2014 All-Russian Grade 10 #6

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13 Jan 2016

Let M be the midpoint of the side AC of $\triangle ABC$. Let $P \in AM$ and $Q \in CM$ be such that $PQ = \frac{AC}{2}$. Let (ABQ) intersect with BC at $X \neq B$ and (BCP) intersect with BA at $Y \neq B$. Prove that the quadrilateral $BXMY$ is cyclic.

By Power of a Point, $AY \cdot AB = AP \cdot AC$ and $CX \cdot CB = CQ \cdot CA$. Let $AC = b$ and $AP = w$. By the condition, $CQ = \frac{b}{2} - w$. Thus, $AY \cdot AB = wb$ and $CX \cdot CB = \left(\frac{b}{2} - w\right) b$. Now, let the circumcircle of BXY intersect AC at J and K (possibly complex). By Power of a Point, $AY \cdot AB = AJ \cdot AK$ and $CX \cdot CB = CJ \cdot CK$. Note that $CJ = b - AJ$ and $CK = b - AK$, so $CX \cdot CB = (b - AJ)(b - AK)$. Substituting in what we know, $AJ \cdot AK = wb$ and $(b - AJ)(b - AK) = \left(\frac{b}{2} - w\right) b$. There is a unique monic quadratic $P(x)$ with roots AJ and AK. We would have $P(0) = wb$ and $P(b) = \left(\frac{b}{2} - w\right) b$. But notice $P(x) = \left(x - \frac{b}{2}\right)(x - 2w)$ satisfies these same conditions, so this is $P(x)$. But $\frac{b}{2}$ is a root of $P(x)$, so either AJ or AK is $\frac{b}{2}$, meaning either J or K is M and thus M lies on the circumcircle of BXY and $BXMY$ is cyclic. ■