 1 \end{vmatrix}}{\begin{vmatrix} a & a^{-1} & 1 \\ d & d^{-1} & 1 \\ x & x^{-1} & 1 \end{vmatrix}} = \frac{(|d|^2-1)(a-x)}{\frac{4}{i}[ADX]}.$$

Taking the magnitude, we have that

$$OO_1 = \frac{\rho(D,\omega)\cdot AX}{4[ADX]} = \frac{\rho(D,\omega)}{2DP_1}$$

as desired. The other result follows by symmetry. $\square$

Note that

$$\frac{1}{2}DP_1 \cdot DP_2 \cdot \sin\angle C = [DP_1P_2] = \frac{\rho(D,\omega)\cdot[XAB]}{4R^2}$$

since $\triangle DP_1P_2$ is the pedal triangle of D w.r.t. $\triangle XAB$. Then

$$OO_1 \cdot OO_2 \propto \frac{\rho(D,\omega)^2}{DP_1 \cdot DP_2} \propto \frac{\rho(D,\omega)}{[XAB]} \propto \frac{DA\cdot DB}{XA \cdot XB} = \frac{\sin\angle DXA}{\sin\angle XDA}\cdot\frac{\sin\angle DXB}{\sin\angle XDB} = \frac{\sin\angle B \sin\angle A}{\sin^2\angle ADX}$$

so it suffices to maximize $\sin\angle ADX$. This occurs when $CX \perp AB$. $\blacksquare$