Poker Probabilities

Tristan Shin

vSDMC

28 Mar 2020

Overview

- Poker
- Counting
    - Fundamental Counting Principle
    - Permutations and combinations
    - Casework
    - Complementary counting
- Probability
    - Definition
    - Applying counting tools
    - Conditional probability
    - Expected value
- Poker Questions

Poker cards

52 cards: 13 **ranks** and 4 **suits**

13 ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A (in increasing order of value)

4 suits: clubs (♣), diamonds (♦), hearts (♥), spades (♠)

All suits have the same value!

Put them together: 8♦, J♥, 2♣

Poker hands

Five cards together form a **hand**.

Examples

- 8♦, J♥, 2♣, 3♣, A♦
- 3♦, 4♣, 5♠, 6♦, 7♦
- 4♣, J♣, 4♠, 4♥, J♦

Poker hands

Five cards together form a **hand**.

Examples

- 8♦, J♥, 2♣, 3♣, A♦(high card)
- 3♦, 4♣, 5♠, 6♦, 7♦(straight)
- 4♣, J♣, 4♠, 4♥, J♦(full house)

Poker hands

All poker hands can be classified into one of 9 groups:

Definition

- High card: K♠, Q♦, J♣, 10♥, 5♠
- Pair: A♦, A♠, 8♣, 5♠, 4♥
- Two pair: A♥, A♣, K♥, K♠, 9♦
- Triple: 5♦, Q♣, 2♥, 2♠, 2♦
- Straight: 9♣, 8♦, 7♠, 6♦, 5♠; also 5♦, 4♦, 3♣, 2♥, A♣
- Flush: K♥, J♥, 9♥, 6♥, 3♥
- Full house: J♥, J♠, J♦, Q♣, Q♦
- Quads: 7♣, 7♥, 7♠, 7♦, 9♠
- Straight flush: 9♦, 8♦, 7♦, 6♦, 5♦

During a hand

No-limit hold'em

- Players sit in a circle
- Each player dealt 2 cards
- Round of betting
- Three cards dealt to middle (flop)
- Second round of betting
- One more card dealt to middle (turn)
- Third round of betting
- One last card dealt to middle (river)
- Last round of betting
- Among all remaining players, best hand wins the pot

Betting

When the betting comes to you, you have a few options:

- If no one before you has bet:
    - You can **check** (adding no money to the pot).
    - You can **bet** (adding any amount of money that you have).
    - You can **fold** (giving up; you can no longer win any money).
- If someone before you has bet:
    - You can **call** (adding the current bet).
    - You can **raise** (adding any amount of money greater than current bet).
    - You can **fold**.

Betting continues in the circle until everyone has either matched the last bet or folded.

For the pedants, there are a few deviations from the normal rules — they won't be relevant today.

End goal

Problem

You are dealt A♠ and 7♠. The flop comes out as 8♠, 6♠, 5♥. The pot is currently at 1000 chips. The player before you bets 500 chips. You have 5 seconds to act. Should you call or fold? (Assume you do not want to raise.) What if they bet 1000 chips? 2000 chips?

Some relevant information:

- Flushes are better than straights.
- A flush with an Ace is the best flush.

Counting

Most of this will be review for some of you.

Theorem (Fundamental Counting Principle, a.k.a. multiplication)

If there are m ways to do one thing and n ways to do another, then there are $m \cdot n$ ways of doing both.

Examples

- If I have 5 shirts and 6 pants, then I have $5 \cdot 6 = 30$ shirt-pant outfits.
- There are $13 \cdot 4 = 52$ cards in a poker deck.
- There are $5! := 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to arrange 5 people in a line.
- In general, there are $n! := n \cdot (n-1) \cdots 2 \cdot 1$ ways to permute n things. We say "n **factorial**" for $n!$ in English.

Permutations

What if we want to choose one thing, then the next?

Problem

Five students are running a race. In how many ways can three of the five students place first, second, and third?

By FCP, answer is $5 \cdot 4 \cdot 3 = \mathbf{60}$ ways.

In general, to choose k things from n things in order, the answer is

$$_n\mathrm{P}_k \coloneqq n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

Combinations

What if we want to choose k things at the same time?

Problem

After winning the aforementioned race, Roger gets to choose 4 different prizes from a set of 9. How many ways can he choose his prizes?

If he chose them in order, there would be ${}_9\mathrm{P}_4 = \frac{9!}{5!}$ ways. But if he chooses prizes ABCD, then there are $4!$ different times that we counted this choice: BADC, CABD, and all the other permutations of ABCD. So we need to divide by $4!$ to get $\frac{9!}{4!5!} = \mathbf{126}$ ways.

In general, to choose k things from n things *not* in order, the answer is

$$\binom{n}{k} := \frac{{}_n\mathrm{P}_k}{k!} = \frac{n!}{k!(n-k)!}.$$

Permutations and combinations

${}_n\mathrm{P}_k = n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}$

$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$

Key relationship:

$$k! \cdot \binom{n}{k} = {}_n\mathrm{P}_k$$

Casework

Sometimes it is easier to count something by breaking it into cases.

Problem

How many ways are there to choose 2 poker cards that are either the same rank or the same suit?

- First case: same rank. There are 13 ways to choose the rank. After that, there are $\binom{4}{2} = 6$ ways to choose the cards. So 78 ways.
- Second case: same suit. There are 4 ways to choose the suit. After that, there are $\binom{13}{2} = 78$ ways to choose the cards. So 312 ways.

Combining these gives an answer of $\mathbf{390}$ ways.

Note that we can break into these cases because they are **disjoint**.

Complementary counting

Sometimes it is easier to count something by counting everything else.

Problem

How many ways are there to choose 2 poker cards that are not consecutive?

Note: two cards are consecutive if their ranks are consecutive, including Ace and 2 (so A♥ and 2♦ are consecutive).

We instead count the number that *are* consecutive. There are 13 consecutive ranks. Then we choose the suits of the cards in $4 \cdot 4$ ways. So 208 consecutive cards. There are $\binom{52}{2} = 1326$ ways to choose the 2 cards, so $1326 - 208 = \mathbf{1118}$ ways to choose 2 non-consecutive cards.

Definition of probability

In non-specific terms:

Definition (Probability)

The **probability** of E happening when doing S is the ratio of the number of ways to do E to the number of ways to do S. We denote this by $\mathbb{P}(E)$.

Problem

What is the probability of flipping heads with a fair coin?

Answer is clearly $\frac{1}{2}$ chance.

A probability example

Problem

A fair coin is flipped 3 times. If the first coin is heads, what is the probability that they all are heads?

The first coin flip does not matter for the second and third. So the probability is $\frac{1}{2} \cdot \frac{1}{2} = \mathbf{\frac{1}{4}}$ chance.

Another probability example

Problem

What is the probability that you are dealt 2 cards that are either the same rank or the same suit?

$$\frac{390}{\binom{52}{2}} = \frac{390}{1326} = \mathbf{\frac{5}{17}}$$

Another way of thinking about it: Pick the first card arbitrarily. Then out of the 51 possible cards for the second card, 3 are the same rank and 12 are the same suit for 15 cards that work. So the probability is $\frac{15}{51} = \frac{5}{17}$ which is the same as before.

Counting tools applied to probability

The same tools apply here:

- Fundamental Counting Principle
- Casework
- Complementary counting

Problem

What is the probability that you are dealt 2 cards that are not consecutive?

Pick the first card arbitrarily. Then out of the 51 possible cards for the second card, 8 are consecutive to the first card. So the probability of consecutive cards is $\frac{8}{51}$, which means that the probability of non-consecutive cards is $1 - \frac{8}{51} = \mathbf{\frac{43}{51}}$ chance.

Yet another probability example

Problem

Suppose you are dealt K♥ and K♠, while your opponent is dealt A♥ and J♥. The flop is Q♠, 10♥, and 6♣. The turn is 7♠. What is the probability that your opponent completes a straight on the river?

There are 44 unknown cards that could come up, among which 2 complete your opponent's straight (K♣ and K♦). So the probability of completing the straight is $\frac{2}{44} = \mathbf{\frac{1}{22}}$.

A note about FCP

Note that we need to be careful about how we apply FCP.

Problem

Darren rolls two fair standard 6-sided dice. What is the probability that their sum is 7 and their product is 12?

Here is a **bad solution**: The probability of the sum being 7 is $\frac{6}{36} = \frac{1}{6}$ and the probability of the product being 12 is $\frac{4}{36} = \frac{1}{9}$, so the total probability is $\frac{1}{54}$.

This is bad because the sum and product are **dependent**. The value of the sum restricts the possible values of the products.

Instead, we can solve this by noting that the dice must have rolled $\{3, 4\}$ so the probability is $\frac{2}{36} = \mathbf{\frac{1}{18}}$.

A note about FCP

We can use FCP to multiply probabilities if they are **independent**.

Problem

Darren rolls two fair standard 6-sided dice. What is the probability that both dice show up as 6?

The answer is $\frac{1}{6} \cdot \frac{1}{6} = \mathbf{\frac{1}{36}}$.

Even another probability example

Problem

Three numbers are chosen uniformly at random from $\{1, \ldots, 2020\}$. What is the probability that the sum of their units digits is even?

This problem seems hard — how do we even grasp it?

We can observe that the sum of the units digits being even is the same as the sum of the three numbers being even.

They key claim is the following: No matter what the first two integers are, there is a $\frac{1}{2}$ chance that the third integer gives that the sum is even.

To see this, consider what happens if the first two sum to an odd number. Then the third number must be odd, so there is a $\frac{1}{2}$ chance. The same happens if the first two sum to an even number, but the third number must be even. So the total probability is $\mathbf{\frac{1}{2}}$.

Some notation

For two events A and B, we write $A \cap B$ for the event of both A and B happening.

Then FCP becomes:

Theorem (FCP for probability)

If A and B are independent, then $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.

Likewise, we write $A \cup B$ for the event of either A or B happening (possibly both).

Then it turns out that:

Theorem (Principle of Inclusion-Exclusion)

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Conditional probability

Sometimes, we are given that an event (B) has happened and want to compute the probability that A will happen. We write this as $A|B$, or "A **given** B".

Then it turns out that

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Conditional probability example

Problem

A fair coin is flipped 3 times. If at least one coin is heads, what is the probability that they all are heads?

We can set A to be "all 3 are heads" and B to be "at least 1 is heads". Then

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/8}{7/8} = \frac{\mathbf{1}}{\mathbf{7}}.$$

Question

Why is this different from the $\frac{1}{4}$ we calculated earlier?

Bayes' theorem

We now take a digression to talk about Bayes' theorem.

Observe:

$$\mathbb{P}(A|B) = \frac{P(A \cap B)}{P(B)}$$
$$\mathbb{P}(B|A) = \frac{P(B \cap A)}{P(A)}$$

So the following is true:

Theorem (Bayes' theorem)

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A) \cdot \mathbb{P}(A)}{\mathbb{P}(B)}$$

Bayes' theorem example

Problem

Suppose Ashton believes he has a 30% chance of having salmonella. He takes a test that is 95% accurate and tests negative. That is, 95% of people with salmonella will test positive, and 95% who do not have it will test negative. What should he update his probability of having salmonella to?

Here, we set A to be "Ashton has salmonella" and B to be "Ashton tests negative".

Bayes' theorem example

Here, we set A to be "Ashton has salmonella" and B to be "Ashton tests negative".

Then
$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A) \cdot \mathbb{P}(A)}{\mathbb{P}(B)}.$$

From the given, $\mathbb{P}(B|A) = 0.05$.

From Ashton's assumption, we should assume $\mathbb{P}(A) = 0.3$

We can compute $\mathbb{P}(B)$ with casework — Ashton has salmonella with probability 0.3 and tests negative with probability 0.05, while he doesn't have salmonella with probability 0.7 and tests negative with probability 0.95. This combines for a $0.3 \cdot 0.05 + 0.7 \cdot 0.95 = 0.68$ chance of testing negative.

Bayes' theorem example

Combining this, we get that

$$\mathbb{P}(A|B) = \frac{0.05 \cdot 0.3}{0.68} \approx 0.022.$$

So Ashton should update to a $\mathbf{2.2}\%$ chance of having salmonella.

What does this tell us?

Ashton's initial evaluation of 30% chance was too high.

Bayesian inference

This is an example of **Bayesian inference**, a process in which Bayes' theorem is used to **update** an assumed probability of an event.

For example, if we wanted more information, we could have Ashton take the test again. Say he turns up negative again. Running similar calculations using our updated probability of Ashton having salmonella gives us that we should update to a 0.12% chance of having salmonella.

The method works the other way too. If Ashton were to have tested positive twice, he should update his probabilities as $30\% \mapsto 89\% \mapsto 99\%$.

Exercise

If Ashton had tested positive then negative, he should not update his probability. Similarly for if he tested negative then positive. Can you show that this is true in general (no matter what his original assumption was)?

One more probability problem

Problem

You are dealt A♠ and 7♠. The flop comes out as 8♠, 6♠, 5♥. What is the probability that you complete a flush on either the turn or the river?

We will use complementary counting. Instead of solving this problem, we will compute the probability that neither of the remaining two cards is a spade (equivalent to no flush).

There are 9 spades left, so the probability of the next card being a spade is $\frac{9}{47}$. So the probability that the turn is *not* a spade is $1 - \frac{9}{47}$.

Similarly, the probability that the river is *not* a spade is $1 - \frac{9}{46}$.

One more probability problem

So the probability that no spade comes up is
$(1-\frac{9}{47})(1-\frac{9}{46}) = 1-\frac{9}{47}-\frac{9}{46}+\frac{9^2}{47\cdot 46}$.

Thus the probability that a spade comes up to complete the flush is

$$1-(1-\frac{9}{47}-\frac{9}{46}+\frac{9^2}{47\cdot 46}) = \frac{9}{47}+\frac{9}{46}-\frac{9^2}{47\cdot 46} \approx 35\%.$$

But what if we want a fast way to estimate this?

Outs

At any point in the game, there are certain cards that would make your hand a lot stronger. For example, in the previous problem, any of the 9 remaining spades would instantly boost your hand. We call these cards **outs**. If you "hit an out," that means that the out appears on the board.

Proposition

If you have n outs after the flop, you have around a $4n\%$ chance of hitting an out.

If you have n outs after the turn, you have around a $2n\%$ chance of hitting an out.

Outs

Proposition

If you have n outs after the flop, you have around a $4n\%$ chance of hitting an out.

Proof.

By the same logic as before, the probability that you hit an out is

$$1-(1-\frac{n}{47})(1-\frac{n}{46})=\frac{n}{47}+\frac{n}{46}-\frac{n^2}{47\cdot 46}.$$

It turns out that n is necessarily small (you will rarely deal with situations with more than 10 outs). As a result, $\frac{n^2}{47\cdot 46}\approx 0$. In addition, we can round 47 and 46 to 50. This gives a probability of $\frac{2n}{50}=4n\%$. $\square$

The margin of error is $\pm 2\%$ for $n\leq 10$. For any reasonable value of n, this is a close enough estimate to work with in real games.

Outs

The proof is even simpler for after the turn.

Proposition

If you have n outs after the turn, you have around a $2n\%$ chance of hitting an out.

Proof.

The probability of hitting an out on the river is $\frac{n}{46} \approx \frac{n}{50} = 2n\%$. $\square$

This estimate is $\pm 2\%$ for $n \leq 11$.

Expected Value

The **expected value** of a random number X is the average value of X. We denote this as $\mathbb{E}[X]$. A common misconception is that the expected value is the most likely outcome. Let me ingrain this in you right now: **Expected value is not necessarily the most likely outcome.**

Examples

- The expected value of a dice roll is $\frac{1+2+3+4+5+6}{6} = 3.5$.
- The expected value of the sum of 100 dice roll is 350.

Linearity of Expectation

Theorem (Linearity of Expectation)

If X and Y are random numbers, then

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

regardless of if X and Y are dependent or not.

Linearity example

Problem

In a math class, everyone has a name tag. The teacher accidentally shuffles the name tags and hands them out randomly. Let F be the number of students who get their own name tag. Show that the expected value of F is 1.

Remark

This result is particularly surprising because I didn't even specify how many students there are!

Linearity example

Easy to verify for $1, 2, 3$ students. Even 4 is doable by hand:

A	B	C	D	Σ
A	**B**	**C**	**D**	4
A	**B**	D	C	2
A	C	B	**D**	2
A	C	D	B	1
A	D	B	C	1
A	D	**C**	B	2
B	A	**C**	**D**	2
B	A	D	C	0
B	C	A	**D**	1
B	C	D	A	0
B	D	A	C	0
B	D	**C**	A	1
C	A	B	**D**	1
C	A	D	B	0
C	**B**	A	**D**	2
C	**B**	D	A	1
C	D	A	B	0
C	D	B	A	0
D	A	B	C	0
D	A	**C**	B	1
D	**B**	A	C	1
D	**B**	**C**	A	2
D	C	A	B	0
D	C	B	A	0
				24

Linearity example

If you average by permutation, it's hard. But if you sum by column . . .

A	B	C	D	Σ
A	**B**	**C**	**D**	4
A	**B**	D	C	2
A	C	B	**D**	2
A	C	D	B	1
A	D	B	C	1
A	D	**C**	B	2
B	A	**C**	**D**	2
B	A	D	C	0
B	C	A	**D**	1
B	C	D	A	0
B	D	A	C	0
B	D	**C**	A	1
C	A	B	**D**	1
C	A	D	B	0
C	**B**	A	**D**	2
C	**B**	D	A	1
C	D	A	B	0
C	D	B	A	0
D	A	B	C	0
D	A	**C**	B	1
D	**B**	A	C	1
D	**B**	**C**	A	2
D	C	A	B	0
D	C	B	A	0
6	6	6	6	24

Linearity example

This can be encapsulated in a linearity argument. Let X_i denote the indicator variable for if the ith person gets their own name tag.

$$X_i = \begin{cases} 1 & \text{if person } i \text{ gets own} \\ 0 & \text{if else} \end{cases}$$

Then $F = X_1 + X_2 + \cdots + X_n$ where n is the number of students.

By linearity,

$$\mathbb{E}[F] = \mathbb{E}[X_1 + X_2 + \cdots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n].$$

But each $\mathbb{E}[X_i]$ is easy to compute as $\frac{1}{n} \cdot 1 + \frac{n-1}{n} \cdot 0 = \frac{1}{n}$.

So $\mathbb{E}[F] = n \cdot \frac{1}{n} = 1$.

Back to the main problem

Problem

You are dealt A♠ and 7♠. The flop comes out as 8♠, 6♠, 5♥. The pot is currently at 1000 chips. The player before you bets 500 chips. You have 5 seconds to act. Should you call or fold? (Assume you do not want to raise.) What if they bet 1000 chips? 2000 chips?

Some relevant information:

- Flushes are better than straights.
- A flush with an Ace is the best flush.

Main problem

Flop: 8♠, 6♠, 5♥; Your hand: A♠, 7♠

Pot: 1500; Bet: 500

Essentially, the situation boils down to:

- If a spade appears, you win.
- Otherwise, you will probably lose.

Question

What about the chance that your opponent gets quads or a full house?

This probability is low enough that we will ignore it — it requires insane luck.

Main problem

Earlier, we computed the probability of hitting a spade to be around 35%. Now, we compute the expected value of winnings based on the current pot and bet if you call (future plays are irrelevant):

- A spade appears with probability 0.35, bringing you $+1500$ chips.
- No spade appears with probability 0.65, bringing you -500 chips.

Thus your expected value of winnings is $0.35 \cdot 1500 - 0.65 \cdot 500 = 200$, so you expect to gain 200 chips. Given this, should you call?

Answer: **YES**

Main problem

Remember that expectation is linear. If you play this scenario out 100 times, you expect to gain $200 \cdot 100 = 20000$ chips. Furthermore,

Theorem (Law of Large Numbers)

Repeat a random process that outputs a number many times and average the outputs. As you repeat the process more and more times, the averages converge towards the expected value.

Because of this, you should (almost) always make a decision which leads to positive expected value.

Main problem

Now, let's solve the rest of the problem. First we tackle the bet of 2000.

Pot: 3000; Bet: 2000

- A spade appears with probability 0.35, bringing you $+3000$ chips.
- No spade appears with probability 0.65, bringing you -2000 chips.

Thus your expected value of winnings is $0.35 \cdot 3000 - 0.65 \cdot 2000 = -250$. So you should *not* call.

Main problem

What about the bet of 1000?

Pot: 2000; Bet: 1000

- A spade appears with probability 0.35, bringing you $+2000$ chips.
- No spade appears with probability 0.65, bringing you -1000 chips.

Thus your expected value of winnings is $0.35 \cdot 2000 - 0.65 \cdot 1000 = 50$. So by our previous discussion, you *should* call.

But is this really what you want to do? Your expected winnings is very small — it pales in comparison to the size of the amount you could lose.

Main problem

In situations like this, it is tough to make a solid decision. We call this situation **high variance** because compared to the size of the expected winnings, your potential winnings and potential losses are large.

In high variance situations, it is common to take less risks and fold instead, even though your expected winnings is technically positive.

Pot odds

The expected value calculation that we did here is an example of computing **pot odds**. If the current pot size is P and the current bet is B, we say that the pot odds are $P : B$ (as a ratio).

For example, the pot odds in the scenario where your opponent bet 500 are $1500 : 500 = 3 : 1$.

Why are pot odds important?

Proposition

Suppose that your probability of winning is p. The pot odds are $P : B$ when the betting comes to you. Then you have a positive expected value of winnings if $p > \frac{B}{P+B}$ and a negative expected value if $p < \frac{B}{P+B}$.

Proof.

See solution to main problem. □

Last remarks

Question

Suppose there are 9 players in a poker game. Further suppose that all 9 players are "equally skilled." What "should" be the winning probability of each player?

Answer: $\frac{1}{9}$

This means that in larger tables, you should be **folding pre-flop** in a lot of hands. Lots of analysis can be done to see which hands you should play and which you should fold.

Another takeaway from poker is that you can use probability to make informed and educated decisions. An example of this is Bayesian inference. Even though we did not go over a poker example, the same concept applies when you get more into the details of playing poker against opponents.

Last remarks

Thank you for coming to this lesson about poker probabilities!

I hope that you learned some facts about poker and probability today.

This presentation can be found at

`http://www.mit.edu/~shint/handouts/vSDMC/poker.pdf`

For any questions or comments, feel free to contact me at `shint@mit.edu`.