Inellipses

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1 Background Info

Theorem 1.1: Ceva's Theorem

Let D, E, F on BC, CA, AB. Then AD, BE, CF concur iff

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Lemma 1.2: Ratio Lemma

Suppose D is on line BC. Then

$$\frac{BD}{DC} = \frac{BA}{AC} \cdot \frac{\sin \angle BAD}{\sin \angle DAC}.$$

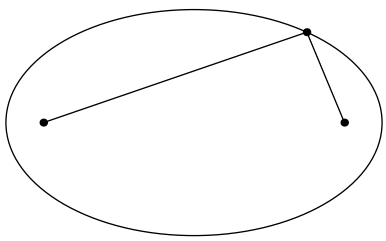
Corollary 1.3: Trig Ceva

Let D, E, F on BC, CA, AB. Then AD, BE, CF concur iff

$$\frac{\sin \angle BAD}{\sin \angle DAC} \cdot \frac{\sin \angle CBE}{\sin \angle EBA} \cdot \frac{\sin \angle ACF}{\sin \angle FCB} = 1.$$

Definition. In $\triangle ABC$, call two points P and Q (in the interior of $\triangle ABC$) *isogonal conjugates* if $\angle BAP = \angle CAQ, \angle CBP = \angle ABQ, \angle ACP = \angle BCQ$. If we have two of these angle relations, the third follows by Trig Ceva.

Definition. An *ellipse*, defined by a constant d and two points F_1, F_2, is the set of points P such that $PF_1 + PF_2 = d$.



Note that by definition, there is exactly one ellipse with given foci passing through a given point.

Lemma 1.4

For any point P on an ellipse $\mathcal{E}$ with foci F_1, F_2, the tangent to $\mathcal{E}$ at P is the exterior angle bisector of $\angle F_1PF_2$.

Proof. Extend ray F_1P to a point Q such that $PQ = PF_2$. Then for any point R on the angle bisector of $\angle F_2PQ$, we have that

$$PF_1 + PF_2 = PF_1 + PQ = F_1Q \leq RF_1 + RQ = RF_1 + RF_2,$$

with equality iff $P = R$. This implies that the angle bisector of $\angle F_2PQ$ is tangent to $\mathcal{E}$ at P. ■

Lemma 1.5

There is exactly one ellipse with given foci and a given tangent line.

Proof. Let the foci be F_1, F_2 and the tangent be ℓ. Consider the tangency point P and let X_1, X_2 be the projections of F_1, F_2 on ℓ. If P is outside segment X_1X_2 (WLOG closer to X_1 than X_2), then ℓ cannot be the exterior angle bisector of $\angle F_1PF_2$ since the outside angle between X_1P and ℓ is greater than $90°$. Thus, P lies between X_1 and X_2. But observe that the function $f(P) = \angle F_1PX_1 - \angle F_2PX_2$ is strictly decreasing on the line segment from X_1 to X_2, going from a positive number to a negative number, meaning that there is exactly one choice of P. Since there is exactly one ellipse with foci F_1, F_2 that passes through this choice of P, we deduce the claim. ■

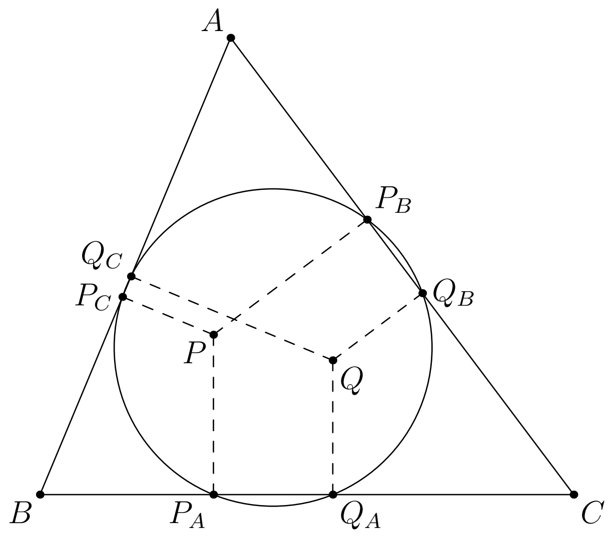
2 Isogonal Conjugates and Inellipses

Definition. In $\triangle ABC$, take P and project onto sides to get P_A, P_B, P_C. $\triangle P_AP_BP_C$ is the *pedal triangle* of P.

Lemma 2.1

If P and Q are isogonal conjugates, then the pedal triangles of P, Q share a circumcircle.

Proof. Draw the diagram as shown:



With $\angle BAP = \angle CAQ = \theta$ and $\angle QAB = \angle PAC = \phi$, we deduce that

$$AP_C \cdot AQ_C = AP\cos\theta \cdot AQ\cos\phi = AP\cos\phi \cdot AP\cos\theta = AP_B \cdot AQ_B,$$

so $P_BQ_BP_CQ_C$ cyclic. Similarly, $P_AQ_AP_BQ_B$ cyclic. But both of their circumcenters are the midpoint of PQ, so these two circles are the same and hence $P_AQ_AP_BQ_BP_CQ_C$ cyclic. ■

Lemma 2.2

Reflect Q over sides to get a triangle with circumcenter P.

Proof. Take homothety centered at Q with ratio 2; this sends the circle in Lemma 2.1 to the circumcircle of the reflected versions of Q, also sends midpoint of PQ to P. ■

Theorem 2.3

There exists an inellipse of $\triangle ABC$ with foci P, Q.

Proof. Let Q'_A be the reflection of Q over BC and $X = PQ'_A \cap BC$ (similarly define Y, Z), then

$$PX + QX = PY + QY = PZ + QZ$$

is the radius of the circle in Lemma 2.2, so there is an ellipse with foci P, Q that passes through X, Y, Z. Also $\angle PXB = \angle QXC$ by definition, so this ellipse is tangent to the sides. ■

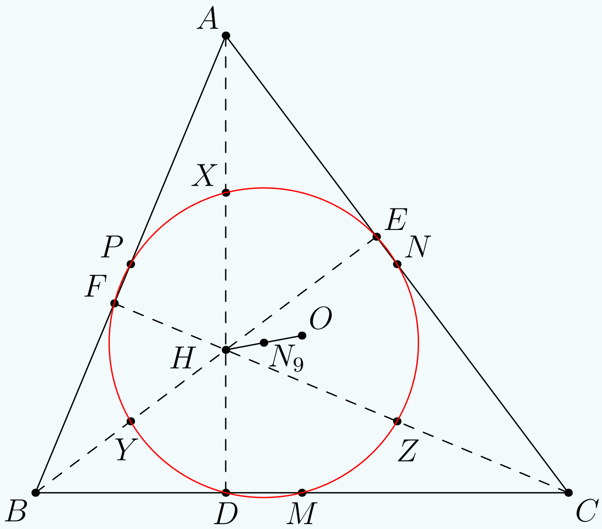
Theorem 2.4

Any inellipse of $\triangle ABC$ has foci that are isogonal conjugates.

Proof. Let the foci be P, Q. Reflect P over AB to get P', Q over AC to get Q'. If the inellipse is tangent to the sides at X, Y, Z, then we see that PYQ' and QZP' collinear because of the angles. Also, $PQ' = QP'$ by the sum definition of an ellipse. Then $\triangle AP'Q \cong \triangle APQ'$ by SSS, so we deduce that $\angle P'AQ = \angle PAQ'$. This implies that AP, AQ are isogonal. By symmetry, this result holds for any vertex, so P, Q are isogonal conjugates. ■

Example 2.5: Nine-Point Circle

In $\triangle ABC$ with circumcenter O, orthocenter H, medial triangle MNP, orthic triangle DEF, let the midpoints of AH, BH, CH be X, Y, Z. Then $M, N, P, D, E, F, X, Y, Z$ lie on a circle called the **nine-point circle** of $\triangle ABC$. The **nine-point center** is the center of the nine-point circle and is the midpoint of OH.



Proof. Observe that the orthocenter and circumcenter of a triangle are isogonal conjugates, so by Lemma 2.1, we see that the medial triangle and orthic triangle of any triangle share a circumcircle. Apply this to $\triangle ABC, \triangle BHC, \triangle CHA, \triangle AHB$ to get the result. ■

Lemma 2.6

Consider triangle ABC and one of its inellipses which touches the sides at D, E, F. Then AD, BE, CF concur.

Proof. Take an affine transformation (stretch) sending the inellipse to a circle, then they concur by Gergonne point (use Ceva's Theorem). ■

3 Marden-Linfield Theorem

First, we begin with a classical theorem relating the roots of functions from a class of functions and the roots of their derivatives.

Theorem 3.1: Gauss-Lucas Theorem

Given a function P that is the product of linear terms to positive real powers, the roots of P' lie in the convex hull of the roots of P.

Proof. Let $P(z) = \prod_r (z-r)^k$. Considering $\ln P$ and differentiating, we see that

$$\frac{P'(z)}{P(z)} = \sum_r \frac{k}{z-r}.$$

Consider a root s of P'. If s is a root of P, then clearly s lies in the convex hull of the roots of P. Otherwise, we have that

$$0 = \frac{P'(s)}{P(s)} = \sum_r \frac{k}{s-r} = \sum_r \frac{k(\overline{s}-\overline{r})}{|s-r|^2},$$

so

$$\overline{s} = \frac{\sum_r \frac{k\overline{r}}{|s-r|^2}}{\sum_r \frac{k}{|s-r|^2}}.$$

Conjugating this gives that

$$s = \frac{\sum_r \frac{kr}{|s-r|^2}}{\sum_r \frac{k}{|s-r|^2}},$$

meaning that s can be written as a linear combination of the r with positive coefficients which sum to 1, meaning that s lies in the convex hull of the r by definition. ■

In particular, if there are only three linear terms of P, then the roots of P form a triangle, and the roots of P' lie in the interior of the triangle.

Lemma 3.2

Given $T(z) = az + b$ a linear transformation ($a \neq 0$), function

$$P(z) = (z-z_1)^d (z-z_2)^e (z-z_3)^f.$$

Consider function

$$Q(z) = (z-T(z_1))^d (z-T(z_2))^e (z-T(z_3))^f.$$

Then the roots of P' map to the roots of Q' under T.

Proof. See that

$$Q\left(T\left(z\right)\right) = a^{d+e+f} P\left(z\right),$$

so differentiating this with respect to z, we observe that

$$aQ'\left(T\left(z\right)\right) = a^{d+e+f} P'\left(z\right),$$

implying the result. ■

Consider a triangle $Z_1Z_2Z_3$ in the complex plane (corresponding complex numbers of z_1, z_2, z_3). Choose positive real constants k_1, k_2, k_3 and consider an inellipse $\mathcal{E}$ of the triangle which touches the side opposite Z_i at Y_i such that $\frac{Z_{i+1}Y_i}{Y_iZ_{i+2}} = \frac{k_{i+1}}{k_{i+2}}$, where indices are taken modulo 3 (this is achievable by Ceva's Theorem and Lemma 2.6). Let

$$P\left(z\right) = \left(z - z_1\right)^{k_1}\left(z - z_2\right)^{k_2}\left(z - z_3\right)^{k_3}.$$

Define

$$Q\left(z\right) = \frac{P'\left(z\right)\left(z - z_1\right)\left(z - z_2\right)\left(z - z_3\right)}{P\left(z\right)}$$

(a quadratic whose roots are the roots of P' that are not also roots of P). Let the roots of Q be r_1, r_2 (points R_1, R_2 in the plane).

Lemma 3.3

The ellipse passing through the point X_1 on Z_2Z_3 with $\frac{Z_2X_1}{X_1Z_3} = \frac{k_2}{k_3}$ with foci at the roots of Q is tangent to Z_2Z_3.

Proof. Map the triangle to $\omega, -k_2, k_3$, where ω is some complex number in the upper half-plane. Then X_1 becomes the origin. We have that

$$P\left(z\right) = \left(z - \omega\right)^{k_1}\left(z + k_2\right)^{k_2}\left(z - k_3\right)^{k_3},$$

so

$$Q\left(z\right) = k_1\left(z + k_2\right)\left(z - k_3\right) + k_2\left(z - \omega\right)\left(z - k_3\right) + k_3\left(z - \omega\right)\left(z + k_2\right).$$

The product of the roots of Q then becomes $-\frac{k_1k_2k_3}{k_1+k_2+k_3}$ upon expanding, meaning that $\arg r_1 + \arg r_2 = \pi$. By Gauss-Lucas, R_1, R_2 are in the triangle. This implies that $\angle R_1X_1Z_2 = \angle R_2X_1X_3$, so the ellipse with foci R_1, R_2 and passing through X_1 is tangent to Z_2Z_3. ■

Lemma 3.4

R_1 and R_2 are isogonal conjugates in $\triangle Z_1Z_2Z_3$.

Proof. Map the triangle to $\omega, 0, k_2 + k_3$, where ω is some complex number in the upper half-plane. Then we have

$$P\left(z\right) = \left(z - \omega\right)^{k_1} z^{k_2}\left(z - \left(k_2 + k_3\right)\right)^{k_3},$$

so

$$Q(z) = k_1 z(z-(k_2+k_3)) + k_2(z-\omega)(z-(k_2+k_3)) + k_3 z(z-\omega).$$

Then the product of the roots of Q is $k_2(k_2+k_3)\omega$, meaning that $\arg r_1 + \arg r_2 = \arg\omega$. By Gauss-Lucas, R_1, R_2 are in the triangle. This implies that $\angle Z_3Z_2R_2 = \angle Z_1Z_2R_1$, so Z_2R_1, Z_2R_2 isogonal. By symmetry, the other pairs of lines that we need are isogonal, so R_1, R_2 are isogonal conjugates in $\triangle Z_1Z_2Z_3$. ■

Theorem 3.5: Marden-Linfield Theorem

Let $Z_1Z_2Z_3$ be a triangle with inellipse $\mathcal{E}$ which is tangent to $Z_{i+1}Z_{i+2}$ at Y_i (indices taken modulo 3).

(a) There exist positive real constants k_1, k_2, k_3 such that $\frac{Z_{i+1}Y_i}{Y_iZ_{i+2}} = \frac{k_{i+1}}{k_{i+2}}$.

(b) If we put the triangle and ellipse on the complex plane with number z_i corresponding to point Z_i and define the function

$$P(z) = (z-z_1)^{k_1}(z-z_2)^{k_2}(z-z_3)^{k_3},$$

then there are two roots of P' that are not roots of P, and they are the foci of $\mathcal{E}$.

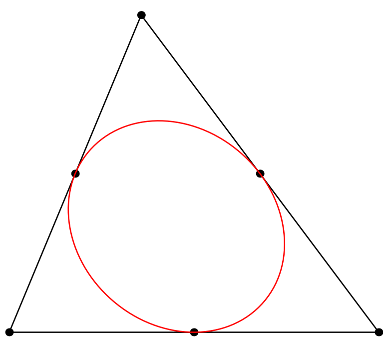
Proof. We have already proven (a).

For (b), we wish to look at the roots of Q.

Let $\mathcal{E}_i$ be the ellipse generated by Lemma 3.3, taking the side opposite Z_i (so we have three ellipses $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$). By Lemma 3.4, we see that R_1, R_2 are isogonal conjugates in $\triangle Z_1Z_2Z_3$. By Theorem 2.3, there exists an inellipse $\mathcal{E}'$ of $\triangle Z_1Z_2Z_3$ with foci at R_1, R_2. But by Lemma 1.5, there is only one ellipse with given foci that is tangent to a given line, so $\mathcal{E}' = \mathcal{E}_i$ for each i, implying that $\mathcal{E}'$ is the inellipse of $\triangle Z_1Z_2Z_3$ which is tangent to $Z_{i+1}Z_{i+2}$ at Y_i, thus $\mathcal{E}' = \mathcal{E}$ and we deduce that the roots of P' that are not roots of P are the foci of $\mathcal{E}$. ■

The special case when the tangency points are the midpoints is Marden's Theorem.

Definition. The **Steiner Inellipse** of a triangle is the inellipse tangent to the sides at their midpoints.



Theorem 3.6: Marden's Theorem

The foci of the Steiner Inellipse are the roots of the derivative of the cubic polynomial whose roots are the vertices of the triangle.

Example 3.7

Draw the figure above in the definition of the Steiner Inellipse (the triangle is a 13-14-15 triangle).

Example 3.8

The locus of centers of inellipses of a triangle is the interior of the medial triangle.

Linfield's theorem also admits a vast generalization about n-gons and curves of class $n-1$ — in fact, Linfield's original paper (found here) was the generalized version.

Theorem 3.9: Linfield's Theorem

Consider the function

$$P(z) = \prod_{i=1}^{n} (z - z_i)^{k_i},$$

where the k_i are real numbers. The roots of P' that are not roots of P are the foci of a curve of class $n-1$ which touches the line through z_i and z_j at a point $y_{i,j}$ such that $\frac{z_i - y_{i,j}}{y_{i,j} - z_j} = \frac{k_i}{k_j}$.

The proof of this theorem is very technical and uses projective facts and is omitted. If you are interested, you can find it in Linfield's paper.

4 Problems

Here are some contest problems involving inellipses (section added 29 Sep 2018).

1. (2000 ISL G3) Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that

$$OD + DH = OE + EH = OF + FH$$

and the lines AD, BE, and CF are concurrent.

2. (2017 BMT I19) Let T be the triangle in the xy-plane with vertices $(0,0)$, $(3,0)$, and $\left(0, \frac{3}{2}\right)$. Let E be the ellipse inscribed in T which meets each side of T at its midpoint. Find the distance from the center of E to $(0,0)$.

3. (2017 CCAMB I15) Let ABC, $AB < AC$ be an acute triangle inscribed in circle Γ with center O. The altitude from A to BC intersects Γ again at A_1. OA_1 intersects BC at A_2 Similarly define B_1, B_2, C_1, and C_2. Then $B_2C_2 = 2\sqrt{2}$. If B_2C_2 intersects AA_2 at X and BC at Y, then $XB_2 = 2$ and $YB_2 = k$. Find k^2.

4. (2015 Romania TST Day 4 #1) Let ABC and ABD be coplanar triangles with equal perimeters. The internal angle bisectors of $\angle CAD$ and $\angle CBD$ meet at P. Show that $\angle APC = \angle BPD$.

5. (2018 CMIMC G9) Suppose $\mathcal{E}_1 \neq \mathcal{E}_2$ are two intersecting ellipses with a common focus X; let the common external tangents of $\mathcal{E}_1$ and $\mathcal{E}_2$ intersect at a point Y. Further suppose that X_1 and X_2 are the other foci of $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, such that $X_1 \in \mathcal{E}_2$ and $X_2 \in \mathcal{E}_1$. If $X_1X_2 = 8$, $XX_2 = 7$, and $XX_1 = 9$, what is XY^2?

6. (2011 ELMOSL G4) Prove that for any convex pentagon $A_1A_2A_3A_4A_5$, there exists a unique pair of points $\{P, Q\}$ (possibly with $P = Q$) such that $\measuredangle PA_iA_{i-1} = \measuredangle A_{i+1}A_iQ$ for $1 \leq i \leq 5$, where indices are taken $\pmod 5$ and angles are directed $\pmod \pi$.

7. (2018 ELMOSL G4) Let $ABCDEF$ be a hexagon inscribed in a circle Ω such that triangles ACE and BDF have the same orthocenter. Suppose that segments BD and DF intersect CE at X and Y, respectively. Show that there is a point common to Ω, the circumcircle of DXY, and the line through A perpendicular to CE.

8. (OMO Spring 2018 #28) In $\triangle ABC$, the incircle ω has center I and is tangent to $\overline{CA}$ and $\overline{AB}$ at E and F respectively. The circumcircle of $\triangle BIC$ meets ω at P and Q. Lines AI and BC meet at D, and the circumcircle of $\triangle PDQ$ meets $\overline{BC}$ again at X. Suppose that $EF = PQ = 16$ and $PX + QX = 17$. Then BC^2 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.