Easier than Vladimir OI

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Convex quadrilateral $ABCD$ is inscribed in a circle Ω centered at O, and circumscribed about a circle ω with center I. Let ω be tangent to side AB at E and side CD at F. Suppose the exterior angle bisectors of $\angle DAB$ and $\angle ABC$ meet at a point X, while the exterior angle bisectors of $\angle BCD$ and $\angle CDA$ meet at a point Y. Prove that lines XE, YF, and OI are concurrent.

Let W be the intersection of the A- and D-exterior angle bisectors, and Z the intersection of the B- and C-exterior angle bisectors. Let ω touch AD at G and BC at H.

The key claim is that $GEHF$ and $WXZY$ are homothetic. The sides are clearly parallel because, for example, $GE \perp AI$ and $WX \perp AI$. Note that $AIBX$ is cyclic because of opposite right angles. In addition, X, I, Y are collinear on the interior angle bisector of the angle formed by lines AD and BC. Thus

$$\measuredangle GEF = \measuredangle DGF = \frac{\pi}{2} - \measuredangle IDA = \measuredangle ABI = \measuredangle AXI = \measuredangle WXY$$

so since $GE \parallel WX$ we have that $EF \parallel XY$. Similarly, $GH \parallel WZ$, so the diagonals are parallel too. So the quadrilaterals are homothetic as claimed.

Now, let the diagonals of $GEHF$ meet at P. The diagonals of $WXZY$ meet at I since X, I, Y and W, I, Z are collinear from before. Thus the homothety implies that XE, YF, and IP are concurrent. But it is well-known (by polars) that I, P, O are collinear, so the conclusion follows. ■