Proving Irrationality

Tristan Shin

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Disclaimer

The following slides are live-TeX'ed, so there may be typos and errors. Sorry in advance.

Irrational Numbers

Rational numbers: $\frac{a}{b}$ where a, b are integers

Example: $\frac{3}{5}, 2.72, \frac{2}{5} + \frac{4}{7} = \frac{34}{35}$

Irrational number: any real number that is not rational

Example: $\sqrt{2}, \pi, \sqrt{3}, e$ (Euler's constant)

$\sqrt{n}$ where n is not a perfect square

$\sqrt{2}$

Proof by contradiction: we want to prove a statement X. Instead, we assume that X is false, derive a contradiction.

X is: $\sqrt{2}$ is irrational

ASSUME $\sqrt{2}$ is rational; that is, we can write $\sqrt{2} = \frac{m}{n}$. Can assume that $\frac{m}{n}$ is a common fraction (e.g. $\frac{3}{9}$ is NOT a common fraction).

$\sqrt{2} \cdot n = m \implies 2n^2 = m^2$, so m is even. $m = 2a$

$2n^2 = m^2 = (2a)^2 = 4a^2 \implies n^2 = 2a^2$, so n is even.
CONTRADICTION!

So $\sqrt{2}$ is irrational.

What about square root of other numbers?

Rephrased version of $\sqrt{2}$ proof: Assume $\sqrt{2} = \frac{m}{n}$ (common fraction). $2n^2 = m^2$. Now, instead of doing divisibility by 2, we look at the *prime factorizations* of $2n^2$ and m^2. First, let 2^k is the power of 2 in the prime factorization of n. Similarly, let 2^ℓ be the power of 2 in the prime factorization of m.

Prime factorization: $2016 = 2^5 \cdot 3^2 \cdot 7$

Then the power of 2 in $2n^2$ is 2^{2k+1} and power of 2 in m^2 is $2^{2\ell}$. So $2k + 1 = 2\ell$, contradiction!

$\sqrt[k]{n}$ is irrational when "you expect it to be" — whenever it is not an integer

Some other irrationals

Suppose $\sqrt{2}+\sqrt{3}=r$ (r is rational). Then squaring, we get $2+3+2\sqrt{6}=r^2$. So $\sqrt{6}=\frac{r^2-5}{2}$ is rational. Contradiction! So $\sqrt{2}+\sqrt{3}$ are also irrational.

What about $\sqrt{2}+\sqrt{3}+\sqrt{5}$? Try squaring, you get $10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}$. This doesn't work!

$\sqrt{a_1}+\sqrt{a_2}+\sqrt{a_3}+\cdots+\sqrt{a_m}$ is irrational "when you expect it to be"

Even further: $\sqrt[k_1]{a_1}\pm\sqrt[k_2]{a_2}\pm\cdots\pm\sqrt[k_m]{a_m}$ are irrational "when you expect it" (can choose $k=\operatorname{lcm}(k_1,k_2,\ldots,k_m)$)

E.g. $\sqrt{2}+\sqrt[3]{5}$ is irrational, this is the same as $\sqrt[6]{8}+\sqrt[6]{25}$ is irrational

Some NOT irrational numbers

Assume a and b are irrational.

- $a + b$ is not necessarily irrational (e.g. $\sqrt{2}$ and $-\sqrt{2}$)
- ab is not necessarily irrational (e.g. $\sqrt{2}$ and $\sqrt{2}$)
- a^b is not necessarily irrational

a^b is not necessarily irrational

$\sqrt{2}^{\sqrt{2}}$ (is actually irrational)

If this is rational, then this is our counterexample.

If this is irrational, then let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. Then

$$a^b = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2$$

so we have a counterexample.

a^b is not necessarily irrational, again

$a = \sqrt{2}$ and $b = \log_{\sqrt{2}} 3$

Logarithms: $\log_c x$ is the exponent d such that $c^d = x$

Example: $\log_2 16 = 4$

Why is b irrational? Well, if $\log_{\sqrt{2}} 3 = \frac{m}{n}$ then
$\sqrt{2}^{m/n} = 3 \implies 2^{m/n} = 9 \implies 2^m = 9^n \implies (m, n) = (0, 0)$,
contradiction!

$a^b = \sqrt{2}^{\log_{\sqrt{2}} 3} = 3$

Euler's constant

$e \approx 2.718281828459045\ldots$

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

$e^{i\pi} + 1 = 0$

$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$

$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

Proof that e is irrational

Assume that $e = \frac{m}{n}$. Multiply by $n!$ to get

$$n! \cdot e = m \cdot (n-1)! = \text{integer}$$

$$\color{blue}{n! \cdot e = \frac{n!}{0!} + \frac{n!}{1!} + \cdots + \frac{n!}{n!}} + \color{red}{\frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + \frac{n!}{(n+3)!} + \dots}$$

$$\begin{aligned} \color{red}{\text{red}} &= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots \\ &< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \dots \\ &= \frac{1/(n+1)}{1 - 1/(n+1)} = \frac{1}{n} \end{aligned}$$

so

$$0 < \text{red} < \frac{1}{n} \leq 1, \quad \text{contradiction!}$$

e^2 is irrational

Assume $e^2 = \frac{m}{n}$. So $n \cdot e = m \cdot e^{-1}$.

$e^{-1} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$

Multiply by $A!$, where $A = 2(m+n)$

$$A! \cdot n \cdot e = \frac{A! \cdot n}{0!} + \cdots + \frac{A! \cdot n}{A!} + \frac{A! \cdot n}{(A+1)!} + \frac{A! \cdot n}{(A+2)!} + \frac{A! \cdot n}{(A+3)!} + \dots$$

So by argument from last slide, $A! \cdot n \cdot e = \text{integer} + (< \frac{n}{A})$

e^2 is irrational, continued

$$A! \cdot m \cdot \frac{1}{e} = \frac{A! \cdot m}{0!} - \cdots + \frac{A! \cdot m}{A!} - \frac{A! \cdot m}{(A+1)!} + \frac{A! \cdot m}{(A+2)!} - \frac{A! \cdot m}{(A+3)!} + \dots$$

$$\text{tail} > -\frac{A! \cdot m}{(A+1)!} = -\frac{m}{A+1} > -\frac{m}{A}$$

$A! \cdot n \cdot e = \text{integer} + (< \frac{n}{A})$

$A! \cdot m \cdot \frac{1}{e} = \text{integer}_1 + (> -\frac{m}{A}) = \text{integer}_2 + (> 1 - \frac{m}{A})$

Thus e^2 is irrational!

Feedback

Thank you for coming! Hope you enjoyed!

Slides will be posted at
`https://www.mit.edu/~shint/handouts/vSDMC/irrational.pdf`

For any questions or comments, feel free to contact me at
`shint@mit.edu`.

If you have feedback, please give it to us at

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