SDMO Review

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MS #1

Problem (2020 SDMO MS #1)

Every bear attending the bear feast consumes exactly 6 quarts of liquid, divided between porridge and honey. Each bear consumes a positive amount of porridge and a positive amount of honey. The newest bear, Algebeara, drank $\frac{2}{17}$ of the total amount of porridge and $\frac{1}{10}$ of the total amount of honey. At the end of the feast, all of the food has been consumed. How much porridge did Algebeara drink?

Let  be total amount of porridge,  be total amount of honey.

$\frac{2}{17}$  $+ \frac{1}{10}$  $= 6$

Let  be # of bears at feast.

MS #1

Let  be total amount of porridge,  be total amount of honey.

$$\frac{2}{17}$$  $$+ \frac{1}{10}$$  $$= 6$$

Let  be # of bears at feast.

 $+$  $= 6$ 

Key idea: bounding

MS #1

$$\frac{20}{17}🥣 + 🍯 = 6 \cdot 10$$

$$\frac{20}{17}🥣 + 🍯 > 🥣 + 🍯 = 6🧸 \implies 🧸 < 10$$

$$\frac{16}{17}🥣 + \frac{8}{10}🍯 = 6 \cdot 8$$

$$\frac{16}{17}🥣 + \frac{8}{10}🍯 < 🥣 + 🍯 = 6🧸 \implies 🧸 > 8$$

So 🧸 $= 9$. So we can solve the system from previous slide. 🥣 $= 34$, so $\frac{2}{17}$🥣 $= \boxed{4}$.

MS #2

Problem (2020 SDMO MS #2)

How many positive integers are there less than $1,000,000$ whose digits have a sum of 11?

Consider all "six digit integers" (allowing leading zeros), e.g. 007281.

$\underline{abcdef}$ with $0 \leq a, b, c, d, e, f \leq 9$ and $a + b + c + d + e + f = 11$

Stars and bars:

$$\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot/////$$

$$\cdot\cdot\cdot/\cdot\cdot\cdot\cdot//\cdot\cdot\cdot/\cdot/$$

340310

$\binom{16}{5} = 4368$

MS #2

$10 + 1 + 0 + 0 + 0 + 0$ or $11 + 0 + 0 + 0 + 0 + 0$

$30 + 6$ ways that we overcounted. So answer should be $4368 - 36 = \boxed{4332}$.

MS #3

Problem (2020 SDMO MS #3)

What is the greatest possible product for a set of positive integers whose sum is 25?

Claim

Max product is achieved when all our nums are $\{2, 3\}$

Proof.

What happens if 1 is one of our nums? If we take any k as our num, then replacing $k, 1$ with $k + 1$ gives larger product. $4 + 1 + 1 + 1 \to 5 + 1 + 1$
What happens if $m \geq 4$ ("large integer") is one of our nums? Then replace m with $2, m - 2$. Increases product because $m \leq 2 \cdot (m - 2)$. So we can get rid of m.
So can keep on shrinking until no more large integers. Yay! $\square$

MS #3

Let a be # of 2's, b be # of 3's, then we want to maximize $2^a \cdot 3^b$ given $2a + 3b = 25$. Write $a = \frac{25-3b}{2}$.

$$2^a 3^b = 2^{\frac{25-3b}{2}} \cdot 3^b = 2^{25/2} \cdot \left(\frac{3}{2^{3/2}}\right)^b$$

So choose $b = 7, a = 2$. Gives answer of $2^2 \cdot 3^7 = \boxed{8748}$.

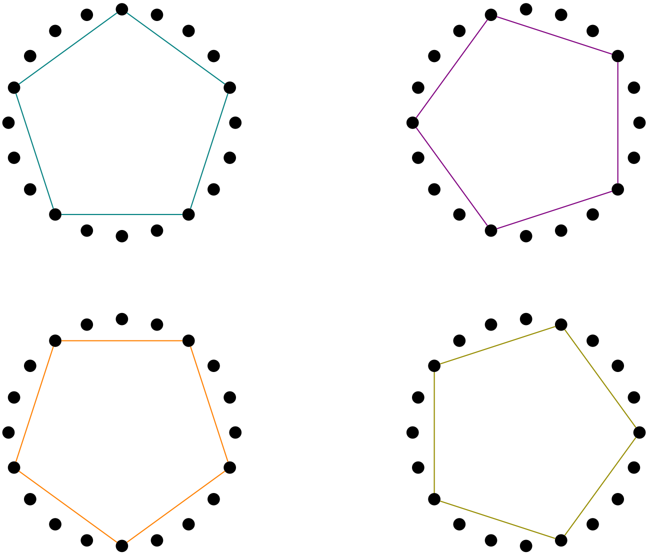
HS #4

Problem (2020 SDMO HS #4)

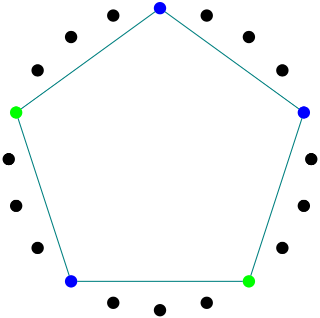
Suppose regular polygon with 20 sides has vertices colored red, blue, and green, such that there are exactly three red vertices. Prove that there are three vertices A, B, and C of the polygon having the same color such that triangle ABC is isosceles.

- counting/parity (if $\#$ of something is odd, then ≥ 1)
  - counting is hard because 20 is even (c.f. 2016 ISL C3)
  - solution exists along these lines, but requires being meticulous
- probabilistic method
  - unlikely to work directly, essentially reduces to counting argument
- pigeonhole
  - need to pass to a substructure which guarantees monochromatic isosceles $\triangle$

HS #4



HS #4



HS #4

Don't even need condition that there are 3 red vtx.

HS #5

Problem (2020 SDMO HS #5)

The length of each side of a convex quadrilateral $ABCD$ is a positive integer. The sum of the lengths of any three sides is divisible by the length of the remaining side. Prove that two of the quadrilateral's sides have the same length.

Assume $a < b < c < d$ are the side lengths.

$$a + b + c > d$$
$$a + b + c < d + d + d = 3d$$

$a + b + c$ is divisible by d. So $a + b + c = 2d$ (equiv $a + b + c + d = 3d$).

HS #5

$\frac{3d}{a}, \frac{3d}{b}, \frac{3d}{c}$ are all integers

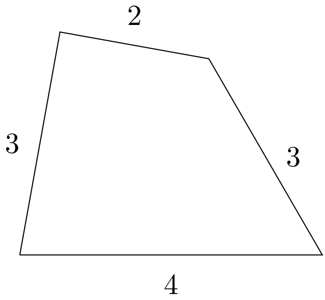
$$3 < \frac{3d}{c} = \frac{3 \cdot \frac{a+b+c}{2}}{c} < \frac{3 \cdot \frac{3c}{2}}{c} = 4.5$$

so $\frac{3d}{c} = 4$. So $a + b + c = 2d = \frac{8c}{3}$, so $a + b = \frac{5c}{3} = \frac{5d}{4}$.

$$4 = \frac{3d}{c} < \frac{3d}{b} = \frac{3 \cdot \frac{4}{5}(a+b)}{b} < \frac{3 \cdot \frac{4}{5} \cdot 2b}{b} = 4.8$$

Contradiction! So our assumption is wrong, so two sides are equal.

HS #5



All (a, b, c, d) that work are $(1, 1, 1, 1), (1, 1, 2, 2), (1, 3, 4, 4), (2, 3, 3, 4)$ up to similarity.