Arrow's Theorem

A Difficulty in the Concept of Social Welfare

Tristan Shin

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Election

Want to determine an election.

Usual methods:

- majority rule
- ranked choice voting
- round-by-round

Condorcet

- Voter 1: $a \succ b \succ c$
- Voter 2: $c \succ a \succ b$
- Voter 3: $b \succ c \succ a$

Condorcet

- $a \succ b \succ c$: 18 votes
- $a \succ c \succ b$: 11 votes
- $b \succ a \succ c$: 25 votes
- $b \succ c \succ a$: 9 votes
- $c \succ a \succ b$: 24 votes
- $c \succ b \succ a$: 13 votes

Majority rule: c wins with 37 votes (over $b = 34$ and $a = 29$)

Round-by-Round: b wins second round with 52 votes (over $c = 48$)

Head-to-Head: a over b with 53, a over c with 54

Condorcet

Take 10 of the $c \succ b \succ a$ candidates, change to $b \succ c \succ a$

- $a \succ b \succ c$: 18 votes
- $a \succ c \succ b$: 11 votes
- $b \succ a \succ c$: 25 votes
- $b \succ c \succ a$: 19 votes
- $c \succ a \succ b$: 24 votes
- $c \succ b \succ a$: 3 votes

Round-by-Round (old): b wins second round with 52 votes (over $c = 48$)

Round-by-Round: c eliminated, a wins second round with 53 votes (over $b = 47$)

Fairness

Theorem (Arrow 1950 (HEAVILY paraphrased))

"No voting system is fair!" (don't take this at face value)

Candidates

We have a set $\mathcal{C}$ of **candidates**, n people voting on them

Each person has **preference**, i.e. "strict total order on $\mathcal{C}$"

For example $a \succ d \succ b \succ c$ if $\mathcal{C} = \{a, b, c, d\}$

Let $\mathcal{R}$ be the set of all possible preferences ($|\mathcal{R}| = |\mathcal{C}|!$)

A **social preference function** is a function $E \colon \mathcal{R}^n \to \mathcal{R}$

Social Preference Functions

A **social preference function** is a function $E\colon \mathcal{R}^n \to \mathcal{R}$

For convenience, say that E takes in $\succ_1, \succ_2, \ldots, \succ_n$ (call these $\mathcal{P}$)

Then say that $\succ$ is $E(\mathcal{P})$

If we use $\succ_1', \succ_2', \ldots, \succ_n'$, then we use $\mathcal{P}'$ and $\succ'$

Social Preference Functions

A **social preference function** is a function $E\colon \mathcal{R}^n \to \mathcal{R}$

We say E satisfies **Unanimity** if $x \succ_i y$ for all i implies $x \succ y$

We say two profiles $\mathcal{P}$ and $\mathcal{P}'$ **rank x and y the same** if $x \succ_i y$ if and only if $x \succ_i' y$

We say E satisfies **Independence of Irrelevant Alternatives (IIA)** if:

if $\mathcal{P}$ and $\mathcal{P}'$ rank x and y the same, then $x \succ y$ if and only if $x \succ' y$

Social Preference Functions

A **social preference function** is a function $E\colon \mathcal{R}^n \to \mathcal{R}$

We say that i **is decisive for** x **over** y if: $x \succ_i y$ implies $x \succ y$

We say that i is a **dictator** if i is decisive for x over y, for all $x \neq y$

(equivalently, $\succ$ and $\succ_i$ are the same)

Arrow's Theorem

Theorem (Arrow 1950)

If a social preference function E satisfies Unanimity and IIA, then there is a dictator.

Sketch

due to Fey 2014

1. Fix candidates a, b. Identify potential dictator.
2. decisive for b over c ($c \notin \{a, b\}$)
3. decisive for a over c
4. decisive for c over a
5. decisive for c over b
6. decisive for a over b and b over a
7. decisive for c over d ($c, d \notin \{a, b\}$)

Main structure

Say trying to prove k is decisive for x over y

Take arbitrary $\mathcal{P} \in \mathcal{R}^n$ with $x \succ_k y$

Construct $\mathcal{P}'$ that ranks x and y the same as $\mathcal{P}$ SUCH THAT $x \succ' y$

By IIA, $x \succ y$

Sometimes, will need to introduce another auxiliary profile $\mathcal{P}^*$

Proof very much like FE

Identify dictator

Fix $a, b \in \mathcal{C}$ distinct.

For $j = 0, \ldots, n$ define $\mathcal{P}^{(j)} = (\succ_1^{(j)}, \ldots, \succ_n^{(j)})$ with

$$\succ_1^{(j)} = \cdots = \succ_j^{(j)} = (b, a, \ldots)$$
$$\succ_{j+1}^{(j)} = \cdots = \succ_n^{(j)} = (a, b, \ldots)$$

$\mathcal{P}^{(0)}$ satisfies $\succ_i^{(0)} = (a, b, \ldots)$, we have $a \succ^{(0)} b$

$\mathcal{P}^{(n)}$ satisfies $\succ_i^{(n)} = (b, a, \ldots)$, we have $b \succ^{(n)} a$

Define k to be smallest positive integer such that $b \succ^{(k)} a$

Claim: k is our dicatator

Decisive for b over c

Take arbitrary $\mathcal{P}$ with $b \succ_k c$.

Consider $\mathcal{P}^*$ with

$$\begin{aligned} \succ_1^* = \cdots = \succ_{k-1}^* &= (b, c, a, \dots) \\ \succ_k^* = \cdots = \succ_n^* &= (a, b, c, \dots) \end{aligned}$$

- $\mathcal{P}^{(k-1)}$ and $\mathcal{P}^*$ rank a and b the same $\implies a \succ^* b$
- Unanimity $\implies b \succ^* c \implies a \succ^* c$

Decisive for b over c

Take arbitrary $\mathcal{P}$ with $b \succ_k c$. If $b \succ_1 c$, then set $\succ'_1 = (b, c, a, \dots)$. If $c \succ_1 b$, then set $\succ'_1 = (c, b, a, \dots)$.

$$\begin{aligned} \succ'_1, \dots, \succ'_{k-1} &= (\{b, c\}, a, \dots) \\ \succ'_k &= (b, a, c, \dots) \\ \succ'_{k+1}, \dots, \succ'_n &= (a, \{b, c\}, \dots) \end{aligned}$$

- $\mathcal{P}^{(k)}$ and $\mathcal{P}'$ rank a and b the same $\implies b \succ' a$
- $\mathcal{P}^*$ and $\mathcal{P}'$ rank a and c the same $\implies a \succ' c \implies b \succ' c$
- $\mathcal{P}'$ and $\mathcal{P}$ rank b and c the same $\implies b \succ c$

Decisive for a over c

$$\begin{aligned} \succ'_1, \dots, \succ'_{k-1}, \succ'_{k+1}, \dots, \succ'_n &= (\{a, c\}, b, \dots) \\ \succ'_k &= (a, b, c, \dots) \end{aligned}$$

- k is decisive for b over $c \implies b \succ' c$
- Unanimity $\implies a \succ' b \implies a \succ' c$

Decisive for c over a

$$\begin{aligned} \succ_1^* = \cdots = \succ_{k-1}^* &= (b, c, a, \dots) \\ \succ_k^* = \cdots = \succ_n^* &= (c, a, b, \dots) \end{aligned}$$

$c \succ^* b$

$$\begin{aligned} \succ_1', \dots, \succ_{k-1}' &= (b, \{a, c\}, \dots) \\ \succ_k' &= (c, b, a, \dots) \\ \succ_{k+1}', \dots, \succ_n' &= (\{a, c\}, b, \dots) \end{aligned}$$

$c \succ' a$

Other decisiveness

Use very similar profiles

Afterthoughts

Q: How can we resolve this?

A: weaken IIA (ranked choice voting)

A: cardinal voting

A: approval voting (game theory)

Q: Why do we care about ranking them? We only want a winner

A: Gibbard-Satterthwaite 1973 (strategyproof)

Reny 2001 provides a unified approach to Arrow and GS

Main takeaway: Strategy-proofness and construction of voting schemes is hard! Can use math to analyze political structures!

Relevant papers

Arrow's 1950 paper: `https://www.stat.uchicago.edu/~lekheng/meetings/mathofranking/ref/arrow.pdf`

Fey proof of Arrow: `https://www.rochester.edu/college/faculty/markfey/papers/ArrowProof3.pdf`

Reny unified approach is sadly locked behind paywall/institutional access; email me if you want it.

Feedback

Thank you very much for attending! Hope you enjoyed the presentation on Arrow's theorem and related topics!

Slides will be posted at `mit.edu/~shint/handouts/vSDMC/arrow.pdf`

For any questions or comments, feel free to contact me at `shint@mit.edu`.

If you have feedback, please give it to us at

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