2020 USOJMO #4

Tristan Shin

30 June 2020

Let $ABCD$ be a cyclic quadrilateral. Let the perpendicular bisector of CA meet CD at E and the perpendicular bisector of BD meet BA at F. Then $EF \parallel BC$.

We proceed using complex numbers with $(ABCD)$ as the unit circle, where lowercase letters denote the usual. We require

$$e + cd\bar{e} = c + d$$
$$e - ac\bar{e} = 0$$

so $e = \frac{ac+ad}{a+d}$. By symmetry, $f = \frac{bd+ad}{a+d}$, so

$$\frac{b-c}{e-f} = \frac{(a+d)(b-c)}{ac-bd} = \frac{ab-cd}{ac-bd} - 1$$

which is equal to its own conjugate, so $EF \parallel BC$ as desired. ■

Remark. A direct application of Pascal to $HDCBAG$ also implies the result, after a quick arc-chasing.