1980 USAMO #3

Tristan Shin

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Let $F_r = x^r \sin rA + y^r \sin rB + z^r \sin rC$, where x, y, z, A, B, C are real and $A + B + C$ is an integral multiple of π. Prove that if $F_1 = F_2 = 0$, then $F_r = 0$ for all positive integral r.

Let $\alpha = xe^{iA}, \beta = ye^{iB}, \gamma = ze^{iC}$, $P_n = \alpha^n + \beta^n + \gamma^n$, and S_n be the nth symmetric sum of α, β, γ. By Newton sums,

$$P_n = S_1 P_{n-1} - S_2 P_{n-2} + S_3 P_{n-3}$$

for $n \geq 3$. Note that $F_n = \operatorname{Im} P_n$, so P_0, P_1, P_2 are real. By Newton sums, S_1, S_2 are real. Furthermore,

$$S_3 = \alpha\beta\gamma = xyze^{i(A+B+C)} \in \mathbb{R}\,.$$

Thus by induction, P_n is real for all $n \in \mathbb{N}$, implying that $F_n = 0$ for all $n \in \mathbb{N}$ as desired. ■