Powerpoint

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For a circle $\mathcal{C}$ centered at O with radius R and point P, define the **power** of P w.r.t. $\mathcal{C}$ to be

$$\rho(P, \mathcal{C}) \coloneqq OP^2 - R^2.$$

Here are some useful facts about power:

- (Power of a Point) Let a line ℓ through P intersect $\mathcal{C}$ at two points A, B. Then $PA \cdot PB = \rho(P, \mathcal{C})$, a constant in ℓ.
  - ∗ The converse also holds: Let PAB and PCD be lines. If $PA \cdot PB = PC \cdot PD$, then $ABCD$ is cyclic.
- (Radical Axis) The locus of points with equal power w.r.t two circles is a line perpendicular to the line through their centers.
- (Radical Center) The pairwise radical axes of three circles concur (possibly at infinity).
- (Coaxial Lemma) Along a circle coaxial to $\mathcal{C}_1, \mathcal{C}_2$, the ratio $\rho(P, \mathcal{C}_1) : \rho(P, \mathcal{C}_2)$ is constant. The converse is also true.
- (Linearity) Along a line, the difference $\rho(P, \mathcal{C}_1) - \rho(P, \mathcal{C}_2)$ is linear.

Theory Problems

1. (La Hire) Let Γ be a circle with center O and radius R. Given a point P, the **inverse** of P w.r.t. Γ is the point P^* on ray $\overrightarrow{OP}$ for which $OP^* \cdot OP = R^2$. The **polar** of P w.r.t. Γ is the line through P^* perpendicular to OP. Prove that P is on the polar of Q if and only if Q is on the polar of P.
2. (Euler) Let ABC be a triangle with circumcenter O, incenter I, circumradius R, and inradius r. Prove that $OI^2 = R(R - 2r)$.
3. (Area of Pedal Triangle) Given a point P, the **pedal triangle** of P w.r.t. $\triangle ABC$ is the triangle $\triangle DEF$ where D, E, F are the projections of P onto the sides of $\triangle ABC$. Prove that the area of the pedal triangle of P is constant on any circle concentric to the circumcircle of $\triangle ABC$.
4. (Newton-Gauss line) Let $ABCD$ be a quadrilateral and $P = AD \cap BC, Q = AB \cap CD$. Then the midpoints of AC, BD, PQ are collinear.
5. (Classical) Let ABC be a triangle with orthic triangle DEF. Prove that the points $EF \cap BC, FD \cap CA, DE \cap AB$ lie on a line perpendicular to the Euler line of $\triangle ABC$.

Olympiad Problems

1. (1998 USAMO #2) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$, one draws the tangent AB to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio $\frac{AM}{MC}$.

2. Let $PAB, AQB, ABR, XBA, BYA, BAZ$ be six similar triangles with P, Q, R, X, Y, Z all on the same side of line AB. Prove that P, Q, R, X, Y, Z lie on a single circle.

3. (2009 USAMO #1) Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R, and S lie on a circle then the center of this circle lies on line XY.

4. (first half of 2018 CCAMB I15) In a triangle ABC, let the B-excircle touch CA at E, C-excircle touch AB at F. If M is the midpoint of BC, then let the angle bisector of $\angle BAC$ meet BC, EF, ME, MF at D, P, E', F'. Suppose that the circumcircles of $\triangle EPE'$ and $\triangle FPF'$ meet again at a point Q and the circumcircle of $\triangle DPQ$ meets line EF again at X.

   Prove that $\frac{XF}{XE} = \frac{DF'}{DE'}$.

5. (2007 Bundeswettbewerb Mathematik Round 1 #3) In triangle ABC points E and F lie on sides AC and BC such that segments AE and BF have equal length, and circles formed by A, C, F and by B, C, E, respectively, intersect at point C and another point D. Prove that the line CD bisects $\angle ACB$.

6. (2011 Iran TST #1) In acute triangle ABC, $\angle B$ is greater than $\angle C$. Let M is midpoint of BC and let E and F be the feet of the altitudes from B and C, respectively. Let K and L be the midpoints of ME and MF, respectively. If KL intersects the line through A parallel to BC at T, prove that $TA = TM$.

7. (1995 IMO #1) Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

8. (2008 IMO #1) An acute-angled triangle ABC has orthocentre H. The circle passing through H with centre the midpoint of BC intersects the line BC at A_1 and A_2. Similarly, the circle passing through H with centre the midpoint of CA intersects the line CA at B_1 and B_2, and the circle passing through H with centre the midpoint of AB intersects the line AB at C_1 and C_2. Show that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle.

9. (2014 All-Russian Grade 10 #6) Let M be the midpoint of the side AC of $\triangle ABC$. Let $P \in AM$ and $Q \in CM$ be such that $PQ = \frac{AC}{2}$. Let (ABQ) intersect with BC at $X \neq B$ and (BCP) intersect with BA at $Y \neq B$. Prove that the quadrilateral $BXMY$ is cyclic.

10. (2007 Poland Second Round #5) We are given a cyclic quadrilateral $ABCD$ with $AB \neq CD$. Quadrilaterals $AKDL$ and $CMBN$ are rhombi with equal sides. Prove that $KLMN$ is cyclic.

11. (2012 Japan MO Finals #4) Let PAB and PCD be triangles such that $PA = PB$, $PC = PD$, and PAC and BPD are both lines in this order. A circle S_1 passing through A, C intersects a circle S_2 passing through B, D at distinct points X, Y. Prove that the circumcenter of $\triangle PXY$ is the midpoint of the centers of S_1, S_2.

12. (2006 USAMO #6) Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

13. (2012 IMO #5) Let ABC be a triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK.

    Show that $MK = ML$.

14. (2012 ISL G8) Let ABC be a triangle with circumcircle ω and ℓ a line without common points on ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP, and CZP have a common point different from P or are mutually tangent at P.

15. (2010 Romania TST Day 5 #2) Let ℓ be a line, and let γ and γ' be two circles. The line ℓ meets γ at points A and B, and γ' at points A' and B'. The tangents to γ at A and B meet at point C, and the tangents to γ' at A' and B' meet at point C'. The lines ℓ and CC' meet at point P. Let λ be a variable line through P and let X be one of the points where λ meets γ, and X' be one of the points where λ meets γ'. Prove that the point of intersection of the lines CX and $C'X'$ lies on a fixed circle.

16. (2013 ELMOSL G7) Let ABC be a triangle inscribed in circle ω, and let the medians from B and C intersect ω at D and E respectively. Let O_1 be the center of the circle through D tangent to AC at C, and let O_2 be the center of the circle through E tangent to AB at B. Prove that O_1, O_2, and the nine-point center of ABC are collinear.