Polynomials in Combinatorics

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1 Theory

Polynomials encode a lot of information about combinatorial objects, especially additive information.

Example 1.1

Suppose we roll two weighted dice with faces $1, \ldots, 6$ (possibly with different weights), and write down the values on the top face as x and y. Is it possible that $\mathbb{P}(x + y = k)$ is constant for $k = 2, \ldots, 12$?

In general, we can take advantage of the additive structure of convolutions, as well as the fact that the fundamental counting principle gives multiplicative structure to variables, to use polynomials to solve combinatorics problems.

Example 1.2: Cayley

Prove that there are n^{n-2} labelled trees on n vertices.

The following theorem is a generalization of the fact that degree n single variable polynomials have at most n roots.

Theorem 1.3: Combinatorial Nullstellensatz

Let $f \in K[x_1, \ldots, x_n]$ be a polynomial over the field K. Suppose that there is some non-zero term $C \prod x_i^{t_i}$ of f such that $\deg P = \sum t_i$. If $S_1, \ldots, S_n$ are subsets of K with $|S_i| > t_i$, then there exist $s_i \in S_i$ for which $f(s_1, \ldots, s_n) \neq 0$.

Example 1.4: Chevalley

Let $f_1, \ldots, f_k$ be n-variable polynomials over $\mathbb{F}_p$ such that

$$\deg f_1 + \cdots + \deg f_k < n.$$

Suppose that the f_i have a common root. Prove that they have another common root.

Example 1.5: Cauchy-Davenport

Let A, B be non-empty subsets of $\mathbb{F}_p$. Prove that

$$|A + B| \geq \min\{p, |A| + |B| - 1\}.$$

2 Problems

1. (2016 HMMT C7) Kelvin the Frog has a pair of standard fair 8-sided dice (each labelled from 1 to 8). Alex the sketchy Kat also has a pair of fair 8-sided dice, but whose faces are labelled differently (the integers on each Alex's dice need not be distinct). To Alex's dismay, when both Kelvin and Alex roll their dice, the probability that they get any given sum is equal!

   Suppose that Alex's two dice have a and b total dots on them, respectively. Assuming that $a \neq b$, find all possible values of $\min\{a, b\}$.

2. (2015 BMO Round 2 #2) At an odd high school, there are an odd number of disjoint classes, each consisting of an odd number of pupils. One pupil from each class will be chosen to form the school council. Say that a pupil is "odd" if their birthday falls on an odd-numbered date, and "not-odd" if otherwise. Prove that the following are equivalent:

   - There are more ways to form a school council which includes an odd number of odd pupils than ways to form a school council which includes an odd number of not-odd pupils.
   - There are an odd number of classes which contain more odd pupils than not-odd pupils.

3. (2007 All-Russian Grade 11 #5) To each vertex of a convex 100-gon we attach a pair of distinct numbers. Prove that one can select a number from each vertex so that the numbers attached to any two adjacent vertices differ.

4. (San Diego) How many permutations of $\{1, \dots, 8\}$ have $0 \pmod 5$ inversions?

5. Choose $2n$ distinct numbers $a_1, \dots, a_n, b_1, \dots, b_n$ and let $M = \{m_{i,j}\}_{1 \le i,j \le n}$ be the matrix with $m_{i,j} = a_i + b_j$. Prove that if the product of each column of M is the same, then the product of each row is the same.

6. (2018 Putnam B6) Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$

7. (Erdős) Let $A \neq B$ be n-element sets of positive integers. Suppose that any integer can be written as a sum of two distinct elements of A in the same number of ways as it can be written as a sum of two distinct elements of B. Determine all possible values of n.

8. (2011 TSTST #9) Let n be a positive integer. Suppose we are given $2^n + 1$ distinct sets, each containing finitely many objects. Place each set into one of two categories,

the red sets and the blue sets, so that there is at least one set in each category. We define the *symmetric difference* of two sets as the set of objects belonging to exactly one of the two sets. Prove that there are at least 2^n different sets which can be obtained as the symmetric difference of a red set and a blue set.

9. (2018 TSTST #6) Let $S = \{1, \ldots, 100\}$, and for every positive integer n define

$$T_n = \{(a_1, \ldots, a_n) \in S^n \mid a_1 + \ldots + a_n \equiv 0 \pmod{100}\}.$$

Determine which n have the following property: if we color any 75 elements of S red, then at least half of the n-tuples in T_n have an even number of coordinates with red elements.

10. Let G be a simple graph with average degree greater than $2p-2$ and maximum degree equal to $2p-1$, where p is a fixed prime. Prove that G contains a p-regular subgraph.

11. (2007 IMO #6) Let n be a positive integer. Consider

$$S = \{(x, y, z) \mid x, y, z \in \{0, 1, \ldots, n\}, x + y + z > 0\}$$

as a set of $(n+1)^3 - 1$ points in the three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

12. (Erdős-Ginzburg-Ziv) Given $2p-1$ integers (not necessarily distinct), prove that there are p of them whose sum is $0 \pmod{p}$.

13. (2015 ISL C6) Let S be a nonempty set of positive integers. We say that a positive integer n is *clean* if it has a unique representation as a sum of an odd number of distinct elements from S. Prove that there exist infinitely many positive integers that are not clean.

14. Prove that K_n cannot be decomposed into fewer than $n-1$ complete bipartite graphs.

15. (Hard, research problem, don't try at home) A *cap set* is a set of points in $(\mathbb{Z}/3\mathbb{Z})^n$ with no three elements in a line. Prove that the largest cap set has size at most $C \cdot 2.76^n$ for some constant C.