Inequalities

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If you have any questions/comments or find any mistakes, please contact me at trshin@ucsd.edu.
This handout is linked at mathweb.ucsd.edu/~trshin/handouts/ineq-ms.pdf.

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1 Toolkit

- Trivial inequality: $x^2 \geq 0$ for real numbers x.
- Triangle inequality: If a, b, c are the side lengths of a triangle, then $b + c > a$ (and other permutations). The converse is also true.

  If x and y are complex numbers, then $|x| + |y| \geq |x + y|$.
- AM-GM: For nonnegative real numbers $a_1, a_2, \ldots, a_n$, we have that
$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$
- Cauchy–Schwarz: For real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$, we have that
$$(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) \geq (x_1 y_1 + \cdots + x_n y_n)^2.$$

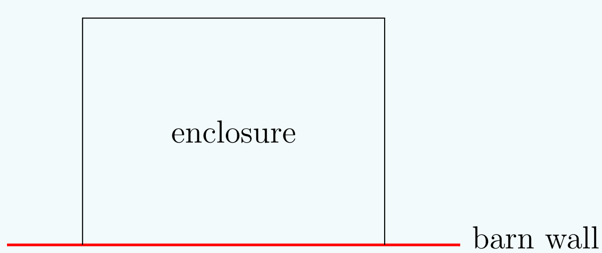
2 Examples

Example 2.1

Find the least possible value of $x^2 - 4x + 8$ for real x.

Example 2.2

John has 100 feet of fence to create an enclosure for his goat. The enclosure must be a rectangle adjacent to his barn, with one side being the wall of the barn and the other three sides made out of the fence. Compute the greatest possible area of the enclosure.



Example 2.3: Nesbitt

Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Example 2.4: Harmonic series

Prove that

$$\frac{\log_2(n)}{2} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \log_2(n) + 2.$$

3 Problems

1. Find the least possible value of $x + \frac{2}{x}$ for positive real x. What value(s) of x minimises this quantity?

2. (MATHCOUNTS) How many different combinations of three numbers can be selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ so that the numbers could represent the side lengths of a triangle?

3. In a right triangle, let a and b be the lengths of the legs, and let c be the length of the hypotenuse. What is the greatest possible value of $\frac{a+b}{c}$?

4. Jennifer wants to make a cardboard bin to store some objects. The bin should be 432 cubic inches in volume, and should be in the shape of a rectangular prism but with an open top (so it has five faces). What is the least amount of cardboard needed to make such a bin?

5. (Jeffrey Kwan) Find all ordered pairs of real numbers (x, y) such that
$$(4x^2 + 4x + 3)(y^2 - 6y + 13) = 8.$$

6. Let a, b, c be real numbers. Prove that
$$a^2 + b^2 + c^2 \geq bc + ca + ab.$$

7. Find the least possible value of $x^4 - 4x^3 + 8x^2 - 8x + 6$ for real x. What value(s) of x minimises this quantity?

8. Suppose x and y are complex numbers such that $|x| \leq |y|$. Prove that $|x + 2y| \geq |y|$.

9. (ARML) Compute the least integer value of the function
$$f(x) = \frac{x^4 - 6x^3 + 2x^2 - 6x + 2}{x^2 + 1}$$
whose domain is the set of all real numbers.

10. (ARML) An (a, r, m, ℓ)-*trapezoid* is a trapezoid with bases of length a and r, and other sides of length m and ℓ. Compute the number of positive integer values of ℓ such that there exists a $(20, 5, 15, \ell)$-trapezoid.

11. (AMC) Real numbers a and b are chosen with $1 < a < b$ such that no triangle with positive area has side lengths 1, a, and b or $\frac{1}{b}$, $\frac{1}{a}$, and 1. What is the smallest possible value of b?

12. (MATHCOUNTS) What is the minimum value of $8x^3 + 36x + \frac{54}{x} + \frac{27}{x^3}$ for positive real numbers x?

13. Let n be a positive integer. Prove that
$$\frac{1}{n+1} < \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + \cdots < \frac{1}{n}.$$

14. Let a, b, c be nonnegative real numbers such that $(a+1)(b+1)(c+1) = 8$. Compute the maximum possible value of abc.

15. Suppose the polynomial $f(x) = x^3 - 3x^2 + bx + c$ has three positive real roots r, s, t. Compute the least possible value of $f(-6)$.

16. Let n be a positive integer. Prove that
$$\log_2(1)^2 + \log_2(2)^2 + \log_2(3)^2 + \cdots + \log_2(n)^2 < 2n(\log_2(n) + 1)^2.$$

3.1 Challenge problems

17. (AMC) How many ordered pairs of positive real numbers (a, b) satisfy the equation
$$(1 + 2a)(2 + 2b)(2a + b) = 32ab?$$

18. (CCAMB) As a, b, c range over all real numbers, let m be the smallest possible value of
$$2(a+b+c)^2 + (ab-4)^2 + (bc-4)^2 + (ca-4)^2$$
and n be the number of ordered triplets (a, b, c) such that the above quantity is minimized. Compute $m + n$.

19. Let n be a positive integer. Prove that
$$\log_2(n)^2 + \log_2(n/2)^2 + \log_2(n/3)^2 + \cdots + \log_2(n/n)^2 < 6n + (\log_2(n) + 2)^3.$$

20. Alex will flip 300 fair coins. Let p be the probability that Alex will see at most 100 heads. Prove that $p < (27/32)^{100}$.

4 Hints

1. You want to go from an expression with x and $1/x$ to an expression with no x. Multiplying x and $1/x$ together does that. So AM-GM is worth a shot.
2. Casework on the largest side length.
3. Use the Pythagorean theorem to replace c.
4. If the dimensions of the bin are ℓ, w, h, then the amount of cardboard used is some expression involving $\ell w, \ell h, wh$.
5. What is an equation doing on an inequalities handout?! Try showing that one side is always at least the other side, and analyse the equality case.
6. How can we use some of a^2, b^2, c^2 to compare against bc? Do the same for ca and ab.
7. A simple completion of the square doesn't quite work, but a more involved one does.
8. An alternate form of the triangle inequality is that $|x-y| \geq |x| - |y|$.
9. Divide out the fraction.
10. Draw one line segment to split the trapezoid into a parallelogram and a triangle.
11. Why would there be no triangle with given side lengths?
12. A direct application of AM-GM gives the wrong answer. Instead, try simplifying the expression first.
13. Use the fact that $\frac{1}{k^2} < \frac{1}{k(k-1)}$. Write down a similar inequality for the lower bound.
14. Expand out the product and use AM-GM to force abc to show up.
15. Write b and c in terms of r, s, t and apply inequalities to force $r+s+t$ to show up.
16. Group the terms into the ranges $2^{i-1} < k \leq 2^i$.
17. Another equation showed up, so once again try showing that one side is always at least the other side.
18. A direct application of the trivial inequality gives the wrong answer. Instead, first expand the squares, then try rewriting as a different sum of squares.
19. Group the terms into the ranges $2^{i-1} < n/k \leq 2^i$.
20. We can compute $p = \frac{1}{2^{300}} \sum\limits_{k=0}^{100} \binom{300}{k}$. Expand out $(1+x)^{300}$ for some $0 < x < 1$ and compare to p.