# 2015 ISL C1

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In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a *left bulldozer* (put to the left of the town and facing left) and a *right bulldozer* (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let $A$ and $B$ be two towns, with $B$ to the right of $A$. We say that town $A$ can *sweep* town $B$ *away* if the right bulldozer of $A$ can move over to $B$ pushing off all bulldozers it meets. Similarly town $B$ can sweep town $A$ away if the left bulldozer of $B$ can move over to $A$ pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town that cannot be swept away by any other one.

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Let $S$ be a set of towns, and define a **capital** of $S$ to be a town that cannot be swept away by any other town in $S$. Proceed by induction on $n$, with the base case being trivial. Label the towns as $1, \ldots, n$ from left to right.

Consider the largest non-endpoint bulldozer, and WLOG let be the right bulldozer of town $k$. By the inductive hypothesis, there is a unique capital $C$ of $\{1, \ldots, k\}$. It turns out that $C$ is also the unique capital of $\{1, \ldots, n\}$. To see why, note that $C$ cannot be swept away by any town to the right of $k$ for size reasons, so $C$ is a capital of $\{1, \ldots, n\}$. Furthermore, $k$ can sweep away any town to the right of $k$, so any capitals of $\{1, \ldots, n\}$ must be in $\{1, \ldots, k\}$. Since $C$ is unique, this implies the result. ■