Bijections and Friends

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If you have any questions/comments or find any mistakes, please contact me at trshin@ucsd.edu.
This handout is linked at mathweb.ucsd.edu/~trshin/handouts/bijections_and_friends.pdf.
Many problems sourced from Yufei Zhao's Putnam course and Jack Wesley's combinatorics course.

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1 Reimagination

Example 1.1

How many tuples $(a_1, \ldots, a_k)$ of positive integers are there such that $a_1 + \cdots + a_k = n$?

Example 1.2

Show that $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$.

Example 1.3

Show that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$.

Example 1.4

Show that $1 + 1 + 2 + 4 + \cdots + 2^{n-1} = 2^n$.

A **partition** of n is a tuple $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers in nonincreasing order such that $\lambda_1 + \cdots + \lambda_\ell = n$. Let number of partitions of n be $p(n)$.

Example 1.5

Show that the number of partitions of n into k parts is equal to the number of partitions of n with $\lambda_1 = k$.

Example 1.6

Show that $p(n)^2 < p(n^2 + 2n)$.

1.1 Problems

1. Compute $\binom{n}{1} + 4\binom{n}{2} + 9\binom{n}{3} + \cdots + n^2\binom{n}{n}$.

2. (Vandermonde identity) Show that $\binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \cdots + \binom{m}{k}\binom{n}{0} = \binom{m+n}{k}$.

3. (ARML) In a Taekwondo class of 21 students, the students compete in "sparring teams" of three students. Suppose that there are 56 different sparring teams that have ever been formed by the students in the class (here, two teams are the same if they consist of the same three students and different otherwise). Given that each student has been, on average, a part of N different sparring teams, compute N.

4. (Putnam) Let k be a nonnegative integer. Evaluate
$$\sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j}.$$

5. Show that $1 + 0 \cdot 0! + 1 \cdot 1! + 2 \cdot 2! + \cdots + (n-1) \cdot (n-1)! = n!$.

6. Show that $1 + k + k(k+1) + k(k+1)^2 + \cdots + k(k+1)^{n-1} = (k+1)^n$.

7. How many partitions of any number are there into 6 parts, each of size at most 7?

8. (AoPS/jlammy) Let $m \geq n$ be two positive integers. Compute
$$\binom{m}{n}\binom{n}{0} - \binom{m-1}{n}\binom{n}{1} + \binom{m-2}{n}\binom{n}{2} - \cdots + (-1)^n\binom{m-n}{n}\binom{n}{n}.$$

9. (Ruzsa triangle inequality) Let A, B, C be finite sets of integers. Prove that
$$|A| \cdot |B - C| \leq |A - B| \cdot |A - C|.$$
Here, $A - B = \{a - b : a \in A, b \in B\}$ and likewise for $B - C$ and $A - C$.

1.2 More problems

10. (Hockey stick identity) Show that $\binom{m}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+n}{n} = \binom{m+n+1}{n}$.

11. Show that $\frac{d-1}{d} + \frac{d-1}{d^2} + \frac{d-1}{d^3} + \cdots + \frac{d-1}{d^n} = 1 - \frac{1}{d^n}$.

12. Let c_n denote the number of tuples $(a_1, \ldots, a_k)$ of positive integers such that $2a_1 + a_2 + \cdots + a_k = n$. Show that $c_n + c_{n+1} = 2^{n-1}$, and use this to find a formula for c_n.

13. Show that the number of partitions of n with all λ_i odd is equal to the number of partitions of n with all λ_i distinct.

2 Uniformity

Example 2.1

Slips of paper with the numbers from 1 to 99 are placed in a hat. Five numbers are randomly drawn out of the hat one at a time (without replacement). What is the probability that the numbers are chosen in increasing order?

Example 2.2

Let $x_1, \ldots, x_{2025}$ be independently chosen uniformly at random from $[0, 1]$. Compute the expected value of $\min\{x_1, \ldots, x_{2025}\}$.

Example 2.3

Choose a random permutation σ of $\{1, \ldots, 2025\}$. Compute the probability that 1 and 2 are in the same cycle (i.e. $\sigma^k(1) = 2$ for some k).

Example 2.4: Daniel Litt

You have 100 urns, each with 99 balls. In 99 of the urns, one ball is red and the rest are green. In the last urn, all 99 balls are red. You choose one of the urns uniformly at random, then pick a ball randomly out of the urn. It is red, and you discard it. You pick another ball randomly out of the urn. What is the probability that it is red?

2.1 Problems

1. Choose a random permutation σ of $\{1, \ldots, 2025\}$. Compute the probability that 1 is in a cycle of length k (i.e. k is the least positive integer j for which $\sigma^j(1) = 1$).

2. (TARML) Thomas and Sophia are on a row of squares labelled $0, 1, \ldots, 35$. They both start at square 0. A fair coin is repeatedly flipped. If it turns up heads, Thomas moves to the next square; if it turns up tails, Sophia moves to the next square. Furthermore, if Sophia moves to a multiple of 5, then Thomas moves back to square 0. Compute the probability that Thomas reaches square 5 before Sophia reaches square 35.

3. (Daniel Litt) Let n be chosen uniformly at random from $\{1, \ldots, 100\}$. An urn has n red balls and $100 - n$ green balls. You pick a ball randomly out of the urn. It is red, and you discard it. You pick another ball randomly out of the urn. What is the probability that it is red?

4. Let $x_1, \ldots, x_{2025}$ be points on a circle, in that order. A person starts at x_1. Every turn, they walk to one of the two neighboring points with probability $1/2$ each. For each i, compute the probability that x_i is the last of the 2025 points to be visited.

   [You may assume that the walk reaches every point with probability 1.]

5. (Laplace's rule of succession) Let p be chosen uniformly at random from $[0, 1]$. A coin is biased to flip heads with probability p. You flip the coin n times, of which k show up heads. What is the probability that the next flip turns up heads?

2.2 More problems

1. An alpine ski team has 20 members. They descend a particular slope one by one every day in a random order, and no two of them ever record identical times. On an average day, how many times will the best record of the day be broken?

2. (SDHMMT) Choose a random permutation σ of $\{1, \ldots, 8\}$. Compute the probability that $\#\{(i, j) : i < j \text{ and } \sigma(i) > \sigma(j)\}$ is a multiple of 5.

3. A plane has 2025 seats and 2025 passengers, each with a different assigned seat. The first passenger to board sits in the wrong seat. Thereafter, each passenger either sits in their set if unoccupied or otherwise sits in a random unoccupied seat. What is the probability that the last passenger sits in their own seat?

4. There are n parking spaces labelled $1, \ldots, 2025$ in that order on a one-way street. Cars $C_1, \ldots, C_n$ enter the street in that order and try to park. Each car C_i has a preferred space a_i. A car will drive to its preferred space. If the space is unoccupied, it will park there. Otherwise, the car will keep on driving and park at the next available space. If the car gets to the end of the parking spaces, it leaves the street and does not park.

   We say that $(a_1, \ldots, a_{2025})$ is a **parking function** if every car parks. Show that the number of parking functions is 2026^{2024}.