San Diego Online Contest

in Mathematics

May 24–29, 2020

If you have any questions, please email us at sandiegoonlinecontest@gmail.com.

Format

- The contest consists of **20 short-answer problems and 3 proof problems**.
- For the short-answer problems, only the answer to the problem is required. For the proof problems, a full proof with mathematical justification is required for full credit.
- The contest will be **scored out of** 100, with each short-answer problem being worth 3.5 points and each proof problem being worth 10 points. Partial credit will be awarded on proof problems using an Olympiad grading scale. There is no partial credit on short-answer problems. Ties will be broken by proof score, then the hardest problem solved on the short-answer section. Here, "hardest" refers to fewer solves (if that is not enough then the later numbered problem is harder).
- Submissions are due at 11:59 PM on May 29.

Rules

- Communication about the contest during the competition window is prohibited. This includes posting on online forums, private messaging people, and breaking shelter-in-place rules to meet up with someone and discuss.
- Computational aids, including calculators, computer programs, WolframAlpha, and Geogebra are prohibited. References, including print and electronic publications, are prohibited.
- Construction tools such as rulers, compasses, protractors, etc. are allowed.
- Writeups for the proof problems can be either handwritten or typed (e.g. using LaTeX or other text processors). In either case, the writeup must be legible.
  - If you use LaTeX to type solutions, we encourage you to use our template which can be found here: bit.ly/SDOC-prooftemplate
  - You are permitted to use software to draw diagrams, but the software must not provide a computational advantage.
- **As part of the submission process, you will be asked to agree that you did not violate any of these rules.**

Once again—if you have any questions, please email us!

Submission

To submit your answers and writeups, please do the following BEFORE 11:59 PM on May 29. **We encourage you to submit well before (30 minutes before) the deadline!**

- First, make sure that you have signed up for the contest. If you did not sign up via the Google Form but are still interested in participating, please email us.
- To submit answers for the short-answer problems, please fill out the Google Form at `bit.ly/SDOC-submission`
  - Submission instructions for the puzzles will be in the Google Form.
  - Follow ARML conventions when simplifying answers.
  - Your answer will be interpreted with order of operations.
  - Use parentheses to group things together.
  - Use sqrt for square root, and ˆ for exponentiation, e.g. 2sqrt(2) and 7ˆ(1/3).
  - Use / for fractions, e.g. 8/15.
  - Use , for ordered pairs, e.g. (3, 5).
  - Use pi for $\pi = 3.1415926\ldots$ and e for $e = 2.718281828\ldots$
  - If your answer is ambiguous, you may be marked incorrect as a default—avoid this by following these instructions!
  - In particular, don't submit 5/8pi. Instead, use whichever of 5pi/8 or 5/(8pi) is correct.
  - As another example, the expression 1+sqrt(5)/2 will be interpreted as $1 + \frac{\sqrt{5}}{2}$. If you intend to mean the golden ratio $\frac{1+\sqrt{5}}{2}$, you should write (1+sqrt(5))/2.
  - Any confusion—email us!
- In this Google Form, you will be asked to agree that you did not violate any of the rules in the contest, following an honor code.
- You will also be asked to state how many pages you used for each proof problem.
- To submit writeups for the proof problems, please email us at our contact email of `sandiegoonlinecontest@gmail.com`.
  - Your email should have the EXACT title "[YOURNAME] SDOC Proof Submission" where you replace "[YOURNAME]" with the name that you provided in the registration form.
  - You should attach your submissions in PDF form, doing either of the following:
    * Submit a single PDF with your submissions, in order.
    * Submit a separate PDF for each problem that you are submitting.
- Once you have completed all of these, you are set!

Short Answer Problems

Each problem in this section is worth 3.5 points.

Here is a printer-friendly version of Problems 9 and 10: bit.ly/SDOC-puzzles

If you are stuck on Problem 8, feel free to ask for a hint by emailing us at our contact email of sandiegoonlinecontest@gmail.com. The same hint will be given to every student who asks. We encourage you to first try solving Problem 8 without the hint, though!

1. Compute the smallest positive integer n such that $\frac{n}{2020}$ is a terminating decimal.

2. Compute the number of tuples (a, b, c) of integers with $1 \leq a, b, c \leq 2020$ such that $2^a + 2^b + 2^c$ is a power of 2.

3. Let $ABCDEF$ be a regular hexagon. If $\triangle ABE$ has area 14, compute the area of $\triangle ABC$.

4. The arithmetic mean of a and b is $\frac{a+b}{2}$ (the average). The harmonic mean of a and b is $\frac{2}{\frac{1}{a}+\frac{1}{b}}$ (the reciprocal of the average of the reciprocals). There is a unique positive integer n such that n and 5 are distinct positive integers whose arithmetic mean and harmonic mean are both integers. Compute n.

5. If it rains today, there is a 50% chance that it will rain tomorrow. Otherwise, there is only a 20% chance. If it rains on a day, there is a 70% chance that the wind that day will be over 20 mph. Otherwise, there is only a 20% chance.

   Suppose that there is a 50% chance that the wind will be over 20 mph today. Compute the probability that the wind will be over 20 mph tomorrow.

6. Define a sequence by $a_1 = 2$ and $a_{k+1} = a_k^{a_k}$ for all positive integers k. Compute the smallest positive integer n such that a_n has at least 10^{700} digits.

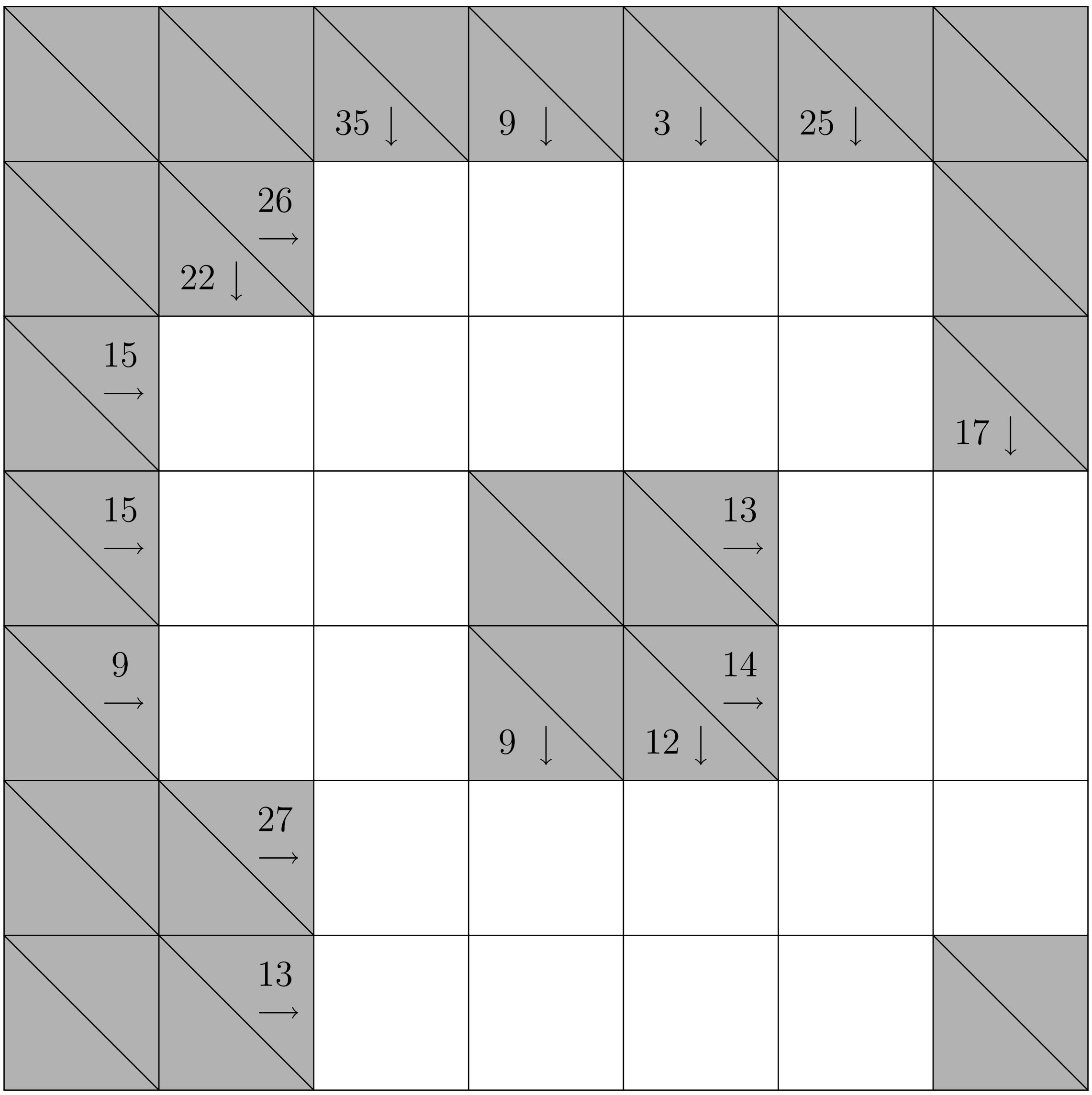
7. Rectangle $\mathcal{R}$ is inscribed in triangle ABC such that $\mathcal{R}$ has two vertices on side BC and one on each of the other two sides. Triangle ABC is partitioned into four distinct sections by $\mathcal{R}$. Three of the sections are triangles with areas of 5, 7, and 18 in some order. Compute the largest possible area of $\mathcal{R}$.

8. There are 150 pairs of twins that attend the South Dakota Oak College. On National Twin Day, all of these twins gather in the gymnasium and are seated (in a uniformly random order) in a circle with 300 chairs. A prize is given to each pair of twins that sit next to each other. Compute the expected value of the number of pairs of twins that win a prize.

9. Fill in the following (currently blank) grid so that:

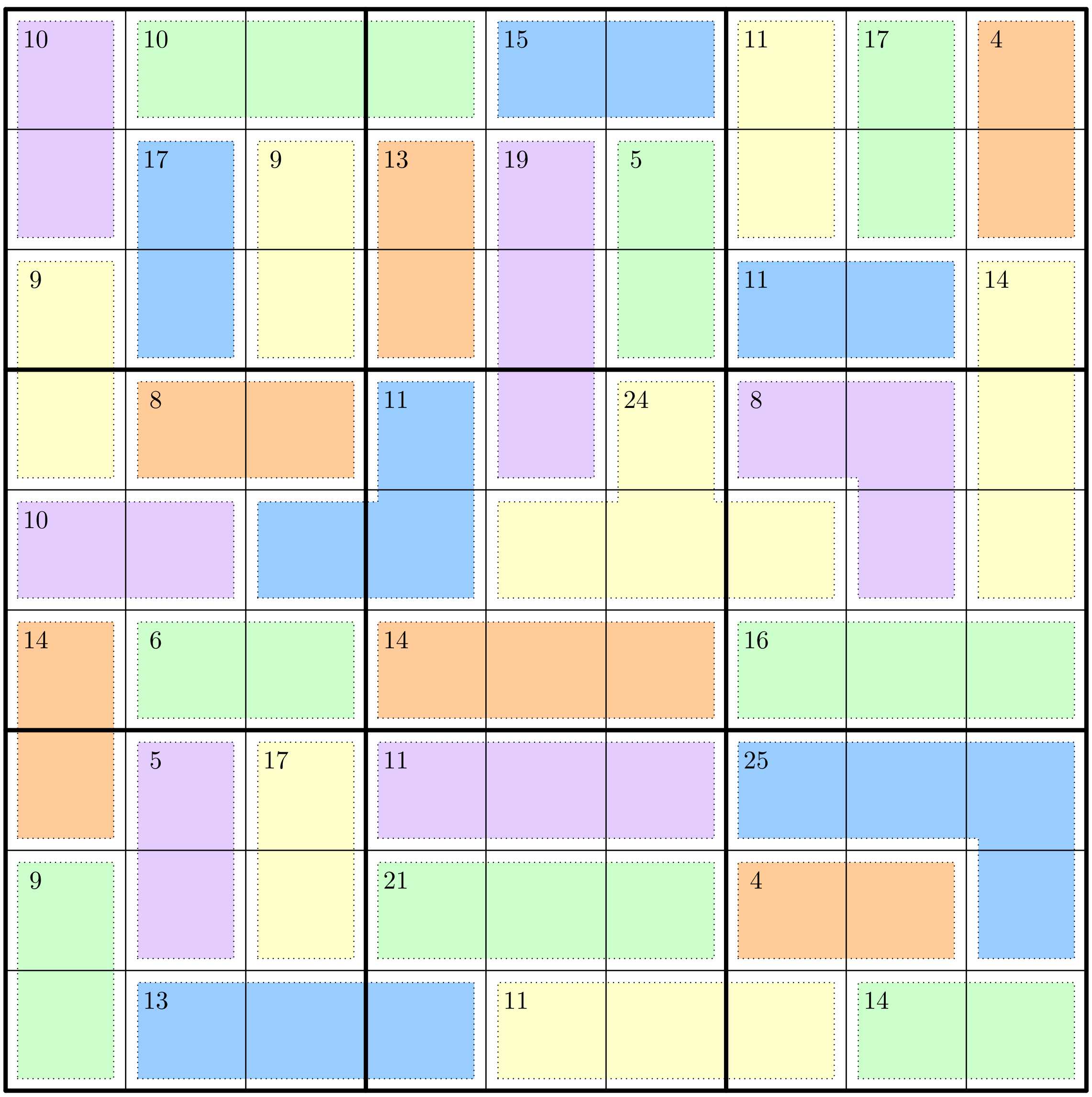
- Each box contains a single digit in the set $\{1, \ldots, 9\}$.
- A ***segment*** is the set of boxes in a row or column from one black box to another, with no black boxes in between. There are 16 segments in this grid.

  If two boxes are in the same segment, then they contain different digits.
- The digits in each segment sum to the number labelling the segment.



10. Fill in the following (currently blank) grid so that:
    - Each box contains a single digit in the set $\{1, \ldots, 9\}$.
    - Each of the nine rows contain each digit exactly once. Each of the nine columns contain each digit exactly once.
    - Each of the nine 3×3 squares (marked by the darker lines) contain each digit exactly once.
    - In each dashed shape, no two boxes contain the same digit. Furthermore, the digits in each dashed shape sum to the number labelling the shape.

    The colors are only provided for convenience of distinguishing dashed shapes (they do not indicate anything; dashed shapes of the same color have no relationship).



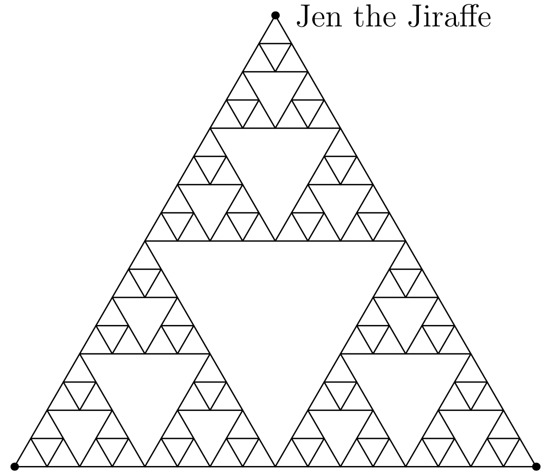
11. Let $f\colon \{1,\dots,2020\} \to \{1,\dots,2020\}$ such that for all $x, y, z \in \{1,\dots,2020\}$ with $x+y-z$ divisible by 2020,

$$2xy + f(x) + f(y) - f(z) \quad \text{is divisible by 2020.}$$

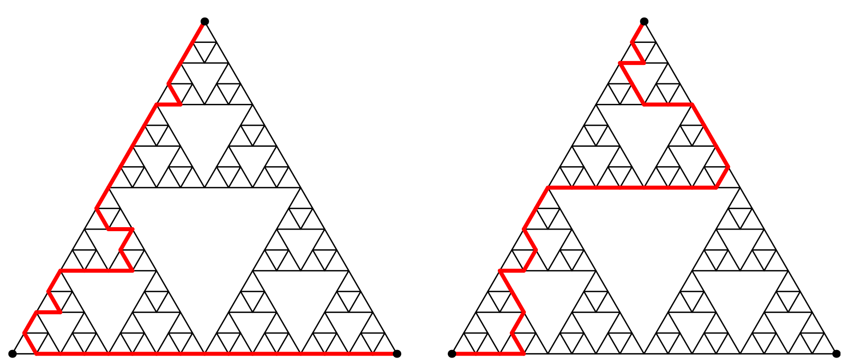
Compute the maximum possible value of $f(150)$.

12. Let ABC be a triangle in which the angle bisector of $\angle BAC$ meets side BC at D. Suppose that the distances from A to the incenters of $\triangle ABD$, $\triangle ACD$, and $\triangle ABC$ are 16, 18, and 20, not necessarily in order. Compute AD.

13. Below is the fourth iteration of Sierpiński's triangle with side length 16 units. Jen the Jiraffe starts at the top corner of the triangle and is trying to get to either of the two bottom corners.



Every second, she travels 1 unit along one of the segments in the diagram. She will only travel downwards, unless there is no option to travel downwards. In this case, she will either move left until she can travel downwards again, or move right until she can travel downwards again. She stops when she reaches either of the two bottom corners. Here are some possible paths that Jen the Jiraffe could take:



Compute the number of ways there are for Jen the Jiraffe to reach either of the two bottom corners.

14. Bill the Belociraptor is at the top corner of the fourth iteration of Sierpiński's triangle with side length 16 and trying to get to either of the two bottom corners (see the previous problem for a diagram). However, he has no sense of direction. Every second, he uniformly randomly chooses a segment at the intersection that he is at (possibly even the segment which he just travelled on) and travels 1 unit along the segment. Compute the expected value of the number of seconds that it will take Bill the Belociraptor to reach either of the two bottom corners.

15. Let n be a positive integer. Define $\varphi(n)$ to be the number of positive integers in $\{1, \ldots, n\}$ that are relatively prime to n and $\sigma(n)$ to be the sum of all positive integers that divide n. Compute the **number of** positive integers $n < 100$ for which

$$\varphi(n) + \sigma(n) \leq 2n + 1.$$

16. There is a unique polynomial P with degree 8 such that

$$P\left(\frac{1}{n}\right) = \frac{1}{n+1} \quad \text{for } n = 1, 2, \ldots, 9.$$

Given that P has no double roots, compute the sum of the reciprocals of the roots of P.

17. Let ABC be an acute triangle. The ***A-excircle*** is the unique circle that is tangent to all three sides of $\triangle ABC$ but is not tangent to segments AB and AC. Suppose that the A-excircle is centered at I_A and touches side BC at D. If $AD = 9$, $DI_A = 8$, and $AI_A = 15$, compute the product $BC \cdot CA \cdot AB$.

18. Triangle ABC has side lengths $AB = 7, BC = 8, CA = 9$. Points J and K lie on side BC such that J lies on the incircle of $\triangle ABC$ and $\angle BAJ = \angle CAK$. The circumcircle of $\triangle AJK$ meets sides AB and AC again at X and Y respectively. Line JY meets line AB at M, and line KX meets line AC at N. Line MN meets line BC at P. If the altitudes from B and C meet their opposite sides at E and F respectively, compute the area of $\triangle PEF$.

19. Compute the **sum of** all prime numbers $p < 100$ for which there exist positive integers a, b such that

$$\begin{aligned} &a^6 - 45a^4b^2 + 135a^2b^4 - 27b^6 - a^3 + 9ab^2 + 1 \quad \text{and} \\ &6a^5b - 60a^3b^3 + 54ab^5 - 3a^2b + 3b^3 \end{aligned}$$

are both divisible by p.

20. Compute

$$\lim_{n\to\infty} \sqrt{n} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin^{-1}\left(\sqrt{x + \frac{1}{2}}\right) (1 - x^2)^n \, \mathrm{d}x.$$

You may use the fact that $\lim\limits_{n\to\infty} \frac{\sqrt{n}}{4^n} \cdot \binom{2n}{n} = \frac{1}{\sqrt{\pi}}$.

Proof Problems

Each problem in this section is worth 10 points.

For the following proof problems, you will submit writeups for these problems rather than answers. Writeups for complete solutions should be mathematically rigorous. Writeups can also be for incomplete solutions; you should write up the progress that you made if you did not solve the problem.

If you are stuck on Problem 21, feel free to ask for a hint by emailing us at our contact email of `sandiegoonlinecontest@gmail.com`. The same hint will be given to every student who asks. We encourage you to first try solving Problem 21 without the hint, though!

21. A positive integer is called ***sweet*** if it can be written as the sum of four positive integers, each pair of which share a common divisor besides 1. If a positive integer is not sweet, then it is ***bitter***. Determine, with proof, all bitter integers.

22. Let ABC be a triangle with incircle ω and circumcircle Γ. Define I to be the center of ω and D, E, F to be the points where ω touches BC, CA, AB respectively. Let line AI meet BC at J and Γ again at M, and let N be the point diametrically opposite M on Γ. Suppose that K is a point on ID such that JK is parallel to MD. Let the perpendicular to AI passing through K intersect IN at Q. Prove that $DQ \perp EF$.

23. For a set A of positive integers, let $D(A)$ be the number of distinct positive integers that can be written as the sum of two (not necessarily distinct) elements of A. Also let $E(A)$ be the number of tuples (a, b, c, d) of elements of A such that $a + b = c + d$. For example, if $A = \{1, 2, 4, 5, 7, 8\}$, then $D(A) = 15$ and $E(A) = 114$.

    Let S be a set of 6 positive integers such that $D(S) \leq 18$. Determine, with proof, the minimum possible value of $E(S)$.